RESEARCH STATEMENT

ASILATA BAPAT

My research focuses on the topology of algebraic varieties, their singularities, and connections to representation theory. It is related to and deploys techniques from several areas of mathematics: representation theory, algebraic geometry, homological algebra, and the theory of differential equations. The two closely-linked theories of $\mathscr{D}$-modules and of perverse sheaves are at the heart of my current research.

Modules over the ring $\mathscr{D}$ of differential operators on a given space are called *$\mathscr{D}$-modules*. Studying $\mathscr{D}$-modules gives an algebraic approach to linear partial differential equations and their solutions. Moreover, information about the singularities of a space can be captured by various $\mathscr{D}$-modules associated to the space.

Bernstein–Sato polynomials are invariants of singularities of hypersurfaces that arise from the theory of $\mathscr{D}$-modules. They were first considered by Sato–Shintani [31] and Bernstein [7] in the early 1970s. They are subtle invariants that are very hard to compute, even for specific examples. A part of my research is aimed at computing the Bernstein–Sato polynomials for *Weyl arrangements* (hyperplane arrangements arising from root systems of semisimple Lie algebras). Bernstein–Sato polynomials are related to several other singularity invariants, such as the monodromy of the Milnor fiber and jumping coefficients, including the log canonical threshold. I proved the Strong Monodromy Conjecture for Weyl arrangements, which shows that their Bernstein–Sato polynomials are related to yet another singularity invariant, namely the local topological zeta function. A special example of a Weyl arrangement is the *braid arrangement*, which is the complement of the ordered configuration space of n points on a line. I computed an upper bound for the Bernstein–Sato polynomial of the braid arrangement. More details are discussed in Section 1.

Perverse sheaves are fundamental objects that provide key geometric insight into many problems of representation theory. Introduced by Beilinson–Bernstein–Deligne–Gabber [5] and Goresky–MacPherson [19, 20] in the early 1980s, they are also related to regular holonomic $\mathscr{D}$-modules via the Riemann–Hilbert correspondence. Many important results in this area translate representations of an algebraic structure into perverse sheaves on a related geometric object. Examples of this include proofs of the Kazhdan–Lusztig conjectures, the generalized Springer correspondence, and the geometric Satake isomorphism. These translations send irreducible representations to *IC sheaves* or *intersection cohomology sheaves*, which are special examples of perverse sheaves. As another part of my research, I found a Künneth-type formula to compute the cohomology of tensor products of IC sheaves (and equivariant IC sheaves) on varieties equipped with a torus action. I am interested in other problems of this flavor, and its connections to representation theory. More details are discussed in Section 2.

1. Bernstein–Sato polynomials

Overview. Let f be a polynomial in n variables, and let $\mathscr{D}$ be the ring of differential operators in n variables. The *Bernstein–Sato polynomial* or *b-function* of f is the minimal monic polynomial $b_f(s)$ that satisfies $L(s)f^{s+1} = b_f(s)f^s$ for some $L(s) \in \mathscr{D}[s] = \mathscr{D} \otimes_{\mathbb{C}} \mathbb{C}[s]$.

From the definition, it is not clear that b-functions always exist, but this was proved by Bernstein in 1972 [7]. It was shown by Kashiwara [23] and Malgrange [28] that all roots of $b_f(s)$ are negative rational numbers. The b-function of f is an invariant of the singularities of the zero locus $V(f)$ of f, and heuristically, complicated singularities have complicated b-functions. In fact, $V(f)$ is smooth if and only if $b_f(s) = (s+1)$.

It is a difficult problem to compute the b-functions of general hypersurfaces. Precise computations are only known in scattered cases. For example, Sato's original definition and subsequent work computes the b-function of semi-invariants of an algebraic group action on a prehomogeneous vector space. Another example is due to Opdam [30]: he explicitly computes the b-functions of certain Weyl-group invariant polynomial functions on the Cartan subalgebra of a semisimple Lie algebra. In my work, I have used Opdam's formula to prove the Strong Monodromy Conjecture for some closely related polynomials on the Cartan subalgebra. I have also found upper bounds on the b-functions of some of these polynomials.

Other singularity invariants and the Strong Monodromy Conjecture. Some relationships between Bernstein–Sato polynomials and other singularity invariants are known, while others are conjectured. We explain some of these, and how my work fits in to this context.

The *Milnor fiber* of a function f is the set $f^{-1}(\epsilon)$ for some ϵ close to zero. For all sufficiently small ϵ, the corresponding Milnor fibers are homeomorphic. Their cohomology, along with the action of natural monodromy operators, provides information about the singularities of $V(f)$. Malgrange [27, 28] and Kashiwara [23] have shown that exponentials of the roots of the b-function are eigenvalues of the monodromy of the Milnor fiber. More precisely, they relate the roots to eigenvalues of monodromy operators on the perverse sheaves of vanishing and nearby cycles.

Let f be a regular function on a smooth variety X, and let $\mu\colon Y \to X$ be a log resolution of $(X, V(f))$. Let $K_{Y/X}$ be the relative canonical divisor on Y, and let F be the proper transform of $V(f)$. For any $c \in \mathbb{Q}^{\geq 0}$, the corresponding *multiplier ideal* is defined to be $\mathscr{J}(f,c) = \mu_* \mathscr{O}_Y(K_{Y/X} - \lfloor cF \rfloor)$. Equivalently, $\mathscr{J}(f,c)$ is the ideal of all algebraic functions g on X such that $|g|^2/|f|^{2c}$ is a locally integrable function. The assignment $c \mapsto \mathscr{J}(f,c)$ is a decreasing step function (by containment of ideals).

A number c is called a *jumping coefficient* if for every $\epsilon > 0$, we have $\mathscr{J}(f, c+\epsilon) \subsetneq \mathscr{J}(f,c)$. That is, if it is one of the jumps of the step function. The smallest non-zero jumping coefficient is known as the *log canonical threshold*. It is a theorem of Kollár [25] that the log canonical threshold of f is equal to the negative of the maximal root of $b_f(s)$. More generally, it was shown by Ein–Lazarsfeld–Smith–Varolin [15] that the negatives of all jumping coefficients in the interval $(0, 1]$ are roots of $b_f(s)$.

The *local topological zeta function* is a singularity invariant first defined by Denef–Loeser [14]. It is a power series constructed from a resolution of singularities, using multiplicities of the exceptional divisors in the preimage of the hypersurface and in the relative canonical divisor.

Conjecture (Strong Monodromy Conjecture, Denef–Loeser [14])**.** *Every pole of the local topological zeta function of a hypersurface $V(f)$ is a root of the b-function of f.*

In a joint work with Robin Walters, I proved the Strong Monodromy Conjecture (SMC) for all Weyl arrangements, which are the hyperplane arrangements cut out by the roots of a semisimple Lie algebra.

Theorem 1.1 (Bapat–Walters, [4]). *The Strong Monodromy Conjecture holds for any Weyl arrangement ξ of d planes in n-space: every pole of the local topological zeta function of ξ is also a root of $b_\xi(s)$.*

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r. Let $\mathfrak{h}$ a Cartan subalgebra, and W its Weyl group. Let ξ be the product of all the positive roots of $\mathfrak{g}$. Then ξ belongs to $\mathbb{C}[\mathfrak{h}]$ and ξ^2 belongs to $\mathbb{C}[\mathfrak{h}]^W$, the space of W-invariants. Recall that $\mathbb{C}[\mathfrak{h}]^W$ is isomorphic to a polynomial ring in r variables, generated by homogeneous polynomials of degrees $d_1, \dots, d_r$ respectively (the fundamental invariants). We will denote this polynomial ring by $\mathbb{C}[\mathfrak{h}/W]$. Let ρ be the image of ξ^2 in $\mathbb{C}[\mathfrak{h}/W]$.

In the paper [30], Opdam explicitly computed the b-function of ρ to be

$$b_\rho(s) = \prod_{i=1}^{r} \prod_{j=1}^{d_i - 1} \left(s + \frac{1}{2} + \frac{j}{d_i} \right). \tag{1}$$

Opdam's method involved explicitly computing a suitable differential operator via hypergeometric shift operators. Although the functions ρ and ξ are related to each other, their b-functions are not obviously related, because the differential operators on $\mathbb{C}[\mathfrak{h}]$ and on $\mathbb{C}[\mathfrak{h}/W]$ are quite different.

We proved the following theorem that relates the two.

Theorem 1.2 (Bapat–Walters, [4]). *Let ρ be the image of ξ^2 in $\mathbb{C}[\mathfrak{h}/W]$. Then*

$$b_\rho(s) \mid b_\xi(2s+1).$$

The key tool used in the proof is the *Harish-Chandra homomorphism* from $\mathscr{D}(\mathfrak{g})^G$ to $\mathscr{D}(\mathfrak{h})^W$ (defined in [21]). By the Chevalley restriction theorem, the W-invariant function ξ^2 on $\mathfrak{h}$ can be lifted to a G-invariant function on $\mathfrak{g}$, which allows us to use the Harish-Chandra homomorphism to compare their b-functions.

It was shown by Budur–Mustaţă–Teitler [11] that proving the SMC for indecomposable and central hyperplane arrangements is equivalent to proving the "n/d conjecture" for each such arrangement. This conjecture states that $-n/d$ is a root of the b-function of any indecomposable and central hyperplane arrangement of d planes in n space. We proved the n/d conjecture for Weyl arrangements using Opdam's formula (1) and the above theorem, which implies the SMC for Weyl arrangements (Theorem 1.1).

The b-function of the Vandermonde determinant. I am interested in computing the b-functions of ξ and ξ^2 explicitly. We expect them to be much more complicated than the b-function of ρ. In the remainder of this section, we only discuss the case of type A, for which we have an upper bound as well as a conjectured answer.

When $\mathfrak{g} = sl_n$, the function ξ is simply the Vandermonde determinant on n variables, which is defined as

$$\mathrm{VM}_n(x_1, \dots, x_n) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \le j < i \le n} (x_i - x_j).$$

The equation $\mathrm{VM}_n(x_1, \ldots, x_n)$ cuts out the braid arrangement in $\mathbb{C}^n$, and its complement is the ordered configuration space of n points in $\mathbb{C}$. Surprisingly, its b-function is not known. We have proved the following (recursive) upper bound on $b_{\mathrm{VM}_n}(s)$.

Theorem 1.3 (Bapat–Walters, in preparation)**.** *For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, denoted $\lambda \vdash n$, let $b_\lambda(s)$ denote the product of the b-functions of VM_{λ_i} for each i. Then we have that*

$$b_{\mathrm{VM}_n}(s) \,\Big|\, \operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s) \cdot \prod_{i=(n-1)}^{(n-1)^2} \left(s + \frac{i}{\binom{n}{2}} \right).$$

In the paper [23], Kashiwara proved the rationality of the roots of b-functions using the method of resolution of singularities. For the braid arrangement, an explicit resolution was introduced by Fulton and MacPherson [17]. For other arrangements, a similar algorithm has been given by DeConcini and Procesi [13]. In their papers [12, 29], Calderón Moreno and Narváez Macarro proved that the $\mathscr{D}$-modules $\mathscr{D}[s]f^s$ and $\mathscr{D}[s]f^{-s-1}$ are dual in certain cases, including for all Weyl arrangements ξ. This implies that $b_\xi(s) = \pm b_\xi(-s-2)$, meaning that the roots of $b_\xi(s)$ are symmetric about the point -1. Our proof uses this fact and Kashiwara's result.

2. Intersection cohomology sheaves

Let X be a smooth projective variety equipped with the action of an algebraic torus T. Fix a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to T$ such that the $\mathbb{G}_m$-fixed set is equal to the T-fixed set X^T. Let $F_1, \ldots, F_k$ be the irreducible components of the fixed set $X^{\mathbb{G}_m}$. In this situation, Białynicki-Birula [8] described two canonical decompositions of X into k locally-closed, $\mathbb{G}_m$-invariant subvarieties, called the *plus decomposition* consisting of *attracting sets* ($\{S_1^+, \ldots, S_k^+\}$) and the *minus decomposition* consisting of *repelling sets* ($\{S_1^-, \ldots, S_k^-\}$):

$$S_i^+ = \{x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x \in F_i\}, \text{ and } S_i^- = \{x \in X \mid \lim_{t \to \infty} \lambda(t) \cdot x \in F_i\}.$$

Note that each S_i^+ and S_i^- map to F_i (by sending a point to its limiting point), and the fibers of these maps are affine spaces. In particular, we consider the case in which each F_i is a point, and then each $S_i^\pm$ is itself an affine space.

Consider (say) the plus decomposition, and group the pieces S_i^+ by dimension. In general, the resulting configuration does not give a topological stratification, but we focus on the case when it does. Moreover, we assume that for each fixed point w, there is some one-parameter subgroup $\mu : \mathbb{G}_m \to T$ such that the attracting set of μ on w has dimension n. The cell decomposition of X so described is suitable both for studying intersection cohomology sheaves and also the topological (singular) cohomology.

I have proved the following theorem.

Theorem 2.1 (Bapat, [2])**.** *Let X be a smooth projective complex algebraic variety equipped with an action of an algebraic torus, with assumptions as before. Let $(\mathscr{L}_1, \ldots, \mathscr{L}_m)$ be an m-tuple of IC sheaves on X constant along the Białynicki-Birula strata, and let $\otimes$ denote the (derived) tensor product. Then the cup-product map induces the following isomorphism:*

$$H^\bullet(\mathscr{L}_1) \underset{H^\bullet(X)}{\otimes} \cdots \underset{H^\bullet(X)}{\otimes} H^\bullet(\mathscr{L}_m) \xrightarrow{\cong} H^\bullet(\mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_m).$$

The same theorem holds for T-equivariant simple perverse sheaves on X and their T-equivariant cohomology, as modules over the T-equivariant cohomology ring of X.

Theorem 2.1 is used in a forthcoming work of Braverman–Finkelberg–Nakajima [9] to study tensor products of complexes on the affine Grassmannian. The most direct motivation for Theorem 2.1 comes from Ginzburg's paper [18], which itself is related to the work of Soergel [32] and Beilinson–Ginzburg–Soergel [6]. Under the hypotheses from the previous subsection, Ginzburg showed that if $\mathscr{L}_1$ and $\mathscr{L}_2$ are two IC sheaves on X constant along the Białynicki-Birula strata, then $\mathrm{Ext}^\bullet_{D^b_c(X)}(\mathscr{L}_1, \mathscr{L}_2) \cong \mathrm{Hom}^\bullet_{H^\bullet(X)}(H^\bullet(\mathscr{L}_1), H^\bullet(\mathscr{L}_2))$. Our theorem is the analog of the theorem above (which discusses the derived Hom functor) for the derived tensor product functor.

A main tool in the proof is the interplay between the two Białynicki-Birula decompositions; one can use Morse theory to show that the plus and minus cells intersect transversally. Using this, one can use induction on the strata to compute the action of the "minus" cohomology classes on the cohomology of $\mathscr{L}_i$ for each i. Using this method to compare the cohomology of $\mathscr{L}_1 \otimes \mathscr{L}_2 \otimes \cdots \otimes \mathscr{L}_m$ with the tensor product of the cohomologies of the $\mathscr{L}_i$ yields our theorem.

3. Ongoing and future projects

3.1. **Bernstein–Sato polynomials of finite Coxeter arrangements.** Following on from the results of Section 1, I am working on determining an explicit description of the b-function of the Vandermonde determinant (joint with Robin Walters). We have the following conjecture for the Vandermonde determinant, which states that the division in Theorem 1.3 is an equality.

Conjecture (Bapat–Walters, [3]). *The b-function of the Vandermonde determinant has the following formula:*

$$b_{\mathrm{VM}_n}(s) = \operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s) \cdot \prod_{i=(n-1)}^{(n-1)^2} \left(s + \frac{i}{\binom{n}{2}} \right).$$

This conjecture is supported by computer experiments using *Macaulay2* and *Singular*, as well as some heuristics. We would also like to find the b-function of the square of the Vandermonde determinant, as well as of the corresponding polynomial on sl_n, namely the discriminant of the characteristic polynomial of an $n \times n$ matrix.

A more ambitious problem is to study generalizations of this setup, such as other finite Coxeter arrangements, as well as noncrystallographic hyperplane arrangements. Related to our proof of the SMC, we would like to investigate the Harish-Chandra map on differential operators on symmetric spaces.

3.2. **A compactification of Calogero–Moser space.** The nth *Calogero–Moser space*, denoted $\mathscr{C}_n$, is defined as

$$\mathscr{C}_n \stackrel{\text{def}}{=} \{(X, Y) \in \mathrm{Mat}_n \times \mathrm{Mat}_n \mid ([X, Y] + I_n) \text{ has rank } 1\} /\!\!/ PGL_n,$$

where PGL_n acts diagonally. The space $\mathscr{C}_n$ is a $2n$-dimensional smooth affine symplectic algebraic variety defined by Kazhdan–Kostant–Sternberg [24] to study the Calogero–Moser integrable system. The *Drinfeld compactification* of $\mathscr{C}_n$, which we denote by $\overline{\mathscr{C}_n}$, was constructed by Wilson [33] and Finkelberg–Ginzburg [16] using an important embedding of $\mathscr{C}_n$ into an *adelic Grassmannian* discovered by Wilson [33].

Another important problem is to find an intrinsic, geometric description of $\overline{\mathscr{C}_n}$. It is known that the complement of $\mathscr{C}_n$ in $\overline{\mathscr{C}_n}$ is an irreducible divisor that induces the following filtration on the coordinate ring of $\mathscr{C}_n$: a function f lies in the kth filtered piece if it has a pole of order at most k at the boundary divisor. Recall that $\mathscr{C}_n$ is a quotient by PGL_n of a subvariety of

$\mathrm{Mat}_n \times \mathrm{Mat}_n$. I am working on the following problem: determine the existence of a degree function on the coordinate ring of $\mathrm{Mat}_n \times \mathrm{Mat}_n$ such that the degree filtration, when restricted to the coordinate ring of $\mathscr{C}_n$, equals the filtration induced by the boundary divisor.

3.3. **Tensor products of IC sheaves.** The results described in Section 2 suggest a natural question: to find a representation-theoretic interpretation of tensor product of IC sheaves on the flag variety, and also to study equivariant analogs of the theorems of Ginzburg and Soergel. Furthermore, I would like to explicitly compute the cohomology of the tensor product of IC sheaves in other examples of interesting spaces with $\mathbb{C}^*$ actions. For example, in the case of the Hilbert scheme of points on a plane, the combinatorics of this computation seems to be interesting.

3.4. **Equivariant multiplicities and smoothness.** Let X be a complex algebraic variety of dimension n, together with an action of an algebraic torus T with finitely many fixed points. The localization theorem states that one can recover the equivariant cohomology of a space from the equivariant cohomology of X^T, together with the local action of T around each fixed point. In particular, the fundamental class of X in equivariant Borel–Moore homology corresponds to a certain collection of rational functions $\{f_x/g_x \mid x \in X^T\}$. For each $x \in X^T$, both f_x and g_x are elements of $H_T^\bullet(\mathrm{pt}, \mathbb{Z})$, and g_x is the product of characters of the T-action on the tangent space at x. The element f_x is called the *equivariant multiplicity* of X at x.

In many cases, it is known that properties of f_x are related to smoothness of X at x. For example, Kumar [26] showed that if X is a Schubert variety, then $f_x = 1$ if and only if X is smooth at x. Kumar [26], Arabia [1], and Brion [10] showed that under suitable conditions, X is *rationally smooth* at x if and only if $f_x \in \mathbb{Z}$. Likewise, Juteau–Williamson [22] showed that X is *p-smooth* at x (for prime p) if and only if $f_x \in \mathbb{Z}$ and $p \nmid f_x$.

Even for Schubert varieties, there are examples of points at which the variety is not rationally smooth, which means that f_x is not an integer. It would be interesting to find a precise statement relating f_x and the behavior of X at x, particularly in the case when f_x is not an integer.

References

[1] A. Arabia, *Classes d'Euler équivariantes et points rationnellement lisses*, Ann. Inst. Fourier (Grenoble), 48 (1998), pp. 861–912.

[2] A. Bapat, *Torus actions and tensor products of intersection cohomology*, Pacific J. Math., 276 (2015), pp. 19–34.

[3] A. Bapat and R. Walters, *The Bernstein-Sato b-function of the Vandermonde determinant*, arXiv:1503.01055 [math.AG], (2015).

[4] ———, *The strong topological monodromy conjecture for Coxeter hyperplane arrangements*. http://math.uchicago.edu/~asilata/papers/bapat-walters-2015.pdf, to appear in Mathematical Research Letters, 2015.

[5] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, in Analysis and topology on singular spaces, I (Luminy, 1981), vol. 100 of Astérisque, Soc. Math. France, Paris, 1982, pp. 5–171.

[6] A. Beilinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc., 9 (1996), pp. 473–527.

[7] J. Bernstein, *Analytic continuation of generalized functions with respect to a parameter*, Funkcional. Anal. i Priložen., 6 (1972), pp. 26–40.

[8] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2), 98 (1973), pp. 480–497.

[9] A. Braverman, M. Finkelberg, and H. Nakajima, *Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II*. In preparation, 2015.

[10] M. Brion, *Equivariant cohomology and equivariant intersection theory*, in Representation theories and algebraic geometry (Montreal, PQ, 1997), vol. 514 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, 1998, pp. 1–37. Notes by Alvaro Rittatore.

[11] N. Budur, M. Mustaţă, and Z. Teitler, *The monodromy conjecture for hyperplane arrangements*, Geom. Dedicata, 153 (2011), pp. 131–137.

[12] F. Calderón Moreno and L. Narváez Macarro, *The module Df^s for locally quasi-homogeneous free divisors*, Compositio Math., 134 (2002), pp. 59–74.

[13] C. De Concini and C. Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.), 1 (1995), pp. 459–494.

[14] J. Denef and F. Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc., 5 (1992), pp. 705–720.

[15] L. Ein, R. Lazarsfeld, K. E. Smith, and D. Varolin, *Jumping coefficients of multiplier ideals*, Duke Math. J., 123 (2004), pp. 469–506.

[16] M. Finkelberg and V. Ginzburg, *Calogero-Moser space and Kostka polynomials*, Adv. Math., 172 (2002), pp. 137–150.

[17] W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2), 139 (1994), pp. 183–225.

[18] V. Ginzburg, *Perverse sheaves and $\mathbf{C}^*$-actions*, J. Amer. Math. Soc., 4 (1991), pp. 483–490.

[19] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology, 19 (1980), pp. 135–162.

[20] ———, *Intersection homology. II*, Invent. Math., 72 (1983), pp. 77–129.

[21] Harish-Chandra, *Invariant differential operators and distributions on a semisimple Lie algebra*, Amer. J. Math., 86 (1964), pp. 534–564.

[22] D. Juteau and G. Williamson, *Kumar's criterion modulo p*, Duke Math. J., 163 (2014), pp. 2617–2638.

[23] M. Kashiwara, *B-functions and holonomic systems. Rationality of roots of B-functions*, Invent. Math., 38 (1976/77), pp. 33–53.

[24] D. Kazhdan, B. Kostant, and S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure Appl. Math., 31 (1978), pp. 481–507.

[25] J. Kollár, *Singularities of pairs*, in Algebraic geometry—Santa Cruz 1995, vol. 62 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1997, pp. 221–287.

[26] S. Kumar, *The nil Hecke ring and singularity of Schubert varieties*, Invent. Math., 123 (1996), pp. 471–506.

[27] B. Malgrange, *Le polynôme de Bernstein d'une singularité isolée*, in Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974), Springer, Berlin, 1975, pp. 98–119. Lecture Notes in Math., Vol. 459.

[28] ———, *Polynômes de Bernstein-Sato et cohomologie évanescente*, in Analysis and topology on singular spaces, II, III (Luminy, 1981), vol. 101 of Astérisque, Soc. Math. France, Paris, 1983, pp. 243–267.

[29] L. Narváez Macarro, *A duality approach to the symmetry of Bernstein-Sato polynomials of free divisors*. arXiv preprint, 2012.

[30] E. M. Opdam, *Some applications of hypergeometric shift operators*, Invent. Math., 98 (1989), pp. 1–18.

[31] M. Sato and T. Shintani, *On zeta functions associated with prehomogeneous vector spaces*, Proc. Nat. Acad. Sci. U.S.A., 69 (1972), pp. 1081–1082.

[32] W. Soergel, *Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc., 3 (1990), pp. 421–445.

[33] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, Invent. Math., 133 (1998), pp. 1–41. With an appendix by I. G. Macdonald.