GEOMETRIC OBJECTS IN REPRESENTATION THEORY

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INTRODUCTION

Interesting geometric objects arise from the representation theory of Lie algebras and non-commutative algebras. The aim of geometric representation theory is to construct dictionaries that link representation theory to geometry, so that geometric properties can be translated into representation-theoretic data. This technique has been key to proving many fundamental theorems in this subject. However, many of these geometric structures remain mysterious.

I propose to study the geometry of several of these varieties occurring naturally in representation theory, which have been actively studied in recent years. I raise several questions related to the structure of these varieties (specifically, Calogero–Moser spaces, Hilbert schemes, Hessenberg varieties, the exotic nilpotent cone, and reflection arrangements of Coxeter type), with the hope that a better understanding of these spaces may shed light on the related representation theory. I explain my approaches towards these questions, as well as some initial progress using the techniques outlined.

The first project involves Calogero–Moser spaces. More specifically, we aim to construct compactifications of this space using using geometric invariant theory as well as quiver representations. We also discuss connections to the Hilbert scheme of points on the plane. Details are discussed in Section 1.

The second project is about analogs of Hessenberg varieties for the exotic nilpotent cone. We give a preliminary definition, and propose to apply approaches in the classical theory of Hessenberg varieties to study these analogs. Details are discussed in Section 2.

The third project aims to understand the Bernstein–Sato polynomials of hyperplane arrangements cut out by reflections in a Weyl group (or more generally, a Coxeter group of not necessarily Weyl type). We propose to use Lie theory to connect of the Bernstein–Sato polynomial of these arrangements with other invariants of singularities. Details are discussed in Section 3.

CONTENTS

The remainder of this section contains more detailed introductions to each project.

Project 1: Compactifications of Calogero–Moser spaces. The Calogero–Moser space (denoted $\mathscr{C}_n$) can be thought of as the space of pairs of $n \times n$ matrices (up to simultaneous conjugation) whose commutator is as close as possible to the identity matrix. The precise definition is in Section 1. It is a smooth, affine, symplectic algebraic variety. It can be studied from many different points of view, through its connections with several areas of algebra and geometry. For example, points of $\mathscr{C}_n$ are known to be in one-to-one correspondence with irreducible representations of a rational Cherednik algebra [20]. Moreover, $\mathscr{C}_n$ is a deformation of the Hilbert scheme of n points on a plane (see, e.g. [29]), and it can be viewed as a rotation of the hyper-Kähler structure on the Hilbert scheme.

The goals of this project are as follows.

(1) Find compactifications of $\mathscr{C}_n$ and interpret the boundary in terms of representation theory.
(2) Understand the geometry of $\mathscr{C}_n$ in the context of its relationship with the Hilbert scheme of points.

More details about this project can be found in Section 1.

Project 2: Exotic Hessenberg varieties and variations. Hessenberg varieties are subvarieties of the full flag variety of a reductive algebraic group, depending on the choice of an element and a certain subspace of the Lie algebra. They are natural generalizations of Springer fibers as well as the Peterson variety, which is important in the study the quantum cohomology of partial flag varieties (see, e.g. [25, 34]).

Hessenberg varieties have connections to many areas, including geometric representation theory, combinatorics, They have been actively studied, especially in cases where the chosen element of the Lie algebra is special (e.g regular semisimple, regular nilpotent, etc).

Classically, Hessenberg varieties are defined in terms of the adjoint representation and the Springer resolution. They have been actively studied (see, e.g. [15, 38, 5, 31, 2, 13, 11, 32, 1, 33]). Often, the element of the Lie algebra is chosen to be special (e.g. regular semisimple, regular nilpotent, minimal nilpotent, etc).

We propose to understand Hessenberg-type generalizations of variations of Springer fibers, such as the exotic nilpotent cone for the symplectic group [24] and the enhanced nilpotent cone [3, 4]. This project is joint with William Graham. The goals are as follows.

(1) Construct and study Hessenberg-type varieties for the exotic representation of the symplectic group.
(2) Define notions analogous to regular nilpotent, regular semisimple, minimal nilpotent, etc, for the exotic representation. Prove analogs of known results for each special family.
(3) Analyze Hessenberg-type varieties for other variations on the nilpotent cone, such as the enhanced nilpotent cone, and to extend known relationships between these nilpotent cones to their Hessenberg varieties.

More details about this project can be found in Section 2.

Project 3: Bernstein–Sato polynomials and monodromy conjectures. The Bernstein–Sato polynomial of a hypersurface singularity is a subtle invariant of the singularity, defined via actions of differential operators on its local equation. It is known to be related to other singularity invariants, such as the monodromy operators on the cohomology of the Milnor fiber [26], the log-canonical threshold, and multiplier ideals [19]. It is also conjectured to be related to the topological zeta function of Denef–Loeser via the monodromy conjecture [16]. On the other hand, it is very difficult to compute explicitly, except in special cases.

As a long-term goal, we would like to completely describe the Bernstein–Sato polynomials of the reflection hyperplane arrangements of Coxeter groups. We tackle this problem by analyzing (for these cases) various singularity invariants related to the Bernstein–Sato polynomial. The specific goals of this project, which continues joint work with Robin Walters, are as follows.

(1) Use Lie theory to explore the connections of Bernstein–Sato polynomials with the topological and motivic zeta functions of Denef–Loeser for reflection hyperplane arrangements.
(2) Understand roots of the Bernstein–Sato polynomials for reflection hyperplane arrangements for Coxeter groups.

More details about this project can be found in Section 3.

1. Compactifications of Calogero–Moser spaces

The Calogero–Moser space is defined as

$$\mathscr{C}_n = \{(X,Y) \in \mathrm{Mat}_n(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C}) \mid ([X,Y]-I) \text{ has rank } 1\}/PGL_n,$$

where PGL_n acts diagonally. The action of PGL_n is a free action. It has been shown in [40] that the geometric quotient $\mathscr{C}_n$ is a smooth, irreducible, affine, symplectic variety of dimension $2n$. We now outline several projects, each of which highlights a different feature of $\mathscr{C}_n$.

1.1. **Embeddings into Grassmannians and GIT compactifications.** In [40], G. Wilson proved many fundamental results about $\mathscr{C}_n$. His main result was an embedding of $\mathscr{C}_n$ into the adelic Grassmannian, which is an infinite-dimensional Grassmannian of subspaces of rational functions on $\mathbb{P}^1_{\mathbb{C}}$ whose elements satisfy certain pole order and vanishing conditions.

Let $\pi\colon \mathscr{C}_n \to \mathbb{A}^n/S_n$ be the map that sends the class of a pair (X,Y) to the unordered collection of eigenvalues of X. In [21], Finkelberg–Ginzburg reinterpret the results of [40] as an embedding of $\mathscr{C}_n$ into a relative Grassmannian $\mathscr{G}_n$, as well as an extension of the map π to $\mathscr{G}_n$. Although both sides of the embedding from [21] are algebraic objects, it is not known whether the embedding itself is realized by an algebraic map.

Further, Finkelberg–Ginzburg define the relative Drinfeld compactification of $\mathscr{C}_n$, denoted $\overline{\mathscr{C}_n}$, to be the closure of its image inside $\mathscr{G}_n$. Since $\mathscr{G}_n$ is projective over $\mathbb{A}^n/S_n$, the fibers of $\overline{\mathscr{C}_n} \to \mathbb{A}^n/S_n$ are compact. In particular, each fiber of $\overline{\mathscr{C}_n}$ is a closed subspace of a Grassmannian. The relative compactification $\overline{\mathscr{C}_n}$ has useful properties: its boundary is an irreducible divisor, and over each fiber of π, it can be expressed as a product of Schubert cells of smaller Grassmannians. However, because the embedding into $\mathscr{G}_n$ is not known to be algebraic, we do not know of an algebraic description of $\overline{\mathscr{C}_n}$.

The broad, long-term goals of this project include: (1) to determine whether the embedding of $\mathscr{C}_n$ into $\mathscr{G}_n$ is algebraic, and (2) to understand $\overline{\mathscr{C}_n}$ more intrinsically, as proposed in [21].

Recall from [20] that points of $\mathscr{C}_n$ parametrize irreducible representations of the rational Cherednik algebra at parameters $t = 0$ and $c = 1$. We plan to investigate whether the boundary of $\overline{\mathscr{C}_n}$ has an interpretation in terms of representation theory of similar objects, e.g. rational Cherednik algebras for smaller values of n. More generally, we hope to construct other compactifications of $\mathscr{C}_n$ with representation-theoretic interpretations, and to relate them to $\overline{\mathscr{C}_n}$. This forms the starting point of our project.

Goal 1.1. Use Geometric Invariant Theory (GIT) to construct compactifications of $\mathscr{C}_n$, and investigate the relationship between these compactifications and $\overline{\mathscr{C}_n}$.

Let $V = \mathbb{C}^n$. Define the space C_n as

$$C_n = \{(X, Y, v, w) \in \mathfrak{gl}(V) \times \mathfrak{gl}(V) \times V \times V^* \mid [X, Y] + v \otimes w = I\}.$$

Then $\mathscr{C}_n \cong C_n/GL(V)$, where $g \in GL(V)$ acts on (X, Y, v, w) as $(gXg^{-1}, gYg^{-1}, gv, (g^{-1})^t w)$. The isomorphism sends the class of a tuple (X, Y, v, w) in $C_n/GL(V)$ to the class of (X, Y) in $\mathscr{C}_n$. The following example explains some initial progress towards the kinds of compactifications we would like to obtain.

Example 1.2. Let $\overline{C_n}$ be the simultaneous projective completion of C_n along the coordinates of Y and w. Explicitly, consider the (n^2+n)-dimensional projective space with homogeneous coordinates $[Y_{ij} : w_k : s]$, where $1 \le i, j, k \le n$. Define $\overline{C_n}$ as

$$\overline{C_n} = \{(X, v, [Y_{ij} : w_k : s]) \in \mathfrak{gl}(V) \times V \times \mathbb{P}^{n^2+n} \mid [X, Y] + v \otimes w = sI\}.$$

Then C_n embeds into $\overline{C_n}$ by sending (X, Y, v, w) to $(X, v, [Y_{ij} : w_k : 1])$. Note that C_n maps to $V \times \mathfrak{gl}(V)$ by sending the point (X, Y, v, w) to (v, X). We call this map p_n (or just p when there is no ambiguity). It can be checked that $(V \times \mathfrak{gl}(V)) /\!\!/ GL(V)) \cong \mathbb{A}^n/S_n$. We have the following commutative diagram, where the horizontal maps are quotient maps.

$$\begin{tikzcd} C_n \arrow[r] \arrow[d, "p_n"'] & \mathscr{C}_n \arrow[d, "\pi"] \\ V \times \mathfrak{gl}(V) \arrow[r] & \mathbb{A}^n/S_n \end{tikzcd}$$

The first vertical map extends to a proper map $p \colon \overline{C_n} \to V \times \mathfrak{gl}(V)$. Taking a suitable GIT quotient of $\overline{C_n}$ by $GL(V)$, we propose to extend p to the GIT quotient of $\overline{C_n}$, thereby obtaining a (possibly partial) compactification of $\mathscr{C}_n$ over $\mathbb{A}^n/S_n$.

The GIT quotient can be computed with a suitably chosen $GL(V)$-linearized line bundle. We will always choose the line bundle $\mathscr{O}(1)$, and the choice of linearization coresponds to the choice of a character $GL(V) \to \mathbb{C}^*$. We conclude this example by discussing the semistability condition for the linearization corresponding to the character $\det^{-1} \colon GL(V) \to \mathbb{C}^*$.

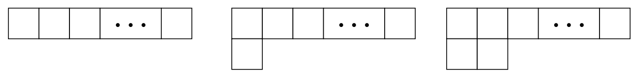
Proposition 1.3 (B.). *A point $(X, v, [Y_{ij} : w_k : s])$ of $\overline{C_n}$ is $(\det^{-1})$-semistable if and only if both the following conditions hold.*

1. *(1) The vector v is cyclic for X and Y together. In other words, there is no non-trivial vector subspace of V containing v that is invariant under the actions of both X and Y. (This condition is well-defined even though we consider Y up to scaling.)*
2. *(2) The vector space spanned by $\{v, Xv, X^2v, \ldots\}$ has codimension at most 1 (equivalently, dimension at least $n-1$).*

This proposition is a good first step towards obtaining an intrinsic description of $\overline{\mathscr{C}_n}$, for the following reason. Recall from results of G. Wilson [40] that the fiber of π over a point $(a, a, \ldots, a) \in \mathbb{A}^n/S_n$ is isomorphic to the union of all n-dimensional Schubert cells in $\mathrm{Gr}(n, 2n)$. Over any other point in $\mathbb{A}^n/S_n$, one can recover the fiber of π by "factorization": it is the product of the fibers described above for smaller values of n, corresponding to the blocks of coincident coordinates in the given point of $\mathbb{A}^n/S_n$. In particular, the generic fiber of π is isomorphic to an n-fold product of $\mathbb{A}^1$s.

The semistability criterion described above recovers a dense open subset of $\mathscr{C}_n$, which is described explicitly in the next proposition.

Proposition 1.4 (B.)**.** *Over a point $(a,\dots,a) \in \mathbb{A}^n/S_n$, the $(\det^{-1})$-semistable subset of the fiber of π corresponds to exactly those n-dimensional Schubert cells of $\mathrm{Gr}(n,2n)$ described by a partition of one of the following three types.*



Over any other point in $\mathbb{A}^n/S_n$, the $(\det^{-1})$-semistable subset of the fiber of π can be recovered by factorization.

The boundary of the partial compactification described in the above example can be interpreted as a subset of a certain GIT quotient of $\mathrm{Hilb}^n(\mathbb{A}^2)$. The connection with $\mathrm{Hilb}^n(\mathbb{A}^2)$ is explored further in Subsection 1.3. Other GIT compactifications constructed as in Example 1.2 will have a similar description at the boundary. We plan to characterize these boundaries in terms of the geometry of $\mathrm{Hilb}^n(\mathbb{A}^2)$.

Goal 1.5. Describe the boundary of the compactification from Example 1.2, as well as the boundaries of other similar compactifications, in terms of the geometry of $\mathrm{Hilb}^n(\mathbb{A}^2)$.

1.2. **An approach towards a compactification of $\mathscr{C}_n$ via quiver representations.** Recall the projection $p_n\colon C_n \to V \times \mathfrak{gl}(V)$ that sends (X,Y,v,w) to (v,X). This map has the following "factorization" property. Given a fixed $(X,Y,v,w) \in C_n$, we can decompose V into generalized eigenspaces of X. We call these subspaces $V_1,\dots,V_k$, with dimensions $d_1,\dots,d_k$ and eigenvalues $a_1,\dots,a_k$ respectively. In this basis, we can decompose (v,X) as follows:

$$(v,X) = \bigoplus_{i=1}^{k}(v_i,X_i) \in \bigoplus_{i=1}^{k}(V_i \times \mathfrak{gl}(V_i)).$$

Then $p_n^{-1}(v,X)$ can be uniquely reconstructed from the collection of $p_{d_i}^{-1}(v_i,X_i)$ for each i. Moreover, $p_{d_i}^{-1}(v_i,X_i)$ is naturally isomorphic to $p_{d_i}^{-1}(v_i,(X_i - a_iI_i))$. In other words, $p_{d_i}(v_i,X_i)$ is independent of the eigenvalue a_i. Therefore, it suffices to restrict to the case where the eigenvalues of X are all zero (i.e, where X is nilpotent). Set C_n^0 to be the subset of C_n that consists of tuples (X,Y,v,w) for which X is nilpotent. For the remainder of the section, we only consider elements of C_n^0.

Let $\mathscr{N}(V)$ be the nilpotent cone inside $\mathfrak{gl}(V)$. Since X is nilpotent, we may think of (v,X) as an element of $V \times \mathscr{N}(V)$, which is known as the enhanced nilpotent cone. It was shown in [3, 37] that the group $GL(V)$ acts on $V \times \mathscr{N}(V)$ with finitely many orbits, which are indexed by the set of bi-partitions of n. This set, which we denote BP_n, is a key object in this project. It consists of pairs (λ,μ) where λ is a partition of k and μ is a partition of $(n-k)$ for some $k \le n$.

In the recent preprint of Bellamy–Boos [9], the authors construct a non-trivial bijection from BP_n to itself, which we denote β. Following [9], we indicate the construction of β. A central object in this bijection is the framed Jordan quiver Q. This quiver has two vertices, with an arrow from the first to the second and a loop from the second to itself. A Q-representation of dimension vector (k,l) is specified by an element (M_{lk},M_{ll}) of $\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^l) \times \mathrm{Hom}(\mathbb{C}^l,\mathbb{C}^l)$ as follows.

$$\begin{tikzcd} \bullet \arrow[r, "M_{lk}"] & \bullet \arrow[loop right, "M_{ll}"] \end{tikzcd}$$

We say that a Q-representation is *admissible* if it satisfies the following two conditions.

(1) The dimension vector (k, l) satisfies $k \leq 1$ and $l \leq n$.
(2) The element $M_{ll} \in \mathrm{Hom}(\mathbb{C}^l, \mathbb{C}^l)$ is nilpotent.

Given $(v, X) \in V \times \mathcal{N}(V)$, we can think of this pair as an admissible Q-representation of dimension vector $(1, n)$. On one hand, the $\mathfrak{gl}(V)$-orbits in $V \times \mathcal{N}(V)$ are parametrized by BP_n, which means that (v, X) corresponds to some bi-partition (λ, μ). On the other hand, as shown in [9], the isomorphism classes of admissible representations of Q of dimension vector $(1, n)$ are indexed by BP_n in a different way. Therefore (v, X) gives rise to a new element of BP_n that the authors define to be $\beta(\lambda, \mu)$.

Once again, let $(X, Y, v, w) \in C_n^0$. The projection p_n does not hit all the $GL(V)$-orbits on $V \times \mathcal{N}(V)$. Let $BP_n^{\neq} \subset BP_n$ consist of bi-partitions (λ, μ) with the following properties.

(1) Both λ and μ have strictly decreasing parts.
(2) If $|\lambda|$ and $|\mu|$ denote the number of parts of λ and μ respectively, then either $|\lambda| = |\mu|$ or $|\lambda| = |\mu| + 1$.

The following proposition, which combines several results from the literature, characterizes in several ways exactly which $GL(V)$-orbits appear.

Proposition 1.6 ([18, 9, 40])**.** *Let $(v, X) \in V \times \mathcal{N}(V)$, and let (λ, μ) be the bi-partition indexing the $GL(V)$-orbit of (v, X) in $V \times \mathcal{N}(V)$. The following conditions are equivalent.*

(1) $(v, X) \in p_n(C_n^0)$.
(2) $(\lambda, \mu) \in BP_n^{\neq}$.
(3) $\beta(\lambda, \mu)$ is of the form $(\rho, \emptyset)$.
(4) The (admissible) Q-representation given by (v, X) is indecomposable.

Recall that there is a $\mathbb{C}^*$-action on $\mathscr{C}_n$ with finitely many fixed points. If $(X, Y, v, w) \in C_n$ such that its image in $\mathscr{C}_n$ is $\mathbb{C}^*$-fixed, then (X, Y, v, w) is automatically lies in C_n^0. The following proposition rephrases a result from [40] in our notation.

Proposition 1.7 (Wilson [40])**.** *The $\mathbb{C}^*$-fixed points of $\mathscr{C}_n$ are naturally indexed by $BP_n^{\neq}$. Let $(X, Y, v, w) \in C_n^0$ such that its image in $\mathscr{C}_n$ is $\mathbb{C}^*$-fixed. Then the element of $BP_n^{\neq}$ indexing this fixed point is exactly the bi-partition corresponding to the $GL(V)$-orbit of (v, X) in $V \times \mathcal{N}(V)$.*

There is a flat one-parameter family $\mathfrak{X} \to \mathbb{A}^1$ with special fiber $\mathrm{Hilb}^n(\mathbb{A}^2)$ and general fiber $\mathscr{C}_n$. More details about this family are mentioned in Subsection 1.3. A two-dimensional torus $\mathbb{C}^* \times \mathbb{C}^* = T_1 \times T_2$ acts on $\mathfrak{X}$, with T_1 acting "along the base" and T_2 "along the fibers". The special fiber of $\mathfrak{X}$ (isomorphic to $\mathrm{Hilb}^n(\mathbb{A}^2)$) is invariant under $T_1 \times T_2$, with finitely many fixed points indexed by partitions of n. The general fiber of $\mathfrak{X}$ (isomorphic to $\mathscr{C}_n$) is invariant under T_2, with finitely many fixed points indexed by $BP_n^{\neq}$ as explained above. Therefore the action of T_1 flows the T_2-fixed point on $\mathscr{C}_n$ to a $(T_1 \times T_2)$-fixed point on $\mathrm{Hilb}^n(\mathbb{A}^2)$. We rephrase the following proposition (explained in Bellamy–Ginzburg [10, Section 4], e.g.) in our notation.

Proposition 1.8. *Consider a torus-fixed point in $\mathscr{C}_n$ corresponding to a bi-partition $(\lambda,\mu) \in BP_n^{\neq}$. Then it flows to the torus-fixed point on* $\mathrm{Hilb}^n(\mathbb{A}^2)$ *corresponding to the partition ρ of n, where $\beta(\lambda,\mu) = (\rho,\emptyset)$.*

To summarize, the subset $BP_n^{\neq}$ of BP_n indexes several related objects in a natural way, as illustrated in the figure below.

$$\begin{tikzcd} & \text{indecomposable, admissible } Q\text{-reps of} \\ & \text{dimension vector } (1,n)\text{, upto isomorphism} \arrow[d, leftrightarrow, "\cong"] & \\ GL(V)\text{-orbits in } p_n(C_n^0) \arrow[r, leftrightarrow, "\cong"] & BP_n^{\neq} \arrow[d, "\beta"', "\cong"] & T_2\text{-fixed points of } \mathscr{C}_n \arrow[l, leftrightarrow, "\cong"'] \\ & (T_1 \times T_2)\text{-fixed points of } \mathrm{Hilb}^n(\mathbb{A}^2) & \end{tikzcd}$$

It can be checked that $p_n(C_n^0)$ is dense in $V \times \mathscr{N}(V)$. Similarly, the space of indecomposable admissible representations of Q of dimension vector $(1,n)$ is dense in the space of all admissible Q-representations with dimension vector $(1,n)$. This indicates that $BP_n^{\neq}$ can be interpreted as a "dense" subset of BP_n. The rough goal of this project is to complete all components of the above picture so that the indexing set consists of all of BP_n.

This could be achieved by finding a space that extends $p_n\colon C_n^0 \to V \times \mathscr{N}(V)$ to a surjective map that captures all the $GL(V)$-orbits on $V \times \mathscr{N}(V)$. Equivalently, the extension of p_n would capture all (possibly decomposable) admissible representations of Q.

Goals 1.9.

(1) Find a space $\widetilde{C_n^0}$ with an action of $GL(V) \times T_2$ that has the following properties.
   (a) $\widetilde{C_n^0}$ contains C_n^0 as a dense open $(GL(V) \times T_2)$-invariant subset.
   (b) The map $p_n\colon C_n^0 \to V \times \mathscr{N}(V)$ extends to a surjective, proper, $(GL(V) \times T_2)$-equivariant map $p_n\colon \widetilde{C_n^0} \to V \times \mathscr{N}(V)$.

(2) Extend this construction to C_n by factorization, to get a space $\widetilde{C_n}$ that maps surjectively via p_n to $V \times \mathfrak{gl}(V)$.

(3) Take an appropriate GIT quotient by $GL(V)$ of the projection $p_n\colon \widetilde{C_n} \to V \times \mathfrak{gl}(V)$ to construct a relative compactification $\widetilde{\mathscr{C}_n}$ of $\mathscr{C}_n$ over $\mathbb{A}^n/S_n$.

We also aim to extend this construction to the family $\mathfrak{X}$.

Goal 1.10. Extend the construction of $\widetilde{\mathscr{C}_n}$ to the family $\mathfrak{X}$ to get a new family $\widetilde{\mathfrak{X}}$ with the following properties, which extend the known picture for the family $\mathfrak{X}$.

(1) The action of $(T_1 \times T_2)$ extends to $\widetilde{\mathfrak{X}}$, where T_1 acts along the base and T_2 acts along the fibers.

(2) Over the general fiber of $\widetilde{\mathfrak{X}}$, the T_2-fixed points are indexed by BP_n.

(3) Over the special fiber of $\widetilde{\mathfrak{X}}$, the $(T_1 \times T_2)$-fixed points are also indexed by BP_n.

(4) Given a T_2-fixed point in a general fiber corresponding to a bi-partition (λ,μ), the action of T_1 flows it to the $(T_1 \times T_2)$-fixed point of the special fiber corresponding to $\beta(\lambda,\mu)$.

1.3. **Relationship with the Hilbert scheme of points on a plane.** Both $\mathscr{C}_n$ and the Hilbert scheme of n points on the plane (denoted $\mathrm{Hilb}^n(\mathbb{A}^2)$) are examples of Nakajima quiver varieties for the same quiver (the Jordan quiver) with different choices of parameters. This interpretation gives rise to a flat one-parameter family $\mathfrak{X} \to \mathbb{A}^1$ whose general fibers are isomorphic to $\mathscr{C}_n$ and whose special fiber is isomorphic to $\mathrm{Hilb}^n(\mathbb{A}^2)$.

Recall the action of the two-dimensional torus $(T_1 \times T_2)$ on $\mathfrak{X}$. As explained in Proposition 1.7 from the previous subsection, it was proved by Wilson [40] that the T_2-fixed points on $\mathscr{C}_n$ are parametrized by partitions of n. Proposition 1.8 from the previous subsection (stated in [10, Section 4]) is a beautiful and non-trivial consequence of this theorem. It states that the T_2-fixed point on $\mathscr{C}_n$ corresponding to a partition λ flows to the $(T_1 \times T_2)$-fixed point on $\mathrm{Hilb}^n(\mathbb{A}^2)$ corresponding to the same partition λ. The known proof is via computation of characters of each torus on the fixed points. A more direct proof, e.g. by an explicit degeneration, seems to be unknown. Finding such a degeneration would aid in understanding the GIT compactifications of $\mathscr{C}_n$ that we aim to construct. It would also be a good step towards solving a conjecture of Bellamy–Ginzburg about embedding the Grothendieck–Springer resolution of $\mathfrak{sl}_2$ into the degenerating family ([10, Conjecture 4.7]).

Goal 1.11. Find an explicit degeneration of each torus-fixed point on $\mathscr{C}_n$ to the corresponding torus-fixed point on $\mathrm{Hilb}^n(\mathbb{A}^2)$.

Example 1.12. As a simple case towards this goal, let λ be a one-hook partition of n, so that $\lambda = (r, 1, \ldots, 1)$ for some $r \leq n$. Suppose that the image of some tuple (X, Y, v, w) in $\mathscr{C}_n$ is the T_2-fixed point corresponding to λ. Assume that each of (X, Y, v, w) is in normal form, as explained in [40, Section 6]. Recall that for each s, the action of T_1 flows this point to some point in the fiber of $\mathfrak{X}$ over s. In particular, for every $s \neq 0$, the point (X, Y, v, w) flows to (sX, sY, sv, sw) in the fiber of $\mathfrak{X}$ over s.

Let P_s be the diagonal matrix with entries $(s^r, s^{r-1}, \ldots, s^1, s^2, \ldots, s^{n-r+1})$. Then the tuple formed by conjugating (sX, sY, sv, sw) by P_s is a different representative in the GL_n-orbit of (sX, sY, sv, sw) in the fiber. In fact, it can be checked that the limit of this new tuple exists as $s \to 0$, and that the resulting $(T_1 \times T_2)$-fixed point in $\mathrm{Hilb}^n(\mathbb{A}^2)$ is precisely the one corresponding to the partition λ.

2. Exotic Hessenberg varieties and variations

This is a joint project with William Graham.

2.1. **Preliminaries.** Let G be a reductive algebraic group over $\mathbb{C}$. Let B be a Borel subgroup. Let $\mathfrak{g}$ and $\mathfrak{b}$ be the Lie algebras of G and B respectively. Then G/B is the flag variety. Suppose that $H \subset \mathfrak{g}$ is a $\mathfrak{b}$-stable vector subspace. We have the map $G \times^B H \to \mathfrak{g}$ given by $[(g, \xi)] \mapsto g \cdot \xi$. For a fixed element $x \in \mathfrak{g}$, the Hessenberg variety corresponding to H and x, denoted $\mathscr{B}(H, x)$, is defined to be the fiber of this map:

$$\mathscr{B}(H, x) = \{[(g, \xi)] \in G \times^B H \mid g \cdot \xi = x\}.$$

Under the projection $G \times^B H \to G/B$, the variety $\mathscr{B}(H, x)$ maps isomorphically to its image. This gives us the following alternate definition of $\mathscr{B}(H, x)$:

$$\mathscr{B}(H, x) = \{g \cdot \mathfrak{b} \in G/B \mid g^{-1} \cdot x \in H\}.$$

We recall a well-known example, which shows that Hessenberg varieties generalize Springer fibers.

Example 2.1. Let $H = \mathfrak{b}$, and let x be an element of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$. Using the second definition, we can write

$$\mathscr{B}(H,x) = \{g \cdot \mathfrak{b} \in G/B \mid g^{-1} \cdot x \in \mathfrak{b}\} \cong \{\mathfrak{b}' \in G/B \mid x \in \mathfrak{b}'\}.$$

Recall the Springer resolution $\widetilde{\mathcal{N}}$ of $\mathcal{N}$, defined as

$$\widetilde{\mathcal{N}} = \{(x,\mathfrak{b}') \in \mathcal{N} \times G/B \mid x \in \mathfrak{b}'\}.$$

The fibers of the projection $\widetilde{\mathcal{N}} \to \mathcal{N}$ are called Springer fibers. Therefore in the case described above, $\mathscr{B}(H,x)$ is simply the Springer fiber over x.

2.2. **Proposed construction of exotic Hessenberg varieties.** Now let G be the symplectic group $\mathrm{Sp}(2n,\mathbb{C})$. In [24], S. Kato defined a variety $\mathfrak{N}$, called the exotic nilpotent cone for G. There is a version of the Springer resolution for $\mathfrak{N}$, called the exotic Springer resolution. The exotic Springer theory for $\mathfrak{N}$ has better behavior than the usual Springer theory for G. The following goal now seems natural.

Goal 2.2. Construct exotic Hessenberg varieties, namely analogs of Hessenberg varieties for the exotic Springer resolution.

We briefly recall facts about $\mathfrak{N}$, and then give a provisional definition of exotic Hessenberg varieties. Let $V \cong \mathbb{C}^{2n}$ be the standard representation of $G = \mathrm{Sp}(2n,\mathbb{C})$, where $\langle \cdot,\cdot \rangle$ is the chosen symplectic form. We follow the expositions of [3, 4]. Let

$$S = \{x \in \mathrm{End}(V) \mid \langle xv, v\rangle = 0 \text{ for every } v \in V\},$$

and let $\mathcal{N}(V)$ be the nilpotent cone inside $\mathrm{End}(V)$. Then

$$\mathfrak{N} = V \times (S \cap \mathcal{N}(V)).$$

Setting $\mathbb{V} = V \oplus S$, we can check that the weights of the maximal torus of G on $\mathbb{V}$ are $\{\pm\epsilon_i\} \cup \{\pm\epsilon_i \pm \epsilon_j\}$ where i and j range from 1 to n and $i < j$. In contrast, the weights of the maximal torus on the adjoint representation are $\{\pm 2\epsilon_i\} \cup \{\pm\epsilon_i \pm \epsilon_j\}$. The positive part of $\mathbb{V}$, denoted $\mathbb{V}^+$, is the direct sum of the weight spaces spanned by the strictly positive weights, namely $\{\epsilon_i\} \cup \{\epsilon_i \pm \epsilon_j\}$. Let $\widetilde{\mathfrak{N}} = G \times^B \mathbb{V}^+$. The action map $\widetilde{\mathfrak{N}} \to \mathbb{V}$, which sends $[(g,v)] \to g \cdot v$, has image exactly $\mathfrak{N}$. The exotic Springer resolution is defined to be the projection $\widetilde{\mathfrak{N}} \to \mathfrak{N}$.

We give the following provisional definition for exotic Hessenberg varieties, analogous to the classical case.

Definition 2.3. Given $x \in \mathbb{V}$, and a B-stable vector subspace $H \subset \mathbb{V}$, we define the exotic Hessenberg variety $\mathfrak{B}(H,x)$ as follows:

$$\mathfrak{B}(H,x) = \{g \cdot \mathfrak{b} \in G/B \mid g^{-1} \cdot x \in H\}.$$

2.3. **Special classes of exotic Hessenberg varieties.** In the classical case, useful results are known about Hessenberg varieties $\mathscr{B}(H,x)$ with restrictions on x. For example, in the paper of De Mari–Procesi–Shayman [15] where Hessenberg varieties were first defined, the authors construct a Białynicki-Birula decomposition for $\mathscr{B}(H,x)$ when x is a regular semisimple element. Explicit combinatorial formulas for cohomology classes given by this decomposition were found in Anderson–Tymoczko [5]. Similarly, affine pavings of Hessenberg varieties have been studied in the cases where x is regular nilpotent (Tymoczko [38]), regular (Precup [31]), minimal nilpotent (Abe–Crooks [1]), etc.

In many of the above cases, the affine pavings found are compatible with the Schubert cell decomposition of G/B. We aim to investigate analogs of these results for special classes of exotic Hessenberg varieties.

Goals 2.4.

(1) Define notions analogous to regular, semisimple, nilpotent, etc. for elements of $\mathbb{V}$.
(2) Understand the geometry and cohomology of special classes of exotic Hessenberg varieties for each of the above classes. In particular, determine whether they have affine pavings.

2.4. **Relationship with the enhanced nilpotent cone.** Recall the enhanced nilpotent cone already discussed in Subsection 1.2. Given a vector space W of dimension n, its enhanced nilpotent cone is the variety $W \times \mathcal{N}(W)$. Recall that the $GL(W)$-orbits on $W \times \mathcal{N}(W)$ are in bijection with the set BP_n of bi-partitions of n. It was shown by Kato [24] that the $\mathrm{Sp}(2n, \mathbb{C})$-orbits on $\mathfrak{N}$ are also in bijection with BP_n. It is known [3] that we have the following embeddings:

$$W \times \mathcal{N}(W) \xhookrightarrow{\varphi_1} \mathfrak{N} \xhookrightarrow{\varphi_2} V \times \mathcal{N}(V).$$

These embeddings have extremely useful properties. For example, they are compatible with the parametrization of orbits by BP_n in the following way. If $(w, y) \in W \times \mathcal{N}(W)$ lies in the $GL(W)$-orbit of (λ, μ), then $\varphi((w, y)) \in \mathfrak{N}$ lies in the $\mathrm{Sp}(2n, \mathbb{C})$-orbit of (λ, μ). Similarly if $(v, x) \in \mathfrak{N}$ lies in the $\mathrm{Sp}(2n, \mathbb{C})$-orbit of (λ, μ), then $\varphi_2((v, x)) \in V \times \mathcal{N}(V)$ lies in the $GL(V)$-orbit of the "doubled" partition $(\lambda \cup \lambda, \mu \cup \mu)$. Moreover, the closure orderings are respected.

Since the enhanced nilpotent cone was originally introduced as an easier version of the exotic nilpotent cone, it is natural to extend our study of Hessenberg varieties to the enhanced nilpotent cone. We expect this theory to be close to the classical theory of Hessenberg varieties. For example, it has been shown by Mautner [27] that Springer-type fibers for the enhanced nilpotent cone have affine pavings. At the same time, we expect the Hessenberg variety with parameter in a specified $GL(W)$-orbit of the enhanced nilpotent cone to give useful information about the exotic Hessenberg variety for the same $\mathrm{Sp}(2n, \mathbb{C})$-orbit. To this end, we have the following goals.

Goals 2.5.

(1) Study Hessenberg varieties for particular GL-orbits in the enhanced nilpotent cone.
(2) Extend the embeddings φ_1, φ_2 to the corresponding Hessenberg varieties.

3. Bernstein–Sato polynomials and monodromy conjectures

This is a joint project with Robin Walters.

3.1. **Definitions and background.** Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and let $\mathscr{D}$ be the ring of differential operators in n variables with polynomial coefficients. Let $\mathscr{D}[s]$ be the ring obtained by attaching the formal variable s, namely $\mathscr{D}[s] = \mathscr{D} \otimes_{\mathbb{C}} \mathbb{C}[s]$. We can define the standard action of $\mathscr{D}[s]$ on the formal symbols f^{s+j} for any j, as follows: any polynomial $g \in \mathbb{C}[x_1, \ldots, x_n, s]$ acts by formal multiplication, while a differential operator acts by "differentiating" f^{s+j} using the Leibniz rule. For example,

$$\partial_{x_i} f^{s+j} = (s+j)(\partial_{x_i} f) f^{s+j-1}.$$

The Bernstein–Sato polynomial of f, denoted $b_f(s)$, is defined to be the monic polynomial of minimal degree that satisfies the following equation for some $P(s) \in \mathscr{D}[s]$:

$$P(s)f^{s+1} = b_f(s)f^s.$$

The Bernstein–Sato polynomial of f is also called the b-function of f.

Let $\mathscr{D}[s]f^s$ be the cyclic $\mathscr{D}[s]$-module generated by the symbol f^s, where $\mathscr{D}[s]$ acts as defined above. Similarly, consider $\mathscr{D}[s]f^{s+1}$, which is a $\mathscr{D}[s]$-submodule of $\mathscr{D}[s]f^s$. Then the Bernstein–Sato polynomial of f can also be defined as the (monic) minimal polynomial of the action of multiplication by s on the quotient $\mathscr{D}[s]$-module $\mathscr{D}[s]f^s/\mathscr{D}[s]f^{s+1}$.

More generally, the construction of the b-function can be localized near any point to yield the local b-function near that point. Several connections are known between the (local) b-function of f and the singularities of the hypersurface cut out by f, denoted $V(f)$. For example, $V(f)$ is smooth at a given point if and only if the local b-function of f near that point equals $(s+1)$. On the other hand, explicitly computing b-functions is a very hard problem in general, so we restrict to special cases where we can exploit some additional symmetry.

3.2. **Weyl arrangements and other hyperplane arrangements.** Let $\mathfrak{g}$ be a semi-simple Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be its Cartan subalgebra, let R be its root system, and let W be the Weyl group. Also fix a choice R^+ of positive roots in R. Consider the following polynomial, which is the product of the positive roots:

$$\xi = \left(\prod_{\alpha \in R^+} \alpha\right) \in \mathbb{C}[\mathfrak{h}].$$

We define the corresponding Weyl arrangement $V(\xi)$ to be the hyperplane arrangement cut out in $\mathfrak{h}$ by ξ. Surprisingly, the b-functions of these arrangements are unknown in general.

Goal 3.1. Understand the b-function of the hyperplane arrangement ξ.

We explain the approach and some partial work towards this goal. Several pieces of information about the Bernstein–Sato polynomial of a hypersurface can be extracted from a resolution of singularities of the hypersurface to normal crossings divisors. The b-function of a normal crossings divisor is easy to compute. Kashiwara [23] proved that the b-function of a (singular) hypersurface always divides a product of integer shifts of the b-function of its resolution of singularities. By this method, one can get access to an upper bound on the b-function of the hypersurface. Although it is not usually known how many integer shifts are needed, a result of Saito [35] implies that for hyperplane arrangements, a shift by at most two is sufficient.

An explicit resolution of singularities can be worked out for Weyl arrangements, using methods of Fulton–MacPherson [22] and De Concini–Procesi [14]. This resolution is combinatorially complicated to analyze. However, it is constructed inductively by a series of consecutive blow-ups, which are combinatorially more manageable. Our first aim is to find a concrete upper bound on the b-function of ξ by step-wise computations for each consecutive blow-up.

Example 3.2. For type A_n, the function ξ is simply the Vandermonde determinant of n variables:

$$\xi_n = \prod_{1 \le i < j \le n} (x_i - x_j).$$

The strategy explained above gives us the following (inductive) upper bound for the b-function of ξ_n.

Proposition 3.3 (B.–Walters, [8]). *For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, denoted $\lambda \vdash n$, let $b_\lambda(s)$ denote the product of the b-functions of ξ_{λ_i} for each i. We have the following divisibility relation:*

$$b_{\xi_n}(s) \,\Big|\, \operatorname*{lcm}_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} b_\lambda(s) \cdot \prod_{i=(n-1)}^{(n-1)^2} \left(s + \frac{i}{\binom{n}{2}} \right).$$

Another approach towards the problem of computing b-functions of Weyl arrangements is to use known relationships between the b-function and other known singularity invariants, such as multiplier ideals and jumping numbers. Recall that for a fixed rational number s, the multiplier ideal corresponding to f and s, denoted $\mathscr{I}(f,s)$, is the ideal consisting of all functions g such that gf^{-s} is locally integrable. When $s < 0$, the multiplier ideal is the unit ideal. However, as s increases, the multiplier ideal shrinks. A number s is called a jumping number for f if it is a value at which the multiplier ideal changes size. That is, for every $\epsilon > 0$, we have $\mathscr{I}(f,s) \subsetneq \mathscr{I}(f, s-\epsilon)$. It was proved in Ein–Lazarsfeld–Smith–Varolin [19] that if s is a jumping number for f and if $0 < s \leq 1$, then $b_f(-s) = 0$. Although this does not exhaust all roots of the b-function, it would be fruitful to compute the jumping numbers of ξ, for example by using the methods of [28, 36].

A motivating result towards finding the b-function of ξ is the computation of a related b-function carried out by Opdam [30]. Recall that ξ^2 is a W-invariant function on $\mathfrak{h}$. By the Chevalley–Shephard–Todd theorem, we know that $\mathbb{C}[\mathfrak{h}]^W \cong \mathbb{C}[\mathfrak{h}/W]$ is a polynomial ring in the same number of variables as $\mathbb{C}[\mathfrak{h}]$. Fix a set of generators of the polynomial ring $\mathbb{C}[\mathfrak{h}/W]$, which map to homogeneous polynomials in $\mathbb{C}[\mathfrak{h}]^W$. Then $\xi^2 \in \mathbb{C}[\mathfrak{h}]^W$ corresponds to an element $g \in \mathbb{C}[\mathfrak{h}/W]$. The following theorem gives a beautiful closed formula for the b-function of g.

Theorem 3.4 (Opdam [30]). *The b-function of g has the following formula:*

$$b_g(s) = \prod_{i=1}^{\mathrm{rk}(\mathfrak{g})} \prod_{j=1}^{d_i - 1} \left(s + \frac{1}{2} + \frac{i}{d_i} \right),$$

where the numbers d_i are the fundamental invariants of $\mathfrak{g}$, also realized as the degrees of the homogeneous generators in $\mathbb{C}[\mathfrak{h}]^W$.

This theorem does not immediately yield information about the b-function of ξ^2 or that of ξ, because the two rings of differential operators $\mathscr{D}(\mathfrak{h})^W$ and $\mathscr{D}(\mathfrak{h}/W)$ are quite different. On the other hand, we have proved the following theorem connecting the two.

Theorem 3.5 (B.–Walters, [7]). *The b-functions of g and of ξ have the following relationship:*

$$b_g(s) \mid b_\xi(2s+1).$$

Since Opdam's theorem describes $b_g(s)$ purely in terms of the data of the Lie algebra and the root system, we would like to find a similar description for $b_\xi(s)$. The previous theorem was proved via a computation using the Harish-Chandra map for differential operators, which maps $\mathscr{D}(\mathfrak{g})^G$ to $\mathscr{D}(\mathfrak{h})^W$. We hope to leverage this map further to get more information about $b_\xi(s)$.

Goal 3.6. Describe $b_\xi(s)$ in terms of the data of $\mathfrak{g}$ and the root system.

3.3. **Monodromy conjectures.** As has already been discussed, it is known that b-function of a polynomial f is related to other singularity invariants of the hypersurface $V(f)$. Two singularity invariants that we particularly focus on in this section are: (1) the eigenvalues of the monodromy operator on the cohomology of the Milnor fiber, and (2) the local topological zeta function. We give a brief overview.

Suppose that $p \in V(f)$, so that $f(p) = 0$. Let $\delta, \epsilon > 0$. Let $B(p, \epsilon)$ be the closed ball of radius ϵ around p, and let $B(0, \delta)$ be the closed ball of radius δ around 0. Then for any sufficiently small δ and ϵ, the pieces $B(p, \epsilon) \cap f^{-1}(\partial B(0, \delta))$ are all homeomorphic and form a fibration over $\partial(B(0, \delta))$. Each fiber is called a Milnor fiber. Traveling once around $\partial B(0, \delta)$ induces an automorphism of the Milnor fiber, which gives rise to the monodromy map on cohomology. It is an old result of Malgrange [26] that if r is a root of the b-function of f, then $\exp(2\pi i r)$ is an eigenvalue of a monodromy operator on the cohomology of the Milnor fiber.

The local topological zeta function (LTZF) was defined by Denef–Loeser in [16]. The LTZF of a polynomial f is an invariant of the singularities of $V(f)$ constructed from a resolution of singularities of $V(f)$ (although it is independent of the chosen resolution). It is a rational function of one variable, and has a formula in terms of the multiplicities of the various exceptional divisors in the resolution.

The monodromy conjecture of Denef–Loeser states that if r is a pole of the LTZF, then $\exp(2\pi i r)$ is an eigenvalue of the monodromy operator on the cohomology of the Milnor fiber. A stronger version of the monodromy conjecture states that if r is a pole of the LTZF of f, then r is also a root of the b-function of f. The monodromy conjecture follows from the strong monodromy conjecture because of Malgrange's theorem.

The monodromy conjecture has been proved for hyperplane arrangements [12]. The strong monodromy conjecture has been proved in special cases, but not for all hyperplane arrangements. We have the following result in this direction, which is a consequence of Theorem 3.5 together with the results of [12].

Theorem 3.7 (B.–Walters, [7])**.** *The strong monodromy conjecture is true for all Weyl hyperplane arrangements.*

A more general version (but with a different proof strategy) was proved in [39] to include several other cases for which the strong monodromy conjecture holds, including the reflection arrangements of all Coxeter groups (of not necessarily Weyl type). As a consequence of this theorem, one can find several roots of the b-function by computing poles of the LTZF. We aim to use this method to investigate the b-functions for Coxeter arrangements.

Goal 3.8. Find roots of the b-functions of Coxeter hyperplane arrangements by computing poles of the local topological zeta function.

The local topological zeta function can be obtained by specializing a more general invariant defined by Denef–Loeser, called the motivic zeta function [17]. Although defined as a certain motivic integral, this object was shown by Denef–Loeser to be a rational function over the Grothendieck ring of varieties. There are versions of the monodromy conjectures for the motivic zeta function as well. These seem to be unexplored in the case of Weyl arrangements.

Goal 3.9. Investigate the motivic zeta functions of Weyl arrangements, and prove the motivic monodromy conjectures in this case.

Since both the LTZF as well as the motivic zeta functions can be described in explicit formulas using a resolution of singularities, they can be computed in small cases (e.g. with the help of a computer). Further, the hyperplane arrangements we consider have the extra structure of the root data, and combinatorial descriptions of resolutions of singularities are known [22, 14]. So we expect to be able to simplify the known formulas, and make explicit calculations towards both of the above goals.

Broader impacts

3.4. **Organization.** I have been active in organizing research and educational activities at various levels. I plan to organize the following activities in the near future.

(1) *High School Varsity Math Tournament* at the University of Georgia (October 2017): I will assist with the organization of this annual math competition for high school students in Athens, GA.
(2) *MathCamp* at the University of Georgia (June 2017): I will help to arrange the academic structure of this camp. In particular, I will help to match faculty mentors with groups of undergraduate students to work on exploratory projects.

I currently organize the following.

(3) The weekly *Algebra Seminar* at the University of Georgia, together with Paul Sobaje (current): The goal of this seminar is to exchange ideas about recent and ongoing research projects in algebra and related areas. The speakers of this seminar are members of the Algebra group at UGA, as well as invited speakers from outside departments.

In the recent past, I have organized the following.

(4) I was *Academic Coordinator* at *Canada/USA Mathcamp* together with Ruthi Hortsch (Summer 2015): We curated the academic schedule of the 5-week camp for high school students based on proposals from the instructors, balancing prerequisites and "core" courses against more advanced ones. We invited visiting speakers to give colloquium talks, and arranged academic advising as well as mentoring of junior instructors by senior instructors.
(5) I was *Vice-President* and *Colloquium Coordinator* of the University of Chicago chapter of the *Association for Women in Mathematics* (2013–2015): I helped organize quarterly AWM meetings, and various AWM activities for the undergraduate and graduate students at the University of Chicago. As colloquium coordinator, I invited and hosted distinguished women mathematicians to give colloquium talks in the department.
(6) I organized the *Ramanujan Mathematics Talent Nurture Camp* in Pune, India, together with Aniruddha Bapat, Kalyani Katariya, and Niranjan Toradmal (2012). This was a one-day mathematics workshop for talented middle-school students from underprivileged backgrounds.
(7) I organized the *Student Representation Theory Seminar* at the University of Chicago (2012–2014): This was a weekly graduate student learning seminar on fundamental work in representation theory, as well as topics of current research.
(8) I organized the *Pizza Seminar* at the University of Chicago, together with Simion Filip (2011–2012): This was a weekly graduate student seminar on mathematics of general interest.

3.5. **Education and mentoring.** I am enthusiastic about teaching and mentoring. I have taught at various levels over the past several years, and I plan to remain actively involved in mathematics education. In addition to my regular teaching duties, I plan to carry out the following educational activities in the near future.

(1) I plan to give expository lectures in the University of Georgia's *Undergraduate Mathematics Club*.
(2) I plan to be a faculty mentor to a group of high school students in the University of Georgia's *MathCamp* (June 2017).

In the recent past, I have been involved in the following educational activities.

(3) I was an instructor at *Canada/USA Mathcamp*, where I taught several mini-courses in advanced undergraduate-level mathematics to high-school students (2012, 2013, 2015).
(4) I supervised four undergraduate reading projects at the University of Chicago, as part of their *Directed Reading Program* (2012–2014).
(5) I supervised two undergraduate reading projects at the University of Chicago's summer *Research Experience for Undergraduates* (REU) program (2011).
(6) I gave an expository talk to undergraduates at the University of Chicago's *Undergraduate Mathematics Club* (2015).
(7) I gave expository lectures to first-year graduate students as part of the University of Chicago's *Warmup Program for Entering Graduate Students* (2012, 2015).

3.6. **Software dissemination.** In my past work [6, 8] as well as in many of the projects mentioned in this proposal, I have extensively used computer algebra systems such as *Macaulay2, Magma, Sage*, and *Singular*. I am enthusiastic about the role of software in mathematics research, and I am a proponent of open source code.

In the near future, I plan to refactor my code into pieces that will be useful to the wider mathematical community. I plan to make my code publicly available through services such as *Github* and *Bitbucket*.

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