A THURSTON COMPACTIFICATION OF THE SPACE OF STABILITY CONDITIONS: RANK TWO CASES

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Abstract. We define a continuous map from the space of Bridgeland stability conditions on a triangulated category $\mathcal{C}$ to an infinite projective space. We conjecture that under some assumptions on the triangulated category $\mathcal{C}$, this map is a homeomorphism onto its image, and the closure is a compact real manifold with boundary. Thus, we obtain a compactification of the space of stability conditions. Both the definition and expected properties of this compactification are analogous in many respects to Thurston's compactification of the Teichmüller space of a hyperbolic surface. In this paper we study in detail the case when $\mathcal{C}$ is the 2-Calabi-Yau category associated to the A_2 quiver, and prove our conjectures in that case.

1. Introduction

A series of recent papers have established a fascinating analogy between the Teichmüller space of a surface and the space of Bridgeland stability conditions on a triangulated category [3, 4, 6]. In this analogy, a curve on the surface corresponds to an object of the category, the topological intersection number corresponds to the dimension of the hom spaces, a metric corresponds to a stability condition, and the length of a curve prescribed by a metric corresponds to the mass prescribed by the corresponding stability condition.

Using this analogy, we can hope to transport powerful tools from geometry to homological algebra. The goal of this paper is to outline one aspect of such a program. We define a compactification of the space of Bridgeland stability conditions on suitable triangulated categories, and propose a conjectural description of its boundary. In the present paper, we work out this compactification explicitly for the smallest non-trivial cases, namely the case of the 2-Calabi–Yau categories associated to the A_2 quiver and the $\widehat{A}_1$ quiver. We will extend our results to more general triangulated categories in future work.

1.1. Mass functions.

The key ingredient in Thurston's compactification of the Teichmüller space is its embedding in an infinite projective space. Roughly speaking, this embedding sends a metric μ to the real-valued function on the set of curves defined by the length with respect to μ. By following the analogy, we are led to the following construction.

Let $\mathbf{k}$ be a field, and $\mathcal{C}$ a $\mathbf{k}$-linear triangulated category. Denote by $\mathbb{R}^{\mathcal{C}}$ the space of functions from the set of objects of $\mathcal{C}$ to $\mathbb{R}$, endowed with the product topology. A stability condition σ on $\mathcal{C}$ yields a function $m_\sigma \in \mathbb{R}^{\mathcal{C}}$ defined by

$$m_\sigma \colon x \mapsto m_\sigma(x),$$

where $m_\sigma(x)$ is the Harder–Narasimhan mass of x with respect to the stability condition σ. More explicitly, let

$$0 \to x_0 \to \cdots \to x_n = x$$

be the Harder–Narasimhan filtration of x with respect to σ and

$$z_i = \mathrm{Cone}(x_{i-1} \to x_i)$$

the Harder–Narasimhan factors. Denote by $Z \colon K(\mathcal{C}) \to \mathbb{C}$ the central charge associated to σ. Then the mass of x with respect to σ is the sum

$$m_\sigma(x) = \sum_i |Z(z_i)|.$$

The association $\sigma \mapsto m_\sigma$ yields a map

$$m \colon \mathrm{Stab}(\mathcal{C}) \to \mathbb{R}^{\mathcal{C}}.$$

From the definition of the topology on $\mathrm{Stab}(\mathcal{C})$, it is immediate that m is continuous.

Recall that we have an action of $\mathbb{C}$ on $\mathrm{Stab}(\mathcal{C})$ in which a complex number $a + ib$ acts by scaling the central charge by $\exp(a)$, and shifting the slicing by b/π. In particular, the subgroup $i\mathbb{R} \subset \mathbb{C}$ acts only by shifting the slicing, and hence leaves the function m unchanged. As a result, the map m induces

$$m \colon \mathrm{Stab}(\mathcal{C})/i\mathbb{R} \to \mathbb{R}^{\mathcal{C}}.$$

Furthermore, the action by an arbitrary complex number changes m only by simultaneous scaling. Therefore, if we denote by $\mathbb{P}^{\mathcal{C}}$ the infinite projective space $\mathbb{P}^{\mathcal{C}} = \left(\mathbb{R}^{\mathcal{C}} \setminus 0\right)/\mathbb{R}^\times$, we get a continuous map

$$m \colon \mathrm{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^{\mathcal{C}}.$$

Guided by the analogy from Teichmüller theory, we may hope for the following.

Expectation 1. *The map $m \colon \mathrm{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^{\mathcal{C}}$ is injective and a homeomorphism onto its image.*

Expectation 2. *The image $m\left(\mathrm{Stab}(\mathcal{C})/\mathbb{C}\right) \subset \mathbb{P}^{\mathcal{C}}$ is pre-compact (that is, its closure is compact).*

If these two expectations hold, then we obtain a compactification of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ given by the closure of $m(\mathrm{Stab}(\mathcal{C})/\mathbb{C})$ in $\mathbb{P}^{\mathcal{C}}$.

We cannot hope for Expectation 1 and Expectation 2 to hold in general, as one can construct easy examples that break injectivity. Nevertheless, they do seem to hold in a number of cases of interest, such as for the Calabi–Yau categories associated to (connected) quivers.

1.2. **Hom functions.** Assuming that the procedure outlined in § 1.1 gives a compactification of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$, the next natural step is to describe the boundary. In Teichmüller theory, the boundary has a beautiful modular interpretation as the space of so-called projective measured foliations. At present, we do not know an appropriate categorical analogue of this notion.

In Teichmüller theory, there is an alternate, more indirect, description of the functions appearing on the boundary. Each (simple, closed) curve γ on a surface gives rise to a function on the set of all curves defined by the topological intersection number with γ. These functions form a dense subset of the boundary of the Teichmüller space.

The intersection functions have a natural categorical analogue. Assume that for any two objects $x, y \in \mathcal{C}$, the vector space

$$\mathrm{Hom}^*(x, y) = \bigoplus_n \mathrm{Hom}(x, y[n])$$

is finite dimensional. Let $x \in \mathcal{C}$ be an object. We have a function $\hom(x) \in \mathbb{R}^{\mathcal{C}}$ defined by

$$\hom(x) \colon y \mapsto \dim_{\mathbf{k}} \mathrm{Hom}^*(x, y).$$

Expectation 3. *There is a suitable class of objects* $\mathbf{S} \subset \mathcal{C}$ *such that the classes of functions* $\hom(x)$ *in* $\mathbb{P}^{\mathcal{C}}$ *form a dense subset of the boundary of* $m(\operatorname{Stab}(\mathcal{C})/\mathbb{C})$ *in* $\mathbb{P}^{\mathcal{C}}$*.*

In the case of the 2-Calabi–Yau categories associated to quivers, we expect the set $\mathbf{S}$ to be the set of spherical objects of $\mathcal{C}$. We prove this is indeed the case for the A_2 quiver.

1.3. **Geometry of the compactification.** Assume that m yields an embedding of $\operatorname{Stab}(\mathcal{C})/\mathbb{C}$ in $\mathbb{P}^{\mathcal{C}}$, and denote the image of the embedding by M. Let $\overline{M}$ be the closure of M and set $\partial M = \overline{M} \setminus M$. Since M is a manifold, it is natural to expect the following.

Expectation 4. *The pair* $(\overline{M}, \partial M)$ *is a manifold with boundary.*

Furthermore, in the cases where M is known to be an open ball (for example, the 2-Calabi–Yau categories of ADE-quivers), we expect $\overline{M}$ to be the closed ball and ∂M to be the sphere. We show that this is indeed the case for the A_2 quiver.

We conjecture that all of the above expectations hold when $\mathcal{C}$ is the 2-Calabi-Yau category associated to a finite connected quiver. It is an interesting question to find suitable general hypotheses on $\mathcal{C}$ that ensure that these expectations hold.

1.4. **The A_2 case.** Let $\mathcal{C}$ be the 2-Calabi–Yau category associated with the A_2 quiver (see § 2 for a detailed description). The main theorem of the paper is that all the expectations listed above hold for $\mathcal{C}$. We view this as a proof-of-concept result that suggests that these expectations are reasonable, at least when restricted to a suitable yet interesting class of triangulated categories.

Denote by $\mathbf{S} \subset \mathcal{C}$ the set of spherical objects. For both the mass and the hom functions, it suffices to take the smaller space $\mathbb{P}^{\mathbf{S}}$ as the target. For reasons that are not completely clear to us, we must modify the hom functions along the diagonal. For a spherical x, define the *reduced hom function* $\overline{\hom}(x)\colon \mathbf{S} \to \mathbb{R}$ by the formula

$$\overline{\hom}(x)\colon y \mapsto \begin{cases} 0, & x \cong y[n] \text{ for some } n, \\ \hom(x,y) & \text{otherwise.} \end{cases}$$

(Even for the 2-Calabi–Yau categories associated to arbitrary quivers, we expect the reduced hom functions, and not the hom functions, to appear at the boundary).

Recall that in the A_2 case, the manifold $\operatorname{Stab}(\mathcal{C})/\mathbb{C}$ is homeomorphic to the open unit disk D in $\mathbb{C}$ [2].

Theorem 1.1 (Main)**.** *Let* $\mathcal{C}$ *be the 2-Calabi–Yau category associated with the* A_2 *quiver.*

(1) The map $m\colon \operatorname{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^{\mathbf{S}}$ *is injective and a homeomorphism onto its image.*
(2) Let $M \subset \mathbb{P}^{\mathbf{S}}$ *be the image of* m*. The closure* $\overline{M}$ *of* M *is compact.*
(3) The set $\{\overline{\hom}(x) \text{ for } x \in \mathbf{S}\}$ *forms a dense subset of* $\partial M = \overline{M} \setminus M$*.*
(4) The pair $(\overline{M}, \partial M)$ *is homeomorphic to* $(\overline{D}, \partial D)$*, where* $\overline{D} \subset \mathbb{C}$ *is the closed unit disk and* $\partial D = S^1$ *is its boundary.*

See Figure 1 for a sketch of the picture.

1.5. **Dynamics of equivalences.** One of the applications of our compactification of M, at least when $\overline{M}$ is homeomorphic to a Euclidean ball, is a dynamical classification of autoequivalences which parallels the Nielsen–Thurston classification of mapping classes as periodic, reducible, or pseudo-Anosov. We first state the theorems here in the case of A_2, and then discuss the more general situation.

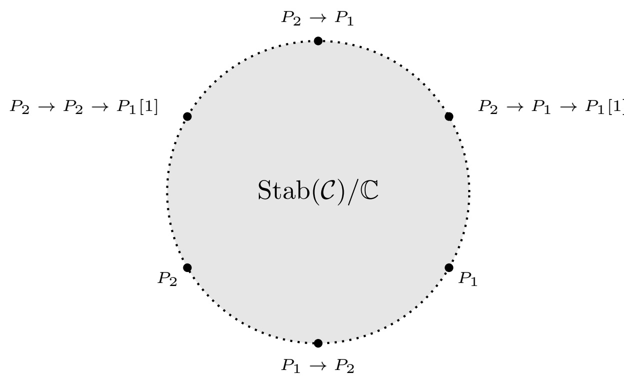


FIGURE 1. The compactification of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$. The spherical objects appear as a dense subset of the boundary.

Let B_3 be the 3-strand braid group, which acts by autoequivalences on $\mathcal{C}$, the 2-Calabi–Yau category associated to the A_2 quiver.

Proposition 1.2. *Every braid $\beta \in B_3$ satisfies exactly one of the following.*

(1) The autoequivalence associated to β fixes a point in M. In this case we say that β is periodic.

(2) The autoequivalence associated to β fixes a unique point in ∂M. In this case we say that β is reducible.

(3) The autoequivalence associated to β fixes exactly two points on ∂M. In this case we say that β is pseudo-Anosov.

It turns out that β is periodic precisely when β is central in B_3. Since the centre of B_3 acts trivially on M, it follows that periodic elements have finite order in their action on M. If β is not periodic, then β is reducible precisely when β acts on $\mathcal{C}$ as the twist in a spherical object x; the function $\overline{\hom}(x) \in \partial M$ is then the unique fixed point of β. Every braid which is not periodic or reducible is pseudo-Anosov.

The terminology introduced in Proposition 1.2 is motivated by the theory of mapping class groups, and for B_3 our terminology matches that coming from geometry. For $\beta \in B_3$, denote by $\overline{\beta}$ the image of β in $\mathrm{PSL}_2(\mathbb{R})$. There is an isomorphism of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ with the upper half plane $\mathbb{H}$ such that the action of B_3 by autoequivalences on $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ is intertwined with the action of B_3 on $\mathbb{H}$ by fractional linear transformations [2]. Then one can check that the dynamical classification of B_3 by autoequivalences in Proposition 1.2 matches the Nielsen–Thurston classification of B_3 as mapping classes of the punctured disc. We note that the compactification $\overline{M}$ of M plays an important role in this categorical formulation of the Nielsen–Thurston classification, as the action of a non-periodic braid on the space M itself will not have any fixed points.

There are further dynamical notions whose categorical and geometric incarnations can be shown to coincide in the A_2 example. For instance, there are two notions of entropy that one can associate to a braid $\beta \in B_3$: its topological entropy as a mapping class on the punctured disc, and its categorical entropy in the sense of [4]. By a result of Ikeda [8, Theorem 3.14], the categorical entropy of an autoequivalence β can be computed from the mass growth of the objects $\{\beta^n(x)\}_{n\in\mathbb{Z}}$. As a consequence of our description of the action of B_3 on M, this mass growth—like the growth of curves on the punctured disc—can be read directly from the corresponding matrix $\overline{\beta} \in \mathrm{PSL}_2(\mathbb{Z})$. It follows that the categorical and topological entropies of β coincide.

Motivated in part by the previous discussion, we introduce some general definitions. Let $\mathcal{C}$ be a triangulated category, $f \in \mathrm{Aut}(\mathcal{C})$, and $(M, \partial M)$ the associated compactification. If $\overline{M}$ is homeomorphic to a closed Euclidean ball, then the Brauer fixed point theorem implies that either there is a stability condition σ such that $f(\sigma) = z \cdot \sigma$ for some $z \in \mathbb{C}$, or there is a point on the boundary ∂M which is fixed by f. If $f(\sigma) = z \cdot \sigma$ for some $z \in \mathbb{C}$, we say that f is *periodic*. An immediate point to note is that, as the masses of the σ-stable objects are bounded away from 0, then the eigenvalue z lies on the unit circle. Furthermore, the set of phases of σ-stable objects is preserved by the action of z. So if these phases are not dense, the rotation z must have finite order. In particular, we obtain the following.

Corollary 1.3. *Let f be a periodic autoequivalence of a triangulated category $\mathcal{C}$ in the above sense (that is, f fixes a point $\sigma \in \mathrm{Stab}(\mathcal{C})/\mathbb{C}$). Suppose that*

1. *there are at most finitely many stable objects of any phase $\phi \in S^1$ (this holds, for example, whenever σ is a stability condition with finite-length heart);*
2. *f does not act trivially on the set of σ-stable objects;*
3. *the phases of the σ-stable objects are not dense in the unit circle.*

Then there exists a positive integer k such that f^k is a power of the triangulated shift.

When $\overline{M}$ is a Euclidean ball, autoequivalences f that are not periodic have fixed points on the boundary. Here there are two basic cases. The first is when f fixes the function $\overline{\hom}(x)$ (up to scaling) for some object $x \in \mathcal{C}$. In this case we say that f is *reducible*. The rationale for this terminology is as follows. First, without loss of generality, we may assume that f fixes (up to scaling) the hom functional of some indecomposable object $x \in \mathcal{C}$. Under suitable assumptions on $\mathcal{C}$, this implies that $f(x)$ is isomorphic to a shift of x. In that case, the orthogonal hom complement to x will be preserved by f, and the dynamical study of f is reduced to the study of its action on this subcategory.

If f is neither periodic nor reducible, then the fixed points of f on ∂M will consist of functions which are not represented by objects of $\mathcal{C}$. (Finding a moduli interpretation of such functions, analogous to Thurston's description of the space of projective measured foliations, is an important problem we do not directly address here.) We expect that, under some reasonable assumptions on $\mathcal{C}$, the autoequivalence f will have a dense orbit on ∂M, and that f will have two fixed points which exhibit sink/source dynamics. We refer to autoequivalences which are neither periodic nor reducible as *pseudo-Anosov*, and conjecture that they are pseudo-Anosov in the sense of [5].

When $\mathcal{C}$ is the 2-Calabi-Yau category associated to a finite connected quiver, the above discussion can be described in detail, resulting in a dynamical classification of spherical twist groups akin to the Nielsen–Thurston classification of mapping classes. We plan to address this in future work.

1.6. q-analogue.

The entire story above admits a q-analogue for any positive real number $q > 0$. Namely, we fix $q > 0$, and deform the masses of the stable objects z_i as follows:

$$m_{q,\sigma}(x) = \sum_i |q^{\phi(z_i)} Z(z_i)|.$$

The q-mass function gives a continuous map

$$m_q \colon \mathrm{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^{\mathcal{C}},$$

which agrees with our original embedding when $q = 1$. We expect that for all $q > 0$, the map $m_q\colon \mathrm{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^{\mathcal{C}}$ is injective and a homeomorphism onto its image. We thus obtain a family of compactifications of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ given by the closures of $m_q(\mathrm{Stab}(\mathcal{C})/\mathbb{C})$ in $\mathbb{P}^{\mathcal{C}}$. We conjecture that the closures $\{\overline{M}_q\}_{q>0}$ are all homeomorphic to each other.

There are some interesting differences between the $q = 1$ and $q \neq 1$ cases, even in the case of the A_2 quiver. For example, at $q = 1$, the functions $\overline{\hom}(x)$ for spherical x are dense in the boundary: under the identifications $M_{q=1} \cong \mathbb{H}$, $\partial M_{q=1} = \mathbb{R} \cup \infty$, these functions are identified with the rational points $\mathbb{Q} \cup \infty$ of the boundary. When $q \neq 1$, however, this is no longer the case. Instead, the $\hom(x)$ functions are identified with a non-dense (in fact, fractal) subset of $\mathbb{R} \cup \infty$, arising as an orbit of the Burau matrices of $\mathrm{SL}_2(\mathbb{R})$. Equally compelling, and even less understood at present, is the behaviour of $\overline{M}_q$ as $q \to \infty$ or $q \to 0$.

1.7. **Organisation.** The paper is organised as follows. In § 2, we recall the 2-Calabi–Yau category $\mathcal{C}$ associated to the A_2 quiver and study Harder–Narasimhan filtrations in it. The highlight of this section is the construction of a finite automaton that describes the dynamics of Harder–Narasimhan filtrations. In § 3, we analyse the embedding of the set of spherical objects of $\mathcal{C}$ in the infinite projective space $\mathbb{P}^{\mathbf{S}}$ In § 4, we show that $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ embeds homeomorphically into $\mathbb{P}^{\mathbf{S}}$. In § 5, we show that the closure of $\mathrm{Stab}(\mathcal{C})$ is the union of itself and the closure of the set of spherical objects. Finally, in § 6, we describe an explicit homeomorphism of the compactification to a disk.

2. The A_2 category

Let $\mathcal{C}$ be the 2-Calabi–Yau category associated to the A_2 quiver. There are several equivalent constructions of $\mathcal{C}$. Before giving one, we recall the salient (and in fact, characterising) properties of $\mathcal{C}$.

The category $\mathcal{C}$ is a triangulated $\mathbf{k}$-linear category characterised by the following properties (see [2, § 1.1]):

(1) $\mathcal{C}$ is 2-Calabi–Yau. That is, for a pair of objects $x, y \in \mathcal{C}$, we have natural isomorphisms
$$\mathrm{Hom}(x, y) \cong \mathrm{Hom}(y, x[2])^*.$$
(2) $\mathcal{C}$ is classically generated by two objects P_1 and P_2 satisfying
$$\mathrm{Hom}(P_i, P_i[n]) = \begin{cases} \mathbf{k} & \text{if } n = 0, 2, \\ 0 & \text{otherwise;} \end{cases}$$
$$\mathrm{Hom}(P_i, P_j[n]) = \begin{cases} \mathbf{k} & \text{if } n = 1, \\ 0 & \text{otherwise, for } i \neq j. \end{cases}$$

One way to construct $\mathcal{C}$ is as follows (from [10, § 2]). Let A be the zig-zag algebra of the A_2 quiver. This is the quotient of the path algebra of the doubled A_2 quiver by the ideal generated by all length 3 paths. It admits a grading by path length. Let $\mathcal{K}$ be the homotopy category of the category of bounded complexes of graded projective left A-modules. The category $\mathcal{K}$ is bi-graded, with one grading coming from the homological degree and one from the internal degree. The category $\mathcal{C}$ is the orbit category of $\mathcal{K}$ where we collapse this bi-grading to a single grading (see [9] for orbit categories). That is,
$$\mathcal{C} = \mathcal{K}/(x \sim x\langle -1\rangle[1]),$$

where $\langle - \rangle$ denotes the simultaneous internal grading shift on all the objects in x and $[-]$ the homological shift on the complex x. In this avatar of $\mathcal{C}$, the generators P_1 and P_2 are given by

$$P_i = Ae_i, \quad \text{in homological degree } 0,$$

where e_i is the idempotent of A given by the path of length 0 based at the vertex i.

The extension closure of P_1 and P_2 in $\mathcal{C}$ is an abelian category; it is the heart of a (bounded) t-structure. We refer to it as the *standard heart*. It has two simple objects, P_1 and P_2, and two additional indecomposable objects, denoted by $P_1 \to P_2$ and $P_2 \to P_1$. The object $P_i \to P_j$ is the unique extension of P_i by P_j. In terms of complexes, it is the complex

$$P_i \to P_j = Ae_i\langle -1\rangle \xrightarrow{f_{ij}} Ae_j, \quad \text{in homological degrees } -1 \text{ and } 0$$

where f_{ij} is right multiplication by the path $i \to j$.

2.1. **Spherical objects and twists.** Both P_1 and P_2 are *spherical* in the sense of [12, Definition 2.9]. In particular, $\operatorname{Hom}^*(x, x) \cong \mathbf{k}[t]/t^2$ as a $\mathbf{k}$-algebra. Any spherical object x gives rise to an autoequivalence $\sigma_x \colon \mathcal{C} \to \mathcal{C}$ called the *spherical twist* in x (see [12]). The twists in P_1 and P_2 satisfy the usual braid relation $\sigma_{P_1}\sigma_{P_2}\sigma_{P_1} \cong \sigma_{P_2}\sigma_{P_1}\sigma_{P_2}$.

Let B_3 denote the 3-strand braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

We have a homomorphism

$$B_3 \to \operatorname{Aut}(\mathcal{C})$$

defined by

$$\sigma_i \mapsto \sigma_{P_i}.$$

Via this homomorphism, we have an action of B_3 on $\mathcal{C}$. This action is faithful [12].

Let $\mathbf{S}$ be the set of spherical objects of $\mathcal{C}$, up to shift. (An object and its triangulated shift will play identical roles in most of our analysis, so it is convenient to not distinguish them). It turns out that B_3 acts transitively on the set of all the spherical objects of $\mathcal{C}$, and hence on $\mathbf{S}$.

The centre of B_3 is generated by $(\sigma_2\sigma_1)^3$. The central element $(\sigma_2\sigma_1)^3$ acts by a triangulated shift, precisely by $x \mapsto x[-2]$. The action of B_3 on $\mathbf{S}$ therefore descends to an action of $B_3/Z(B_3)$ on $\mathbf{S}$.

We have an isomorphism $B_3/Z(B_3) \to \mathrm{PSL}_2(\mathbb{Z})$ given by

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$\sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The stabiliser of an element $x \in \mathbf{S}$ is the subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ generated by the image of σ_x. In particular, the stabiliser of P_1 is the matrix σ_1. As a result, we have a $\mathrm{PSL}_2(\mathbb{Z})$-equivariant bijection

$$\mathbb{P}^1(\mathbb{Z}) \to \mathbf{S} \tag{1}$$

defined uniquely by the choice

$$[1:0] \mapsto P_1.$$

For example, we have $[0 : 1] = \sigma_1(\sigma_2([1 : 0]))$, and so $[0 : 1]$ maps to $\sigma_1(\sigma_2(P_1)) = P_2$. More generally, we can calculate the image in $\mathbf{S}$ of a point $[a : c] \in \mathbb{P}^1(\mathbb{Z})$ for $c \neq 0$ as follows. Write the rational number a/c as a continued fraction with an odd number of terms:

$$\frac{a}{c} = n_0 + \cfrac{1}{n_2 + \cfrac{1}{\ddots + \cfrac{1}{n_{2k}}}}.$$

Here, each n_i is an integer, with $n_i > 0$ for $i = 1, \ldots, 2k$. When viewed as fractional linear transformations, the matrices σ_1 and σ_2 transform a rational number α by

$$\alpha \xmapsto{\sigma_1} 1 + \alpha, \quad \alpha \xmapsto{\sigma_2} \frac{1}{-1 + 1/\alpha}.$$

We deduce that the image of $[a : c]$ in $\mathbf{S}$ is the object

$$\sigma_1^{n_0}\sigma_2^{-n_1} \cdots \sigma_1^{n_{2k}}(P_2). \tag{2}$$

2.2. **Harder–Narasimhan filtrations.** Let τ be a stability condition such that the objects P_1, P_2, and $X = P_2 \to P_1$ are τ semi-stable. This implies that, up to rotation, we can arrange the phases ϕ so that

$$0 = \phi(P_1) \leq \phi(P_2) \leq 1.$$

We say that such a stability condition is *standard* (see Figure 2). If moreover we have

$$0 = \phi(P_1) < \phi(P_2) < 1,$$

we say that τ is *strictly standard*. Let $\Lambda \subset \mathrm{Stab}(\mathcal{C})/\mathbb{C}$ be the set of standard stability conditions, and let $\Lambda^\circ \subset \Lambda$ be the subset of strictly standard stability conditions. The set Λ is a closed subset of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ that tessellates $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ under the action of B_3. More precisely, Λ satisfies the following properties (see, e.g., [2, Proposition 4.2]).

(1) Each point of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ lies in the B_3-orbit of a point of Λ.
(2) The stabiliser of Λ is the subgroup generated by $\gamma = \sigma_2\sigma_1$ in B_3.
(3) For any $g \in B_3$ not in the stabiliser, the interiors of Λ and $g\Lambda$ have empty intersection.

The set of strictly standard stability conditions is an open subset of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$.

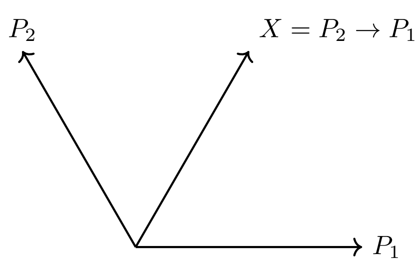


FIGURE 2. Central charges of semistable objects in a standard stability condition, up to rotation.

Fix a strictly standard stability condition τ. Our next goal is to get a precise understanding of the Harder–Narasimhan filtration of all spherical objects with respect to τ. In the rest of the subsection, the Harder–Narasimhan filtrations are with respect to τ.

In Appendix A we investigate conditions under which a filtration of an object can be refined and rearranged to produce the Harder–Narasimhan filtration. A sufficient condition is when the filtration is *geodesic*, as in Definition A.8.

Definition 2.1. Let x be an object and let σ be a triangulated autoequivalence. Suppose that

$$0 \to x_0 \to x_1 \to \cdots \to x_n = x$$

is the Harder–Narasimhan filtration of x with stable sub-quotients z_i. By applying σ, we obtain

$$0 \to \sigma(x_0) \to \sigma(x_1) \to \cdots \to \sigma(x_n) = \sigma(x), \tag{3}$$

which has sub-quotients $\sigma(z_i)$. We say that σ is a *geodesic* autoequivalence for x if the filtration (3) is a geodesic filtration.

For these observations to be of much use, we need a rich class of geodesic equivalences. Denote by $\sigma_X \in B_3$ the element

$$\sigma_X = \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^{-1}.$$

It acts on $\mathcal{C}$ by the spherical twist in X (justifying the abuse of notation). Set

$$\gamma = \sigma_2\sigma_1 = \sigma_X\sigma_2 = \sigma_1\sigma_X.$$

Note that γ acts on the stable objects P_2, P_1, X by a cyclic rotation, lowering the phases. Explicitly, we have

$$\begin{aligned}\gamma\colon P_2 &\mapsto P_1,\\ \gamma\colon P_1 &\mapsto X[-1],\\ \gamma\colon X &\mapsto P_2[-1].\end{aligned}$$

Therefore applying γ or γ^{-1} to an HN filtration gives an HN filtration. As a result of Remark A.9, γ and γ^{-1} are geodesic for *every* object.

On the other hand, the spherical twists σ_1, σ_2, and σ_X are geodesic only for certain objects, and we organise this information as follows. Figure 3 shows a directed graph, which we call the *Harder–Narasimhan automaton.* Its vertices are labelled by pairs of stable objects. We say that an object x is *supported by* a vertex v if, up to shift, all the Harder–Narasimhan factors of x are among the labels of v. The edges of the graph are labelled by autoequivalences. The graph is designed to satisfy the following property.

Proposition 2.2. *Let $x \in \mathcal{C}$ be an object supported by a vertex v in Figure 3. Let e be an edge with source v and label σ. Then σ is a geodesic autoequivalence for x, and $\sigma(x)$ is supported by the target of e.*

Proof. Recall that γ and γ^{-1} are geodesic for any object. By first applying γ or γ^{-1}, we may assume that x is supported at $[P_1, P_2]$. We must now check that when we apply σ_1 or σ_X to the Harder–Narasimhan filtration of x, we obtain a geodesic filtration. We check that in both cases, the resulting filtrations satisfy the conditions of Proposition A.15.

Let $i < j$. If (z_i, z_j) is a pair of subquotients in the Harder–Narasimhan filtration of x such that the induced map $z_j \to z_i[1]$ is zero, then it remains zero after applying any auto-equivalence. It remains to check the non-overlapping condition for pairs (z_i, z_j) that admit a non-zero map

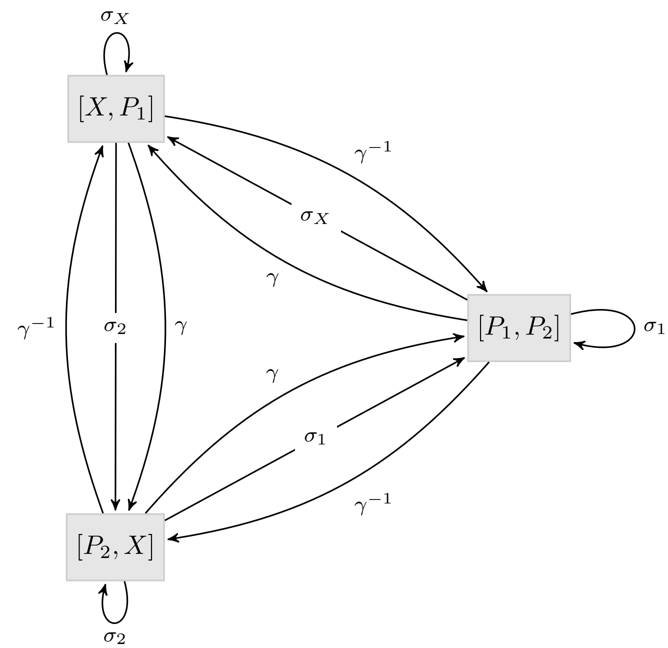


FIGURE 3. An automaton describing the dynamics of Harder–Narasimhan filtrations in a stability condition with stable objects P_1, P_2, and $X = P_2 \to P_1$.

$z_j \to z_i[1]$. Up to twist, such pairs are exactly $(P_1[1], P_1)$, $(P_2[1], P_2)$, and (P_2, P_1). The first two are easy; we check the last one. We have

$$\begin{aligned} (P_2, P_1) &\xrightarrow{\sigma_1} (P_1 \to P_2, P_1[-1]) \\ &\xrightarrow{\sigma_X} (P_1[1], X[-1] \to P_1). \end{aligned} \tag{4}$$

Observe that each pair on the right satisfies the non-overlapping phase condition, and is supported by the correct vertex. □

An important consequence of this property is that each edge is a linear transformation on the multiplicities of the Harder–Narasimhan subquotients. Given an object whose Harder–Narasimhan subquotients consist of a copies of A and b copies of B, we say that $(a, b)^t$ is its *multiplicity vector* at $[A, B]$.

Proposition 2.3. *Let $e\colon [A, B] \xrightarrow{\sigma} [C, D]$ be an edge of the Harder–Narasimhan automaton corresponding to an autoequivalence σ. There exists a 2×2 integer matrix M_e with the following property. If x is an object supported at $[A, B]$ with multiplicity vector $(a, b)^t$, the multiplicity vector of $\sigma(x)$ at $[C, D]$ is $M_e \cdot (a, b)^t$.*

Proof. By Proposition 2.2, σ is geodesic for x. Hence, the Harder–Narasimhan filtration of $\sigma(x)$ is a refinement of σ applied to the Harder–Narasimhan filtration of x. The matrix M_e is then simply the matrix whose columns are the multiplicity vectors of $\sigma(A)$ and $\sigma(B)$ at $[C, D]$. □

The following example illustrates the proposition.

Example 2.4. Let $e\colon [P_2, X] \xrightarrow{\sigma_1} [P_1, P_2]$. The associated matrix M_e is

$$M_e = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note that the columns of M_e are the multiplicity vectors of $\sigma_1(P_2)$ and $\sigma_1(X)$ in $[P_1, P_2]$. Let x be the object $\sigma_2^2\gamma P_2$. Explicitly, we have

$$x = P_2[-1] \to X, \quad \sigma_1 x = P_1[-1] \to P_2[-1] \to P_2.$$

The multiplicity vector of x at $[P_2, X]$ is $(1,1)^t$, and the multiplicity vector of $\sigma_1 x$ at $[P_1, P_2]$ is $(1,2)^t$. Indeed, $(1,2)^t = M_e \cdot (1,1)^t$.

We have seen that it is particularly easy to understand how Harder–Narasimhan filtrations change along each edge of the automaton. The next proposition shows that *every* braid can be written as a composition of edges along the automaton. This makes the automaton truly useful.

Proposition 2.5. *Any braid $\beta \in B_3$ can be written as*

$$\beta = \sigma_{a_1}^{m_1}\sigma_{a_2}^{m_2}\cdots\sigma_{a_k}^{m_k}\gamma^n,$$

where n is an integer, k is a non-negative integer, the m_i are positive integers, and the sequence

$$(a_1, a_2 \dots, a_k)$$

is a contiguous subsequence of the infinite cyclic sequence

$$(\dots, X, 1, 2, X, 1, 2, \dots).$$

Proof. We repeatedly use the commutation relations

$$\gamma\sigma_2\gamma^{-1} = \sigma_1, \quad \gamma\sigma_X\gamma^{-1} = \sigma_2, \quad \gamma\sigma_1\gamma^{-1} = \sigma_X.$$

Begin by writing β as any product of the generators σ_1 and σ_2, along with their inverses. Eliminate the inverses of the generators by rewriting as follows:

$$\sigma_1^{-1} = \sigma_X\gamma^{-1}, \quad \sigma_2^{-1} = \sigma_1\gamma^{-1}.$$

Next, use the commutation relations to rewrite

$$\gamma^i\sigma_X = \sigma_2\gamma^i, \quad \gamma^i\sigma_1 = \sigma_X\gamma^i, \quad \gamma^i\sigma_2 = \sigma_1\gamma^i,$$

and thus move instances of γ^{-1} to the right as a single power of γ^{-1}. The rest of β is now a product of elements from $\{\sigma_1, \sigma_2, \sigma_X\}$.

Finally, replace any occurrences of $\sigma_2\sigma_1$, $\sigma_1\sigma_X$, or $\sigma_X\sigma_2$ by γ and move γ to the right, again using the commutation relations. Each such operation decreases the length of the braid in the elements $\{\sigma_1, \sigma_2, \sigma_X, \gamma^{\pm}\}$. Therefore, the procedure terminates and β reaches the desired form. □

We call the writing described by Proposition 2.5 an *admissible cyclic writing*. Armed with an admissible cyclic writing, we can use the Harder–Narasimhan automaton to find the Harder–Narasimhan filtration of any spherical object. Indeed, write

$$x = \beta P_2$$

for some braid β. Write β as an admissible cyclic expression. Starting at an appropriate vertex supporting P_2, simply apply β one letter at a time, always staying within the automaton.

Recall that we have an identification $\mathbf{S} = \mathbb{P}^1(\mathbb{Z})$ via the map (1). Note that $\mathbf{S} \subset \mathbb{P}^1(\mathbb{R})$ are simply the rational points. The division of $\mathbf{S}$ according to the support corresponds nicely to a geometric division of the circle $\mathbb{P}^1(\mathbb{R})$, which we now describe. The three objects P_1, P_2, and X in $\mathbb{P}^1(\mathbb{Z}) \subset$

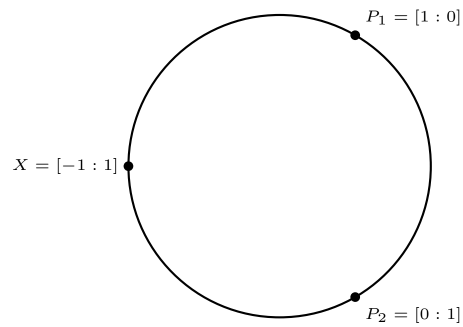


FIGURE 4. The points P_1, P_2, and $X = P_2 \to P_1$ divide $\mathbb{P}^1(\mathbb{R})$ into three arcs. The Harder–Narasimhan pieces of an object only include the two endpoints of the arc on which the object lies.

$\mathbb{P}^1(\mathbb{R}) \cong S^1$ divide $\mathbb{P}^1(\mathbb{R})$ into three closed arcs (see Figure 4). We denote these arcs by $[P_1, P_2]$, $[P_2, X]$, and $[X, P_1]$. Explicitly, the arc $[P_1, P_2]$ is the closure of the unique connected component of $\mathbb{P}^1(\mathbb{R}) \setminus \{P_1, P_2\}$ that does not contain X, and likewise for the other two arcs. Alternatively, recalling that the points P_1, P_2, and X in coordinates are $[1:0]$, $[0:1]$, and $[1:-1]$, we see that the three arcs are

$$\begin{aligned}
[P_1, P_2] &= \{[a:c] \mid 0 \leq a/c \leq \infty\}, \\
[P_2, X] &= \{[a:c] \mid -1 \leq a/c \leq 0\}, \text{ and} \\
[X, P_1] &= \{[a:c] \mid -\infty \leq a/c \leq -1\}.
\end{aligned}$$

Although we have written ∞ and $-\infty$ distinctly, this is only for the purpose of writing sensible inequalities; the two represent the same point on $\mathbb{P}^1(\mathbb{R})$.

Proposition 2.6. *The objects of* $\mathbf{S}$ *corresponding to the points of the arc* $[P_1, P_2]$ *are supported on the vertex* $[P_1, P_2]$*, and likewise for the other two arcs.*

Proof. The statement is clearly true for the objects P_1, P_2, and X. For the general case, it suffices to show that if we replace each vertex in the Harder–Narasimhan automaton by the corresponding arc, and the autoequivalences by the corresponding transformations in $\mathrm{PSL}_2(\mathbb{Z})$, then the transformation takes the source arc to the target arc. This is easily verified. $\square$

Proposition 2.7. *Let* x *be a spherical object corresponding to* $[a:c] \in \mathbb{P}^1(\mathbb{Z})$*, where* a, c *are relatively prime integers. We have the following equalities.*

1. *The minimal complex of projective modules representing* x *has exactly* $|a|$ *occurrences of* P_1 *and* $|c|$ *occurrences of* P_2*.*
2. $\overline{\hom}(x, P_2) = |a|$ *and* $\overline{\hom}(x, P_1) = |c|$*.*

The first part of the proposition is [11, Proposition 4.8]. Using the Harder–Narasimhan automaton gives a new and simpler proof.

Proof. We begin by proving the first assertion. Write

$$x = \beta P_2,$$

for a braid β, and express β in an admissible cyclic form. Recall that P_2 corresponds to $a = 0$ and $c = 1$, and hence satisfies the conclusion. By Proposition 2.3, we see that each edge in the automaton changes the number of occurrences of P_1 and P_2 in the minimal complex by a linear transformation. It suffices to verify that this linear transformation is the same as the linear transformation on (a, c), up to sign. We carry out the verification for the vertex $[P_1, P_2]$, leaving the other two vertices to the reader. The matrices of $\gamma, \gamma^{-1}, \sigma_1, \sigma_X \in \mathrm{PSL}_2(\mathbb{Z})$ are

$$\gamma = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_X = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

On the other hand, the autoequivalences change the number of occurrences of P_1 and P_2 from (m, n) to

$$(m,n) \xrightarrow{\gamma} (m+n, m), \quad (m,n) \xrightarrow{\gamma^{-1}} (n, m+n),$$
$$(m,n) \xrightarrow{\sigma_1} (m+n, n), \quad (m,n) \xrightarrow{\sigma_X} (2m+n, m).$$

Up to sign, the two transformations agree.

Let us now prove the second assertion. We prove it for $\overline{\hom}(x, P_1)$; the statement for $\overline{\hom}(x, P_2)$ follows by applying γ.

If $x = P_1$, then the statement is evident. Suppose x is supported at $[X, P_2]$. Consider the Harder–Narasimhan filtration of x:

$$0 = x_0 \to \cdots \to x_n = x,$$

with sub-quotients P_2 and X, up to shift. By applying $\mathrm{Hom}(P_1, -)$, we obtain a filtration

$$0 = \mathrm{Hom}(P_1, y_0) \to \cdots \to \mathrm{Hom}(P_1, y_n) = \mathrm{Hom}(P_1, y),$$

in the bounded derived category of (graded) vector spaces. The sub-quotients of this filtration are obtained by applying $\mathrm{Hom}(P_1, -)$ to the sub-quotients of the original filtration. Since both $\mathrm{Hom}(P_1, P_2)$ and $\mathrm{Hom}(P_1, X)$ are one-dimensional, these sub-quotients are one-dimensional vector spaces. The boundary maps between these sub-quotients are induced by the boundary maps between the sub-quotients in the filtration of x. The filtration of x has three possible non-zero boundary maps:

$$P_2 \to P_2[2], \quad X \to X[2], \text{ and } P_2 \to X[2].$$

All three are killed by $\mathrm{Hom}(P_1, -)$ for degree reasons. As a result, we conclude that

$$\overline{\hom}(P_1, x) = \overline{\hom}(x, P_1) = n = \text{number of } P_2\text{'s in the minimal complex of } x.$$

To prove the general case, it suffices to show that for any object x (other than a twist of P_1), there exists an r for which $\sigma_1^r x$ is supported by $[P_2, X]$. Indeed, by the first part of the proposition, x and $\sigma_1^n x$ have the same number of occurrences of P_2, and plainly, both have the same value for $\overline{\hom}(P_1)$. Say $x = \beta P_1$, and suppose β has the admissible cyclic expression

$$\beta = \sigma_{a_1}^{m_1} \cdots \sigma_{a_k}^{m_k} \gamma^n.$$

We induct on the quantity $\ell(\beta) = \sum m_i$. If $\ell(\beta) = 0$, then $x = P_2$ or $x = X$, and it is already supported at $[P_2, X]$. If $a_1 = 2$, then x is already supported at $[P_2, X]$. If $a_1 = 1$, then $\ell(\sigma_1^{-1}\beta) < \ell(\beta)$. If $a_1 = X$, then $\ell(\sigma_1 \beta) < \ell(\beta)$. To see this, note that

$$\sigma_1 \beta = \gamma \sigma_{a_1}^{m_1 - 1} \sigma_{a_2}^{m_2} \cdots \sigma_{a_k}^{m_k} \gamma^n.$$

By reducing as described in Proposition 2.5, we see that β' has smaller ℓ than β. The proof is now complete. $\square$

Let x be a spherical object corresponding to $[a : c] \in \mathbb{P}^1(\mathbb{Z})$ for relatively prime integers a, c.

Corollary 2.8. *The τ-mass of x is given by*

$$m_\tau(x) = \begin{cases} |a|m_\tau(P_1) + |c|m_\tau(P_2) & \text{if } x \in [P_1, P_2], \\ |c|m_\tau(X) + (|a| - |c|)m_\tau(P_1) & \text{if } x \in [P_2 \to P_1, P_1], \\ |a|m_\tau(X) + (|c| - |a|)m_\tau(P_2) & \text{if } x \in [P_2, X]. \end{cases}$$

Proof. Count the number of Harder–Narasimhan pieces using the counts of P_1 and P_2. □

Remark 2.9. The automaton also allows us to compute the categorical entropy of a braid β, and to show that it is the Perron-Frobenius eigenvalue of the matrix of absolute values of $\overline{\beta} \in \mathrm{PSL}_2(\mathbb{Z})$. In particular, the categorical and topological entropies are equal.

Corollary 2.10. *The values of the basic hom functions on x are given by*

(1) $\overline{\hom}(P_1, x) = |c|$,
(2) $\overline{\hom}(P_2, x) = |a|$,
(3) $\overline{\hom}(X, x) = |a + c|$,
(4) $\overline{\hom}(P_1 \to P_2, x) = |a - c|$.

Proof. The first two follow directly from Proposition 2.7. For the last two, use

$$\overline{\hom}(X, x) = \overline{\hom}(P_1, \sigma_2^{-1}x),$$

and

$$\overline{\hom}(P_1 \to P_2, x) = \overline{\hom}(P_2, \sigma_1^{-1}x).$$

The result follows by using the action of $\sigma_i^{\pm 1}$ on (a, c) and applying Proposition 2.7. □

3. The boundary

We now have all the tools to analyse the map $h\colon \mathbf{S} \to \mathbb{P}^{\mathbf{S}}$ defined in the introduction. We begin by recalling the definition. Consider the map

$$\overline{\hom}\colon \mathbf{S} \to \mathbb{R}^{\mathbf{S}}$$

defined by

$$\overline{\hom}(x)\colon y \mapsto \overline{\hom}(x, y).$$

The map $h\colon \mathbf{S} \to \mathbb{P}^{\mathbf{S}}$ is the composition of $\overline{\hom}$ and the projection $\mathbb{R}^{\mathbf{S}} \setminus 0 \to \mathbb{P}^{\mathbf{S}}$.

Consider $\mathbf{S}$ as a subset of $\mathbb{P}^1(\mathbb{R})$ via the identification $\mathbf{S} = \mathbb{P}^1(\mathbb{Z})$ as defined in (1).

Proposition 3.1. *The map $h\colon \mathbf{S} \to \mathbb{P}(\mathbb{R}^{\mathbf{S}})$ extends to a continuous map $h\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^{\mathbf{S}}$. The extension maps $\mathbb{P}^1(\mathbb{R})$ homeomorphically onto its image.*

Proof. Let $s \in \mathbf{S}$ be a spherical object (up to shift). Write $s = \beta P_2$ for some braid β. Let $\overline{\beta}$ be the image of β in $\mathrm{PSL}_2(\mathbb{Z})$. Then $\overline{\beta}$ is uniquely determined up to right multiplication by powers of $\overline{\sigma}_2$. Consider a point $[a : c] \in \mathbb{P}^1(\mathbb{Z})$, where a and c are relatively prime integers, and let $t \in \mathbf{S}$ be the corresponding spherical object. We have

$$\begin{aligned} \overline{\hom}(s, t) &= \overline{\hom}(\beta P_2, t) \\ &= \overline{\hom}(P_2, \beta^{-1}t) \\ &= |(1, 0) \cdot \overline{\beta}^{-1} \cdot (a, c)^t| \text{ by Proposition 2.7.} \end{aligned}$$

We define $h\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}(\mathbb{R}^{\mathbf{S}})$ by using the final expression for an arbitrary $[a : c]$. That is, we set $h([a : c])$ to be the (projectivised) function whose value at $s = \beta P_2$ is

$$|(1,0) \cdot \overline{\beta}^{-1} \cdot (a,c)^t|.$$

Plainly, $h\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^{\mathbf{S}}$ is a continuous extension of the original map $h\colon \mathbf{S} \to \mathbb{P}^{\mathbf{S}}$.

We now check that the extended map is a homeomorphism onto its image. Since the domain $\mathbb{P}^1(\mathbb{R})$ is compact and the target $\mathbb{P}(\mathbb{R}^{\mathbf{S}})$ is Hausdorff, it suffices to check that it is injective. Let

$$T = \{P_1, P_2, X\}.$$

Consider the restricted map $h_T\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}(\mathbb{R}^T)$ obtained as the composition of h and the projection to $\mathbb{P}^T$. From Corollary 2.10, h_T is given in coordinates by

$$h_T\colon [a : c] \mapsto [|a| : |c| : |a + c|].$$

This map is injective, and hence so is h. □

4. The interior

We now study the map $m\colon \operatorname{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^{\mathbf{S}}$ defined in the introduction. To recall, m sends a stability condition τ (modulo the $\mathbb{C}$ action) to the function $m_\tau\colon \mathbf{S} \to \mathbb{R}$ (well-defined up to scaling) defined by

$$m_\tau\colon x \mapsto m_\tau(x).$$

The goal of this section is to prove that the map m is a homeomorphism onto its image. We first show that the map m is injective.

Proposition 4.1 (Injectivity)**.** *The map $m\colon \operatorname{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^{\mathbf{S}}$ is injective.*

Proof. Recall the set Λ of standard stability conditions, as defined in § 2.2. Since Λ tessellates $\operatorname{Stab}(\mathcal{C})/\mathbb{C}$ under the braid group action, it is sufficient to prove injectivity of m on Λ. Let $f \in \mathbb{P}^{\mathbf{S}}$ be an element in the image of m, say $f = m(\tau)$. By applying a braid, assume that $\tau \in \Lambda$. We must show that if $f = m(\tau')$ for some other stability condition τ', then $\tau = \tau'$.

Since τ is standard, P_1, P_2, and $X = P_2 \to P_1$ are τ-semistable. Therefore, they have minimal τ-mass among the objects of S of classes $[P_1]$, $[P_2]$, and $[X]$ respectively in the Grothendieck group. Therefore, if $f = m(\tau')$, then they also have minimal τ' mass among the objects of the respective classes in the Grothendieck group. As a result, P_1, P_2, and X are also τ'-semistable, and hence τ' is standard. It remains to show that, up to the $\mathbb{C}$ action, τ and τ' have the same central charge.

By changing τ and τ' by a rotation, we may assume that

$$0 = \phi_\tau(P_1) \leq \phi_\tau(P_2) \leq 1,$$

and

$$0 = \phi_{\tau'}(P_1) \leq \phi_{\tau'}(P_2) \leq 1.$$

Furthermore, after scaling one of them, we may assume that $m_\tau = m_{\tau'}$.

Consider the three vectors $v_1 = Z(P_1)$, $v_2 = Z(X)$, and $v_3 = Z(P_2)$ in the upper half plane, where Z denotes the central charge of either of the two stability conditions. We know that these vectors are oriented anti-clockwise, and have the same lengths for both stability conditions. Since the vectors also satisfy $v_2 = v_1 + v_3$, we may think of them as the vectors corresponding to three sides of a triangle. As the lengths of the three sides of a triangle determine the triangle up to isometry, it follows that the three vectors v_i must be the same for τ and τ'. In other words, τ and τ' have the same central charge. □

Remark 4.2. Essentially the same proof shows that m is injective for the 2-Calabi–Yau category of any finite, connected quiver. Indeed, looking at the mass-minimising objects in the mass function lets us find the (semi-)stable objects. By using the linear relationships among their classes in the Grothendieck group, we can reconstruct Z.

The following proposition gives an important criterion to separate strictly standard stability conditions from all others.

Proposition 4.3 (Degeneracy)**.** *Let τ be a stability condition that is not strictly standard. Then the triangle inequalities for the three real numbers $m_\tau(P_1)$, $m_\tau(P_2)$, and $m_\tau(X)$ are degenerate. That is, one number is the sum of the other two. If τ is strictly standard, none of the three triangle inequalities are degenerate.*

Proof. If τ is strictly standard, it is clear that none of the three triangle inequalities degenerate. If τ is standard but not strictly standard, it is again clear that one of the three triangle inequalities degenerates. Suppose now that τ is non-standard.

Write $\tau = \beta\tau'$, where τ' is standard. Then $m_\tau(-) = m_{\tau'}(\beta^{-1}-)$. Write β^{-1} in a (slightly modified) admissible cyclic form

$$\beta^{-1} = \gamma^r \sigma_{a_1}^{m_1} \cdots \sigma_{a_r}^{m_r}.$$

We apply β^{-1} letter by letter to P_1, P_2, and X and keep track of their Harder–Narasimhan filtrations using the Harder–Narasimhan automaton. After the first step, we have the objects

$$\sigma_{a_r} P_1, \quad \sigma_{a_r} P_2, \quad \sigma_{a_r} X.$$

The key observation is that all three objects are supported by the *same* vertex—the one with σ_{a_r} as a loop. Furthermore, the Harder–Narasimhan multiplicity vector of one is the sum of the other two. Indeed, we have the following, where $[-]$ denotes the multiplicity vector:

$$\begin{aligned} [\sigma_1(P_2)] &= [\sigma_1(P_1)] + [\sigma_1(X)] \\ [\sigma_2(X)] &= [\sigma_2(P_1)] + [\sigma_2(P_2)] \\ [\sigma_X(P_1)] &= [\sigma_X(P_2)] + [\sigma_X(X)]. \end{aligned}$$

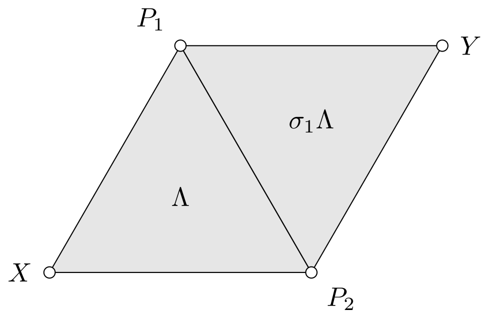
By Proposition 2.3, all further transformations preserve this linear relationship. The proof is now complete. ∎

Remark 4.4. Consider the set of stability conditions in $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ that are not strictly standard; that is, the complement of Λ°. This set has three connected components, corresponding to the three connected components of the exchange graph minus the central vertex. The component containing $\sigma_i\Lambda$, for $i = 1, 2, X$, contains $\beta\Lambda$ for β which have an admissible cyclic writing beginning with σ_i.

From the proof of Proposition 4.3, we see that the three components also correspond to the three ways in which the triangle inequality among the masses of P_1, P_2, and X degenerates. On the component containing $\sigma_1\Lambda$, the mass of X is the sum of the masses of P_1 and P_2, and likewise for the other two.

Proposition 4.5. *The map $m\colon \mathrm{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^{\mathbf{S}}$ is a homeomorphism onto its image.*

Proof. Consider the set $T = \{P_1, P_2, X, Y\}$, where $X = P_2 \to P_1$ and $Y = P_1 \to P_2$. Say that a stability condition is *semi-standard* if its semi-stable objects, up to twist, are among the objects in T. If τ is semi-standard, then up to rotation, its heart is the standard heart. Also observe that the set of semi-standard stability conditions is the union $\Pi = \Lambda \cup \sigma_1\Lambda$, as shown in the figure below.



Note that the interior Π° of Π is an open subset of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ homeomorphic to the interior of the unit square.

Consider the map $m_T\colon \mathrm{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^T$ obtained by composing m with the standard projection. Consider the subset $\Omega \subset \mathbb{P}^T$ consisting of points $[x, y, z, w]$ such that:

- $x, y, z, w \geq 0$;
- the triples (x, y, z) and (y, z, w) satisfy the triangle inequalities; and
- $z = (x + y)$ or $w = (x + y)$.

By Proposition 4.3, the image of m_T is contained in Ω. Furthermore, the stability conditions that are not semi-standard map to the boundary of Ω. As a consequence, the pre-image of Ω° is Π°. The map $m_T\colon \Pi^\circ \to \Omega^\circ$ is clearly a homeomorphism. As the translates of Π° under the braid group cover $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$, we conclude that m is a homeomorphism onto its image. □

5. Closure of the interior

The goal of this section is to show that the closure of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ in $\mathbb{P}^{\mathbf{S}}$ is compact, and to identify this closure. We begin by proving that the closure is compact. This is a fairly general phenomenon, so let us work generally for the moment. Let $\mathcal{C}$ be any triangulated category that admits a heart with finitely many simple objects $P_1, \dots, P_n$. Let $\mathbf{S} \subset \mathcal{C}$ be any set that contains $P_1, \dots, P_n$. Define the map $m\colon \mathrm{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^{\mathbf{S}}$ as before.

Proposition 5.1 (Pre-compactness)**.** *In the setup above, the closure of the image of m is compact.*

Proof. Let $\widetilde{m}\colon \mathrm{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{R}^{\mathbf{S}}$ be the lift of m characterised by

$$\sum \widetilde{m}(P_i) = 1.$$

Let $\widetilde{B}$ be the closure in $\mathbb{R}^{\mathbf{S}}$ of the image of $\widetilde{m}$.

Recall that for an exact triangle

$$x \to y \to z \xrightarrow{+1},$$

we have the following triangle inequality [8, Proposition 3.3]:

$$\widetilde{m}_\tau(y) \leq \widetilde{m}_\tau(x) + \widetilde{m}_\tau(z).$$

For every $s \in \mathbf{S}$, there exists an $n = n(s)$ and a filtration

$$0 = s_0 \to s_1 \to \cdots \to s_n = s,$$

where the sub-quotients are twists of P_i. By the triangle inequality, and the normalisation $\sum \widetilde{m}(P_i) = 1$, we have

$$\widetilde{m}_\tau(s) \leq n = n(s).$$

In other words, $\widetilde{m}$ maps $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ to the product $\prod_{s\in S}[0,n(s)]$. By the Tychonoff theorem, the product is compact, and hence $\widetilde{B}$ is compact. Thanks to the normalisation $\sum \widetilde{m}(P_i) = 1$, we see that $\widetilde{B}$ is contained in $\mathbb{R}^{\mathbf{S}} \setminus 0$. Finally, note that the closure of the image of m in $\mathbb{P}^{\mathbf{S}}$ is contained in the image of $\widetilde{B}$ under the projection map, which is a compact set. Hence, the closure of the image of m is also compact. $\square$

We now return to the case at hand: let $\mathcal{C}$ and $\mathbf{S}$ once again be as defined for the A_2 case in § 2. Set

$$\begin{aligned} P &= \overline{h(\mathbf{S})} = h(\mathbb{P}^1(\mathbb{R})) \subset \mathbb{P}^{\mathbf{S}} \\ M &= m\left(\mathrm{Stab}(\mathcal{C})/\mathbb{C}\right) \subset \mathbb{P}^{\mathbf{S}}. \end{aligned}$$

We prove that $\overline{M} = M \cup P$.

Let τ be a standard stability condition. Since the three positive real numbers $m_\tau(P_1)$, $m_\tau(P_2)$, and $m_\tau(X)$ satisfy the triangle inequalities, there exist non-negative real numbers x, y, z such that

$$m_\tau(P_1) = y + z, \quad m_\tau(P_2) = z + x, \quad m_\tau(X) = x + y.$$

We call the x, y, z, the *Gromov coordinates* of τ. Note that if τ is strictly standard, then the Gromov coordinates are all positive. Otherwise, one of the coordinates is zero. Two of the coordinates cannot be zero.

Recall that $m_\tau \colon \mathbf{S} \to \mathbb{R}$ is the mass function associated to τ and $\overline{\hom}(s)$ the reduced hom functional associated to an object s.

Proposition 5.2 (Linearity)**.** *We have*

$$m_\tau = x\,\overline{\hom}(P_1) + y\,\overline{\hom}(P_2) + z\,\overline{\hom}(X).$$

Proof. Let $s \in \mathbf{S}$ be a spherical object. Observe that the Harder–Narasimhan filtration of s is the same for all strictly standard stability conditions, and its pieces are P_1, P_2, and X, up to shift. Denoting by $a(s)$, $b(s)$, and $c(s)$ the multiplicities of these three in the Harder–Narasimhan filtration, we have

$$m_\tau(s) = (y+z)a(s) + (x+z)b(s) + (x+y)c(s).$$

In particular, $m_\tau(s)$ is linear in the Gromov coordinates. Using Proposition 2.7, we have

$$\begin{aligned} b(s) + c(s) &= \text{number of occurrences of } P_2 \text{ in the minimal complex of } s \\ &= \overline{\hom}(P_1, s), \\ a(s) + c(s) &= \text{number of occurrences of } P_1 \text{ in the minimal complex of } s \\ &= \overline{\hom}(P_2, s), \text{ and} \\ a(s) + b(s) &= \overline{\hom}(X, s). \end{aligned}$$

The first two equalities are evident. The third is obtained by applying γ to the second. We conclude that

$$m_\tau(s) = x\,\overline{\hom}(P_1, s) + y\,\overline{\hom}(P_2, s) + z\,\overline{\hom}(X, s).$$

Suppose that τ is standard but not strictly standard, such that $0 = \phi(P_1) = \phi(P_2)$. In this case, the stable objects in the standard heart are precisely P_1 and P_2, and the Gromov coordinate corresponding to X is zero. The previous argument extends to this case if we take a refinement of the Harder–Narasimhan filtration consisting of stable objects (shifts of P_1 and P_2).

Since every stability condition is in the B_3-orbit of either a strictly standard stability condition or one in which $0 = \phi(P_1) = \phi(P_2)$, linearity holds for all of them. Indeed, let τ be an arbitrary stability condition. Then there are three τ semi-stable objects, say A, B, C, whose classes in the Grothendieck group are (up to sign) the classes of P_1, P_2, and X. These are obtained simply by applying an appropriate braid to P_1, P_2, and X. Define the Gromov coordinates x, y, z for τ by the condition

$$m_\tau(A) = y + z, \quad m_\tau(B) = x + z, \quad m_\tau(C) = x + y.$$

With the above notation, we have

$$m_\tau = x\,\overline{\hom}(A) + y\,\overline{\hom}(B) + z\,\overline{\hom}(C).$$

□

We now use the Gromov coordinates to get a simple geometric picture of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$. Let Δ denote the following clipped triangle:

$$\begin{aligned}\Delta &= \{(x, y, z) \in \mathbb{R}^3_{\geq 0} \mid \text{at least two coordinates non-zero}\}/\,\mathbb{R}_{>0}\\ &\cong \text{a closed planar triangle minus the three vertices.}\end{aligned}$$

Let $\Lambda \subset \mathrm{Stab}(\mathcal{C})/\mathbb{C}$ be the closed subset of standard stability conditions. The Gromov coordinates allow us to identify Λ with the clipped triangle Δ. Since Λ tessellates for the action of $\mathrm{PSL}_2(\mathbb{Z})$, we obtain the complete picture of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ as unions of copies of Δ.

Let us describe this picture in more detail. Recall that the element $\gamma = \sigma_2\sigma_1$ generates the stabiliser of Λ in $\mathrm{PSL}_2(\mathbb{Z})$. So, we can label the various (distinct) translates of Λ by $\langle\gamma\rangle$-cosets. Two translates of Λ are either disjoint or intersect along an edge, which is a copy of the open interval. The three translates that intersect Λ along its three edges are $\sigma_1\Lambda$, $\sigma_X\Lambda$, and $\sigma_2\Lambda$. Consequently, the three translates that intersect $\beta\Lambda$ are $\beta\sigma_1\Lambda$, $\beta\sigma_X\Lambda$, and $\beta\sigma_2\Lambda$. We can encode the translates and their intersections in a graph (called the *exchange graph* in [2]). Its vertices are left γ-cosets in $\mathrm{PSL}_2(\mathbb{Z})$, and a coset $\beta\langle\gamma\rangle$ is connected by an edge with $\beta\sigma_1\langle\gamma\rangle$, $\beta\sigma_X\langle\gamma\rangle$, and $\beta\sigma_2\langle\gamma\rangle$ (see Figure 5).

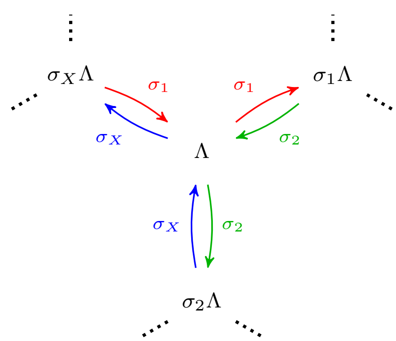


FIGURE 5. Exchange graph showing the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the set of standard stability conditions $\Lambda \subset \mathrm{Stab}(\mathcal{C})/\mathbb{C}$.

Let

$$\overline{\Delta} = \left(\mathbb{R}^3_{\geq 0} \setminus 0\right) / \,\mathbb{R}_{>0}\,.$$

Then $\overline{\Delta}$ is homeomorphic to a closed planar triangle. An immediate consequence of the linearity (from Proposition 5.2) is the following.

Proposition 5.3 (Closure of Λ). *The closure of Λ in $\mathbb{P}^{\mathbf{S}}$ is*

$$\overline{\Lambda} = \Lambda \cup \left\{\overline{\hom}(P_1), \overline{\hom}(P_2), \overline{\hom}(X)\right\}.$$

This closure is homeomorphic to $\overline{\Delta}$.

Proof. Identify Λ with Δ using the Gromov coordinates. From Proposition 5.2, we see that the map $m\colon \Delta \to \mathbb{P}^{\mathbf{S}}$ extends to a continuous map $\overline{\Delta} \to \mathbb{P}^{\mathbf{S}}$. Its image is closed, and equals the union of Λ and the three additional points described above. The map from $\overline{\Delta}$ is injective, and hence an isomorphism onto its image. □

Remark 5.4. Proposition 5.3 shows how the $\overline{\hom}$ functions appear in the closure of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$. There is another way in which $\overline{\hom}(x)$, for a spherical x, can be obtained as a limit of mass functions of stability conditions, which holds more generally. Let τ be any stability condition. Then the limit of the mass functions of $\sigma_x^n\tau$ in $\mathbb{P}^{\mathbf{S}}$, as $n \to \infty$ is the function $\overline{\hom}(x)$. Since we will not use this fact, we omit the proof. This is an analogue of the fact in Teichmüller theory that in the Thurston compactification, one approaches the PMF given by a simple closed curve γ by repeatedly applying Dehn twists in γ.

It is convenient to have a measure of closeness for two elements of a projective space. Given two non-zero vectors in $\mathbb{R}^n$ for some positive integer n, denote by $\sphericalangle(v, w)$ the (acute) angle between them. By a slight abuse of notation, we use $\sphericalangle(v, w)$ also for two points $v, w \in \mathbb{P}^{n-1}$; this is just the angle between any two representatives in $\mathbb{R}^n$.

Given a finite subset $T \subset \mathbf{S}$, let $h_T\colon \mathbf{S} \to \mathbb{P}^T$ be the composition of $h\colon \mathbf{S} \to \mathbb{P}^{\mathbf{S}}$ and the projection onto the T-coordinates.

Proposition 5.5 (Group action contracts). *Fix two elements $A, B \in \mathbf{S}$ and a finite subset $T \subset \mathbf{S}$. Given an $\epsilon > 0$, for all but finitely many elements L of $\mathrm{PSL}_2(\mathbb{Z})$, we have*

$$\sphericalangle(h_T(LA), h_T(LB)) < \epsilon.$$

Proof. In the identification $S = \mathbb{P}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{R})$, let A and B be represented by two integer vectors a and b in $\mathbb{Z}^2$. The map $h_T\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^T$ is continuous, and hence uniformly continuous. Therefore, it suffices to check that for all but finitely many L, the angle between La and Lb in $\mathbb{R}^2$ is small. To see this, observe that we have

$$\sin\left(\sphericalangle(La, Lb)\right) = \frac{|\det L| \cdot |\det(a \mid b)|}{|La| \cdot |Lb|} = \frac{|\det(a \mid b)|}{|La| \cdot |Lb|}.$$

Since L is required to have integer entries, the quantity $|L|\cdot|Lb|$ is large—greater than $\epsilon|\det(a \mid b)|$—for all but finitely many L. The claim follows. □

Proposition 5.6 (Closure of M). *The sets M and P are disjoint and their union is the closure of M.*

Proof. The mass of every object in a stability condition is positive. On the other hand, $\overline{\hom}(x, x) = 0$, by definition. Therefore, M and P are disjoint.

By Proposition 5.3, we know that P is contained in the closure of M. We now prove that $M \cup P$ is closed. Let τ_n be a sequence in $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ whose images $m(\tau_n)$ in M approach a limit $z \in \mathbb{P}^{\mathbf{S}}$. We must show that z lies in $M \cup P$.

Write $\tau_n = \beta_n\tau_n'$, where β_n is a braid and τ_n' is a standard stability condition. Let $\overline{\beta_n}$ be the image of β_n in $\mathrm{PSL}_2(\mathbb{Z})$. We have two possibilities: the set $\{\overline{\beta}_n\}$ is finite or infinite.

If $\{\overline{\beta}_n\}$ is finite, then the sequence $\{\tau_n\}$ is contained in the union of finitely many translates of the set of standard stability conditions Λ. By Proposition 5.3, the closure of Λ is contained in $M \cup P$. Hence, the closure of the union of finitely many translates of Λ is also contained in $M \cup P$. So the limit z lies in $M \cup P$.

Suppose $\{\overline{\beta}_n\}$ is infinite. Set $A_n = \beta_n(P_1)$, $B_n = \beta_n(P_2)$, and $C_n = \beta_n(X)$. Since P is compact, we may assume, after passing to a subsequence if necessary, that $h(A_n)$, $h(B_n)$, and $h(C_n)$ have limits in P, say A, B, and C.

We first observe that $A = B = C$. To see this, let $T \subset \mathbf{S}$ be an arbitrary finite set, and use the subscript T to denote projections to $\mathbb{P}^T$. It suffices to show that $A_T = B_T = C_T$. Since $\{\overline{\beta}_n\}$ is infinite, by Proposition 5.5, we note that the angles between $h_T(A_n)$, $h_T(B_n)$, and $h_T(C_n)$ approach 0 as n approaches ∞. Therefore, the three sequences have the same limits. But these limits are A_T, B_T, and C_T.

Next, we claim that $z = h(A) = h(B) = h(C)$. Indeed, by linearity (Proposition 5.2), we know that there exist non-negative real numbers x_n, y_n, z_n such that

$$m_T(\tau_n) = x_n h_T(A_n) + y_n h_T(B_n) + z_n h_T(C_n).$$

Taking the limit as $n \to \infty$ in projective space yields

$$z = h(A) = h(B) = h(C).$$

In particular, z lies in $M \cup P$. The proof is thus complete. □

6. Homeomorphism to a disk

We take up the final part of the main theorem, namely that $(\overline{M}, P)$ is a manifold with boundary homeomorphic to the unit disk. We explicitly construct a homeomorphism from $\overline{M}$ to the disk, using the unprojectivised, but suitably normalised, mass and hom functions.

Fix

$$T = \{P_1, P_2, X = P_2 \to P_1\}.$$

Define a map

$$\mu\colon \operatorname{Stab}(\mathcal{C})/i\,\mathbb{R} \to \mathbb{R}^T = \mathbb{R}^3$$

by

$$\mu\colon \tau \mapsto (m_\tau(P_1), m_\tau(P_2), m_\tau(X)).$$

Similarly, define

$$\eta\colon \mathbf{S} \to \mathbb{R}^T = \mathbb{R}^3$$

by

$$\eta(s) = (\overline{\hom}(s, P_1), \overline{\hom}(s, P_2), \overline{\hom}(s, X)).$$

Thinking of $\mathbf{S}$ as $\mathbb{P}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{R})$, it is easy to see that η extends to a continuous map

$$\eta\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{R}^3\,.$$

Indeed, using Corollary 2.10, we get

$$\eta\colon [a : c] \mapsto (|c|, |a|, |a + c|).$$

Let τ be a stability condition with (semi)-stable objects A, B, and C of class $[P_1]$, $[P_2]$, and $[X]$ in the Grothendieck group. Let x, y, z be the Gromov coordinates of τ, namely the non-negative real numbers such that

$$m_\tau(A) = y + z, \quad m_\tau(B) = z + x, \quad m_\tau(C) = x + y.$$

Denote by $|-|$ the standard Euclidean norm in $\mathbb{R}^3$. We say that τ is *normalised* if

$$x|\eta(A)| + y|\eta(B)| + z|\eta(C)| = 1. \tag{5}$$

Remark 6.1. We point out one subtle aspect of the definition. If τ is degenerate—that is, it has strictly semi-stable objects—then one of the A, B, or C is not uniquely determined, even up to shift. In this case, however, the corresponding Gromov coordinate is 0, and hence (5) is well-posed.

The normalisation yields a continuous section of

$$\operatorname{Stab}(\mathcal{C})/i\,\mathbb{R} \to \operatorname{Stab}(\mathcal{C})/\mathbb{C}.$$

That is, for every stability condition τ up to rotation and scaling, there is a unique normalised stability condition τ^ν up to rotation, and furthermore the map $\tau \to \tau^\nu$ is continuous.

We now identify $\operatorname{Stab}(\mathcal{C})/\mathbb{C}$ with its image M in $\mathbb{P}^{\mathbf{S}}$ and $\mathbb{P}^1(\mathbb{R})$ with its image P in $\mathbb{P}^{\mathbf{S}}$. Define

$$\pi\colon \overline{M} = M \cup P \to \mathbb{R}^3 \tag{6}$$

as follows. For $\tau \in M$, set

$$\pi(\tau) = \mu(\tau^\nu), \tag{7}$$

and for $s \in P$, set

$$\pi(s) = \eta(s)/|\eta(s)|.$$

By construction, $\pi(a:c)$ is the unit vector in the direction of

$$\eta(a:c) = (|a|, |c|, |a+c|).$$

That is, π maps P to the unit sphere in $\mathbb{R}^3$. It is easy to see that $\pi|_P$ is injective, and hence a homeomorphism onto its image. The image consists of three circular arcs, one for each pair of end-points from the three points

$$\frac{1}{\sqrt{2}}(0,1,1), \quad \frac{1}{\sqrt{2}}(1,0,1), \text{ and } \frac{1}{\sqrt{2}}(1,1,0).$$

The arc joining each pair is a geodesic arc on the unit sphere; that is, the plane it spans passes through the origin.

Equation 5 and the triangle inequality imply that for every stability condition τ, we have $|\pi(\tau)| \leq 1$. In fact, for every stability condition, at least two of the Gromov coordinates are non-zero. Therefore, we actually have a strict inequality $|\pi(\tau)| < 1$.

To get a better understanding of π, let us study it on the translates of the fundamental domain Λ. Let β be a braid. Consider the translate $\beta\overline{\Lambda}$. Set

$$A = \beta(P_1), \quad B = \beta(P_2), \quad C = \beta(X).$$

These are the (semi)-stable objects for the stability conditions in $\beta\Lambda$. Recall that $\beta\overline{\Lambda}$ is homeomorphic to the projectivised octant $\overline{\Delta} = \left(\mathbb{R}^3_{\geq 0} \setminus 0\right)/\mathbb{R}_{>0}$; the homeomorphism is given by the Gromov coordinates. The normalisation condition (5) is a linear condition on the Gromov coordinates. It cuts out an affine hyperplane slice of the octant, and gives a section of the projectivisation map. Since π is linear in the Gromov coordinates for normalised stability conditions, it maps $\beta\overline{\Lambda}$ linearly, and homeomorphically, onto the triangle in $\mathbb{R}^3$ with vertices $\pi(A)$, $\pi(B)$, and $\pi(C)$.

Let $\Phi \subset \mathbb{R}^3$ be the union of the triangle

$$\{(x,y,z) \mid x+y+z = \sqrt{2},\ x+y-z \geq 0,\ y+z-x \geq 0,\ z+x-y \geq 0\},$$

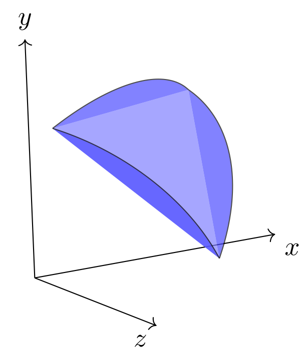


FIGURE 6. The homeomorphic image $\Phi \subset \mathbb{R}^3$ of the compactified stability manifold.

and the three circular segments, each bounded by an edge of the triangle above and the arc in the image of $\pi(P)$ with the same end-points as the edge (see Figure 6).

Observe that π maps $\overline{\Lambda}$, the triangle formed by the standard stablity conditions, to the central triangle of Φ. On the other hand, the translates of $\overline{\Lambda}$ on the three sides of the identity on the exchange graph (Figure 5) are mapped to the three circular segments. Indeed, the segment to which π sends a non-standard stability condition τ depends on how the triangle inequality degenerates among the τ-masses of P_1, P_2, and X, and this in turn, is determined by where τ lies on the exchange graph.

Proposition 6.2. *The map $\pi\colon \overline{M} \to \Phi$ is a homeomorphism, where $\overline{M}$ is given the subspace topology from $\mathbb{P}^{\mathbf{S}}$.*

Proof. Let us first show that π is continuous. Since π is linear on the translates of $\overline{\Lambda}$, and these translates cover M, it follows that π is continuous on M. The restriction of π to P is given by

$$\pi\colon [a:c] \mapsto \frac{1}{\sqrt{a^2+c^2+(a+c)^2}}\left(|a|,|c|,|a+c|\right),$$

which is continuous. We must show that π is continuous on $\overline{M}$ at a point $p \in P$.

Let τ_n be a sequence of normalised stability conditions converging to $p \in P$. We already know that $m_T(\tau_n)$ converges to $h_T(p)$ in the projective space $\mathbb{P}^T$. Therefore, it suffices to show that $\pi(\tau_n)$ approaches a vector of norm 1 in $\mathbb{R}^3$.

Let $\epsilon > 0$ be given. Write $\tau_n = \beta_n \tau_n'$ for some braid β_n and standard stability condition τ_n. Denote by $\overline{\beta}_n$ the image of β_n in $\mathrm{PSL}_2(\mathbb{Z})$. If the set $\{\overline{\beta}_n\}$ is finite, then the sequence τ_n lies in finitely many translates of Λ. Without loss of generality, we may assume that it lies in one translate, say $\beta\Lambda$. Recall that $\pi\colon \overline{\beta}\Lambda \to \mathbb{R}^3$ is linear in the Gromov coordinates and hence continuous. Therefore, $\pi(\tau_n) \to \pi(p)$ as $n \to \infty$.

The harder case is when the set $\{\overline{\beta}_n\}$ is infinite. Set $A_n = \beta_n(P_1)$, $B_n = \beta_n(P_2)$, $C_n = \beta_n(X)$, and let x_n, y_n, z_n be the Gromov coordinates. But in this case, we know by Proposition 5.5 that the angle between $\eta(A_n)$, $\eta(B_n)$, and $\eta(C_n)$ approaches 0 as n approaches ∞. Therefore, the difference between

$$|x_n\eta(A_n) + y_n\eta(B_n) + z_n\eta(C_n)| \text{ and } x_n|\eta(A_n)| + y_n|\eta(B_n)| + z_n|\eta(C_n)|$$

approaches 0 as n approaches ∞. Since τ_n is normalised, the right-hand quantity is 1, and the left-hand quantity is $|\pi(\tau_n)|$.

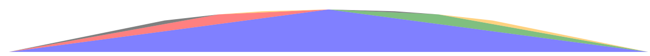


FIGURE 7. The image under π of translates of Λ give the Farey triangulation of a circular segment.

We have now proved that $\pi\colon \overline{M} \to \Phi$ is continuous. Since $\overline{M}$ is compact, π is a homeomorphism once we know that it is a bijection. We know that the map

$$\pi\colon P \to \partial\Phi = \Phi \cap \{v \mid |v| = 1\}$$

is a bijection and π maps M to

$$\Phi^\circ = \Phi \cap \{v \mid |v| < 1\}.$$

Recall that M is the union of the translates of the fundamental domains $\beta\Lambda$, and each fundamental domain is homeomorphic to a clipped triangle (a planar triangle minus the vertices). The map π maps each translate bijectively to a (clipped) triangle in Φ°. To check that π is an injection, we must check that the clipped triangles $\pi(\beta\Lambda)$ have disjoint interiors, and two of them, say $\pi(\beta\Lambda)$ and $\pi(\beta'\Lambda)$, intersect along an edge if and only if β and β' are adjacent in the exchange graph. To check that π is a surjection, we must check that the union $\bigcup_\beta \pi(\beta\Lambda)$ is Φ°.

Take $\beta = \mathrm{id}$. Then $\pi(\Lambda)$ is the central triangle of Φ°. The only other triangles that intersect this triangle are $\pi(\sigma_1\Lambda)$, $\pi(\sigma_2\Lambda)$, and $\pi(\sigma_X\Lambda)$, and the intersections are along the three edges, as required.

The exchange graph divides the non-trivial translates of Λ into three connected components, and the translates in each of the three components map to the three distinct circular segments in Φ°. So it suffices to restrict our attention to one component and the corresponding circular segment. Since γ permutes the components, it suffices to look at only one of them.

Let us consider the component containing $\sigma_1\Lambda$. The translates in this component are $A\Lambda$ where $A \in \mathrm{PSL}_2(\mathbb{Z})$ is a matrix that has an admissible cyclic writing that starts with σ_1. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\pi(A\Lambda)$ is the clipped triangle with vertices

$$\pi([a : c]), \quad \pi([b : d]), \quad \pi([a - b : c - d]). \tag{8}$$

Since we are considering the translates in the $\sigma_1\Lambda$ component, the points $[a : c]$, $[b : d]$, and $[a - b : c - d]$ lie in the arc $[P_1, P_2]$ of $\mathbb{P}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{R})$. (So, under the bijection $\mathbb{P}^1(\mathbb{Z}) = \mathbb{Q} \cup \{\infty\}$ given by $[a : c] \to a/c$, they correspond to $\mathbb{Q}_{\geq 0} \cup \{\infty\}$). By construction, the points $\pi([a : c])$ and $\pi([b : d])$ form an edge of a clipped triangle if and only if $|ad - bc| = 1$. Thus, the triangles (8) form the Farey tessellation [1, Chapter 8] of the circular segment, which has the intersection properties as dictated by the exchange graph (see Figure 7). $\square$

7. The $\widehat{A}_1$ category

Let $\mathcal{C}$ be the 2-Calabi–Yau category associated to the $\widehat{A}_1$ quiver. It is generated by two objects P_0 and P_1, satisfying

$$\mathrm{Hom}(P_i, P_i[n]) = \begin{cases} \mathbf{k} & \text{if } n = 0, 2, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathrm{Hom}(P_i, P_j[n]) = \begin{cases} \mathbf{k}^2 & \text{if } n = 1, \\ 0 & \text{otherwise, for } i \neq j. \end{cases}$$

Let T be the set of spherical objects of the form $nP_1 \to (n \pm 1)P_0$. There is a bijection $\mathbb{Z} \to T$ given by

$$n \mapsto |n|P_1 \to |n-1|P_0.$$

Let P_n denote the object $|n|P_1 \to |n-1|P_0$.

The group $\mathrm{Aut}(\mathcal{C})$ contains the spherical twists $\sigma_0 = \sigma_{P_0}$ and $\sigma_1 = \sigma_{P_1}$. Let γ be the element $\sigma_1\sigma_0$. It is easy to check that γ preserves T, and in fact acts by translation by 2 up to shift:

$$\gamma(P_k) = \begin{cases} P_{k+2}, & k \neq 0, -1, \\ P_{k+2}[-1], & k = 0, -1. \end{cases}$$

Let G be the group generated by σ_0 and σ_1. Regard $\mathbb{Z}$ as the subset of $\mathbb{P}^1(\mathbb{Z})$ consisting of points of the form $[k:1]$. We can write a homomorphism $G \to \mathrm{PSL}_2(\mathbb{Z})$ compatible with the action of γ on $\mathbb{Z}$, as follows:

$$\sigma_0 \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \sigma_1 \mapsto \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}.$$

Indeed, under the homomorphism above, γ is the matrix of translation by 2:

$$\gamma \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathbf{S}$ be the G-orbit of $\{P_0, P_1\}$. By transporting the G-action on $T \subset \mathbf{S}$ to the $\mathrm{PSL}_2(\mathbb{Z})$-action on $\mathbb{P}^1(\mathbb{Z})$, we obtain an injection $i \colon \mathbf{S} \hookrightarrow \mathbb{P}^1(\mathbb{Z})$.

7.1. **Harder–Narasimhan filtrations and the Harder–Narasimhan automaton.** Let τ be a stability condition on $\mathcal{C}$ such that all objects of T are τ-semistable. This implies that up to rotation, we can arrange the phases ϕ so that

$$0 = \phi(P_0) \leq \phi(P_1) \leq 1.$$

We say that such a stability condition is *standard*. If moreover we have

$$0 = \phi(P_0) < \phi(P_1) < 1,$$

we say that τ is *strictly standard*. If τ is standard but not strictly standard, we say that τ is *weakly standard*. In a strictly standard stability condition τ, the central charges of the objects of T are arranged as shown in Figure 8. The dotted line is the direction of the sum of the central charges of P_0 and P_1. For convenience set P_∞ to be any one of the indecomposable extensions of P_0 by P_1. Then the dotted line is also the direction of the central charge of P_∞. We now construct a Harder–Narasimhan automaton for the $\widehat{A}_1$ category $\mathcal{C}$ analogous to Figure 3. Once again we need a collection of geodesic equivalences, and these will be certain twists in the objects P_n of T. We

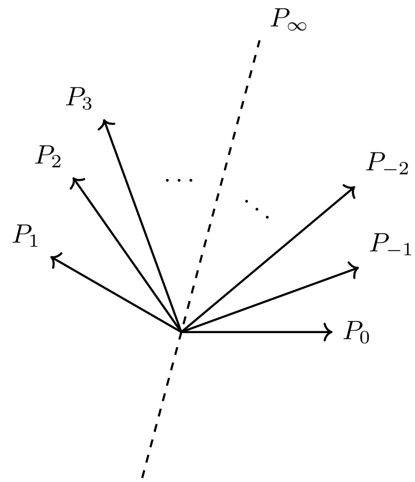


FIGURE 8. The central charges of the objects of T in a strictly standard stability condition.

begin with $\gamma = \sigma_1\sigma_0$, which acts on the objects of T by an infinite-order cyclic rotation by two steps, lowering the phases. More precisely, recall that

$$\gamma\colon P_n \mapsto P_{n+2} \text{ for } n \neq 0, -1, \infty,$$
$$\gamma\colon P_n \mapsto P_{n+2}[-1] \text{ for } n = 0, -1,$$
$$\gamma\colon P_\infty \mapsto P_\infty.$$

For each $k \in \mathbb{Z}$, let σ_k denote the spherical twist in the object P_k. We have

$$\sigma_{2k} = \gamma^k\sigma_0\gamma^{-k} \text{ and } \sigma_{2k+1} = \gamma^k\sigma_1\gamma^{-k}.$$

Let V_k denote the vertex consisting of objects whose Harder–Narasimhan pieces are among $\{P_k, P_{k+1}\}$. We draw an edge labelled σ_j from V_k to V_l if and only if every object supported at V_k is sent to an object supported at V_l under the action of σ_j. The following proposition proves that σ_j is edge from V_k to V_j unless $j = k+1$, and that each edge satisfies the geodesic property. The resulting automaton is shown in Figure 9.

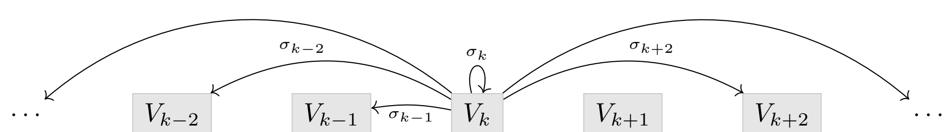


FIGURE 9. The Harder–Narasimhan automaton for $\widehat{A}_1$, showing the outgoing edges from V_k.

Proposition 7.1. *Let x be an object of $\mathcal{C}$ supported at V_k for some $k \in \mathbb{Z}$.*

(1) If $j \neq k+1$, the object $\sigma_j(x)$ is supported at V_j, and σ_j is a geodesic autoequivalence for x. The object $\sigma_{k+1}(x)$ is supported either at V_k or at V_{k+1}.

(2) If $j \neq k$, the object $\sigma_j^{-1}(x)$ is supported at V_{j-1}, and σ_j^{-1} is a geodesic autoequivalence for x. The object $\sigma_k^{-1}(x)$ is supported either at V_k or at V_{k-1}.

Proof. We first prove the proposition for $k = 0$. Note that for $j \geq 2$, we have

$$\begin{aligned} \sigma_j P_0 &= P_j^{\oplus(j-1)}[-1] \to P_{j+1}^{\oplus j}, \text{ and} \\ \sigma_j P_1 &= P_j^{\oplus(j-2)} \to P_{j+1}^{\oplus(j-1)}[1]. \end{aligned} \tag{9}$$

Similarly, for $j \leq 0$, we have

$$\begin{aligned} \sigma_j P_0 &= P_j^{\oplus(1-j)}[-1] \to P_{j+1}^{\oplus -j}, \text{ and} \\ \sigma_j P_1 &= P_j^{\oplus(2-j)} \to P_{j+1}^{\oplus(1-j)}[1]. \end{aligned} \tag{10}$$

Suppose that $j \neq 1$. The calculation above shows that $\sigma_j(P_0)$ and $\sigma_j(P_1)$ are both supported in V_j. By an argument similar to the one in Proposition 2.2, σ_j is geodesic for objects supported in V_0. Consequently if x is any object supported in V_0, then $\sigma_j(x)$ is supported in V_j.

Now we prove the proposition for an arbitrary k. For any integer $n \in \mathbb{Z}$, set

$$q(n) = \lfloor n/2 \rfloor, \quad r(n) = n - 2q(n).$$

Note that we have

$$P_k = \gamma^{q(k)} P_{r(k)}.$$

Using our calculations above, we see that

$$\sigma_j P_k = \gamma^{q(k)} \sigma_{j-2q(k)} P_{r(k)}.$$

Now we can use equations (9) and (10) to compute explicit formulas for the action of σ_j on P_k and P_{k+1}. After some casework based on the parity of k and the value of $j-k$, we compute that unless $j = k+1$, we have the following:

$$\begin{aligned} \sigma_j P_k &= P_j^{\oplus|j-k-1|}[-1] \to P_{j+1}^{\oplus|j-k|}, \\ \sigma_j P_{k+1} &= P_j^{\oplus|j-k-2|} \to P_{j+1}^{\oplus|j-k-1|}. \end{aligned} \tag{11}$$

As before, these formulas show that unless $j = k+1$, the autoequivalence σ_j is geodesic for objects supported in V_k. Also if x is supported in V_k, then $\sigma_j(x)$ is supported in V_j. The proof is thus complete. □

We immediately deduce the following analogue of Proposition 2.3.

Proposition 7.2. *Let $e\colon V_i \xrightarrow{\sigma} V_j$ be an edge of the Harder–Narasimhan automaton corresponding to an autoequivalence σ. There exists a 2×2 integer matrix M_e with the following property. If x is an object supported at V_i with multiplicity vector $(a, b)^t$, the multiplicity vector of $\sigma(x)$ at V_j is $M_e \cdot (a, b)^t$.*

Proof. From the proof of Proposition 7.1, precisely the equations (11), we see that the matrix for the edge σ_j from V_k to V_j is given by

$$M_{\sigma_j\colon V_k \to V_j} = \begin{pmatrix} |j-k-1| & |j-k-2| \\ |j-k| & |j-k-1| \end{pmatrix}.$$

Since γ merely shifts the stable objects and preserves the phase ordering, it is geodesic for any object. The Harder–Narasimhan transformation matrix for γ is the identity:

$$M_{\gamma\colon V_k \to V_{k+2}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

Proposition 7.3. *Every $g \in G$ has an expression of the following form:*

$$g = \sigma_{k_1}\sigma_{k_2}\dots\sigma_{k_n}\gamma^l$$

for integers $k_1, k_2, \dots, k_n, l$ such that $k_i - k_{i+1} \neq 1$ for each i.

Proof. First write g as any expression in the generators $\{\sigma_0^{\pm}, \sigma_1^{\pm}\}$. For any $k \in \mathbb{Z}$, we have

$$\sigma_k^{-1} = \sigma_{k-1}\gamma^{-1}, \quad \gamma^{-1}\sigma_k = \sigma_{k-2}\gamma^{-1}, \quad \gamma = \sigma_k\sigma_{k-1}.$$

Using these relations, we first rewrite g solely in terms of the elements σ_k for $k \in \mathbb{Z}$ and $\gamma^{\pm}$. Then we successively replace any instances of $\sigma_k\sigma_{k-1}$ as γ, and commute all powers of γ to the right. This process terminates, resulting in an expression of the desired form. □

Recall that we have an embedding i of $\mathbf{S}$ into $\mathbb{P}^1(\mathbb{Z})$, under which P_k maps to $k = [k : 1] \in \mathbb{P}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{R})$. For any $k \in \mathbb{Z}$, let $[k, k+1]$ be the closed interval in $\mathbb{R}$, which we identify with a subset of $\mathbb{P}^1(\mathbb{R})$. The union of all of these closed intervals is precisely $\mathbb{P}^1(\mathbb{R}) \setminus \{\infty\}$.

Proposition 7.4. *An object $x \in \mathbf{S}$ is supported at V_k if and only if $i(x) \in [k, k+1]$.*

Proof. The statement is clearly true for the objects P_k. For the general case, it suffices to show that if we replace each vertex in the Harder–Narasimhan automaton (Figure 9) by the corresponding arc, and the autoequivalences by the corresponding transformations in $\mathrm{PSL}_2(\mathbb{Z})$, then the transformation takes the source arc to the target arc. This is easily verified. □

Proposition 7.5. *Let $x \in \mathbf{S}$ such that $i(x) = [a : c] \in \mathbb{P}^1(\mathbb{Z})$, with a and c relatively prime integers. Then the minimal complex of projective modules representing x has exactly $|c - a|$ occurrences of P_0 and $|a|$ occurrences of P_1.*

Proof. The assertion is clear for the objects P_k for each $k \in \mathbb{Z}$. Write $x = g(P)$ for some $g \in G$ and $P \in \{P_0, P_1\}$. Express g as an admissible cyclic word as in Proposition 7.3:

$$g = \sigma_{k_1}\cdots\sigma_{k_n}\gamma^l.$$

We induct on n: if $n = 0$, then $x = P_k$ for some k and we are done.

Let us assume the statement for $y = \sigma_{k_2}\cdots\sigma_{k_n}\gamma^l$, and prove it for $x = \sigma_{k_1}y$. For an object P, denote by $v(P)$ the vector whose coordinates are the number of occurrences of P_0 and the number of occurrences of P_1 in the minimal complex of P. For example, we have $v(P_k) = (|k-1|, |k|)^t$. Now, if P is supported at the vertex V_k and $(\alpha, \beta)^t$ is its Harder–Narasimhan multiplicity vector, then

$$v(P) = \begin{pmatrix} |k-1| & |k| \\ |k| & |k+1| \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

For simplicity, assume that neither k_1 nor k_2 is zero. By Proposition 7.2, we have

$$\begin{aligned}
v(x) &= \begin{pmatrix} |k_1-1| & |k_1| \\ |k_1| & |k_1+1| \end{pmatrix} M_{\sigma_{k_1}\colon V_{k_2}\to V_{k_1}} \begin{pmatrix} |k_2-1| & |k_2| \\ |k_2| & |k_2+1| \end{pmatrix}^{-1} \\
&= \pm \begin{pmatrix} k_1-1 & k_1 \\ k_1 & k_1+1 \end{pmatrix} \begin{pmatrix} k_1-k_2-1 & k_1-k_2-2 \\ k_1-k_2 & k_1-k_2-1 \end{pmatrix} \begin{pmatrix} k_2+1 & -k_2 \\ -k_2 & k_2-1 \end{pmatrix} v(y) \\
&= \pm \begin{pmatrix} 2k_1^2-2k_1+1 & -2k_1^2+4k_1-2 \\ 2k_1^2 & -2k_1^2+2k_1+1 \end{pmatrix} v(y) \\
&= \pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+2k_1 & -2k_1^2 \\ 2 & 1-2k_1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} v(y).
\end{aligned}$$

In the calculation above, we used the assumption $k_1 \neq 0$ and $k_2 \neq 0$ in eliminating the absolute value signs from the first and the third matrices on the first line. By the inductive assumption, we have

$$i(y) = \pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} v(y),$$

and the matrix $\begin{pmatrix} 1+2k_1 & -2k_1^2 \\ 2 & 1-2k_1 \end{pmatrix}$ is the image in $\mathrm{PSL}_2(\mathbb{Z})$ of σ_k. Hence, we have

$$\begin{aligned}
i(x) &= \pm \begin{pmatrix} 1+2k_1 & -2k_1^2 \\ 2 & 1-2k_1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} v(y) \\
&= (a, c).
\end{aligned}$$

and therefore

$$\begin{aligned}
v(x) &= \pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} \\
&= \begin{pmatrix} |a-c| \\ |a| \end{pmatrix},
\end{aligned}$$

as desired. If $k_1 = 0$ or $k_2 = 0$, the calculation above needs small modifications to make sure the signs are correct, but we omit these details. The inductive step is then complete. □

Proposition 7.6. *Let $x \in \mathbf{S}$ such that any minimal complex of x has exactly n_0 occurrences of P_0 and n_1 occurrences of P_1. Then $\overline{\hom}(x, P_1) = 2n_0$ and $\overline{\hom}(x, P_0) = 2n_1$.*

Proof. We prove the assertion for $\overline{\hom}(x, P_1)$. The proof for $\overline{\hom}(x, P_0)$ is similar. If $x = P_1$, the statement is evident. Fix a strictly standard stability condition. Suppose that x is supported at V_k for $k \neq 0, 1$. Consider the Harder–Narasimhan filtration of x:

$$0 = x_0 \to x_1 \to \cdots \to x_n = x,$$

with Harder–Narasimhan factors P_k and P_{k+1}, up to shift. By applying $\mathrm{Hom}(P_1, -)$, we obtain the following filtration in the bounded derived category of graded vector spaces:

$$0 = \mathrm{Hom}(P_1, x_0) \to \cdots \to \mathrm{Hom}(P_1, x_n) = \mathrm{Hom}(P_1, x).$$

The sub-quotients of this filtration are obtained by applying $\mathrm{Hom}(P_1, -)$ to the sub-quotients of the original filtration. the boundary maps between these sub-quotients are induced by the boundary maps between the sub-quotients in the filtration of x. The filtration of x has three possible non-zero boundary maps:

$$P_k \to P_k[2], \quad P_{k+1} \to P_{k+1}[2], \quad P_k \to P_{k+1}[1].$$

Every map from P_1 to P_k has positive degree for $k \neq 1$. The first two boundary maps have degree two, so they are obviously killed by $\mathrm{Hom}(P_1, -)$ for degree reasons. The last boundary map has degree one, but factors as

$$P_k \to \bigoplus_{i=1}^{|k|} P_1 \to \bigoplus_{i=1}^{|k|} P_0 \to P_{k+1}[1].$$

Any map from P_1 to P_k on the minimal complexes is either a combination of degree two loop map to the $|k|$ copies of P_1, or a combination of degree-one maps to the $|k-1|$ copies of P_0. Composing either of these with the factoring above yields zero, and therefore any map from P_k to $P_{k+1}[1]$ is killed by $\mathrm{Hom}(P_1, -)$.

We conclude that $\mathrm{Hom}(P_1, x)$ is a direct sum of $\mathrm{Hom}(P_1, y_i)$ where y_i are the Harder–Narasimhan factors. Since we are working with a strictly standard stability condition, the objects P_0 and P_1 are precisely the simple objects of the standard heart. The Harder–Narasimhan filtration is a refinement of the cohomology filtration of x, and on each cohomology factor it is a coarsening of a Jordan–Hölder filtration of x. Therefore the Hardar–Narasimhan filtration is realised in a minimal complex of x, and counting the constituent pieces of the Harder–Narasimhan factors is equivalent to counting the constituents of the minimal complex of x.

Recall that when $k \neq 1$, we have $\overline{\mathrm{hom}}(P_1, P_k) = 2|k-1|$. So every Harder–Narasimhan factor that is a shift of P_k contributes $2|k-1|$ to the dimension, while every Harder–Narasimhan factor that is a shift of P_{k+1} contributes $2|k|$ to the total dimension. On the other hand, every shift of P_k contains exactly $|k-1|$ copies of P_0.

To prove the general case, it suffices to prove that for any object x (other than a twist of P_1), there exists an r for which $\sigma_1^r x$ is supported by a vertex V_k with $k \neq 0, 1$. Indeed, by Proposition 7.5, the objects x and $\sigma_1^r x$ have the same number of occurrences of P_0. Moreover both x and $\sigma_1^r x$ have the same value for $\overline{\mathrm{hom}}(P_1, -)$, because σ_1 is an autoequivalence that juts shifts P_1 in degree. Suppose that $x = \beta P_j$ for $j = 0$ or $j = 1$, where β has a cyclic expression

$$\beta = \sigma_{k_1} \sigma_{k_2} \dots \sigma_{k_n} \gamma^l.$$

We induct on the quantity $\ell(\beta) = n$. If $n = 0$, then $x = P_i$ for some i. Since $x \neq P_1$, it is already supported on some V_k with $k \neq 0, 1$, and we are done. Recall that x is supported on V_{k_1}. If $k_1 \neq 0, 1$, then we are done by the previous argument. If $k_1 = 1$, then $\ell(\sigma_1^{-1}\beta) < \ell(\beta)$. If $k_1 = 0$, then $\ell(\sigma_1 \beta) < \ell(\beta)$. To see this, note that

$$\sigma_1 \beta = \gamma \sigma_{k_2} \dots \sigma_{k_n} \gamma^l.$$

By reducing as described in Proposition 7.3, we see that $\ell(\sigma_1\beta)$ is smaller than $\ell(\beta)$. The proof is now complete. □

Proposition 7.7 (Degeneracy)**.** *For any stability condition τ and any $i \in \mathbb{Z}$, the three positive numbers*

$$m_\tau(P_{i-1}), \quad 2m_\tau(P_i), \quad m_\tau(P_{i+1})$$

form the sides of a triangle. If τ is a strictly standard stability condition, then none of these triangles are degenerate. That is, all three triangle inequalities are strict. If τ is not strictly standard, all of these triangles are degenerate. That is, in each triangle, one of the quantities is the sum of the other two.

Proof. We have distinguished triangles

$$P_1[-1] \to P_0 \oplus P_0 \to P_{-1} \xrightarrow{+},$$
$$P_2 \to P_1 \oplus P_1 \to P_0[1] \xrightarrow{+} .$$

Applying γ^i, we obtain the triangles

$$\begin{aligned} P_{2i+1}[-1] \to P_{2i} \oplus P_{2i} \to P_{2i-1} \xrightarrow{+}, \\ P_{2i+2} \to P_{2i+1} \oplus P_{2i+1} \to P_{2i}[1] \xrightarrow{+} . \end{aligned} \tag{12}$$

Using the triangle inequality [8, Proposition 3.3] on each of these triangles, we deduce that for each i, the numbers

$$m_\tau(P_{i-1}), \quad 2m_\tau(P_i), \quad m_\tau(P_{i+1})$$

form the sides of a triangle. If τ is strictly standard, it is clear that these triangles are non-degenerate.

Now suppose that τ is not strictly standard. If τ is standard, then we either have $\phi(P_0) = \phi(P_1)$ or $\phi(P_1) = 1$. In either of these cases, the central charges of all semistable objects lie on a single line through the origin, and any distinguished triangles formed by semistable objects are necessarily degenerate.

If τ is non-standard, write $\tau = \beta\tau'$ where τ' is standard. Note that $m_\tau(-) = m_{\tau'}(\beta^{-1}(-))$. Write β^{-1} in a modified admissible cyclic form as

$$\beta^{-1} = \gamma^l \sigma_{k_1} \cdots \sigma_{k_n}.$$

Note that $n \geq 1$ because τ is non-standard. We apply β^{-1} letter by letter to each P_i, keeping track of the masses along the way. The key observation is that the objects $\sigma_{k_n} P_i$ are all supported on the *same* vertex, namely V_{k_n}. Consequently the Harder–Narasimhan filtrations of these objects with respect to any standard stability condition only contain the objects P_{k_n} and P_{k_n+1}. Recall from (11) that

$$\sigma_{k_n} P_i = P_{k_n}^{\oplus|k_n-i-1|} + P_{k_n+1}^{\oplus|k_n-i|}.$$

Furthermore in each of the distinguished triangles (12), the Harder–Narasimhan multiplicity vector of one of the objects is the sum of the other two. Indeed, we have the following, where $[-]$ denotes the multiplicity vector:

$$\begin{aligned} [\sigma_{k_n}(P_i \oplus P_i)] &= [\sigma_{k_n}(P_{i-1})] + [\sigma_{k_n}(P_{i+1})] \quad \text{if } i \neq k_n, k_n - 1, \\ [\sigma_{k_n}(P_{i+1})] &= [\sigma_{k_n}(P_{i-1})] + [\sigma_{k_n}(P_i \oplus P_i)] \quad \text{if } i = k_n, \\ [\sigma_{k_n}(P_{i-1})] &= [\sigma_{k_n}(P_i \oplus P_i)] + [\sigma_{k_n}(P_{i+1})] \quad \text{if } i = k_n - 1, \end{aligned}$$

By Proposition 7.2, all further transformations preserve this linear relationship. The proof is now complete. ☐

Let $\Lambda \subset \mathrm{Stab}(\widehat{A}_1)/\mathbb{C} = \mathrm{Stab}(\mathcal{C})/\mathbb{C}$ be the subset consisting of standard stability conditions. Let Λ° be the subset of strictly standard stability conditions. We know that Λ is a closed subset of $\mathrm{Stab}(\widehat{A}_1)/\mathbb{C}$. Let $\mathrm{Stab}^\circ(\mathcal{C})/\mathbb{C}$ be the connected component of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ that contains Λ. By an argument similar to the proof of [7, Proposition 4.13], one can show that Λ tessellates $\mathrm{Stab}^\circ(\mathcal{C})/\mathbb{C}$ under the action of G.

Proposition 7.8. *The map* $m\colon \mathrm{Stab}^\circ(\mathcal{C})/\mathbb{C} \to \mathbb{P}^{\mathbf{S}}$ *is a homeomorphism onto its image.*

Proof. Say that a stability condition τ is semi-standard if any τ-semistable object, up to twist, is either P_i or $\sigma_0(P_i)$. If τ is semi-standard, then up to rotation, its heart is the standard heart. Note that if Π denotes the subset of $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ of semi-standard stability conditions, then $\Pi = \Lambda \cup \sigma_0\Lambda$. The interior of Π is the union of the Λ° and $\sigma_0\Lambda^\circ$, together with the set of standard stability conditions for which $0 = \phi(P_0) = \phi(P_1)$.

Let $E = \sigma_0(P_1)$, and let T be the following subset of $\mathbf{S}$:

$$T = \{P_{-1}, P_0, P_1, E\}.$$

Note that

$$\sigma_0(P_{-1}) = P_1, \quad \sigma_0(P_0) = P_0[-1], \quad \sigma_0(P_1) = E.$$

Note also that we have the following distinguished triangles:

$$\Delta_1 : P_0 \oplus P_0 \to P_{-1} \to P_1 \xrightarrow{+},$$

$$\Delta_2 = \sigma_0\Delta_1 : (P_0 \oplus P_0)[-1] \to P_1 \to E \xrightarrow{+1} .$$

Consider the map $m_T\colon \mathrm{Stab}(\mathcal{C})/\mathbb{C} \to \mathbb{P}^T$ obtained by composing m with the standard projection onto the T coordinates. Let $\Omega \subset \mathbb{P}^T$ be the subset consisting of points $[x : y : z : w]$ such that:

- $x, y, z, w \geq 0$;
- the triples $(x, 2y, z)$ and $(2y, z, w)$ satisfy the triangle inequalities; and
- $x = 2y + z$ or $w = 2y + z$.

Observe that Ω is a union of two closed triangles along a common edge. The common edge is precisely the points $[x : y : z : w]$ with $x = w = 2y + z$. The boundary of Ω is the union of four line segments that are the other two edges of the two triangles. On the boundary, both triangle inequalities degenerate into equalities. If τ is strictly standard, then $\sigma_0^{-1}\tau$ cannot be standard. By Proposition 7.7, the triangle Δ_1 degenerates for $\sigma_0^{-1}\tau$. Equivalently, the triangle Δ_2 degenerates for τ, and the following equation holds:

$$m_\tau(E) = 2m_\tau(P_0) + m_\tau(P_1).$$

We also see that the Δ_1 does not degenerate for τ. Similarly if $\sigma_0^{-1}\tau$ is strictly standard, then τ cannot be standard. By Proposition 7.7, the triangle Δ_1 degenerates for τ, and the following equation holds:

$$m_\tau(P_{-1}) = 2m_\tau(P_0) + m_\tau(P_1).$$

By applying σ_0, we also see that the triangle Δ_2 does not degenerate for τ. These calculations show that the image of Π° is contained in the interior of Ω.

Now suppose that neither τ nor $\sigma^{-1}\tau$ are strictly standard. Then by Proposition 7.7, Δ_1 degenerates for both τ and $\sigma_0^{-1}\tau$. Equivalently, both Δ_1 and Δ_2 degenerate for τ. Note that the degeneration of the triangle inequalities is such that m_T maps τ to the boundary of Ω.

We see that the preimage under m_T of Ω° is exactly Π°. The map $m_T\colon \Pi^\circ \to \Omega^\circ$ is clearly a homeomorphism. Since the translates of Π° under G cover $\mathrm{Stab}^\circ(\mathcal{C})/\mathbb{C}$, we conclude that m is a homeomorphism onto its image. □

7.2. **Gromov coordinates.**

Definition 7.9 (Standard Gromov coordinates)**.** Let τ be a standard stability condition. For each integer $i \in \mathbb{Z}$, let $x_i(\tau)$ be the rational number defined as follows:

$$x_i(\tau) = \frac{m_\tau(P_{i-1}) + m_\tau(P_{i+1}) - 2m_\tau(P_i)}{2}.$$

We call the numbers $(x_i(\tau))$ the *Gromov coordinates* of τ.

Using the triangle inequality [8, Proposition 3.3] on the triangles (12), we deduce that all Gromov coordinates of τ are non-negative. If τ is strictly standard, they are all positive.

Proposition 7.10 (Linearity). *Let τ be a standard stability condition with Gromov coordinates (x_i). For any object $X \in \mathbf{S}$, we have*

$$m_\tau(X) = \frac{1}{2}\sum_{i\in\mathbb{Z}} x_i \,\overline{\hom}(P_i, X).$$

Proof. Recall from Proposition 7.6 that for any object X, the quantities $\overline{\hom}(P_0, X)$ and $\overline{\hom}(P_1, X)$ are twice the number of instances of P_1 and P_0 respectively in the minimal complex of X. Recall that for any $i \in \mathbb{Z}$, we have

$$P_{2i} = \gamma^i P_0, \quad P_{2i+1} = \gamma^i P_1.$$

Therefore $\overline{\hom}(P_{2i}, X)$ and $\overline{\hom}(P_{2i+1}, X)$ are twice the number of instances of P_1 and P_0 respectively in the minimal complex of $\gamma^{-i}X$.

Since X is spherical, X is supported at some vertex V_k of the Harder–Narasimhan automaton. Suppose that the Harder–Narasimhan filtration of X consists of α copies of P_k and β copies of P_{k+1}. Then $\gamma^{-i}X$ is supported on V_{k-2i}, and its HN filtration contains α copies of P_{k-2i} and β copies of P_{k-2i+1}. In particular, for any $i \in \mathbb{Z}$, the number of instances of P_1 and P_0 in the minimal complex of $\gamma^{-i}X$ are exactly

$$(\alpha|k-2i| + \beta|k-2i+1|) \text{ and } (\alpha|k-2i-1| + \beta|k-2i|)$$

respectively. We deduce that for any $j \in \mathbb{Z}$, we have

$$\overline{\hom}(P_j, X) = 2\left(\alpha|k-j| + \beta|k-j+1|\right). \tag{13}$$

In the expression on the right hand side of the equality in the statement of the proposition, the coefficient of $m_\tau(P_j)$ is exactly

$$\frac{\overline{\hom}(P_{j-1}, X) + \overline{\hom}(P_{j+1}, X) - 2\,\overline{\hom}(P_j, X)}{4}.$$

Using (13), we compute this to be

$$\frac{\alpha\left(|k-j+1| + |k-j-1| - 2|k-j|\right) + \beta\left(|k-j+2| + |k-j| - 2|k-j+1|\right)}{2}.$$

The quantity above evaluates to zero unless $j = k$ or $j = k+1$. If $j = k$, it evaluates to α. If $j = k+1$, it evaluates to β.

Therefore we see that

$$\frac{1}{2}\sum_{i\in\mathbb{Z}} x_i \,\overline{\hom}(P_i, X) = \alpha \cdot m_\tau(P_k) + \beta \cdot m_\tau(P_{k+1}),$$

which is precisely the τ-mass of X. $\square$

Next we study the topology of the space Λ, and compute the closure of $m(\Lambda)$ in $\mathbb{P}^{\mathbf{S}}$. Let $\overline{\Delta}$ be a closed disk, which is the one-point compactification of the closed upper half plane. Regard the upper half plane as a subset of $\overline{\Delta}$.

Proposition 7.11. *Let Z be a central charge function such that $Z(P_0) = 0$ and $Z(P_1)$ lies in the closed upper half plane. Then Z is the central charge of a unique standard stability condition if and only if $Z(P_1)$ avoids the following set:*

$$L = \{-1, \infty\} \cup \left\{ \frac{1-n}{n} \mid n \in \mathbb{Z} \setminus \{0\} \right\}.$$

Consequently, Λ can be identified with the set $\Delta = \overline{\Delta} \setminus L$.

Proof. Observe that any point of Λ is uniquely specified by specifying the central charges of P_0 and P_1 up to scaling by $\mathbb{C}$, subject to the following further conditions.

S1: Up to rotation, both $Z(P_0)$ and $Z(P_1)$ lie in the closed upper half plane except the origin.

S2: Up to rotation, the phases satisfy

$$0 = \phi(P_0) \leq \phi(P_1) \leq 1.$$

S3: All indecomposable objects have non-zero central charge.

Fix a central charge function Z. We investigate the conditions under which Z satisfies S1, S2, and S3. Assume by rotation and scaling that $Z(P_0) = 1$.

If $Z(P_1)$ lies in the open upper half plane, then S1 and S2 hold automatically. Since the objects P_0 and P_1 are simple in the $[0, 1)$ heart, S3 holds as well. In this case, Z specifies a strictly standard stability condition. Therefore the set Λ° can be identified with the open upper half plane via $Z(P_1)$.

Now suppose that $Z(P_1)$ lies on the real axis. The conditions S1 and S2 are again automatic; it remains to see when S3 holds. If $\phi(P_1) = 0$, then the objects P_0 and P_1 are again the simple objects in the $[0, 1)$-heart of the t-structure, and S3 holds as above.

It remains to consider the case when $\phi(P_1) = 1$. In this case, the $[0, 1)$ heart of the t-structure contains P_0 and $P_1[-1]$. First recall that we have a distinguished triangle as follows:

$$P_0 \to P_\infty \to P_1 \xrightarrow{+1}.$$

We see that

$$Z(P_\infty) = Z(P_0) + Z(P_1) = 1 + Z(P_1).$$

Since P_∞ is an indecomposable object, we conclude that S3 cannot hold if $Z(P_1) = -1$ Assuming that $Z(P_1) \neq -1$, let us compute the central charges of P_0 and P_1 after alternately applying σ_1^{-1} and σ_0^{-1} to the objects in succession. Let $Z_0 = Z$, and let Z_n be the central charge after the nth such step. Explicitly, we have

$$Z_{2n+1}(x) = Z_{2n}(\sigma_1^{-1}x), \quad Z_{2n}(x) = Z_{2n-1}(\sigma_0^{-1}x).$$

Recall that

$$\sigma_1^{-1}(P_1) = P_1[1], \quad \sigma_1^{-1}(P_0) = P_0 \to P_1^{\oplus 2},$$
$$\sigma_0^{-1}(P_1) = P_1 \to P_0^{\oplus 2}, \quad \sigma_0^{-1}(P_0) = P_0[1].$$

We compute that

$$Z_{2n+1}(P_1) = -Z_{2n}(P_1), \quad Z_{2n+1}(P_0) = Z_{2n}(P_0) + 2Z_{2n}(P_1),$$
$$Z_{2n}(P_1) = Z_{2n-1}(P_1) + 2Z_{2n-1}(P_0), \quad Z_{2n}(P_0) = -Z_{2n-1}(P_0).$$

Solving this system, we obtain the following equalities:

$$Z_{2n+1}(P_1) = (-2n-1)Z_0(P_1) - 2n, \quad Z_{2n+1}(P_0) = (2n+2)Z_0(P_1) + (2n+1),$$
$$Z_{2n}(P_1) = (2n+1)Z_0(P_1) + 2n, \quad Z_{2n}(P_0) = -2nZ_0(P_1) + (1-2n).$$

If Z is the central charge of a bona fide stability condition, each Z_k must be as well. In particular, $Z_k(P_1)$ and $Z_k(P_0)$ cannot be zero. We see that S3 cannot hold if $Z_0(P_1) = (1-k)/k$ for some $k \in \mathbb{Z}_{>0}$. On the other hand, suppose there is some $k \in \mathbb{Z}_{>0}$ for which

$$\frac{1-(k+1)}{k+1} < Z_0(P_1) < \frac{1-k}{k}.$$

Then both $Z_k(P_0)$ and $Z_k(P_1)$ are positive real numbers, and therefore give rise to a bona fide stability condition as discussed above. Going back, the condition S3 must hold for the original central charge $Z = Z_0$.

Carrying out a similar calculation by alternately applying σ_0 and then σ_1, we see that S3 cannot hold if $Z(P_0) = (-1-k)/k$ for some $k \in \mathbb{Z}_{>0}$. On the other hand, if there is some $k \in \mathbb{Z}_{>0}$ for which

$$\frac{-1-k}{k} < Z_0(P_1) < \frac{-1-(k+1)}{k+1},$$

then S3 holds for $Z = Z_0$. The proof is now complete. $\square$

We use our identification of Λ with Δ to prove the following proposition.

Proposition 7.12 (Closure of Λ)**.** *The closure of Λ in $\mathbb{P}(\mathbb{R}^{\mathbf{S}})$ is precisely*

$$\overline{\Lambda} = \Lambda \cup \{\overline{\hom}(P_i) \mid i \in \mathbb{Z}\} \cup \{\overline{\hom}(P_1) - \overline{\hom}(P_0)\}.$$

This closure is homeomorphic to $\overline{\Delta}$.

Proof. Identify Λ with Δ via Proposition 7.11. If $\tau \in \Lambda$ corresponds to $\omega_\tau \in \Delta$, we know (up to a complex scalar) that

$$Z_\tau(P_0) = 1, \quad Z_\tau(P_1) = \omega_\tau.$$

We see that

$$Z_\tau(P_i) = |i|\omega_\tau + |i-1|.$$

We compute what happens to the Gromov coordinates of τ as ω_τ approaches one of points of L. First suppose that ω_τ approaches the point $(1-n)/n$ for some $n \neq 0$. In this case,

$$Z_\tau(P_i) \to \frac{|i|(1-n) + |i-1|n}{n}.$$

Using Definition 7.9, we compute that the vector of Gromov coordinates approaches a single spike:

$$x_j(\tau) \to \begin{cases} 0, & j \neq n, \\ 1, & j = n \end{cases}.$$

$Z_\tau(P_i)$ approaches $|i-1|$. Therefore $x_j(\tau)$ approaches 0 unless $j = n$, and $x_n(\tau)$ approaches 1. Similarly, as ω_τ approaches ∞, we can simultaneously rescale to assume that $Z_\tau(P_0) \to 0$, keeping $Z_\tau(P_1)$ constant. We conclude again that when $\omega_\tau \to \infty$, we have

$$x_j(\tau) \to \begin{cases} 0, & j \neq 0, \\ 1, & j = 0 \end{cases}.$$

By Proposition 7.10, the mass m_τ of any object is an infinite linear combination of the Gromov coordinates with fixed coefficients. As a result, as ω_τ approaches any point in L of the form $(1-n)/n$ (including the point ∞ corresponding to $n = 0$), the map $m\colon \Delta \to \mathbb{P}^{\mathbf{S}}$ extends continuously. The extension to the limiting point $(1-n)/n$ is precisely the $\overline{\hom}$ functional from the object P_n.

Finally suppose that ω_τ approaches $-1 \in L$. In this case,

$$Z_\tau(P_j) \to -|i| + |i-1| = \begin{cases} -1, & i > 0, \\ 1, & i \leq 0 \end{cases}.$$

We compute that

$$x_j(\tau) \to \begin{cases} 0, & i < 0 \text{ or } i > 1, \\ -1, & i = 0, \\ 1, & i = 1. \end{cases}$$

Once again the map $m\colon \Delta \to \mathbb{P}^{\mathbf{S}}$ extends continuously to the limiting point -1. In this case, the extension is precisely

$$m_\tau(X) = \frac{1}{2}(\overline{\hom}(P_1, X) - \overline{\hom}(P_0, X))$$

up to a simultaneous scalar. We see that the mass map extends continuously to all the points of L, giving precisely the values of the hom functionals. Since $\overline{\Delta}$ is compact, no other points are in the closure. □

Remark 7.13. The point -1 is already a limit point of the set L because the sequence $(1-n)/n$ converges to -1 for $n \in \mathbb{N}$, as $n \to \infty$. Therefore the additional point in the closure, namely the functional $\overline{\hom}(P_1) - \overline{\hom}(P_0)$, is already a limit point of the set $\{\overline{\hom}(P_i) \mid i \in \mathbb{Z}\}$.

It is now easy to identify the closure of $\operatorname{Stab}(\mathcal{C})/\mathbb{C}$, using the following contraction property, analogous to Proposition 5.5. Fix a metric on $\mathbb{P}^1(\mathbb{R}) = S^1$.

Proposition 7.14. *Let $\epsilon > 0$ and a compact subset I of $\mathbb{P}^1(\mathbb{R})$ be given. Then for all but finitely many elements $g \in G$, the image $g(I)$ has diameter less than ϵ.*

Proof. The action of G on $\mathbb{P}^1(\mathbb{R})$ is via the inclusion $G \subset \mathrm{PSL}_2(\mathbb{Z})$. The claim is true for $\mathrm{PSL}_2(\mathbb{Z})$, as shown in the proof of Proposition 5.5, and hence also for G. □

Let M and P be the images of $\operatorname{Stab}(\mathcal{C})/\mathbb{C}$ and $\mathbf{S}$ in $\mathbb{P}^{\mathbf{S}}$, respectively.

Proposition 7.15 (Closure of M)**.** *The sets M and P are disjoint, and their union is the closure of M.*

Proof. The proof is entirely analogous to the proof of Proposition 5.6. □

We conjecture that the union $M \cup P$ is homeomorphic to a closed disk, as in the A_2 case, but we do not yet have a complete proof.

Appendix A. Geodesic filtrations

Our goal in this note is to understand filtrations that can be re-arranged to the Harder–Narasimhan (HN) filtration. All the objects and morphisms are in a triangulated category that is equipped with a stability condition.

Before filtrations, we study the same question for distinguished triangles. Consider a distinguished triangle

$$A \to C \to B \to A[1].$$

For the HN-filtration C to be obtained from the HN-filtrations of A and B, the following property of the map $B \to A[1]$ turns out to be crucial.

Definition A.1 (Rectifiable map)**.** We say that the map $B \to A[1]$ is *rectifiable* if for every $\alpha \in \mathbb{R}$, the induced map

$$B_{>\alpha} \to A_{\leq\alpha}[1]$$

vanishes.

Example A.2. A direct sum (the zero map $B \to A[1]$) is rectifiable.

Example A.3. Suppose $\lfloor A \rfloor \geq \lceil B \rceil$. Then any map $B \to A[1]$ is rectifiable. Indeed, in this case either $B_{>\alpha}$ or $A_{\leq\alpha}$ is zero.

Example A.4. If both $B \to A_1[1]$ and $B \to A_2[1]$ are rectifiable, then so is the direct sum $B \to (A_1 \oplus A_2)[1]$.

Proposition A.5. *Suppose $B \to A[1]$ is rectifiable. Then for every $C \to B$ and $A \to D$, the induced map $C \to D[1]$ is rectifiable.*

Proof. The map $C_{>\alpha} \to D_{\leq\alpha}[1]$ factors as

$$C_{>\alpha} \to B_{>\alpha} \to A_{\leq\alpha}[1] \to D_{\leq\alpha}[1].$$

Since $B \to A[1]$ is rectifiable, the map $B_{>\alpha} \to A_{\leq\alpha}[1]$ vanishes. Hence, the map $C_{>\alpha} \to D_{\leq\alpha}[1]$ vanishes too. ☐

Proposition A.6. *Suppose $e\colon B \to A[1]$ is rectifiable. Set $C = \operatorname{Cone}(e)[-1]$, so that we have the triangle*

$$A \to C \to B \xrightarrow{+1}.$$

Then we have exact triangles

$$A_{\leq\alpha} \to C_{\leq\alpha} \to B_{\leq\alpha} \xrightarrow{+1}, \tag{14}$$

and

$$A_{>\alpha} \to C_{>\alpha} \to B_{>\alpha} \xrightarrow{+1}, \tag{15}$$

where the maps are induced from e. Furthermore, the maps in (14) *and* (15) *are rectifiable.*

Proof. We refine the triangle

$$A \to C \to B \xrightarrow{+1}$$

to the filtration

$$\begin{tikzcd}
0 \arrow[rr] & & C_1 \arrow[rr] \arrow[dl] & & C_2 \arrow[rr] \arrow[dl] & & C_3 \arrow[rr] \arrow[dl] & & C_4 = C. \arrow[dl] \\
& A_{>\alpha} \arrow[ul, dashed, "+1"] & & A_{\leq\alpha} \arrow[ul, dashed, "+1"] & & B_{>\alpha} \arrow[ul, dashed, "+1"] & & B_{\leq\alpha} \arrow[ul, dashed, "+1"] &
\end{tikzcd}$$

Since $B \to A[1]$ is rectifiable, the middle map $B_{>\alpha} \to A_{\leq\alpha}[1]$ vanishes. Hence, we can obtain a new filtration where the middle two cones are swapped:

$$\begin{tikzcd}
0 \arrow[rr] & & C_1 \arrow[rr] \arrow[dl] & & C_2' \arrow[rr] \arrow[dl] & & C_3' \arrow[rr] \arrow[dl] & & C_4 = C. \arrow[dl] \\
& A_{>\alpha} \arrow[ul, dashed, "+1"] & & B_{>\alpha} \arrow[ul, dashed, "+1"] & & A_{\leq\alpha} \arrow[ul, dashed, "+1"] & & B_{\leq\alpha} \arrow[ul, dashed, "+1"] &
\end{tikzcd}$$

We now see that $C_{>\alpha} = C_2'$ and $C_{\leq\alpha} = \operatorname{Cone}(C_2' \to C_4)$, and these are extensions of $A_{>\alpha}, B_{>\alpha}$ and of $A_{\leq\alpha}, B_{\leq\alpha}$, as desired.

Let us understand the map $B_{>\alpha} \to A_{>\alpha}[1]$. Consider the composite of $B_{>\alpha} \to B$ and $B \to A[1]$, say $e\colon B_{>\alpha} \to A[1]$. The composite of e with $A[1] \to A_{\leq\alpha}[1]$ vanishes, by assumption, and hence e induces the map

$$B_{>\alpha} \to A_{>\alpha}[1].$$

The map above induces the map $B_{\leq\alpha} \to A_{\leq\alpha}[1]$.

Finally, we check that the map (14) is rectifiable. We need to check that for every β, the map

$$B_{(\beta,\alpha]} \to A_{\leq \min(\alpha,\beta)}[1]$$

vanishes. We have the diagram

$$\begin{tikzcd} B_{>\max(\alpha,\beta)} \arrow[r] \arrow[d] & B_{>\beta} \arrow[r] \arrow[d] & B_{(\beta,\alpha]} \arrow[d] \\ A_{(\alpha,\beta]}[1] \arrow[r] & A_{\leq\beta}[1] \arrow[r] & A_{\leq\min(\alpha,\beta)}[1], \end{tikzcd}$$

where the rows are distinguished triangles. Since $B \to A[1]$ is rectifiable, the left and the middle vertical maps vanish. Hence, the right vertical map also vanishes.

The proof that the map (15) is rectifiable is similar. □

Proposition A.7. *Let $A \to C \to B \xrightarrow{e} A[1]$ be a distinguished triangle where e is rectifiable. Then the HN filtration of C is a rearrangement of the concatenation of the HN filtrations of A and B. In particular, we have*

$$\operatorname{mass}(C) = \operatorname{mass}(A) + \operatorname{mass}(B).$$

Proof. We induct on the HN length of C. The baes case—C being semi-stable—is easy. In general, choose an α such that $C_{>\alpha}$ and $C_{\leq\alpha}$ have smaller HN lengths than C. The HN filtration of C is the concatenation of the HN filtrations of $C_{>\alpha}$ and $C_{\leq\alpha}$. From the inductive assumption and the triangles (14) and (15) in Proposition A.6, we get that the HN filtration of $C_{>\alpha}$ (resp. $C_{\leq\alpha}$) is a re-arrangement of the concatenation of the HN filtrations of $A_{>\alpha}$ and $B_{>\alpha}$ (resp. $A_{\leq\alpha}$ and $B_{\leq\alpha}$). The result follows. □

We now take up filtrations.

Definition A.8 (Geodesic filtration)**.** Consider a filtration

$$X_0 \to X_1 \to \cdots \to X_n.$$

We inductively define what it means for such a filtration to be *geodesic*. All filtrations for $n = 0$ and 1 are geodesic. For $n \geq 2$, the filtration is geodesic if

(1) the filtration $X_1 \to \cdots \to X_n$ is geodesic, and
(2) the map $\operatorname{Cone}(X_1 \to X_n) \to \operatorname{Cone}(X_0 \to X_1)[1]$ is rectifiable.

Remark A.9. Observe that Harder–Narasimhan filtrations are obviously geodesic.

Remark A.10. The definition of a geodesic filtration is "translation invariant." That is, a filtration

$$X_0 \to X_1 \to \cdots \to X_n$$

is geodesic if and only if the filtration

$$0 = X'_0 \to X'_1 \to \cdots \to X'_n$$

is, where $X'_i = \operatorname{Cone}(X_0 \to X_i)$.

Proposition A.11. *Suppose*

$$X_0 \to \cdots \to X_n$$

is geodesic. Then for every i, j with $i < j$, the truncation

$$X_i \to \cdots \to X_j$$

is geodesic

Proof. This is an easy consequence of Proposition A.5. □

Proposition A.12. *Consider a filtration*

$$X_0 \to X_1 \to \cdots \to X_n.$$

The following are equivalent:

(1) $X_0 \to X_1 \to \cdots \to X_n$ is geodesic.

(2) for all i, both $X_0 \to \cdots \to X_i$ and $X_i \to \cdots \to X_n$ are geodesic, and the map

$$\mathrm{Cone}(X_i, X_n) \to \mathrm{Cone}(X_0, X_i)[1]$$

is rectifiable.

(3) for some i, both $X_0 \to \cdots \to X_i$ and $X_i \to \cdots \to X_n$ are geodesic, and the map

$$\mathrm{Cone}(X_i, X_n) \to \mathrm{Cone}(X_0, X_i)[1]$$

is rectifiable.

Proof. Let us show that (1) implies (2). We induct on i. The case $i = 1$ follows from the definition of a geodesic filtration. Assume $i \geq 2$. By Proposition A.11, we get that both $X_0 \to \cdots \to X_i$ and $X_i \to \cdots \to X_n$ are geodesic. For brevity, set $X(i, j) = \mathrm{Cone}(X_i \to X_j)$. We have to prove that the map $X(i, n) \to X(0, i)[1]$ is rectifiable, that is the map

$$X(i, n)_{>\alpha} \to X(0, i)_{\leq\alpha}[1]$$

vanishes. By Proposition A.6, we have the triangle

$$X(0, 1)_{\leq\alpha} \to X(0, i)_{\leq\alpha} \to X(1, i)_{\leq\alpha}.$$

By the inductive hypothesis applied to the geodesic filtration $X_1 \to \cdots \to X_n$ for $(i - 1)$, we get that the map

$$X(i, n) \to X(1, i)[1] \tag{16}$$

is rectifiable. Therefore, the composite of

$$X(i, n)_{>\alpha} \to X(0, i)_{\leq\alpha}[1]$$

with $X(0, i)_{\leq\alpha}[1] \to X(1, i)_{\leq\alpha}[1]$ already vanishes, and hence we get an induced map

$$X(i, n)_{>\alpha} \to X(0, 1)_{\leq\alpha}[1].$$

By Proposition A.6 applied to the rectifiable map in (16), we have the triangle

$$X(1, i)_{>\alpha} \to X(1, n)_{>\alpha} \to X(i, n)_{>\alpha} \xrightarrow{+1}.$$

The map $X(1, n)_{>\alpha} \to X(0, 1)_{\leq\alpha}[1]$ vanishes since $X_0 \to \cdots \to X_n$ is geodesic. The induced map $X(1, i)_{>\alpha}[1] \to X(0, 1)_{\leq\alpha}[1]$ vanishes for phase reasons. Therefore, the map $X(i, n)_{>\alpha} \to X(0, 1)_{\leq\alpha}[1]$ vanishes, as required.

That (2) implies (3) is a tautology.

Let us show that (3) implies (1). We again induct on i. The case $i = 1$ is the definition. Assume $i \geq 2$. By using Proposition A.11 and the inductive hypothesis, we conclude that

$$X_1 \to \cdots \to X_n$$

is geodesic. It remains to show that the map $X(1,n) \to X(0,1)[1]$ is rectifiable. We have the rectifiable map

$$X(1,i) \to X(1,n) \to X(i,n) \xrightarrow{+1},$$

which by Proposition A.6 gives the exact triangle

$$X(1,i)_{>\alpha} \to X(1,n)_{>\alpha} \to X(i,n)_{>\alpha} \xrightarrow{+1} .$$

Since $X_0 \to \cdots \to X_i$ is geodesic, the map $X(1,i)_{>\alpha} \to X(0,1)_{\leq\alpha}[1]$ vanishes. We thus get an induced map

$$X(i,n)_{>\alpha} \to X(0,1)_{\leq\alpha}[1],$$

which we must show vanishes. Since $X_0 \to \cdots \to X_i$ is geodesic, we have the triangle

$$X(0,1)_{\leq\alpha} \to X(0,i)_{\leq\alpha} \to X(1,i)_{\leq\alpha} \xrightarrow{+1} .$$

The map $X(i,n)_{>\alpha} \to X(0,i)_{\leq\alpha}[1]$ vanishes since $X(i,n) \to X(0,i)[1]$ is rectifiable. The induced map $X(i,n)_{>\alpha} \to X(1,i)_{\leq\alpha}$ vanishes for phase reasons. Therefore, the map $X(i,n)_{>\alpha} \to X(0,1)_{\leq\alpha}[1]$ vanishes. □

Proposition A.13. *Suppose*

$$0 = X_0 \to \cdots \to X_n = X$$

is geodesic. Set $A_i = \mathrm{Cone}(X_{i-1} \to X_i)$*. Then the HN filtration of* X *is a rearrangement of the concatenation of the HN filtrations of the* A_i*. In particular, we have*

$$\mathrm{mass}(X) = \sum \mathrm{mass}(A_i).$$

Proof. Apply Proposition A.7 repeatedly. □

Proposition A.14. *Suppose*

$$X_0 \to \cdots \to X_{i-1} \to X_i \to X_{i+1} \to \cdots \to X_n$$

is rectifiable, and the map $\mathrm{Cone}(X_i, X_{i+1}) \to \mathrm{Cone}(X_{i-1}, X_i)[1]$ *is zero; that is*

$$\mathrm{Cone}(X_{i-1}, X_{i+1}) = \mathrm{Cone}(X_{i-1}, X_i) \oplus \mathrm{Cone}(X_i, X_{i+1}).$$

Then the flipped filtration

$$X_0 \to \cdots \to X_{i-1} \to X_i' \to X_{i+1} \to \cdots \to X_n = X,$$

where $X_i' = \mathrm{Cone}(X_{i+1} \to \mathrm{Cone}(X_{i-1}, X_i))[-1]$*, is also rectifiable.*

Proof. We may assume $i - 1 = 0$, and $X_0 = 0$. By Proposition A.12 applied with $i = 2$, it suffices to prove that the map

$$\mathrm{Cone}(X_2, X_n) \to X_2[1]$$

is rectifiable. Note that this map is unchanged after the swap. But this map is rectifiable by the hypothesis. □

Proposition A.15. *Consider a filtration*

$$0 = X_0 \to \cdots \to X_n = X.$$

Set $A_i = \mathrm{Cone}(X_{i-1} \to X_i)$*. Suppose for every* i, j *with* $i < j$*, we have*

$$\mathrm{Hom}(A_j, A_i[1]) = 0 \text{ or } \lfloor A_i \rfloor \geq \lceil A_j \rceil.$$

Then the filtration is geodesic.

Proof. We induct on n. The base case $n = 0$ is trivial. Assume $n \geq 1$. By applying Proposition A.14 repeatedly, we may assume that

$$\lceil A_1 \rceil \geq \cdots \geq \lceil A_n \rceil.$$

As before, set $X(i, j) = \mathrm{Cone}(X_i \to X_j)$. Let i be the largest such that for all $j > i$, we have

$$\lfloor A_1 \rfloor \geq \lceil A_j \rceil.$$

By the assumption, we have $\mathrm{Hom}(A_j, A_1[1]) = 0$ for all $j \leq i$. Therefore, we get $X_i = A_1 \oplus X(1, i)$. By the inductive assumption and the translation invariance (Remark A.10),

$$X_0 \to \cdots \to X_i$$

and

$$X_i \to \cdots \to X_n$$

are both a geodesic filtrations with sub-quotients $A_1, \ldots A_i$ and $A_{i+1}, \ldots, A_n$, respectively. By Proposition A.12, it suffices to show that the map

$$X(i, n) \to X_i[1]$$

is rectifiable. But $X_i = A_1 \oplus X(1, i)$. By the inductive assumption, the filtration

$$0 \to X(1, 2) \to \cdots \to X(1, n)$$

is geodesic, and hence the map $X(i, n) \to X(1, i)[1]$ is rectifiable. The map $X(i, n) \to A_1[1]$ is rectifiable by construction, since $\lfloor A_1 \rfloor \geq \lceil X(i, n) \rceil$. Hence, $X(i, n) \to X_i[1] = (A_1 \oplus X(1, i))[1]$ is rectifiable. The proof is now complete. $\square$

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