

CATEGORICAL  $q$ -DEFORMED RATIONAL  
NUMBERS & COMPACTIFICATIONS OF  
STABILITY SPACE

Asilata Bapat (ANU)

+      Louis Becker  
Anand Deopurkar  
Anthony Licata

## The big-picture

$$B_r \subset V \xrightarrow{\text{categorify}} B_r \subset \mathcal{E}$$

## The big-picture

$$B_r \subset V_r \xrightarrow{\text{categorify}} B_r \subset \mathcal{E}_r \xleftarrow{\text{(triangulated)}}$$

## The big-picture

$$B_r \subset V_r \xrightarrow{\text{categorify}} B_r \subset \mathcal{E}_r \xleftarrow{\text{(triangulated)}}$$

$$\begin{matrix} \text{Stab } \mathcal{E}_r \\ \cup \\ B_r \end{matrix} \xrightarrow{\text{compactify}} \overline{\text{Stab } \mathcal{E}_r} \begin{matrix} \cup \\ B_r \end{matrix}$$

## The big-picture

$$B_r \subset V_r \xrightarrow{\text{categorify}} B_r \subset \mathcal{E}_r \xleftarrow{\text{(triangulated)}}$$

$$\begin{matrix} \text{Stab } \mathcal{E}_r \\ \cup \\ B_r \end{matrix} \xrightarrow{\text{compactify}} \begin{matrix} \overline{\text{Stab}}^3 \mathcal{E}_r \\ \cup \\ B_r \end{matrix}$$

Q: What is the topology of  $\text{Stab } \mathcal{E}_r$ ?

Q: What can we read off about  $B_r$  from its action on  $\overline{\text{Stab}}^3 \mathcal{E}_r$ ?

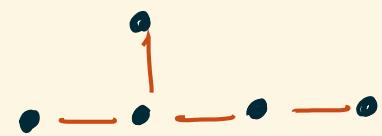
## Tensor categories (aside)

Unfortunately, not really an ingredient  
in this talk!

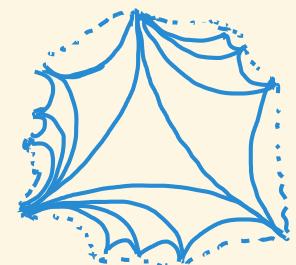
However, if  $\mathcal{C}$  carries an action by a  
fusion category  $\mathcal{F}$ , it is fruitful to  
consider stability conditions respecting  
the action of  $\mathcal{F}$  [e.g. work of E. Henry]

## Plan

① Generalities on  $C_r$ , Stab,  
and the  $B_r$ -action



② The family of compactifications



③ The three strand braid group



## Categorical $B_r$ action

$\mathcal{C}_r = 2\text{-CY category of connected graph } \Gamma$

[categorifies Burau rep of  $B_r$ ]

## Categorical $B_r$ action

$\mathcal{C}_r = 2\text{-CY category of connected graph } \Gamma$

[categorifies Burau rep of  $B_r$ ]

Constructed via zig-zag algebra of  $\Gamma$   
(a quotient of path algebra of  $\Gamma^{\text{dbr}}$ ),  
considered as dga.

$\mathcal{C}_r = \text{homotopy category of projective modules}$

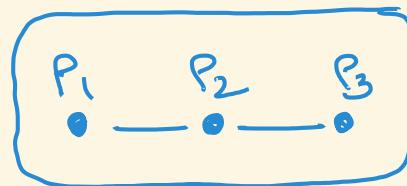
## Categorical $B_r$ action

$\mathcal{C}_r = 2\text{-CY category of connected graph } \Gamma$

[categorifies Burau rep of  $B_r$ ]

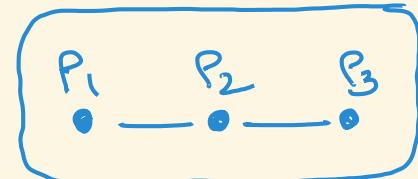
### Important features:

- $\mathcal{C}_r = \langle P_i \mid i \text{ vertex} \rangle$
- Lots of spherical objects  
 $\Rightarrow$  lots of auto-equivalences.



## Categorical $B_r$ action

In particular, each  $P_i$  is spherical.



- $\sigma_{P_i} : \mathcal{C}^* \mathcal{L}_r$  is an autoequivalence;
  - $\sigma_{P_i}$  satisfy the braid relations. (of  $r$ )
- $\Rightarrow B_r \in \mathcal{C}^* \mathcal{L}_r$  (and yields Burau rep on Grothendieck group)

## Bridgeland stability conditions & $B_r$ -action

A stability condition  $\tau$  is data on  $\mathcal{C}_r$  that yields a family of metrics on  $\mathcal{C}_r$ : each arrow in  $\mathcal{C}_r$  has a  $(\tau, g)$ -length.

## Bridgeland stability conditions & $B_r$ -action

A stability condition  $\tau$  is data on  $\mathcal{C}_r$  that yields a family of metrics on  $\mathcal{C}_r$ : each arrow in  $\mathcal{C}_r$  has a  $(\tau, g)$ -length.

The size of  $X \in \text{ob } \mathcal{C}_r$  is measured by "pulling tight to a geodesic"  $0 \rightarrow X$ .

## Bridgeland stability conditions & $B_T$ -action

The size of  $X \in \text{ob } \mathcal{C}_T$  is measured by  
"pulling tight to a geodesic"  $0 \rightarrow X$ .

This is called the " $q$ -mass" of  $X$  wrt  $T$ .

## Bridgeland stability conditions & $B_\Gamma$ -action

More precisely,  $\tau$  yields, for any  $X \in \text{ob } \mathcal{C}_\Gamma$ ,  
a canonical Harder-Narasimhan filtration

$$0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n = X$$
$$\begin{matrix} \tau_{\leq i} \downarrow & \tau_{\leq i+1} \downarrow & & \tau_{\leq n+1} \downarrow \\ A_1 & A_2 & & A_n \end{matrix}$$

with  $\tau$ -semistable pieces  $A_i$

## Bridgeland stability conditions & $B_\tau$ -action

More precisely,  $\tau$  yields, for any  $X \in \text{ob } \mathcal{C}_\tau$ ,  
a canonical Harder-Narasimhan filtration

$$0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n = X$$
$$\begin{matrix} \tau_{\leq i} \downarrow & \tau_{\leq i+1} \downarrow & & \tau_{\leq n+1} \downarrow \\ A_1 & A_2 & & A_n \end{matrix}$$

with  $\tau$ -semistable pieces  $A_i$

Each semistable  $A$  has a modulus  $|A|_\tau \in \mathbb{R}_{>0}$   
and a phase  $\phi_\tau(A) \in \mathbb{R}$ .

## Bridgeland stability conditions & Br-action

The size of  $X \in \text{ob } \mathcal{C}_\tau$  is measured by "pulling tight to a geodesic"  $0 \rightarrow X$ .

This is called the "q-mass" of  $X$  wrt  $\tau$ .

$$0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n = X$$

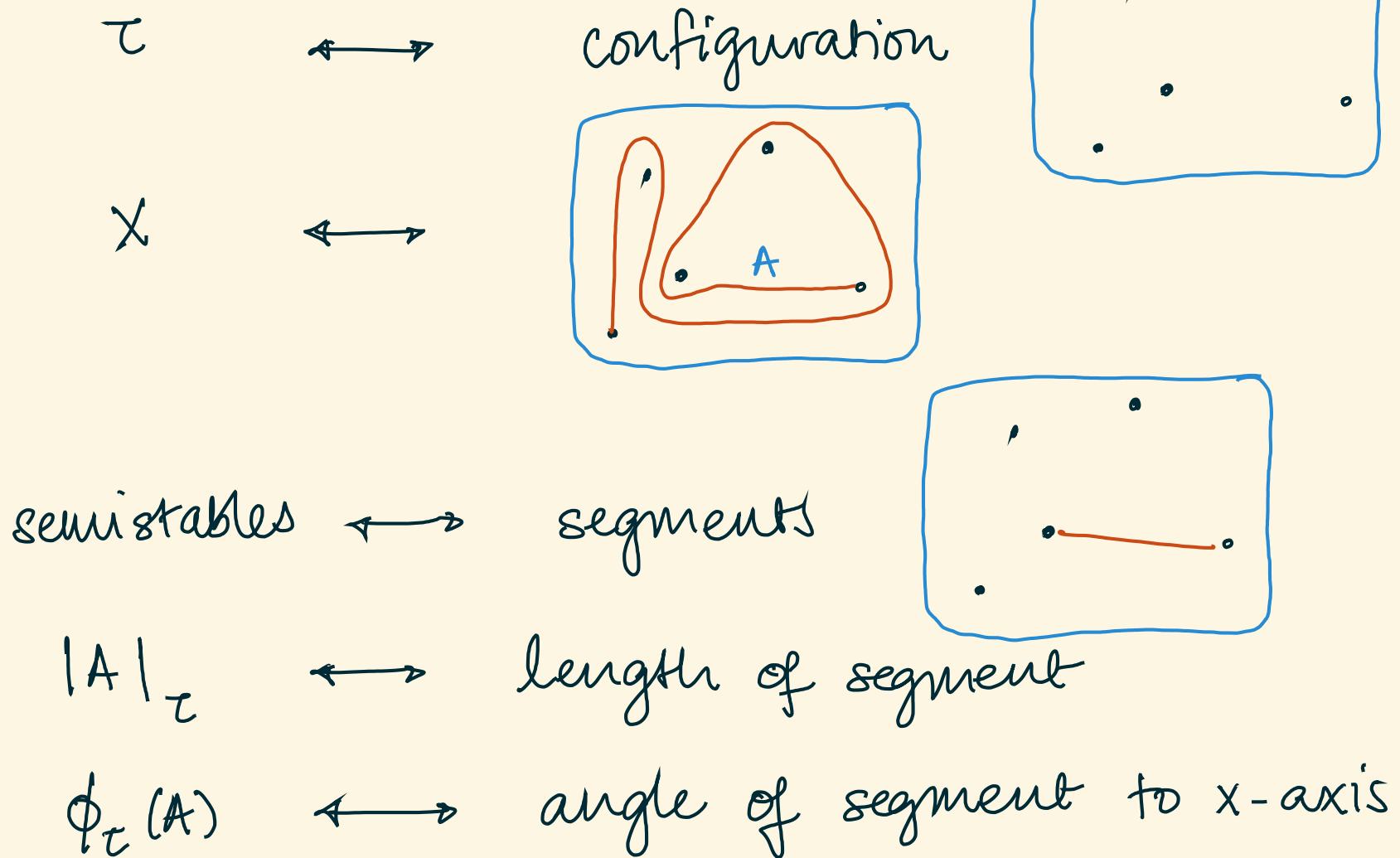
$\begin{matrix} \uparrow \tau_{11} & \uparrow \tau_{21} & & \downarrow \tau_{n1} \\ A_1 & A_2 & & A_n \end{matrix}$

$\Rightarrow$

$$\begin{aligned} m_{q, \tau}(X) := \\ \sum q^{\phi_\tau(A_i)} \cdot |A_i|_\tau \end{aligned}$$

# Bridgeland stability conditions & Br-action

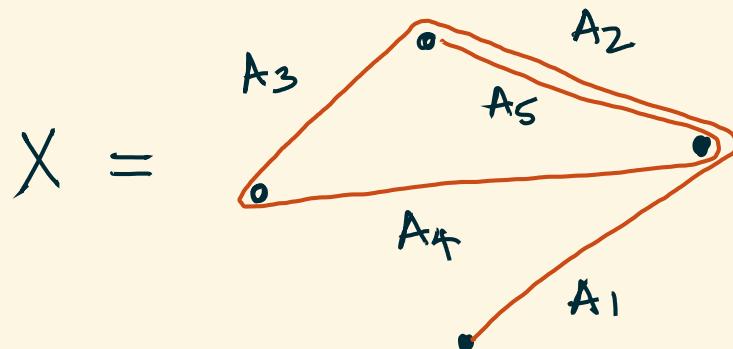
Illustration (accurate in type A!)



## Bridgeland stability conditions & Br-action

The size of  $X \in \text{ob } \mathcal{C}_\tau$  is measured by "pulling tight to a geodesic"  $0 \rightarrow X$ .

This is called the "q-mass" of  $X$  wrt  $\tau$ .



$$m_{q, \tau}(X) := \sum q^{\phi_\tau(A_i)} \cdot |A_i|_\tau$$

## Bridgeland stability conditions & $B_r$ -action

[Bridgeland]  $\text{Stab } \mathcal{C}_r$  is a complex manifold.

Since  $B_r \subset \mathcal{C}_r$ , we also have

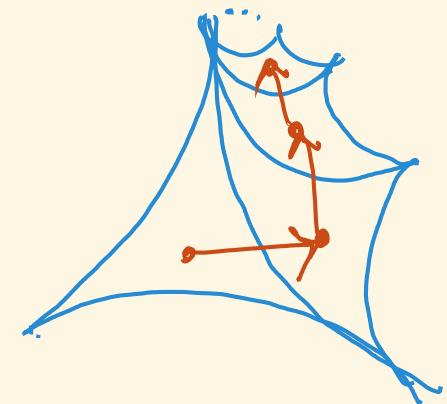
$$B_r \subset \text{Stab } \mathcal{C}_r.$$

Limiting operations on  $\text{Stab } \Phi_r$

## Limiting operations on $\text{Stab } \mathcal{C}_r$

① Fix  $\beta \in B_r$  and  $\tau \in \text{Stab } \mathcal{C}_r$ .

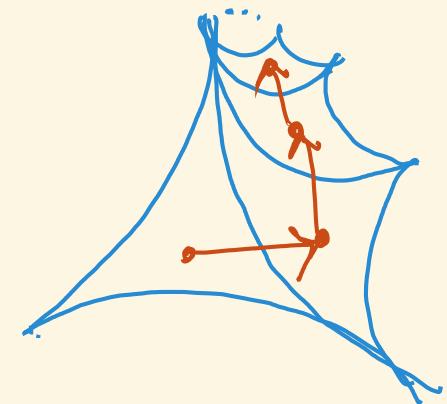
Consider  $\lim_{n \rightarrow \infty} \beta^n \tau$ .



## Limiting operations on $\text{Stab } \mathcal{C}_r$

① Fix  $\beta \in B_r$  and  $\tau \in \text{Stab } \mathcal{C}_r$ .

Consider  $\lim_{n \rightarrow \infty} \beta^n \tau$ .



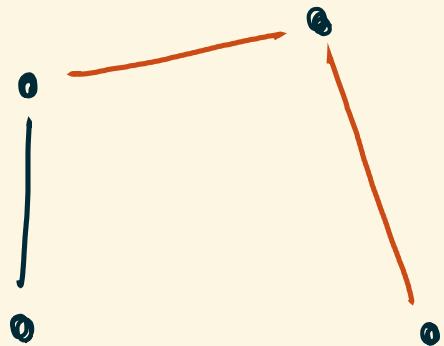
[BDL, BBL] Taking  $\beta = \delta_x$  for  $x$  spherical :

$$\lim_{n \rightarrow \infty} m_{\beta^n \tau, q} (Y) = q\text{-dim Hom}(X, Y)$$

up to simultaneous scalar

## Limiting operations on $\text{Stab } \mathfrak{t}_r$

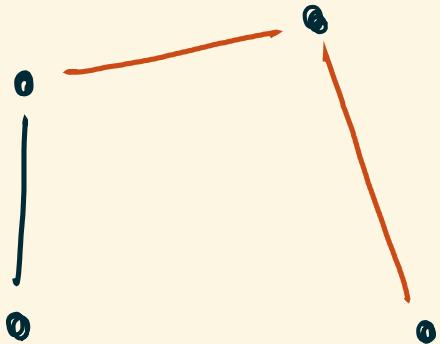
②



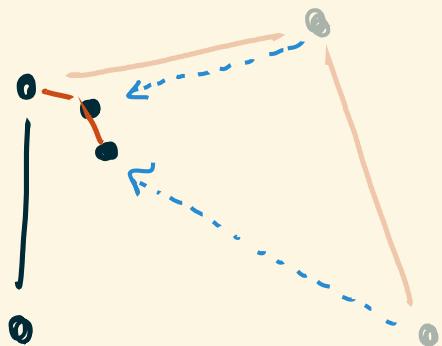
Shrink all but  
one of the simple  
semistables to zero

## Limiting operations on $\text{Stab } \mathfrak{t}_r$

②



Shrink all but  
one of the simple  
semistables to zero



In the limit, the  
 $q$ -mass counts the  
“ $q$ -occurrences” of the  
remaining semistable  
in any given object.

## Limiting operations on $\text{Stab } \mathcal{C}_r$

Moral : Limits may not make sense as stability conditions, but their q-masses make sense.

## Limiting operations on $\text{Stab } \mathcal{C}_r$

Moral : Limits may not make sense as stability conditions, but their q-masses make sense.

## Mass map

$$\begin{aligned} \text{Stab } \mathcal{C}_r &\longrightarrow \mathbb{P} \mathbb{R}^S \\ \tau &\longmapsto [x \mapsto m_{q,\tau}(x)]/\sim \end{aligned}$$

## Mass map & compactification

$$\begin{aligned} \text{Stab } \mathcal{C}_r &\longrightarrow \mathbb{P} \mathbb{R}^S \\ \tau &\longmapsto [x \mapsto m_{q,\tau}(x)]/\sim \end{aligned}$$

- [BDL, BBL] The mass map is injective, and  $\overline{\text{Stab}}^g \mathcal{C}_r$  is compact.

## Mass map & compactification

- [BDL, BBL] The mass map is injective, and  $\overline{\text{Stab}}_{\mathcal{C}_r}^q$  is compact.
- In the boundary, we see :

$$\text{hom} := \lim_{n \rightarrow \infty} M_{\beta^n, q} \text{ for } \beta = \text{spherical twist}$$

occ :=  $q$ -occurrences of a fixed semistable

## General conjectures & questions

Q:  $\overline{\text{Stab}}^g \mathcal{C}_r \simeq$  closed ball ?

Q: how & occ [+ linear combinations]  
recover a dense subset of the boundary  
sphere ?

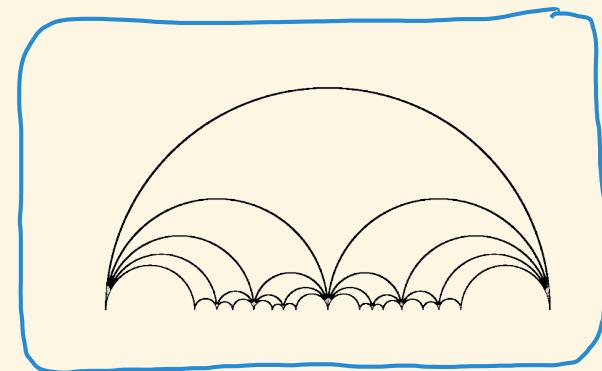
## General conjectures & questions

Q:  $\overline{\text{Stab}}^g \mathcal{C}_\Gamma \simeq$  closed ball ?

Q: how & occ [+ linear combinations]  
recover a dense subset of the boundary  
sphere?

Q: What does this tell us about  $B_\Gamma$ ?  
What are the other points on the boundary?

# The story of the 3-strand braid group

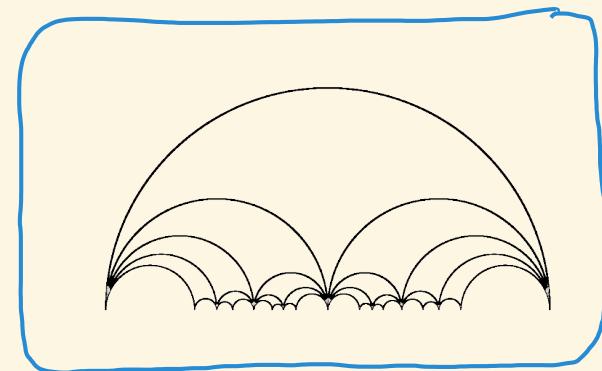


# The story of the 3-strand braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$$

$$B_3 \rightarrow PSL_2(\mathbb{Z})$$

$$\sigma_1 \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \mapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

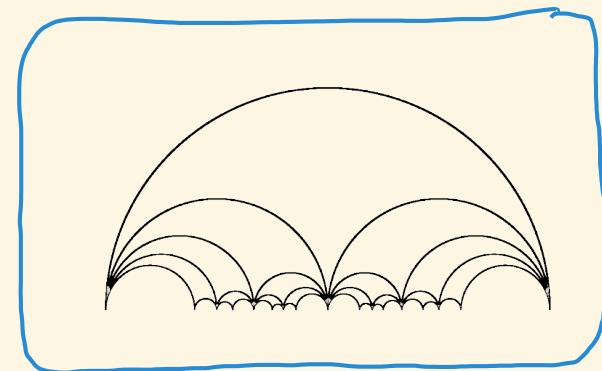


# The story of the 3-strand braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$$

$$B_3 \rightarrow PSL_2(\mathbb{Z})$$

$$\sigma_1 \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \mapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



- $PSL_2(\mathbb{Z})$ , and hence  $B_3$ , acts on  $\mathbb{C} \cup \{\infty\}$  by fractional linear transformations
- Action preserves  $\mathbb{H}$  and  $\mathbb{R} \cup \{\infty\}$

## The story of the 3-strand braid group

For the remainder of the talk, take

$$\mathcal{C} = \mathcal{C}(\bullet - \bullet) = \langle P_1, P_2 \rangle \circlearrowleft B_3$$

Fact :

$$\begin{array}{ccc} \text{Stab } \mathcal{C} & \simeq & \mathbb{H} \\ \mathcal{C} & & \mathcal{O} \\ B_3 & & B_3 \text{ via } \mathrm{PSL}_2(\mathbb{Z}) \end{array}$$

## The story of the 3-strand braid group

Take  $\mathcal{C} = \mathcal{C}(\bullet - \bullet) = \langle P_1, P_2 \rangle \wr B_3$

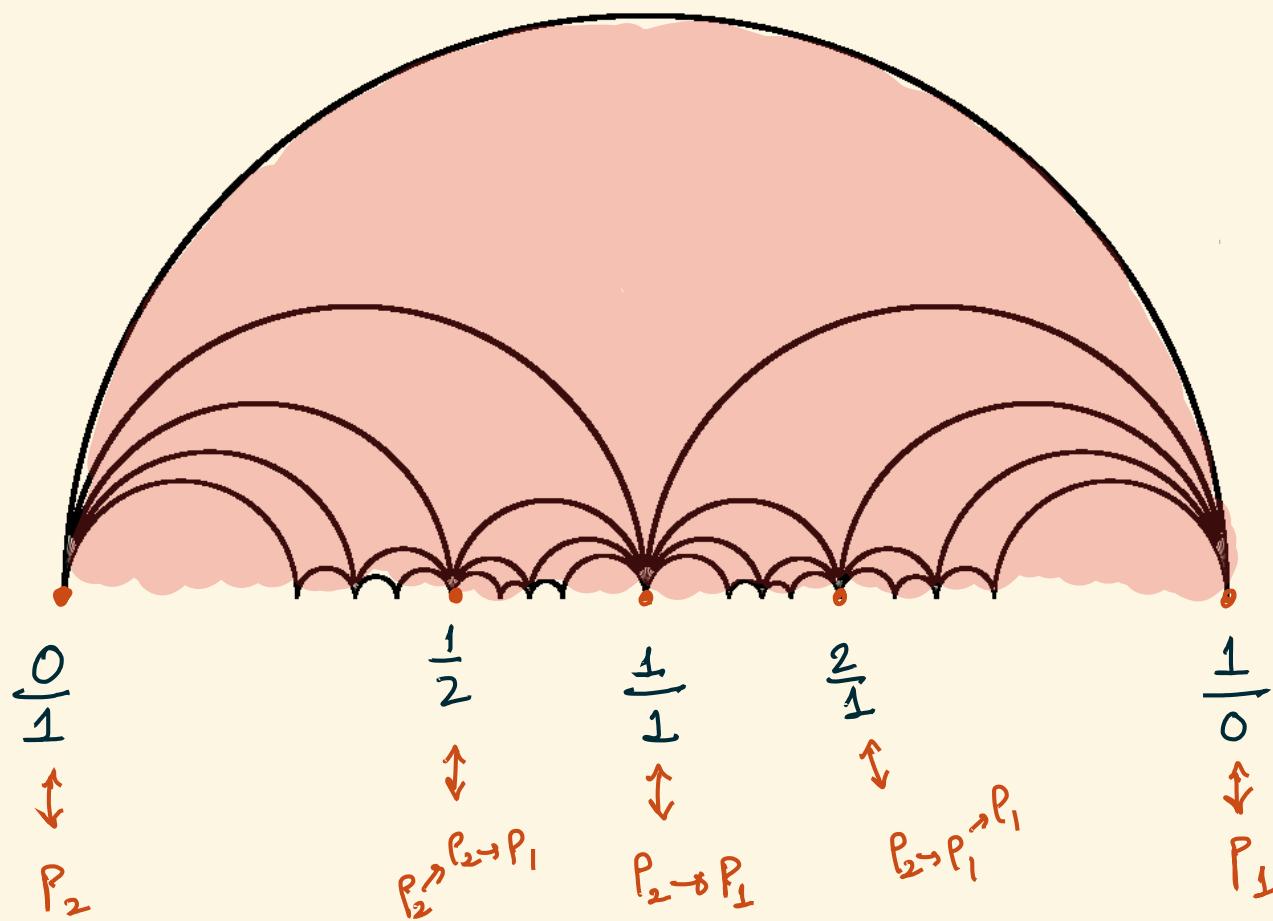
Thm [BDL]: For  $q=1$ :

- ①  $\overline{\text{hom}}$  and  $\text{occ}$  coincide.
- ②  $\overleftarrow{\text{hom}}_X \mapsto \pm \frac{\overline{\text{hom}}(X, P_2)}{\overline{\text{hom}}(X, P_1)}$  is a

$B_3$ -equivariant bijection from the spherical objects of  $\mathcal{C}$  to  $\mathbb{Q} \cup \{\infty\}$

## The $\overline{\text{hom}}$ functionals as rationals

Pictorially, at  $g=1$ :



## The $q$ -deformed story for $B_3$

Thm [BBL] For an indeterminate  $q$ :

$$\textcircled{1} \quad \overline{\hom}_X \mapsto \pm q^{\epsilon} \begin{matrix} \overline{\hom}_q(X, P_2) \\ \overline{\hom}_q(X, P_1) \end{matrix} \quad \text{and}$$

$$\text{occ}_X \mapsto \pm q^{\epsilon} \frac{\text{occ}(P_2, X)}{\text{occ}(P_1, X)} \quad \text{are } B_3\text{-equivariant.}$$

## The $q$ -deformed story for $B_3$

Thm [BBL] For an indeterminate  $q$ :

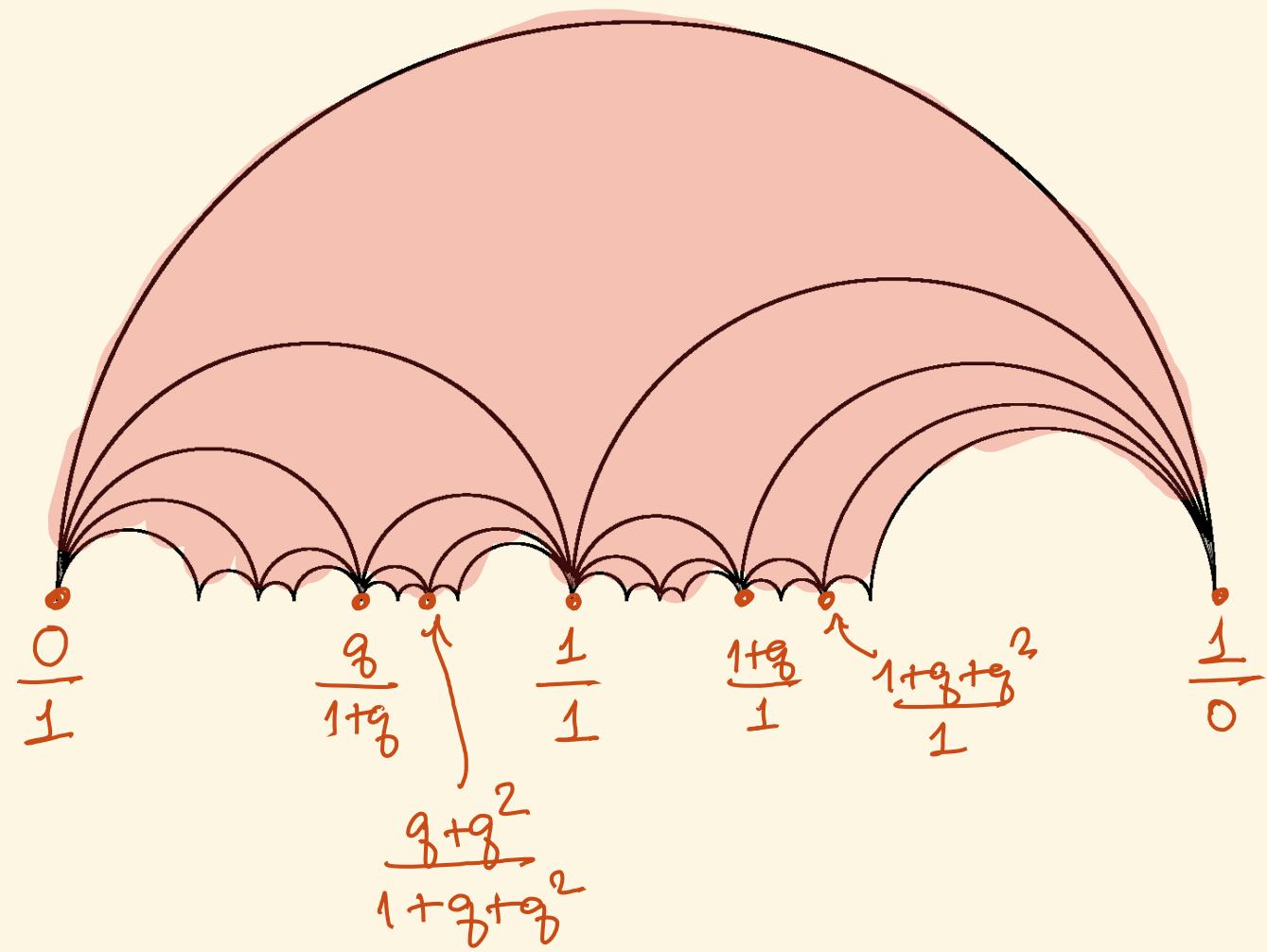
$$\textcircled{1} \quad \overline{\hom}_X \mapsto \pm q^{\epsilon} \begin{matrix} \overline{\hom}_q(X, P_2) \\ \overline{\hom}_q(X, P_1) \end{matrix} \quad \text{and}$$

$$\text{occ}_X \mapsto \pm q^{\epsilon} \frac{\text{occ}(P_2, X)}{\text{occ}(P_1, X)} \quad \text{are } B_3\text{-equivariant.}$$

The  $B_3$ -action on the right is by fractional linear transformations via Burau matrices.

## The $q$ -deformed story for $B_3$

Pictorially, at  $q \neq 1$ :



## The $q$ -deformed story for $B_3$

Thm [cont'd]

②  $\pm q^{(1)} \frac{\text{occ}(P_2, x)}{\text{occ}(P_1, x)}$  are exactly the  $q$ -deformed rationals of Morier-Genoud - Ovsienko.

③  $\pm q^{(1)} \frac{\overline{\text{hom}}(x, P_2)}{\overline{\text{hom}}(x, P_1)}$  give a new  $q$ -deformation of  $\mathbb{Q} \cup \{\infty\}$ .

## The $q$ -deformed story for $B_3$

For  $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$  corresponding to the spherical object  $X$ , set :

$$\textcircled{1} \quad \left[ \frac{r}{s} \right]_q^{\#} := \pm q^{l'} \frac{\text{occ}(P_2, X)}{\text{occ}(P_1, X)} \quad \begin{matrix} \text{right } q\text{-deformed} \\ \text{rational} \end{matrix}$$

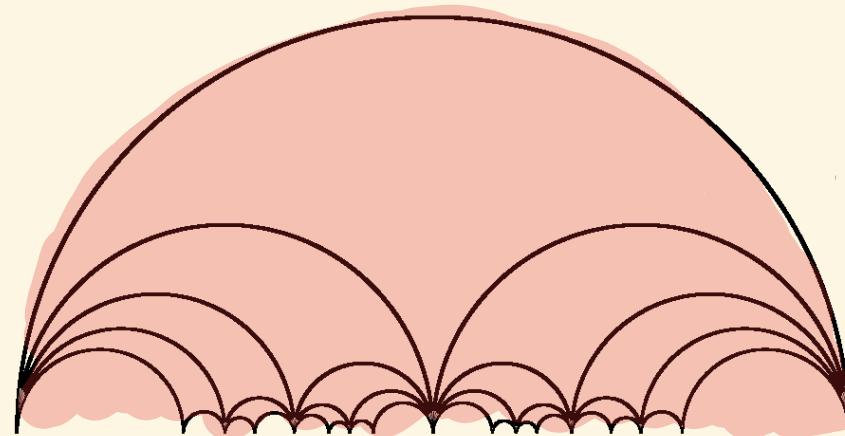
$$\textcircled{2} \quad \left[ \frac{r}{s} \right]_q^b := \pm q^{l'} \frac{\overline{\text{hom}}(X, P_2)}{\overline{\text{hom}}(X, P_1)} \quad \begin{matrix} \text{left } q\text{-deformed} \\ \text{rational} \end{matrix}$$

Specialising  $q_0$

Now fix  $0 < q < 1$ .

Consider the ideal triangle with vertices  $0, 1, \infty$ .  
[corresponds to a piece of stability space]

The  $\text{PSL}_2(\mathbb{Z})$ -orbit:



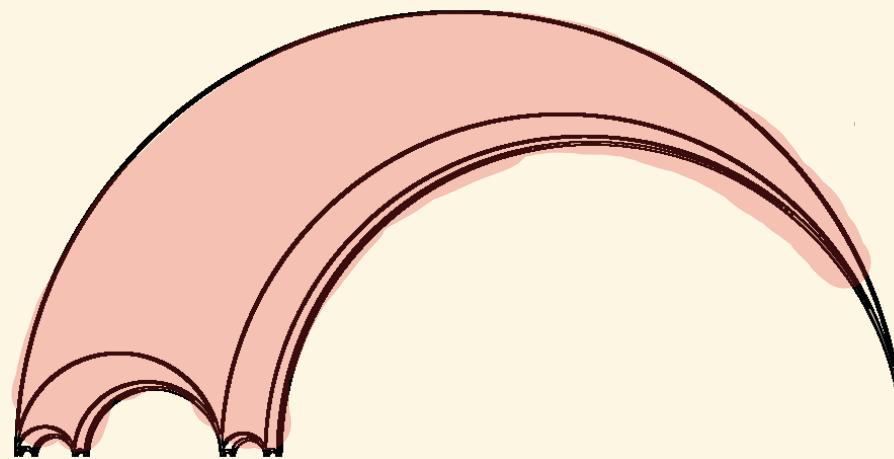
$[q_0 = 1]$

Specialising  $q$

Now fix  $0 < q < 1$ .

Consider the ideal triangle with vertices  $0, 1, \infty$ .

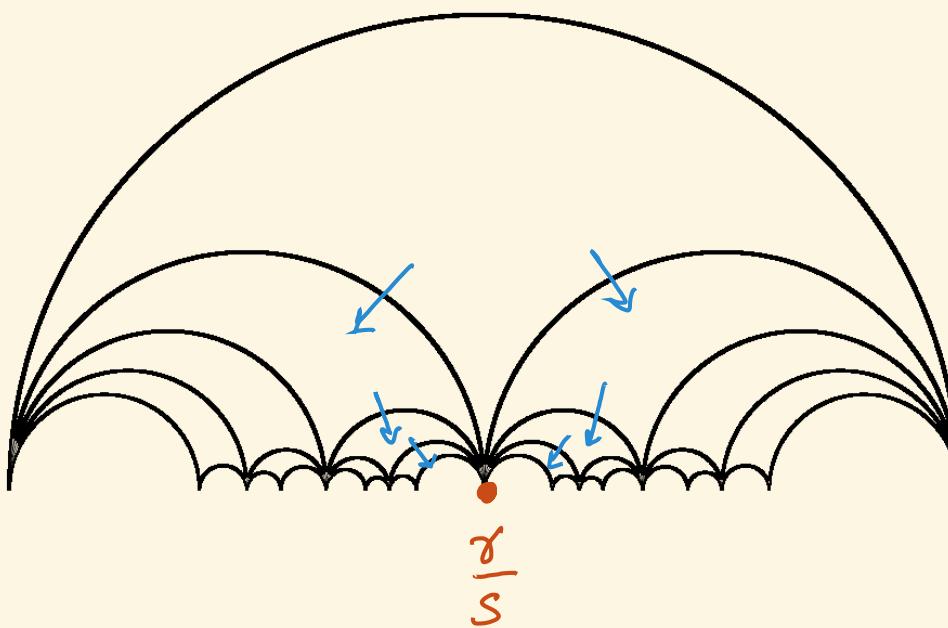
The  $\text{PSL}_{2,q}(\mathbb{Z})$ -orbit:



[ $q = 0.3$ ]

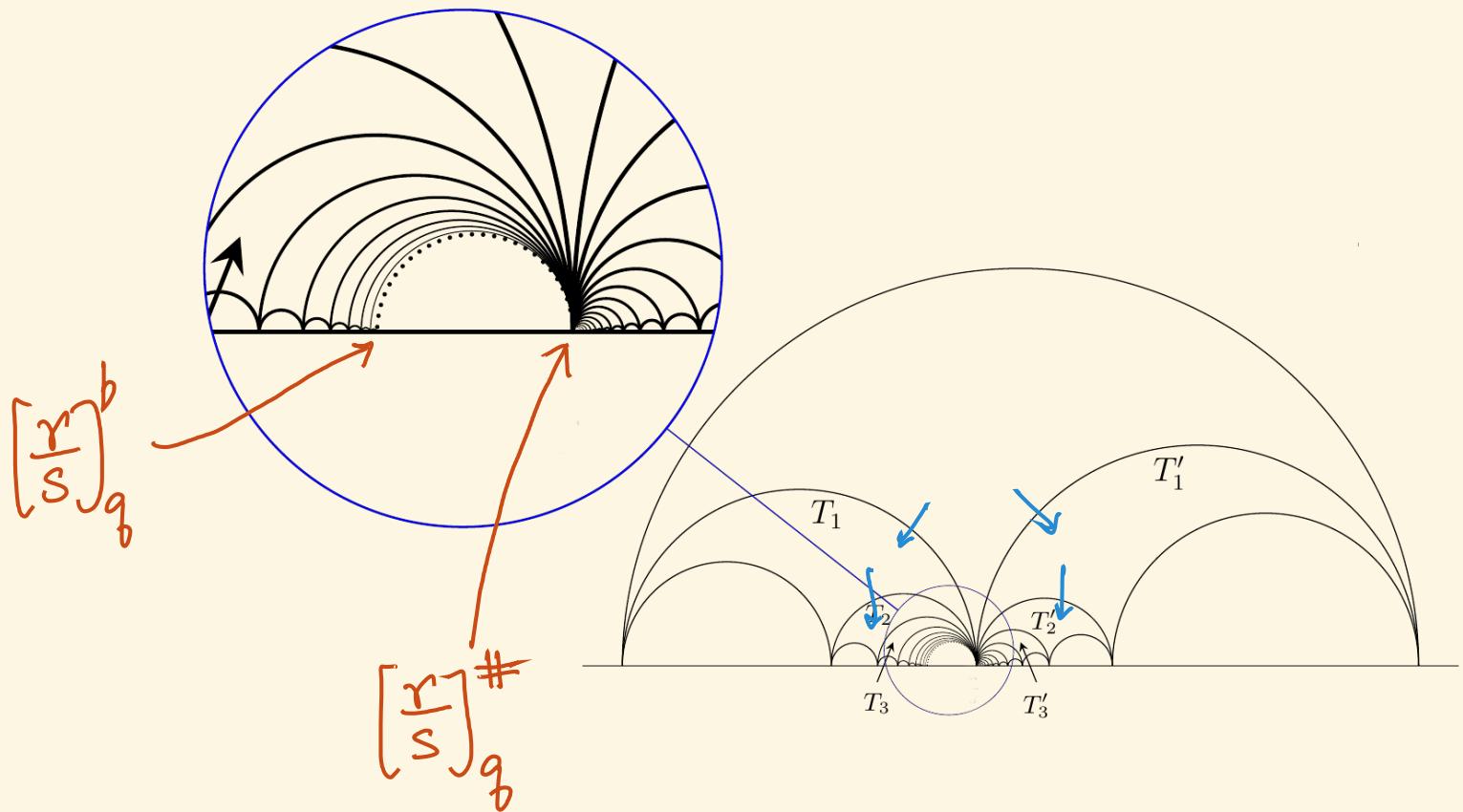
## Specialising $q$

At  $q=1$ , left & right limits of Farey triangles agree.



## Specialising $g$

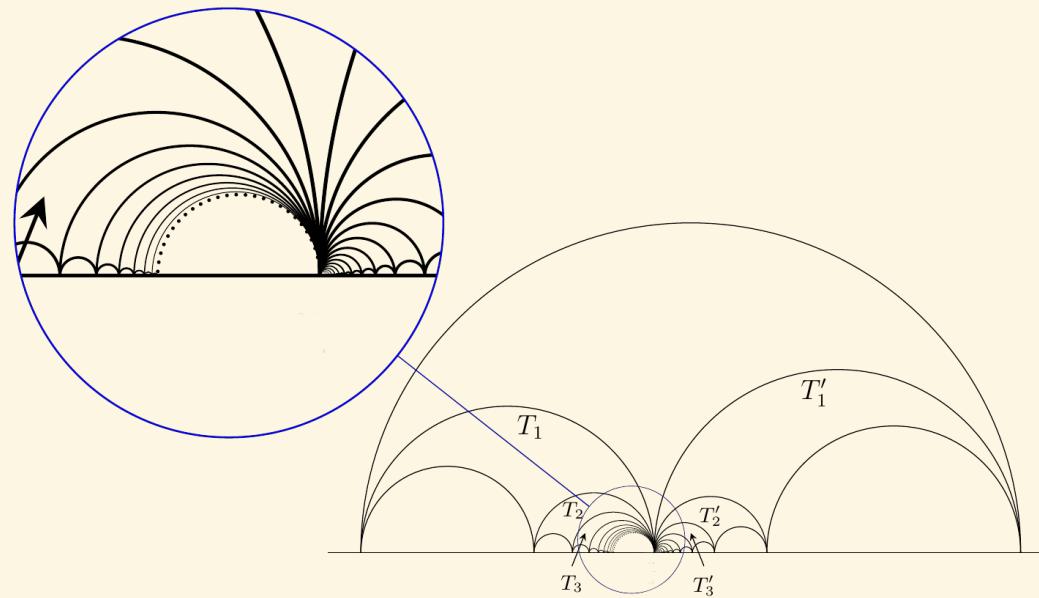
At  $g \neq 1$ , the left & right limits of Farey triangles do not agree — we get  $[\frac{r}{s}]_g^b$  &  $[\frac{r}{s}]_g^\#$ !



## Specialising $g$

At  $g \neq 1$ , the left & right limits of Farey triangles do not agree — we get  $[\frac{r}{s}]_g^b$  &  $[\frac{r}{s}]_g^{\#}$ !

Moreover, the entire semicircle connecting them lies in the limit.



$\overline{\text{Stab}}^q \mathcal{C}$  at a fixed positive  $q$

Thm [B-Becker-Licata]

- ① The union of the closed semicircles  $\left[ \left[ \frac{r}{s} \right]_q^b, \left[ \frac{r}{s} \right]_q^\# \right]$  is dense in the boundary of  $\overline{\text{Stab}}^q \mathcal{C}$
- ② The remaining points of the boundary are exactly the "q-irrationals".
- ③ The boundary is homeomorphic to  $S^1$ .

Thank you!