RECOLLEMENT FOR PERVERSE SHEAVES ON REAL HYPERPLANE ARRANGEMENTS

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Abstract. We consider a hyperplane arrangement in $\mathbb{C}^n$ defined over $\mathbb{R}$, and the associated natural stratification of $\mathbb{C}^n$. The category of perverse sheaves smooth with respect to this stratification was described by Kapranov and Schechtman in terms of quiver representations. Using work of Weissman, we reinterpret this category as the category of finite-dimensional modules over an explicit algebra. We also describe *recollement* (open-closed decomposition) of perverse sheaves in terms of this module category. As an application, we identify the modules associated to all intersection cohomology complexes.

We also compute recollement for W-equivariant perverse sheaves for the reflection arrangement of a finite Coxeter group W. We identify the equivariant intersection cohomology sheaves arising as intermediate extensions of local systems on the open stratum, thereby answering a question of Weissman.

1. Introduction

Let X be a smooth complex algebraic variety equipped with an algebraic stratification $\mathscr{S}$. Let k be a fixed field of coefficients. The category of perverse sheaves smooth with respect to the stratification $\mathscr{S}$ is a certain abelian subcategory of the bounded derived category of cohomologically $\mathscr{S}$-constructible sheaves of k-vector spaces on X [1]. This category is typically denoted $\mathrm{Perv}(X,\mathscr{S})$. For simplicity, assume that there is a unique, connected open stratum. Then $\mathrm{Perv}(X,\mathscr{S})$ is a natural generalisation of the category of finite-rank k-local systems on the open stratum, which has an algebraic description as the category of representations of the fundamental group of the open stratum. A general question is to realise $\mathrm{Perv}(X,\mathscr{S})$ as the category of representations of an explicit algebra, extending the above algebraic description of the category of local systems on the open stratum. There are some general strategies to answer this question [2, 14], and complete answers are known in some cases [10, 11, 5, 4]. Our main focus is the algebraic description of perverse sheaves on real hyperplane arrangements due to Kapranov–Schechtman [12], as well as its equivariant analogue for Coxeter arrangements due to Weissman [19].

In general, $\mathrm{Perv}(X,\mathscr{S})$ can be reconstructed from the categories of perverse sheaves on the open stratum and on its complement. This setup is called gluing or *recollement* [1]. Details about recollements on abelian categories are in Section 3. One of the goals of this paper is to extend the algebraic descriptions of perverse sheaves from [12] and [19] to their open/closed recollements, as well as to the subcategories of sheaves supported on closed unions of strata. We now introduce the setup.

Let $\mathscr{H}$ be an arrangement of linear hyperplanes in $\mathbb{R}^n$. Let $(\mathscr{C},\leq)$ be the partially ordered set of faces of $\mathscr{H}$, ordered by closure-inclusion. Let $\mathscr{H}_{\mathbb{C}}$ denote the complexification of $\mathscr{H}$ in the complex vector space $X=\mathbb{C}^n$. Then $\mathscr{H}_{\mathbb{C}}$ determines a natural stratification $\mathscr{S}$ on X. More details are in Subsection 2.1. Kapranov and Schechtman [12] describe $\mathrm{Perv}(X,\mathscr{S})$ as a certain subcategory of the category of representations of the double quiver of the poset $(\mathscr{C},\leq)$. If $\mathscr{H}$ is the reflection arrangement of a finite Coxeter group

Date: October 31, 2018.

W, then Weissman [19] extended the above description to the category $\mathrm{Perv}_W(X, \mathscr{S})$, the category of W-equivariant perverse sheaves. Further, Weissman constructed an explicit algebra such that $\mathrm{Perv}_W(X, \mathscr{S})$ is equivalent to the finite-dimensional module category over this algebra. In Subsection 2.2, we construct a similar explicit algebra R associated to any real arrangement $\mathscr{H}$.

The main results of the paper are summarised as follows.

(1) Theorem 2.7 gives an equivalence of categories between $\mathrm{Perv}(X, \mathscr{S})$ and the category of finite-dimensional R-modules.
(2) Theorem 4.7 proves that the recollement of $\mathrm{Perv}(X, \mathscr{S})$ into its open and closed pieces is equivalent to a certain recollement for R-modules. Corollary 4.9 identifies the R-modules associated to the intersection cohomology sheaves on X arising as intermediate extensions of local systems on the open stratum.
(3) Theorem 4.10 describse the category of perverse sheaves supported on a closed union of strata of $\mathscr{S}$ as the category of finite-dimensional modules over a certain quotient of R. Corollary 4.13 identifies the R-modules associated to all intersection cohomology sheaves on X.
(4) Section 5 discusses the W-equivariant setup for the reflection arrangement of a finite Coxeter group W. Theorem 5.2 is the W-equivariant analogue of Theorem 4.7. Corollary 5.3 is the W-equivariant analogue of Corollary 4.9.

Acknowledgements. I am indebted to Martin Weissman for helpful conversations and clarifications. I am grateful to Thomas Gobet for a useful discussion, and to Corey Jones for suggesting the idea of the proof of Proposition 6.1.

2. Perverse sheaves on real hyperplane arrangements: an alternate description

The main theorem of [12] states that the category of perverse sheaves on $\mathbb{C}^n$ constructible with respect to $\mathscr{S}$ is equivalent to a particular subcategory of representations of a quiver. It will be more transparent for us to describe this subcategory as the module category of an explicit algebra. The aim of this section is to construct this algebra, and then to translate the data of one of the above admissible quiver representations into the data of a module over this algebra. We denote this algebra by R.

The results of this section should be thought of as parallel to a similar translation from [19, Section 4.3], which focuses on equivariant perverse sheaves on Coxeter arrangements. The main theorem in this section is Theorem 2.7, which obtains an equivalence of categories between $\mathrm{Perv}(X, \mathscr{S})$ and finite-dimensional R-modules.

2.1. Real hyperplane arrangements and double quiver representations.

This subsection recalls some definitions following [12] and [19], as well as the main theorem of [12]. For much of the paper, fix $\mathscr{H}$ to be a hyperplane arrangement in $\mathbb{R}^n$, and let $\mathscr{H}_{\mathbb{C}}$ be its complexification in $X = \mathbb{C}^n$. For each $H \in \mathscr{H}$, also fix a real linear polynomial f_H that cuts out H.

For each $x \in \mathbb{R}^n$, its real sign vector $\sigma(x) \in \{+, -, 0\}^{\mathscr{H}}$ is defined as follows. For any $H \in \mathscr{H}$, the sign $\sigma(x)_H$ at that hyperplane is either $+$, $-$, or 0, depending on whether $f_H(x)$ is positive, negative, or zero respectively. We then have an equivalence relation on $\mathbb{R}^n$, where $x \sim y$ if $\sigma(x) = \sigma(y)$. Equivalence classes in $\mathbb{R}^n$ with respect to this equivalence relation are called the *faces* of $\mathscr{H}$, and they form the *face poset* $\mathscr{C}$. The partial order on $\mathscr{C}$ is closure-inclusion: if $C', C \in \mathscr{C}$, then $C' \leq C$ if $C' \subseteq \overline{C}$. We will denote by $\mathscr{C}^i$ the subset of $\mathscr{C}$ consisting of faces of real codimension i. In particular, elements of $\mathscr{C}^0$ are sometimes called chambers.

For each $x \in X$, its complex sign vector $\sigma_{\mathbb{C}}(x) \in \{*, 0\}^{\mathscr{H}}$ is defined as follows. For any $H \in \mathscr{H}_{\mathbb{C}}$, the complex sign $\sigma_{\mathbb{C}}(x)_H$ at that hyperplane either equals $*$ or 0 depending on whether $f_H(x)$ is non-zero or zero respectively. If $x, y \in X$, we say that $x \sim_{\mathbb{C}} y$ if $\sigma_{\mathbb{C}}(x) = \sigma_{\mathbb{C}}(y)$. The set of equivalence classes in X with respect to this equivalence relation forms a stratification of X, denoted by $\mathscr{S}$. We consider the category $\mathrm{Perv}(X, \mathscr{S})$ of perverse sheaves on $X = \mathbb{C}^n$ that are smooth with respect to $\mathscr{S}$.

Definition 2.1. Let A and B be faces in $\mathscr{C}^d$, with $d \geq 1$. We say that A *opposes* B through C if the following conditions hold.
(1) The real linear spans of A and B are equal.
(2) There is some face $C \in \mathscr{C}^{(d-1)}$ such that $C \leq A$, $C \leq B$, and A and B lie on opposite sides of C. More precisely, for every hyperplane $H \in \mathscr{H}$ such that $\sigma(C)_H = 0$, we have $\sigma(A)_H = -\sigma(B)_H$.

Definition 2.2. We say that three faces A, B, and C in $\mathscr{C}$ are *collinear* if a straight line segment can be drawn in $\mathbb{R}^n$ that intersects the faces A, B, and C in that order.

Definition 2.3. A double representation E of $\mathscr{C}$ consists of the following data.
(1) A vector space E_C for each $C \in \mathscr{C}$.
(2) Maps $\gamma_{C'C} \colon E_{C'} \to E_C$ and $\delta_{CC'} \colon E_C \to E_{C'}$ for every $C' \leq C$.
For each $C \in \mathscr{C}$, we require $\gamma_{CC} = \delta_{CC} = \mathrm{id}_{E_C}$. Additionally, for each C_1, C_2, C_3 in $\mathscr{C}$ such that $C_1 \leq C_2 \leq C_3$, we require

$$\gamma_{C_2C_3}\gamma_{C_1C_2} = \gamma_{C_1C_3} \text{and } \delta_{C_2C_1}\delta_{C_3C_2} = \delta_{C_3C_1}.$$

A morphism $V \to W$ of double representations consists of vector space maps $V_C \to W_C$ for each $C \in \mathscr{C}$, that intertwine with each $\gamma_{C'C}$ and $\delta_{CC'}$. Double representations of $\mathscr{C}$ form an abelian category.

Definition 2.4. The category $\mathscr{A}$ is defined to be the full subcategory of double representations of $\mathscr{C}$ consisting of objects E satisfying the following three properties.
(1) Monotonicity: for every $C' \leq C$, we have $\gamma_{C'C}\delta_{CC'} = \mathrm{id}_{E_C}$. As a consequence, we have well-defined maps $\varphi_{AB} \colon E_A \to E_B$ for any $A, B \in \mathscr{C}$, defined as follows. For any $C \in \mathscr{C}$ satisfying $C \leq A$ and $C \leq B$, set $\varphi_{AB} = \gamma_{CB}\delta_{AC}$.
(2) Transitivity: for any $A, B, C \in \mathscr{C}$ that are collinear (in that order), we have $\varphi_{AC} = \varphi_{BC}\varphi_{AB}$.
(3) Invertibility: for any $A, B \in \mathscr{C}$ that oppose each other, the map φ_{AB} is an isomorphism.

Theorem 2.5 ([12, Theorem 8.1]). *There is an equivalence of categories*

$$\mathbf{Q} \colon \mathrm{Perv}(X, \mathscr{S}) \to \mathscr{A}.$$

2.2. **The algebra** R. We construct an algebra R to encode the information of the category $\mathscr{A}$. First define R_0 to be the algebra generated freely over $\mathbb{C}$ by the set $\{e_C \mid C \in \mathscr{C}\}$, subject to the following relations.
(R1) Every generator is idempotent. That is, for any face C, we have $e_C^2 = e_C$.
(R2) For any three collinear faces A, B, and C, we have:

$$e_A e_C = e_A e_B e_C.$$

(R3) Whenever $A \leq B$, we have

$$e_A e_B = e_B = e_B e_A.$$

Then the idempotent corresponding to the unique smallest face (corresponding to the origin of $\mathbb{R}^n$) is the unit of R_0. We define R to be the noncommutative localisation of R_0

at the multiplicative subset generated by the following elements, for any two opposing faces A and B:

$$e_A e_B e_A + (1 - e_A).$$

If A and B are two opposing faces, let s_{AB} denote the multiplicative inverse of the element $e_A e_B e_A + (1 - e_A)$, which means that

$$(e_A e_B e_A + (1 - e_A))s_{AB} = s_{AB}(e_A e_B e_A + (1 - e_A)) = 1.$$

Since e_A is an idempotent, $(1-e_A)$ is also an idempotent. Moreover, $e_A(1-e_A) = (1-e_A)e_A = 0$. Multiplying the above equation on the left and right by $(1 - e_A)$ and e_A respectively, we conclude that

$$(1 - e_A)s_{AB} = s_{AB}(1 - e_A) = (1 - e_A),$$
$$e_A e_B e_A s_{AB} = s_{AB} e_A e_B e_A = e_A.$$

The first equation also implies that

$$e_A s_{AB} = s_{AB} e_A = e_A s_{AB} e_A. \tag{1}$$

2.3. **Double quiver representations and R-modules.** As before, let $\mathscr{A}$ be the category of monotonic, transitive, and invertible double representations of $\mathscr{C}$. Let R-mod be the category of finite-dimensional (left) R-modules. We now define functors $\mathbf{M}\colon \mathscr{A} \to R$-mod and $\mathbf{N}\colon R$-mod $\to \mathscr{A}$, and show that they are mutually inverse equivalences.

Let $E = (E_C, \gamma_{C'C}, \delta_{CC'})$ be an object of $\mathscr{A}$. Let Z be the unique smallest face in $\mathscr{C}$. Set the underlying vector space of $\mathbf{M}(E)$ to be E_Z. For each generator e_C of R, set $e_C\colon E_Z \to E_Z$ to be the map $\delta_{CZ}\gamma_{ZC}$.

Proposition 2.6. *Given an object E of $\mathscr{A}$, the vector space E_Z is an R-module via the actions $e_C(v) = \delta_{CZ}\gamma_{ZC}(v)$ for each $C \in \mathscr{C}$.*

Proof. We check that the actions defined for the generators e_C satisfy the relations of R, and also that each element of the localised subset acts invertibly on E_Z.

(R1) Since $\gamma_{ZC}\delta_{CZ} = \mathrm{id}_{E_C}$ for each C, the action of every e_C is idempotent.

(R2) Let A, B, and C be three collinear faces. Then for each $v \in E_Z$, we have

$$e_A e_B e_C(v) = \delta_{AZ}\gamma_{ZA}\delta_{BZ}\gamma_{ZB}\delta_{CZ}\gamma_{ZC}(v).$$

Recall that we have

$$\gamma_{ZA}\delta_{BZ}\gamma_{ZB}\delta_{CZ} = \varphi_{BA}\varphi_{CB} = \varphi_{CA} = \gamma_{ZA}\delta_{CZ},$$

where the second equality is by the transitivity property of E. All together we have

$$e_A e_B e_C(v) = \delta_{AZ}\gamma_{ZA}\delta_{CZ}\gamma_{ZC}(v) = e_A e_C(v).$$

(R3) Suppose that $A \leq B$. Then for each $v \in E_Z$ we have

$$e_A e_B(v) = \delta_{AZ}\gamma_{ZA}\delta_{BZ}\gamma_{ZB}(v).$$

Since $A \leq B$, we have $\delta_{BZ} = \delta_{AZ}\delta_{BA}$. Moreover, $\gamma_{ZA}\delta_{AZ} = \mathrm{id}_{E_A}$. So

$$e_A e_B(v) = \delta_{AZ}\delta_{BA}\gamma_{ZB}(v) = \delta_{BZ}\gamma_{ZB}(v) = e_B(v).$$

A similar computation in the other direction (using $\gamma_{ZB} = \gamma_{AB}\gamma_{ZA}$) shows that $e_B(v) = e_B e_A(v)$.

Now suppose that A and B oppose each other. Note that $E_Z = e_A E_Z \oplus (1-e_A)E_Z$, and the element $e_A e_B e_A + (1-e_A)$ acts by $e_A e_B e_A$ on $e_A E_Z$ and by $(1-e_A)$ on $(1-e_A)E_Z$. The latter action is by the identity map, and so obviously an isomorphism.

So checking that the map $e_A e_B e_A + (1-e_A)$ is invertible on E_Z is equivalent to checking that $e_A e_B e_A$ is invertible on $e_A E_Z$. Since E_Z is finite-dimensional, it is sufficient to check that $e_A e_B e_A$ is injective on $e_A E_Z$.

Suppose that for some $e_A(v) \in e_A E_Z$, we have $e_A e_B e_A(v) = 0$. Recall that

$$e_A e_B e_A(v) = \delta_{AZ}\gamma_{ZA}\delta_{BZ}\gamma_{ZB}\delta_{AZ}\gamma_{ZA}(v) = \delta_{AZ}\varphi_{BA}\varphi_{AB}\gamma_{ZA}(v).$$

Since A and B oppose each other, the maps φ_{AB} and φ_{BA} are both isomorphisms. Since $Z \leq A$, the map δ_{AZ} is injective. This means that if $e_A e_B e_A(v) = 0$, then $\gamma_{ZA}(v) = 0$. Therefore $\delta_{AZ}\gamma_{ZA}(v) = e_A(v) = 0$. This calculation implies that $e_A e_B e_A$ is injective (and hence an isomorphism) on $e_A E_Z$. Consequently, $e_A e_B e_A + (1-e_A)$ is an isomorphism on E_Z. These checks prove that E_Z acquires the structure of an R-module via the actions defined. □

We now define $\mathbf{M}$ on morphisms of $\mathscr{A}$. Given $\alpha' = (E'_C, \gamma'_{C'C}, \delta'_{CC'})$ and a morphism $f : \alpha \to \alpha'$, set $\mathbf{M}(f)$ to be the map $f_Z : E_Z \to E'_Z$. For any $C \in \mathscr{C}$, observe that

$$e_C f_Z = \delta_{CZ}\gamma_{ZC} f_Z = \delta_{CZ} f_C \gamma'_{ZC} = f_Z \delta'_{CZ}\gamma'_{ZC} = f_Z e_C.$$

Since f commutes with all generators, it is a map of R-modules.

Having defined $\mathbf{M} : \mathscr{A} \to R$-mod, we now define $\mathbf{N} : R$-mod $\to \mathscr{A}$. Let V be an R-module. Define $\mathbf{N}(V) = (V_C, \gamma'_{C'C}, \delta'_{CC'})$ as follows. For every $C \in \mathscr{C}$, set $V_C = e_C V$. Recall that for every $C' \leq C$, we have $e_C = e_{C'} e_C$. Set $\gamma'_{C'C} : e_{C'}V \to e_C V$ to be multiplication by e_C. Set $\delta'_{CC'} : e_C V \to e_{C'} V$ to be the natural inclusion map given by $e_C v \mapsto e_{C'} e_C v$. We can think of $\delta'_{CC'}$ as being multiplication by $e_{C'}$. To ensure that $\mathbf{N}(V)$ is an object of $\mathscr{A}$, we check the following three conditions.

Monotonicity: Let $C' \leq C$. Then for any $e_C v \in e_C V$, we have

$$\gamma'_{C'C}\delta'_{CC'}(e_C v) = \gamma'_{C'C}(e_{C'} e_C v) = e_C e_{C'} e_C v = e_C v.$$

Therefore $\gamma'_{C'C}\delta'_{CC'} = \mathrm{id}_{V_C}$.

Transitivity: Let A, B, and C be collinear faces. For any $e_A v \in e_A V$, we have

$$\begin{aligned}
\varphi_{BC}\varphi_{AB}(e_A V) &= \gamma'_{ZC}\delta'_{BZ}\gamma'_{ZB}\delta'_{AZ}(e_A v) \\
&= e_C e_Z e_B e_Z e_A v \\
&= e_C e_B e_A v \\
&= e_C e_A v \quad \text{(by the relation in } R\text{)} \\
&= e_C e_Z e_A v = \varphi_{AC} v
\end{aligned}$$

Invertibility: Let A and B be two faces that oppose each other. Observe that

$$\varphi_{AB}\varphi_{BA} = e_B e_A e_B,$$

which is invertible on $e_B V$. Similarly $\varphi_{BA}\varphi_{AB} = e_A e_B e_A$, which is invertible on $e_A V$. The first relation implies that φ_{AB} is surjective and φ_{BA} is injective. The second relation implies that φ_{BA} is surjective and φ_{AB} is injective. So both φ_{AB} and φ_{BA} are isomorphisms.

Therefore $\mathbf{N}$ sends R-modules to objects in $\mathscr{A}$. On a morphism $f : V \to W$ of R-modules, we set $\mathbf{N}(f)_C : e_C V \to e_C W$ to be $\mathbf{N}(f)_C = e_C f = f e_C$.

2.4. **Equivalence of categories.** Combining the constructions above with Kapranov–Schechtman's theorem, we obtain the following theorem.

Theorem 2.7. *The functors* $\mathbf{M}$ *and* $\mathbf{N}$ *give mutually inverse equivalences of categories between* R-mod *and* $\mathscr{A}$*. Composing with the equivalence* $\mathbf{Q}\colon \mathrm{Perv}(\mathbb{C}^n, \mathscr{H}) \to \mathscr{A}$*, we conclude that* $\mathrm{Perv}(\mathbb{C}^n, \mathscr{H})$ *and* R-mod *are equivalent via the composition* $\mathbf{M} \circ \mathbf{Q}$.

Proof. It is easy to see that $\mathbf{M} \circ \mathbf{N}$ is the identity functor on R-mod. For the other composition, we show that $\mathbf{N} \circ \mathbf{M}$ is isomorphic to the identity functor on $\mathscr{A}$. We need isomorphisms $\iota_\alpha\colon \alpha \to \mathbf{N}(\mathbf{M}(\alpha))$ for each object α of $\mathscr{A}$, such that if $f : \alpha \to \beta$ is any morphism in $\mathscr{A}$, then the following diagram commutes.

$$\begin{tikzcd} \alpha \arrow[r, "\iota_\alpha"] \arrow[d, "f"'] & \mathbf{N}(\mathbf{M}(\alpha)) \arrow[d, "\mathbf{N}(\mathbf{M}(f))"] \\ \beta \arrow[r, "\iota_\beta"] & \mathbf{N}(\mathbf{M}(\beta)) \end{tikzcd} \tag{2}$$

Let $\alpha = (E_C, \gamma_{C'C}, \delta_{CC'})$ and let $\mathbf{N}(\mathbf{M}(\alpha)) = (e_C E_Z, \gamma'_{C'C}, \delta'_{CC'})$. Recall that $\varphi_{CZ}\colon E_C \to E_Z$ can be rewritten as follows:

$$\varphi_{CZ} = \gamma_{ZZ}\delta_{CZ} = \delta_{CZ} = \delta_{CZ}\gamma_{ZC}\delta_{CZ} = e_C\delta_{CZ}.$$

So the image of φ_{CZ} lies in $e_C E_Z$. We also have the following equalities.

$$\begin{aligned} \varphi_{CZ} \circ \varphi_{ZC} &= \gamma_{ZZ}\delta_{CZ}\gamma_{ZC}\delta_{ZZ} = e_C = \mathrm{id}_{e_C E_Z} \\ \varphi_{ZC} \circ \varphi_{CZ} &= \gamma_{ZC}\delta_{ZZ}\gamma_{ZZ}\delta_{CZ} = \gamma_{ZC}\delta_{CZ} = \mathrm{id}_{E_Z}. \end{aligned}$$

So $\varphi_{CZ}\colon E_C \to e_C E_Z$ is an isomorphism, with inverse φ_{ZC}. We also check the following for every $C' \leq C$.

$$\begin{aligned} \gamma'_{C'C} \circ \varphi_{C'Z} &= e_C \circ \gamma_{ZZ}\delta_{C'Z} = \delta_{CZ}\gamma_{ZC}\delta_{C'Z} \\ &= \delta_{CZ}\gamma_{C'C}\gamma_{ZC'}\delta_{C'Z} = \delta_{CZ}\gamma_{C'C} = \gamma_{ZZ}\delta_{CZ}\gamma_{C'C} \\ &= \varphi_{CZ} \circ \gamma_{C'C}. \\ \delta'_{CC'} \circ \varphi_{CZ} &= e_{C'}\gamma_{ZZ}\delta_{CZ} = \delta_{C'Z}\gamma_{ZC'}\gamma_{ZZ}\delta_{CZ} = \delta_{C'Z}\gamma_{ZC'}\delta_{CZ} \\ &= \delta_{C'Z}\gamma_{ZC'}\delta_{C'Z}\delta_{CC'} = \delta_{C'Z}\delta_{CC'} = \gamma_{ZZ}\delta_{C'Z}\delta_{CC'} \\ &= \varphi_{C'Z} \circ \delta_{CC'}. \end{aligned}$$

Therefore the following diagrams commute.

$$\begin{tikzcd} E_{C'} \arrow[r, "\gamma_{C'C}"] \arrow[d, "\varphi_{C'Z}"'] & E_C \arrow[d, "\varphi_{CZ}"] \\ e_{C'}E_Z \arrow[r, "\gamma'_{C'C}"] & e_C E_Z \end{tikzcd} \qquad \begin{tikzcd} E_C \arrow[r, "\delta_{CC'}"] \arrow[d, "\varphi_{CZ}"'] & E_{C'} \arrow[d, "\varphi_{C'Z}"] \\ e_C E_Z \arrow[r, "\delta'_{CC'}"] & e_{C'}E_Z \end{tikzcd}$$

So the family of maps $\{\varphi_{CZ} \mid C \in \mathscr{C}\}$ defines an $\mathscr{A}$-isomorphism from α to $\mathbf{N}(\mathbf{M}(\alpha))$. We set ι_α to be this isomorphism.

Let $\beta = (F_C, \gamma_{C'C}, \delta_{CC'})$, and let $f : \alpha \to \beta$ be any $\mathscr{A}$-morphism. Checking the commutativity of the diagram in (2) is equivalent to checking the commutativity of the following diagram for each C.

$$\begin{tikzcd} E_C \arrow[r, "(\iota_\alpha)_C"] \arrow[d, "f_C"'] & e_C E_Z \arrow[d, "e_C f_Z"] \\ F_C \arrow[r, "(\iota_\beta)_C"] & e_C F_Z \end{tikzcd}$$

This check is completed via the following calculation.

$$\begin{aligned} e_C f_Z (\iota_\alpha)_C &= \delta_{CZ}\gamma_{ZC} f_Z \gamma_{ZZ}\delta_{CZ} \\ &= \delta_{CZ}\gamma_{ZC}\delta_{CZ} f_C \\ &= \delta_{CZ} f_Z = \gamma_{ZZ}\delta_{CZ} f_C \\ &= (\iota_\beta) f_C. \end{aligned}$$

Therefore $\mathbf{N}\circ\mathbf{M}$ is isomorphic to the identity functor on $\mathscr{A}$, and the proof is complete. □

3. Recollement structures on abelian categories

We recall some general definitions, which originally appeared in [1, Section 1.4]. Our definitions follow the more recent reference [9].

Definition 3.1. Let $\mathscr{A}$, $\mathscr{A}'$, and $\mathscr{A}''$ be abelian categories. Suppose that we have functors i_*, i^*, $i^!$, j^*, $j_!$ and j_* that fit into the following diagram.

$$\begin{tikzcd} \mathscr{A}' \arrow[r, "i_*"] & \mathscr{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathscr{A}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Then these functors are said to form a *recollement* if the following conditions hold.

(1) There are adjunctions $(j_!, j^*)$, (j^*, j_*), (i^*, i_*), and $(i_*, i^!)$.
(2) The unit morphisms $\mathrm{id}_{\mathscr{A}'} \to i^! i_*$ and $\mathrm{id}_{\mathscr{A}''} \to j^* j_!$ are isomorphisms.
(3) The counit morphisms $i^* i_* \to \mathrm{id}_{\mathscr{A}'}$ and $j^* j_* \to \mathrm{id}_{\mathscr{A}''}$ are isomorphisms.
(4) The functor i_* is an embedding onto the full subcategory of $\mathscr{A}$ of objects A such that $j^* A = 0$. Consequently, the compositions $j^* i_*$, $i^* j_!$, and $i^! j_*$ are zero.

3.1. **Recollement of perverse sheaves.** Let $\mathrm{Perv}(X, \mathscr{S})$ be the category of perverse sheaves on a space X with respect to a given stratification $\mathscr{S}$. Let $U \in \mathscr{S}$ be an open stratum, and let $V = X \setminus U$. Let $j\colon U \to X$ and $i\colon V \to X$ be the inclusion maps. We have a recollement as follows (see, e.g. [1]).

$$\begin{tikzcd} \mathrm{Perv}(V, \mathscr{S}|_V) \arrow[r, "i_*"] & \mathrm{Perv}(X, \mathscr{S}) \arrow[l, bend right, "{}^p\mathrm{H}^0(i^*)"'] \arrow[l, bend left, "{}^p\mathrm{H}^0(i^!)"] \arrow[r, "j^*"] & \mathrm{Perv}(U, \mathscr{S}|_U) \arrow[l, bend right, "{}^p\mathrm{H}^0(j_!)"'] \arrow[l, bend left, "{}^p\mathrm{H}^0(j_*)"] \end{tikzcd}$$

3.2. **Recollement induced by an idempotent on a module category.** Let A be a ring and e an idempotent. We recall the standard recollement induced on A-mod by e in this situation (see, e.g., [6, Section 1] or [18, Section 4.1]). Define the functors Ind_e and Coind_e from eAe-mod to A-mod as follows:

$$\mathrm{Ind}_e(M) = Ae \otimes_{eAe} M, \quad \mathrm{Coind}_e(M) = \mathrm{Hom}_{eAe}(eA, M).$$

Define Res_e from A-mod to eAe-mod by

$$\mathrm{Res}_e(N) = eN \cong \mathrm{Hom}_A(Ae, N) \cong eA \otimes_A N.$$

Define endofunctors T_e and N_e on A-mod as follows:

$$T_e(M) = AeM, \quad N_e(M) = \{m \in M \mid eAm = 0\}.$$

Finally, let Inc from A/AeA-mod to A-mod be the functor that upgrades A/AeA-modules to A-modules via the quotient $A \to A/AeA$. The following theorem is well-known (see, e.g. [15]).

Theorem 3.2. *Let A be a ring and $e \in A$ an idempotent. Consider the following functors:*

$$i_* = \mathrm{Inc}, \quad i^* = (-)/T_e(-), \quad i^! = N_e,$$
$$j^* = \mathrm{Res}_e, \quad j_! = \mathrm{Ind}_e, \quad j_* = \mathrm{Coind}_e\,.$$

These six functors fit into the following recollement diagram.

$$\begin{tikzcd} A/AeA\text{-mod} \arrow[r, "i_*"] & A\text{-mod} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & eAe\text{-mod} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Note that if $M \in eAe$-mod, then there is a natural homomorphism of A-modules from $j_!M$ to j_*M, described as follows. Given $a \in A$ and $m \in M$, send $ae \otimes m \in j_!M$ to the element of $j_*(M) = \mathrm{Hom}_{eAe}(eA, M)$ given by $eb \mapsto (ebae)m$. It is easy to check that this is a well-defined map of A-modules. Motivated by the analogous construction in the case of perverse sheaves, we give the following definition.

Definition 3.3. The assignment $M \in eAe$-mod to the image of the A-module homomorphism $j_!(M) \to j_*(M)$ defines a functor from eAe-mod to A-mod. We call this functor the *intermediate extension functor*, and denote it $j_{!*}$.

4. Comparisons between R-modules and perverse sheaves

The aim of this section is to extend the equivalence obtained in Theorem 2.7 to categories related to $\mathrm{Perv}(X, \mathscr{S})$ and specific objects of $\mathrm{Perv}(X, \mathscr{S})$. In particular, we describe an equivalent for R-modules of the open/closed recollement of $\mathrm{Perv}(X, \mathscr{S})$. We also describe an equivalent for R-modules for the full subcategories of perverse sheaves supported on closed unions of strata. We use these descriptions to identify intersection cohomology sheaves on X in terms of R-modules.

4.1. An equivalence of recollements.

Lemma 4.1. *Let $A, F_1, \ldots, F_k, B$ be a sequence of faces in $\mathscr{C}$ of the same dimension related by successive oppositions. That is, each face opposes the next one in order. Then the ideal generated by e_A in R contains e_B.*

Proof. By induction, it is enough to prove the lemma in the case where A opposes B. This amounts to showing that the image of e_B is zero in R/Re_AR.

Recall that $s = e_Be_Ae_B + (1 - e_B)$ is invertible in R. So for some $t \in R$, we have $st = ts = 1$. Let $\overline{s}$ and $\overline{t}$ denote the images of s and t in R/Re_AR. We have $\overline{st} = \overline{ts} = 1$ in R/Re_AR. Note that $\overline{s} = (1 - \overline{e}_B)$. We also know that e_B is idempotent, and so $e_B(1 - e_B) = 0$. Multiplying the equation $1 = \overline{st}$ on the left by $\overline{e}_B$, we see that $\overline{e}_B = 0$. □

Corollary 4.2. *Let $A, F_1, \ldots, F_k, B$ be a sequence of faces in $\mathscr{C}$ of the same dimension related by successive oppositions. That is, each face opposes the next one in order. Then the categories e_ARe_A-mod and e_BRe_B-mod are equivalent.*

Proof. By Theorem 3.2, we have the following two recollements:

$$\begin{tikzcd}
R/Re_AR\text{-mod} \arrow[r] & R\text{-mod} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & e_ARe_A\text{-mod} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
R/Re_BR\text{-mod} \arrow[r] & R\text{-mod} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & e_BRe_B\text{-mod} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

By Lemma 4.1, we know that $Re_AR = Re_BR$,and so the categories on the left of both recollements are equal. Consequently, the three pairs of corresponding functors on the left are also equal in both cases. By general properties of recollements, the categories e_ARe_A-mod and e_BRe_B-mod are both equivalent to the quotient category of R-mod by the image of e_ARe_A-mod (or equivalently e_BRe_B-mod). □

We recall the following definition [16, Definition 4.1].

Definition 4.3. Two recollements $(\mathscr{A}', \mathscr{A}, \mathscr{A}'')$ and $(\mathscr{B}', \mathscr{B}, \mathscr{B}'')$ are said to be *equivalent* if we have equivalences of categories $F' \colon \mathscr{A}' \to \mathscr{B}'$, $F \colon \mathscr{A} \to \mathscr{B}$ and $F'' \colon \mathscr{A}'' \to \mathscr{B}''$ that commute with the six recollement functors up to natural equivalence, as shown in Figure 1.

$$\begin{tikzcd}
\mathscr{A}' \arrow[r, shift left=2, "i_*"] \arrow[d, "F'"'] & \mathscr{A} \arrow[l, shift left=2, "i^*"] \arrow[l, shift right=2, "i^!"'] \arrow[r, shift left=2, "j^*"] \arrow[d, "F"'] & \mathscr{A}'' \arrow[l, shift left=2, "j_!"] \arrow[l, shift right=2, "j_*"'] \arrow[d, "F''"] \\
\mathscr{B}' \arrow[r, shift left=2, "i_*"] & \mathscr{B} \arrow[l, shift left=2, "i^*"] \arrow[l, shift right=2, "i^!"'] \arrow[r, shift left=2, "j^*"] & \mathscr{B}'' \arrow[l, shift left=2, "j_!"] \arrow[l, shift right=2, "j_*"']
\end{tikzcd}$$

FIGURE 1. An equivalence of two recollements.

In fact, Lemma 4.2 of [16] proves that two recollements as above are equivalent if and only if there are equivalences of categories $F \colon \mathscr{A} \to \mathscr{B}$ and $F'' \colon \mathscr{A}'' \to \mathscr{B}''$ such that the functors j^*F and $F''j^*$ are naturally equivalent.

Lemma 4.4. *Suppose that $(\mathscr{A}', \mathscr{A}, \mathscr{A}'')$ and $(\mathscr{B}', \mathscr{B}, \mathscr{B}'')$ are two recollements. Suppose that there are functors $F \colon \mathscr{A} \to \mathscr{B}$, $F' \colon \mathscr{A}' \to \mathscr{B}'$, and $F'' \colon \mathscr{A}' \to \mathscr{B}'$ such that the following conditions hold.*

(1) The functors F and F' are equivalences of categories.
(2) The functors j^*F and $F''j^*$ are naturally equivalent.
(3) The functors i_*F' and Fi_* are naturally equivalent.

Then the functor F'' is full and essentially surjective.

Proof. Let $(\mathscr{A}', \mathscr{A}, \mathscr{A}'')$ and $(\mathscr{B}', \mathscr{B}, \mathscr{B}'')$ be two recollements, and supppose that the conditions of the lemma hold. First we show that F'' is full. Let A_1'' and A_2'' be objects of $\mathscr{A}''$, and let $B_i'' = F''(A_i'')$ for $i \in \{1, 2\}$. Suppose that there is a morphism $f'' \colon B_1'' \to B_2''$ in $\mathscr{B}''$. We will exhibit a lift of f'' to a morphism from A_1'' to A_2'' in $\mathscr{A}''$.

Since j^*j_* is isomorphic to $\mathrm{id}_{\mathscr{A}''}$ on $\mathscr{A}''$, we can find objects A_1 and A_2 of $\mathscr{A}$ such that $j^*(A_i) = A_i''$. Let $B_i = F(A_i)$ for $i \in \{1, 2\}$. Then $j^*B_i = j^*F(A_i)$ is naturally isomorphic to $B_i'' = F''j^*(A_i)$ for each i. Let $g'' \colon j^*B_1 \to j^*B_2$ denote the map corresponding to f'' under

the natural isomorphism. Since j^* is a full functor on $\mathscr{B}''$, the morphism g'' has a lift $g: B_1 \to B_2$ in $\mathscr{B}$. Similarly since F is an equivalence, g has a lift $\widetilde{g}: A_1 \to A_2$ in $\mathscr{A}$. Now we can check that $j^*\widetilde{g}: A_1'' \to A_2''$ is a morphism in $\mathscr{A}''$ that lifts $f'': B_1'' \to B_2''$.

Next we check that F'' is essentially surjective. Let B'' be any object of $\mathscr{B}''$. Recall that the counit morphism $j^*j_*B'' \to B''$ is an isomorphism. Since F is essentially surjective, there is some $A \in \mathscr{A}$ such that $F(A) \cong j_*B''$. Let $A'' = j^*A$. Then we see that

$$F''(A'') \cong j^*F(A) \cong j^*j_*B'' \cong B''.$$

□

Proposition 4.5. *Fix $F \in \mathscr{C}^0$ and let $e = e_F$. Let U be the open stratum in $\mathscr{S}$, which is also the complex span of F. Let $\pi_1(U,e)$ denote the fundamental group of U with the basepoint chosen to be some point of F. There is a ring homomorphism $\iota: \mathbb{C}[\pi_1(U,e)] \to eRe$.*

Proof. To prove this proposition, we use the Salvetti presentation of the fundamental groupoid $\pi_1(U)$, as described originally in [17] and also in [12, Proposition 9.12].

We recall this presentation here. There is one object x_A in $\pi_1(U)$ for each $A \in \mathscr{C}^0$. The generating morphisms are $\psi_{AB}: x_A \to x_B$ for each ordered pair (A,B) of opposing objects in $\mathscr{C}^0$. Let $\psi_{AB}^-: x_A \to x_B$ denote the inverse of the generating morphism $\psi_{BA}: B \to A$. The relations are as follows. Let F be any face in $\mathscr{C}^2$ (i.e., a face of codimension two). As discussed in [12, Example 7.9], let A and C be two elements of $\mathscr{C}^0$ such that $F < A$, $F < C$, and $C = -A$ is the opposite chamber to A. Then we can label the chambers around F by $A = B_1, B_2, \ldots, B_{m+1} = C, B_{m+2}, \ldots, B_{2m}$, where any two successive chambers oppose each other, and B_{2m} opposes $A = B_1$. For each such instance, we have the *Zifferblatt relation* in $\pi_1(U)$:

$$\psi_{B_m,C}\psi_{B_{m-1},B_m}\cdots\psi_{A,B_2} = \psi_{B_{m+2},C}\psi_{B_{m+3},B_{m+2}}\cdots\psi_{A,B_{2m}}.$$

Now if we fix a basepoint in the real chamber $F \in \mathscr{C}^0$, we can consider the fundamental group $\pi_1(U,e)$. The generators and relations of $\pi_1(U,e)$ can be deduced from the generators and relations of $\pi_1(U)$. Elements of $\pi_1(U,e)$ consist of all composable words in the letters ψ_{AB} and ψ_{AB}^- that begin and end at x_F, with the relations generated by the Zifferblatt relations discussed above. This means that locally within any word, any subword consisting of a minimal path between a chamber and its negative can be substituted by the subword formed by the other instance of the minimal path between them. Moreover, any instances of $\psi_{AB}\psi_{BA}^-$ or $\psi_{AB}^-\psi_{BA}$ can be cancelled.

Now define a function $\iota: \pi_1(U,e) \to eRe$ as follows. Let $w = \psi_{A_k,F}^{\sigma_k}\psi_{A_{k-1},A_k}^{\sigma_{k-1}}\cdots\psi_{F,A_1}^{\sigma_1}$ be any word in $\pi_1(U,e)$. We replace any positively-signed letter ψ_{AB} to the ring element e_Be_A. We replace any negatively-signed letter ψ_{AB}^- by the ring element $e_Bs_{BA}e_B = s_{BA}e_B = e_Bs_{BA}$. Then it is easy to see that any word w as above gets sent to an element $\iota(w)$ of eRe. We check that the relations are satisfied.

(1) Suppose that $w = w_1\psi_{AB}\psi_{BA}^-w_2$. Then we know that $\iota(w_1) = \iota(w_1)e_B$, since w_1 ends at x_B. Therefore

$$\iota(w) = \iota(w_1)e_Be_Ae_Bs_{BA}\iota(w_2) = \iota(w_1)e_B\iota(w_2) = \iota(w_1)\iota(w_2) = \iota(w_1w_2).$$

Similarly if $w = w_1\psi_{AB}^-\psi_{BA}w_2$, then

$$\iota(w) = \iota(w_1)s_{BA}e_Be_Ae_B\iota(w_2) = \iota(w_1)e_B\iota(w_2) = \iota(w_1)\iota(w_2) = \iota(w_1w_2).$$

(2) Suppose that $A, C \in \mathscr{C}^0$ such that $C = -A$, and F is a codimension two face such that $F < A$. Number the faces around F starting at A as $B_1, \ldots, B_{2m}$, as in the Zifferblatt relations. Let $w = w_1\psi_{B_m,C}\cdots\psi_{A,B_2}w_2$. Then

$$\iota(w) = \iota(w_1)e_Ce_{B_m}e_{B_m}e_{B_{m-1}}\cdots e_{B_2}e_A\iota(w_2).$$

Since $B_1, B_2, \ldots, B_m$ are successive opposing chambers, any three consecutive ones are collinear. So we can telescope the expression $u = e_c e_{B_m} e_{B_m} e_{B_{m-1}} \cdots e_{B_2} e_A$ to $e_C e_A$. Telescoping in reverse for the other path $A = B_1, B_{2m}, \ldots, B_{m+1} = C$, we see that

$$u = e_C e_{B_{m+2}} e_{B_{m+2}} e_{B_{m+3}} \cdots e_{B_{2m}} e_A = \iota(\psi_{B_{m+2},C} \psi_{B_{m+3},B_{m+2}} \cdots \psi_{B_{2m},A}).$$

This proves the Zifferblatt relation:

$$\begin{aligned} \iota(w) &= \iota(w_1)\iota(\psi_{B_{m+2},C} \psi_{B_{m+3},B_{m+2}} \cdots \psi_{B_{2m},A})\iota(w_2) \\ &= \iota(w_1 \psi_{B_{m+2},C} \psi_{B_{m+3},B_{m+2}} \cdots \psi_{B_{2m},A} w_2). \end{aligned}$$

So we have showed that $\iota\colon \pi_1(U,e) \to eRe$ maps the group $\pi_1(U,e)$ into the ring eRe. We extend by linearity to a ring homomorphism $\iota\colon \mathbb{C}[\pi_1(U,e)] \to eRe$, and the proposition is proved. □

Lemma 4.6. *Let $F\colon R\text{-mod} \to \mathrm{Perv}(X,\mathscr{S})$ denote the (inverse) equivalence from Theorem 2.7. Fix any $A \in \mathscr{C}^0$, and let $e = e_A$. There is a faithful functor $F''_A\colon eRe\text{-mod} \to Perv(U, \mathscr{S}|_U)$, such that j^*F and $F''_A j^*$ are naturally equivalent.*

Proof. Recall that $\mathrm{Perv}(U,\mathscr{S}|_U)$ is the category of perverse sheaves on U whose cohomology sheaves are locally constant on U. So it is equivalent to the abelian category of local systems on U, shifted by the complex dimension of U. In turn, the category of (shifted) local systems on U is equivalent to the category of finite-dimensional representations of its fundamental group.

As per Proposition 4.5, there is a ring homomorphism $\iota\colon \mathbb{C}[\pi_1(U,e)] \to eRe$, which induces a restriction functor $\iota^*\colon eRe\text{-mod} \to \mathbb{C}[\pi_1(U,e)]\text{-mod}$. Define the functor F''_A to be the composition of the equivalence $\mathbb{C}[\pi_1(U,e)]\text{-mod} \to \mathrm{Perv}(U,\mathscr{S}|_U)$ with the restriction $eRe\text{-mod} \to \mathbb{C}[\pi_1(U,e)]\text{-mod}$.

We now show that the functors j^*F and $F''_A j^*$ are naturally equivalent. The functor F sends an R-module M to a complex of sheaves $\mathscr{E}^\bullet(M)$, which is perverse with respect to $\mathscr{S}$. Therefore $j^*(\mathscr{E}^\bullet(M))$ is quasi-isomorphic to a (shifted) local system on U, namely its left-most homology sheaf. The construction from [12] produces an explicit form for $\mathscr{E}^\bullet(M)$ as follows.

$$\mathscr{E}^\bullet(M) = \left\{ \bigoplus_{\mathrm{codim}(C)=0} \mathscr{E}_C(M) \otimes \mathrm{or}(C) \to \bigoplus_{\mathrm{codim}(C)=1} \mathscr{E}_C(M) \otimes \mathrm{or}(C) \to \cdots \right\} \tag{3}$$

In the above complex, the sheaves $\mathscr{E}_C(M)$ are locally constant on a finer stratification of $\mathbb{C}^n$ than $\mathscr{S}$, and $\mathrm{or}(C)$ is the orientation local system of C. We refer the reader to [12, Section 6] for the full construction, and recall details here as necessary.

So the complex $j^*F(M) = j^*(\mathscr{E}^\bullet(M))$ is quasi-isomorphic to the kernel K of the first map in $j^*(\mathscr{E}^\bullet(M))$, as shown below.

$$\begin{tikzcd} & K \arrow[d, "\sim"] & & \\ \bigoplus_{\mathrm{codim}(C)=0} & j^*(\mathscr{E}_C(M) \otimes \mathrm{or}(C)) \arrow[r] & \bigoplus_{\mathrm{codim}(C)=1} j^*(\mathscr{E}_C(M) \otimes \mathrm{or}(C)) \arrow[r] & \cdots \end{tikzcd}$$

On the other hand, the functor $F''_A j^*$ first sends the R-module M to the eRe-module eM, and then to the local system defined by the $\pi_1(U,e)$ representation $\iota^*(eM)$. We now show that K is isomorphic to this local system.

Recall from [12] that the sheaves $\mathscr{E}_C(M)$ are locally constant on cells of the form $iP+Q$, where $P, Q \in \mathscr{C}$. The collection of cells of this form refines the stratification $\mathscr{S}$. So we can

compute K cell-wise, using the description of the cell-wise stalks of $\mathscr{E}_C(M)$ from [12]. If $C, Q \in \mathscr{C}$, let $C \circ Q$ denote the minimal face K such that $K \geq C$ and $K + \mathrm{Span}_{\mathbb{R}}(C) \supset Q$. For any cell of the form $iP + Q$, and any face $C \in \mathscr{C}$, we recall that

$$\mathscr{E}_C(M)|_{iP+Q} = \begin{cases} M_{C \circ Q} & \text{if } P \leq C, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Since we are only concerned with $j^*(\mathscr{E}^\bullet(M))$, we disregard any cells $iP + Q$ that do not lie in the open stratum U. These are precisely those $iP + Q$ such that both P and Q are faces of real codimension greater than or equal to 1. In what follows, we only consider cells $iP + Q$ where either $P \in \mathscr{C}^0$ or $Q \in \mathscr{C}^0$.

Let $P \in \mathscr{C}^0$ be a codimension-zero face, and let $Q \in \mathscr{C}$ be any other face. The description of the stalks in Equation 4 implies that for any $C \in \mathscr{C}$, the stalk of the sheaf $\mathscr{E}_C(M)$ at $iP + Q$ equals M_P if $C = P$, and zero otherwise. In particular, we conclude that the stalk of all of K at $iA + Q$ is precisely $M_A = e_A M$. Since K is a local system, its stalks at all points of U must be isomorphic to $e_A M$. It remains to calculate the monodromy maps.

Consider any non-trivial element ℓ of $\pi_1(U, e)$, which can be written as follows

$$\ell = \psi^{\sigma_{n-1}}_{A_{n-1},A} \psi^{\sigma_{n-1}}_{A_{n-2},A_{n-1}} \cdots \psi^{\sigma_0}_{A,A_1}.$$

Here, $A = A_0, A_1, \ldots, A_{n-1}, A_n = A$ are elements of $\mathscr{C}^0$ such that A_k and A_{k+1} oppose each other through the codimension-one face W_k. Then ℓ can be represented as a loop in U that begins and ends in the cell $iA + \{0\}$. Further, up to homotopy we can choose a representative such that each letter $\psi^{\sigma_k}_{A_k,A_{k+1}}$ is represented by a pair of straight-line segments with the following properties.

(1) The first segment goes from $iA_k + \{0\}$ to $iW_k + Q_k$ for a certain $Q_k \in \mathscr{C}^0$.
(2) The second segment goes from $iW_k + Q_k$ to $iA_{k+1} + \{0\}$.
(3) If the sign σ_k is positive, then $Q_k = A_k$, otherwise $Q_k = A_{k+1}$.

To calculate the monodromy, it is sufficient to compute the transition maps corresponding to each letter in the word ℓ.

Consider the letter $\psi^{\sigma_k}_{A_k,A_{k+1}}$. This corresponds to a transition map from the stalk of K at $iA_k + \{0\}$ to the stalk of K at $iA_{k+1} + \{0\}$, via the stalk at $iW_k + Q_k$. As observed earlier, for $* \in \{k, k+1\}$, the stalk of $\mathscr{E}_C(M)$ at A_* equals M_{A_*} if $C = A_*$, and zero otherwise. Similarly, observe from Equation 4 that

$$\mathscr{E}_C(M)|_{iW_k+Q_k} = \begin{cases} M_{A_*} & \text{if } C = A_* \text{ for } * \in \{k, k+1\} \\ M_{Q_k} & \text{if } C = W_k \\ 0 & \text{otherwise.} \end{cases}$$

So the stalk of K at $iA_k + \{0\}$ is just $M_{A_k} = e_{A_k} M$, and the stalk of K at $iA_{k+1} + \{0\}$ is just $M_{A_{k+1}} = e_{A_{k+1}} M$. The stalk of K at $iW_k + Q_k$ is the kernel of the following two-term complex:

$$e_{A_k} M \oplus e_{A_{k+1}} M \to M_{Q_k},$$

where $M_{Q_k} = e_{A_k}$ if $\sigma = +1$ and $M_{Q_k} = e_{A_{k+1}}$ if $\sigma = -1$. The restriction map $e_{A_k} M \to e_{A_k} M$ is the identity, while the restriction $e_{A_k} M \to e_{A_{k+1}} M$ is the map $e_{A_{k+1}} e_{A_k}$. Similarly the restriction map $e_{A_{k+1}} M \to e_{A_{k+1}} M$ is the identity, while the restriction $e_{A_{k+1}} M \to e_{A_k} M$ is the map $e_{A_k} e_{A_{k+1}}$. Note that we have ignored orientations here—the orientation local systems are trivial because each cell is contractible, and the maps on the orientation local systems are canonical. To compute the kernel, it is only relevant that the restriction maps above have opposite signs, since A_k and A_{k+1} lie on opposite sides of W_k.

If $\sigma = +1$, then the stalk of K at $iW_k + Q_k$ is

$$\{(x,y) \in e_{A_k}M \oplus e_{A_{k+1}}M \mid x = e_{A_k}e_{A_{k+1}}y\}.$$

If $\sigma = -1$, then the stalk of K at $iW_k + Q_k$ is

$$\{(x,y) \in e_{A_k}M \oplus e_{A_{k+1}}M \mid y = e_{A_{k+1}}e_{A_k}x\} = \{(x,y) \in e_{A_k}M \oplus e_{A_{k+1}}M \mid x = e_{A_{k+1}}s_{A_{k+1},A_k}y\}.$$

So the transition map $\psi^{\sigma_k}_{A_k,A_{k+1}} : e_{A_k}M \to e_{A_{k+1}}M$ is

$$\psi^{\sigma_k}_{A_k,A_{k+1}} = \begin{cases} x \mapsto e_{A_k}e_{A_{k+1}}x & \text{if } \sigma_k = +1, \\ x \mapsto e_{A_{k+1}}s_{A_{k+1},A_k}x, & \text{if } \sigma_k = -1. \end{cases}$$

Composing the transition maps obtained for each letter $\psi^{\sigma_k}_{A_k,A_{k+1}}$ in ℓ, we see that this is exactly how we obtained the action of ℓ on $\iota^*(eM)$ as constructed in Proposition 4.5. This argument shows that j^*F is naturally isomorphic to $F''j^*$.

Finally, we show that F'' is faithful. The functor ι^* is faithful because it induces the identity functor on the underlying categories of vector spaces. Since F'' is constructed as the composition of an equivalence of categories with ι^*, it is faithful. □

Theorem 4.7. *Let U be the open stratum of $\mathscr{S}$, namely the complement of all of the complex hyperplanes in $\mathscr{H}$. Then $U \cap \mathbb{R}^n$ is a disjoint union of all the top-dimensional faces of $\mathscr{C}$. Let $V = \mathbb{C}^n \setminus U$. Fix any $A \in \mathscr{C}^0$, and let $e = e_A$. Then we have an equivalence of recollements as follows.*

$$\begin{tikzcd}
R/ReR\text{-mod} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "F'_A"'] & R\text{-mod} \arrow[l] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "F"'] & eRe\text{-mod} \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[d, "F''_A"] \\
\mathrm{Perv}(V, \mathscr{S}|_V) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathrm{Perv}(X, \mathscr{S}) \arrow[l] \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathrm{Perv}(U, \mathscr{S}|_U) \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2]
\end{tikzcd}$$

Proof. From the previous lemma, we already have functors $F : R\text{-mod} \to \mathrm{Perv}(X, \mathscr{S})$ and $F''_A : eRe\text{-mod} \to \mathrm{Perv}(U, \mathscr{S}|_U)$. We now construct an equivalence $F'_A : (R/ReR)\text{-mod} \to \mathrm{Perv}(V, \mathscr{S}|_V)$. Then we show that these functors give an equivalence of recollements as above.

To define F'_A, start with any M in (R/ReR)-mod. We may think of M as an R-module that is annihilated by $e = e_A$. Recall from Lemma 4.1 that if B is any other face in $\mathscr{C}^0$, then $Re_AR = Re_BR$. So we may consider M as an R-module that is annihilated by all idempotents $\{e_B \mid B \in \mathscr{C}^0\}$. The functor $\mathbf{P} \circ \mathbf{N}$ from Theorem 2.7 sends M to the perverse sheaf $\mathscr{E}^\bullet(M)$. For each $B \in \mathscr{C}^0$, we have $V_B = e_BV = 0$. From this it is easy to check that $\mathscr{E}^\bullet(M)$ is term-wise annihilated by j^*, and hence is supported on V. Since the category $\mathrm{Perv}(V, \mathscr{S}|_V)$ is isomorphic to the full subcategory of $\mathrm{Perv}(X, \mathscr{S})$ of objects supported on V, this defines the functor F'_A. By construction, it is clear that $i_*F'_A = Fi_*$.

To check that F'_A is an equivalence, we define an inverse functor in a similar way. If $\mathscr{E}^\bullet$ is any perverse sheaf supported on V, then the functor $\mathbf{Q}$ from Theorem 2.7 sends $\mathscr{E}^\bullet$ to a double quiver representation $(E_C, \gamma_{C'C}, \delta_{CC'})$. Following [12], let $j : \mathbb{R}^n \to \mathbb{C}^n$. Then E_C is defined as the global sections of $j^!\mathscr{E}$ on C. Since $\mathscr{E}$ is supported on the closed stratum, each E_C is manifestly zero for any $C \in \mathscr{C}^0$. The functor $\mathbf{M}$ from Theorem 2.7 sends the double quiver representation $(E_C, \gamma_{CC'}, \delta_{C'C})$ to the R-module E_Z, where $e_CE_Z = E_C$ for each $C \in \mathscr{C}$. By construction, $e_CE_Z = 0$ for every $C \in \mathscr{C}^0$, and so we can regard $(\mathbf{M} \circ \mathbf{Q})(\mathscr{E})$ as an element of $R/(ReR)$-mod. This is the inverse functor to F'_A, and so F'_A is an equivalence of categories.

By the construction above and by Lemma 4.6, we observe that the functors F, F', and F'' satisfy the conditions of Lemma 4.4. Therefore F''_A is full and essentially surjective in

addition to being faithful, and so it is an equivalence. The functors F, F'_A, and F''_A thus induce an equivalence of recollements as desired. □

As an immediate consequence of the fact that F''_A is an equivalence of categories, we obtain the following corollary.

Corollary 4.8. *The ring homomorphism ι of Proposition 4.5 induces the pullback functor $\iota^* eRe$-mod $\to$: $\mathbb{C}[\pi_1(U,e)]$ on the respective categories of finite-dimensional modules. This functor is an equivalence of categories.*

Since the finite-dimensional module categories of $\mathbb{C}[\pi_1(U,e)]$ and eRe are isomorphic, it is natural to ask whether the map ι is an isomorphism. We discuss this question in Section 6.

If $\mathscr{L}$ is a local system on the open stratum U, then the perverse sheaf $j_{!*}(\mathscr{L})$ is supported on X and smooth with respect to $\mathscr{S}$. It is known as the corresponding *intersection cohomology sheaf*, denoted $\mathrm{IC}(\mathscr{L})$. Using the equivalence of recollements obtained above, we also deduce the following corollary.

Corollary 4.9. *Let U be the open stratum of X, and let $\mathscr{L}$ be a local system on U, corresponding to a representation L of $\pi_1(U,e)$. Let $M \in eRe$-mod be the image of L under the inverse equivalence of ι^*. Under the equivalence* $\mathrm{Perv}(X,\mathscr{S}) \cong R$-mod*, the intersection cohomology sheaf* $\mathrm{IC}(\mathscr{L})$ *maps to the R-module $j_{!*}(M)$, where $j_{!*}$ is the intermediate extension functor from Definition 3.3.*

4.2. **Perverse sheaves supported on closed unions of strata.** In this subsection, we extend the previous results to describe the category of perverse sheaves smooth with respect to $\mathscr{S}$ that are supported on the closure of the union of some of the strata. As an application, we give a description of all intersection cohomology sheaves on X smooth with respect to $\mathscr{S}$.

For the remainder of the section, let Z be a closed union of some strata of $\mathscr{S}$. That is, $Z = \overline{Z}$. Note that Z is a union of vector subspaces of X defined over $\mathbb{R}$. Let $\mathscr{C}_Z$ denote the restriction of the real face poset to Z:

$$\mathscr{C}_Z = \{C \in \mathscr{C} \mid C \subset Z_{\mathbb{R}}\}.$$

Theorem 4.10. *Let I_Z be the ideal of R generated by the set $\{e_C \mid C \notin \mathscr{C}_Z\}$. Then there is an equivalence of categories (R/I_Z)-mod $\to \mathrm{Perv}(Z, \mathscr{S}|_Z)$.*

Proof. Since Z is a closed subset of X, the category $\mathrm{Perv}(Z,\mathscr{S}|_Z)$ is isomorphic to the full subcategory of $\mathrm{Perv}(X,\mathscr{S})$ consisting of objects supported on Z. Similarly, we may think of the category (R/I_Z)-mod as the full subcategory of R-mod consisting of objects annihilated by I_Z. The construction of the functor (R/I_Z)-mod $\to \mathrm{Perv}(Z,\mathscr{S}|_Z)$ is similar to the construction of the functor F'_A in the proof of Theorem 4.7.

Let $M \in R$-mod be an R-module that is annihilated by I_Z. The functor $\mathbf{P} \circ \mathbf{N}$ from Theorem 2.7 sends M to a perverse sheaf $\mathscr{E}^\bullet(M)$. Using the description of the stalks of the component sheaves $\mathscr{E}_C(M)$ at cells $iP+Q$ from Equation 4, we conclude that $\mathscr{E}_C(M)|_{iP+Q} = 0$ unless $P \subset C$ and $C \subset Z_{\mathbb{R}}$. Moreover, even if $C \subset Z_{\mathbb{R}}$, the stalk $\mathscr{E}_C(M)$ at a cell $iP+Q$ equals zero unless $C \circ Q \subset Z_{\mathbb{R}}$. Let $Z'_{\mathbb{R}} \subset Z_{\mathbb{R}}$ be the maximal real vector space that contains C. Then the cell $C \circ Q$ is a subset of Z if and only if $Q \subset Z'_{\mathbb{R}}$. This means that both P and Q are contained in a common vector space $Z'_{\mathbb{R}} \subset Z_{\mathbb{R}}$, which is true if and only if $iP+Q \subset Z$. We conclude that the stalks of the component sheaves of $\mathscr{E}^\bullet(M)$ are zero at all points outside Z. In other words, $\mathscr{E}^\bullet(M)$ is supported on Z, and may be thought of as

an object of $\mathrm{Perv}(Z, \mathscr{S}|_Z)$. The inverse equivalence is constructed in a similar manner to the inverse of F'_A in the proof of Theorem 4.7.

□

4.3. **Perverse sheaves supported on the closure of a single stratum.** As a special case of the previous section, let Z be the closure of a single stratum of $\mathscr{S}$. Then $\mathscr{H}_{\mathbb{C}}^Z$, the restriction of $\mathscr{H}_{\mathbb{C}}$ to Z, is a hyperplane arrangement in Z defined over $\mathbb{R}$. The stratification on Z obtained via $\mathscr{H}_{\mathbb{C}}^Z$ coincides with $\mathscr{S}|_Z$. Let R_Z be the algebra defined as in Subsection 2.2 for the arrangement $\mathscr{H}_{\mathbb{C}}^Z$ on Z. Let I_Z be the ideal of R generated by the set $\{e_C \mid C \notin \mathscr{C}_Z\}$, as above.

By Theorem 4.10, there is an equivalence of categories $(R/I_Z)\text{-mod} \to \mathrm{Perv}(Z, \mathscr{S}|_Z)$. On the other hand by Theorem 2.7, we also have an equivalence $R_Z\text{-mod} \to \mathrm{Perv}(Z, \mathscr{S}|_Z)$. Our next proposition relates these two. It is also natural to ask how the algebras R_Z and R/I_Z are themselves related. We discuss this question in Section 6.

Proposition 4.11.
(1) There is a surjective ring homomorphism $\rho_Z : R/I_Z \to R_Z$, inducing a pullback functor $\rho_Z^* : R_Z\text{-mod} \to (R/I_Z)\text{-mod}$.
(2) The composition

$$R_Z\text{-mod} \xrightarrow{\rho_Z^*} (R/I_Z)\text{-mod} \to \mathrm{Perv}(Z, \mathscr{S}|_Z)$$

coincides with the equivalence of categories constructed in Theorem 2.7. Consequently, ρ_Z^* is an equivalence of categories.

Proof. For clarity, denote the idempotent generators of R_Z by $\{f_C \mid C \in \mathscr{C}_Z\}$. Define a homomorphism $\widetilde{\rho}_Z : R \to R_Z$ as follows:

$$e_C \mapsto \begin{cases} f_C & \text{if } C \subset T_{\mathbb{R}}, \\ 0 & \text{otherwise.} \end{cases}$$

We check the relations from Subsection 2.2.
(R1) The relation (R1) is clear.
(R2) Suppose A, B, and C are three collinear faces. If $A \subset T_{\mathbb{R}}$ and $C \subset T_{\mathbb{R}}$, we must have $B \subset T_{\mathbb{R}}$ because $T_{\mathbb{R}}$ is a real vector space. In this case, $\widetilde{\rho}_Z(e_A) = f_A$, $\widetilde{\rho}_Z(e_B) = f_B$, and $\widetilde{\rho}_Z(e_C) = f_C$ and (R2) clearly holds. If one of the three collinear faces lies outside $T_{\mathbb{R}}$, then at least one other face must also lie outside $T_{\mathbb{R}}$; again because $T_{\mathbb{R}}$ is a vector space. In this case both sides of (R2) are zero.
(R3) Suppose that $A \leq B$. If $A \subset T_{\mathbb{R}}$ and $B \subset T_{\mathbb{R}}$, then again (R3) is clear. If $B \not\subset T_{\mathbb{R}}$ then both sides of (R3) are zero. If $A \not\subset T_{\mathbb{R}}$ then it follows that $B \not\subset T_{\mathbb{R}}$, and again both sides of (R3) are zero.

As for the localisation, suppose that A and B are two faces in $\mathscr{C}$ that share a wall. Since $T_{\mathbb{R}}$ is a vector space, $A \subset T_{\mathbb{R}}$ if and only if $B \subset T_{\mathbb{R}}$. If they are both in $T_{\mathbb{R}}$, then the element $e_A e_B e_A + (1 - e_A)$ maps under $\widetilde{\rho}_Z$ to $f_A f_B f_A + (1 - f_A)$, which is invertible in R_Z. If not, then the element $e_A e_B e_A + (1 - e_A)$ maps under $\widetilde{\rho}_Z$ to $1 \in R_Z$, which is obviously invertible. So the map $\widetilde{\rho}_Z$ as defined on the generators extends to the ring R. Since $\widetilde{\rho}_Z$ vanishes on I_Z by construction, it factors through $\rho_Z : (R/I_Z) \to R_Z$. Moreover, the image of ρ_Z contains all generators of R_Z as well as all the adjoined inverses. Therefore ρ_Z is surjective.

Now consider the composition $R_Z\text{-mod} \to (R/I_Z)\text{-mod} \to \mathrm{Perv}(Z, \mathscr{S}|_Z)$. Suppose that the equivalence from Theorem 2.7 sends $M \in R_Z\text{-mod}$ to the complex $\mathscr{F}^\bullet(M)$. Let $\mathscr{E}^\bullet(M)$ be the complex obtained from the (R/I_Z)-module $\rho_Z^*(M)$, via the equivalence constructed above. Both $\mathscr{F}^\bullet(M)$ and $\mathscr{E}^\bullet(M)$ are complexes of sheaves that are locally constant on cells

of the form $iP+Q$ for $P,Q \in \mathscr{C}$. We compare them stalk-wise to check that they are equal. From the expression in Equation 3, the stalks of $\mathscr{F}^\bullet(M)$ are zero by construction on any point of X outside Z. From the proof above, the stalks of $\mathscr{E}^\bullet(M)$ are zero on any point of X outside Z. Now suppose that $iP+Q$ is a cell where $P \subset Z_{\mathbb{R}}$ and $Q \subset Z_{\mathbb{R}}$. Suppose that $C \in \mathscr{C}$ with $C \subset Z_{\mathbb{R}}$ and $P \leq C$. Let $K = C \circ Q$. In this case, the stalk of $\mathscr{E}_C(M)$ at $iP+Q$ equals $e_K(\rho_Z^*(M)) = f_K M$. The stalk of $\mathscr{F}_C(M)$ at $iP+Q$ also equals $f_K M$. Therefore $\mathscr{F}^\bullet(M) = \mathscr{E}^\bullet(M)$, and the theorem is proved. □

The following proposition says that the composition $R_Z \to (R/I_Z)\text{-mod} \to \mathrm{Perv}(Z, \mathscr{S}|_Z)$ is compatible with open restriction. We omit the proof, which is similar to arguments from previous results.

Proposition 4.12. *Fix some $A \in \mathscr{C}_Z$ of maximal dimension in Z. Let $f = f_A$ be the corresponding idempotent in R_Z, and let e be the image of the idempotent e_A in the quotient R/I_Z. The ring homomorphism ρ_Z restricts to a ring homomorphism $e(R/I_Z)e \to fR_Zf$, and induces an equivalence of categories fR_Zf -mod $\to e(R/I_Z)e$ -mod. Moreover, the following diagram commutes upto natural isomorphism.*

$$\begin{tikzcd}
R_Z\text{-mod} \arrow[r] \arrow[d, shift left, "j^{-1}"] & (R/I_Z)\text{-mod} \arrow[r] \arrow[d, shift left, "j^{-1}"] & \mathrm{Perv}(Z, \mathscr{S}|_Z) \arrow[d, shift left, "j^{-1}"] \\
fR_Zf\text{-mod} \arrow[r] \arrow[u, bend left, "j_!"] \arrow[u, bend right, "j_*"'] & e(R/I_Z)e\text{-mod} \arrow[r] \arrow[u, bend left, "j_!"] \arrow[u, bend right, "j_*"'] & \mathrm{Perv}(Y, \mathscr{S}_Y) \arrow[u, bend left, "j_!"] \arrow[u, bend right, "j_*"']
\end{tikzcd}$$

As a consequence, each local system $\mathscr{L}$ on Y corresponds to an $e(R/I_Z)e$-module. The next proposition is immediate, and gives a description of intersection cohomology sheaves on X coming from local systems on strata.

Corollary 4.13. *Suppose that $\mathscr{L}$ is a local system on Y, corresponding to the $e(R/I_Z)e$-module M. Let $j_{!*}(M)$ be the intermediate extension of M to an (R/I_Z)-module. Under the equivalence R-mod $\to \mathrm{Perv}(X, \mathscr{S})$ from Theorem 4.7, the module $j_{!*}(M)$ (thought of as an R-module) maps to $\mathrm{IC}(\mathscr{L})$.*

5. Application to W-equivariant perverse sheaves

An important example of the kinds of hyperplane arrangements studied above is given by the reflection arrangement of a finite Coxeter group W. In this case we can consider the same setup as in Subsection 2.1. Additionally, the group W acts on this setup. We consider the category of W-equivariant perverse sheaves on $\mathscr{S}$, denoted $\mathrm{Perv}_W(X, \mathscr{S})$.

Weissman [19] defines an algebra analogous to the algebra R defined in Subsection 2.2. Further, Weissman proves that $\mathrm{Perv}_W(X, \mathscr{S})$ is equivalent to the category of finite-dimensional modules over this algebra. This theorem is the W-equivariant analogue of our Theorem 2.7. The aim of this section is to apply the methods from the remainder of the paper to the case of W-equivariant perverse sheaves on Coxeter arrangements.

5.1. **Definition of the algebra R_W.** This section recalls some definitions from [19]. We refer to [19] as well as [3] for details. In the remainder of this paper we fix $\mathscr{H}$ to be the real reflection arrangement of a finite Coxeter group W. Otherwise the setup is identical to that in Subsection 2.1.

Further, we fix a chamber $A \in \mathscr{C}^0$. Let S be the set of reflections in the walls of A. Then (W, S) forms a finite Coxeter system. This means that W is generated by the set S, modulo the following relations.

(1) $s^2 = 1$ for each $s \in S$.

(2) $(st)^{m_{st}} = 1$ for each pair $(s,t) \in S$, where $m_{s,t}$ denotes the order of (st) in W.

Let $\mathscr{C}^+ = \{C \in \mathscr{C} \mid C \leq A\}$. Note that $\overline{A} = \bigcup_{C \in \mathscr{C}^+} C$ is a fundamental domain for the action of W on $\mathbb{C}^n$. The set of subsets of S, denoted Λ, is partially ordered by reverse inclusion. As explained in [19, Proposition 2.1.1], the posets $\mathscr{C}^+$ and Λ are isomorphic. The isomorphism is given by sending any $I \in \Lambda$ to

$$C_I = \{x \in \overline{A} \mid s(x) = x \text{ for all } s \in I, \text{ and } s(x) \neq x \text{ for all } s \notin I\}.$$

For example, $C_\emptyset = A$ and $C_S = \{0\}$.

Suppose that $I, J \in \Lambda$ and $w \in W$. Recall from [19, Lemma 4.1.1] that I is said to oppose J through w (written $I \mid_w J$) if the following hold.

- There is some $K \in \Lambda$ such that $\#I = \#J = \#K - 1$, and $I \cup J \subset K$.
- $w \in W_K$, and $wJw^{-1} = I$.
- There are opposite faces $C_1 \mid_{C_0} C_2$ such that $C_1 = C_I$, $C_0 = C_K$, and $C_2 = w(C_J)$.

We recall the definition of the algebra from [19, Theorem 4.3.1], which we denote as R_W. It is denoted by $\mathscr{A}_W$ in the paper above.

Definition 5.1. Let R_W^0 be the algebra generated freely over $\mathbb{C}$ by the sets $\{e_I : I \in \Lambda\}$ and $\{s : s \in S\}$, subject to the following relations.

(1) For any two $I, J \in \Lambda$, we have $e_I e_J = e_{I \cap J} = e_J e_I$.
(2) If $s \in I$, then $s e_I = e_I s$.
(3) For each $s \in S$, we have $s^2 = 1$.
(4) For any two $s, t \in S$ with $m_{s,t} < \infty$, we have $(st)^{m_{s,t}} = 1$.
(5) Suppose I and J are subsets of S such that $S = I \sqcup J$. Let $A \subset I$ and $B \subset J$. Let w_A and w_B be the longest elements of the sub-Coxeter systems (W_A, A) and (W_B, B). If $w, w_1, w_2 \in W_I$ such that $w = w_2 w_1$, and if

$$l(w w_B w_A) = l(w w_B w^{-1}) + l(w_2) + l(w_1) + l(w_A),$$

then

$$e_{A \cup J} \cdot w_1 \cdot e_J \cdot w_2 \cdot e_{B \cup J} = e_{A \cup J} \cdot w \cdot e_{B \cup J}.$$

Then the algebra R_W is defined to be the localisation of R_W^0 at the multiplicative subset generated by the elements of the form

$$\{e_I w^{-1} e_J w e_I + (1 - e_I) \mid I \mid_w J\}.$$

5.2. **Equivalence of recollements.** We prove the following equivariant analogue of Theorem 4.7.

Theorem 5.2. *Let $\mathscr{H}$ be the real reflection arrangement of a finite Coxeter group W, and let $\mathscr{H}_\mathbb{C}$ be its complexification. Let $\mathscr{S}$ be the stratification of $X = \mathbb{C}^n$ by the complex faces of $\mathscr{H}_\mathbb{C}$. Let $U \in \mathscr{S}$ be the open stratum, and let V be its complement in X. Let $A \in \mathscr{C}$ be a fixed chamber, so that the closure of A is a fundamental domain for the W-action on X. Let $e \in R_W$ be the idempotent corresponding to A (equivalently, to $\emptyset \in \Lambda$). Then we have an equivalence of recollements as follows.*

$$\begin{tikzcd}
R_W/R_W e R_W\text{-mod} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "F'_A"'] & R_W\text{-mod} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, "F"'] & e R_W e\text{-mod} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[d, "F''_A"] \\
\mathrm{Perv}_W(V, \mathscr{S}|_V) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathrm{Perv}_W(X, \mathscr{S}) \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathrm{Perv}_W(U, \mathscr{S}|_U) \arrow[l, shift left=2] \arrow[l, shift right=2]
\end{tikzcd}$$

Proof. First note that $\mathrm{Perv}_W(U, \mathscr{S}|_U)$ is equivalent to the category of W-equivariant local systems on U. Since the action of W on U is free, this category is equivalent to the category of local systems on U/W. Recall that the fundamental group of U/W is the Artin

braid group Γ_W corresponding to W. We already know from [19, Proposition 4.4.2] that there is a ring homomorphism $\iota_W : \mathbb{C}[\Gamma_W] \to eR_We$, given by sending $s \in S$ to $ese \in eR_We$. There is a corresponding pullback functor $\iota_W^* : eR_We\text{-mod} \to \mathbb{C}[\Gamma_W]\text{-mod}$.

We now set F to be the equivalence $R_W\text{-mod} \to \mathrm{Perv}_W(X, \mathscr{S})$ as in [19, Theorem 4.3.1]. We define F_A'' analogously to the definition in Lemma 4.6. The key step of Lemma 4.6 is to show that if $M \in R\text{-mod}$, the action of any $\ell \in \pi_1(U, e)$ on $j^*F(M)$ coincides with the action of ℓ on the eRe-module eM via ι^*. We used the fact that any $\ell \in \pi_1(U, e)$ can be written as a product of "half-monodromies" around hyperplanes in $\mathscr{H}$, starting and ending at the basepoint. In the equivariant case, the half-monodromies starting at the basepoint are now themselves elements of the fundamental group. An analogous calculation goes through in this case, and this is precisely the content of [19, Proposition 4.4.1]. Finally we define $F_A' : R_W/R_WeR_W\text{-mod} \to \mathrm{Perv}_W(V, \mathscr{S}|_V)$ exactly as in Theorem 4.7. The remainder of the proof is analogous to the proof of Theorem 4.7. □

Recall (see, e.g., [19, Section 4.4]) that a W-equivariant local system on the open stratum U is just a representation of the braid group $\Gamma_W = \pi_1(U/W)$. We now have the following analogue of Corollary 4.9.

Corollary 5.3. *Let U be the open stratum of X, and let $\mathscr{L}$ be an equivariant local system on U corresponding to a representation L of Γ_W. Let $e \in R_W$ be the idempotent corresponding to $\emptyset \in \Lambda$. Let $M \in eR_We$-mod be the object corresponding to L under the equivalence $\mathrm{Perv}(U, \mathscr{S}|_U) \cong eR_We$-mod. Then the intersection cohomology sheaf $IC(\mathscr{L})$ corresponds to the R_W-module $j_{!*}(M)$, where $j_{!*}$ is the intermediate extension functor on R_W-modules as defined in Definition 3.3.*

Let Z be any W-stable and closed union of strata of $\mathscr{S}$. Let $\mathscr{C}_Z^+$ be the subposet of $\mathscr{C}^+$ defined as $\{C \in \mathscr{C}^+ \mid C \subset Z_{\mathbb{R}}\}$. Let Λ_Z be the image of $\mathscr{C}_Z^+$ under the identification of posets $\mathscr{C}^+ \cong \Lambda$. As in Subsection 4.2, we obtain a description in terms of R_W-modules for the W-equivariant perverse sheaves supported on Z.

Proposition 5.4. *Let I_Z be the ideal of R_W generated by $\{e_I \mid I \notin \Lambda_Z\}$. Then there is an equivalence of categories as follows:*

$$(R_W/I_Z)\text{-mod} \to \mathrm{Perv}_W(Z, \mathscr{S}|_Z).$$

Proof. Since Z is closed and W-stable, the category $\mathrm{Perv}_W(Z, \mathscr{S}|_Z)$ is isomorphic to the full subcategory of $\mathrm{Perv}(X, \mathscr{S})$ consisting of objects supported on Z. Similarly, the category (R_W/I_Z)-mod is isomorphic to the full subcategory of R_W-mod consisting of objects annihilated by I_Z. The remainder of the proof is analogous to the proof of Theorem 4.10, using the results of Sections 2.2, 3.3, and Theorem 4.3.1 of [19] to translate between W-equivariant perverse sheaves and R_W-modules. □

6. Further observations and questions

6.1. **Algebras with equivalent finite-dimensional module categories.** Recall that two rings are said to be Morita equivalent if their module categories are equivalent. In the previous sections, we have discussed several pairs of algebras whose finite-dimensional module categories are equivalent via pullback maps induced from homomorphisms between them. It is natural to ask whether in these cases this structure induces an isomorphism, or at least a Morita equivalence between these algebras.

If A is an algebra over a field k, let A-Mod be the category of all left A-modules, including those that are infinite-dimensional over k. Recall that we use A-mod to denote the category of k-finite-dimensional left A-modules.

6.1.1. *The algebras associated to perverse sheaves on the open stratum.* First consider the W-equivariant case for the reflection arrangement of a finite Coxeter group W. Recall from [19, Proposition 4.4.2] that there is an algebra homomorphism $\iota\colon \mathbb{C}[\Gamma_W] \to eR_We$, where Γ_W and eR_We are as defined in Section 5. We see as a corollary of Theorem 5.2 that the pullback functor $\iota^*\colon eR_We\text{-mod} \to \mathbb{C}[\Gamma_W]\text{-mod}$ on the finite-dimensional module categories is an equivalence. It is not clear from the definition of ι whether it has any other nice properties. We prove the following proposition.

Proposition 6.1. *The ring homomorphism $\iota\colon \mathbb{C}[\Gamma_W] \to eR_We$ is injective.*

Proof. Suppose that $K = \ker(\iota)$. Since the pullback functor $\iota^*\colon eR_We\text{-mod} \to \mathbb{C}[\Gamma_W]\text{-mod}$ is an equivalence, we see that each element of K must act by zero on every finite-dimensional module of $\mathbb{C}[\Gamma_W]$. To show that K is trivial, it is enough to show that for each non-zero $r \in \mathbb{C}[\Gamma_W]$, there is some finite-dimensional module of $\mathbb{C}[\Gamma_W]$ on which r does not act by zero.

Let r be any element of $\mathbb{C}[\Gamma_W]$. We will find a finite-dimensional representation of Γ_W on which r does not act by zero. Write r as a finite linear combination

$$r = \sum_{g\in\Gamma_W} c_g g,$$

where each $c_g \in \mathbb{C}$ for each $g \in \Gamma_W$, and $c_g = 0$ for all but finitely many $g \in \Gamma_W$. It is well-known that Γ_W is a linear group [13, 8, 7]. In particular, there is an embedding $\Gamma_W \overset{\rho}{\hookrightarrow} GL(V)$ for some finite-dimensional complex vector space V. Consider the representations $\operatorname{Sym}^k V$ as k varies over the positive integers. For each $k \in \mathbb{N}$ consider $\operatorname{Sym}^k \rho(r)$, which is the element of $GL(\operatorname{Sym}^k(V))$ corresponding to the action of r on $\operatorname{Sym}^k V$. If $\rho(r) = \operatorname{Sym}^1 \rho(r)$ is nonzero, we are done.

Otherwise, by choosing a basis $\{v_1, \dots, v_n\}$ of V, we see that each matrix entry of $\rho(r)$ is zero. So for each (i, j), we have the following equation:

$$\sum_{g\in\Gamma_W} c_g(\rho(g))_{(i,j)} = 0.$$

The matrix entries of $\operatorname{Sym}^k(\rho(r))$ corresponding to the basis vectors $v_1^k, \dots, v_n^k$ contain as a subset the kth powers of the matrix entries of $\rho(r)$. If $\operatorname{Sym}^k(\rho(r))$ were zero for each k, then in particular we would have

$$\sum_{g\in\Gamma_W} c_g(\rho(g))^k_{(i,j)} = 0$$

for each positive integer k. Since all of the above sums are finite, the above equations hold if and only if whenever $c_g \neq 0$, we have $\rho(g)_{(i,j)} = 0$ for all (i, j). This means that whenever $c_g \neq 0$, we have $\rho(g) = 0$. However, g ranges over Γ_W, and $\rho\colon \Gamma_W \to GL(V)$ is an embedding. So for each $g \in \Gamma_W$, we have $\rho(g) \neq 0$. We conclude that $c_g = 0$ for each $g \in \Gamma_W$, which means that $r = 0$. This is a contradiction.

We conclude that for some positive integer k, the action of r on the finite-dimensional representation $\operatorname{Sym}^k V$ is nonzero. Therefore $K = \ker(\iota)$ is trivial, and the proof is complete. □

We have the following questions.

Questions 6.2.
(1) Is the homomorphism $\iota\colon \mathbb{C}[\Gamma_W] \to eR_We$ an isomorphism?
(2) Is the pullback functor $\iota^*\colon eR_We\text{-Mod} \to \mathbb{C}[\Gamma_W]\text{-Mod}$ an equivalence of categories?

We have a similar situation in the non-equivariant case. From Proposition 4.5 and Corollary 4.8, we see that the homomorphism $\iota\colon \mathbb{C}[\pi_1(U,e)] \to eRe$ induces an equivalence

$$\iota^*\colon eRe\text{-mod} \to \mathbb{C}[\pi_1(U,e)]\text{-mod}\,.$$

If the group $\pi_1(U,e)$ is linear, then we can use the same argument as in Proposition 6.1 to see that the above map ι is injective. In general, we have the following questions.

Questions 6.3.
(1) Is the algebra homomorphism $\iota\colon \mathbb{C}[\pi_1(U,e)] \to eRe$ injective? Is it surjective?
(2) Is the pullback functor $\iota^*\colon eRe$-Mod $\to \mathbb{C}[\pi_1(U,e)]$-Mod an equivalence?

6.1.2. *The algebras associated to perverse sheaves on the closure of a stratum.* Recall the setup of Subsection 4.3. We have the algebras R_Z and R/I_Z, with a surjective ring homomorphism $\rho_Z\colon R/I_Z \to R_Z$ that induces an equivalence $\rho_Z^*\colon (R/I_Z)\text{-mod} \to R_Z\text{-mod}$. We again have the following questions.

Questions 6.4.
(1) Is the algebra homomorphism $\rho_Z\colon (R/I_Z) \to R_Z$ injective?
(2) Is the pullback functor $\rho_Z^*\colon (R/I_Z)$-Mod $\to R_Z$-Mod an equivalence?

6.2. **W-equivariant IC sheaves supported on closures of smaller orbits.** Let Y be the W-orbit of a single stratum in the stratification $\mathscr{S}$ associated to a finite Coxeter arrangement. Given a W-equivariant local system on Y, we can apply the intermediate extension functor to obtain a W-equivariant perverse sheaf on $\overline{Y}$, which extends by zero to a W-equivariant perverse sheaf on X.

Question 6.5. *Is there an equivariant analogue of Corollary 4.13? In other words, can we describe the restriction functor from $\mathrm{Perv}_W(\overline{Y}, \mathscr{S}|_{\overline{Y}}) \to \mathrm{Perv}_W(Y, \mathscr{S}_Y)$ in terms of rings as the restriction via an idempotent?*

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