

# A THURSTON COMPACTIFICATION OF THE SPACE OF STABILITY CONDITIONS: THE $A_2$ CASE

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**ABSTRACT.** We define a continuous map from the space of Bridgeland stability conditions on a triangulated category  $\mathcal{C}$  to an infinite projective space. We conjecture that under some assumptions on the triangulated category  $\mathcal{C}$ , this map is a homeomorphism onto its image, and the closure is a compact real manifold with boundary. Thus, we obtain a compactification of the space of stability conditions. Both the definition and expected properties of this compactification are analogous in many respects to Thurston's compactification of the Teichmüller space of a hyperbolic surface. In this paper we study in detail the case when  $\mathcal{C}$  is the 2-Calabi-Yau category associated to the  $A_2$  quiver, and prove our conjectures in that case.

## 1. INTRODUCTION

A series of recent papers have established a fascinating analogy between the Teichmüller space of a surface and the space of Bridgeland stability conditions on a triangulated category [3, 4, 6]. In this analogy, a curve on the surface corresponds to an object of the category, the topological intersection number corresponds to the dimension of the hom spaces, a metric corresponds to a stability condition, and the length of a curve prescribed by a metric corresponds to the mass prescribed by the corresponding stability condition.

Using this analogy, we can hope to transport powerful tools from geometry to homological algebra. The goal of this paper is to outline one aspect of such a program. We define a compactification of the space of Bridgeland stability conditions on suitable triangulated categories, and propose a conjectural description of its boundary. In the present paper, we work out this compactification explicitly for the simplest non-trivial case, namely the case of the 2-Calabi–Yau category associated to the  $A_2$  quiver [2]. We will extend our results to more general triangulated categories in future work.

**1.1. Mass functions.** The key ingredient in Thurston's compactification of the Teichmüller space is its embedding in an infinite projective space. Roughly speaking, this embedding sends a metric  $\mu$  to the real-valued function on the set of curves

defined by the length with respect to  $\mu$ . By following the analogy, we are led to the following construction.

Let  $\mathbf{k}$  be a field, and  $\mathcal{C}$  a  $\mathbf{k}$ -linear triangulated category. Denote by  $\mathbb{R}^{\mathcal{C}}$  the space of functions from the set of objects of  $\mathcal{C}$  to  $\mathbb{R}$ , endowed with the product topology. A stability condition  $\sigma$  on  $\mathcal{C}$  yields a function  $m_{\sigma} \in \mathbb{R}^{\mathcal{C}}$  defined by

$$m_{\sigma}: x \mapsto m_{\sigma}(x),$$

where  $m_{\sigma}(x)$  is the Harder–Narasimhan mass of  $x$  with respect to the stability condition  $\sigma$ . More explicitly, let

$$0 \rightarrow x_0 \rightarrow \cdots \rightarrow x_n = x$$

be the Harder–Narasimhan filtration of  $x$  with respect to  $\sigma$  and

$$z_i = \text{Cone}(x_{i-1} \rightarrow x_i)$$

the Harder–Narasimhan factors. Denote by  $Z: K(\mathcal{C}) \rightarrow \mathbb{C}$  the central charge associated to  $\sigma$ . Then the mass of  $x$  with respect to  $\sigma$  is the sum

$$m_{\sigma}(x) = \sum_i |Z(z_i)|.$$

The association  $\sigma \mapsto m_{\sigma}$  yields a map

$$m: \text{Stab}(\mathcal{C}) \rightarrow \mathbb{R}^{\mathcal{C}}.$$

From the definition of the topology on  $\text{Stab}(\mathcal{C})$ , it is immediate that  $m$  is continuous.

Recall that we have an action of  $\mathbb{C}$  on  $\text{Stab}(\mathcal{C})$  in which a complex number  $a + ib$  acts by scaling the central charge by  $\exp(a)$ , and shifting the slicing by  $b/\pi$ . In particular, the subgroup  $i\mathbb{R} \subset \mathbb{C}$  acts only by shifting the slicing, and hence leaves the function  $m$  unchanged. As a result, the map  $m$  induces

$$m: \text{Stab}(\mathcal{C})/i\mathbb{R} \rightarrow \mathbb{R}^{\mathcal{C}}.$$

Furthermore, the action by an arbitrary complex number changes  $m$  only by simultaneous scaling. Therefore, if we denote by  $\mathbb{P}^{\mathcal{C}}$  the infinite projective space  $\mathbb{P}^{\mathcal{C}} = (\mathbb{R}^{\mathcal{C}} \setminus 0)/\mathbb{R}^{\times}$ , we get a continuous map

$$m: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{P}^{\mathcal{C}}.$$

Guided by the analogy from Teichmüller theory, we may hope for the following.

**Expectation 1.** *The map  $m: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{P}^{\mathcal{C}}$  is injective and a homeomorphism onto its image.*

**Expectation 2.** *The image  $m(\text{Stab}(\mathcal{C})/\mathbb{C}) \subset \mathbb{P}^{\mathcal{C}}$  is pre-compact (that is, its closure is compact).*

If these two expectations hold, then we obtain a compactification of  $\text{Stab}(\mathcal{C})/\mathbb{C}$  given by the closure of  $m(\text{Stab}(\mathcal{C})/\mathbb{C})$  in  $\mathbb{P}^{\mathcal{C}}$ .

We cannot hope for Expectation 1 and Expectation 2 to hold in general, as one can construct easy examples that break injectivity. Nevertheless, they do seem to hold in a number of cases of interest, such as for the Calabi–Yau categories associated to (connected) quivers.

**1.2. Hom functions.** Assuming that the procedure outlined in § 1.1 gives a compactification of  $\text{Stab}(\mathcal{C})/\mathbb{C}$ , the next natural step is to describe the boundary. In Teichmüller theory, the boundary has a beautiful modular interpretation as the space of so-called projective measured foliations. At present, we do not know an appropriate categorical analogue of this notion.

In Teichmüller theory, there is an alternate, more indirect, description of the functions appearing on the boundary. Each (simple, closed) curve  $\gamma$  on a surface gives rise to a function on the set of all curves defined by the topological intersection number with  $\gamma$ . These functions form a dense subset of the boundary of the Teichmüller space.

The intersection functions have a natural categorical analogue. Assume that for any two objects  $x, y \in \mathcal{C}$ , the vector space

$$\text{Hom}^*(x, y) = \bigoplus_n \text{Hom}(x, y[n])$$

is finite dimensional. Let  $x \in \mathcal{C}$  be an object. We have a function  $\text{hom}(x) \in \mathbb{R}^{\mathcal{C}}$  defined by

$$\text{hom}(x): y \mapsto \dim_{\mathbf{k}} \text{Hom}^*(x, y).$$

**Expectation 3.** *There is a suitable class of objects  $\mathbf{S} \subset \mathcal{C}$  such that the classes of functions  $\text{hom}(x)$  in  $\mathbb{P}^{\mathcal{C}}$  form a dense subset of the boundary of  $m(\text{Stab}(\mathcal{C})/\mathbb{C})$  in  $\mathbb{P}^{\mathcal{C}}$ .*

In the case of the 2-Calabi–Yau categories associated to quivers, we expect the set  $\mathbf{S}$  to be the set of spherical objects of  $\mathcal{C}$ . We prove this is indeed the case for the  $A_2$  quiver.

**1.3. Geometry of the compactification.** Assume that  $m$  yields an embedding of  $\text{Stab}(\mathcal{C})/\mathbb{C}$  in  $\mathbb{P}^{\mathcal{C}}$ , and denote the image of the embedding by  $M$ . Let  $\overline{M}$  be the closure of  $M$  and set  $\partial M = \overline{M} \setminus M$ . Since  $M$  is a manifold, it is natural to expect the following.

**Expectation 4.** *The pair  $(\overline{M}, \partial M)$  is a manifold with boundary.*

Furthermore, in the cases where  $M$  is known to be an open ball (for example, the 2-Calabi–Yau categories of  $ADE$ -quivers), we expect  $\overline{M}$  to be the closed ball and  $\partial M$  to be the sphere. We show that this is indeed the case for the  $A_2$  quiver.

We conjecture that all of the above expectations hold when  $\mathcal{C}$  is the 2-Calabi–Yau category associated to a finite connected quiver. It is an interesting question to find suitable general hypotheses on  $\mathcal{C}$  that ensure that these expectations hold.

**1.4. The  $A_2$  case.** Let  $\mathcal{C}$  be the 2-Calabi–Yau category associated with the  $A_2$  quiver (see § 2 for a detailed description). The main theorem of the paper is that all the expectations listed above hold for  $\mathcal{C}$ . We view this as a proof-of-concept result that suggests that these expectations are reasonable, at least when restricted to a suitable yet interesting class of triangulated categories.

Denote by  $\mathbf{S} \subset \mathcal{C}$  the set of spherical objects. For both the mass and the hom functions, it suffices to take the smaller space  $\mathbb{P}^{\mathbf{S}}$  as the target. For reasons that are not completely clear to us, we must modify the hom functions along the diagonal. For a spherical  $x$ , define the *reduced hom function*  $\overline{\text{hom}}(x): \mathbf{S} \rightarrow \mathbb{R}$  by the formula

$$\overline{\text{hom}}(x): y \mapsto \begin{cases} 0, & x \cong y[n] \text{ for some } n, \\ \text{hom}(x, y) & \text{otherwise.} \end{cases}$$

(Even for the 2-Calabi–Yau categories associated to arbitrary quivers, we expect the reduced hom functions, and not the hom functions, to appear at the boundary).

Recall that in the  $A_2$  case, the manifold  $\text{Stab}(\mathcal{C})/\mathbb{C}$  is homeomorphic to the open unit disk  $D$  in  $\mathbb{C}$  [2].

**Theorem 1.1 (Main).** *Let  $\mathcal{C}$  be the 2-Calabi–Yau category associated with the  $A_2$  quiver.*

- (1) *The map  $m: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{P}^{\mathbf{S}}$  is injective and a homeomorphism onto its image.*
- (2) *Let  $M \subset \mathbb{P}^{\mathbf{S}}$  be the image of  $m$ . The closure  $\overline{M}$  of  $M$  is compact.*
- (3) *The set  $\{\overline{\text{hom}}(x) \text{ for } x \in \mathbf{S}\}$  forms a dense subset of  $\partial M = \overline{M} \setminus M$ .*
- (4) *The pair  $(\overline{M}, \partial M)$  is homeomorphic to  $(\overline{D}, \partial D)$ , where  $\overline{D} \subset \mathbb{C}$  is the closed unit disk and  $\partial D = S^1$  is its boundary.*

See Figure 1 for a sketch of the picture.

**1.5. Dynamics of equivalences.** One of the applications of our compactification of  $M$ , at least when  $\overline{M}$  is homeomorphic to a Euclidean ball, is a dynamical classification of autoequivalences which parallels the Nielsen–Thurston classification of mapping classes as periodic, reducible, or pseudo-Anosov. We first state the theorems here in the case of  $A_2$ , and then discuss the more general situation.

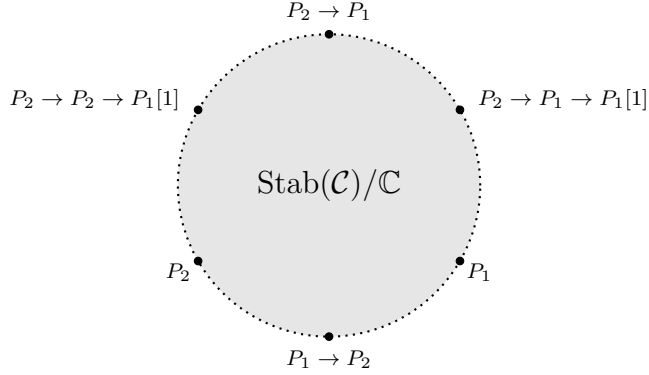


FIGURE 1. The compactification of  $\text{Stab}(\mathcal{C})/\mathbb{C}$ . The spherical objects appear as a dense subset of the boundary.

Let  $B_3$  be the 3-strand braid group, which acts by autoequivalences on  $\mathcal{C}$ , the 2-Calabi–Yau category associated to the  $A_2$  quiver.

**Proposition 1.2.** *Every braid  $\beta \in B_3$  satisfies exactly one of the following.*

- (1) *The autoequivalence associated to  $\beta$  fixes a point in  $M$ . In this case we say that  $\beta$  is periodic.*
- (2) *The autoequivalence associated to  $\beta$  fixes a unique point in  $\partial M$ . In this case we say that  $\beta$  is reducible.*
- (3) *The autoequivalence associated to  $\beta$  fixes exactly two points on  $\partial M$ . In this case we say that  $\beta$  is pseudo-Anosov.*

It turns out that  $\beta$  is periodic precisely when  $\beta$  is central in  $B_3$ . Since the centre of  $B_3$  acts trivially on  $M$ , it follows that periodic elements have finite order in their action on  $M$ . If  $\beta$  is not periodic, then  $\beta$  is reducible precisely when  $\beta$  acts on  $\mathcal{C}$  as the twist in a spherical object  $x$ ; the function  $\overline{\text{hom}}(x) \in \partial M$  is then the unique fixed point of  $\beta$ . Every braid which is not periodic or reducible is pseudo-Anosov.

The terminology introduced in Proposition 1.2 is motivated by the theory of mapping class groups, and for  $B_3$  our terminology matches that coming from geometry. For  $\beta \in B_3$ , denote by  $\bar{\beta}$  the image of  $\beta$  in  $\text{PSL}_2(\mathbb{R})$ . There is an isomorphism of  $\text{Stab}(\mathcal{C})/\mathbb{C}$  with the upper half plane  $\mathbb{H}$  such that the action of  $B_3$  by autoequivalences on  $\text{Stab}(\mathcal{C})/\mathbb{C}$  is intertwined with the action of  $B_3$  on  $\mathbb{H}$  by fractional linear transformations [2]. Then one can check that the dynamical classification of  $B_3$  by autoequivalences in Proposition 1.2 matches the Nielsen–Thurston classification of  $B_3$  as mapping classes of the punctured disc. We note that the compactification  $\bar{M}$  of  $M$  plays an important role in this categorical formulation of the Nielsen–Thurston classification, as the action of a non-periodic braid on the space  $M$  itself will not have any fixed points.

There are further dynamical notions whose categorical and geometric incarnations can be shown to coincide in the  $A_2$  example. For instance, there are two notions of entropy that one can associate to a braid  $\beta \in B_3$ : its topological entropy as a mapping class on the punctured disc, and its categorical entropy in the sense of [4]. By a result of Ikeda [7, Theorem 3.14], the categorical entropy of an autoequivalence  $\beta$  can be computed from the mass growth of the objects  $\{\beta^n(x)\}_{n \in \mathbb{Z}}$ . As a consequence of our description of the action of  $B_3$  on  $M$ , this mass growth—like the growth of curves on the punctured disc—can be read directly from the corresponding matrix  $\bar{\beta} \in \mathrm{PSL}_2(\mathbb{Z})$ . It follows that the categorical and topological entropies of  $\beta$  coincide.

Motivated in part by the previous discussion, we introduce some general definitions. Let  $\mathcal{C}$  be a triangulated category,  $f \in \mathrm{Aut}(\mathcal{C})$ , and  $(M, \partial M)$  the associated compactification. If  $\bar{M}$  is homeomorphic to a closed Euclidean ball, then the Brauer fixed point theorem implies that either there is a stability condition  $\sigma$  such that  $f(\sigma) = z \cdot \sigma$  for some  $z \in \mathbb{C}$ , or there is a point on the boundary  $\partial M$  which is fixed by  $f$ . If  $f(\sigma) = z \cdot \sigma$  for some  $z \in \mathbb{C}$ , we say that  $f$  is *periodic*. An immediate point to note is that, as the masses of the  $\sigma$ -stable objects are bounded away from 0, then the eigenvalue  $z$  lies on the unit circle. Furthermore, the set of phases of  $\sigma$ -stable objects is preserved by the action of  $z$ . So if these phases are not dense, the rotation  $z$  must have finite order. In particular, we obtain the following.

**Corollary 1.3.** *Let  $f$  be a periodic autoequivalence of a triangulated category  $\mathcal{C}$  in the above sense (that is,  $f$  fixes a point  $\sigma \in \mathrm{Stab}(\mathcal{C})/\mathbb{C}$ ). Suppose that*

- (1) *there are at most finitely many stable objects of any phase  $\phi \in S^1$  (this holds, for example, whenever  $\sigma$  is a stability condition with finite-length heart);*
- (2)  *$f$  does not act trivially on the set of  $\sigma$ -stable objects;*
- (3) *the phases of the  $\sigma$ -stable objects are not dense in the unit circle.*

*Then there exists a positive integer  $k$  such that  $f^k$  is a power of the triangulated shift.*

When  $\bar{M}$  is a Euclidean ball, autoequivalences  $f$  that are not periodic have fixed points on the boundary. Here there are two basic cases. The first is when  $f$  fixes the function  $\overline{\mathrm{hom}}(x)$  (up to scaling) for some object  $x \in \mathcal{C}$ . In this case we say that  $f$  is *reducible*. The rationale for this terminology is as follows. First, without loss of generality, we may assume that  $f$  fixes (up to scaling) the hom functional of some indecomposable object  $x \in \mathcal{C}$ . Under suitable assumptions on  $\mathcal{C}$ , this implies that  $f(x)$  is isomorphic to a shift of  $x$ . In that case, the orthogonal hom complement to  $x$  will be preserved by  $f$ , and the dynamical study of  $f$  is reduced to the study of its action on this subcategory.

If  $f$  is neither periodic nor reducible, then the fixed points of  $f$  on  $\partial M$  will consist of functions which are not represented by objects of  $\mathcal{C}$ . (Finding a moduli interpretation of such functions, analogous to Thurston's description of the space of projective

measured foliations, is an important problem we do not directly address here.) We expect that, under some reasonable assumptions on  $\mathcal{C}$ , the autoequivalence  $f$  will have a dense orbit on  $\partial M$ , and that  $f$  will have two fixed points which exhibit sink/source dynamics. We refer to autoequivalences which are neither periodic nor reducible as *pseudo-Anosov*, and conjecture that they are pseudo-Anosov in the sense of [5].

When  $\mathcal{C}$  is the 2-Calabi-Yau category associated to a finite connected quiver, the above discussion can be described in detail, resulting in a dynamical classification of spherical twist groups akin to the Nielsen–Thurston classification of mapping classes. We plan to address this in future work.

### 1.6. $q$ -analogue.

The entire story above admits a  $q$ -analogue for any positive real number  $q > 0$ . Namely, we fix  $q > 0$ , and deform the masses of the stable objects  $z_i$  as follows:

$$m_{q,\sigma}(x) = \sum_i |q^{\phi(z_i)} Z(z_i)|.$$

The  $q$ -mass function gives a continuous map

$$m_q: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{P}^{\mathcal{C}},$$

which agrees with our original embedding when  $q = 1$ . We expect that for all  $q > 0$ , the map  $m_q: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{P}^{\mathcal{C}}$  is injective and a homeomorphism onto its image. We thus obtain a family of compactifications of  $\text{Stab}(\mathcal{C})/\mathbb{C}$  given by the closures of  $m_q(\text{Stab}(\mathcal{C})/\mathbb{C})$  in  $\mathbb{P}^{\mathcal{C}}$ . We conjecture that the closures  $\{\overline{M}_q\}_{q>0}$  are all homeomorphic to each other.

There are some interesting differences between the  $q = 1$  and  $q \neq 1$  cases, even in the case of the  $A_2$  quiver. For example, at  $q = 1$ , the functions  $\text{hom}(x)$  for spherical  $x$  are dense in the boundary: under the identifications  $M_{q=1} \cong \mathbb{H}$ ,  $\partial M_{q=1} = \mathbb{R} \cup \infty$ , these functions are identified with the rational points  $\mathbb{Q} \cup \infty$  of the boundary. When  $q \neq 1$ , however, this is no longer the case. Instead, the  $\text{hom}(x)$  functions are identified with a non-dense (in fact, fractal) subset of  $\mathbb{R} \cup \infty$ , arising as an orbit of the Burau matrices of  $\text{SL}_2(\mathbb{R})$ . Equally compelling, and even less understood at present, is the behaviour of  $\overline{M}_q$  as  $q \rightarrow \infty$  or  $q \rightarrow 0$ .

**1.7. Organisation.** The paper is organised as follows. In § 2, we recall the 2-Calabi–Yau category  $\mathcal{C}$  associated to the  $A_2$  quiver and study Harder–Narasimhan filtrations in it. The highlight of this section is the construction of a finite automaton that describes the dynamics of Harder–Narasimhan filtrations. In § 3, we analyse the embedding of the set of spherical objects of  $\mathcal{C}$  in the infinite projective space  $\mathbb{P}^{\mathcal{S}}$ . In § 4, we show that  $\text{Stab}(\mathcal{C})/\mathbb{C}$  embeds homeomorphically into  $\mathbb{P}^{\mathcal{S}}$ . In § 5, we show that the closure of  $\text{Stab}(\mathcal{C})$  is the union of itself and the closure of the set of

spherical objects. Finally, in § 6, we describe an explicit homeomorphism of the compactification to a disk.

## 2. THE $A_2$ CATEGORY

Let  $\mathcal{C}$  be the 2-Calabi–Yau category associated to the  $A_2$  quiver. There are several equivalent constructions of  $\mathcal{C}$ . Before giving one, we recall the salient (and in fact, characterising) properties of  $\mathcal{C}$ .

The category  $\mathcal{C}$  is a triangulated  $\mathbf{k}$ -linear category characterised by the following properties (see [2, § 1.1]):

- (1)  $\mathcal{C}$  is 2-Calabi–Yau. That is, for a pair of objects  $x, y \in \mathcal{C}$ , we have natural isomorphisms

$$\mathrm{Hom}(x, y) \cong \mathrm{Hom}(y, x[2])^*.$$

- (2)  $\mathcal{C}$  is classically generated by two objects  $P_1$  and  $P_2$  satisfying

$$\begin{aligned} \mathrm{Hom}(P_i, P_i[n]) &= \begin{cases} \mathbf{k} & \text{if } n = 0, 2, \\ 0 & \text{otherwise;} \end{cases} \\ \mathrm{Hom}(P_i, P_j[n]) &= \begin{cases} \mathbf{k} & \text{if } n = 1, \\ 0 & \text{otherwise, for } i \neq j. \end{cases} \end{aligned}$$

One way to construct  $\mathcal{C}$  is as follows (from [9, § 2]). Let  $A$  be the zig-zag algebra of the  $A_2$  quiver. This is the quotient of the path algebra of the doubled  $A_2$  quiver by the ideal generated by all length 3 paths. It admits a grading by path length. Let  $\mathcal{K}$  be the homotopy category of the category of bounded complexes of graded projective left  $A$ -modules. The category  $\mathcal{K}$  is bi-graded, with one grading coming from the homological degree and one from the internal degree. The category  $\mathcal{C}$  is the orbit category of  $\mathcal{K}$  where we collapse this bi-grading to a single grading (see [8] for orbit categories). That is,

$$\mathcal{C} = \mathcal{K} / (x \sim x\langle -1 \rangle[1]),$$

where  $\langle - \rangle$  denotes the simultaneous internal grading shift on all the objects in  $x$  and  $[-]$  the homological shift on the complex  $x$ . In this avatar of  $\mathcal{C}$ , the generators  $P_1$  and  $P_2$  are given by

$$P_i = Ae_i, \quad \text{in homological degree 0,}$$

where  $e_i$  is the idempotent of  $A$  given by the path of length 0 based at the vertex  $i$ .

The extension closure of  $P_1$  and  $P_2$  in  $\mathcal{C}$  is an abelian category; it is the heart of a (bounded)  $t$ -structure. We refer to it as the *standard heart*. It has two simple objects,  $P_1$  and  $P_2$ , and two additional indecomposable objects, denoted by  $P_1 \rightarrow P_2$



and  $P_2 \rightarrow P_1$ . The object  $P_i \rightarrow P_j$  is the unique extension of  $P_i$  by  $P_j$ . In terms of complexes, it is the complex

$$P_i \rightarrow P_j = Ae_i\langle -1 \rangle \xrightarrow{f_{ij}} Ae_j, \quad \text{in homological degrees } -1 \text{ and } 0$$

where  $f_{ij}$  is right multiplication by the path  $i \rightarrow j$ .

**2.1. Spherical objects and twists.** Both  $P_1$  and  $P_2$  are *spherical* in the sense of [11, Definition 2.9]. In particular,  $\text{Hom}^*(x, x) \cong \mathbf{k}[t]/t^2$  as a  $\mathbf{k}$ -algebra. Any spherical object  $x$  gives rise to an autoequivalence  $\sigma_x: \mathcal{C} \rightarrow \mathcal{C}$  called the *spherical twist* in  $x$  (see [11]). The twists in  $P_1$  and  $P_2$  satisfy the usual braid relation  $\sigma_{P_1}\sigma_{P_2}\sigma_{P_1} \cong \sigma_{P_2}\sigma_{P_1}\sigma_{P_2}$ .

Let  $B_3$  denote the 3-strand braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

We have a homomorphism

$$B_3 \rightarrow \text{Aut}(\mathcal{C})$$

defined by

$$\sigma_i \mapsto \sigma_{P_i}.$$

Via this homomorphism, we have an action of  $B_3$  on  $\mathcal{C}$ . This action is faithful [11].

Let  $\mathbf{S}$  be the set of spherical objects of  $\mathcal{C}$ , up to shift. (An object and its triangulated shift will play identical roles in most of our analysis, so it is convenient to not distinguish them). It turns out that  $B_3$  acts transitively on the set of all the spherical objects of  $\mathcal{C}$ , and hence on  $\mathbf{S}$ .

The centre of  $B_3$  is generated by  $(\sigma_2\sigma_1)^3$ . The central element  $(\sigma_2\sigma_1)^3$  acts by a triangulated shift, precisely by  $x \mapsto x[-2]$ . The action of  $B_3$  on  $\mathbf{S}$  therefore descends to an action of  $B_3/Z(B_3)$  on  $\mathbf{S}$ .

We have an isomorphism  $B_3/Z(B_3) \rightarrow \text{PSL}_2(\mathbb{Z})$  given by

$$\begin{aligned} \sigma_1 &\mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\ \sigma_2 &\mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

The stabiliser of an element  $x \in \mathbf{S}$  is the subgroup of  $\text{PSL}_2(\mathbb{Z})$  generated by the image of  $\sigma_x$ . In particular, the stabiliser of  $P_1$  is the matrix  $\sigma_1$ . As a result, we have a  $\text{PSL}_2(\mathbb{Z})$ -equivariant bijection

$$(1) \quad \mathbb{P}^1(\mathbb{Z}) \rightarrow \mathbf{S}$$

defined uniquely by the choice

$$[1 : 0] \mapsto P_1.$$

For example, we have  $[0 : 1] = \sigma_1(\sigma_2([1 : 0]))$ , and so  $[0 : 1]$  maps to  $\sigma_1(\sigma_2(P_1)) = P_2$ . More generally, we can calculate the image in  $\mathbf{S}$  of a point  $[a : c] \in \mathbb{P}^1(\mathbb{Z})$  for  $c \neq 0$  as

follows. Write the rational number  $a/c$  as a continued fraction with an odd number of terms:

$$\frac{a}{c} = n_0 + \frac{1}{n_2 + \frac{1}{\ddots + \frac{1}{n_{2k}}}}.$$

Here, each  $n_i$  is an integer, with  $n_i > 0$  for  $i = 1, \dots, 2k$ . When viewed as fractional linear transformations, the matrices  $\sigma_1$  and  $\sigma_2$  transform a rational number  $\alpha$  by

$$\alpha \xrightarrow{\sigma_1} 1 + \alpha, \quad \alpha \xrightarrow{\sigma_2} \frac{1}{-1 + 1/\alpha}.$$

We deduce that the image of  $[a : c]$  in  $\mathbf{S}$  is the object

$$(2) \quad \sigma_1^{n_0} \sigma_2^{-n_1} \cdots \sigma_1^{n_{2k}}(P_2).$$

**2.2. Harder–Narasimhan filtrations.** Let  $\tau$  be a stability condition such that the objects  $P_1$ ,  $P_2$ , and  $X = P_2 \rightarrow P_1$  are  $\tau$  semi-stable. This implies that, up to rotation, we can arrange the phases  $\phi$  so that

$$0 = \phi(P_1) \leq \phi(P_2) < 1.$$

We say that such a stability condition is *standard* (see Figure 2). The set  $\Lambda \subset \text{Stab}(\mathcal{C})/\mathbb{C}$  of standard stability conditions is a closed subset that tessellates  $\text{Stab}(\mathcal{C})/\mathbb{C}$  under the action of  $B_3$ . More precisely,  $\Lambda$  satisfies the following properties (see, e.g., [2, Proposition 4.2]).

- (1) Each point of  $\text{Stab}(\mathcal{C})/\mathbb{C}$  lies in the  $B_3$ -orbit of a point of  $\Lambda$ .
- (2) The stabiliser of  $\Lambda$  is the subgroup generated by  $\gamma = \sigma_2 \sigma_1$  in  $B_3$ .
- (3) For any  $g \in B_3$  not in the stabiliser, the interiors of  $\Lambda$  and  $g\Lambda$  have empty intersection.

If we have  $\phi(P_1) < \phi(P_2)$ , then we say that  $\tau$  is *non-degenerate*. The set of non-degenerate standard stability conditions is an open subset.

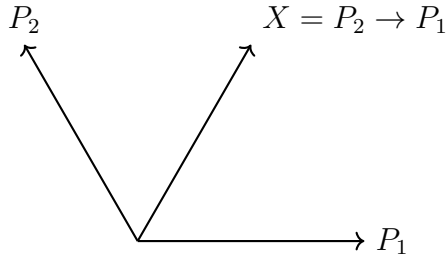


FIGURE 2. Central charges of semistable objects in a standard stability condition, up to rotation.

Fix a non-degenerate standard stability condition  $\tau$ . Our next goal is to get a precise understanding of the Harder–Narasimhan filtration of all spherical objects with respect to  $\tau$ . In the rest of the subsection, the Harder–Narasimhan filtrations are with respect to  $\tau$ .

We begin by making a few easy observations. Let  $x$  be an object with a filtration

$$(3) \quad 0 \rightarrow x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = x.$$

Let  $z_i$  be the sub-quotients, defined by the triangles

$$\begin{array}{ccc} x_{i-1} & \xrightarrow{\quad} & x_i \\ & \nwarrow \text{dashed} & \nearrow \\ & z_i & \end{array} \quad \begin{array}{c} +1 \\ \end{array}.$$

Denote by  $\lfloor - \rfloor$  (resp.  $\lceil - \rceil$ ) the Harder–Narasimhan pieces of lowest (resp. highest) phase. We say that the filtration (3) is *non-overlapping* if for every  $i$ , we have

$$\phi(\lfloor z_i \rfloor) \geq \phi(\lceil z_{i+1} \rceil).$$

By pulling back the Harder–Narasimhan filtrations of the sub-quotients  $z_i$ , we can refine any filtration (3) to one where the sub-quotients are stable. If (3) is non-overlapping, then the stable factors in the refined filtration appear in non-increasing order of phase, and therefore yield the Harder–Narasimhan filtration of  $x$ .

Now suppose that

$$0 \rightarrow x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = x$$

is the Harder–Narasimhan filtration of  $x$  with stable sub-quotients  $z_i$ . Let  $\sigma$  be a triangulated autoequivalence. By applying  $\sigma$  to the filtration, we obtain

$$(4) \quad 0 \rightarrow \sigma(x_0) \rightarrow \sigma(x_1) \rightarrow \cdots \rightarrow \sigma(x_n) = \sigma(x),$$

which has sub-quotients  $\sigma(z_i)$ . The new sub-quotients may not be stable, but if the filtration (4) is non-overlapping, then the Harder–Narasimhan filtration of  $\sigma(x)$  is just a refinement of this filtration. If this is the case, we say that  $\sigma$  is a *non-overlapping* autoequivalence for  $x$ .

For these observations to be of much use, we need a rich class of non-overlapping equivalences. Denote by  $\sigma_X \in B_3$  the element

$$\sigma_X = \sigma_1^{-1} \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2^{-1}.$$

It acts on  $\mathcal{C}$  by the spherical twist in  $X$  (justifying the abuse of notation). Set

$$\gamma = \sigma_2 \sigma_1 = \sigma_X \sigma_2 = \sigma_1 \sigma_X.$$

Note that  $\gamma$  acts on the stable objects  $P_2, P_1, X$  by a cyclic rotation, lowering the phases. Explicitly, we have

$$\begin{aligned}\gamma: P_2 &\mapsto P_1, \\ \gamma: P_1 &\mapsto X[-1], \\ \gamma: X &\mapsto P_2[-1].\end{aligned}$$

As a result,  $\gamma$  and  $\gamma^{-1}$  are non-overlapping for *every* object.

On the other hand, the spherical twists  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_X$  are non-overlapping only for certain objects, and we organise this information as follows. Figure 3 shows a directed graph, which we call the *Harder–Narasimhan automaton*. Its vertices are labelled by pairs of stable objects. We say that an object  $x$  is *supported by* a vertex  $v$  if, up to shift, all the Harder–Narasimhan factors of  $x$  are among the labels of  $v$ . The edges of the graph are labelled by autoequivalences. The graph is designed to satisfy the following non-overlapping property.

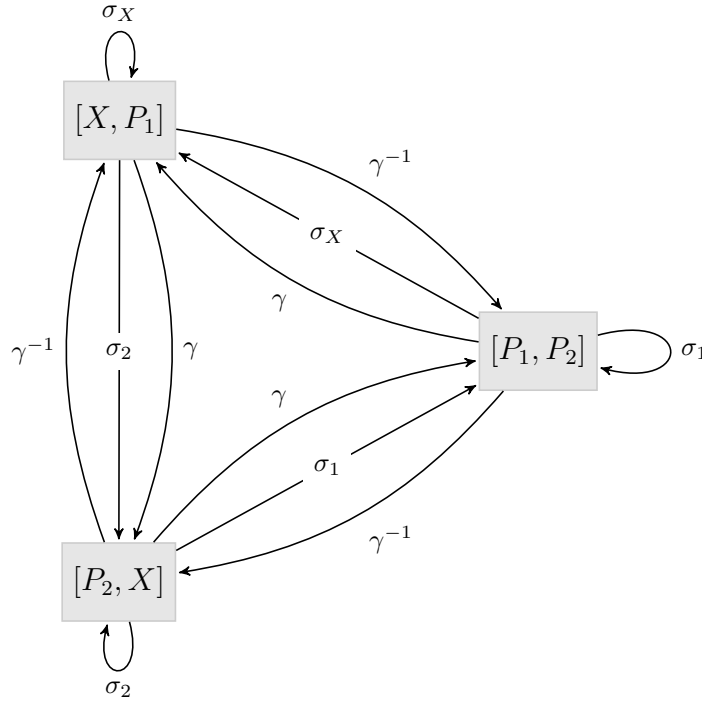


FIGURE 3. An automaton describing the dynamics of Harder–Narasimhan filtrations in a stability condition with stable objects  $P_1$ ,  $P_2$ , and  $X = P_2 \rightarrow P_1$ .

**Proposition 2.1.** *Let  $x \in \mathcal{C}$  be an object supported by a vertex  $v$  in Figure 3. Let  $e$  be an edge with source  $v$  and label  $\sigma$ . Then  $\sigma$  is a non-overlapping autoequivalence for  $x$ , and  $\sigma(x)$  is supported by the target of  $e$ .*

*Proof.* Recall that  $\gamma$  and  $\gamma^{-1}$  are non-overlapping for any object. By first applying  $\gamma$  or  $\gamma^{-1}$ , we may assume that  $x$  is supported at  $[P_1, P_2]$ . We must now check that when we apply  $\sigma_1$  or  $\sigma_X$  to two consecutive Harder–Narasimhan pieces of  $x$ , we avoid any phase overlaps. The three possible consecutive pairs are  $(P_1[1], P_1)$ ,  $(P_2[1], P_2)$ , and  $(P_2, P_1)$ . The first two are easy; we check the last one. We have

$$(5) \quad \begin{aligned} (P_2, P_1) &\xrightarrow{\sigma_1} (P_1 \rightarrow P_2, P_1[-1]) \\ &\xrightarrow{\sigma_X} (P_1[1], X[-1] \rightarrow P_1). \end{aligned}$$

Observe that each pair on the right satisfies the non-overlapping condition, and is supported by the correct vertex.  $\square$

The following makes the automaton truly useful.

**Proposition 2.2.** *Any braid  $\beta \in B_3$  can be written as*

$$\beta = \sigma_{a_1}^{m_1} \sigma_{a_2}^{m_2} \cdots \sigma_{a_k}^{m_k} \gamma^n,$$

where  $n$  is an integer,  $k$  is a non-negative integer, the  $m_i$  are positive integers, and the sequence

$$(a_1, a_2, \dots, a_k)$$

is a contiguous subsequence of the infinite cyclic sequence

$$(\dots, X, 1, 2, X, 1, 2, \dots).$$

*Proof.* We repeatedly use the commutation relations

$$\gamma \sigma_2 \gamma^{-1} = \sigma_1, \quad \gamma \sigma_X \gamma^{-1} = \sigma_2, \quad \gamma \sigma_1 \gamma^{-1} = \sigma_X.$$

Begin by writing  $\beta$  as any product of the generators  $\sigma_1$  and  $\sigma_2$ , along with their inverses. Eliminate the inverses of the generators by rewriting as follows:

$$\sigma_1^{-1} = \sigma_X \gamma^{-1}, \quad \sigma_2^{-1} = \sigma_1 \gamma^{-1}.$$

Next, use the commutation relations to rewrite

$$\gamma^i \sigma_X = \sigma_2 \gamma^i, \quad \gamma^i \sigma_1 = \sigma_X \gamma^i, \quad \gamma^i \sigma_2 = \sigma_1 \gamma^i,$$

and thus move instances of  $\gamma^{-1}$  to the right as a single power of  $\gamma^{-1}$ . The rest of  $\beta$  is now a product of elements from  $\{\sigma_1, \sigma_2, \sigma_X\}$ .

Finally, replace any occurrences of  $\sigma_2 \sigma_1$ ,  $\sigma_1 \sigma_X$ , or  $\sigma_X \sigma_2$  by  $\gamma$  and move  $\gamma$  to the right, again using the commutation relations. Each such operation decreases the length of the braid in the elements  $\{\sigma_1, \sigma_2, \sigma_X, \gamma^{\pm}\}$ . Therefore, the procedure terminates and  $\beta$  reaches the desired form.  $\square$

We call the writing described by Proposition 2.2 an *admissible cyclic writing*. Armed with an admissible cyclic writing, we can use the Harder–Narasimhan automaton to find the Harder–Narasimhan filtration of any spherical object. Indeed, write

$$x = \beta P_2$$

for some braid  $\beta$ . Write  $\beta$  as an admissible cyclic expression. Starting at an appropriate vertex supporting  $P_2$ , simply apply  $\beta$  one letter at a time, always staying within the automaton.

An important feature of this procedure is that each edge is a linear transformation on the multiplicities of the Harder–Narasimhan subquotients. Given an object whose Harder–Narasimhan subquotients consist of  $a$  copies of  $A$  and  $b$  copies of  $B$ , we say that  $(a, b)^t$  is its *multiplicity vector* at  $[A, B]$ .

**Proposition 2.3.** *Let  $e: [A, B] \xrightarrow{\sigma} [C, D]$  be an edge of the Harder–Narasimhan automaton corresponding to an autoequivalence  $\sigma$ . There exists a  $2 \times 2$  integer matrix  $M_e$  with the following property. If  $x$  is an object supported at  $[A, B]$  with multiplicity vector  $(a, b)^t$ , the multiplicity vector of  $\sigma(x)$  at  $[C, D]$  is  $M_e \cdot (a, b)^t$ .*

*Proof.* By Proposition 2.1,  $\sigma$  is non-overlapping for  $x$ . Hence, the Harder–Narasimhan filtration of  $\sigma(x)$  is a refinement of  $\sigma$  applied to the Harder–Narasimhan filtration of  $x$ . The matrix  $M_e$  is then simply the matrix whose columns are the multiplicity vectors of  $\sigma(A)$  and  $\sigma(B)$  at  $[C, D]$ .  $\square$

The following example illustrates the proposition.

**Example 2.4.** Let  $e: [P_2, X] \xrightarrow{\sigma_1} [P_1, P_2]$ . The associated matrix  $M_e$  is

$$M_e = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note that the columns of  $M_e$  are the multiplicity vectors of  $\sigma_1(P_2)$  and  $\sigma_1(X)$  in  $[P_1, P_2]$ . Let  $x$  be the object  $\sigma_2^2 \gamma P_2$ . Explicitly, we have

$$x = P_2[-1] \rightarrow X, \quad \sigma_1 x = P_1[-1] \rightarrow P_2[-1] \rightarrow P_2.$$

The multiplicity vector of  $x$  at  $[P_2, X]$  is  $(1, 1)^t$ , and the multiplicity vector of  $\sigma_1 x$  at  $[P_1, P_2]$  is  $(1, 2)^t$ . Indeed,  $(1, 2)^t = M_e \cdot (1, 1)^t$ .

Recall that we have an identification  $\mathbf{S} = \mathbb{P}^1(\mathbb{Z})$  via the map (1). Note that  $\mathbf{S} \subset \mathbb{P}^1(\mathbb{R})$  are simply the rational points. The division of  $\mathbf{S}$  according to the support corresponds nicely to a geometric division of the circle  $\mathbb{P}^1(\mathbb{R})$ , which we now describe. The three objects  $P_1$ ,  $P_2$ , and  $X$  in  $\mathbb{P}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{R}) \cong S^1$  divide  $\mathbb{P}^1(\mathbb{R})$  into three closed arcs (see Figure 4). We denote these arcs by  $[P_1, P_2]$ ,  $[P_2, X]$ , and  $[X, P_1]$ . Explicitly, the arc  $[P_1, P_2]$  is the closure of the unique connected component of  $\mathbb{P}^1(\mathbb{R}) \setminus \{P_1, P_2\}$  that does not contain  $X$ , and likewise for the other two arcs. Alternatively, recalling

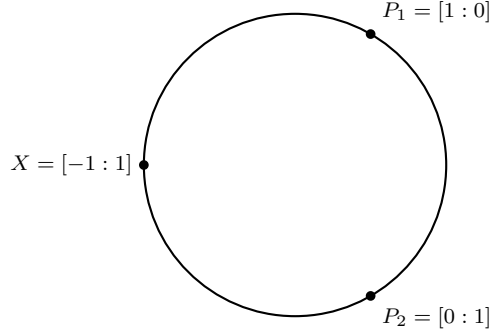


FIGURE 4. The points  $P_1$ ,  $P_2$ , and  $X = P_2 \rightarrow P_1$  divide  $\mathbb{P}^1(\mathbb{R})$  into three arcs. The Harder–Narasimhan pieces of an object only include the two endpoints of the arc on which the object lies.

that the points  $P_1$ ,  $P_2$ , and  $X$  in coordinates are  $[1 : 0]$ ,  $[0 : 1]$ , and  $[1 : -1]$ , we see that the three arcs are

$$\begin{aligned} [P_1, P_2] &= \{[a : c] \mid 0 \leq a/c \leq \infty\}, \\ [P_2, X] &= \{[a : c] \mid -1 \leq a/c \leq 0\}, \text{ and} \\ [X, P_1] &= \{[a : c] \mid -\infty \leq a/c \leq -1\}. \end{aligned}$$

Although we have written  $\infty$  and  $-\infty$  distinctly, this is only for the purpose of writing sensible inequalities; the two represent the same point on  $\mathbb{P}^1(\mathbb{R})$ .

**Proposition 2.5.** *The objects of  $\mathbf{S}$  corresponding to the points of the arc  $[P_1, P_2]$  are supported on the vertex  $[P_1, P_2]$ , and likewise for the other two arcs.*

*Proof.* The statement is clearly true for the objects  $P_1$ ,  $P_2$ , and  $X$ . For the general case, it suffices to show that if we replace each vertex in the Harder–Narasimhan automaton by the corresponding arc, and the autoequivalences by the corresponding transformations in  $\mathrm{PSL}_2(\mathbb{Z})$ , then the transformation takes the source arc to the target arc. This is easily verified.  $\square$

Let  $x$  be a spherical object corresponding to  $[a : c] \in \mathbb{P}^1(\mathbb{Z})$ , where  $a, c$  are relatively prime integers.

**Proposition 2.6.** *We have the following equalities.*

- (1) *The minimal complex of projective modules representing  $x$  has exactly  $|a|$  occurrences of  $P_1$  and  $|c|$  occurrences of  $P_2$ .*
- (2)  *$\overline{\mathrm{hom}}(x, P_2) = |a|$  and  $\overline{\mathrm{hom}}(x, P_1) = |c|$ .*

The first part of the proposition is [10, Proposition 4.8]. Using the Harder–Narasimhan automaton gives a new and simpler proof.

*Proof.* We begin by proving the first assertion. Write

$$x = \beta P_2,$$

for a braid  $\beta$ , and express  $\beta$  in an admissible cyclic form. Recall that  $P_2$  corresponds to  $a = 0$  and  $c = 1$ , and hence satisfies the conclusion. By Proposition 2.3, we see that each edge in the automaton changes the number of occurrences of  $P_1$  and  $P_2$  in the minimal complex by a linear transformation. It suffices to verify that this linear transformation is the same as the linear transformation on  $(a, c)$ , up to sign. We carry out the verification for the vertex  $[P_1, P_2]$ , leaving the other two vertices to the reader. The matrices of  $\gamma, \gamma^{-1}, \sigma_1, \sigma_X \in \mathrm{PSL}_2(\mathbb{Z})$  are

$$\gamma = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_X = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

On the other hand, the autoequivalences change the number of occurrences of  $P_1$  and  $P_2$  from  $(m, n)$  to

$$\begin{aligned} (m, n) &\xrightarrow{\gamma} (m+n, m), & (m, n) &\xrightarrow{\gamma^{-1}} (n, m+n), \\ (m, n) &\xrightarrow{\sigma_1} (m+n, n), & (m, n) &\xrightarrow{\sigma_X} (2m+n, m). \end{aligned}$$

Up to sign, the two transformations agree.

Let us now prove the second assertion. We prove it for  $\overline{\mathrm{hom}}(x, P_1)$ ; the statement for  $\overline{\mathrm{hom}}(x, P_2)$  follows by applying  $\gamma$ .

If  $x = P_1$ , then the statement is evident. Suppose  $x$  is supported at  $[X, P_2]$ . Consider the Harder–Narasimhan filtration of  $x$ :

$$0 = x_0 \rightarrow \cdots \rightarrow x_n = x,$$

with sub-quotients  $P_2$  and  $X$ , up to shift. By applying  $\mathrm{Hom}(P_1, -)$ , we obtain a filtration

$$0 = \mathrm{Hom}(P_1, y_0) \rightarrow \cdots \rightarrow \mathrm{Hom}(P_1, y_n) = \mathrm{Hom}(P_1, y),$$

in the bounded derived category of (graded) vector spaces. The sub-quotients of this filtration are obtained by applying  $\mathrm{Hom}(P_1, -)$  to the sub-quotients of the original filtration. Since both  $\mathrm{Hom}(P_1, P_2)$  and  $\mathrm{Hom}(P_1, X)$  are one-dimensional, these sub-quotients are one-dimensional vector spaces. The boundary maps between these sub-quotients are induced by the boundary maps between the sub-quotients in the filtration of  $x$ . The filtration of  $x$  has three possible non-zero boundary maps:

$$P_2 \rightarrow P_2[2], \quad X \rightarrow X[2], \quad \text{and} \quad P_2 \rightarrow X[2].$$

All three are killed by  $\mathrm{Hom}(P_1, -)$  for degree reasons. As a result, we conclude that

$$\overline{\mathrm{hom}}(P_1, x) = \overline{\mathrm{hom}}(x, P_1) = n = \text{number of } P_2\text{'s in the minimal complex of } x.$$



To prove the general case, it suffices to show that for any object  $x$  (other than a twist of  $P_1$ ), there exists an  $r$  for which  $\sigma_1^r x$  is supported by  $[P_2, X]$ . Indeed, by the first part of the proposition,  $x$  and  $\sigma_1^n x$  have the same number of occurrences of  $P_2$ , and plainly, both have the same value for  $\overline{\text{hom}}(P_1)$ . Say  $x = \beta P_1$ , and suppose  $\beta$  has the admissible cyclic expression

$$\beta = \sigma_{a_1}^{m_1} \cdots \sigma_{a_k}^{m_k} \gamma^n.$$

We induct on the quantity  $\ell(\beta) = \sum m_i$ . If  $\ell(\beta) = 0$ , then  $x = P_2$  or  $x = X$ , and it is already supported at  $[P_2, X]$ . If  $a_1 = 2$ , then  $x$  is already supported at  $[P_2, X]$ . If  $a_1 = 1$ , then  $\ell(\sigma_1^{-1}\beta) < \ell(\beta)$ . If  $a_1 = X$ , then  $\ell(\sigma_1\beta) < \ell(\beta)$ . To see this, note that

$$\sigma_1\beta = \gamma\sigma_{a_1}^{m_1-1}\sigma_{a_2}^{m_2}\cdots\sigma_{a_k}^{m_k}\gamma^n.$$

By reducing as described in Proposition 2.2, we see that  $\beta'$  has smaller  $\ell$  than  $\beta$ . The proof is now complete.  $\square$

Let  $x$  be a spherical object corresponding to  $[a : c] \in \mathbb{P}^1(\mathbb{Z})$  for relatively prime integers  $a, c$ .

**Corollary 2.7.** *The  $\tau$ -mass of  $x$  is given by*

$$m_\tau(x) = \begin{cases} |a|m_\tau(P_1) + |c|m_\tau(P_2) & \text{if } x \in [P_1, P_2], \\ |c|m_\tau(X) + (|a| - |c|)m_\tau(P_1) & \text{if } x \in [P_2 \rightarrow P_1, P_1], \\ |a|m_\tau(X) + (|c| - |a|)m_\tau(P_2) & \text{if } x \in [P_2, X]. \end{cases}$$

*Proof.* Count the number of Harder–Narasimhan pieces using the counts of  $P_1$  and  $P_2$ .  $\square$

*Remark 2.8.* The automaton also allows us to compute the categorical entropy of a braid  $\beta$ , and to show that it is the Perron-Frobenius eigenvalue of the matrix of absolute values of  $\bar{\beta} \in \text{PSL}_2(\mathbb{Z})$ . In particular, the categorical and topological entropies are equal.

**Corollary 2.9.** *The values of the basic hom functions on  $x$  are given by*

- (1)  $\overline{\text{hom}}(P_1, x) = |c|$ ,
- (2)  $\overline{\text{hom}}(P_2, x) = |a|$ ,
- (3)  $\overline{\text{hom}}(X, x) = |a + c|$ ,
- (4)  $\overline{\text{hom}}(P_1 \rightarrow P_2, x) = |a - c|$ .

*Proof.* The first two follow directly from Proposition 2.6. For the last two, use

$$\overline{\text{hom}}(X, x) = \overline{\text{hom}}(P_1, \sigma_2^{-1}x),$$

and

$$\overline{\text{hom}}(P_1 \rightarrow P_2, x) = \overline{\text{hom}}(P_2, \sigma_1^{-1}x).$$

The result follows by using the action of  $\sigma_i^{\pm 1}$  on  $(a, c)$  and applying Proposition 2.6.  $\square$

### 3. THE BOUNDARY

We now have all the tools to analyse the map  $h: \mathbf{S} \rightarrow \mathbb{P}^{\mathbf{S}}$  defined in the introduction. We begin by recalling the definition. Consider the map

$$\overline{\text{hom}}: \mathbf{S} \rightarrow \mathbb{R}^{\mathbf{S}}$$

defined by

$$\overline{\text{hom}}(x): y \mapsto \overline{\text{hom}}(x, y).$$

The map  $h: \mathbf{S} \rightarrow \mathbb{P}^{\mathbf{S}}$  is the composition of  $\overline{\text{hom}}$  and the projection  $\mathbb{R}^{\mathbf{S}} \setminus 0 \rightarrow \mathbb{P}^{\mathbf{S}}$ .

Consider  $\mathbf{S}$  as a subset of  $\mathbb{P}^1(\mathbb{R})$  via the identification  $\mathbf{S} = \mathbb{P}^1(\mathbb{Z})$  as defined in (1).

**Proposition 3.1.** *The map  $h: \mathbf{S} \rightarrow \mathbb{P}(\mathbb{R}^{\mathbf{S}})$  extends to a continuous map  $h: \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{P}^{\mathbf{S}}$ . The extension maps  $\mathbb{P}^1(\mathbb{R})$  homeomorphically onto its image.*

*Proof.* Let  $s \in \mathbf{S}$  be a spherical object (up to shift). Write  $s = \beta P_2$  for some braid  $\beta$ . Let  $\bar{\beta}$  be the image of  $\beta$  in  $\text{PSL}_2(\mathbb{Z})$ . Then  $\bar{\beta}$  is uniquely determined up to right multiplication by powers of  $\bar{\sigma}_2$ . Consider a point  $[a : c] \in \mathbb{P}^1(\mathbb{Z})$ , where  $a$  and  $c$  are relatively prime integers, and let  $t \in \mathbf{S}$  be the corresponding spherical object. We have

$$\begin{aligned} \overline{\text{hom}}(s, t) &= \overline{\text{hom}}(\beta P_2, t) \\ &= \overline{\text{hom}}(P_2, \beta^{-1}t) \\ &= |(1, 0) \cdot \bar{\beta}^{-1} \cdot (a, c)^t| \text{ by Proposition 2.6.} \end{aligned}$$

We define  $h: \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{P}(\mathbb{R}^{\mathbf{S}})$  by using the final expression for an arbitrary  $[a : c]$ . That is, we set  $h([a : c])$  to be the (projectivised) function whose value at  $s = \beta P_2$  is

$$|(1, 0) \cdot \bar{\beta}^{-1} \cdot (a, c)^t|.$$

Plainly,  $h: \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{P}^{\mathbf{S}}$  is a continuous extension of the original map  $h: \mathbf{S} \rightarrow \mathbb{P}^{\mathbf{S}}$ .

We now check that the extended map is a homeomorphism onto its image. Since the domain  $\mathbb{P}^1(\mathbb{R})$  is compact and the target  $\mathbb{P}(\mathbb{R}^{\mathbf{S}})$  is Hausdorff, it suffices to check that it is injective. Let

$$T = \{P_1, P_2, X\}.$$

Consider the restricted map  $h_T: \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{P}(\mathbb{R}^T)$  obtained as the composition of  $h$  and the projection to  $\mathbb{P}^T$ . From Corollary 2.9,  $h_T$  is given in coordinates by

$$h_T: [a : c] \mapsto [|a| : |c| : |a + c|].$$

This map is injective, and hence so is  $h$ .  $\square$

#### 4. THE INTERIOR

We now study the map  $m: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{P}^{\mathbf{S}}$  defined in the introduction. To recall,  $m$  sends a stability condition  $\tau$  (modulo the  $\mathbb{C}$  action) to the function  $m_\tau: \mathbf{S} \rightarrow \mathbb{R}$  (well-defined up to scaling) defined by

$$m_\tau: x \mapsto m_\tau(x).$$

We begin by proving that the image of  $m$  is pre-compact. This is a fairly general phenomenon, so let us work generally for the moment. Let  $\mathcal{C}$  be any triangulated category that admits a heart with finitely many simple objects  $P_1, \dots, P_n$ . Let  $\mathbf{S} \subset \mathcal{C}$  be any set that contains  $P_1, \dots, P_n$ . Define the map  $m: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{P}^{\mathbf{S}}$  as before.

**Proposition 4.1** (Pre-compactness). *In the setup above, the closure of the image of  $m$  is compact.*

*Proof.* Let  $\tilde{m}: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{R}^{\mathbf{S}}$  be the lift of  $m$  characterised by

$$\sum \tilde{m}(P_i) = 1.$$

Let  $\tilde{B}$  be the closure in  $\mathbb{R}^{\mathbf{S}}$  of the image of  $\tilde{m}$ .

Recall that for an exact triangle

$$x \rightarrow y \rightarrow z \xrightarrow{+1},$$

we have the following triangle inequality [7, Proposition 3.3]:

$$\tilde{m}_\tau(y) \leq \tilde{m}_\tau(x) + \tilde{m}_\tau(z).$$

For every  $s \in \mathbf{S}$ , there exists an  $n = n(s)$  and a filtration

$$0 = s_0 \rightarrow s_1 \rightarrow \dots \rightarrow s_n = s,$$

where the sub-quotients are twists of  $P_i$ . By the triangle inequality, and the normalisation  $\sum \tilde{m}(P_i) = 1$ , we have

$$\tilde{m}_\tau(s) \leq n = n(s).$$

In other words,  $\tilde{m}$  maps  $\text{Stab}(\mathcal{C})/\mathbb{C}$  to the product  $\prod_{s \in \mathbf{S}} [0, n(s)]$ . By the Tychonoff theorem, the product is compact, and hence  $\tilde{B}$  is compact. Thanks to the normalisation  $\sum \tilde{m}(P_i) = 1$ , we see that  $\tilde{B}$  is contained in  $\mathbb{R}^{\mathbf{S}} \setminus 0$ . Finally, note that the closure of the image of  $m$  in  $\mathbb{P}^{\mathbf{S}}$  is contained in the image of  $\tilde{B}$  under the projection map, which is a compact set. Hence, the closure of the image of  $m$  is also compact.  $\square$

We now return to the case at hand: let  $\mathcal{C}$  and  $\mathbf{S}$  once again be as defined for the  $A_2$  case in § 2. Having proved pre-compactness, the next easiest part of the main theorem is the injectivity.

**Proposition 4.2** (Injectivity). *The map  $m: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{P}^{\mathbf{S}}$  is injective.*

*Proof.* Let  $f \in \mathbb{P}^{\mathbf{S}}$  be an element in the image of  $m$ , say  $f = m(\tau)$ . By applying a braid, we may assume that  $\tau$  is a standard stability condition as defined in § 2.2. We must show that if  $f = m(\tau')$  for some other stability condition  $\tau'$ , then  $\tau = \tau'$ .

Since  $\tau$  is standard,  $P_1$ ,  $P_2$ , and  $X = P_2 \rightarrow P_1$  are  $\tau$ -semistable. Therefore, they have minimal  $\tau$ -mass among the objects of  $S$  of classes  $[P_1]$ ,  $[P_2]$ , and  $[X]$  respectively in the Grothendieck group. Therefore, if  $f = m(\tau')$ , then they also have minimal  $\tau'$  mass among the objects of the respective classes in the Grothendieck group. As a result,  $P_1$ ,  $P_2$ , and  $X$  are also  $\tau'$ -semistable, and hence  $\tau'$  is standard. It remains to show that, up to the  $\mathbb{C}$  action,  $\tau$  and  $\tau'$  have the same central charge.

By changing  $\tau$  and  $\tau'$  by a rotation, we may assume that

$$0 = \phi_{\tau}(P_1) \leq \phi_{\tau}(P_2) < 1,$$

and

$$0 = \phi_{\tau'}(P_1) \leq \phi_{\tau'}(P_2) < 1.$$

Furthermore, after scaling one of them, we may assume that  $m_{\tau} = m_{\tau'}$ .

Consider the three vectors  $v_1 = Z(P_1)$ ,  $v_2 = Z(X)$ , and  $v_3 = Z(P_2)$  in the upper half plane, where  $Z$  denotes the central charge of either of the two stability conditions. We know that these vectors are oriented anti-clockwise, and have the same lengths for both stability conditions. Since the vectors also satisfy  $v_2 = v_1 + v_3$ , we may think of them as the vectors corresponding to three sides of a triangle. As the lengths of the three sides of a triangle determine the triangle up to isometry, it follows that the three vectors  $v_i$  must be the same for  $\tau$  and  $\tau'$ . In other,  $\tau$  and  $\tau'$  have the same central charge.  $\square$

*Remark 4.3.* Essentially the same proof shows that  $m$  is injective for the 2-Calabi–Yau category of any finite, connected quiver. Indeed, looking at the mass-minimising objects in the mass function lets us find the (semi-)stable objects. By using the linear relationships among their classes in the Grothendieck group, we can reconstruct  $Z$ .

To move beyond pre-compactness and injectivity, we must understand the structure of mass functions in more detail.

Let  $\tau$  be a standard stability condition. Since the three positive real numbers  $m(P_1)$ ,  $m(P_2)$ , and  $m(X)$  satisfy the triangle inequalities, there exist non-negative real numbers  $x$ ,  $y$ ,  $z$  such that

$$m_{\tau}(P_1) = y + z, \quad m_{\tau}(P_2) = z + x, \quad m_{\tau}(X) = x + y.$$

We call the  $x$ ,  $y$ ,  $z$ , the *Gromov coordinates* of  $\tau$ . Note that if one of the coordinates is zero, then  $\tau$  is degenerate. Two of the coordinates cannot be zero.

Recall that  $m_{\tau}: \mathbf{S} \rightarrow \mathbb{R}$  is the mass function associated to  $\tau$  and  $\overline{\text{hom}}(s)$  the reduced hom functional associated to an object  $s$ .

**Proposition 4.4** (Linearity). *We have*

$$m_\tau = x \overline{\text{hom}}(P_1) + y \overline{\text{hom}}(P_2) + z \overline{\text{hom}}(X).$$

*Proof.* Let  $s \in \mathbf{S}$  be a spherical object. Observe that the Harder–Narasimhan filtration of  $s$  is the same for all standard stability conditions, and its pieces are  $P_1$ ,  $P_2$ , and  $X$ , up to shift. Denoting by  $a(s)$ ,  $b(s)$ , and  $c(s)$  the multiplicities of these three in the Harder–Narasimhan filtration, we have

$$m_\tau(s) = (y + z)a(s) + (x + z)b(s) + (x + y)c(s).$$

In particular,  $m_\tau(s)$  is linear in the Gromov coordinates. We have

$$\begin{aligned} b(s) + c(s) &= \text{number of occurrences of } P_2 \text{ in the minimal complex of } s \\ &= \overline{\text{hom}}(P_1, s), \\ a(s) + c(s) &= \text{number of occurrences of } P_1 \text{ in the minimal complex of } s \\ &= \overline{\text{hom}}(P_2, s), \text{ and} \\ a(s) + b(s) &= \overline{\text{hom}}(X, s). \end{aligned}$$

The first two equalities are evident. The third is obtained by applying  $\gamma$  to the second. We conclude that

$$m_\tau(s) = (y + z) \overline{\text{hom}}(P_1, s) + (x + z) \overline{\text{hom}}(P_2, s) + (x + y) \overline{\text{hom}}(X, s).$$

□

Since every stability condition is in the  $B_3$ -orbit of a standard one, linearity holds for all of them. Indeed, let  $\tau$  be an arbitrary stability condition. Then there are three  $\tau$  semi-stable objects, say  $A$ ,  $B$ ,  $C$ , whose classes in the Grothendieck group are (up to sign) the classes of  $P_1$ ,  $P_2$ , and  $X$ . These are obtained simply by applying an appropriate braid to  $P_1$ ,  $P_2$ , and  $X$ . Define the Gromov coordinates  $x$ ,  $y$ ,  $z$  for  $\tau$  by the condition

$$m_\tau(A) = y + z, \quad m_\tau(B) = x + z, \quad m_\tau(C) = x + y.$$

With the above notation, we have

$$m_\tau = x \overline{\text{hom}}(A) + y \overline{\text{hom}}(B) + z \overline{\text{hom}}(C).$$

We now use the Gromov coordinates to get a simple geometric picture of  $\text{Stab}(\mathcal{C})/\mathbb{C}$ . Let  $\Delta$  denote the following clipped triangle:

$$\begin{aligned} \Delta &= \{(x, y, z) \in \mathbb{R}_{\geq 0}^3 \mid \text{at least two coordinates non-zero}\} / \mathbb{R}_{> 0} \\ &\cong \text{a closed planar triangle minus the three vertices.} \end{aligned}$$

Let  $\Lambda \subset \text{Stab}(\mathcal{C})/\mathbb{C}$  be the closed subset of standard stability conditions. The Gromov coordinates allow us to identify  $\Lambda$  with the clipped triangle  $\Delta$ . Since  $\Lambda$  tessellates

for the action of  $\mathrm{PSL}_2(\mathbb{Z})$ , we obtain the complete picture of  $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$  as unions of copies of  $\Delta$ .

Let us describe this picture in more detail. Recall that the element  $\gamma = \sigma_2\sigma_1$  generates the stabiliser of  $\Lambda$  in  $\mathrm{PSL}_2(\mathbb{Z})$ . So, we can label the various (distinct) translates of  $\Lambda$  by  $\langle\gamma\rangle$ -cosets. Two translates of  $\Lambda$  are either disjoint or intersect along an edge, which is a copy of the open interval. The three translates that intersect  $\Lambda$  along its three edges are  $\sigma_1\Lambda$ ,  $\sigma_X\Lambda$ , and  $\sigma_2\Lambda$ . Consequently, the three translates that intersect  $\beta\Lambda$  are  $\beta\sigma_1\Lambda$ ,  $\beta\sigma_X\Lambda$ , and  $\beta\sigma_2\Lambda$ . We can encode the translates and their intersections in a graph (called the *exchange graph* in [2]). Its vertices are left  $\gamma$ -cosets in  $\mathrm{PSL}_2(\mathbb{Z})$ , and a coset  $\beta\langle\gamma\rangle$  is connected by an edge with  $\beta\sigma_1\langle\gamma\rangle$ ,  $\beta\sigma_X\langle\gamma\rangle$ , and  $\beta\sigma_2\langle\gamma\rangle$  (see Figure 5).

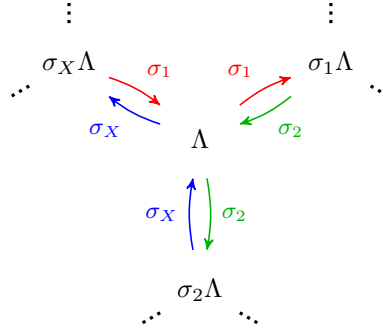


FIGURE 5. Exchange graph showing the action of  $\mathrm{PSL}_2(\mathbb{Z})$  on the set of standard stability conditions  $\Lambda \subset \mathrm{Stab}(\mathcal{C})/\mathbb{C}$ .

Let

$$\overline{\Delta} = (\mathbb{R}_{\geq 0}^3 \setminus \{0\}) / \mathbb{R}_{>0}.$$

Then  $\overline{\Delta}$  is homeomorphic to a closed planar triangle. An immediate consequence of the linearity (from Proposition 4.4) is the following.

**Proposition 4.5** (Closure of  $\Lambda$ ). *The closure of  $\Lambda$  in  $\mathbb{P}^{\mathbf{S}}$  is*

$$\overline{\Lambda} = \Lambda \cup \{\overline{\mathrm{hom}}(P_1), \overline{\mathrm{hom}}(P_2), \overline{\mathrm{hom}}(X)\}.$$

*This closure is homeomorphic to  $\overline{\Delta}$ .*

*Proof.* Identify  $\Lambda$  with  $\Delta$  using the Gromov coordinates. From Proposition 4.4, we see that the map  $m: \Delta \rightarrow \mathbb{P}^{\mathbf{S}}$  extends to a continuous map  $\overline{\Delta} \rightarrow \mathbb{P}^{\mathbf{S}}$ . Its image is closed, and equals the union of  $\Lambda$  and the three additional points described above. The map from  $\overline{\Delta}$  is injective, and hence an isomorphism onto its image.  $\square$

*Remark 4.6.* Proposition 4.5 shows how the  $\overline{\mathrm{hom}}$  functions appear in the closure of  $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$ . There is another way in which  $\overline{\mathrm{hom}}(x)$ , for a spherical  $x$ , can be obtained

as a limit of mass functions of stability conditions, which holds more generally. Let  $\tau$  be any stability condition. Then the limit of the mass functions of  $\sigma_x^n \tau$  in  $\mathbb{P}^{\mathbf{S}}$ , as  $n \rightarrow \infty$  is the function  $\overline{\text{hom}}(x)$ . Since we will not use this fact, we omit the proof. This is an analogue of the fact in Teichmüller theory that in the Thurston compactification, one approaches the PMF given by a simple closed curve  $\gamma$  by repeatedly applying Dehn twists in  $\gamma$ .

The following proposition gives an important criterion to separate standard and non-standard stability conditions.

**Proposition 4.7** (Degeneracy). *Let  $\tau$  be a non-standard stability condition. Then the triangle inequalities for the three real numbers  $m_\tau(P_1)$ ,  $m_\tau(P_2)$ , and  $m_\tau(X)$  are degenerate. That is, one number is the sum of the other two.*

*Proof.* Write  $\tau = \beta\tau'$ , where  $\tau'$  is standard. Then  $m_\tau(-) = m_{\tau'}(\beta^{-1}-)$ . Write  $\beta^{-1}$  in a (slightly modified) admissible cyclic form

$$\beta^{-1} = \gamma^r \sigma_{a_1}^{m_1} \cdots \sigma_{a_r}^{m_r}.$$

We apply  $\beta^{-1}$  letter by letter to  $P_1$ ,  $P_2$ , and  $X$  and keep track of their Harder–Narasimhan filtrations using the Harder–Narasimhan automaton. After the first step, we have the objects

$$\sigma_{a_r} P_1, \quad \sigma_{a_r} P_2, \quad \sigma_{a_r} X.$$

The key observation is that all three objects are supported by the *same* vertex—the one with  $\sigma_{a_r}$  as a loop. Furthermore, the Harder–Narasimhan multiplicity vector of one is the sum of the other two. Indeed, we have the following, where  $[-]$  denotes the multiplicity vector:

$$\begin{aligned} [\sigma_1(P_2)] &= [\sigma_1(P_1)] + [\sigma_1(X)] \\ [\sigma_2(X)] &= [\sigma_2(P_1)] + [\sigma_2(P_2)] \\ [\sigma_X(P_1)] &= [\sigma_X(P_2)] + [\sigma_X(X)]. \end{aligned}$$

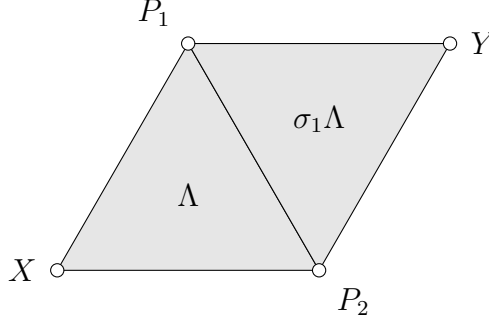
By Proposition 2.3, all further transformations preserve this linear relationship. The proof is now complete.  $\square$

*Remark 4.8.* Consider the set of non-standard stability conditions in  $\text{Stab}(\mathcal{C})/\mathbb{C}$ ; that is, the complement of  $\Lambda$ . This set has three connected components, corresponding to the three connected components of the exchange graph minus the central vertex. The component containing  $\sigma_i \Lambda$ , for  $i = 1, 2, X$ , contains  $\beta \Lambda$  for  $\beta$  which have an admissible cyclic writing beginning with  $\sigma_i$ .

From the proof of Proposition 4.7, we see that the three components also correspond to the three ways in which the triangle inequality among the masses of  $P_1$ ,  $P_2$ , and  $X$  degenerates. On the component containing  $\sigma_1 \Lambda$ , the mass of  $X$  is the sum of the masses of  $P_1$  and  $P_2$ , and likewise for the other two.

**Proposition 4.9.** *The map  $m: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{P}^{\mathbf{S}}$  is a homeomorphism onto its image.*

*Proof.* Consider the set  $T = \{P_1, P_2, X, Y\}$ , where  $X = P_2 \rightarrow P_1$  and  $Y = P_1 \rightarrow P_2$ . Say that a stability condition is *semi-standard* if its semi-stable objects, up to twist, are among the objects in  $T$ . If  $\tau$  is semi-standard, then up to rotation, its heart is the standard heart. Also observe that the set of semi-standard stability conditions is the union  $\Pi = \Lambda \cup \sigma_1 \Lambda$ , as shown in the figure below.



Note that the interior  $\Pi^\circ$  of  $\Pi$  is an open subset of  $\text{Stab}(\mathcal{C})/\mathbb{C}$  homeomorphic to the interior of the unit square.

Consider the map  $m_T: \text{Stab}(\mathcal{C})/\mathbb{C} \rightarrow \mathbb{P}^T$  obtained by composing  $m$  with the standard projection. Consider the subset  $\Omega \subset \mathbb{P}^T$  defined by

$$\Omega = \{[x, y, z, x + y] \mid x, y, z \geq 0\} \cup \{[x, y, x + y, z] \mid x, y, z \geq 0\}.$$

By Proposition 4.7, the image of  $m_T$  is contained in  $\Omega$ . Furthermore, the stability conditions that are not semi-standard map to the boundary of  $\Omega$ . As a consequence, the pre-image of  $\Omega^\circ$  is  $\Pi^\circ$ . The map  $m_T: \Pi^\circ \rightarrow \Omega^\circ$  is clearly a homeomorphism. As the translates of  $\Pi^\circ$  under the braid group cover  $\text{Stab}(\mathcal{C})/\mathbb{C}$ , we conclude that  $m$  is a homeomorphism onto its image.  $\square$

## 5. CLOSURE OF THE INTERIOR

From Proposition 4.1, we know that the closure of  $\text{Stab}(\mathcal{C})/\mathbb{C}$  in  $\mathbb{P}^{\mathbf{S}}$  is compact. The goal of this section is to identify this closure. Set

$$P = \overline{h(\mathbf{S})} = h(\mathbb{P}^1(\mathbb{R})) \subset \mathbb{P}^{\mathbf{S}}$$

$$M = m(\text{Stab}(\mathcal{C})/\mathbb{C}) \subset \mathbb{P}^{\mathbf{S}}.$$

We prove that  $\overline{M} = M \cup P$ .

It is convenient to have a measure of closeness for two elements of a projective space. Given two non-zero vectors in  $\mathbb{R}^n$  for some positive integer  $n$ , denote by  $\angle(v, w)$  the (acute) angle between them. By a slight abuse of notation, we use



$\angle(v, w)$  also for two points  $v, w \in \mathbb{P}^{n-1}$ ; this is just the angle between any two representatives in  $\mathbb{R}^n$ .

Given a finite subset  $T \subset \mathbf{S}$ , let  $h_T: \mathbf{S} \rightarrow \mathbb{P}^T$  be the composition of  $h: \mathbf{S} \rightarrow \mathbb{P}^{\mathbf{S}}$  and the projection onto the  $T$ -coordinates.

**Proposition 5.1** (Group action contracts). *Fix two elements  $A, B \in \mathbf{S}$  and a finite subset  $T \subset \mathbf{S}$ . Given an  $\epsilon > 0$ , for all but finitely many elements  $L$  of  $\mathrm{PSL}_2(\mathbb{Z})$ , we have*

$$\angle(h_T(LA), h_T(LB)) < \epsilon.$$

*Proof.* In the identification  $S = \mathbb{P}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{R})$ , let  $A$  and  $B$  be represented by two integer vectors  $a$  and  $b$  in  $\mathbb{Z}^2$ . The map  $h_T: \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{P}^T$  is continuous, and hence uniformly continuous. Therefore, it suffices to check that for all but finitely many  $L$ , the angle between  $La$  and  $Lb$  in  $\mathbb{R}^2$  is small. To see this, observe that we have

$$\sin(\angle(La, Lb)) = \frac{|\det L| \cdot |\det(a \mid b)|}{|La| \cdot |Lb|} = \frac{|\det(a \mid b)|}{|La| \cdot |Lb|}.$$

Since  $L$  is required to have integer entries, the quantity  $|L| \cdot |Lb|$  is large—greater than  $\epsilon |\det(a \mid b)|$ —for all but finitely many  $L$ . The claim follows.  $\square$

**Proposition 5.2** (Closure of  $M$ ). *The sets  $M$  and  $P$  are disjoint and their union is the closure of  $M$ .*

*Proof.* The mass of every object in a stability condition is positive. On the other hand,  $\overline{\mathrm{hom}}(x, x) = 0$ , by definition. Therefore,  $M$  and  $P$  are disjoint.

By Proposition 4.5, we know that  $P$  is contained in the closure of  $M$ . We now prove that  $M \cup P$  is closed. Let  $\tau_n$  be a sequence in  $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$  whose images  $m(\tau_n)$  in  $M$  approach a limit  $z \in \mathbb{P}^{\mathbf{S}}$ . We must show that  $z$  lies in  $M \cup P$ .

Write  $\tau_n = \beta_n \tau'_n$ , where  $\beta_n$  is a braid and  $\tau'_n$  is a standard stability condition. Let  $\overline{\beta_n}$  be the image of  $\beta_n$  in  $\mathrm{PSL}_2(\mathbb{Z})$ . We have two possibilities: the set  $\{\overline{\beta_n}\}$  is finite or infinite.

If  $\{\overline{\beta_n}\}$  is finite, then the sequence  $\{\tau_n\}$  is contained in the union of finitely many translates of the set of standard stability conditions  $\Lambda$ . By Proposition 4.5, the closure of  $\Lambda$  is contained in  $M \cup P$ . Hence, the closure of the union of finitely many translates of  $\Lambda$  is also contained in  $M \cup P$ . So the limit  $z$  lies in  $M \cup P$ .

Suppose  $\{\overline{\beta_n}\}$  is infinite. Set  $A_n = \beta_n(P_1)$ ,  $B_n = \beta_n(P_2)$ , and  $C_n = \beta_n(X)$ . Since  $P$  is compact, we may assume, after passing to a subsequence if necessary, that  $h(A_n)$ ,  $h(B_n)$ , and  $h(C_n)$  have limits in  $P$ , say  $A$ ,  $B$ , and  $C$ .

We first observe that  $A = B = C$ . To see this, let  $T \subset \mathbf{S}$  be an arbitrary finite set, and use the subscript  $T$  to denote projections to  $\mathbb{P}^T$ . It suffices to show that  $A_T = B_T = C_T$ . Since  $\{\overline{\beta_n}\}$  is infinite, by Proposition 5.1, we note that the angles

between  $h_T(A_n)$ ,  $h_T(B_n)$ , and  $h_T(C_n)$  approach 0 as  $n$  approaches  $\infty$ . Therefore, the three sequences have the same limits. But these limits are  $A_T$ ,  $B_T$ , and  $C_T$ .

Next, we claim that  $z = h(A) = h(B) = h(C)$ . Indeed, by linearity (Proposition 4.4), we know that there exist non-negative real numbers  $x_n, y_n, z_n$  such that

$$m_T(\tau_n) = x_n h_T(A_n) + y_n h_T(B_n) + z_n h_T(C_n).$$

Taking the limit as  $n \rightarrow \infty$  in projective space yields

$$z = h(A) = h(B) = h(C).$$

In particular,  $z$  lies in  $M \cup P$ . The proof is thus complete.  $\square$

## 6. HOMEOMORPHISM TO A DISK

We take up the final part of the main theorem, namely that  $(\overline{M}, P)$  is a manifold with boundary homeomorphic to the unit disk. We explicitly construct a homeomorphism from  $\overline{M}$  to the disk, using the unprojectivised, but suitably normalised, mass and hom functions.

Fix

$$T = \{P_1, P_2, X = P_2 \rightarrow P_1\}.$$

Define a map

$$\mu: \text{Stab}(\mathcal{C})/i\mathbb{R} \rightarrow \mathbb{R}^T = \mathbb{R}^3$$

by

$$\mu: \tau \mapsto (m_\tau(P_1), m_\tau(P_2), m_\tau(X)).$$

Similarly, define

$$\eta: \mathbf{S} \rightarrow \mathbb{R}^T = \mathbb{R}^3$$

by

$$\eta(s) = (\overline{\text{hom}}(s, P_1), \overline{\text{hom}}(s, P_2), \overline{\text{hom}}(s, X)).$$

Thinking of  $\mathbf{S}$  as  $\mathbb{P}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{R})$ , it is easy to see that  $\eta$  extends to a continuous map

$$\eta: \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{R}^3.$$

Indeed, using Corollary 2.9, we get

$$\eta: [a : c] \mapsto (|c|, |a|, |a + c|).$$

Let  $\tau$  be a stability condition with (semi)-stable objects  $A, B$ , and  $C$  of class  $[P_1]$ ,  $[P_2]$ , and  $[X]$  in the Grothendieck group. Let  $x, y, z$  be the Gromov coordinates of  $\tau$ , namely the non-negative real numbers such that

$$m_\tau(A) = y + z, \quad m_\tau(B) = z + x, \quad m_\tau(C) = x + y.$$

Denote by  $|\cdot|$  the standard Euclidean norm in  $\mathbb{R}^3$ . We say that  $\tau$  is *normalised* if

$$(6) \quad x|\eta(A)| + y|\eta(B)| + z|\eta(C)| = 1.$$

*Remark 6.1.* We point out one subtle aspect of the definition. If  $\tau$  is degenerate—that is, it has strictly semi-stable objects—then one of the  $A$ ,  $B$ , or  $C$  is not uniquely determined, even up to shift. In this case, however, the corresponding Gromov coordinate is 0, and hence (6) is well-posed.

The normalisation yields a continuous section of

$$\mathrm{Stab}(\mathcal{C})/i\mathbb{R} \rightarrow \mathrm{Stab}(\mathcal{C})/\mathbb{C}.$$

That is, for every stability condition  $\tau$  up to rotation and scaling, there is a unique normalised stability condition  $\tau^\nu$  up to rotation, and furthermore the map  $\tau \rightarrow \tau^\nu$  is continuous.

We now identify  $\mathrm{Stab}(\mathcal{C})/\mathbb{C}$  with its image  $M$  in  $\mathbb{P}^{\mathbf{S}}$  and  $\mathbb{P}^1(\mathbb{R})$  with its image  $P$  in  $\mathbb{P}^{\mathbf{S}}$ . Define

$$(7) \quad \pi: \overline{M} = M \cup P \rightarrow \mathbb{R}^3$$

as follows. For  $\tau \in M$ , set

$$(8) \quad \pi(\tau) = \mu(\tau^\nu),$$

and for  $s \in P$ , set

$$\pi(s) = \eta(s)/|\eta(s)|.$$

By construction,  $\pi(a : c)$  is the unit vector in the direction of

$$\eta(a : c) = (|a|, |c|, |a + c|).$$

That is,  $\pi$  maps  $P$  to the unit sphere in  $\mathbb{R}^3$ . It is easy to see that  $\pi|_P$  is injective, and hence a homeomorphism onto its image. The image consists of three circular arcs, one for each pair of end-points from the three points

$$\frac{1}{\sqrt{2}}(0, 1, 1), \quad \frac{1}{\sqrt{2}}(1, 0, 1), \quad \text{and} \quad \frac{1}{\sqrt{2}}(1, 1, 0).$$

The arc joining each pair is a geodesic arc on the unit sphere; that is, the plane it spans passes through the origin.

Equation 6 and the triangle inequality imply that for every stability condition  $\tau$ , we have  $|\pi(\tau)| \leq 1$ . In fact, for every stability condition, at least two of the Gromov coordinates are non-zero. Therefore, we actually have a strict inequality  $|\pi(\tau)| < 1$ .

To get a better understanding of  $\pi$ , let us study it on the translates of the fundamental domain  $\Lambda$ . Let  $\beta$  be a braid. Consider the translate  $\beta\Lambda$ . Set

$$A = \beta(P_1), \quad B = \beta(P_2), \quad C = \beta(X).$$

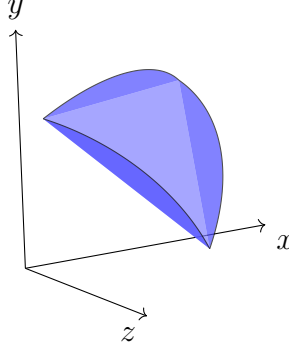


FIGURE 6. The homeomorphic image  $\Phi \subset \mathbb{R}^3$  of the compactified stability manifold.

These are the (semi)-stable objects for the stability conditions in  $\beta\Lambda$ . Recall that  $\beta\bar{\Lambda}$  is homeomorphic to the projectivised octant  $\bar{\Delta} = (\mathbb{R}_{\geq 0}^3 \setminus 0) / \mathbb{R}_{>0}$ ; the homeomorphism is given by the Gromov coordinates. The normalisation condition (6) is a linear condition on the Gromov coordinates. It cuts out an affine hyperplane slice of the octant, and gives a section of the projectivisation map. Since  $\pi$  is linear in the Gromov coordinates for normalised stability conditions, it maps  $\beta\bar{\Lambda}$  linearly, and homeomorphically, onto the triangle in  $\mathbb{R}^3$  with vertices  $\pi(A)$ ,  $\pi(B)$ , and  $\pi(C)$ .

Let  $\Phi \subset \mathbb{R}^3$  be the union of the triangle

$$\{(x, y, z) \mid x + y + z = \sqrt{2}, x + y - z \geq 0, y + z - x \geq 0, z + x - y \geq 0\},$$

and the three circular segments, each bounded by an edge of the triangle above and the arc in the image of  $\pi(P)$  with the same end-points as the edge (see Figure 6).

Observe that  $\pi$  maps  $\bar{\Lambda}$ , the triangle formed by the standard stability conditions, to the central triangle of  $\Phi$ . On the other hand, the translates of  $\bar{\Lambda}$  on the three sides of the identity on the exchange graph (Figure 5) are mapped to the three circular segments. Indeed, the segment to which  $\pi$  sends a non-standard stability condition  $\tau$  depends on how the triangle inequality degenerates among the  $\tau$ -masses of  $P_1$ ,  $P_2$ , and  $X$ , and this in turn, is determined by where  $\tau$  lies on the exchange graph.

**Proposition 6.2.** *The map  $\pi: \bar{M} \rightarrow \Phi$  is a homeomorphism, where  $\bar{M}$  is given the subspace topology from  $\mathbb{P}^S$ .*

*Proof.* Let us first show that  $\pi$  is continuous. Since  $\pi$  is linear on the translates of  $\bar{\Lambda}$ , and these translates cover  $M$ , it follows that  $\pi$  is continuous on  $M$ . The restriction of  $\pi$  to  $P$  is given by

$$\pi: [a : c] \mapsto \frac{1}{\sqrt{a^2 + c^2 + (a + c)^2}} (|a|, |c|, |a + c|),$$

which is continuous. We must show that  $\pi$  is continuous on  $\overline{M}$  at a point  $p \in P$ .

Let  $\tau_n$  be a sequence of normalised stability conditions converging to  $p \in P$ . We already know that  $m_T(\tau_n)$  converges to  $h_T(p)$  in the projective space  $\mathbb{P}^T$ . Therefore, it suffices to show that  $\pi(\tau_n)$  approaches a vector of norm 1 in  $\mathbb{R}^3$ .

Let  $\epsilon > 0$  be given. Write  $\tau_n = \beta_n \tau'_n$  for some braid  $\beta_n$  and standard stability condition  $\tau'_n$ . Denote by  $\overline{\beta}_n$  the image of  $\beta_n$  in  $\mathrm{PSL}_2(\mathbb{Z})$ . If the set  $\{\overline{\beta}_n\}$  is finite, then the sequence  $\tau_n$  lies in finitely many translates of  $\Lambda$ . Without loss of generality, we may assume that it lies in one translate, say  $\beta\Lambda$ . Recall that  $\pi: \overline{\beta}\Lambda \rightarrow \mathbb{R}^3$  is linear in the Gromov coordinates and hence continuous. Therefore,  $\pi(\tau_n) \rightarrow \pi(p)$  as  $n \rightarrow \infty$ .

The harder case is when the set  $\{\overline{\beta}_n\}$  is infinite. Set  $A_n = \beta_n(P_1)$ ,  $B_n = \beta_n(P_2)$ ,  $C_n = \beta_n(X)$ , and let  $x_n, y_n, z_n$  be the Gromov coordinates. But in this case, we know by Proposition 5.1 that the angle between  $\eta(A_n)$ ,  $\eta(B_n)$ , and  $\eta(C_n)$  approaches 0 as  $n$  approaches  $\infty$ . Therefore, the difference between

$$|x_n \eta(A_n) + y_n \eta(B_n) + z_n \eta(C_n)| \text{ and } x_n |\eta(A_n)| + y_n |\eta(B_n)| + z_n |\eta(C_n)|$$

approaches 0 as  $n$  approaches  $\infty$ . Since  $\tau_n$  is normalised, the right-hand quantity is 1, and the left-hand quantity is  $|\pi(\tau_n)|$ .

We have now proved that  $\pi: \overline{M} \rightarrow \Phi$  is continuous. Since  $\overline{M}$  is compact,  $\pi$  is a homeomorphism once we know that it is a bijection. We know that the map

$$\pi: P \rightarrow \partial\Phi = \Phi \cap \{v \mid |v| = 1\}$$

is a bijection and  $\pi$  maps  $M$  to

$$\Phi^\circ = \Phi \cap \{v \mid |v| < 1\}.$$

Recall that  $M$  is the union of the translates of the fundamental domains  $\beta\Lambda$ , and each fundamental domain is homeomorphic to a clipped triangle (a planar triangle minus the vertices). The map  $\pi$  maps each translate bijectively to a (clipped) triangle in  $\Phi^\circ$ . To check that  $\pi$  is an injection, we must check that the clipped triangles  $\pi(\beta\Lambda)$  have disjoint interiors, and two of them, say  $\pi(\beta\Lambda)$  and  $\pi(\beta'\Lambda)$ , intersect along an edge if and only if  $\beta$  and  $\beta'$  are adjacent in the exchange graph. To check that  $\pi$  is a surjection, we must check that the union  $\bigcup_\beta \pi(\beta\Lambda)$  is  $\Phi^\circ$ .

Take  $\beta = \mathrm{id}$ . Then  $\pi(\Lambda)$  is the central triangle of  $\Phi^\circ$ . The only other triangles that intersect this triangle are  $\pi(\sigma_1\Lambda)$ ,  $\pi(\sigma_2\Lambda)$ , and  $\pi(\sigma_X\Lambda)$ , and the intersections are along the three edges, as required.

The exchange graph divides the non-trivial translates of  $\Lambda$  into three connected components, and the translates in each of the three components map to the three distinct circular segments in  $\Phi^\circ$ . So it suffices to restrict our attention to one component and the corresponding circular segment. Since  $\gamma$  permutes the components, it suffices to look at only one of them.



FIGURE 7. The image under  $\pi$  of translates of  $\Lambda$  give the Farey triangulation of a circular segment.

Let us consider the component containing  $\sigma_1\Lambda$ . The translates in this component are  $A\Lambda$  where  $A \in \mathrm{PSL}_2(\mathbb{Z})$  is a matrix that has an admissible cyclic writing that starts with  $\sigma_1$ . Suppose  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then  $\pi(A\Lambda)$  is the clipped triangle with vertices

$$(9) \quad \pi([a : c]), \quad \pi([b : d]), \quad \pi([a - b : c - d]).$$

Since we are considering the translates in the  $\sigma_1\Lambda$  component, the points  $[a : c]$ ,  $[b : d]$ , and  $[a - b : c - d]$  lie in the arc  $[P_1, P_2]$  of  $\mathbb{P}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{R})$ . (So, under the bijection  $\mathbb{P}^1(\mathbb{Z}) = \mathbb{Q} \cup \{\infty\}$  given by  $[a : c] \rightarrow a/c$ , they correspond to  $\mathbb{Q}_{\geq 0} \cup \{\infty\}$ ). By construction, the points  $\pi([a : c])$  and  $\pi([b : d])$  form an edge of a clipped triangle if and only if  $|ad - bc| = 1$ . Thus, the triangles (9) form the Farey tessellation [1, Chapter 8] of the circular segment, which has the intersection properties as dictated by the exchange graph (see Figure 7).  $\square$

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