

# A THURSTON COMPACTIFICATION OF THE SPACE OF STABILITY CONDITIONS

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**ABSTRACT.** We propose a compactification of the moduli space of Bridgeland stability conditions of a triangulated category. For 2-Calabi–Yau quiver categories, we conjecture that this compactification is a real manifold with boundary, on which the action of the autoequivalence group extends continuously. The key ingredient in the compactification is an embedding of the stability space into an infinite-dimensional projective space. We study this embedding in detail in the case of 2-Calabi–Yau categories associated to quivers, and prove our conjectures in the  $A_2$  and  $\widehat{A}_1$  cases. Central to our analysis is a detailed understanding of Harder–Narasimhan multiplicities and how they transform under auto-equivalences. We achieve this by introducing a structure called a Harder–Narasimhan automaton and constructing examples for  $A_2$  and  $\widehat{A}_1$ .

## 1. INTRODUCTION

Many interesting triangulated categories arise as Fukaya categories, which may be studied using geometric techniques. Purely categorical formulations of these techniques, independent of the geometric origins, may be useful in studying more general triangulated categories. Exemplified in the work of Khovanov, Seidel, Thomas (and others) [17, 18, 21], this philosophy has had spectacular success.

In recent years, the connection between geometry and homological algebra has been further strengthened by relating Teichmüller theory and Bridgeland stability [7, 8, 9, 11, 13, 16, 22, 23]. An exciting prospect of this connection is a theory of dynamics in triangulated categories parallel to the theory of dynamics in surface geometry. To realise this possibility, we should consider not only spaces of Bridgeland stability conditions but also their modular compactifications. In this paper, we propose one such compactification, inspired by Thurston’s compactification of Teichmüller space. We work out this compactification explicitly for two small but highly non-trivial cases, namely the 2-Calabi–Yau categories associated to the  $A_2$  quiver and the  $\widehat{A}_1$  quiver. In future work, we will generalise our results and explore their applications to dynamics of triangulated categories.

**1.1. A projective embedding of the stability manifold.** The key idea in Thurston’s compactification of Teichmüller space is to embed it in an infinite projective space. This embedding sends a point of Teichmüller space, represented by a metric  $\mu$ , to the real-valued function on the set of curves defined by the length with respect to  $\mu$ .

In our categorical analogue of Thurston’s construction, we regard Bridgeland stability conditions as metrics and spherical objects as curves. On any triangulated category, a Bridgeland stability condition does indeed satisfy a triangle inequality [16]. On certain triangulated categories, spherical objects correspond to curves on a surface, as described by Khovanov and Seidel [17]. Motivated by their work, we regard spherical objects as replacements for curves more generally. A categorical analogue of the length of a curve is

then the length of a spherical object with respect to the metric on the category. The latter is usually called the mass of the object.

Let us now give the precise definitions. Let  $\mathbf{k}$  be a field, and  $\mathcal{C}$  a  $\mathbf{k}$ -linear triangulated category. Given a stability condition  $\tau$  on  $\mathcal{C}$  with central charge  $Z$ , the *mass* of an object  $x$  is defined as

$$(1) \quad m_\tau(x) = \sum_i |Z(z_i)|,$$

where  $z_i$  are the semistable Harder–Narasimhan factors of  $x$ .

Let  $\mathbf{S}$  be a subset of the set of objects of  $\mathcal{C}$ . Denote by  $\mathbf{R}^{\mathbf{S}}$  the space of functions from  $\mathbf{S}$  to  $\mathbf{R}$ , endowed with the product topology. The association  $\tau \mapsto m_\tau$  yields a continuous map

$$m: \text{Stab}(\mathcal{C}) \rightarrow \mathbf{R}^{\mathbf{S}}.$$

We have an action of  $\mathbf{C}$  on  $\text{Stab}(\mathcal{C})$ . Let  $\mathbf{P}^{\mathbf{S}}$  denote the quotient  $(\mathbf{R}^{\mathbf{S}} \setminus \{0\})/\mathbf{R}^\times$ . Then the mass map descends to a continuous map

$$m: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}}.$$

Guided by the analogy with Teichmüller theory, we hope for the following.

**Expectation 1.** *For a suitable set of objects  $\mathbf{S}$ , the map  $m: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}}$  is injective and a homeomorphism onto its image.*

Let  $M$  be the image of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  in  $\mathbf{P}^{\mathbf{S}}$ . Let  $\overline{M}$  be the closure of  $M$  and let  $\partial M$  be the boundary of  $\overline{M}$ .

**Expectation 2.** *The space  $\overline{M}$  is compact.*

If these two expectations hold, then we obtain a compactification  $\overline{M}$  of  $\text{Stab}(\mathcal{C})/\mathbf{C}$ . For 2-Calabi–Yau quiver categories, we conjecture that both of these expectations hold when  $\mathbf{S}$  is the set of spherical objects of  $\mathcal{C}$ . See § 1.3 for the results in this direction.

**1.2. The boundary of the projective embedding.** The next natural step is to describe the boundary. In Teichmüller theory, the boundary can be described as follows. Each simple closed curve  $\gamma$  on a surface gives rise to a function on the set of all curves defined by the topological (unsigned) intersection number with  $\gamma$ . The boundary is the closure of the set of such functions.

These intersection functions have a natural categorical analogue. Assume that the hom spaces in  $\mathcal{C}$  are finite-dimensional. For each object  $x \in \mathcal{C}$ , define  $\text{hom}(x) \in \mathbf{R}^{\mathcal{C}}$  as

$$\text{hom}(x): y \mapsto \dim_{\mathbf{k}} \text{Hom}^*(x, y).$$

**Expectation 3.** *There is a suitable class of objects  $\mathbf{T} \subset \mathcal{C}$  such that the set*

$$\{\text{hom}(x) \in \mathbf{P}^{\mathbf{S}} \text{ for } x \in \mathbf{T}\}$$

*is dense in  $\partial M$ .*

For 2-Calabi–Yau categories associated to quivers, we prove that the points  $\text{hom}(x)$ , where  $x$  is a spherical object, lie in  $\overline{M}$ . We expect that these points lie in  $\partial M$  and together form a dense subset of  $\partial M$ . That is, for 2-Calabi–Yau quiver categories, taking  $\mathbf{T}$  to be the set of all spherical objects should satisfy Expectation 3.

**Expectation 4.** *The pair  $(\overline{M}, \partial M)$  is a manifold with boundary.*

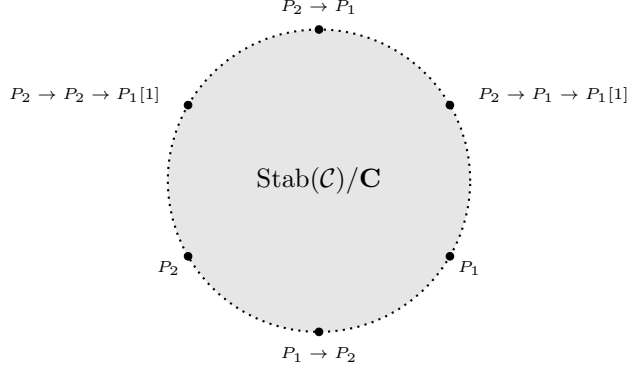


FIGURE 1. The compactification of  $\text{Stab}(\mathcal{C})/\mathbf{C}$ . The spherical objects appear as a dense subset of the boundary.

Furthermore, in finite (ADE) type, we expect  $\overline{M}$  to be the closed ball and  $\partial M$  to be the sphere. We warn the reader that even in these cases, the interior of  $\overline{M}$  may strictly contain  $M$ ; see § 1.6.

**1.3. Summary of results.** We conjecture that all expectations hold when  $\mathcal{C}$  is the 2-Calabi–Yau category associated to a finite connected quiver. We prove parts of this conjecture in various levels of generality in this paper.

Let  $\mathcal{C}$  be the 2-Calabi–Yau category associated to a finite connected quiver  $\Gamma$  (see § 2 for a detailed description), and  $\mathbf{S}$  the set of spherical objects in  $\mathcal{C}$ . We prove the following statements.

- (1) The map  $m: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}}$  is injective (part of Expectation 1).
- (2) The closure of the image of  $m$  is compact (Expectation 2).
- (3) The reduced hom functionals corresponding to spherical objects are in the boundary of the image of  $m$  (part of Expectation 3).
- (4) All four expectations hold when  $\Gamma$  is the  $A_2$  quiver, resulting in a complete description of a compactification of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  in this case. See Figure 1 for a sketch of the resulting picture. We also prove the first three expectations when  $\Gamma$  is the  $\hat{A}_1$  quiver.

**1.4. Harder–Narasimhan automata.** The 2-Calabi–Yau category  $\mathcal{C}$  associated to a quiver  $\Gamma$  admits the action of a large symmetry group—the Artin–Tits braid group  $B_\Gamma$ . Our analysis of the stability manifold and its compactification heavily uses this symmetry. To do so, we must understand how the mass of an object, or better, its Harder–Narasimhan (HN) filtration, transforms under an auto-equivalence.

Fix a stability condition on  $\mathcal{C}$ , and let  $\Sigma$  be the set of indecomposable semistable objects up to shift. For an object  $x \in \mathcal{C}$ , let  $HN(x) \in \mathbf{Z}^\Sigma$  denote the HN multiplicity vector of  $x$ . Let a group  $G$  act on  $\mathcal{C}$  by auto-equivalences. The action of  $G$  on  $\mathcal{C}$  does not usually induce a compatible linear action of  $G$  on  $\mathbf{Z}^\Sigma$ . Nevertheless, we prove in the  $A_2$  and  $\hat{A}_1$  cases that the action is “piecewise linear”. Roughly speaking, the objects of  $\mathcal{C}$  may be partitioned into subsets, called *states*. For each pair of states, there are certain allowable elements of  $G$  that send objects of the first state to objects of the second, and induce linear transformations on

the HN multiplicities. We formalise this phenomenon using structures called HN automata, in § 3.

As an application of this construction, we recover a theorem of Rouquier–Zimmermann [20, Proposition 4.8] in the type  $A_2$  case and prove a new analogue of this theorem in the type  $\hat{A}_1$  case. More generally, we anticipate that automata should play an important role in analysing group actions on categories.

### 1.5. $q$ -analogue.

The entire story admits a  $q$ -analogue for any positive real number  $q > 0$ . Let  $\tau$  be a stability condition with central charge  $Z$  and phase function  $\phi$ . Define the  $q$ -mass of an object  $x$  as follows (deforming (1)):

$$m_{q,\tau}(x) = \sum_i |q^{\phi(z_i)} Z(z_i)|.$$

We get a continuous map

$$m_q: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}},$$

which agrees with our original map  $m$  when  $q = 1$ . We thus obtain a family of compactifications of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  given by the closures of  $m_q(\text{Stab}(\mathcal{C})/\mathbf{C})$  in  $\mathbf{P}^{\mathbf{C}}$ . We conjecture that the closures  $\{\bar{M}_q\}_{q>0}$  are all homeomorphic to each other.

There are some interesting differences between the  $q = 1$  and  $q \neq 1$  cases, even in the case of the  $A_2$  quiver. As we prove in this paper, the functions  $\text{hom}(x)$  for spherical  $x$  are dense in the boundary for  $q = 1$ . In fact, we have identifications

$$M_{q=1} \cong \mathbf{H}^2 \text{ and } \partial M_{q=1} = \mathbf{R} \cup \{\infty\},$$

under which the  $\text{hom}$ -functions are identified with the rational points  $\mathbf{Q} \cup \{\infty\}$  of the boundary. When  $q \neq 1$ , the picture is much more intricate, and is studied in detail in [1]. In this case, the  $\text{hom}(x)$  functions are identified with a non-dense (in fact, fractal) subset of  $\mathbf{R} \cup \{\infty\}$ . See [1] for a complete description of this set and the entire boundary. Equally compelling, and even less understood at present, is the behaviour of  $\bar{M}_q$  as  $q \rightarrow \infty$  or  $q \rightarrow 0$ .

**1.6. Comparison with Teichmüller theory.** Although our construction is motivated by Thurston’s compactification of the Teichmüller space, the resulting picture differs from it in interesting ways.

Let us restrict to the 2-Calabi–Yau category of the  $A_n$  quiver. In this case, both the Bridgeland moduli space and the Teichmüller space are universal covers of the complement of the hyperplane arrangement of type  $A_n$ . For  $n > 2$ , however, no homeomorphism between them extends to a homeomorphism between the respective compactifications. In Thurston’s construction, the Teichmüller space maps homeomorphically onto the interior of the compactification. In our case, this is not true. For  $n \geq 3$ , the interior of our compactification contains additional points, some of which can be identified with degenerate stability conditions constructed by Bolognese [3].

In Teichmüller theory, the boundary has a modular interpretation as the space of projective measured foliations. At present, we do not know an appropriate categorical analogue of this notion.

On the other hand, our theory has a natural  $q$ -analogue, as described in § 1.5. This constructs a family of compactifications of the space of stability conditions, depending on a positive real parameter  $q$ . This is in contrast to existing literature on quantizations of Teichmüller space (e.g. [12]), which pertains to quantizing algebras of functions.

It seems likely that a closer geometric analogue of our construction is the compactification of the moduli space of flat metrics on a surface, constructed, for example, by Duchin, Leininger, and Rafi [10]. It would be interesting to relate the two constructions in the  $A_n$  case.

**1.7. Organisation.** In § 2 we give precise definitions of the categories we work with, and recall some background on Bridgeland stability conditions. In § 3 we introduce the definition of Harder–Narasimhan automata. In § 4 we describe the embedding of the space of stability conditions into an infinite-dimensional projective space via mass functionals. We prove several general results about this embedding: that it is injective and pre-compact, and that reduced hom functionals lie in the boundary of this embedding. We also prove that this embedding is a homeomorphism onto its image if we have access to a suitable HN automaton.

In § 5 and § 6 we explicitly construct suitable HN automata for the cases of the  $A_2$  and  $\widehat{A}_1$  quivers. Using these automata, we prove the results mentioned in § 1.3. Along the way we also obtain new proofs of existing results in the literature. For instance, we construct an explicit isomorphism of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  with an open disk in the case of the  $A_2$  quiver. Another isomorphism has been constructed in [7]. We recover a result of Rouquier–Zimmermann [20, Proposition 4.8] for the  $A_2$  case, and prove an analogue of this theorem in the  $\widehat{A}_1$  case.

In Appendix A we establish general results about rearrangements of Harder–Narasimhan filtrations in a stability condition.

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## 2. BACKGROUND ON CATEGORIES AND BRIDGELAND STABILITY CONDITIONS

**2.1. The category  $\mathcal{C}_\Gamma$ .** Let  $\Gamma$  be a connected graph without loops and multiple edges. Let  $B_\Gamma$  be the Artin–Tits braid group associated to  $\Gamma$ . This group is generated by  $\sigma_i$ , where  $i$  ranges over the set of vertices of  $\Gamma$ , modulo the following relations:

$$\begin{aligned}\sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j \text{ if there is an edge between } i \text{ and } j, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \text{ otherwise.}\end{aligned}$$

We describe a category  $\mathcal{C}_\Gamma$  with a (weak) action of  $B_\Gamma$  that categorifies the standard representation of the associated Coxeter group. The construction is via the zig-zag algebra, which was introduced in [14]. A construction for type  $A_n$  graphs is described in [21, 23], and a detailed construction for all  $\Gamma$  may be found in [2, §2.3].

Let  $\Gamma^{\text{dbl}}$  denote the doubled quiver, which has the same vertices as  $\Gamma$  and two edges for each edge of  $\Gamma$ . More precisely, each edge  $f$  of  $\Gamma$  is replaced in  $\Gamma^{\text{dbl}}$  by two oriented edges connecting the same vertices but with opposite orientations. Let  $\text{Path}(\Gamma^{\text{dbl}})$  denote the path algebra of  $\Gamma^{\text{dbl}}$ . As a vector space over  $\mathbf{k}$ , the algebra  $\text{Path}(\Gamma^{\text{dbl}})$  is spanned by all oriented paths in  $\Gamma^{\text{dbl}}$ , and the multiplication is given by concatenation. The natural length

function on paths, which declares edges to have length one, induces a non-negative grading on  $\text{Path}(\Gamma^{\text{dbl}})$ .

Since  $\Gamma$  does not have multiple edges, we may express a path unambiguously by listing the vertices it travels through in order. So, for example, if  $i$  and  $j$  are neighbouring vertices, the element  $(i|j|i)$  denotes the length two path which begins at  $i$ , travels along the unique oriented edge from  $i$  to  $j$ , and then travels back to  $i$  along the unique oriented edge from  $j$  to  $i$ . Suppose  $\Gamma$  has at least two vertices. We let  $A(\Gamma)$  denote the quotient algebra of the path algebra by the two-sided ideal generated by the following elements:

- (1) all length three paths,
- (2) length two paths  $(i|j|k)$  with  $i \neq k$ , and
- (3)  $(i|j|i) - (i|k|i)$  when  $i$  is connected to both  $j$  and  $k$ .

These relations are all homogeneous, so that the natural grading on the path algebra descends to a grading on  $A(\Gamma)$ . If  $\Gamma$  has a single vertex, (type  $A_1$ ), we set  $A(\Gamma) = \mathbb{C}[x]/x^2$ , graded by setting  $\deg(x) = 2$ . Following [14], we call  $A(\Gamma)$  the zigzag algebra of  $\Gamma$ .

The minimal idempotents of  $A(\Gamma)$  are the length zero paths  $(i)$  for  $i \in \Gamma$ . The modules  $P_i = A(\Gamma)(i)$  are indecomposable projective left  $A(\Gamma)$  modules. Any finitely-generated graded projective left  $A(\Gamma)$  module is isomorphic to a direct sum of grading shifts of the modules  $P_i$ .

The category considered in [14] is the homotopy category of graded projective left  $A(\Gamma)$ -modules. This admits two shift functors, namely the homological shift and the grading shift. As in [21, § 4] and [23, § 3], we work in a simpler setting where the two shifts are identified. We consider  $A(\Gamma)$  as a differential graded algebra (dga), where all the differentials are zero. Let  $\text{DGM}_\Gamma$  be the category of differential graded modules (dgm) over the dga  $A(\Gamma)$ . Objects of  $\text{DGM}_\Gamma$  are dgms, and morphisms are chain maps, compatible with the action of  $A(\Gamma)$ . Let  $\mathcal{D}_\Gamma$  be the category obtained from  $\text{DGM}_\Gamma$  by inverting the quasi-isomorphisms. Note that each graded  $A(\Gamma)$ -module can be considered as an element of  $\text{DGM}_\Gamma$  (and hence of  $\mathcal{D}_\Gamma$ ); we simply take all differentials to be zero. We let  $\mathcal{C}_\Gamma \subset \mathcal{D}_\Gamma$  be the full triangulated subcategory generated by the objects  $P_i$ .

The categories considered in [21] and [23] are described in a slightly different way. However, [2, Proposition 2.1] shows that they coincide with our category  $\mathcal{C}_\Gamma$ . Having given the construction, we state the salient properties of  $\mathcal{C}_\Gamma$ .

- (1)  $\mathcal{C}_\Gamma$  is a  $\mathbf{k}$ -linear triangulated category.
- (2)  $\mathcal{C}_\Gamma$  is 2-Calabi–Yau [2, Proposition 2.3]. That is, for a pair of objects  $x, y \in \mathcal{C}$ , we have a natural isomorphism

$$\text{Hom}(x, y) \cong \text{Hom}(y, x[2])^*.$$

- (3)  $\mathcal{C}_\Gamma$  is classically generated by the objects  $P_i$ , satisfying

$$\begin{aligned} \text{Hom}(P_i, P_i[n]) &= \begin{cases} \mathbf{k} & \text{if } n = 0 \text{ or } n = 2, \\ 0 & \text{otherwise;} \end{cases} \\ \text{Hom}(P_i, P_j[n]) &= \begin{cases} \mathbf{k} & \text{if } n = 1 \text{ and } i \text{ and } j \text{ are neighbours,} \\ 0 & \text{otherwise, for } i \neq j. \end{cases} \end{aligned}$$

The Grothendieck group  $K_\Gamma$  of  $\mathcal{C}_\Gamma$  is a free  $\mathbf{Z}$  module generated by the classes  $\{[P_i]\}$ .

**2.2. Spherical objects and twists.** The indecomposable projective modules  $P_i$  are *spherical* in the sense of [21, Definition 2.9]. In particular,  $\text{Hom}^*(x, x) \cong \mathbf{k}[t]/t^2$  as a graded  $\mathbf{k}$ -algebra. Any spherical object  $x$  gives rise to an autoequivalence  $\sigma_x: \mathcal{C}_\Gamma \rightarrow \mathcal{C}_\Gamma$  called the

*spherical twist* in  $x$  (see [21]). The twists in the  $P_i$  satisfy the defining relations of the Artin–Tits braid group  $B_\Gamma$  associated to  $\Gamma$ . That is,

$$\begin{aligned}\sigma_{P_i}\sigma_{P_j}\sigma_{P_i} &\cong \sigma_{P_j}\sigma_{P_i}\sigma_{P_j} \text{ when } i \text{ and } j \text{ are connected by an edge of } \Gamma, \\ \sigma_{P_i}\sigma_{P_j} &\cong \sigma_{P_j}\sigma_{P_i} \text{ when } i \text{ and } j \text{ are not connected by an edge of } \Gamma.\end{aligned}$$

Via the homomorphism  $B_\Gamma \rightarrow \text{Aut}(\mathcal{C}_\Gamma)$  defined by  $\sigma_i \mapsto \sigma_{P_i}$ , we have a (weak) action of  $B_\Gamma$  on  $\mathcal{C}_\Gamma$ . This action is known to be faithful in some cases—for example when  $\Gamma$  is an ADE Dynkin diagram [5, 17]—and is conjecturally faithful for all  $\Gamma$ . The induced action on  $K_\Gamma$  is the standard (geometric) representation of the Coxeter group associated to  $\Gamma$ .

**2.3. Stability conditions.** We assume some familiarity with the theory of Bridgeland stability conditions (see [6]), but also recall the basic definitions. A stability condition  $\tau = (\mathcal{P}, Z)$  is specified by two compatible structures:

- (1) the *slicing*  $\mathcal{P}$  consists of additive subcategories  $\mathcal{P}(\phi)$  for each real number  $\phi$ ;
- (2) the *central charge*  $Z$  is a homomorphism of additive groups from the Grothendieck group  $K(\mathcal{C})$  to  $\mathbf{C}$ .

These structures satisfy the following compatibility conditions:

- (1)  $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ ;
- (2) if  $A \in \mathcal{P}(\phi)$  and  $B \in \mathcal{P}(\psi)$  with  $\phi > \psi$ , then  $\text{Hom}(A, B) = 0$ ;
- (3) every nonzero object  $X \in \mathcal{C}$  can be filtered as

$$\begin{array}{ccccccc} 0 = X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \cdots & X_{n-1} \longrightarrow X_n = X, \\ & \nwarrow \text{dashed} & \swarrow & & \nwarrow \text{dashed} & & \swarrow \\ & +1 & & & +1 & & +1 \\ & A_1 & & & A_2 & \cdots & A_n \end{array}$$

where each triangle is distinguished, and  $A_i \in \mathcal{P}(\phi_i)$  with  $\phi_1 > \cdots > \phi_n$ ;

- (4) for every  $A \in \mathcal{P}(\phi)$ , there is some positive real number  $m(A)$  such that

$$Z([A]) = m(A) \cdot e^{i\pi\phi}.$$

Objects in  $\mathcal{P}(\phi)$  are called *semistable of phase*  $\phi$ . The filtration described in the compatibility condition (3) is called the *Harder–Narasimhan (HN) filtration* of  $X$ . It is unique up to isomorphisms (see, e.g. [6, §3]). The objects  $A_i$  appearing in this filtration are called the *HN pieces* of  $X$ . The quantity  $m(A)$  appearing in the compatibility condition (4) is called the *HN mass* of the semistable object  $A$ . For any other object  $X$ , we set the HN mass of  $X$  to be the sum of the HN masses of the HN pieces of  $X$ .

The collection of all stability conditions is denoted  $\text{Stab}(\mathcal{C})$ . For all the categories  $\mathcal{C}$  considered in this paper, the Grothendieck group  $K(\mathcal{C})$  is free of finite rank. Furthermore, it follows from the results of [6] that the forgetful map

$$\text{Stab}(\mathcal{C}) \rightarrow \text{Hom}(K(\mathcal{C}), \mathbf{C})$$

sending  $(Z, \mathcal{P})$  to  $Z$  is a local homeomorphism. It follows that  $\text{Stab}(\mathcal{C})$  has the structure of a complex manifold.

There is an action of the complex numbers  $\mathbf{C}$  on  $\text{Stab}(\mathcal{C})$  lifting the scaling action of  $\mathbf{C}^*$  on  $\text{Hom}(K(\mathcal{C}), \mathbf{C})$ . Explicitly, if  $\tau = (\mathcal{P}, Z)$  and  $\omega = a + i\pi b$ , then

$$(\omega \cdot \mathcal{P})(\phi) = \mathcal{P}(\phi - b) \text{ and } \omega \cdot Z = e^\omega Z.$$

It follows that the induced forgetful map  $\text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \text{Hom}(K(\mathcal{C}), \mathbf{C})/\mathbf{C}^*$  is a local homeomorphism, and hence  $\text{Stab}(\mathcal{C})/\mathbf{C}$  is also a complex manifold.

### 3. HARDER–NARASIMHAN (HN) AUTOMATA

A Harder–Narasimhan automaton (or HN automaton) is a device that allows for an effective computation of HN multiplicities. This is an example of a more general structure, which we call automata, that we now introduce.

Fix a group  $G$ . We think of  $G$  as a category with one object and invertible arrows, and denote this category by  $\underline{G}$ . A  $G$ -automaton is a category  $\Theta$  with a functor  $\sigma: \Theta \rightarrow \underline{G}$ . We refer to objects of  $\Theta$  as *states* and arrows of  $\Theta$  as *transitions*. To ease notation, we often drop  $\sigma$ .

Note that each arrow  $e$  in  $\Theta$  is labelled with an element of  $G$ , namely  $\sigma(e)$ . We can thus represent  $\Theta$  by a  $G$ -labelled graph (we usually show only a subset of arrows that generates the category). Conversely, a  $G$ -labelled graph represents an automaton, whose states are the vertices of the graph, whose transitions are (free) compositions of the edges in the graph, and where the functor to  $\underline{G}$  is defined by the edge labels.

We say that an element  $g$  of  $G$  is *recognised by*  $\Theta$  if  $g = \sigma(e)$  for some arrow  $e$  of  $\Theta$ . Likewise, we say that a writing  $g = g_1 \cdots g_n$  is *recognised by*  $\Theta$  if  $g_i = \sigma(e_i)$  where  $e_1, \dots, e_n$  is a sequence of composable arrows in  $\Theta$ .

Just as a  $G$ -set is a functor  $\underline{G} \rightarrow \mathbf{Set}$ , we define an  $\Theta$ -set to be a functor  $\Theta \rightarrow \mathbf{Set}$ . Likewise, an  $\Theta$ -representation is a functor  $\Theta \rightarrow \mathbf{Ab}$  or  $\Theta \rightarrow \mathbf{Vect}_k$  for some field  $k$ . Morphisms of sets and representations are natural transformations between the respective functors, as expected. Keeping with the usual terminology, we refer to these as  $\Theta$ -equivariant maps.

Note that a  $G$ -set gives a  $\Theta$ -set, obtained by pre-composing with  $\Theta \rightarrow \underline{G}$ . Thus, the idea of an  $\Theta$ -set is weaker than that of a  $G$ -set.

Let  $S$  be a  $\Theta$ -set. Suppose we have a subset  $T(v) \subset S(v)$  for every state  $v$ . The  $\Theta$ -set *generated by*  $T$ , denoted  $\langle T \rangle$ , is the smallest  $\Theta$ -subset of  $S$  such that  $\langle T \rangle(v)$  contains  $T(v)$  for all  $v$ . Explicitly,  $\langle T \rangle(v)$  is the set consisting of  $e(x)$  for all arrows  $e: w \rightarrow v$  and  $x \in T(w)$ .

Let  $\mathcal{C}$  be a triangulated category and  $G$  a group acting on  $\mathcal{C}$  by triangulated auto-equivalences. Let  $\text{ob } \mathcal{C}$  be the set of isomorphism classes of objects of  $\mathcal{C}$ . Then  $\text{ob } \mathcal{C}$  is a  $G$ -set. Fix a stability condition  $\tau$  on  $\mathcal{C}$ . Let  $\Sigma \subset \text{ob } \mathcal{C}$  be the set of indecomposable  $\tau$ -semistable objects. For  $x \in \text{ob } \mathcal{C}$ , let  $HN_\tau(x) \in \mathbf{Z}^\Sigma$  be the  $\tau$ -HN multiplicity vector of  $x$  and  $m_\tau(x)$  the  $\tau$ -mass of  $x$ .

**Definition 3.1.** A  $\tau$ -HN automaton consists of a  $G$ -automaton  $\Theta$  along with the following additional data:

- (1) a sub  $\Theta$ -set  $S \subset \text{ob } \mathcal{C}$ ,
- (2) a  $\Theta$ -representation  $M$  and an  $\Theta$ -equivariant map  $i: S \rightarrow M$ ,
- (3) for every state  $v$ , a linear map  $HN_v: M(v) \rightarrow \mathbf{Z}^\Sigma$ .

The data is required to satisfy the following for every  $s \in S(v)$ :

$$HN_\tau(s) = HN_v(i(s)).$$

A  $\tau$ -mass automaton is analogous, with  $HN_v$  replaced by  $m_v: M(v) \rightarrow \mathbf{R}$  that satisfies

$$m_\tau(s) = m_v(i(s)).$$

*Remark 3.2.* A  $\tau$ -HN automaton yields a  $\tau$ -mass automaton. Indeed, we take  $m_v$  to be the composite of  $HN_v$  and the linear function  $\mathbf{Z}^\Sigma \rightarrow \mathbf{R}$  that sends a semi-stable object to its mass.

Abusing notation, we denote an HN (or mass) automaton by the same name as the underlying automaton. We also leave out  $i: S \rightarrow M$  from the notation, and instead write  $[x]$  for  $i(x)$ . We refer to an element of  $\text{ob } \mathcal{C}$  contained in  $S(v)$  as an object *supported at* the



state  $v$ . Note that an object may be supported at multiple states. We say that an object is *recognised by*  $\Theta$  if it is supported at some state.

HN automata give a partial linearisation of the action of  $G$  on HN multiplicities. To see how, suppose for simplicity that the maps  $HN_v: M(v) \rightarrow \mathbf{Z}^\Sigma$  are injective. Consider a state  $v$  and an arrow  $e: v \rightarrow w$ . We have a linear transformation

$$M(e): M(v) \rightarrow M(w)$$

that agrees with the transformation

$$HN_\tau(s) \mapsto HN_\tau(e \cdot s)$$

for every  $s \in S(v)$ . This linearisation is only partial for two reasons: first, it only applies to objects recognised by the automaton; and second, even for an object recognised by the automaton, say for an object supported at  $v$ , it only applies to the arrows originating from  $v$ .

A useful automaton would recognise a rich class of objects and transformations. In the  $A_2$  and  $\widehat{A}_1$  cases, we construct automata that recognise all braid actions and all spherical objects that are in the braid group orbit of a standard spherical object. It also has additional nice properties—for example, it is “mass degenerating”, as explained in § 4.2. These properties allow us to prove structural results about stability conditions on  $\mathcal{C}$ .

#### 4. THE PROJECTIVE EMBEDDING AND ITS BOUNDARY

For this section, let  $\mathcal{C}$  be the 2-CY category associated to any finite connected quiver, as defined in § 2. Fix  $\mathbf{S}$  to be a set of objects of  $\mathcal{C}$  satisfying the following properties.

- (1) The set  $\mathbf{S}$  contains the objects  $\{P_i \mid i \in \Gamma\}$ .
- (2) The set  $\mathbf{S}$  is closed under the action of the Artin–Tits braid group  $B_\Gamma$ .

Given a stability condition  $\tau$  and an object  $x$  of  $\mathcal{C}$ , the  $\tau$ -mass of  $x$  is a non-negative real number. Acting on  $\tau$  be the natural action of  $\mathbf{C}$  on  $\text{Stab}(\mathcal{C})$  rescales the  $\tau$ -mass of every object  $x$  by the same constant. Therefore we can define a *mass map*

$$m: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}} = \mathbf{P}(\mathbf{R}^{\mathbf{S}}),$$

sending the class of a stability condition  $\tau$  to the projectivised infinite vector with coordinates  $m_\tau(x)$  for each  $x \in \mathbf{S}$ .

The aim of this section is to prove some general properties about the mass map.

**4.1. Injectivity.** We first show that the map  $m: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}}$  is injective. Let  $\heartsuit_{\text{std}}$  be the heart of the standard  $t$ -structure on  $\mathcal{C}$ . Say that a stability condition  $\tau = (Z, P)$  is standard if there is some  $\alpha \in \mathbf{R}$  such that the abelian category  $P([\alpha, \alpha+1))$  is equal to  $\heartsuit_{\text{std}}$ . Note that the property of being standard is invariant under the action of  $\mathbf{C}$  on  $\text{Stab}(\mathcal{C})$ .

Any standard stability condition can be continuously deformed to any other standard stability condition: we can first match up the standard hearts by rotation, and then deform the central charge without changing the standard heart from one stability condition to the other. Let  $\text{Stab}^\circ(\mathcal{C})/\mathbf{C}$  be the connected component of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  containing the standard stability conditions.

**Proposition 4.1** (Injectivity). *Let  $\mathcal{C}$  be the 2CY category associated to a finite connected quiver. Then the mass map  $m: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}}$  is injective on  $\text{Stab}^\circ(\mathcal{C})/\mathbf{C}$ .*

*Proof.* Let  $\tilde{m}: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{R}^{\mathbf{S}} \setminus \{0\}$  be the lift of the map  $m$  characterised by

$$\sum \tilde{m}(P_i) = 1.$$

We show that  $\tilde{m}$  is injective on  $\text{Stab}^\circ(\mathcal{C})/\mathbf{C}$ , which is equivalent to showing that  $m$  is injective.

Let  $\tau$  be a stability condition. By [15, Proposition 4.13],  $\tau$  is in the braid group orbit of a standard stability condition. (Although [15] treats the case of preprojective algebras, the same proof works in our setting.) So by applying a braid if necessary, suppose that  $\tau$  is standard. By applying a phase shift if necessary, suppose that  $P_\tau([0, 1)) = \heartsuit_{\text{std}}$ . In particular, the objects  $\{P_i \mid i \in Q_0\}$  are stable objects in  $P_\tau([0, 1))$ .

Assume that the central charge  $Z_\tau$  is scaled so that for any semistable object  $x$ , we have

$$\tilde{m}_\tau(x) = |Z_\tau(x)|.$$

In fact, the semistable objects of  $\tau$  are characterised by this property.

Let  $\tau$  and  $\tau'$  be two stability conditions such that  $\tilde{m}_\tau = \tilde{m}_{\tau'}$ , with central charges rescaled as above. Let  $x$  be a stable object of  $\tau$ , which means in particular that

$$\tilde{m}_\tau(x) = |Z_\tau(x)|.$$

If  $x$  is not  $\tau'$ -stable, then it has a filtration (a refinement of the Harder–Narasimhan filtration) with stable pieces  $y_1, \dots, y_k$ , such that

$$\tilde{m}_{\tau'}(x) = \sum_{i=1}^k \tilde{m}_{\tau'}(y_i).$$

Since  $\tilde{m}_\tau = \tilde{m}_{\tau'}$ , the same equality holds for  $\tau$ :

$$|Z_\tau(x)| = \tilde{m}_\tau(x) = \sum_{i=1}^k \tilde{m}_\tau(y_i).$$

By the triangle inequality for the vectors  $Z_\tau(x)$  and  $Z_\tau(y_i)$  for each  $i$ , we see that all these vectors lie on the same real ray in the complex plane. A non-trivial equality as above cannot hold if  $x$  is  $\tau$ -stable, because a stable object of phase  $\phi$  is simple in the abelian category  $P_\tau(\phi)$ . Therefore  $x$  is also  $\tau'$ -stable.

We see that  $\tau$  and  $\tau'$  have the same collections of stable objects. We now show that up to the action of  $\mathbf{C}$ , they have identical central charges and slicings.

Let us show inductively that up to a simultaneous  $\mathbf{C}$ -action, we have  $Z_{\tau'}(P_i) = Z_\tau(P_i)$  and  $\phi_{\tau'}(P_i) = \phi_\tau(P_i)$  for each  $i \in Q_0$ . Order  $Q_0$  such that for each  $i > 1$  there is some  $j < i$  such that the sub-quiver  $\{i, j\}$  forms a doubled quiver of type  $A_2$ .

For the base case, by rotating and scaling  $\tau'$  if necessary, assume that  $Z_{\tau'}(P_1) = Z_\tau(P_1)$ , and  $\phi_{\tau'}(P_1) = \phi_\tau(P_1)$ . For the induction step, consider some  $i > 1$ . Fix  $j < i$  such that the sub-quiver  $\{i, j\}$  is a doubled quiver of type  $A_2$ . By the induction hypothesis, we have  $Z_{\tau'}(P_j) = Z_\tau(P_j)$  and  $\phi_{\tau'}(P_j) = \phi_\tau(P_j)$ .

First suppose that  $\phi_\tau(P_i) = \phi_\tau(P_j)$ . In this case, the objects  $P_j \rightarrow P_i$  and  $P_i \rightarrow P_j$  are both  $\tau$ -semistable, and moreover

$$\tilde{m}_\tau(P_j \rightarrow P_i) = \tilde{m}_\tau(P_i \rightarrow P_j) = \tilde{m}_\tau(P_i) + \tilde{m}_\tau(P_j).$$

Note that the same equalities hold for  $\tilde{m}_{\tau'}$ . Let us show that  $\phi_{\tau'}(P_i) = \phi_{\tau'}(P_j)$ . If not, then one of them is smaller than the other; suppose for instance that  $\phi_{\tau'}(P_i) < \phi_{\tau'}(P_j)$ . In this case,  $P_j \rightarrow P_i$  is  $\tau'$ -semistable, and so

$$\tilde{m}_{\tau'}(P_j \rightarrow P_i) = |Z_{\tau'}(P_j \rightarrow P_i)|.$$

By the triangle inequality, we have

$$\tilde{m}_{\tau'}(P_j \rightarrow P_i) < \tilde{m}_{\tau'}(P_i) + \tilde{m}_{\tau'}(P_j),$$

which is a contradiction. So  $\phi_{\tau'}(P_i) = \phi_{\tau'}(P_j)$ , and by the induction hypothesis, we conclude that  $Z_{\tau'}(P_i) = Z_{\tau}(P_i)$ .

Next suppose that  $\phi_{\tau}(P_i) < \phi_{\tau}(P_j)$ ; the other direction is analogous. In this case, the object  $P_j \rightarrow P_i$  is  $\tau$ -stable, and

$$\phi_{\tau}(P_i) < \phi_{\tau}(P_j \rightarrow P_i) < \phi_{\tau}(P_j) < \phi_{\tau}(P_i) + 1.$$

Since the stable objects in  $\tau$  and  $\tau'$  agree,  $P_j \rightarrow P_i$  is also  $\tau'$ -stable. Since we know that

$$\mathrm{Hom}(P_i, P_j \rightarrow P_i) \neq 0, \quad \mathrm{Hom}(P_j \rightarrow P_i, P_j) \neq 0, \quad \text{and} \quad \mathrm{Hom}(P_j, P_i[1]) \neq 0,$$

the same inequalities hold for  $\phi_{\tau'}$ :

$$(2) \quad \phi'_{\tau}(P_i) < \phi'_{\tau}(P_j \rightarrow P_i) < \phi'_{\tau}(P_j) < \phi'_{\tau}(P_i) + 1.$$

We also know that

$$m_{\tau'}(P_i) = m_{\tau}(P_i), \quad m_{\tau'}(P_j \rightarrow P_i) = m_{\tau}(P_j \rightarrow P_i).$$

The three numbers  $m_{\tau'}(P_i)$ ,  $m_{\tau'}(P_j \rightarrow P_i)$ , and  $m_{\tau'}(P_j)$  form the three side lengths of a triangle. Using this together with (2) above, we can uniquely reconstruct the central charge  $Z_{\tau'}(P_i)$ . Therefore we see that  $Z_{\tau'}(P_i) = Z_{\tau}(P_i)$ . The phase  $\phi_{\tau'}(P_i)$  is now uniquely determined up to an integer shift. Once again, (2) fixes this integer shift with respect to  $\phi_{\tau'}(P_j)$ , which is known to equal  $\phi_{\tau}(P_j)$ . Therefore we see that  $\phi_{\tau'}(P_i) = \phi_{\tau}(P_i)$ , and the proof is complete.

The Grothendieck group of  $\mathcal{C}$  is spanned by the classes of  $\{P_i \mid i \in Q_0\}$ . So for any object  $x$  of  $\mathcal{C}$ , we now have  $Z_{\tau'}(x) = Z_{\tau}(x)$ . Consequently,  $\phi_{\tau'}(x)$  is an integer shift of  $\phi_{\tau}(x)$  for each stable  $x$ . It remains to show that  $\phi_{\tau'}(x) = \phi_{\tau}(x)$  for each stable  $x$ .

Let  $x$  be stable in  $\tau$  (equivalently, in  $\tau'$ ). Since  $\tau$  is standard,  $x$  belongs to  $\heartsuit_{\mathrm{std}}[n]$  for some integer  $n$ . Then  $x$  is an iterated extension of  $\{P_i[n] \mid i \in Q_0\}$  in the abelian category  $\heartsuit_{\mathrm{std}}[n]$ . Suppose that we have nonzero maps  $P_i[n] \rightarrow x$  and  $x \rightarrow P_j[n]$  in the abelian category  $\heartsuit_{\mathrm{std}}[n]$ , for some  $i, j \in Q_0$ . These correspond to nonzero maps  $P_i[n] \rightarrow x$  and  $x \rightarrow P_j[n]$  in  $\mathcal{C}$ . We see that

$$n \leq \phi_{\tau'}(P_i[n]) \leq \phi_{\tau'}(x), \quad \phi_{\tau'}(x) \leq \phi_{\tau'}(P_j[n]) < n + 1.$$

These inequalities pin down  $\phi_{\tau'}(x)$  to lie in the interval  $[n, n + 1)$ , and  $\phi_{\tau}(x)$  lies in the same interval. Combined with the fact that  $\phi_{\tau'}(x)$  is an integer shift of  $\phi_{\tau}(x)$ , we have  $\phi_{\tau'}(x) = \phi_{\tau}(x)$ .

We have shown that up to an overall rotation or scaling,  $\tau'$  and  $\tau$  have the same central charges and the same stable objects with the same phases. We conclude that  $\tau = \tau'$  in  $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$ , and this completes the proof of injectivity.  $\square$

**4.2. Homeomorphism when there is an automaton.** Assuming the existence of a suitable HN automaton, we can prove that the mass map  $m: \mathrm{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}}$  is a homeomorphism onto its image. Let  $\tau$  be a stability condition such that there exists a  $\tau$ -mass automaton  $\Theta$  with objects  $S$  and representation  $M$ .

We need to make a further assumption on the automaton, namely that it is a degenerating automaton. We now describe this assumption. Recall that for an object  $x$  supported at a state  $v$ , the symbol  $[x]$  denotes the image of  $x$  in  $M(v)$  under the map  $S \rightarrow M$ .

Let  $x \rightarrow y \rightarrow z \xrightarrow{+1}$  be a distinguished triangle. By [16], we have the triangle inequality

$$m_{\tau}(y) \leq m_{\tau}(x) + m_{\tau}(z).$$

If the inequality is strict, we say that the triangle is *non-degenerate*. Let  $e: v \rightarrow w$  be an arrow of  $\Theta$ . We say that the  $e$  *degenerates* the triangle if:

- (1)  $e \cdot x$ ,  $e \cdot y$ , and  $e \cdot z$  are direct sums of objects supported at  $w$ ,
- (2) we have

$$[e \cdot y] = [e \cdot x] + [e \cdot z].$$

The second condition implies that

$$m_\tau(e \cdot y) = m_\tau(e \cdot x) + m_\tau(e \cdot z).$$

Linearity ensures that this degeneration of the triangle inequality persists: for any arrow  $f: w \rightarrow u$  of  $\Theta$ , we have

$$m_\tau(fe \cdot y) = m_\tau(fe \cdot x) + m_\tau(fe \cdot z).$$

**Definition 4.2.** Suppose  $\Theta$  is a  $\tau$ -mass automaton. We say that  $\Theta$  is *degenerating* if there exists a finite set  $T$  of non-degenerate triangles such that for every  $g \in G$ , either  $\tau$  and  $g^{-1}\tau$  have the same stable objects or  $g = \sigma(e)$  for some arrow  $e$  in  $\Theta$  that degenerates a triangle in  $T$ .

Let  $\mathcal{C}$  be the 2-CY category associated to a connected quiver  $\Gamma$ . Let  $\mathbf{S} \subset \mathcal{C}$  be the set of spherical objects of  $\mathcal{C}$ , and  $\tau$  a standard stability condition.

**Proposition 4.3.** *In the setup above, suppose we have a degenerating  $\tau$ -mass automaton  $\Theta$ . Then the map  $m: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}}$  is a local homeomorphism onto its image at  $\tau$ . That is, there exist open sets  $U \subset \text{Stab}(\mathcal{C})/\mathbf{C}$  containing  $\tau$  and  $V \subset \mathbf{P}^{\mathbf{S}}$  containing  $m(\tau)$  such that*

$$m|_U: U \rightarrow V \cap \text{im}(m)$$

*is a homeomorphism.*

*Proof of Proposition 4.3.* Let  $U \subset \text{Stab}(\mathcal{C})/\mathbf{C}$  be the set of stability conditions with the same stable objects as  $\tau$ . As in the proof of injectivity (Proposition 4.1), order the vertices of  $\Gamma$  so that for every  $i > 1$ , there exists  $j < i$  such that the sub-quiver  $\{i, j\}$  forms a doubled quiver of type  $A_2$ . Let  $S \subset \mathbf{S}$  be the finite set

$$S = \{P_i \mid i \in Q_0\} \cup \{P_j \rightarrow P_i \mid \{i, j\} \cong A_2\}.$$

Consider the restricted mass map  $m: U \rightarrow \mathbf{P}^S$ . In the proof of injectivity, we re-constructed  $\tau \in U$  from  $m(\tau) \in \mathbf{P}^S$ . Observe that the inverse map  $m(\tau) \mapsto \tau$  is continuous. In other words,  $U \rightarrow \mathbf{P}^S$  is a homeomorphism onto its image, and hence, so is  $U \rightarrow \mathbf{P}^{\mathbf{S}}$ .

We have seen that the map

$$m: U \rightarrow m(U)$$

is a homeomorphism. To conclude the proposition, it remains to show that there exists an open subset  $V \subset \mathbf{P}^{\mathbf{S}}$  containing  $m(\tau)$  such that

$$V \cap \text{im}(m) \subset m(U).$$

To construct  $V$ , we use that  $\Theta$  is mass degenerating. Let  $T$  be a finite set of non-degenerate triangles as required for this property. Let  $V \subset \mathbf{P}^{\mathbf{S}}$  be defined by the inequalities

$$m(y) < m(x) + m(z)$$

for a triangle  $x \rightarrow y \rightarrow z \xrightarrow{+1}$  in  $T$ . Then  $V \cap \text{im}(m)$  is contained in  $m(U)$ , as desired.  $\square$

**4.3. Pre-compactness.** Recall that  $\mathcal{C}$  be the 2-CY category associated to a connected quiver  $Q$ , and  $\mathbf{S} \subset \mathcal{C}$  is the set of spherical objects of  $\mathcal{C}$ , and we have the projectivised mass map

$$m: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}}.$$

**Proposition 4.4** (Pre-compactness). *In the setup above, the closure of the image of  $m$  is compact.*

*Proof.* Let  $\tilde{m}: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{R}^{\mathbf{S}}$  be the lift of  $m$  characterised by

$$\sum \tilde{m}(P_i) = 1.$$

Let  $\tilde{B}$  be the closure in  $\mathbf{R}^{\mathbf{S}}$  of the image of  $\tilde{m}$ .

Recall that for an exact triangle

$$x \rightarrow y \rightarrow z \xrightarrow{+1},$$

we have the following triangle inequality [16, Proposition 3.3]:

$$\tilde{m}_\tau(y) \leq \tilde{m}_\tau(x) + \tilde{m}_\tau(z).$$

For every  $s \in \mathbf{S}$ , there exists an  $n = n(s)$  and a filtration

$$0 = s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_n = s,$$

where the sub-quotients are twists of  $P_i$ . By the triangle inequality, and the normalisation  $\sum \tilde{m}(P_i) = 1$ , we have

$$\tilde{m}_\tau(s) \leq n = n(s).$$

In other words,  $\tilde{m}$  maps  $\text{Stab}(\mathcal{C})/\mathbf{C}$  to the product  $\prod_{s \in \mathbf{S}} [0, n(s)]$ . By the Tychonoff theorem, the product is compact, and hence  $\tilde{B}$  is compact. Thanks to the normalisation  $\sum \tilde{m}(P_i) = 1$ , we see that  $\tilde{B}$  is contained in  $\mathbf{R}^{\mathbf{S}} \setminus 0$ . Finally, note that the closure of the image of  $m$  in  $\mathbf{P}^{\mathbf{S}}$  is contained in the image of  $\tilde{B}$  under the projection map, which is a compact set. Hence, the closure of the image of  $m$  is also compact.  $\square$

**4.4. Hom functionals are in the boundary.** Let  $a$  be a spherical object. The goal of this section is to prove that the projectivized functional  $\overline{\text{hom}}(a)$  lies in the closure of the image of  $\text{Stab}(\mathcal{C})$ . The key ingredient is the following.

**Proposition 4.5.** *Let  $\tau$  be a stability condition in which  $a$  is stable. For every  $x \in \mathbf{S}$ , we have*

$$\lim_{n \rightarrow \infty} \frac{m_\tau(\sigma_a^n x)}{n} = m(a) \cdot \overline{\text{hom}}(a, x).$$

**Corollary 4.6.** *In  $\mathbf{P}^{\mathbf{S}}$  we have*

$$\lim_{n \rightarrow \infty} m(\sigma_a^{-n} \tau) = \overline{\text{hom}}(a).$$

*In particular, the projectivized functional  $\overline{\text{hom}}(a)$  is in the closure of the image of  $\text{Stab}(\mathcal{C})$ .*

For the proof of Proposition 4.5, we need some preparatory lemmas.

**Lemma 4.7.** *Let  $x \in \mathcal{C}$  be any object that does not contain any shift of  $a$  as a direct summand. Consider the filtration*

$$x \rightarrow \sigma_a x \rightarrow \sigma_a^2 x$$

*with subquotients  $\text{hom}(a, x) \otimes a[1]$  and  $\text{hom}(a, \sigma_a x) \otimes a[1]$ . Then the connecting map*

$$\text{hom}(a, \sigma_a x) \otimes a[1] = \text{hom}(a, x) \otimes a \rightarrow \text{hom}(a, x) \otimes a[2]$$

*is equal to  $\text{id} \otimes \text{loop}_a$ .*

*Proof.* Let  $x$  be any object. The connecting map

$$(3) \quad \text{hom}(a, x) \otimes a \rightarrow \text{hom}(a, x) \otimes a[2]$$

is adjoint to a map

$$(4) \quad \text{hom}(a, x) \otimes \text{hom}(a, x)^* \otimes a \rightarrow a[2].$$

By the 2-CY property, we have  $\text{hom}(a, x)^* = \text{hom}(x, a)[2]$ , and thus the map in (4) is equivalent to a map

$$(5) \quad \text{hom}(a, x) \otimes \text{hom}(x, a) \otimes a \rightarrow a.$$

We leave it to the reader to check that the map (5) is the composite

$$\text{hom}(a, x) \otimes \text{hom}(x, a) \otimes a \rightarrow \text{hom}(a, a) \otimes a \rightarrow a,$$

where the first map is the composition of the first two factors and the second map is evaluation.

Now suppose that  $x$  does not have shifts of  $a$  as summands. Then, it is easy to check that the image of the composition map

$$\text{hom}(a, x) \otimes \text{hom}(x, a) \rightarrow \text{hom}(a, a)$$

is the one-dimensional subspace spanned by  $\text{loop}_a$ , and that

$$\text{hom}(a, x) \otimes \text{hom}(x, a) \rightarrow \langle \text{loop}_a \rangle$$

is a perfect pairing. By undoing the adjunctions, we conclude that (3) is  $\text{id} \otimes \text{loop}_a$ .  $\square$

**Lemma 4.8.** *Let  $x \in \mathcal{C}$  be any object that does not contain any shift of  $a$  as a direct summand. There exists  $N$  (depending on  $x$  and  $a$ ) such that for every  $n \geq N$ , the filtration*

$$0 \rightarrow \sigma_a^n x \rightarrow \sigma_a^{n+1} x$$

*is geodesic.*

*Proof.* We need to show that the connecting map

$$\text{hom}(a, \sigma_a^n x) \otimes a[1] \rightarrow \sigma_a^n x[1]$$

is rectifiable. After rewriting  $\text{hom}(a, \sigma_a^n x)$  as  $\text{hom}(a, x)[-n]$  and eliminating the shift by  $[1]$ , the map becomes

$$\chi: \text{hom}(a, x) \otimes a[-n] \rightarrow \sigma_a^n x.$$

Define  $\ell_n$  by the exact triangle

$$x \rightarrow \sigma_a^n x \rightarrow \ell_n \xrightarrow{+1}.$$

Thanks to repeated applications of Lemma 4.7, it is easy to see that we have

$$\ell_n = \text{hom}(a, x) \otimes a_n,$$

where  $a_n$  is an iterated extension of twists of  $a$  by loop maps. More precisely, the objects  $a_n$  are defined inductively as follows. Set  $a_1 = a[1]$  and observe that we have the map  $a[-1] \rightarrow$

$a[1]$  given by  $\text{loop}_a$ . Assuming we have defined  $a_{n-1}$  along with a map  $a[-n+1] \rightarrow a_{n-1}$ , define  $a_n$  by the triangle

$$a[-n+1] \rightarrow a_{n-1} \rightarrow a_n \xrightarrow{+1}.$$

To continue the inductive definition, we need to provide a map  $a[-n] \rightarrow a_n$ . Consider the loop map  $a[-n] \rightarrow a[-n+2]$ . It is easy to check that the composite  $a[-n] \rightarrow a[-n+2] \rightarrow a_{n-1}[1]$  vanishes, and thus yields a map  $a[-n] \rightarrow a_n$ .

By definition, we have the filtration

$$a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n \rightarrow a_{n+1}$$

with subquotients  $a, a[-1], \dots, a[-n+1]$ . This filtration is non-overlapping, and hence geodesic. As a result, the filtration

$$(6) \quad \ell_1 \rightarrow \ell_2 \rightarrow \cdots \rightarrow \ell_n \rightarrow \ell_{n+1},$$

obtained by applying  $\text{hom}(a, x) \otimes -$  to the previous filtration, is also geodesic.

Let  $\alpha$  be a real number. We need to show that the map

$$(7) \quad (\text{hom}(a, x) \otimes a[-n])_{\geq \alpha} \rightarrow (\sigma_a^n x)_{< \alpha+1}$$

induced by  $\chi$  is zero. By shifting  $x$  if necessary, assume that all the HN factors of  $x$  have phase greater than 0. Let  $N$  be such that for any  $n \geq N$ , all the HN factors of  $\text{hom}(a, x) \otimes a[-n]$  have phase less than  $-1$ . As a result, if  $\alpha \geq -1$ , then  $(\text{hom}(a, x) \otimes a[-n])_{\geq \alpha} = 0$  and hence the map in (7) is zero. If  $\alpha < -1$ , then  $x_{< \alpha+1} = 0$ , and the triangle

$$x \rightarrow \sigma_a^n x \rightarrow \ell_n$$

induces an isomorphism

$$(8) \quad (\sigma_a^n x)_{< \alpha+1} \xrightarrow{\sim} (\ell_n)_{< \alpha+1}.$$

The map  $(\text{hom}(a, x) \otimes a[-n])_{\geq \alpha} \rightarrow (\ell_n)_{< \alpha+1}$  is zero since the filtration (6) is geodesic. In view of the isomorphism (8), we get that the map in (7) is also zero.  $\square$

*Proof of Proposition 4.5.* If  $x$  is a shift of  $a$ , then both sides are zero. Assume that  $x$  is not a shift of  $a$ . Since  $x$  is indecomposable, it does not contain any shift of  $a$  as a summand. Let  $N$  be as guaranteed by Lemma 4.8. Then, for every  $n \geq N$ , we have

$$m_\tau(\sigma_a^n x) = (n - N)m(a) \text{hom}(a, x) + m_\tau(\sigma_a^N x).$$

Dividing by  $n$  and taking the limit as  $n \rightarrow \infty$  completes the proof.  $\square$

*Remark 4.9* (Degenerate stability conditions). Proposition 4.5 describes one way to approach the boundary of the stability manifold in  $\mathbf{P}^{\mathbf{S}}$ , namely by applying powers of a spherical twist. There are more direct ways to approach the boundary. Start with a stability condition  $\tau$  with heart  $\heartsuit$  and central charge  $Z: K(\mathcal{C}) \rightarrow \mathbf{C}$ . Recall that  $Z$  must map non-zero objects of  $\heartsuit$  to non-zero complex numbers. Take a deformation  $Z_t$  of  $Z$  such that  $Z_0$  sends some objects of  $\heartsuit$  to 0. Then the limit of the stability condition described by  $(\heartsuit, Z_t)$  is not a stability condition; it must be on the boundary. We expect such limits to be degenerate stability conditions in the sense of [3]. We expect these to form a measure zero, but dense subset of the boundary.

In the analogy with the Teichmüller space, applying a spherical twist corresponds to applying a Dehn twist. On the other hand, degenerating  $Z$  corresponds to contracting a (collection of) curve(s). In geometry, both operations yield the same limit, in contrast with the categorical analogues.

5. THE  $A_2$  CASE

The aim of this section is to construct the Thurston compactification of stability space for 2-Calabi–Yau category  $\mathcal{C}_\Gamma$  where  $\Gamma$  is the  $A_2$  quiver.

**5.1. Overview.** Let  $\mathcal{C}$  be the 2-Calabi–Yau category associated to the  $A_2$  quiver, as defined in § 2. It is a graded,  $\mathbf{k}$ -linear triangulated category classically generated by two spherical objects, which we denote  $P_1$  and  $P_2$ .

The extension closure of  $P_1$  and  $P_2$  in  $\mathcal{C}$  is an abelian category; it is the heart of a (bounded)  $t$ -structure. We refer to it as the *standard heart*  $\heartsuit_{\text{std}}$ . It has two simple objects,  $P_1$  and  $P_2$ , and two additional indecomposable objects, denoted by  $P_1 \rightarrow P_2$  and  $P_2 \rightarrow P_1$ . The object  $P_i \rightarrow P_j$  is the unique extension of  $P_i$  by  $P_j$ . In terms of complexes, it is the complex

$$P_i \rightarrow P_j = Ae_i\langle -1 \rangle \xrightarrow{f_{ij}} Ae_j, \quad \text{in homological degrees } -1 \text{ and } 0$$

where  $f_{ij}$  is right multiplication by the path  $i \rightarrow j$ .

**5.2. The spherical objects and the boundary circle.** As discussed in § 2.2, the spherical twists in the objects  $P_i$  satisfy the defining relations of the Artin–Tits braid group of the  $A_2$  quiver. In this case, this group is the three-strand braid group  $B_3$ :

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

Denoting the twist in  $P_i$  as  $\sigma_{P_i}$ , we have the relation  $\sigma_{P_1}\sigma_{P_2}\sigma_{P_1} \cong \sigma_{P_2}\sigma_{P_1}\sigma_{P_2}$ . The weak action of  $B_3$  on  $\mathcal{C}$  via the homomorphism

$$\sigma_i \mapsto \sigma_{P_i}$$

is faithful [21].

Let  $\mathbf{S}$  be the set of spherical objects of  $\mathcal{C}$ , up to shifts. It turns out that  $B_3$  acts transitively on the set of all the spherical objects of  $\mathcal{C}$ , and hence on  $\mathbf{S}$  [2].

The centre of  $B_3$  is generated by  $(\sigma_2\sigma_1)^3$ . The central element  $(\sigma_2\sigma_1)^3$  acts by a triangulated shift, precisely by  $x \mapsto x[-2]$ . The action of  $B_3$  on  $\mathbf{S}$  therefore descends to an action of  $B_3/Z(B_3)$  on  $\mathbf{S}$ .

We have an isomorphism  $B_3/Z(B_3) \rightarrow \text{PSL}_2(\mathbf{Z})$  given by

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The stabiliser of an element  $x \in \mathbf{S}$  is the subgroup of  $\text{PSL}_2(\mathbf{Z})$  generated by the image of  $\sigma_x$ . In particular, the stabiliser of  $P_1$  is generated by  $\sigma_1$ . As a result, we have a  $\text{PSL}_2(\mathbf{Z})$ -equivariant bijection

$$(9) \quad i: \mathbf{S} \rightarrow \mathbf{P}^1(\mathbf{Z})$$

defined uniquely by the choice

$$P_1 \mapsto [1 : 0].$$

Under this bijection, we can calculate the pre-image in  $\mathbf{S}$  of a point  $[a : c] \in \mathbf{P}^1(\mathbf{Z})$  as follows. Assume  $c \neq 0$ . Write the rational number  $a/c$  as a continued fraction with an odd number of terms:

$$\frac{a}{c} = n_0 + \frac{1}{n_2 + \frac{1}{\ddots + \frac{1}{n_{2k}}}}.$$



Here, each  $n_i$  is an integer, with  $n_i > 0$  for  $i = 1, \dots, 2k$ . Then  $[a : c]$  corresponds to the object of  $\mathbf{S}$  given by

$$(10) \quad \sigma_1^{n_0} \sigma_2^{-n_1} \cdots \sigma_1^{n_{2k}}(P_2).$$

For example, we get

$$P_1 \mapsto [1 : 0] \mapsto P_1, \quad P_2 \mapsto [0 : 1], \quad (P_2 \rightarrow P_1) \mapsto [1 : -1], \text{ and } (P_1 \rightarrow P_2) \mapsto [1 : 1].$$

**5.3. Standard stability conditions.** We say that a stability condition  $\tau$  is *standard* if its  $[0, 1)$  heart is the standard heart  $\heartsuit_{\text{std}}$ . More generally, we say that an element of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  is *standard* if one of its representatives is standard. For a standard  $\tau$ , both  $P_1$  and  $P_2$  must be stable. Let  $\phi$  denote the phase function for  $\tau$ . Then we either have  $\phi(P_1) \leq \phi(P_2)$  or  $\phi(P_2) \leq \phi(P_1)$ . In the first case, we say  $\tau$  is standard of *type A*, and in the latter case, of *type B*. In type A (resp. type B), the object  $X = (P_2 \rightarrow P_1)$  (resp.  $X' = (P_1 \rightarrow P_2)$ ) is (semi)-stable (stable if the inequality is strict). If the inequality is strict, we say that  $\tau$  is *non-degenerate*. More generally, a stability condition is *non-degenerate* if it does not have strictly semi-stable objects. See Figure 2 for a sketch of a standard, type A, non-degenerate  $\tau$ .

By [7, Proposition 4.2], every stability condition can be written as  $\beta\tau$ , where  $\beta \in B_3$  and  $\tau$  is a standard stability condition of type A.

The set of standard, type A stability conditions in  $\text{Stab}(\mathcal{C})/\mathbf{C}$  is a locally closed subset. Let  $\Lambda$  be its closure. This includes stability conditions where  $P_1$  and  $P_2$  are stable, and  $\phi(P_2) = \phi(P_1) + 1$ . We continue to call the stability conditions in  $\Lambda$  as of *type A*, reserving the adjective standard for those where  $\phi(P_2) < \phi(P_1) + 1$ . Note that the non-standard type A stability conditions are degenerate.

The set  $\Lambda$  tessellates  $\text{Stab}(\mathcal{C})/\mathbf{C}$  under the action of  $B_3$ . More precisely,  $\Lambda$  satisfies the following properties (see, e.g., [7, Proposition 4.2]).

- (1) Each point of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  lies in the  $B_3$ -orbit of a point of  $\Lambda$ .
- (2) The stabiliser of  $\Lambda$  is the subgroup generated by  $\gamma = \sigma_2\sigma_1$  in  $B_3$ .
- (3) For any  $g \in B_3$  not in the stabiliser, the interiors of  $\Lambda$  and  $g\Lambda$  have empty intersection.

See Figure 6 for a picture of the tessellation of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  by the orbit of  $\Lambda$ . The interior of  $\Lambda$  is the set of non-degenerate standard stability conditions of type A.

**5.4. The automaton.** Fix a non-degenerate standard stability condition  $\tau$  of type A. Define the braid  $\gamma \in B_3$  by

$$\gamma = \sigma_2\sigma_1 = \sigma_x\sigma_2 = \sigma_1\sigma_x.$$

Figure 3 shows a  $\tau$ -HN automaton  $\Theta$  that computes HN multiplicities up to shift. Formally, the automaton  $\Theta$  is defined by the  $G$ -labelled graph with three vertices and three edges from each vertex, as shown in Figure 3. The three states support the objects  $\{X, P_1\}$ ,  $\{P_1, P_2\}$ , and  $\{P_2, X\}$ , and the  $\Theta$ -set  $S \subset \text{ob } \mathcal{C}$  is the one generated by this data. The representation  $M$  associates the free abelian group  $\mathbf{Z}^{\{A, B\}}$  to the state labelled  $[A, B]$ . The linear maps  $M(e)$  are given by  $2 \times 2$  matrices in the standard basis, as shown on  $e$  in Figure 3. (The arrows labelled  $\gamma^\pm$  correspond to the identity matrix, which is not shown.) The map  $S \rightarrow M$  is the unique  $\Theta$ -equivariant map that sends the objects  $\{A, B\}$  supported at the state  $[A, B]$  to the standard basis vectors. Note that  $\Sigma = \{P_1, P_2, X\}$  is the set of indecomposable semistable (= stable) objects of  $\tau$  (up to shift). For  $v = [A, B]$ , the map  $HN_v : \mathbf{Z}^{\{A, B\}} \rightarrow \mathbf{Z}^\Sigma$  is simply the inclusion induced by the inclusion  $\{A, B\} \rightarrow \Sigma$ . It is not

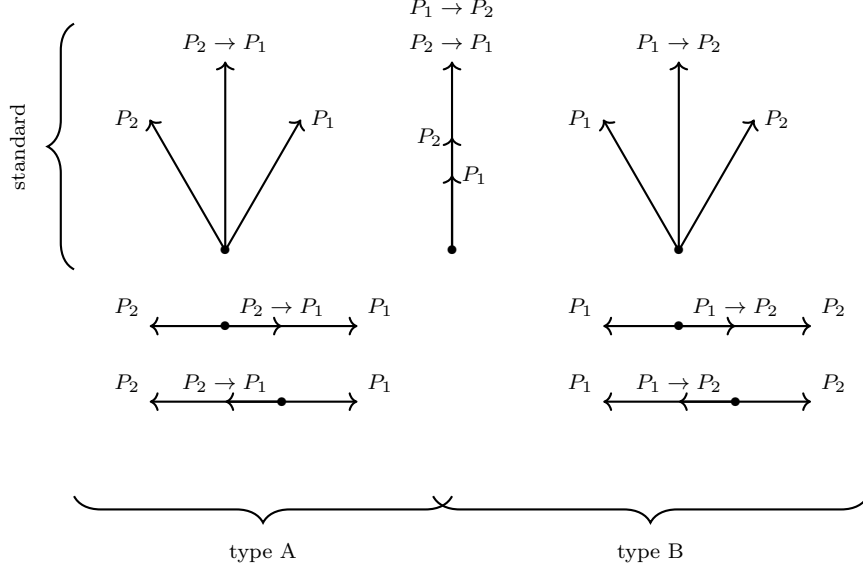


FIGURE 2. Central charges in a stability condition of type A and B

*a priori* clear why  $S \rightarrow M$  is well-defined and why  $\Theta$  computes  $\tau$ -HN multiplicities (up to shift). Both are consequences of Proposition 5.1.

**Proposition 5.1.** *Let  $e: v \rightarrow w$  be an arrow of  $\Theta$ , where  $v = [A, B]$  and  $w = [C, D]$ . Let  $x \in \mathcal{C}$  be any object whose HN filtration consists of shifts of  $A$  and  $B$ . The following hold.*

- (1) *The HN filtration of  $e \cdot x$  consists of shifts of  $C$  and  $D$ .*
- (2) *Consider  $HN_\tau(x) \in \mathbf{Z}^{\{A, B\}}$  and  $HN_\tau(e \cdot x) \in \mathbf{Z}^{\{C, D\}}$ . Then*

$$HN_\tau(e \cdot x) = M(e) \cdot HN_\tau(x).$$

*Proof.* It suffices to treat the case where  $e$  is an arrow shown in Figure 3; the rest follows by composition. Let

$$0 \rightarrow x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = x$$

be an HN-filtration of  $x$  with stable factors  $z_i = \text{Cone}(x_{i-1} \rightarrow x_i)$ . By applying  $\sigma = \sigma(e)$ , we get a filtration

$$(11) \quad 0 \rightarrow \sigma(x_0) \rightarrow \sigma(x_1) \rightarrow \cdots \rightarrow \sigma(x_n) = \sigma(x)$$

with factors  $\sigma(z_i)$ .

We check that the filtration (11) is geodesic in the sense of Definition A.6. Suppose  $\sigma = \gamma$ . Recall that  $\gamma$  simply permutes the three stable objects and preserves the ordering of the phases. So, (11) is already an HN filtration (and hence obviously geodesic, see Remark A.7). For the others, we use Proposition A.14.

We outline the case  $v = [P_1, P_2]$ , leaving the others to the reader. To apply Proposition A.14 to the filtration (11), we must check the following: for  $i < j$ , either  $\text{Hom}(\sigma z_j, \sigma z_i[1]) = 0$  or  $\lfloor \sigma z_i \rfloor \geq \lceil \sigma z_j \rceil$ . By assumption,  $z_i$  and  $z_j$  are shifts of  $P_1$  or  $P_2$ , and since  $i < j$ , we have  $\phi(z_i) > \phi(z_j)$ . The only pairs  $(z_i, z_j)$  with  $\text{Hom}(z_j, z_i[1]) \neq 0$  are  $(P_1[1], P_1)$ ,  $(P_2[1], P_2)$ , and  $(P_2, P_1)$ . The two possible  $\sigma$  at the vertex  $v$  are  $\sigma = \sigma_1$  and  $\sigma = \sigma_X$ . We must now

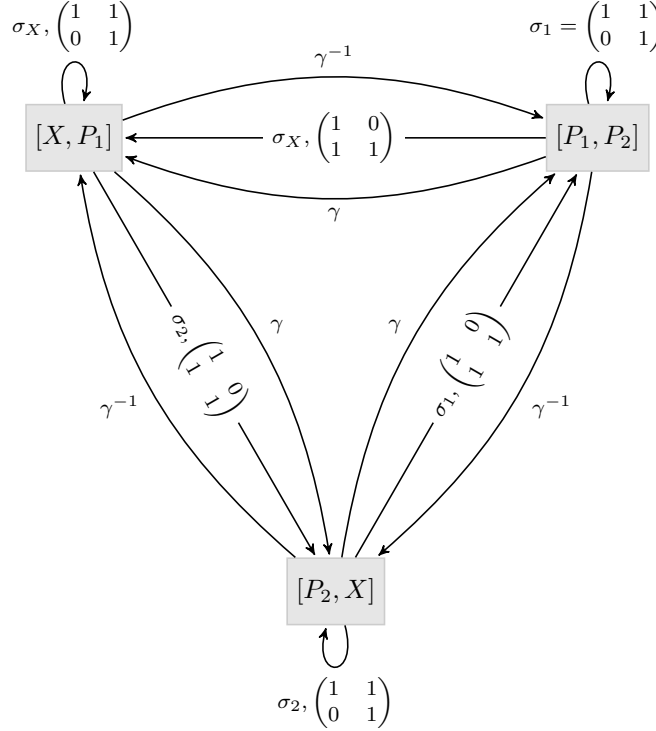


FIGURE 3. An automaton describing the dynamics of Harder–Narasimhan filtrations in a stability condition with stable objects  $P_1$ ,  $P_2$ , and  $X = P_2 \rightarrow P_1$ .

check that  $\lfloor \sigma z_i \rfloor \geq \lceil \sigma z_j \rceil$  holds for six possibilities: 2 for  $\sigma$  and 3 for the pairs  $(z_i, z_j)$ . This is straightforward, so we leave it out.

Since the filtration (11) is geodesic, an HN filtration of  $\sigma(x)$  is simply a rearrangement of the HN filtrations of  $\sigma(z_i)$ . In particular, the HN multiplicity vector of  $\sigma(x)$  is the sum of those of  $\sigma(z_i)$ . As a result, it suffices to prove both parts of the proposition for  $x = P_1$  and  $x = P_2$ . This is another straightforward calculation.  $\square$

The next proposition shows that the automaton recognises every braid and every object of  $\mathbf{S}$ .

**Proposition 5.2.** *Let  $\beta \in B_3$  and let  $s \in \mathbf{S}$  be arbitrary.*

- (1) *The automaton in Figure 3 recognises  $\beta$ . More precisely,  $\beta$  has a recognised expression of the form*

$$\beta = \gamma^n \sigma_{a_1}^{m_1} \sigma_{a_2}^{m_2} \cdots \sigma_{a_k}^{m_k},$$

*where  $n$  is an integer,  $k$  is a non-negative integer, the  $m_i$  are positive integers, and the sequence  $(a_1, a_2, \dots, a_k)$  is a contiguous subsequence of the sequence  $(\dots, X, 1, 2, X, 1, 2, \dots)$ .*

- (2) *The automaton in Figure 3 recognises  $s$ .*  
 (3) *The  $\tau$ -HN filtration of  $s$  consists of at most two objects out of  $\{P_1, P_2, X\}$ .*

We call the writing of  $\beta$  described in Proposition 5.1 an *admissible cyclic writing*.

*Proof.* For (1), we repeatedly use the commutation relations

$$\gamma\sigma_2\gamma^{-1} = \sigma_1, \quad \gamma\sigma_X\gamma^{-1} = \sigma_2, \quad \gamma\sigma_1\gamma^{-1} = \sigma_X.$$

Begin by writing  $\beta$  as any product of the generators  $\sigma_1$  and  $\sigma_2$ , along with their inverses. Eliminate the inverses of the generators by rewriting as follows:

$$\sigma_1^{-1} = \sigma_X\gamma^{-1}, \quad \sigma_2^{-1} = \sigma_1\gamma^{-1}.$$

Next, use the commutation relations to rewrite

$$\gamma^i\sigma_X = \sigma_2\gamma^i, \quad \gamma^i\sigma_1 = \sigma_X\gamma^i, \quad \gamma^i\sigma_2 = \sigma_1\gamma^i,$$

and thus move instances of  $\gamma^{-1}$  to the left as a single power of  $\gamma^{-1}$ . The rest of  $\beta$  is now a product of elements from  $\{\sigma_1, \sigma_2, \sigma_X\}$ .

Finally, replace any occurrences of  $\sigma_2\sigma_1$ ,  $\sigma_1\sigma_X$ , or  $\sigma_X\sigma_2$  by  $\gamma$  and move  $\gamma$  to the left, again using the commutation relations. Each such operation decreases the length of the braid in the elements  $\{\sigma_1, \sigma_2, \sigma_X, \gamma^\pm\}$ . Therefore, the procedure terminates and  $\beta$  reaches the desired form.

For (2), write  $s$  as  $s = \beta P_1$ , and then write

$$\beta = \gamma^n \sigma_{a_1}^{m_1} \sigma_{a_2}^{m_2} \cdots \sigma_{a_k}^{m_k}$$

as in (1). Note that  $P_1$  is supported at two states, namely  $[P_1, P_2]$  and  $[X, P_1]$ . Each of the letters  $\{\sigma_1, \sigma_2, \sigma_X\}$  is applicable at least at one of the two states above, so we can apply  $\sigma_{a_k}$ . The condition on our cyclic writing guarantees that we can apply each subsequent letter. Therefore,  $\beta P_1$  is recognised by the automaton.

Finally, (3) follows from (2) and Proposition 5.1.  $\square$

*Remark 5.3.* Let  $\tau$  be a degenerate stability condition of type A. Then the automaton in Figure 3 does not compute  $\tau$ -HN-multiplicities. Indeed, a degenerate  $\tau$  has additional semi-stable objects than those in  $\Sigma = \{P_1, P_2, X\}$ , which are not accounted for by the automaton. Nevertheless, it *does* compute  $\tau$ -masses, with the mass functions  $m_v: \mathbf{Z}^\Sigma \rightarrow \mathbf{R}$  defined as usual by

$$m_v: s \mapsto m_\tau(s) \text{ for } s \in \Sigma.$$

*Remark 5.4.* Let  $\tau$  be a degenerate standard stability condition of type A. Recall that  $\gamma = \sigma_2\sigma_1$ . Let

$$\gamma' = \sigma_1\sigma_2, \quad w_0 = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.$$

Figure 4 describes an automaton that computes  $\tau$ -HN multiplicities, continuing to recognises all braids and spherical objects. Since the automaton in Figure 3 suffices for mass computations, we omit the details.

*Remark 5.5.* For every non-degenerate stability condition, we have an HN automaton obtained by applying an appropriate braid to the automaton in (3). Likewise, for a degenerate stability condition, we have an HN automaton obtained by applying a braid to the automaton in (4). Thus, we have an automaton at every point of the stability space. Observe that these automata exhibit a wall-and-chamber behaviour: they remain constant in a chamber, get more complicated along a wall, and flip to a different automaton as we cross the wall.

We observe that the automaton in Figure 3 is mass degenerating.

**Proposition 5.6.** *Let  $\tau$  be a standard stability condition of type A, possibly degenerate. The automaton  $\Theta$  is mass degenerating for  $\tau$ .*

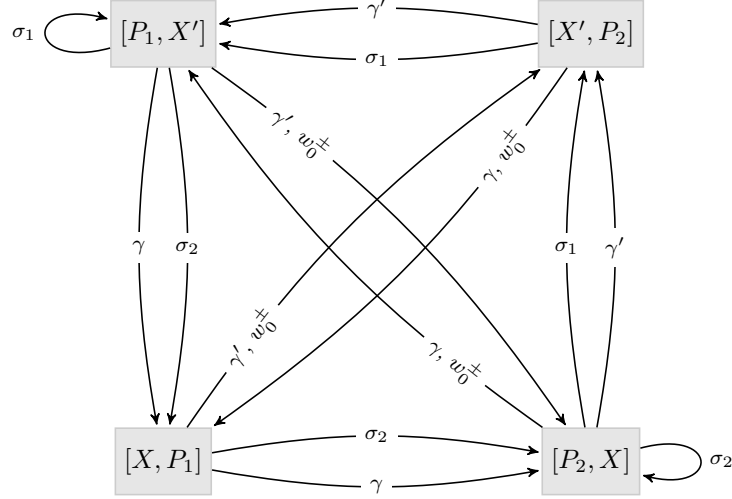


FIGURE 4. An automaton describing the dynamics of HN filtrations in a degenerate stability condition with semistable objects  $P_1$ ,  $P_2$ ,  $X = P_2 \rightarrow P_1$ , and  $X' = P_1 \rightarrow P_2$ .

*Proof.* By Proposition 5.1, every braid has a writing recognised by  $\Theta$  that ends with either  $\sigma_1$ ,  $\sigma_2$ , or  $\sigma_X$ . Therefore, it suffices to find non-degenerate triangles that are made degenerate by each of these letters.

Suppose  $\tau$  is non-degenerate; that is  $\phi(P_1) < \phi(P_2)$ . The non-degenerate triangle

$$P_1 \rightarrow X \rightarrow P_2 \xrightarrow{+1}$$

is made degenerate by  $\sigma_2$ . The non-degenerate triangle

$$X \rightarrow P_2 \rightarrow P_1[1] \xrightarrow{+1}$$

is made degenerate by  $\sigma_1$ . And the non-degenerate triangle

$$P_2[-1] \rightarrow P_1 \rightarrow X \xrightarrow{+1}$$

is made degenerate by  $\sigma_X$ .

If  $\tau$  is degenerate, that is  $\phi(P_1) = \phi(P_2)$ , then  $m_\tau(X) = m_\tau(P_1) + m_\tau(P_2)$ . In this case the first triangle is already degenerate, and must be replaced. Set  $X' = \sigma_1(P_2) = (P_1 \rightarrow P_2)$ . Then the non-degenerate triangle

$$X' \rightarrow P_1 \rightarrow P_2[1]$$

is made degenerate by  $\sigma_2$ . □

**5.5. Consequences of the automaton.** In this section, we use the automaton from § 5.4 to prove results about our projective embedding of  $\text{Stab}(\mathcal{C})/\mathbf{C}$ .

**Proposition 5.7.** *The map  $m: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}}$  is a homeomorphism onto its image.*

*Proof.* We have already shown in Proposition 4.3 that the map  $m$  is a local homeomorphism onto its image at  $\tau$ , for any standard stability condition  $\tau$  of type A. Since every stability condition is in the  $B_3$  orbit of a standard stability condition of type A, and the map  $m$  is  $B_3$ -equivariant, we see that  $m$  is a local homeomorphism onto its image at every point

$\tau \in \text{Stab}(\mathcal{C})/\mathbf{C}$ . By Proposition 4.1, the map  $m$  is also injective. This proves that  $m$  is a homeomorphism onto its image.  $\square$

Recall that we have a surjective map  $B_3 \rightarrow \text{PSL}_2(\mathbf{Z})$ . Via this map, the  $B_3$ -automaton  $\Theta$  becomes a  $\text{PSL}_2(\mathbf{Z})$ -automaton. Consider  $\mathbf{P}^1(\mathbf{Z})$  as a  $\text{PSL}_2(\mathbf{Z})$ -set in the standard way, and hence also as a  $\Theta$ -set. Let  $\mathbf{P}M$  be the projectivisation of the representation  $M$ . Then  $\mathbf{P}M$  is also a  $\Theta$ -set.

**Proposition 5.8.** *We have an isomorphism of  $\Theta$ -sets  $\phi: \mathbf{P}M \rightarrow \mathbf{P}^1(\mathbf{Z})$ .*

*Proof.* The isomorphism  $\phi$  is induced by linear maps  $M(v) \rightarrow \mathbf{Z}^2$  defined at each state as follows

$$\begin{aligned} \phi_{[P_1, P_2]} : P_1 &\mapsto (1, 0) \text{ and } P_2 \mapsto (0, 1), \\ \phi_{[P_2, X]} : P_2 &\mapsto (0, -1) \text{ and } X \mapsto (1, -1), \\ \phi_{[X, P_1]} : X &\mapsto (-1, 1) \text{ and } P_1 \mapsto (-1, 0). \end{aligned}$$

To check that  $\phi$  is  $\Theta$ -equivariant, we must check that for every arrow  $e: v \rightarrow w$  in Figure 3, we have

$$\sigma(e)\phi_v = \phi_w M(e)$$

in  $\text{PSL}_2(\mathbf{Z})$ . We omit this straightforward verification.  $\square$

Recall that we have a  $B_3$ -equivariant map  $i: \mathbf{S} \rightarrow \mathbf{P}^1(\mathbf{Z})$  defined in (9). Thanks to Proposition 5.8, we get another map: send an object  $s \in \mathbf{S}$  supported at the vertex  $v$  to  $\phi([s]) \in \mathbf{P}^1(\mathbf{Z})$  (note that the definition does not create conflicts for objects supported at two states). It is not hard to see that the new map agrees with the old map. Indeed, to see this, note that every object  $s \in \mathbf{S}$  has an expression  $s = g_1 \cdots g_n \cdot P_1$  recognised by  $\Theta$ . Since the map  $s \mapsto \phi([s])$  is  $\Theta$ -equivariant, we have

$$\phi([s]) = g_1 \cdots g_n \phi([P_1]).$$

Since the map  $i$  is  $B_3$ -equivariant, the same equation holds with  $i$  instead of  $\phi$ . Since  $i(P_1) = \phi([P_1])$ , we see that  $i(s) = \phi([s])$ .

The division of  $\mathbf{S}$  according to the support corresponds nicely to a geometric division of the circle  $\mathbf{P}^1(\mathbf{R})$ , which we now describe. The three objects  $P_1$ ,  $P_2$ , and  $X$  divide  $\mathbf{P}^1(\mathbf{R})$  into three closed arcs (see Figure 5). We denote these arcs by  $[P_1, P_2]$ ,  $[P_2, X]$ , and  $[X, P_1]$ .

**Proposition 5.9.** *The objects of  $\mathbf{S}$  supported at the state  $[P_1, P_2]$  lie on the arc  $[P_1, P_2]$ , and likewise for the other two arcs.*

*Proof.* We use that the map  $\mathbf{S} \rightarrow \mathbf{P}^1(\mathbf{R})$  is given by  $s \mapsto \phi([s])$ . For  $s$  supported at  $[P_1, P_2]$ , the vector  $[s]$  lies in the non-negative cone in  $\mathbf{Z}^{\{P_1, P_2\}}$ , which is mapped by  $\phi_{[P_1, P_2]}$  to the arc  $[P_1, P_2]$ . The argument for the other two states is similar.  $\square$

The following proposition is a theorem of Rouquier and Zimmermann [20, Proposition 4.8]. We give a new and simpler proof using the automaton.

**Proposition 5.10.** *Let  $s$  be a spherical object corresponding to  $[a : c] \in \mathbf{P}^1(\mathbf{Z})$ , where  $a, c$  are relatively prime integers. Every minimal complex of projective modules representing  $s$  has exactly  $|a|$  occurrences of  $P_1$  and  $|c|$  occurrences of  $P_2$ .*

*Proof.* Recall that  $\tau$  is a standard stability condition of type A. The objects  $P_1$  and  $P_2$  are the simple objects of the standard heart of  $\tau$ . The  $\tau$ -HN filtration is a refinement of

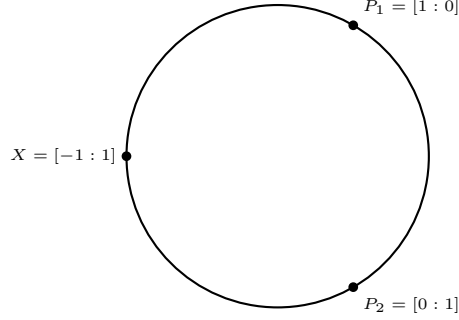


FIGURE 5. The points  $P_1$ ,  $P_2$ , and  $X = P_2 \rightarrow P_1$  divide  $\mathbf{P}^1(\mathbf{R})$  into three arcs. The Harder–Narasimhan pieces of an object only include the two endpoints of the arc on which the object lies.

the cohomology filtration, and on each cohomology factor it is a coarsening of a Jordan–Hölder filtration. Therefore, the number of occurrences of  $P_i$  is the sum of the number of occurrences of  $P_i$  in the  $\tau$ -HN factors.

Suppose  $s$  is supported at the state  $v$  of  $\Theta$ . By Proposition 5.1,  $[s] \in M(v)$  is the  $\tau$ -HN multiplicity vector of  $s$ . By the argument above, the number of occurrences of  $P_i$  in the minimal complex of  $s$  is a linear function of  $[s]$ .

By the discussion following Proposition 5.8, we know that  $[a : c] = \phi([s])$ . The proof of Proposition 5.8 gives an isomorphism  $\tilde{\phi}: M \rightarrow \mathbf{Z}^2$ , which is  $\Theta$ -equivariant up to sign, and sends  $[s]$  to  $(a, c)$ . By looking at the basis vectors, we see that the number of occurrences of  $P_i$  is given by the absolute value of the  $i$ -th coordinate of  $\tilde{\phi}$ . The statement follows.  $\square$

The next proposition relates the minimal complex and the hom functors, as a step towards describing the closure of the image of the mass map  $m$ .

**Proposition 5.11.** *Let  $x$  be a spherical object corresponding to  $[a : c] \in \mathbf{P}^1(\mathbf{Z})$ , where  $a, c$  are relatively prime integers. Then  $\text{hom}(x, P_2) = |a|$  and  $\text{hom}(x, P_1) = |c|$ .*

*Proof.* We prove the assertion for  $\overline{\text{hom}}(x, P_1)$ ; the other case follows by applying  $\gamma$ .

If  $x = P_1$ , then the statement is evident. Suppose  $x$  is supported at  $[P_2, X]$ . Consider the Harder–Narasimhan filtration of  $x$ :

$$0 = x_0 \rightarrow \cdots \rightarrow x_n = x,$$

with sub-quotients  $z_i = \text{Cone}(x_i \rightarrow x_{i+1})$ . Note that  $z_i$  is isomorphic to  $P_2$  or  $X$ , up to shift.

By applying  $\text{Hom}(P_1, -)$ , we obtain a filtration

$$(12) \quad 0 = \text{Hom}(P_1, x_0) \rightarrow \cdots \rightarrow \text{Hom}(P_1, x_n) = \text{Hom}(P_1, x),$$

in the bounded derived category of (graded) vector spaces. We claim that each of the triangles

$$\text{Hom}(P_1, x_i) \rightarrow \text{Hom}(P_1, x_{i+1}) \rightarrow \text{Hom}(P_1, z_i) \xrightarrow{+1}$$

is split; that is, all possible boundary maps between the sub-quotients of (12) are zero. Recall that the filtration of  $x$  has three possible non-zero boundary maps:

$$P_2 \rightarrow P_2[2], \quad X \rightarrow X[2], \quad \text{and} \quad P_2 \rightarrow X[2].$$

It is easy to verify that all three are killed by the functor  $\text{Hom}(P_1, -)$ .

In particular, we have

$$\text{Hom}(P_1, x) = \bigoplus \text{Hom}(P_1, z_i).$$

For  $z_i = P_2$  or  $z_i = X$ , we have

$$\overline{\text{hom}}(z_i, P_1) = \overline{\text{hom}}(P_1, z_i) = 1 = \text{number of } P_2\text{'s in the minimal complex of } z_i.$$

Hence, by linearity, we conclude

$$\overline{\text{hom}}(x, P_1) = \overline{\text{hom}}(P_1, x) = \text{number of } P_2\text{'s in the minimal complex of } x.$$

We now treat the case of  $x$  supported at the other two vertices. By Proposition 5.10,  $x$  and  $\sigma_1^r x$  have the same number of occurrences of  $P_2$ , and both also have the same value for  $\overline{\text{hom}}(P_1)$ . Hence, the proposition for  $x$  implies the proposition for  $\sigma_1^r x$ . If  $x$  is supported at  $[X, P_1]$ , then  $\sigma_1^{-1} x = \sigma_2 \gamma^{-1} x$  is supported at  $[P_2, X]$ ; hence the proposition holds for  $x$ . If  $x$  is supported at  $[P_1, P_2]$ , then it is easy to check that  $x$  has a recognised writing of the form

$$x = \sigma_1^a \beta y,$$

where  $y \in \{P_1, P_2, X\}$  and  $\beta y$  is not supported at  $[P_1, P_2]$ . Then  $y = \sigma_1^{-a} x$  is supported at  $[P_2, X]$  or  $[X, P_1]$ , for which the proposition holds; hence the proposition holds for  $x$ .  $\square$

**Corollary 5.12.** *The values of the basic hom functions on  $x = [a : c]$  are given by*

- (1)  $\overline{\text{hom}}(P_1, x) = |c|$ ,
- (2)  $\overline{\text{hom}}(P_2, x) = |a|$ ,
- (3)  $\overline{\text{hom}}(X, x) = |a + c|$ ,
- (4)  $\overline{\text{hom}}(P_1 \rightarrow P_2, x) = |a - c|$ .

*Proof.* The first two follow directly from Proposition 5.11. For the last two, use

$$\overline{\text{hom}}(X, x) = \overline{\text{hom}}(P_1, \sigma_2^{-1} x),$$

and

$$\overline{\text{hom}}(P_1 \rightarrow P_2, x) = \overline{\text{hom}}(P_2, \sigma_1^{-1} x).$$

The result follows by using the action of  $\sigma_i^{\pm 1}$  on  $(a, c)$  and applying Proposition 5.10.  $\square$

Recall the map  $\overline{\text{hom}}: \mathbf{S} \rightarrow \mathbf{R}^{\mathbf{S}}$  defined by

$$\overline{\text{hom}}(x): y \mapsto \overline{\text{hom}}(x, y).$$

The map  $h: \mathbf{S} \rightarrow \mathbf{P}^{\mathbf{S}}$  is the composition of  $\overline{\text{hom}}$  and the projection  $\mathbf{R}^{\mathbf{S}} \setminus 0 \rightarrow \mathbf{P}^{\mathbf{S}}$ . Consider  $\mathbf{S}$  as a subset of  $\mathbf{P}^1(\mathbf{R})$  via the identification  $\mathbf{S} = \mathbf{P}^1(\mathbf{Z})$  as defined in (9).

**Proposition 5.13.** *The map  $h: \mathbf{S} \rightarrow \mathbf{P}(\mathbf{R}^{\mathbf{S}})$  extends to a continuous map  $h: \mathbf{P}^1(\mathbf{R}) \rightarrow \mathbf{P}^{\mathbf{S}}$ . The extension maps  $\mathbf{P}^1(\mathbf{R})$  homeomorphically onto its image.*

*Proof.* Let  $s \in \mathbf{S}$  be a spherical object (up to shift). Write  $s = \beta P_2$  for some braid  $\beta$ . Let  $\bar{\beta}$  be the image of  $\beta$  in  $\text{PSL}_2(\mathbf{Z})$ . Then  $\bar{\beta}$  is uniquely determined up to right multiplication by powers of  $\bar{\sigma}_2$ . Consider a point  $[a : c] \in \mathbf{P}^1(\mathbf{Z})$ , where  $a$  and  $c$  are relatively prime integers, and let  $t \in \mathbf{S}$  be the corresponding spherical object. We have

$$\begin{aligned} \overline{\text{hom}}(s, t) &= \overline{\text{hom}}(\beta P_2, t) \\ &= \overline{\text{hom}}(P_2, \beta^{-1} t) \\ &= |(1, 0) \cdot \bar{\beta}^{-1} \cdot (a, c)^t| \text{ by Proposition 5.11.} \end{aligned}$$



We define  $h: \mathbf{P}^1(\mathbf{R}) \rightarrow \mathbf{P}(\mathbf{R}^{\mathbf{S}})$  by using the final expression for an arbitrary  $[a : c]$ . That is, we set  $h([a : c])$  to be the (projectivised) function whose value at  $s = \beta P_2$  is

$$|(1, 0) \cdot \bar{\beta}^{-1} \cdot (a, c)^t|.$$

Plainly,  $h: \mathbf{P}^1(\mathbf{R}) \rightarrow \mathbf{P}^{\mathbf{S}}$  is a continuous extension of the original map  $h: \mathbf{S} \rightarrow \mathbf{P}^{\mathbf{S}}$ .

We now check that the extended map is a homeomorphism onto its image. Since the domain  $\mathbf{P}^1(\mathbf{R})$  is compact and the target  $\mathbf{P}(\mathbf{R}^{\mathbf{S}})$  is Hausdorff, it suffices to check that it is injective. Let

$$T = \{P_1, P_2, X\}.$$

Consider the restricted map  $h_T: \mathbf{P}^1(\mathbf{R}) \rightarrow \mathbf{P}(\mathbf{R}^T)$  obtained as the composition of  $h$  and the projection to  $\mathbf{P}^T$ . From Corollary 5.12,  $h_T$  is given in coordinates by

$$h_T: [a : c] \mapsto [|a| : |c| : |a + c|].$$

This map is injective, and hence so is  $h$ . □

**5.6. Gromov coordinates.** Let  $\tau$  be a stability condition of type A. Since the three positive real numbers  $m_\tau(P_1)$ ,  $m_\tau(P_2)$ , and  $m_\tau(X)$  satisfy the triangle inequalities, there exist non-negative real numbers  $x, y, z$  such that

$$m_\tau(P_1) = y + z, \quad m_\tau(P_2) = z + x, \quad m_\tau(X) = x + y.$$

We call the  $x, y, z$ , the *Gromov coordinates* of  $\tau$ . Note that if  $\tau$  is non-degenerate, then the Gromov coordinates are all positive. Otherwise, one of the coordinates is zero. Two of the coordinates cannot be zero.

Recall that  $m_\tau: \mathbf{S} \rightarrow \mathbf{R}$  is the mass function associated to  $\tau$  and  $\overline{\text{hom}}(s)$  the reduced hom functional associated to an object  $s$ .

**Proposition 5.14** (Linearity). *We have*

$$m_\tau = x \overline{\text{hom}}(P_1) + y \overline{\text{hom}}(P_2) + z \overline{\text{hom}}(X).$$

*Proof.* Let  $s \in \mathbf{S}$  be a spherical object. Suppose  $\tau$  is non-degenerate. Observe that the HN filtration of  $s$  is the same for all non-degenerate type A stability conditions, and its stable factors are  $P_1, P_2$ , and  $X$ , up to shift. Denoting by  $a(s)$ ,  $b(s)$ , and  $c(s)$  the multiplicities of these three, we have

$$m_\tau(s) = (y + z)a(s) + (x + z)b(s) + (x + y)c(s).$$

In particular,  $m_\tau(s)$  is linear in the Gromov coordinates. Using Proposition 5.10, we have

$$\begin{aligned} b(s) + c(s) &= \text{number of occurrences of } P_2 \text{ in the minimal complex of } s \\ &= \overline{\text{hom}}(P_1, s), \\ a(s) + c(s) &= \text{number of occurrences of } P_1 \text{ in the minimal complex of } s \\ &= \overline{\text{hom}}(P_2, s), \text{ and} \\ a(s) + b(s) &= \overline{\text{hom}}(X, s). \end{aligned}$$

The first two equalities are evident. The third is obtained by applying  $\gamma$  to the second. We conclude that

$$m_\tau(s) = x \overline{\text{hom}}(P_1, s) + y \overline{\text{hom}}(P_2, s) + z \overline{\text{hom}}(X, s).$$

The case of a degenerate  $\tau$  follows by continuity. □

Let  $\tau$  be an arbitrary stability condition. Then there are three  $\tau$  semi-stable objects, say  $A, B, C$ , whose classes in the Grothendieck group are (up to sign) the classes of  $P_1, P_2$ , and  $X$ . These are obtained simply by applying an appropriate braid to  $P_1, P_2$ , and  $X$ . Define the Gromov coordinates  $x, y, z$  for  $\tau$  by the condition

$$m_\tau(A) = y + z, \quad m_\tau(B) = x + z, \quad m_\tau(C) = x + y.$$

Then Proposition 5.14 implies that we have

$$m_\tau = x \overline{\text{hom}}(A) + y \overline{\text{hom}}(B) + z \overline{\text{hom}}(C).$$

The Gromov coordinates give a nice geometric picture of  $\text{Stab}(\mathcal{C})/\mathbf{C} \subset \mathbf{P}^{\mathbf{S}}$ . Let  $\Delta$  denote the following clipped triangle:

$$\begin{aligned} \Delta &= \{(x, y, z) \in \mathbf{R}_{\geq 0}^3 \mid \text{at least two coordinates non-zero}\} / \mathbf{R}_{>0} \\ &\cong \text{a closed planar triangle minus the three vertices.} \end{aligned}$$

Recall that  $\Lambda \subset \text{Stab}(\mathcal{C})/\mathbf{C}$  is the set of type A stability conditions. We have a homeomorphism  $\Delta \rightarrow \Lambda \subset \mathbf{P}^{\mathbf{S}}$  given by the Gromov coordinates:

$$(x, y, z) \mapsto x \cdot \overline{\text{hom}}(P_1) + y \cdot \overline{\text{hom}}(P_2) + z \cdot \overline{\text{hom}}(X).$$

Note that the homeomorphism extends to the closed (unclipped) triangle  $\overline{\Delta}$ , under which the three vertices are mapped to the points  $P_1, P_2$ , and  $X$ . By applying the  $B_3$  action, we obtain the picture as shown in Figure 6.

Let us describe the transformations indicated in Figure 6 more explicitly. Recall that the element  $\gamma = \sigma_2 \sigma_1$  generates the stabiliser of  $\Lambda$  in  $\text{PSL}_2(\mathbf{Z})$ . So, we can label the various (distinct) translates of  $\Lambda$  by  $\langle \gamma \rangle$ -cosets. Two translates of  $\Lambda$  are either disjoint or intersect along an edge, which is a copy of the open interval. The three translates that intersect  $\Lambda$  along its three edges are  $\sigma_1 \Lambda$ ,  $\sigma_X \Lambda$ , and  $\sigma_2 \Lambda$ . Consequently, the three translates that intersect  $\beta \Lambda$  are  $\beta \sigma_1 \Lambda$ ,  $\beta \sigma_X \Lambda$ , and  $\beta \sigma_2 \Lambda$ . We can encode the translates and their intersections in a graph (called the *exchange graph* in [7]). Its vertices are left  $\gamma$ -cosets in  $\text{PSL}_2(\mathbf{Z})$ , and a coset  $\beta \langle \gamma \rangle$  is connected by an edge with  $\beta \sigma_1 \langle \gamma \rangle$ ,  $\beta \sigma_X \langle \gamma \rangle$ , and  $\beta \sigma_2 \langle \gamma \rangle$ .

*Remark 5.15.* The complement of  $\Lambda$  in  $\text{Stab}(\mathcal{C})/\mathbf{C}$  has three connected components. We can describe these three components in two ways. For the first, write  $\tau'$  in the complement of  $\Lambda$  as  $\tau' = \beta \tau$ , where  $\beta \in B_3$  and  $\tau \in \Lambda$ . Write  $\beta$  in a form recognised by the automaton in Figure 3. Then the three components correspond to the three possible end states of the automaton when it reads the writing of  $\beta$ . For the second, recall from the proof of Proposition 5.6 that the three numbers

$$m_{\tau'}(P_1), m_{\tau'}(P_2), \text{ and } m_{\tau'}(X)$$

satisfy a degenerate triangle inequality—one is the sum of the other two. The three connected components correspond to the ways in which the inequality degenerates.

**5.7. The closure.** We know by Proposition 4.4 that the closure of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  in  $\mathbf{P}^{\mathbf{S}}$  is compact. The goal of this section is to identify this closure. Set

$$P = \overline{h(\mathbf{S})} = h(\mathbf{P}^1(\mathbf{R})) \subset \mathbf{P}^{\mathbf{S}}$$

$$M = m(\text{Stab}(\mathcal{C})/\mathbf{C}) \subset \mathbf{P}^{\mathbf{S}}.$$

The main goal of this section is to prove that  $\overline{M} = M \cup P$ .

Let

$$\overline{\Delta} = (\mathbf{R}_{\geq 0}^3 \setminus \{0\}) / \mathbf{R}_{>0}.$$

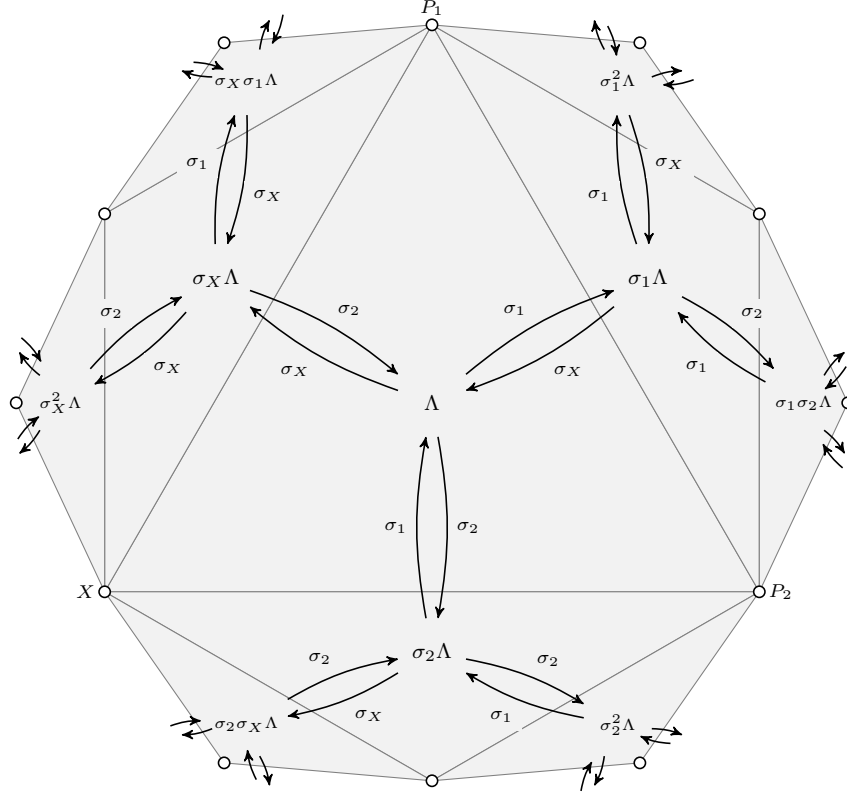


FIGURE 6. The tessellation of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  by the braid group orbit of  $\Lambda$ . The vertices of the triangles correspond to spherical objects.

Then  $\overline{\Delta}$  is homeomorphic to a closed planar triangle. An immediate consequence of the linearity (from Proposition 5.14) is the following.

**Proposition 5.16** (Closure of  $\Lambda$ ). *The closure of  $\Lambda$  in  $\mathbf{P}^{\mathbf{S}}$  is*

$$\overline{\Lambda} = \Lambda \cup \{\overline{\text{hom}}(P_1), \overline{\text{hom}}(P_2), \overline{\text{hom}}(X)\}.$$

*This closure is homeomorphic to  $\overline{\Delta}$ .*

*Proof.* Identify  $\Lambda$  with  $\Delta$  using the Gromov coordinates. From Proposition 5.14, we see that the map  $m: \Delta \rightarrow \mathbf{P}^{\mathbf{S}}$  extends to a continuous map  $\overline{\Delta} \rightarrow \mathbf{P}^{\mathbf{S}}$ . Its image is closed, and equals the union of  $\Lambda$  and the three additional points described above. The map from  $\overline{\Delta}$  is injective, and hence an isomorphism onto its image.  $\square$

It is convenient to have a measure of closeness for two elements of a projective space. Given two non-zero vectors in  $\mathbf{R}^n$  for some positive integer  $n$ , denote by  $\angle(v, w)$  the (acute) angle between them. By a slight abuse of notation, we use  $\angle(v, w)$  also for two points  $v, w \in \mathbf{P}^{n-1}$ ; this is just the angle between any two representatives in  $\mathbf{R}^n$ .

Given a finite subset  $T \subset \mathbf{S}$ , let  $h_T: \mathbf{S} \rightarrow \mathbf{P}^T$  be the composition of  $h: \mathbf{S} \rightarrow \mathbf{P}^{\mathbf{S}}$  and the projection onto the  $T$ -coordinates.

**Proposition 5.17** (Group action contracts). *Fix two elements  $A, B \in \mathbf{S}$  and a finite subset  $T \subset \mathbf{S}$ . Given an  $\epsilon > 0$ , for all but finitely many elements  $L$  of  $\mathrm{PSL}_2(\mathbf{Z})$ , we have*

$$\angle(h_T(LA), h_T(LB)) < \epsilon.$$

*Proof.* In the identification  $S = \mathbf{P}^1(\mathbf{Z}) \subset \mathbf{P}^1(\mathbf{R})$ , let  $A$  and  $B$  be represented by two integer vectors  $a$  and  $b$  in  $\mathbf{Z}^2$ . The map  $h_T: \mathbf{P}^1(\mathbf{R}) \rightarrow \mathbf{P}^T$  is continuous, and hence uniformly continuous. Therefore, it suffices to check that for all but finitely many  $L$ , the angle between  $La$  and  $Lb$  in  $\mathbf{R}^2$  is small. To see this, observe that we have

$$\sin(\angle(La, Lb)) = \frac{|\det L| \cdot |\det(a \mid b)|}{|La| \cdot |Lb|} = \frac{|\det(a \mid b)|}{|La| \cdot |Lb|}.$$

Since  $L$  is required to have integer entries, the quantity  $|L| \cdot |Lb|$  is large—greater than  $\epsilon |\det(a \mid b)|$ —for all but finitely many  $L$ . The claim follows.  $\square$

**Proposition 5.18** (Closure of  $M$ ). *The sets  $M$  and  $P$  are disjoint and their union is the closure of  $M$ .*

*Proof.* The mass of every object in a stability condition is positive. On the other hand,  $\mathrm{hom}(x, x) = 0$ , by definition. Therefore,  $M$  and  $P$  are disjoint.

By Proposition 5.16, we know that  $P$  is contained in the closure of  $M$ . We now prove that  $M \cup P$  is closed. Let  $\tau_n$  be a sequence in  $\mathrm{Stab}(\mathcal{C})/\mathbf{C}$  whose images  $m(\tau_n)$  in  $M$  approach a limit  $z \in \mathbf{P}^{\mathbf{S}}$ . We must show that  $z$  lies in  $M \cup P$ .

Write  $\tau_n = \beta_n \tau'_n$ , where  $\beta_n$  is a braid and  $\tau'_n \in \Lambda$  is a stability condition of type A. Let  $\overline{\beta_n}$  be the image of  $\beta_n$  in  $\mathrm{PSL}_2(\mathbf{Z})$ . We have two possibilities: the set  $\{\overline{\beta_n}\}$  is finite or infinite.

If  $\{\overline{\beta_n}\}$  is finite, then the sequence  $\{\tau_n\}$  is contained in the union of finitely many translates of the set of standard stability conditions  $\Lambda$ . By Proposition 5.16, the closure of  $\Lambda$  is contained in  $M \cup P$ . Hence, the closure of the union of finitely many translates of  $\Lambda$  is also contained in  $M \cup P$ . So the limit  $z$  lies in  $M \cup P$ .

Suppose  $\{\overline{\beta_n}\}$  is infinite. Set  $A_n = \beta_n(P_1)$ ,  $B_n = \beta_n(P_2)$ , and  $C_n = \beta_n(X)$ . Since  $P$  is compact, we may assume, after passing to a subsequence if necessary, that  $h(A_n)$ ,  $h(B_n)$ , and  $h(C_n)$  have limits in  $P$ , say  $A$ ,  $B$ , and  $C$ .

We first observe that  $A = B = C$ . To see this, let  $T \subset \mathbf{S}$  be an arbitrary finite set, and use the subscript  $T$  to denote projections to  $\mathbf{P}^T$ . It suffices to show that  $A_T = B_T = C_T$ . Since  $\{\overline{\beta_n}\}$  is infinite, by Proposition 5.17, we note that the angles between  $h_T(A_n)$ ,  $h_T(B_n)$ , and  $h_T(C_n)$  approach 0 as  $n$  approaches  $\infty$ . Therefore, the three sequences have the same limits. But these limits are  $A_T$ ,  $B_T$ , and  $C_T$ .

Next, we claim that  $z = h(A) = h(B) = h(C)$ . Indeed, by linearity (Proposition 5.14), we know that there exist non-negative real numbers  $x_n, y_n, z_n$  such that

$$m_T(\tau_n) = x_n h_T(A_n) + y_n h_T(B_n) + z_n h_T(C_n).$$

Taking the limit as  $n \rightarrow \infty$  in projective space yields

$$z = h(A) = h(B) = h(C).$$

In particular,  $z$  lies in  $M \cup P$ . The proof is thus complete.  $\square$

**5.8. Homeomorphism to the closed disk.** We take up the final part of the main theorem, namely that  $(\overline{M}, P)$  is a manifold with boundary homeomorphic to the unit disk. We explicitly construct a homeomorphism from  $\overline{M}$  to the disk, using the unprojectivised, but suitably normalised, mass and hom functions.

Fix

$$T = \{P_1, P_2, X = P_2 \rightarrow P_1\}.$$

Define a map

$$\mu: \text{Stab}(\mathcal{C})/i\mathbf{R} \rightarrow \mathbf{R}^T = \mathbf{R}^3$$

by

$$\mu: \tau \mapsto (m_\tau(P_1), m_\tau(P_2), m_\tau(X)).$$

Similarly, define

$$\eta: \mathbf{S} \rightarrow \mathbf{R}^T = \mathbf{R}^3$$

by

$$\eta(s) = (\overline{\text{hom}}(s, P_1), \overline{\text{hom}}(s, P_2), \overline{\text{hom}}(s, X)).$$

Thinking of  $\mathbf{S}$  as  $\mathbf{P}^1(\mathbf{Z}) \subset \mathbf{P}^1(\mathbf{R})$ , it is easy to see that  $\eta$  extends to a continuous map

$$\eta: \mathbf{P}^1(\mathbf{R}) \rightarrow \mathbf{R}^3.$$

Indeed, using Corollary 5.12, we get

$$\eta: [a : c] \mapsto (|c|, |a|, |a + c|).$$

Let  $\tau$  be a stability condition with (semi)-stable objects  $A$ ,  $B$ , and  $C$  of class  $[P_1]$ ,  $[P_2]$ , and  $[X]$  in the Grothendieck group. Let  $x$ ,  $y$ ,  $z$  be the Gromov coordinates of  $\tau$ , namely the non-negative real numbers such that

$$m_\tau(A) = y + z, \quad m_\tau(B) = z + x, \quad m_\tau(C) = x + y.$$

Denote by  $|\cdot|$  the standard Euclidean norm in  $\mathbf{R}^3$ . We say that  $\tau$  is *normalised* if

$$(13) \quad x|\eta(A)| + y|\eta(B)| + z|\eta(C)| = 1.$$

*Remark 5.19.* We point out one subtle aspect of the definition. If  $\tau$  is degenerate—that is, it has strictly semi-stable objects—then one of the  $A$ ,  $B$ , or  $C$  is not uniquely determined, even up to shift. In this case, however, the corresponding Gromov coordinate is 0, and hence (13) is well-posed.

The normalisation yields a continuous section of

$$\text{Stab}(\mathcal{C})/i\mathbf{R} \rightarrow \text{Stab}(\mathcal{C})/\mathbf{C}.$$

That is, for every stability condition  $\tau$  up to rotation and scaling, there is a unique normalised stability condition  $\tau^\nu$  up to rotation, and furthermore the map  $\tau \rightarrow \tau^\nu$  is continuous.

We now identify  $\text{Stab}(\mathcal{C})/\mathbf{C}$  with its image  $M$  in  $\mathbf{P}^{\mathbf{S}}$  and  $\mathbf{P}^1(\mathbf{R})$  with its image  $P$  in  $\mathbf{P}^{\mathbf{S}}$ . Define

$$(14) \quad \pi: \overline{M} = M \cup P \rightarrow \mathbf{R}^3$$

as follows. For  $\tau \in M$ , set

$$(15) \quad \pi(\tau) = \mu(\tau^\nu),$$

and for  $s \in P$ , set

$$\pi(s) = \eta(s)/|\eta(s)|.$$

By construction,  $\pi(a : c)$  is the unit vector in the direction of

$$\eta(a : c) = (|a|, |c|, |a + c|).$$

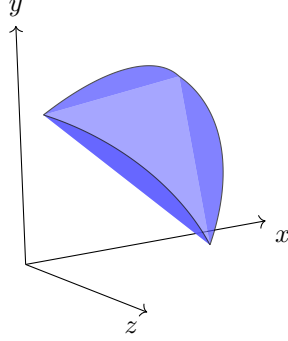


FIGURE 7. The homeomorphic image  $\Phi \subset \mathbf{R}^3$  of the compactified stability manifold.

That is,  $\pi$  maps  $P$  to the unit sphere in  $\mathbf{R}^3$ . It is easy to see that  $\pi|_P$  is injective, and hence a homeomorphism onto its image. The image consists of three circular arcs, one for each pair of end-points from the three points

$$\frac{1}{\sqrt{2}}(0, 1, 1), \quad \frac{1}{\sqrt{2}}(1, 0, 1), \quad \text{and} \quad \frac{1}{\sqrt{2}}(1, 1, 0).$$

The arc joining each pair is a geodesic arc on the unit sphere; that is, the plane it spans passes through the origin.

Equation 13 and the triangle inequality imply that for every stability condition  $\tau$ , we have  $|\pi(\tau)| \leq 1$ . In fact, for every stability condition, at least two of the Gromov coordinates are non-zero. Therefore, we actually have a strict inequality  $|\pi(\tau)| < 1$ .

To get a better understanding of  $\pi$ , let us study it on the translates of the fundamental domain  $\Lambda$ . Let  $\beta$  be a braid. Consider the translate  $\beta\bar{\Lambda}$ . Set

$$A = \beta(P_1), \quad B = \beta(P_2), \quad C = \beta(X).$$

These are the (semi)-stable objects for the stability conditions in  $\beta\bar{\Lambda}$ . Recall that  $\beta\bar{\Lambda}$  is homeomorphic to the projectivised octant  $\bar{\Delta} = (\mathbf{R}_{\geq 0}^3 \setminus 0) / \mathbf{R}_{>0}$ ; the homeomorphism is given by the Gromov coordinates. The normalisation condition (13) is a linear condition on the Gromov coordinates. It cuts out an affine hyperplane slice of the octant, and gives a section of the projectivisation map. Since  $\pi$  is linear in the Gromov coordinates for normalised stability conditions, it maps  $\beta\bar{\Lambda}$  linearly, and homeomorphically, onto the triangle in  $\mathbf{R}^3$  with vertices  $\pi(A)$ ,  $\pi(B)$ , and  $\pi(C)$ .

Let  $\Phi \subset \mathbf{R}^3$  be the union of the triangle

$$\{(x, y, z) \mid x + y + z = \sqrt{2}, x + y - z \geq 0, y + z - x \geq 0, z + x - y \geq 0\},$$

and the three circular segments, each bounded by an edge of the triangle above and the arc in the image of  $\pi(P)$  with the same end-points as the edge (see Figure 7).

Observe that  $\pi$  maps  $\bar{\Lambda}$ , the triangle formed by the standard stability conditions, to the central triangle of  $\Phi$ . On the other hand, the translates of  $\bar{\Lambda}$  on the three sides of the identity on the exchange graph (Figure 6) are mapped to the three circular segments. Indeed, the segment to which  $\pi$  sends a non-standard stability condition  $\tau$  depends on how the triangle inequality degenerates among the  $\tau$ -masses of  $P_1$ ,  $P_2$ , and  $X$ , and this in turn, is determined by where  $\tau$  lies on the exchange graph.

**Proposition 5.20.** *The map  $\pi: \overline{M} \rightarrow \Phi$  is a homeomorphism, where  $\overline{M}$  is given the subspace topology from  $\mathbf{P}^S$ .*

*Proof.* Let us first show that  $\pi$  is continuous. Since  $\pi$  is linear on the translates of  $\overline{\Lambda}$ , and these translates cover  $M$ , it follows that  $\pi$  is continuous on  $M$ . The restriction of  $\pi$  to  $P$  is given by

$$\pi: [a : c] \mapsto \frac{1}{\sqrt{a^2 + c^2 + (a + c)^2}} (|a|, |c|, |a + c|),$$

which is continuous. We must show that  $\pi$  is continuous on  $\overline{M}$  at a point  $p \in P$ .

Let  $\tau_n$  be a sequence of normalised stability conditions converging to  $p \in P$ . We already know that  $m_T(\tau_n)$  converges to  $h_T(p)$  in the projective space  $\mathbf{P}^T$ . Therefore, it suffices to show that  $\pi(\tau_n)$  approaches a vector of norm 1 in  $\mathbf{R}^3$ .

Let  $\epsilon > 0$  be given. Write  $\tau_n = \beta_n \tau'_n$  for some braid  $\beta_n$  and standard stability condition  $\tau'_n$ . Denote by  $\overline{\beta}_n$  the image of  $\beta_n$  in  $\mathrm{PSL}_2(\mathbf{Z})$ . If the set  $\{\overline{\beta}_n\}$  is finite, then the sequence  $\tau'_n$  lies in finitely many translates of  $\Lambda$ . Without loss of generality, we may assume that it lies in one translate, say  $\beta\Lambda$ . Recall that  $\pi: \overline{\beta}\Lambda \rightarrow \mathbf{R}^3$  is linear in the Gromov coordinates and hence continuous. Therefore,  $\pi(\tau_n) \rightarrow \pi(p)$  as  $n \rightarrow \infty$ .

The harder case is when the set  $\{\overline{\beta}_n\}$  is infinite. Set  $A_n = \beta_n(P_1)$ ,  $B_n = \beta_n(P_2)$ ,  $C_n = \beta_n(X)$ , and let  $x_n, y_n, z_n$  be the Gromov coordinates. But in this case, we know by Proposition 5.17 that the angle between  $\eta(A_n)$ ,  $\eta(B_n)$ , and  $\eta(C_n)$  approaches 0 as  $n$  approaches  $\infty$ . Therefore, the difference between

$$|x_n \eta(A_n) + y_n \eta(B_n) + z_n \eta(C_n)| \text{ and } x_n |\eta(A_n)| + y_n |\eta(B_n)| + z_n |\eta(C_n)|$$

approaches 0 as  $n$  approaches  $\infty$ . Since  $\tau_n$  is normalised, the right-hand quantity is 1, and the left-hand quantity is  $|\pi(\tau_n)|$ .

We have now proved that  $\pi: \overline{M} \rightarrow \Phi$  is continuous. Since  $\overline{M}$  is compact,  $\pi$  is a homeomorphism once we know that it is a bijection. We know that the map

$$\pi: P \rightarrow \partial\Phi = \Phi \cap \{v \mid |v| = 1\}$$

is a bijection and  $\pi$  maps  $M$  to

$$\Phi^\circ = \Phi \cap \{v \mid |v| < 1\}.$$

Recall that  $M$  is the union of the translates of the fundamental domains  $\beta\Lambda$ , and each fundamental domain is homeomorphic to a clipped triangle (a planar triangle minus the vertices). The map  $\pi$  maps each translate bijectively to a (clipped) triangle in  $\Phi^\circ$ . To check that  $\pi$  is an injection, we must check that the clipped triangles  $\pi(\beta\Lambda)$  have disjoint interiors, and two of them, say  $\pi(\beta\Lambda)$  and  $\pi(\beta'\Lambda)$ , intersect along an edge if and only if  $\beta$  and  $\beta'$  are adjacent in the exchange graph. To check that  $\pi$  is a surjection, we must check that the union  $\bigcup_\beta \pi(\beta\Lambda)$  is  $\Phi^\circ$ .

Take  $\beta = \mathrm{id}$ . Then  $\pi(\Lambda)$  is the central triangle of  $\Phi^\circ$ . The only other triangles that intersect this triangle are  $\pi(\sigma_1\Lambda)$ ,  $\pi(\sigma_2\Lambda)$ , and  $\pi(\sigma_X\Lambda)$ , and the intersections are along the three edges, as required.

The exchange graph divides the non-trivial translates of  $\Lambda$  into three connected components, and the translates in each of the three components map to the three distinct circular segments in  $\Phi^\circ$ . So it suffices to restrict our attention to one component and the corresponding circular segment. Since  $\gamma$  permutes the components, it suffices to look at only one of them.

Let us consider the component containing  $\sigma_1\Lambda$ . The translates in this component are  $A\Lambda$  where  $A \in \mathrm{PSL}_2(\mathbf{Z})$  is a matrix that has an admissible cyclic writing that starts with  $\sigma_1$ . Suppose  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then  $\pi(A\Lambda)$  is the clipped triangle with vertices

$$(16) \quad \pi([a : c]), \quad \pi([b : d]), \quad \pi([a - b : c - d]).$$

Since we are considering the translates in the  $\sigma_1\Lambda$  component, the points  $[a : c]$ ,  $[b : d]$ , and  $[a - b : c - d]$  lie in the arc  $[P_1, P_2]$  of  $\mathbf{P}^1(\mathbf{Z}) \subset \mathbf{P}^1(\mathbf{R})$ . (So, under the bijection  $\mathbf{P}^1(\mathbf{Z}) = \mathbf{Q} \cup \{\infty\}$  given by  $[a : c] \rightarrow a/c$ , they correspond to  $\mathbf{Q}_{\geq 0} \cup \{\infty\}$ ). By construction, the points  $\pi([a : c])$  and  $\pi([b : d])$  form an edge of a clipped triangle if and only if  $|ad - bc| = 1$ . Thus, the triangles (16) form the Farey tessellation [4, Chapter 8] of the circular segment, which has the intersection properties as dictated by the exchange graph.  $\square$

## 6. THE $\widehat{A}_1$ CASE

The aim of this section is to construct the Thurston compactification of stability space for 2-Calabi–Yau category  $\mathcal{C}_\Gamma$  where  $\Gamma$  is the  $\widehat{A}_1$  quiver.

**6.1. Overview.** Let  $\mathcal{C}$  be the 2-Calabi–Yau category associated to the  $\widehat{A}_1$  quiver, as defined in § 2. It is a graded,  $\mathbf{k}$ -linear triangulated category classically generated by two spherical objects, which we denote  $P_0$  and  $P_1$ .

The extension closure of  $P_0$  and  $P_1$  in  $\mathcal{C}$  is an abelian category; it is the heart of a (bounded)  $t$ -structure. We refer to it as the *standard heart*  $\heartsuit_{\mathrm{std}}$ .

**6.2. The spherical objects and the boundary circle.** As discussed in § 2.2, the spherical twists in the objects  $P_i$  satisfy the defining relations of the Artin–Tits braid group of the  $\widehat{A}_1$  quiver. In this case, this group is the free group on two letters  $F_2$ :

$$F_2 = \langle \sigma_0, \sigma_1 \rangle.$$

Denoting the twist in  $P_i$  as  $\sigma_{P_i}$ , we have a weak action of  $F_2$  on  $\mathcal{C}$  given by

$$\sigma_i \rightarrow \sigma_{P_i}.$$

Let  $G$  be the image of  $F_2$  in  $\mathrm{Aut}(\mathcal{C})$ ; it is precisely the subgroup generated by  $\sigma_{P_0}$  and  $\sigma_{P_1}$ .

Define  $\gamma$  as the following auto-equivalence:

$$\gamma = \sigma_{P_1} \sigma_{P_0}.$$

One can check that for any integer  $n$ , we can write

$$\begin{aligned} \sigma_1 \gamma^n(P_0) &= |2n + 2|P_1 \rightarrow |2n + 1|P_0, \text{ and} \\ \gamma^n P_1 &= |2n + 1|P_1 \rightarrow |2n|P_0. \end{aligned}$$

The objects obtained as above are all spherical; let  $T$  be the set containing all of them. Then  $T$  contains exactly the objects of the form  $nP_1 \rightarrow (n \pm 1)P_0$  for any integer  $n$ . There is a bijection  $\mathbf{Z} \rightarrow T$  given by

$$n \mapsto |n|P_1 \rightarrow |n - 1|P_0.$$

Let  $P_n$  denote the object  $|n|P_1 \rightarrow |n - 1|P_0$ .

It is easy to check that  $\gamma$  preserves  $T$ , and in fact acts by translation by 2 up to shift:

$$\gamma(P_n) = \begin{cases} P_{n+2}, & n \neq 0, -1, \\ P_{n+2}[-1], & n = 0, -1. \end{cases}$$



Regard  $\mathbf{Z}$  as the subset of  $\mathbf{P}^1(\mathbf{Z})$  consisting of points of the form  $[k : 1]$ . We can write a homomorphism  $G \rightarrow \mathrm{PSL}_2(\mathbf{Z})$  compatible with the action of  $\gamma$  on  $\mathbf{Z}$ , as follows:

$$\sigma_{P_0} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \sigma_{P_1} \mapsto \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}.$$

Indeed, under the homomorphism above,  $\gamma$  is the matrix of translation by 2:

$$\gamma \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Let  $\mathbf{S}$  be the  $G$ -orbit of  $\{P_0, P_1\}$ . We have a  $G$ -equivariant map  $i: \mathbf{S} \rightarrow \mathbf{P}^1(\mathbf{Z})$  characterised by  $P_n \mapsto [n : 1]$ . It is easy to check that  $i$  is injective. For convenience, let  $P_\infty$  be any one of the indecomposable extensions of  $P_0$  by  $P_1$ , written as  $P_1 \rightarrow P_0$ . We extend the embedding above to  $\mathbf{S} \cup \{P_\infty\}$  by sending  $P_\infty$  to  $[1 : 0]$ , which is the point at infinity.

The element  $\gamma = \sigma_1 \sigma_0$  acts on the objects of  $T$  by an infinite-order cyclic rotation by two steps, lowering the phases. More precisely,

$$\begin{aligned} \gamma: P_n &\mapsto P_{n+2} \text{ for } n \neq 0, -1, \infty, \\ \gamma: P_n &\mapsto P_{n+2}[-1] \text{ for } n = 0, -1, \\ \gamma: P_\infty &\mapsto P_\infty. \end{aligned}$$

We see that the object  $P_k \in \mathbf{S}$  maps to the point  $k = [k : 1]$  of  $\mathbf{P}^1(\mathbf{Z})$ . For each  $k \in \mathbf{Z}$ , let  $\sigma_k$  denote the spherical twist in the object  $P_k$ . We have

$$\sigma_{2k} = \gamma^k \sigma_0 \gamma^{-k} \text{ and } \sigma_{2k+1} = \gamma^k \sigma_1 \gamma^{-k}.$$

The image of  $\sigma_k$  in  $\mathrm{PSL}_2(\mathbf{Z})$  is given by

$$(17) \quad \sigma_k \mapsto \begin{pmatrix} 2k+1 & -2k^2 \\ 2 & -2k+1 \end{pmatrix}.$$

**6.3. Standard stability conditions.** We say that a stability condition  $\tau$  is *standard* if its heart  $\heartsuit([0, 1])$  is the standard heart  $\heartsuit_{\mathrm{std}}$ . Let  $\phi$  denote the phase function for  $\tau$ . Then we either have  $\phi(P_0) \leq \phi(P_1)$  or  $\phi(P_1) \leq \phi(P_0)$ . In the first case, we say  $\tau$  is standard of *type A*, and in the latter case, of *type B*. Note that, in type A, all objects of  $T$  are  $\tau$ -semistable.

Let  $\Lambda \subset \mathrm{Stab}(\mathcal{C})/\mathbf{C}$  be the closure of the set of standard stability conditions of type A. Then  $\Lambda$  includes non-standard degenerate stability conditions where  $\phi(P_1) = \phi(P_0) + 1$ . We continue to call these stability conditions as of *type A*, reserving the adjective standard for those where  $\phi(P_1) < \phi(P_0) + 1$ . Figure 8 shows the central charges of the objects of  $T$  in a standard non-degenerate stability condition of type A. The dotted line is the direction of the sum of the central charges of  $P_0$  and  $P_1$ , which is also the central charge of  $P_\infty$ .

**6.4. The automaton.** Fix a non-degenerate stability condition  $\tau$  of type A. Figure 9 describes a  $\tau$ -HN-automaton  $\Theta$  by depicting some of the incoming and outgoing edges at one of the vertices.

Here is a more formal description of the automaton.

- (1) The automaton  $\Theta$  is defined by the  $G$ -labelled graph with vertices indexed by  $\mathbf{Z}$  and edges as indicated in Figure 9: an edge from state  $k$  to state  $j$  labelled  $\sigma_{P_j}$  (unless  $j = k + 1$ ); an edge from state  $k$  to state  $(k + 2)$  labelled  $\gamma$ ; and an edge from state  $k$  to state  $(k - 2)$  labelled  $\gamma^{-1}$ .
- (2) The state  $k$  supports the objects  $P_k$  and  $P_{k+1}$ . The  $\Theta$ -set  $S \subset \mathrm{ob} \mathcal{C}$  is the one generated by this data.

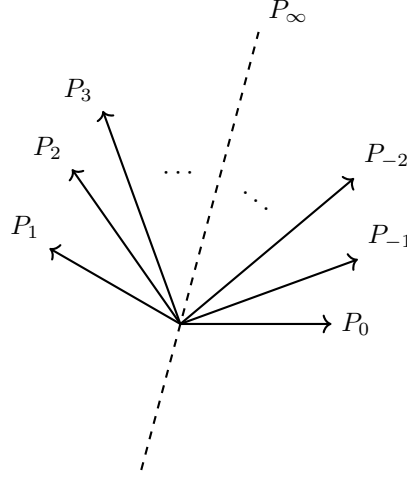


FIGURE 8. The central charges of the objects of  $T$  in a standard non-degenerate stability condition of type A.

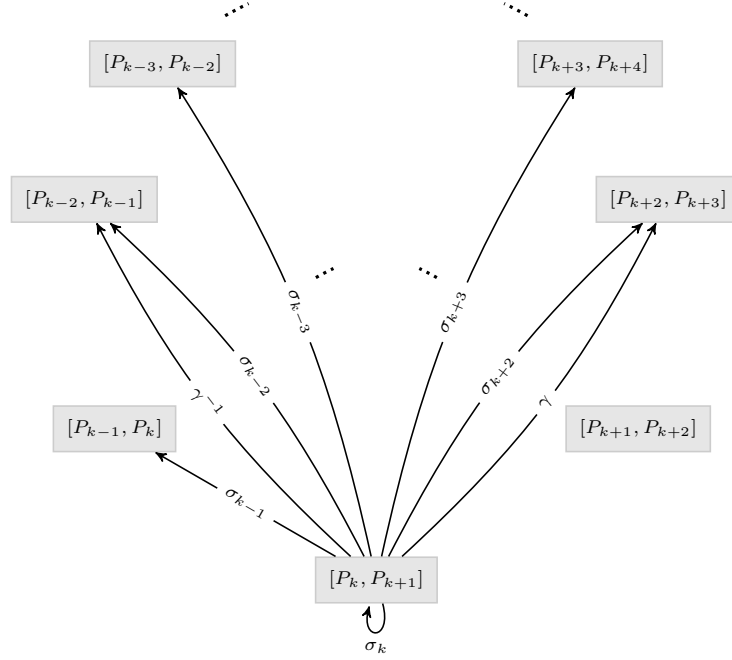


FIGURE 9. An HN automaton for a nondegenerate type A stability condition for  $\hat{A}_1$ , showing the outgoing edges from  $[P_k, P_{k+1}]$ .

- (3) The representation  $M$  associates to the state  $k$  the free abelian group  $\mathbf{Z}^{\{P_k, P_{k+1}\}}$ . The map associated to the edge  $\sigma_{P_j}: k \rightarrow j$  is represented by the matrix

$$(18) \quad \begin{pmatrix} |j-k-1| & |j-k-2| \\ |j-k| & |j-k-1| \end{pmatrix}$$

in the standard basis. The maps associated to the edges labelled  $\gamma^\pm$  are represented by the identity matrix.

- (4) The map  $S \rightarrow M$  is the unique one that sends  $\{P_k, P_{k+1}\}$  to the corresponding standard basis vectors. The map  $HN_k: \mathbf{Z}^{\{P_k, P_{k+1}\}} \rightarrow \mathbf{Z}^\Sigma$  is induced by the inclusion  $\{P_k, P_{k+1}\} \rightarrow \Sigma$ .

Proposition 6.1 shows that  $S \rightarrow M$  is well-defined and that  $HN_k$  indeed computes the HN-multiplicities.

**Proposition 6.1.** *Let  $e: [P_k, P_{k+1}] \rightarrow [P_j, P_{j+1}]$  be an edge of the automaton. Let  $x \in \mathcal{C}$  be an object whose HN filtration consists of shifts of  $P_k$  and  $P_{k+1}$ .*

- (1) *The HN filtration of  $e \cdot x$  consists of shifts of  $P_j$  and  $P_{j+1}$ .*  
(2) *Let  $v \in \mathbf{Z}^{\{P_k, P_{k+1}\}}$  be the HN multiplicity vector of  $x$ . Then  $M(e)v$  is the HN multiplicity vector of  $e \cdot x$ .*

*Proof.* Let  $\sigma$  be the auto-equivalence associated to  $e$ . By an argument similar to Proposition 5.1, the image under  $\sigma$  of the HN filtration of  $x$  is geodesic. So it suffices to prove the proposition for  $x = P_k$  and  $x = P_{k+1}$ .

Suppose  $\sigma = \gamma^\pm$  so that  $j = k \pm 2$ . Since  $\gamma^\pm$  sends  $P_i$  to  $P_{i \pm 2}$  (up to shift), preserving the ordering of the phases, it is clear that  $\sigma(x)$  is supported at  $[P_j, P_{j+1}]$ , and that the HN multiplicities transform by the identity matrix.

Suppose that  $k = 0$ . We can check explicitly that if  $j \neq 1$ , we have

$$(19) \quad \begin{aligned} \sigma_j P_0 &= P_j^{\oplus |j-1|}[-1] \rightarrow P_{j+1}^{\oplus |j|}, \text{ and} \\ \sigma_j P_1 &= P_j^{\oplus |j-2|}[-1] \rightarrow P_{j+1}^{\oplus |j-1|}. \end{aligned}$$

As a result, both  $\sigma_j(P_0)$  and  $\sigma_j(P_1)$  are both supported at state  $j$  and their HN multiplicities transform according to the matrix  $M(e)$ .

Let  $k$  be arbitrary. Observe that we have

$$\sigma_j(P_k) = \begin{cases} \gamma^m \sigma_{j-2m} P_0, & \text{if } k = 2m \text{ is even,} \\ \gamma^m \sigma_{j-2m} P_1, & \text{if } k = 2m + 1 \text{ is odd.} \end{cases}$$

Using the above, we see that for any  $k$  except  $k = j - 1$ , we have

$$(20) \quad \sigma_j P_k = P_j^{\oplus |j-k-1|}[-1] \rightarrow P_{j+1}^{\oplus |j-k|}.$$

As a result,  $\sigma P_k$  and  $\sigma P_{k+1}$  are both supported at state  $j$  and their HN multiplicities transform according to the matrix  $M(e)$ .  $\square$

The next proposition shows that the automaton recognises every braid and every object of  $\mathbf{S}$ .

**Proposition 6.2.** *Let  $\beta \in F_2$  and let  $s \in \mathbf{S}$  be arbitrary.*

- (1) *The automaton in Figure 9 recognises  $\beta$ . More precisely,  $\beta$  has the recognised expression*

$$\beta = \gamma^n \sigma_{a_1} \sigma_{a_2} \cdots \sigma_{a_k},$$

*such that  $a_i - a_{i+1} \neq 1$  for each  $i$ .*

(2) The automaton in Figure 9 recognises  $s$ .

(3) The  $\tau$ -HN filtration of  $s$  consists of at most two objects out of  $T$ , up to shift.

*Proof.* For (1), first write  $\beta$  as any expression in the generators  $\{\sigma_0^{\pm 1}, \sigma_1^{\pm 1}\}$ . For any  $i \in \mathbf{Z}$ , we have

$$\sigma_i^{-1} = \sigma_{i-1}\gamma^{-1}, \quad \gamma^{-1}\sigma_i = \sigma_{i-2}\gamma^{-1}, \quad \gamma = \sigma_i\sigma_{i-1}.$$

Using these relations, we first rewrite  $\beta$  solely in terms of the elements  $\sigma_i$  for  $i \in \mathbf{Z}$  and  $\gamma^{\pm}$ . Then we successively replace any instances of  $\sigma_i\sigma_{i-1}$  by  $\gamma$ , and commute all powers of  $\gamma$  to the left. This process terminates, resulting in an expression of the desired form.

For (2), write  $s = \beta P_i$  where  $i$  is either 0 or 1. Assume that  $i = 0$ ; the other case is similar. Rewrite

$$\beta = \gamma^n \sigma_{a_1} \sigma_{a_2} \cdots \sigma_{a_k}$$

as in (1). Note that the object  $P_0$  is supported at two states, namely  $[P_0, P_1]$  and  $[P_{-1}, P_0]$ . We can apply  $\sigma_{a_k}$  at least at one of these two states. From the condition on our cyclic writing, it is clear that all subsequent letters are applicable. To finish, any power of  $\gamma$  is applicable at any vertex. Therefore  $s$  is recognised by the automaton.

Finally, (3) follows from (2) and from Proposition 6.1  $\square$

*Remark 6.3.* Let  $\tau$  be a degenerate stability condition of type A. Then  $\Theta$  does not compute the  $\tau$ -HN multiplicities, but it does compute  $\tau$ -masses, as explained in Remark 5.3

*Remark 6.4.* Let  $\tau$  be a degenerate standard stability condition of type A. Set  $P'_i$  to be the object  $\sigma_0 P_{i-1}$ . The objects  $P'_i$  are also semistable in  $\tau$ , in addition to the objects  $P_i$ . Note that up to shift, we have  $P'_1 = P_0$  and  $P'_0 = P_1$ . Recall that  $\gamma = \sigma_1 \sigma_0$ . Let  $\gamma' = \sigma_0 \sigma_1$ .

Figure 10 shows an automaton that computes  $\tau$ -HN multiplicities. Only the outgoing arrows labelled  $\sigma_0$  and  $\sigma_1$  are depicted. The remaining possible labels for outgoing arrows are  $\gamma^{\pm}$  and  $\gamma'^{\pm}$ , which act as follows:

$$\begin{aligned} \gamma: [P_i, P_{i+1}] &\rightarrow [P_{i+2}, P_{i+3}] \text{ unless } i = -2, & \gamma: [P'_i, P'_{i+1}] &\rightarrow [P_1, P_2]; \\ \gamma^{-1}: [P_i, P_{i+1}] &\rightarrow [P_{i-2}, P_{i-3}], \text{ unless } i = 2, & \gamma^{-1}: [P'_i, P'_{i+1}] &\rightarrow [P_{-2}, P_{-1}]; \\ \gamma': [P'_i, P'_{i+1}] &\rightarrow [P_{i+2}, P_{i+3}] \text{ unless } i = -2, & \gamma': [P_i, P_{i+1}] &\rightarrow [P'_1, P'_2]; \\ \gamma'^{-1}: [P'_i, P'_{i+1}] &\rightarrow [P'_{i-2}, P'_{i-3}] \text{ unless } i = 2, & \gamma'^{-1}: [P_i, P_{i+1}] &\rightarrow [P'_{-2}, P'_{-1}]. \end{aligned}$$

Since  $\Theta$  suffices for mass computations, we omit further details.

We observe that  $\Theta$  is degenerating.

**Proposition 6.5.** *Let  $\tau$  be a stability condition of type A. Then  $\Theta$  is degenerating for  $\tau$ .*

*Proof.* By Proposition 6.1, every braid has a recognised expression that ends in  $\sigma_i$  for some  $i \in \mathbf{Z}$ . Therefore it suffices to find a finite set of non-degenerate triangles, such that each of these letters makes at least one of the triangles degenerate. Let us exhibit two triangles that suffice.

The non-degenerate distinguished triangle

$$P_1[-1] \rightarrow P_0 \oplus P_0 \rightarrow P_{-1} \xrightarrow{+1}$$

is made degenerate by  $\sigma_j$  for all  $j \in \mathbf{Z} \setminus \{0, 1\}$ . This can be checked directly using the formulas (20). For  $j = 0$  or  $j = 1$ , we can use any  $\gamma$ -translate of the above triangle. In particular, the non-degenerate triangle

$$P_3[-1] \rightarrow P_2 \oplus P_2 \rightarrow P_1 \xrightarrow{+1}$$

is made degenerate by both  $\sigma_0$  and  $\sigma_1$ .  $\square$

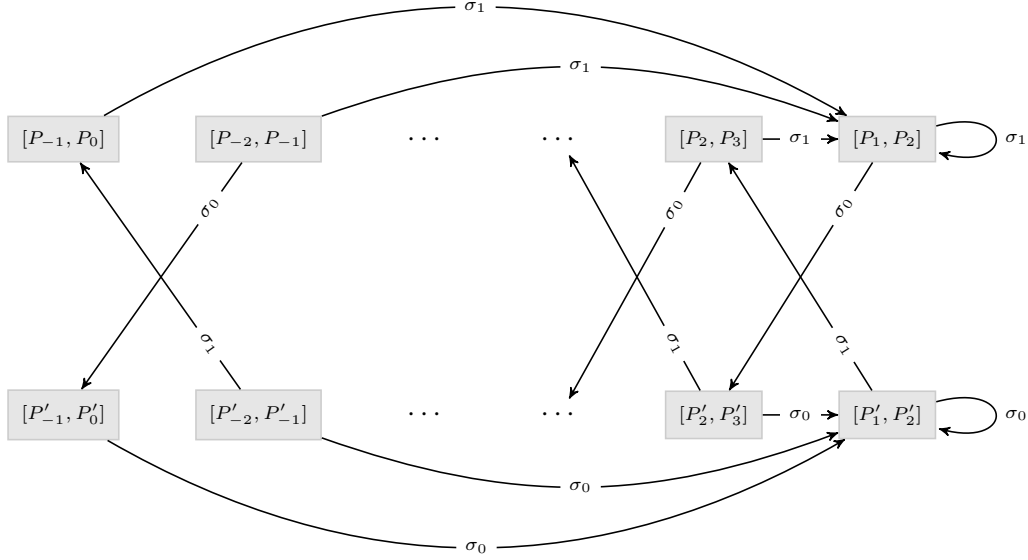


FIGURE 10. An HN automaton for a degenerate standard stability condition of type A, depicting outgoing arrows labelled  $\sigma_0$  and  $\sigma_1$ .

**6.5. Consequences of the automaton.** In this section, we use the automaton from § 6.4 to prove results about our projective embedding of  $\text{Stab}(\mathcal{C})/\mathbf{C}$ .

**Proposition 6.6.** *The map  $m: \text{Stab}(\mathcal{C})/\mathbf{C} \rightarrow \mathbf{P}^{\mathbf{S}}$  is a homeomorphism onto its image.*

*Proof.* The proof is entirely analogous to the proof of Proposition 5.7.  $\square$

As in the  $A_2$ -case, the representation  $M$  of  $\Theta$  is projectively equivalent to the standard representation of  $\text{PSL}_2(\mathbf{Z})$ .

**Proposition 6.7.** *We have an isomorphism of  $\Theta$ -sets  $\phi: \mathbf{P}M \rightarrow \mathbf{P}^1(\mathbf{Z})$ .*

*Proof.* The isomorphism  $\phi$  is given by the linear map  $\mathbf{Z}^{\{P_k, P_{k+1}\}} \rightarrow \mathbf{Z}^2$  defined by

$$\phi_k: P_k \mapsto (k, 1), \quad P_{k+1} \mapsto (k+1, 1).$$

Let us check that  $\phi$  is  $\Theta$ -equivariant. It is evident that  $\phi_k$  intertwines the edges labelled  $\gamma^{\pm}$ . To check that it also intertwines  $e: k \rightarrow j$  labelled  $\sigma_{P_j}$ , we must check that

$$\sigma(P_j) \cdot \phi_k = \phi_j \cdot M(e).$$

Recalling the matrix of  $\sigma_{P_j}$  from (17) and  $M(e)$  from (18), we must check

$$\begin{pmatrix} 2j+1 & -2j^2 \\ 2 & -2j+1 \end{pmatrix} \begin{pmatrix} k & k+1 \\ 1 & 1 \end{pmatrix} = \pm \begin{pmatrix} j-k-1 & j-k-2 \\ j-k & j-k-1 \end{pmatrix} \begin{pmatrix} j & j+1 \\ 1 & 1 \end{pmatrix}.$$

This is straightforward.  $\square$

Recall that we have a  $G$ -equivariant map  $i: \mathbf{S} \rightarrow \mathbf{P}^1(\mathbf{Z})$ . Proposition 6.7 gives another map, defined by  $s \mapsto \phi([s])$ . By the same argument as in the  $A_2$  case, these two maps agree.

The division of  $\mathbf{S}$  according to the HN support corresponds nicely to a geometric division of the circle  $\mathbf{P}^1(\mathbf{R})$ , which we now describe. The objects  $P_i$  divide  $\mathbf{P}^1(\mathbf{R})$  into a collection of closed arcs, namely  $\{[P_i, P_{i+1}] \mid i \in \mathbf{Z}\}$ . Together with  $P_\infty$ , these arcs cover all of  $\mathbf{P}^1(\mathbf{R})$ . We show that this decomposition of the circle is compatible with the automaton.

**Proposition 6.8.** *For each  $i \in \mathbf{Z}$ , the objects supported at state  $i$  are mapped to the arc  $[P_i, P_{i+1}]$ .*

*Proof.* Entirely analogous to that of Proposition 5.9.  $\square$

We obtain a Rouquier–Zimmermann theorem for  $\widehat{A}_1$ .

**Proposition 6.9.** *Let  $x \in \mathbf{S}$  such that  $i(x) = [a : c] \in \mathbf{P}^1(\mathbf{Z})$ , with  $a$  and  $c$  relatively prime integers. Then the minimal complex of projective modules representing  $x$  has exactly  $|a - c|$  occurrences of  $P_0$  and  $|a|$  occurrences of  $P_1$ .*

*Proof.* Entirely analogous to that of Proposition 5.10.  $\square$

**Proposition 6.10.** *Let  $x \in \mathbf{S}$  such that any minimal complex of  $x$  has exactly  $n_0$  occurrences of  $P_0$  and  $n_1$  occurrences of  $P_1$ . Then  $\overline{\text{hom}}(x, P_1) = 2n_0$  and  $\overline{\text{hom}}(x, P_0) = 2n_1$ .*

*Proof.* We prove the assertion for  $\overline{\text{hom}}(x, P_1)$ . The proof for  $\overline{\text{hom}}(x, P_0)$  is similar. Fix a non-degenerate stability condition of type A.

If  $x = P_1$ , the statement is evident. Suppose that  $x$  is supported at  $[P_k, P_{k+1}]$  for  $k \neq 0, 1$ . Consider the Harder–Narasimhan filtration of  $x$ :

$$0 = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = x,$$

with sub-quotients  $z_i = \text{Cone}(x_i \rightarrow x_{i+1})$ . Each  $z_i$  is either  $P_k$  or  $P_{k+1}$ , up to shift.

By applying  $\text{Hom}(P_1, -)$ , we obtain the following filtration in the bounded derived category of graded vector spaces:

$$(21) \quad 0 = \text{Hom}(P_1, x_0) \rightarrow \cdots \rightarrow \text{Hom}(P_1, x_n) = \text{Hom}(P_1, x).$$

We claim that each of the triangles

$$\text{Hom}(P_1, x_i) \rightarrow \text{Hom}(P_1, x_{i+1}) \rightarrow \text{Hom}(P_1, z_i) \xrightarrow{+1}$$

is split; that is, all possible boundary maps between the sub-quotients of (21) are zero. The filtration of  $x$  has three possible non-zero boundary maps:

$$P_k \rightarrow P_k[2], \quad P_{k+1} \rightarrow P_{k+1}[2], \quad P_k \rightarrow P_{k+1}[1].$$

It is easy to verify that all three are killed by the functor  $\text{Hom}(P_1, -)$  when  $k \neq 0, 1$ . In particular, we have

$$\text{Hom}(P_1, x) = \bigoplus \text{Hom}(P_1, z_i).$$

We can compute directly using minimal complexes that  $\overline{\text{hom}}(P_1, P_k) = 2|k - 1|$  if  $k \neq 1$ . So when  $k \neq 0, 1$ , we have for  $z_i = P_k$  that

$$\overline{\text{hom}}(z_i, P_1) = \overline{\text{hom}}(P_1, z_i) = 2|k - 1|,$$

and for  $z_i = P_{k+1}$ , that

$$\overline{\text{hom}}(z_i, P_1) = \overline{\text{hom}}(P_1, z_i) = 2|k|.$$

In each case,  $z_i$  contributes twice the number of occurrences of  $P_0$  in its minimal complex. By linearity, we conclude that  $\overline{\text{hom}}(x, P_1)$  is twice the number of occurrences of  $P_0$  in its minimal complex.

We now treat the cases where  $x$  is supported either at  $[P_0, P_1]$  or at  $[P_1, P_2]$ . By Proposition 6.9, the objects  $x$  and  $\sigma_1^r x$  have the same number of occurrences of  $P_2$ , and both also have the same value for  $\overline{\text{hom}}(-, P_1)$ . Hence the proposition for  $x$  implies the proposition for  $\sigma_1^r x$ .

If  $x$  is supported at  $[P_0, P_1]$ , then it is easy to check that  $x$  has a recognised writing of the form

$$x = \sigma_0^a \beta y,$$

where  $y \in \{P_0, P_1\}$ , and  $\beta y$  is supported neither at  $[P_{-1}, P_0]$  nor at  $[P_0, P_1]$ . Then it is easy to check that  $\sigma_1 x$  is supported neither at  $[P_0, P_1]$  nor at  $[P_1, P_2]$ , hence the proposition follows for  $\sigma_1 x$ , and hence for  $x$ , from the previous case.

If  $x$  is supported at  $[P_1, P_2]$ , then it is easy to check that  $x$  has a recognised writing of the form

$$x = \sigma_1^a \beta y,$$

where  $y \in \{P_0, P_1\}$ , and  $\beta y = \sigma_1^{-a} x$  is not supported at  $[P_1, P_2]$ . Then the proposition follows for  $\beta y$ , and hence for  $x$ , from the previous cases.  $\square$

## 6.6. Gromov coordinates.

**Definition 6.11** (Gromov coordinates). Let  $\tau$  be a stability condition of type A. For each integer  $i \in \mathbf{Z}$ , let  $x_i(\tau)$  be the rational number defined as follows:

$$x_i(\tau) = \frac{m_\tau(P_{i-1}) + m_\tau(P_{i+1}) - 2m_\tau(P_i)}{2}.$$

We call the numbers  $(x_i(\tau))$  the *Gromov coordinates* of  $\tau$ .

Using the triangle inequality [16, Proposition 3.3] on the triangles

$$(22) \quad \begin{aligned} P_{2i+1}[-1] &\rightarrow P_{2i} \oplus P_{2i} \rightarrow P_{2i-1} \xrightarrow{+1}, \\ P_{2i+2} &\rightarrow P_{2i+1} \oplus P_{2i+1} \rightarrow P_{2i}[1] \xrightarrow{+1}, \end{aligned}$$

we deduce that all Gromov coordinates of  $\tau$  are non-negative. If  $\tau$  is non-degenerate, they are all positive.

**Proposition 6.12** (Linearity). *Let  $\tau$  be a stability condition of type A with Gromov coordinates  $(x_i)$ . For any object  $X \in \mathbf{S}$ , we have*

$$(23) \quad m_\tau(X) = \frac{1}{2} \sum_{i \in \mathbf{Z}} x_i \overline{\text{hom}}(P_i, X).$$

*Proof.* Recall that for any  $i \in \mathbf{Z}$ , we have

$$P_{2i} = \gamma^i P_0, \quad P_{2i+1} = \gamma^i P_1.$$

Therefore, we get

$$\overline{\text{hom}}(P_{2i}, X) = \overline{\text{hom}}(P_0, \gamma^{-i} X), \quad \overline{\text{hom}}(P_{2i+1}, X) = \overline{\text{hom}}(P_1, \gamma^{-i} X).$$

Proposition 6.10 allows us to compute  $\overline{\text{hom}}(P_0, \gamma^{-i} X)$  and  $\overline{\text{hom}}(P_1, \gamma^{-i} X)$  from the HN multiplicities of  $\gamma^{-i} X$ . In turn, the automaton Figure 9 allows us to compute the HN multiplicities of  $\gamma^{-i} X$  from the HN multiplicities of  $X$ . Suppose  $X$  is supported at the  $k$ th state of the automaton. Suppose the HN filtration of  $X$  consists of  $\alpha$  copies of  $P_k$  and  $\beta$  copies of  $P_{k+1}$ , up to shift. Then  $\gamma^{-i} X$  is supported on the  $(k - 2i)$ th state, and its HN

filtration contains  $\alpha$  copies of  $P_{k-2i}$  and  $\beta$  copies of  $P_{k+1-2i}$ . In particular, for any  $i \in \mathbf{Z}$ , the number of instances of  $P_1$  and  $P_0$  in the minimal complex of  $\gamma^{-i}X$  are exactly

$$(\alpha|k-2i| + \beta|k-2i+1|) \text{ and } (\alpha|k-2i-1| + \beta|k-2i|)$$

respectively. Using Proposition 6.10, we deduce that for any  $j \in \mathbf{Z}$ , we have

$$(24) \quad \overline{\text{hom}}(P_j, X) = 2(\alpha|k-j| + \beta|k-j+1|).$$

Substitute

$$x_i = \frac{m_\tau(P_{i-1}) + m_\tau(P_{i+1}) - 2m_\tau(P_i)}{2}$$

in

$$\frac{1}{2} \sum_{i \in \mathbf{Z}} x_i \overline{\text{hom}}(P_i, X).$$

Then the coefficient of  $m_\tau(P_j)$  is exactly

$$\frac{\overline{\text{hom}}(P_{j-1}, X) + \overline{\text{hom}}(P_{j+1}, X) - 2\overline{\text{hom}}(P_j, X)}{4}.$$

Using (24), we see that the coefficient is  $\alpha$  if  $j = k$ , is  $\beta$  if  $j = k+1$ , and is 0 otherwise. Therefore we see that

$$\frac{1}{2} \sum_{i \in \mathbf{Z}} x_i \overline{\text{hom}}(P_i, X) = \alpha \cdot m_\tau(P_k) + \beta \cdot m_\tau(P_{k+1}),$$

which is precisely the  $\tau$ -mass of  $X$ .  $\square$

Next we study the topology of the space  $\Lambda$ , and compute the closure of  $m(\Lambda)$  in  $\mathbf{P}^S$ .

**Proposition 6.13.** *A stability condition in  $\Lambda$  is specified uniquely by a central charge  $Z$  for which  $Z(P_1)/Z(P_0)$  lies in the closed upper half plane and avoids the set*

$$L = \{-1\} \cup \left\{ \frac{1-n}{n} \mid n \in \mathbf{Z} \setminus \{0\} \right\}.$$

*Proof.* Recall that  $\Lambda$  is the closure of non-degenerate standard stability conditions of type A, namely those for which we have the phase inequalities

$$\phi(P_0) < \phi(P_1) < \phi(P_0) + 1.$$

It follows that for any stability condition in the closure,  $\omega = Z(P_1)/Z(P_0)$  must lie in the closed upper half plane. Also recall that a stability condition is uniquely specified by its heart and its central charge.

To see why  $\omega$  avoids  $L$ , note that every stability condition in  $\text{Stab}(\mathcal{C})$  contains semi-stable objects of class  $[P_n]$  for  $n \in \mathbf{Z}$  and  $n = \infty$ . Therefore, we must have  $Z([P_n]) \neq 0$  for  $n \in \mathbf{Z} \cup \{\infty\}$ . Since

$$[P_n] = |n|[P_1] + |n-1|[P_0] \text{ and } [P_\infty] = [P_0] + [P_1],$$

we see that  $\omega$  must avoid  $L$ .

We now prove that if  $\omega = Z(P_1)/Z(P_0)$  avoids  $L$  then there is a unique  $\tau \in \text{Stab}(\mathcal{C})/\mathbf{C}$  with  $Z$  as the central charge. If  $\omega$  lies in the (open) upper half plane, then  $\tau$  must be standard and non-degenerate. Since the standard heart is the extension closure of  $P_0$  and  $P_1$ , specifying  $Z(P_0)$  and  $Z(P_1)$  in the upper half plane uniquely determines the stability condition  $\tau$ , and specifying their ratio  $\omega$  determines  $\tau$  uniquely up to the  $\mathbf{C}$  action. Since  $\omega$  is in the upper half plane,  $\tau$  is of type A.

In the rest of the proof, uniquely means uniquely up to the  $\mathbf{C}$  action.



We now treat the degenerate cases, namely those where  $\omega$  is a real number. The set  $\mathbf{R} \setminus L$  is a disjoint union of open intervals of two kinds:

$$\mathbf{R} \setminus L = \bigcup_{n \geq 0} \left( -\frac{n}{n+1}, -\frac{n-1}{n} \right) \cup \bigcup_{n \geq 0} \left( -\frac{n+1}{n}, -\frac{n+2}{n+1} \right).$$

The heart of the stability condition will depend on the interval that contains  $\omega$ .

Suppose  $\omega = Z(P_1)/Z(P_0)$  lies in  $(0, +\infty)$ , which is the interval of the first kind with  $n = 0$ . A type A stability condition with central charge  $Z$  must be standard, which is determined uniquely by  $\omega$ , as explained before.

Suppose  $\omega = Z(P_1)/Z(P_0)$  lies in  $(-1/2, 0)$ , which is the interval of the first kind with  $n = 1$ . Consider the central charge  $Z' = Z \circ \sigma_1$ . Note that  $\sigma_1^{-1}$  takes stability conditions of type A to those of type B. Also, note that  $\omega' = Z'(P_0)/Z'(P_1)$  lies in the interval  $(0, +\infty)$ . The previous argument applied to type B conditions shows that there is a unique  $\tau'$  of type B with central charge  $Z'$ . Then  $\tau = \sigma_1 \tau'$  is the unique stability condition of type A with central charge  $Z$ .

Suppose  $\omega = Z(P_1)/Z(P_0)$  lies in  $\left( -\frac{n}{n+1}, -\frac{n-1}{n} \right)$  for  $n \geq 2$ , which are the rest of the intervals of the first kind. We construct the corresponding stability condition using the action of  $\gamma = \sigma_1 \sigma_0$ . Note that  $\gamma$  preserves the set of stability conditions of type A. Let  $m = \lfloor n/2 \rfloor$  and set  $Z' = Z \circ \gamma^m$ . Then  $\omega' = Z'(P_1)/Z'(P_0)$  lies in the interval  $(-1/2, 0)$  or  $(0, +\infty)$ , depending on the parity of  $n$ . In either case, the previous argument shows that there is a unique stability condition  $\tau'$  of type A with central charge  $Z'$ . Then  $\tau = \gamma^m \tau'$  is the unique stability condition of type A with central charge  $Z$ .

For  $\omega = Z(P_1)/Z(P_0)$  in the intervals of the second kind, we use a similar argument with  $Z' = Z \circ \gamma^{-m}$ . We omit the details.  $\square$

Let  $\overline{\Delta}$  be a closed disk, considered as the one-point compactification of the closed upper half plane. Consider the map

$$\delta: \Lambda \rightarrow \overline{\Delta}$$

defined as follows. Let  $\tau$  be a point of  $\Lambda$  with central charge  $Z$ . Set

$$\delta_\tau = \frac{Z(P_0)}{Z(P_0) + Z(P_1)}.$$

By Proposition 6.13, the map  $\delta$  is injective and a homeomorphism onto its image. It is easily verified that the image of  $\delta$  is

$$\delta(\Lambda) = \overline{\Delta} \setminus (\mathbf{Z} \cup \{\infty\}).$$

Figure 11 shows a sketch of  $\Lambda$  as a subset of the closed disk  $\overline{\Delta}$ .

**6.7. The closure.** We use the embedding  $\Lambda \xrightarrow{\delta} \overline{\Delta}$  to prove the following proposition.

**Proposition 6.14** (Closure of  $\Lambda$ ). *The closure of  $\Lambda$  in  $\mathbf{P}(\mathbf{R}^S)$  is precisely*

$$\overline{\Lambda} = \Lambda \cup \{\overline{\text{hom}}(P_i) \mid i \in \mathbf{Z}\} \cup \{\overline{\text{hom}}(P_1) - \overline{\text{hom}}(P_0)\}.$$

*This closure is homeomorphic to  $\overline{\Delta}$ .*

*Proof.* Via the identification  $\Lambda \xrightarrow{\delta} \overline{\Delta} \setminus (\mathbf{Z} \cup \{\infty\})$ , the set  $\overline{\Delta} \setminus (\mathbf{Z} \cup \{\infty\})$  maps to  $\mathbf{P}(\mathbf{R}^S)$  by the mass map. We now show that this map extends continuously to  $\overline{\Delta}$ . By Proposition 6.12, we can express the mass functional  $m_\tau$  of  $\tau \in \Lambda$  as an infinite linear combination of the Gromov coordinates of  $\tau$ . We now compute the limit of each Gromov coordinate of  $\tau$  as

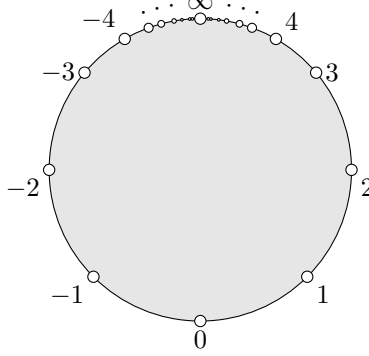


FIGURE 11. The set  $\Lambda$  of stability conditions of type A for the CY2 category associated to  $\hat{A}_1$ . The stability conditions are labelled by the ratio  $\frac{Z(P_0)}{Z(P_0)+Z(P_1)}$ . The interior of the disk represents the upper half plane.

$\delta_\tau$  approaches one of the points in  $\mathbf{Z} \cup \{\infty\}$ , and thereby check that the map  $m_\tau$  extends continuously.

Let  $Z_\tau$  be the central charge of  $\tau$ . We know that up to a complex scalar, we have

$$Z_\tau(P_0) = 1, \quad Z_\tau(P_1) = \frac{1 - \delta_\tau}{\delta_\tau}.$$

We see that

$$Z_\tau(P_i) = \frac{|i|(1 - \delta_\tau) + |i - 1|\delta_\tau}{\delta_\tau}.$$

First suppose that  $\delta_\tau$  approaches some  $n \in \mathbf{Z}$ . In this case,

$$Z_\tau(P_i) \rightarrow \frac{|i|(1 - n) + |i - 1|n}{n}.$$

From Definition 6.11, the limit of each Gromov coordinate is

$$x_j(\tau) \rightarrow \begin{cases} 0, & j \neq n, \\ 1, & j = n. \end{cases}$$

The sum in (23) obviously converges in the limit. Up to a simultaneous scalar, the limit of the mass functional is

$$m_\tau(-) \rightarrow \overline{\text{hom}}(P_n, -).$$

Now suppose that  $\delta_\tau$  approaches  $\infty$ . In this case,

$$Z_\tau(P_j) \rightarrow -|i| + |i - 1| = \begin{cases} -1, & i > 0, \\ 1, & i \leq 0. \end{cases}$$

We compute that

$$x_j(\tau) \rightarrow \begin{cases} 0, & i < 0 \text{ or } i > 1, \\ -1, & i = 0, \\ 1, & i = 1. \end{cases}$$

Again, the sum in (23) obviously converges in the limit. Up to a simultaneous scalar, the limit of the mass functional is

$$m_\tau(-) \rightarrow \overline{\text{hom}}(P_1, -) - \overline{\text{hom}}(P_0, -).$$

We have exhibited a factoring

$$\begin{array}{ccc} \Lambda & \xrightarrow{m} & \mathbf{P}(\mathbf{R}^S) \\ & \searrow \delta & \nearrow d \\ & \overline{\Delta} & \end{array}$$

and identified the additional points in the image of  $\overline{\Delta}$ . Since  $\overline{\Delta}$  is compact, no other points are in the closure.  $\square$

*Remark 6.15.* The point  $\infty$  is already a limit point of the subset  $\mathbf{Z} \cup \{\infty\} \subset \overline{\Delta}$ . Therefore the additional point in the closure, namely the functional  $\overline{\text{hom}}(P_1) - \overline{\text{hom}}(P_0)$ , is already a limit point of the set  $\{\overline{\text{hom}}(P_i) \mid i \in \mathbf{Z}\}$ .

It is now easy to identify the closure of  $\text{Stab}(\mathcal{C})/\mathbf{C}$ , using the following contraction property, analogous to Proposition 5.17. Fix a metric on  $\mathbf{P}^1(\mathbf{R}) = S^1$ .

**Proposition 6.16.** *Let  $\epsilon > 0$  and a compact subset  $I$  of  $\mathbf{P}^1(\mathbf{R})$  be given. Then for all but finitely many elements  $g \in G$ , the image  $g(I)$  has diameter less than  $\epsilon$ .*

*Proof.* The action of  $G$  on  $\mathbf{P}^1(\mathbf{R})$  is via the inclusion  $G \subset \text{PSL}_2(\mathbf{Z})$ . The claim is true for  $\text{PSL}_2(\mathbf{Z})$ , as shown in the proof of Proposition 5.17, and hence also for  $G$ .  $\square$

Let  $M$  and  $P$  be the images of  $\text{Stab}(\mathcal{C})/\mathbf{C}$  and  $\mathbf{S}$  in  $\mathbf{P}^S$ , respectively.

**Proposition 6.17** (Closure of  $M$ ). *The sets  $M$  and  $P$  are disjoint, and their union is the closure of  $M$ .*

*Proof.* The proof is entirely analogous to the proof of Proposition 5.18.  $\square$

We conjecture that the union  $M \cup P$  is homeomorphic to a closed disk, as in the  $A_2$  case, but we do not yet have a complete proof.

## APPENDIX A. GEODESIC FILTRATIONS

Our goal in this note is to understand filtrations that can be re-arranged to the Harder–Narasimhan (HN) filtration. All the objects and morphisms are in a triangulated category that is equipped with a stability condition. Denote by  $[-]$  (resp  $[-]$ ) the bottom (resp. top) sub-quotient in the HN filtration.

We begin by studying distinguished triangles. Consider a distinguished triangle

$$A \rightarrow C \rightarrow B \rightarrow A[1].$$

When is the HN-filtration of  $C$  composed of the HN-filtrations of  $A$  and  $B$ ? The following property of the map  $B \rightarrow A[1]$  gives an answer.

**Definition A.1** (Rectifiable map). We say that a map  $B \rightarrow A[1]$  is *rectifiable* if for every  $\alpha \in \mathbf{R}$ , the induced map

$$B_{>\alpha} \rightarrow A_{\leq\alpha}[1]$$

vanishes.

**Example A.2.** (1) The zero map  $B \rightarrow A[1]$  is rectifiable.

(2) If both  $B \rightarrow A_1[1]$  and  $B \rightarrow A_2[1]$  are rectifiable, then so is the direct sum  $B \rightarrow (A_1 \oplus A_2)[1]$ .

(3) Suppose  $[A] \geq [B]$ . In this case, we say that  $A$  and  $B$  are *non-overlapping*. Then any map  $B \rightarrow A[1]$  is rectifiable. Indeed, either  $B_{>\alpha}$  or  $A_{\leq\alpha}$  is zero.

**Proposition A.3.** *Suppose  $B \rightarrow A[1]$  is rectifiable. Then for every  $C \rightarrow B$  and  $A \rightarrow D$ , the induced map  $C \rightarrow D[1]$  is rectifiable.*

*Proof.* The map  $C_{>\alpha} \rightarrow D_{\leq\alpha}[1]$  factors as

$$C_{>\alpha} \rightarrow B_{>\alpha} \rightarrow A_{\leq\alpha}[1] \rightarrow D_{\leq\alpha}[1].$$

Since  $B \rightarrow A[1]$  is rectifiable, the map  $B_{>\alpha} \rightarrow A_{\leq\alpha}[1]$  vanishes. Hence, the map  $C_{>\alpha} \rightarrow D_{\leq\alpha}[1]$  vanishes too.  $\square$

**Proposition A.4.** *Suppose  $e: B \rightarrow A[1]$  is rectifiable. Set  $C = \text{Cone}(e)[-1]$ , so that we have the triangle*

$$A \rightarrow C \rightarrow B \xrightarrow{+1}.$$

*Then we have exact triangles*

$$(25) \quad A_{\leq\alpha} \rightarrow C_{\leq\alpha} \rightarrow B_{\leq\alpha} \xrightarrow{+1},$$

*and*

$$(26) \quad A_{>\alpha} \rightarrow C_{>\alpha} \rightarrow B_{>\alpha} \xrightarrow{+1},$$

*in which the connecting maps  $B_{\leq\alpha} \rightarrow A_{\leq\alpha}[1]$  and  $B_{>\alpha} \rightarrow A_{>\alpha}[1]$  are induced from  $e$ . Furthermore, these induced maps are also rectifiable.*

*Proof.* We refine the triangle

$$A \rightarrow C \rightarrow B \xrightarrow{+1}$$

to the filtration

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & C_4 = C. \\ & \nearrow^{+1} & \searrow & \nearrow^{+1} & \searrow & \nearrow^{+1} & \searrow & \nearrow^{+1} & \\ & & A_{>\alpha} & & A_{\leq\alpha} & & B_{>\alpha} & & B_{\leq\alpha} \end{array}$$

Since  $B \rightarrow A[1]$  is rectifiable, the middle map  $B_{>\alpha} \rightarrow A_{\leq\alpha}[1]$  vanishes. Hence, we can swap the middle two sub-quotients to obtain a new filtration

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_1 & \longrightarrow & C'_2 & \longrightarrow & C'_3 & \longrightarrow & C_4 = C. \\ & \nearrow^{+1} & \searrow & \nearrow^{+1} & \searrow & \nearrow^{+1} & \searrow & \nearrow^{+1} & \\ & & A_{>\alpha} & & B_{>\alpha} & & A_{\leq\alpha} & & B_{\leq\alpha} \end{array}$$

We see that  $C_{>\alpha} = C'_2$  and  $C_{\leq\alpha} = \text{Cone}(C'_2 \rightarrow C_4)$ , and these are extensions of  $A_{>\alpha}, B_{>\alpha}$  and of  $A_{\leq\alpha}, B_{\leq\alpha}$ , as desired.

Let us describe the connecting maps of (25). Consider the composite of  $B_{>\alpha} \rightarrow B$  and  $B \rightarrow A[1]$ , say  $e: B_{>\alpha} \rightarrow A[1]$ . The composite of  $e$  with  $A[1] \rightarrow A_{\leq\alpha}[1]$  vanishes, by assumption, and hence  $e$  induces the map

$$B_{>\alpha} \rightarrow A_{>\alpha}[1].$$

This is the connecting map of the triangle (25). The connecting map of the triangle (26) is obtained similarly.

Let us check that the connecting map of (25) is rectifiable. We must check that for every  $\beta$ , the map

$$B_{(\beta,\alpha]} \rightarrow A_{\leq\min(\alpha,\beta)}[1]$$

vanishes. We have the diagram

$$\begin{array}{ccccc} B_{>\max(\alpha,\beta)} & \longrightarrow & B_{>\beta} & \longrightarrow & B_{(\beta,\alpha]} \\ \downarrow & & \downarrow & & \downarrow \\ A_{(\alpha,\beta]}[1] & \longrightarrow & A_{\leq\beta}[1] & \longrightarrow & A_{\leq\min(\alpha,\beta)}[1], \end{array}$$

where the rows are distinguished triangles. Since  $B \rightarrow A[1]$  is rectifiable, the left and the middle vertical maps vanish. Hence, the right vertical map also vanishes.

The proof that the connecting map in (26) is rectifiable is similar.  $\square$

**Proposition A.5.** *Let  $A \rightarrow C \rightarrow B \xrightarrow{e} A[1]$  be a distinguished triangle where  $e$  is rectifiable. Then the HN filtration of  $C$  is a rearrangement of the concatenation of the HN filtrations of  $A$  and  $B$ . In particular, we have*

$$\text{mass}(C) = \text{mass}(A) + \text{mass}(B).$$

*Proof.* We induct on the HN length of  $C$ . The base case— $C$  being semi-stable—is easy. In general, choose an  $\alpha$  such that  $C_{>\alpha}$  and  $C_{\leq\alpha}$  have smaller HN lengths than  $C$ . The HN filtration of  $C$  is the concatenation of the HN filtrations of  $C_{>\alpha}$  and  $C_{\leq\alpha}$ . From the inductive assumption and the triangles (25) and (26) in Proposition A.4, we get that the HN filtration of  $C_{>\alpha}$  (resp.  $C_{\leq\alpha}$ ) is a re-arrangement of the concatenation of the HN filtrations of  $A_{>\alpha}$  and  $B_{>\alpha}$  (resp.  $A_{\leq\alpha}$  and  $B_{\leq\alpha}$ ). The result follows.  $\square$

We now take up filtrations.

**Definition A.6** (Geodesic filtration). Consider a filtration

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n.$$

We inductively define what it means for such a filtration to be *geodesic*. All filtrations for  $n = 0$  and  $1$  are geodesic. For  $n \geq 2$ , the filtration is geodesic if

- (1) the filtration  $X_1 \rightarrow \cdots \rightarrow X_n$  is geodesic, and
- (2) the map  $\text{Cone}(X_1 \rightarrow X_n) \rightarrow \text{Cone}(X_0 \rightarrow X_1)[1]$  is rectifiable.

*Remark A.7.* Observe that Harder–Narasimhan filtrations are obviously geodesic.

*Remark A.8.* The definition of a geodesic filtration is “translation invariant.” That is, a filtration

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n$$

is geodesic if and only if the filtration

$$0 = X'_0 \rightarrow X'_1 \rightarrow \cdots \rightarrow X'_n$$

is also geodesic, where  $X'_i = \text{Cone}(X_0 \rightarrow X_i)$ .

*Remark A.9.* We comment on the name “geodesic”. A stability condition on a triangulated category gives a translation invariant metric in the sense of [19], defined as follows. We set the length of a morphism  $f: A \rightarrow B$  to be the HN-mass of  $\text{Cone}(f)$ . We can view a filtration  $X_0 \cdots \rightarrow X_n$  as a path in the category from  $X_0$  to  $X_n$ . By definition, its length is the sum of the masses of the sub-quotients. Suppose we translate so that the path begins at 0, that is, we pass to  $0 = X'_0 \rightarrow \cdots \rightarrow X'_n$ , as above. If the result is the HN-filtration, or a rearrangement of it, then its length is the minimum possible. We show that the Definition A.6 implies that this is the case (Proposition A.5).

**Proposition A.10.** *Suppose*

$$X_0 \rightarrow \cdots \rightarrow X_n$$

*is geodesic. Then for every  $i, j$  with  $i < j$ , the truncation*

$$X_i \rightarrow \cdots \rightarrow X_j$$

*is geodesic*

*Proof.* This is an easy consequence of Proposition A.3. □

**Proposition A.11.** *Consider a filtration*

$$X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n.$$

*The following are equivalent:*

- (1)  $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n$  *is geodesic.*
- (2) *for all  $i$ , both  $X_0 \rightarrow \cdots \rightarrow X_i$  and  $X_i \rightarrow \cdots \rightarrow X_n$  are geodesic, and the connecting map*

$$\text{Cone}(X_i, X_n) \rightarrow \text{Cone}(X_0, X_i)[1]$$

*is rectifiable.*

- (3) *for some  $i$ , both  $X_0 \rightarrow \cdots \rightarrow X_i$  and  $X_i \rightarrow \cdots \rightarrow X_n$  are geodesic, and the connecting map*

$$\text{Cone}(X_i, X_n) \rightarrow \text{Cone}(X_0, X_i)[1]$$

*is rectifiable.*

*Proof.* Let us show that (1) implies (2). We induct on  $i$ . The case  $i = 1$  follows from the definition of a geodesic filtration. Assume  $i \geq 2$ . By Proposition A.10, we get that both  $X_0 \rightarrow \cdots \rightarrow X_i$  and  $X_i \rightarrow \cdots \rightarrow X_n$  are geodesic. For brevity, set  $X(i, j) = \text{Cone}(X_i \rightarrow X_j)$ . We have to prove that the map  $X(i, n) \rightarrow X(0, i)[1]$  is rectifiable, that is the map

$$X(i, n)_{>\alpha} \rightarrow X(0, i)_{\leq\alpha}[1]$$

vanishes. By Proposition A.4, we have the triangle

$$X(0, 1)_{\leq\alpha} \rightarrow X(0, i)_{\leq\alpha} \rightarrow X(1, i)_{\leq\alpha}.$$

By the inductive hypothesis applied to the geodesic filtration  $X_1 \rightarrow \cdots \rightarrow X_n$  for  $(i - 1)$ , we get that the map

$$(27) \quad X(i, n) \rightarrow X(1, i)[1]$$

is rectifiable. Therefore, the composite of

$$X(i, n)_{>\alpha} \rightarrow X(0, i)_{\leq\alpha}[1]$$

with  $X(0, i)_{\leq\alpha}[1] \rightarrow X(1, i)_{\leq\alpha}[1]$  already vanishes, and hence we get an induced map

$$X(i, n)_{>\alpha} \rightarrow X(0, 1)_{\leq\alpha}[1].$$

By Proposition A.4 applied to the rectifiable map in (27), we have the triangle

$$X(1, i)_{>\alpha} \rightarrow X(1, n)_{>\alpha} \rightarrow X(i, n)_{>\alpha} \xrightarrow{+1}.$$

The map  $X(1, n)_{>\alpha} \rightarrow X(0, 1)_{\leq\alpha}[1]$  vanishes since  $X_0 \rightarrow \cdots \rightarrow X_n$  is geodesic. The induced map  $X(1, i)_{>\alpha}[1] \rightarrow X(0, 1)_{\leq\alpha}[1]$  vanishes for phase reasons. Therefore, the map  $X(i, n)_{>\alpha} \rightarrow X(0, 1)_{\leq\alpha}[1]$  vanishes, as required.

That (2) implies (3) is a tautology.

Let us show that (3) implies (1). We again induct on  $i$ . The case  $i = 1$  is the definition. Assume  $i \geq 2$ . By using Proposition A.10 and the inductive hypothesis, we conclude that

$$X_1 \rightarrow \cdots \rightarrow X_n$$

is geodesic. It remains to show that the map  $X(1, n) \rightarrow X(0, 1)[1]$  is rectifiable. We have the rectifiable map

$$X(1, i) \rightarrow X(1, n) \rightarrow X(i, n) \xrightarrow{+1},$$

which by Proposition A.4 gives the exact triangle

$$X(1, i)_{>\alpha} \rightarrow X(1, n)_{>\alpha} \rightarrow X(i, n)_{>\alpha} \xrightarrow{+1}.$$

Since  $X_0 \rightarrow \cdots \rightarrow X_i$  is geodesic, the map  $X(1, i)_{>\alpha} \rightarrow X(0, 1)_{\leq\alpha}[1]$  vanishes. We thus get an induced map

$$X(i, n)_{>\alpha} \rightarrow X(0, 1)_{\leq\alpha}[1],$$

which we must show vanishes. Since  $X_0 \rightarrow \cdots \rightarrow X_i$  is geodesic, we have the triangle

$$X(0, 1)_{\leq\alpha} \rightarrow X(0, i)_{\leq\alpha} \rightarrow X(1, i)_{\leq\alpha} \xrightarrow{+1}.$$

The map  $X(i, n)_{>\alpha} \rightarrow X(0, i)_{\leq\alpha}[1]$  vanishes since  $X(i, n) \rightarrow X(0, i)[1]$  is rectifiable. The induced map  $X(i, n)_{>\alpha} \rightarrow X(1, i)_{\leq\alpha}$  vanishes for phase reasons. Therefore, the map  $X(i, n)_{>\alpha} \rightarrow X(0, 1)_{\leq\alpha}[1]$  vanishes.  $\square$

**Proposition A.12.** *Suppose*

$$0 = X_0 \rightarrow \cdots \rightarrow X_n = X$$

*is geodesic. Set  $A_i = \text{Cone}(X_{i-1} \rightarrow X_i)$ . Then the HN filtration of  $X$  is a rearrangement of the concatenation of the HN filtrations of the  $A_i$ . In particular, we have*

$$\text{mass}(X) = \sum \text{mass}(A_i).$$

*Proof.* Apply Proposition A.5 repeatedly.  $\square$

**Proposition A.13.** *Suppose*

$$X_0 \rightarrow \cdots \rightarrow X_{i-1} \rightarrow X_i \rightarrow X_{i+1} \rightarrow \cdots \rightarrow X_n$$

*is geodesic, and the map  $\text{Cone}(X_i, X_{i+1}) \rightarrow \text{Cone}(X_{i-1}, X_i)[1]$  is zero; that is*

$$\text{Cone}(X_{i-1}, X_{i+1}) = \text{Cone}(X_{i-1}, X_i) \oplus \text{Cone}(X_i, X_{i+1}).$$

*Then the flipped filtration*

$$X_0 \rightarrow \cdots \rightarrow X_{i-1} \rightarrow X'_i \rightarrow X_{i+1} \rightarrow \cdots \rightarrow X_n = X,$$

*where  $X'_i = \text{Cone}(X_{i+1} \rightarrow \text{Cone}(X_{i-1}, X_i))[-1]$ , is also geodesic.*

*Proof.* We may assume  $i - 1 = 0$ , and  $X_0 = 0$ . By Proposition A.11 applied with  $i = 2$ , it suffices to prove that the map

$$\text{Cone}(X_2, X_n) \rightarrow X_2[1]$$

is rectifiable. Note that this map is unchanged after the swap. But this map is rectifiable by the hypothesis.  $\square$

**Proposition A.14.** *Consider a filtration*

$$0 = X_0 \rightarrow \cdots \rightarrow X_n = X.$$

*Set  $A_i = \text{Cone}(X_{i-1} \rightarrow X_i)$ . Suppose for every  $i, j$  with  $i < j$ , we have*

$$\text{Hom}(A_j, A_i[1]) = 0 \text{ or } \lfloor A_i \rfloor \geq \lceil A_j \rceil.$$

*Then the filtration is geodesic.*

*Proof.* We induct on  $n$ . The base case  $n = 0$  is trivial. Assume  $n \geq 1$ . By applying Proposition A.13 repeatedly, we may assume that

$$\lceil A_1 \rceil \geq \cdots \geq \lceil A_n \rceil.$$

As before, set  $X(i, j) = \text{Cone}(X_i \rightarrow X_j)$ . Let  $i$  be the largest such that for all  $j > i$ , we have

$$\lfloor A_1 \rfloor \geq \lceil A_j \rceil.$$

By the assumption, we have  $\text{Hom}(A_j, A_1[1]) = 0$  for all  $j \leq i$ . Therefore, we get  $X_i = A_1 \oplus X(1, i)$ . By the inductive assumption and the translation invariance (Remark A.8),

$$X_0 \rightarrow \cdots \rightarrow X_i$$

and

$$X_i \rightarrow \cdots \rightarrow X_n$$

are both a geodesic filtrations with sub-quotients  $A_1, \dots, A_i$  and  $A_{i+1}, \dots, A_n$ , respectively. By Proposition A.11, it suffices to show that the map

$$X(i, n) \rightarrow X_i[1]$$

is rectifiable. But  $X_i = A_1 \oplus X(1, i)$ . By the inductive assumption, the filtration

$$0 \rightarrow X(1, 2) \rightarrow \cdots \rightarrow X(1, n)$$

is geodesic, and hence the map  $X(i, n) \rightarrow X(1, i)[1]$  is rectifiable. The map  $X(i, n) \rightarrow A_1[1]$  is rectifiable by construction, since  $\lfloor A_1 \rfloor \geq \lceil X(i, n) \rceil$ . Hence,  $X(i, n) \rightarrow X_i[1] = (A_1 \oplus X(1, i))[1]$  is rectifiable. The proof is now complete.  $\square$

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