SPHERICAL OBJECTS AND STABILITY CONDITIONS ON CY2 QUIVER CATEGORIES

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1. Introduction

Let $\mathcal{C}$ be a triangulated category. Suppose $\mathcal{C}$ admits a Bridgeland stability condition τ. We can use τ to endow $\mathcal{C}$ with a rich array of additional structures. These structures can be of a geometric flavour, such as a measure of the mass or the spread of every object, a measure of length of every morphism, or of a combinatorial flavour, such as the counts of multiplicities of semi-stable objects in the Harder–Narasimhan filtration. These additional structure gives new tools to understand old questions about $\mathcal{C}$.

We apply this philosophy to study the 2-Calabi–Yau category $\mathcal{C}_\Gamma$ associated a quiver Γ. As an application, when Γ is of finite type, we obtain a classification of all spherical objects in $\mathcal{C}_\Gamma$.

Theorem 1.1. *Let Γ be a quiver of type A_n, D_n, or E_6, E_7, E_8. The spherical objects of $\mathcal{C}_\Gamma$ lie in the B_Γ orbit of the simple objects of the standard heart.*

We also prove the connectedness of the space $\mathrm{Stab}(\mathcal{C}_\Gamma)$ of stability conditions on $\mathcal{C}_\Gamma$.

Theorem 1.2. *Any stability condition $\tau \in \mathrm{Stab}(\mathcal{C}_\Gamma)$ is in the B_Γ orbit of a standard stability condition. Furthermore, $\mathrm{Stab}(\mathcal{C}_\Gamma)$ is connected.*

We note that Theorem 1.1 and Theorem 1.2 have been proved for type A_n in [2, 3] using the geometry of the minimal resolution of the type A_n Kleinian singularity. The proofs of [2,3] hinge on a key result about type A_n singularities that does not hold in types D and E (namely, that vector bundles on the exceptional fibre split into direct sums of line bundles). Not only do we generalise the results to types A, D, and E, but also our proofs are based on purely categorical arguments, without reference to geometry.

Our key technique is an algorithm that simplifies an object of $\mathcal{C}$ by applying a sequence of spherical twists. To make this precise, let x be an object of $\mathcal{C}$. It admits a unique Harder–Narasimhan filtration

$$0 = x_0 \to x_1 \to \cdots \to x_n = x,$$

whose cones $z_i = \mathrm{Cone}(x_{i-1} \to x_i)$ are τ-semistable and appear in the order of decreasing phase. Let $|x|$ be the *spread of* x, defined as the difference between the phases of z_n and z_1. Under suitable hypotheses on $\mathcal{C}$ and τ, we show that by applying a suitable spherical twist to x, we can decrease its spread. By repeating this procedure, we reach a much simpler x', for example, one which lies in the heart of τ.

We can use the same technique to simplify an arbitrary stability condition σ. Let x be an object. We take the σ-spread of x as a measure of the complexity of σ. By applying spherical twists, we show that the complexity decreases. By choosing a suitable x, we ensure that we in fact reach a standard stability condition.

2. Spherical stable objects

In this section, Γ is an arbitrary quiver, not necessarily of finite type. Let $\mathcal{C}$ be the 2-CY category associated to Γ. We collect some preliminary results about stability conditions and stable objects in $\mathcal{C}$.

The Grothendieck group $K(\mathcal{C})$ with the Hom pairing is naturally identified with the root lattice of Γ. Since $\dim(\mathrm{Hom}^*(X,X)) = 2$ for any spherical object X, its class $[X]$ in $K(\mathcal{C})$ is a real root.

Let τ be a stability condition on $\mathcal{C}$ with central charge $Z\colon K(\mathcal{C}) \to \mathbb{C}$ and slicing $\mathcal{P}$.

Proposition 2.1. *Assume that τ is generic in the following sense: Z maps distinct roots to complex numbers of distinct phase. Suppose X is a τ-semistable spherical object. Then X is τ-stable, and it is the unique τ-stable spherical object of phase $\phi = \phi(X)$.*

Proof. To show that X is stable, we must show that X is simple in the abelian category $\mathcal{P}(\phi)$. Let $S \subset X$ be a non-zero simple sub-object. By [1, Corollary 2.3], S must be spherical. Then the class of S in $K(\mathcal{C})$ is a root. Since $Z(S)$ has the same argument as $Z(X)$, the genericity assumption on τ means that $X = S$ in $K(\mathcal{C})$. But then $X/S = 0$ in $K(\mathcal{C})$, and since X/S is in a heart of τ, this implies that $X = S$.

Next, suppose Y is another τ-stable spherical object of the same phase as X. Again by the genericity of τ, we have $X = Y$ in $K(\mathcal{C})$. But then we get

$$\begin{aligned}\dim \mathrm{Hom}^*(X,Y) &= \dim \mathrm{Hom}^0(X,Y) - \dim \mathrm{Hom}^1(X,Y) + \dim \mathrm{Hom}^2(X,Y)\\ &= X \cdot X = 2,\end{aligned}$$

which implies $\mathrm{Hom}^0(X,Y) \neq 0$. Since X and Y are simple objects of $\mathcal{P}(\phi)$, this forces $X \cong Y$. $\square$

We now give an effective construction of a τ-stable spherical object of every possible class, for a generic standard stability condition τ.

For every simple root v, we have an object $P_v \in \mathcal{C}$, which is spherical and a simple object of the standard heart. Let w be an arbitrary positive root. Write

$$w = s_{v_n} \cdots s_{v_1} v, \tag{1}$$

where v is a simple root and $s_{v_1}, \dots, s_{v_n}$ are reflections in the simple roots $v_1, \dots, v_n$. Set $v_0 = v$. Associate to (1) a sequence of roots $R_0, \dots, R_n$ defined by

$$R_i = s_{v_n} \cdots s_{v_{i+1}}(v_i).$$

Note that $R_0 = w$ and if (1) is a minimal expression for w, then all the roots in the root sequence are positive.

Let $\epsilon_1, \dots, \epsilon_n$ be ± 1. Consider an object X of $\mathcal{C}$ defined by

$$X = \sigma_{v_n}^{\epsilon_n} \circ \cdots \circ \sigma_{v_1}^{\epsilon_1}(P_v), \tag{2}$$

where σ_{v_i} is the spherical twist in P_{v_i}. The ϵ's allow us to divide the root sequence into *positive* and *negative* sub-sequences, defined by

$$\begin{aligned}R_+ &= (R_i \mid \epsilon_i = 1),\\ R_- &= (R_i \mid \epsilon_i = -1).\end{aligned}$$

We call R_0 the *neutral root.*

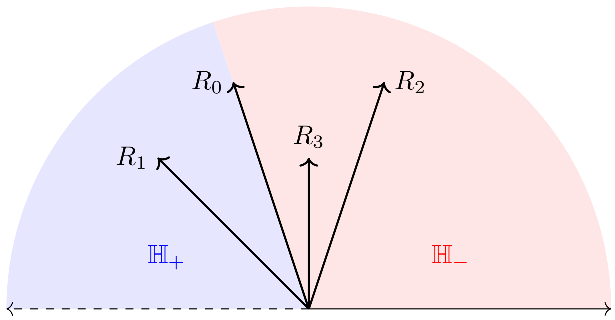

FIGURE 1. Consider the A_3 quiver with simple roots α_i and simple reflections s_i. Consider the root sequence for $w = s_2 s_3 s_1(\alpha_2)$. The central charge chosen for the diagram above maps R_1 to $\mathbb{H}_+$ and R_2, R_3 to $\mathbb{H}_-$. By Proposition 2.2, the stable object of class w is $\sigma_2^{-1}\sigma_3^{-1}\sigma_1(P_2)$.

Let τ be a stability condition on $\mathcal{C}$ such that the $[0,1)$ heart of τ is the standard heart. Let Z be the central charge of τ. Let $\mathbb{H} \subset \mathcal{C}$ be the half-open upper half plane:

$$\mathbb{H} = \{z \mid \Im(z) > 0\} \cup \mathbb{R}_{>0}.$$

Let $\alpha = Z(R_0)$. Since R_0 is a positive root, $Z(R_0)$ lies in $\mathbb{H}$. It divides $\mathbb{H}$ into two pieces

$$\mathbb{H}_+ = \{z \mid \arg z > \arg \alpha\},$$
$$\mathbb{H}_- = \{z \mid \arg z < \arg \alpha\},$$

where arg is taken in $[0, \pi)$.

Figure 1 shows an example of the construction above for a stability condition on the A_3-category.

Proposition 2.2. *With the notation above, the object X defined by (2) is τ-semistable if and only if $Z(R_+) \subset \mathbb{H}_+$ and $Z(R_-) \subset \mathbb{H}_-$.*

Let τ be generic. Proposition 2.2 gives an effective construction of the unique τ-stable object of class $w = R_0$. Indeed, we compute the root sequence R, look at its image in $Z(R) \subset \mathbb{H}$, and take $\epsilon_i = \pm 1$, depending on whether $Z(R_i)$ lies in $\mathbb{H}_\pm$.

We need some preparation to prove Proposition 2.2, including the definitions and basic properties of spherical twists from the beginning of § 3. Let $\mathcal{A} \subset \mathcal{C}$ be the standard heart. Set

$$K = K(\mathcal{A}) = K(\mathcal{C}).$$

Denote by $[X]$, the class in K of an object X.

Lemma 2.3. *Let v be a simple root and $X \in \mathcal{A}$ be any object. The twist $\sigma_{P_v}^{-1} X$ lies in $\mathcal{A}$ if and only if P_v is not a sub of X. Similarly, the twist $\sigma_{P_v} X$ lies in $\mathcal{A}$ if and only if P_v is not a quotient of X.*

Proof. We prove the first statement; the second is analogous. Set $P = P_v$. We have the exact triangle

$$P \otimes \operatorname{Hom}(X, P)^\vee[-1] \to \sigma_P^{-1}(X) \to X \xrightarrow{+1}.$$

Since both P and X lie in the heart, the 2-CY property implies that $\operatorname{Hom}^i(X, P)$ is zero for $i < 0$ and $i > 2$. Thus, $\operatorname{Hom}(X, P)^\vee[-1]$ is a direct sum of $\mathbf{k}[j]$ for $j = -1, 0, 1$.

Assume that P is not a sub of X. Since $P \in \mathcal{A}$ is simple, we must have $\operatorname{Hom}^0(P, X) = 0$. By the 2-CY property, this implies $\operatorname{Hom}^2(X, P) = 0$. Then $\operatorname{Hom}(X, P)^\vee[-1]$ is a direct sum of $\mathbf{k}[j]$ for $j = -1, 0$. As a result, $\sigma_P^{-1}(X)_{<0}$ lies in $\mathcal{C}_{<1}$. We must prove that it lies in $\mathcal{C}_{[0,1)}$. The truncation $\sigma_P^{-1}(X)_{<0}$ is a quotient of a direct sum of copies of $P[-1]$. Since P is simple, it is a direct sum of copies of $P[-1]$. If it were non-zero, then we have a non-zero map $\sigma_P^{-1}(X)_{<0} \to P[-1]$, and hence a non-zero map $\sigma_P^{-1}(X) \to P[-1]$. By applying σ_P, we would then obtain a non-zero map $X \to P[-2]$, which is absurd. We conclude that $\sigma_P^{-1}(X)_{<0} = 0$ and hence $\sigma_P^{-1}X$ lies in $\mathcal{A} = \mathcal{C}_{[0,1)}$.

Conversely, if P is a sub of X, then we have a non-zero map $P \to X$ and hence a non-zero map $\sigma_P^{-1}P = P[1] \to \sigma_P^{-1}X$. It follows that $\sigma_P^{-1}X$ is not in $\mathcal{A}$. $\square$

Consider an object $X \in \mathcal{A}$. We say that a subset $S \subset K$ *envelops* the subs (resp. quotients) of X if for every sub (resp. quotient) object Y of X, the class $[Y]$ can be expressed as non-negative linear combination of the elements of S and $\pm[X]$. Observe that if S envelops the subs of X then $-S$ envelops the quotients of X, and vice-versa.

Lemma 2.4. *Let X be an object of $\mathcal{A}$ and let $v \in K$ be a simple root. Let S be a subset of K and set $S' = s_v(S) \cup \{v\}$.*

(1) If S envelops the subs of X and $\sigma_{P_v}^{-1}X$ lies in $\mathcal{A}$, then S' envelops the subs of $\sigma_{P_v}^{-1}X$.
(2) If $S \subset K$ envelops the quotients of X and $\sigma_{P_v}X$ lies in $\mathcal{A}$, then S' envelops the quotients of $\sigma_{P_v}X$.

Proof. We prove the first assertion; the second is similar.

Set $P = P_v$. Let Y be a sub of $\sigma_P^{-1}X$.

Suppose P is not a quotient of Y. Set

$$Q = \operatorname{coker}(Y \to \sigma_P^{-1}X).$$

Since $\sigma_P\sigma_P^{-1}X$ lies in $\mathcal{A}$, by Lemma 2.3, P is not a quotient of $\sigma_P^{-1}X$. Therefore, P is not a quotient of Q. By applying σ_P to the exact sequence

$$0 \to Y \to \sigma_P^{-1}X \to Q \to 0,$$

we get an exact triangle

$$\sigma_P Y \to X \to \sigma_P Q \xrightarrow{+1},$$

whose terms are in $\mathcal{A}$ by Lemma 2.3. Therefore, it is an exact sequence in $\mathcal{A}$. Since S envelops the subs of X, the class $[\sigma_P Y] = s_v[Y]$ is a non-negative linear combination of elements of S and $\pm[X]$. Equivalently, the class $[Y]$ is a non-negative linear combination of the elements of $s_v(S)$ and $\pm[\sigma_P^{-1}X]$.

It remains to treat the case when P is a quotient of Y. Define $Y' \subset Y$ be such that we have an exact sequence

$$0 \to Y' \to Y \to P^{\oplus n} \to 0$$

for some n and P is not a quotient of Y (such a Y' exists because $\mathcal{A}$ is a finite-length category). By the argument above $[Y']$ is a non-negative linear combination of the elements of $s_v(S)$ and $\pm[\sigma_P^{-1}X]$. But then $[Y]$ is a non-negative linear combination of the elements of $s_v(S) \cup \{v\}$ and $\pm[\sigma_P^{-1}X]$, as desired. $\square$

Recall the definition

$$X = \sigma_{v_n}^{\epsilon_n} \circ \cdots \circ \sigma_{v_1}^{\epsilon_1}(P_v),$$

and the root sequence R_i, divided into a positive sub-sequence R_+, a negative sub-sequence $R-$, and the neutral root R_0.

Lemma 2.5. *In the above setup, suppose there exists a linear functional $\lambda\colon K(\mathcal{C}) \to \mathbb{R}$ such that $\lambda(R_0) = 0$, and $\lambda(R_+) \subset \mathbb{R}_{>0}$, and $\lambda(R_-) \subset \mathbb{R}_{<0}$. Then X lies in the heart $\mathcal{A}$. Furthermore, the R_- (resp. R_+) envelops the subs (resp. quotients) of X.*

Proof. We induct on n. If $n = 0$, then $X = P_v$ is simple, both R_+ and R_- are empty, and the statement holds.

Assume the statement for $(n-1)$. Let

$$X' = \sigma_{v_{n-1}}^{\epsilon_{n-1}} \circ \cdots \circ \sigma_{v_1}^{\epsilon_1}(P_v),$$

and let R' denote the root sequence for X'. Then we have $R'_i = s_{v_n} R_i$ for $i = 0, \dots, n-1$. Note that

$$\lambda' = \lambda \circ s_{v_n} : K(\mathcal{C}) \to \mathbb{R}$$

is a linear functional that vanishes on the neutral root R'_0 for X' and takes opposite signs on the positive and the negative sub-sequences R'_+ and R'_-. By the induction hypothesis, X' lies in the heart $\mathcal{A}$ and its subs (resp. quotients) are enveloped by R'_- (resp. R'_+).

Observe that $R_n = v_n$. Suppose $\epsilon_n = -1$. By our assumption, λ takes opposite signs on $s_{v_n}(R_n) = -R_n$ and R_-. Therefore, λ' takes opposite signs on R_n and R'_-. Since R'_- envelops the subs of X', we conclude that P_{v_n} is not a sub of X'. Hence, by Lemma 2.3, X is in the heart and its subs are enveloped by $s_1(R'_s) \cup \{v_n\} = R_-$. The proof when $\epsilon_n = +1$ is similar. □

We now have the tools to finish the proof of Proposition 2.2. Suppose Z maps R_+ and R_- to $\mathbb{H}_+$ and $\mathbb{H}_-$, respectively. Choose a linear functional $\ell\colon \mathbb{C} \to \mathbb{R}$ that vanishes on $\alpha = Z(R_0)$ and takes positive (resp. negative) values on $\mathbb{H}_+$ (resp. $\mathbb{H}_-$). Set $\lambda = \ell \circ Z$. Then λ satisfies the hypotheses of Lemma 2.5. As a result, X is in the heart $\mathcal{A}$. To show that X is semi-stable, consider a sub $Y \subset X$. Since $\lambda(R_-) \subset \mathbb{R}_{<0}$, we see that $Z(R_-) \subset \mathbb{H}_-$. But R_- envelops the subs of X, that is, $[Y]$ is a non-negative linear combination of elements of R_- and $\pm[X]$. By applying Z, we obtain that $Z(Y)$ is a non-negative linear combination of elements of $Z(R_-)$ and $\pm\alpha$. Since $Z(R_-)$ lies in $\mathbb{H}_-$, we conclude that $Z(Y)$ lies in $\mathbb{H}_- \cup \mathbb{R}_{>0} \cdot \alpha$. In any case, we have $\arg Y \leq \arg X$. Since this is true for any sub $Y \subset X$, we conclude that X is semi-stable.

3. Phase impromevent using spherical twists

The goal of this section is to prove that by applying suitable spherical twists, we can predictably increase/decrease the bottom/top phase of an object. We begin by recalling the notion of spherical objects, spherical twists, and bottom/top phases. Throughout, fix a $\mathbf{k}$-linear 2-CY triangulated category $\mathcal{C}$.

Recall that an object X of $\mathcal{C}$ is spherical if we have an isomorphism of $\mathbf{k}$-algebras

$$\operatorname{Hom}^*(X, X) \cong H^*(S^2, \mathbf{k}).$$

In what follows, we write $\operatorname{Hom}(X, Y)$ for the graded vector space $\operatorname{Hom}^*(X, Y) = \bigoplus_n \operatorname{Hom}^0(X, Y[n])$.

A spherical object X defines a triangulated auto-equivalence of $\mathcal{C}$, called the spherical twist in X, denoted by σ_X. An object Y and its spherical twist $\sigma_X(Y)$ are related by the exact triangle

$$\operatorname{Hom}(X, Y) \otimes X \to Y \to \sigma_X(Y) \xrightarrow{+1}, \tag{3}$$

where the map $\operatorname{Hom}(X,Y)\otimes X \to Y$ is the tautological one. Likewise, Y and the inverse twist $\sigma_X^{-1}(Y)$ are related by the exact triangle

$$\sigma_X^{-1}(Y) \to Y \to X \otimes \operatorname{Hom}(Y,X)^\vee \xrightarrow{+1}, \tag{4}$$

where the map $Y \to X \otimes \operatorname{Hom}(Y,X)^\vee$ is adjoint to the tautological one.

Let τ be a stability condition on $\mathcal{C}$. Denote by $\phi(X)$ the phase of a semi-stable object X. An arbitrary object X has a unique Harder–Narasimhan (HN) filtration

$$0 = X_0 \to X_1 \to \cdots \to X_n = X,$$

where the sub-quotients Z_i, defined by triangles $X_{i-1} \to X_i \to Z_i \xrightarrow{+1}$, are τ-semistable and satisfy

$$\phi(Z_1) > \cdots > \phi(Z_n).$$

With the notation above, set

$$\phi^+(X) = \phi(Z_1), \quad \phi^-(X) = \phi(Z_n),$$
$$\lceil X \rceil = Z_1, \text{ and } \lfloor X \rfloor = Z_n.$$

We call $\phi^+(X)$ (resp. $\phi^-(X)$) the *top* (resp. *bottom*) phase of X. We have triangles

$$\lceil X \rceil \to X \to Q \xrightarrow{+1},$$

with $\phi^+(X) > \phi^+(Q)$, and similarly

$$K \to X \to \lfloor X \rfloor \xrightarrow{+1},$$

with $\phi^-(K) < \phi^-(X)$.

The following is standard.

Lemma 3.1 (Sandwich lemma)**.** *Let $X \to Y \to Z \xrightarrow{+1}$ be an exact triangle. Then*

$$\phi^-(Y) \geq \min\{\phi^-(X), \phi^-(Z)\}, \text{ and}$$
$$\phi^+(Y) \leq \max\{\phi^+(X), \phi^+(Z)\}.$$

We now investigate the effect of applying suitable spherical twists on the bottom and the top phase.

Proposition 3.2. *Let X be a spherical stable object of $\mathcal{C}$ such that X is the unique stable object of its phase. Let Y be any object of $\mathcal{C}$. We have the following.*

1. *If $\phi(X) \leq \phi^-(Y)$, then $\phi(X) < \phi^-(\sigma_X^{-1}(Y))$.*
2. *If $\phi^+(Y) \leq \phi(X)$, then $\phi^+(\sigma_X(Y)) < \phi(X)$.*

Proof. We prove the first inequality; the second is analogous. The key is the exact triangle

$$X \otimes \operatorname{Hom}(Y,X)^\vee[-1] \to \sigma_X^{-1}(Y) \to Y \xrightarrow{+1}, \tag{5}$$

obtained by rotating the triangle in (4).

We first prove the proposition assuming the strict inequality

$$\phi(X) < \phi^{-1}(Y).$$

The inequality above implies that $\operatorname{Hom}^i(Y,X) = 0$ for $i \leq 0$. Therefore $\operatorname{Hom}(Y,X)$ is a direct sum of $\mathbf{k}[j]$ for $j < 0$, and hence its dual $\operatorname{Hom}(Y,X)^\vee$ is a direct sum of $\mathbf{k}[j]$ for $j > 0$. As a result, we have

$$\phi(X) \leq \phi^-(X \otimes \operatorname{Hom}(Y,X)^*[-1]).$$

Applying the sandwich lemma (Lemma 3.1) to the key triangle (5), we see that $\phi(X) \leq \phi^-(\sigma_X^{-1}(Y))$.

To show that the inequality is strict, it suffices to show that $\sigma_X^{-1}Y$ does not have a nonzero map to any stable object of phase $\phi(X)$. By our assumption, the only such stable object is X itself. Suppose $f\colon \sigma_X^{-1}(Y) \to X$ is a map. Applying σ_X gives a map $\sigma_X(f)\colon Y \to X[-1]$. Since $\phi(X[-1]) < \phi^-(Y)$, the map $\sigma_X(f)$ must be zero. Therefore f is zero. The proof is thus complete, assuming $\phi(X) < \phi^-(Y)$.

We now treat the case $\phi(X) = \phi^-(Y)$. By our assumption of X, in this case, $\lfloor Y \rfloor$ must be a direct sum of copies of X. This means that we have an exact triangle

$$Z \to Y \to X^{\oplus n} \xrightarrow{+1}, \tag{6}$$

where $\phi(X) < \phi^-(Z)$. Applying σ_X^{-1} yields

$$\sigma_X^{-1}(Z) \to \sigma_X^{-1}(Y) \to X[1]^{\oplus n} \xrightarrow{+1} . \tag{7}$$

We now apply the previous argument to Z, noting that $\phi(X) < \phi^-(Z)$. By applying the sandwich lemma (Lemma 3.1) to (7), we conclude that $\phi(X) < \phi^-(\sigma_X^{-1}(Y))$. ☐

The following shows that the improvement on one end achieved by Proposition 3.2 does not cause a deterioration at the other end.

Proposition 3.3. *Let X be a spherical stable object of $\mathcal{C}$. Assume that X is the unique stable object of its phase. Let Y be any object of $\mathcal{C}$ such that $\operatorname{Hom}^i(Y,Y) = 0$ for any $i < 0$ and $\phi^+(Y) - \phi^-(Y) \geq 1$. The following hold.*

(1) If $\phi^-(Y) = \phi(X)$, then $\phi^+(Y) \geq \phi^+(\sigma_X^{-1}(Y))$.
(2) If $\phi^+(Y) = \phi(X)$, then $\phi^-(Y) \leq \phi^-(\sigma_X(Y))$.

Proof. We prove the first statement; the second is analogous. Again, the key is the exact triangle

$$X \otimes \operatorname{Hom}(Y,X)^\vee[-1] \to \sigma_X^{-1}(Y) \to Y \xrightarrow{+1}, \tag{8}$$

obtained by rotating the triangle (4). By the sandwich lemma (Lemma 3.1), it suffices to show that

$$\phi^+(Y) \geq \phi^+(X \otimes \operatorname{Hom}(Y,X)^*[-1]). \tag{9}$$

Let ℓ be the largest integer such that $\operatorname{Hom}^\ell(Y,X) \neq 0$. Then $\operatorname{Hom}(Y,X)$ is a direct sum of $k[j]$ with $j \geq -\ell$. Then the dual $\operatorname{Hom}(Y,X)^\vee$ is a direct sum of $\mathbf{k}[j]$ with $j \leq \ell$. Therefore, we get

$$\phi^+(X \otimes \operatorname{Hom}(Y,X)^\vee[-1]) = \phi(X[\ell-1]). \tag{10}$$

Thus, showing (9) is equivalent to showing

$$\phi^+(Y) \geq \phi(X[\ell-1]).$$

Suppose $\ell \leq 2$. Then we have

$$\phi^-(Y) + 1 = \phi(X[1]) \geq \phi(X[\ell-1]).$$

Since $\phi^+(Y) - \phi^-(Y) \geq 1$, we conclude that

$$\phi^+(Y) \geq \phi(X[\ell-1]),$$

as desired.

Suppose $\ell > 2$. Then the 2-CY property implies that

$$\operatorname{Hom}^{2-\ell}(X,Y) \neq 0.$$

By our assumptions on X and Y, the object $\lfloor Y \rfloor$ is a direct sum of copies of X. Therefore, we also have

$$\operatorname{Hom}^{2-\ell}(\lfloor Y \rfloor, Y) \neq 0.$$

Let $f \in \operatorname{Hom}^{2-\ell}(\lfloor Y \rfloor, Y)$ be a non-zero element. Define the object K by the following exact triangle:

$$K \to Y \to \lfloor Y \rfloor \xrightarrow{+1} .$$

The composition of $Y \to \lfloor Y \rfloor$ with f gives a map $Y \to Y[2-\ell]$. Since this is a map of negative degree from Y to itself, it must be zero. Therefore f factors as the composite $\lfloor Y \rfloor \to K[1]$ and a (non-zero) map $g\colon K[1] \to Y[2-\ell]$

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, "0"'] & \lfloor Y \rfloor \arrow[r] \arrow[d, "f"] & K[1] \arrow[r, "+1"] \arrow[dl, "g"] & {} \\ & Y[2-\ell]. & & \end{tikzcd}$$

Since g is non-zero, we get

$$\phi^+(Y[2-\ell]) \geq \phi^-(K[1]).$$

By construction, we have

$$\phi^-(K) > \phi^-(Y) = \phi(X).$$

By combining the last two inequalities, we see that

$$\phi^+(Y[2-\ell]) > \phi(X[1]).$$

Therefore, we get

$$\phi^+(Y) > \phi(X[\ell-1]),$$

as desired. □

The following is an analogue of Proposition 3.4 for Y of small spread.

Proposition 3.4. *Let X be a spherical stable object of $\mathcal{C}$. Assume that X is the unique stable object of its phase. Let Y be any object of $\mathcal{C}$ such that $\phi^+(Y) - \phi^-(Y) < 1$. Assume, furthermore, that* $\operatorname{Hom}^0(Y,Y) = \mathbf{k}$ *and that X is not a direct summand of Y. The following hold.*

(1) If $\phi^-(Y) = \phi(X)$, then $\phi^+(Y) \geq \phi^+(\sigma_X^{-1}(Y))$.
(2) If $\phi^+(Y) = \phi(X)$, then $\phi^-(Y) \leq \phi^-(\sigma_X(Y))$.

Proof. We prove the first statement; the second is analogous. We begin as in the proof of Proposition 3.3. From the triangle

$$X \otimes \operatorname{Hom}(Y,X)^\vee[-1] \to \sigma_X^{-1}(Y) \to Y \xrightarrow{+1}, \tag{11}$$

we see that it suffices to show that

$$\phi^+(Y) \geq \phi^+(X \otimes \operatorname{Hom}(Y,X)^\vee[-1]).$$

Let ℓ be the largest integer such that $\operatorname{Hom}^\ell(Y,X) \neq 0$. Then we must show that

$$\phi^+(Y) \geq \phi(X[\ell-1]).$$

By the 2-CY property, we have $\operatorname{Hom}^\ell(Y,X) \cong \operatorname{Hom}^{2-\ell}(X,Y)^\vee$. Since

$$\phi^+(Y) < \phi^-(Y) + 1 = \phi(X) + 1,$$

there cannot be a non-zero map from X to $Y[k]$ for $k < 0$. As a result, we must have $\ell \leq 2$.

Suppose $\ell \leq 1$. Then we have

$$\phi^+(Y) \geq \phi^-(Y) = \phi(X) \geq \phi(X[\ell - 1]),$$

as desired.

We rule out $\ell = 2$. Let us show that if $\ell = 2$, then X must be a direct summand of Y.

Let $\mathcal{P}$ be the slicing defined by the stability condition τ. Since $\phi^+(Y) - \phi^-(Y) < 1$ and $\phi(X) = \phi^-(Y)$, both X and Y lie in the abelian category $\mathcal{A} = \mathcal{P}[\alpha, \alpha+1)$ for $\alpha = \phi^-(Y)$. By our assumptions, $\lfloor Y \rfloor$ is a direct sum of copies of X, say $\lfloor Y \rfloor = X^{\oplus n}$.

If $\ell = 2$, we have a non-zero map $i \colon X \to Y$. Consider the composite

$$X \xrightarrow{i} Y \to \lfloor Y \rfloor. \tag{12}$$

We show that the composite is non-zero. Equivalently, we must show that i does not factor through the kernel K of $Y \to \lfloor Y \rfloor$. In fact, let us prove that there are no non-zero maps from X to K.

Since $\mathrm{Hom}^0(Y, Y) = \mathbf{k}$, every non-zero map from Y to itself is an isomorphism. A non-zero map $X \to K$ gives a non-zero map $Y \to Y$ that is not an isomorphism, namely the composite

$$Y \twoheadrightarrow \lfloor Y \rfloor = X^{\oplus n} \twoheadrightarrow X \to K \hookrightarrow Y.$$

Therefore, there are no non-zero maps $X \to K$.

Since we have $\lfloor Y \rfloor = X^{\oplus n}$ and the composite in (12) is non-zero, there is a map $\pi \colon Y \to X$ such that $\pi \circ i \colon X \to X$ is non-zero. But X is spherical, so $\pi \circ i$ must be an isomorphism. That is, X is a direct summand of Y, as desired. □

Remark 3.5. In Proposition 3.2, Proposition 3.3, Proposition 3.2, suppose we know *a priori* that the stable factors of Y, and any spherical twist applied to Y, are spherical. This is true, for example, if Y is itself a spherical object, or a direct sum of spherical objects (see [1, Corollary 2.3]). Then we may weaken the uniqueness assumption on X to the following: X is the unique *spherical* stable object of its phase.

4. Applications of phase improvement

In this section, we reap the benefits of the phase improvement results proved in § 3.

Fix the following notation:

- $\mathcal{C}$ a $\mathbf{k}$-linear 2-CY triangulated category,
- τ a stability condition on $\mathcal{C}$,
- Φ the subset of $\mathbb{R}$ consisting of the phases of τ-stable spherical objects
- G the group of auto-equivalences of $\mathcal{C}$ generated by the twists in τ-stable spherical objects.

Proposition 4.1. *With the notation introduced at the beginning of § 4, assume that τ admits at most one spherical stable object of every phase and $\Phi \subset \mathbb{R}$ is discrete. Then every spherical object in $\mathcal{C}$ is in the G-orbit of a τ-stable spherical object.*

Proof. Let Y be any spherical object of $\mathcal{C}$. Define the spread of Y, denoted by $|Y|$, by the formula

$$|Y| = \phi^+(Y) - \phi^-(Y).$$

Note that since the stable factors of Y must be spherical [1, Corollary 2.3], both $\phi^+(Y)$ and $\phi^-(Y)$ lie in Φ, and hence their difference lies in the discrete set $\{a - b \mid a, b \in \Phi, a \geq b\}$.

We induct on $|Y|$. If $|Y| = 0$, then Y is τ-stable, and we are done.

Otherwise, let X be the unique τ-stable spherical object of phase $\phi^-(Y)$. Since Y is spherical and not stable, X is not a direct summand of Y. Consider $Y' = \sigma_X^{-1} Y$. By Proposition 3.2 along with Proposition 3.3 or Proposition 3.4, we have $|Y'| < |Y|$. By the induction hypothesis, Y' lies in the G orbit of P, and hence so does Y.

Note that the same argument works with $Y' = \sigma_Z Y$ where Z is the unique τ-stable spherical object of phase $\phi^+(Y)$. □

Corollary 4.2 (Theorem 1.1)**.** *Let $\mathcal{C}$ be the 2-CY category associated to a quiver of finite (ADE) type. Then every spherical object of $\mathcal{C}$ is in the braid group orbit of a simple object of the standard heart.*

Proof. Choose a stability condition τ on $\mathcal{C}$ with the standard heart and generic central charge $Z\colon K(\mathcal{C}) \to \mathbb{C}$. In particular, assume that Z maps distinct roots to complex numbers of distinct arguments. Then, by Proposition 2.1, there is at most one spherical stable object of every phase.

Let $\Phi \subset \mathbb{R}$ be the set of phases of spherical stable objects. Since the class of a spherical object in $K(\mathcal{C})$ is a root, of which there are only finitely many, the set Φ consists of integer translates of a finite set. In particular, Φ is discrete.

From Proposition 2.2, we see that the τ-stable spherical objects lie in the braid group orbit of the simple objects of the standard heart. Recall that if $Y = \beta X$, where X is a spherical object and σ is an auto-equivalence, then $\sigma_Y = \beta\sigma_X\beta^{-1}$. Thus, the group G generated by twists in the τ-stable spherical objects is no bigger than the group generated by the twists in the simple objects of the standard heart, namely the (image of the) braid group.

We now apply Proposition 4.1 and conclude the result. □

Remark 4.3 (Choice of writing)*.* Note that in Proposition 4.1, we have a choice of applying a positive or a negative twist, leading to different expressions for a given spherical object as a braid image of a simple object. It is an interesting problem to understand these different expressions.

Proposition 4.4. *With the notation introduced at the beginning of § 4, assume that τ admits at most one spherical stable object of every phase and $\Phi \subset \mathbb{R}$ is discrete. Let Y be a direct sum of spherical objects of $\mathcal{C}$ such that $\operatorname{Hom}^i(Y,Y) = 0$ for $i < 0$. Then there exists a stability condition ω in the G-orbit of τ such that Y lies in the $[\alpha, \alpha+1)$-heart of ω for some α.*

Proof. We induct on the spread $|Y| = |Y|_\tau$. Note that this quantity lies in the discrete set $\{a - b \mid a, b \in \Phi\}$.

If $|Y|_\tau < 1$, then we are done. Simply take $\omega = \tau$ and $\alpha = \phi^-(Y)$.

Suppose $|Y|_\tau \geq 1$. Let X be the unique spherical τ-stable object of phase $\phi^-(Y)$. Let $\tau' = \sigma_X\tau$. Note that the group G and the set Φ are unchanged if we replace τ by τ'. By Proposition 3.2 and Proposition 3.3, we have

$$|Y|_{\tau'} = |\sigma_X^{-1}Y|_\tau < |Y|_\tau.$$

We conclude the result by the induction hypothesis. □

Corollary 4.5 (Theorem 1.2)**.** *Let $\mathcal{C}$ be the 2-CY category associated to a quiver of finite (ADE) type. Then the stability manifold of $\mathcal{C}$ is connected. Furthermore, up to rotation, every stability condition is in the braid group orbit of a standard stability condition.*

Proof. Let τ be an arbitrary stability condition on $\mathcal{C}$. Let $Z\colon K(\mathcal{C}) \to \mathbb{C}$ be its central charge. Perturb τ so that Z maps distinct roots to complex numbers of distinct arguments. Note that the perturbed τ lies in the same connected component of the stability manifold as the original τ. There is now a unique spherical τ-stable object of every phase.

Since every spherical object is in the braid group orbit of the simple objects of the standard heart, the subgroup G of $\mathrm{Aut}(\mathcal{C})$ generated by twists in τ-stable spherical objects is a subgroup of the image of the braid group in $\mathrm{Aut}(\mathcal{C})$.

Let $\Phi \subset \mathbb{R}$ be the set of phases of spherical τ-stable objects. Since there are finitely many roots, Φ consists of integer translates of a finite set, and hence is finite.

Let Y be the direct sum of the simple objects in the standard heart of $\mathcal{C}$. Note that Y satisfies the hypotheses of Proposition 4.4. By Proposition 4.4, there is a stability condition ω in the braid group orbit of τ such that Y is in the $[\alpha, \alpha+1)$ heart of ω. Let ω' be the rotation of ω by α, so that Y lies in the $[0,1)$ heart of ω'. Note that ω' is in the same connected component as ω.

The direct summands of Y generate the standard heart of $\mathcal{C}$ (under taking extensions). Therefore, the $[0,1)$ heart of ω' contains the standard heart. Since both hearts are hearts of a t-structure, they must in fact be equal. In other words, ω' is a standard stability condition.

Let $\mathrm{Stab}_0\,\mathcal{C}$ be the connected component of $\mathrm{Stab}\,\mathcal{C}$ that contains the standard stability conditions. We have shown that an arbitrary $\tau \in \mathrm{Stab}\,\mathcal{C}$ is in the braid group orbit of a stability condition ω in $\mathrm{Stab}_0\,\mathcal{C}$. But we know that the braid group preserves the connected component $\mathrm{Stab}_0\,\mathcal{C}$. Hence, we conclude that $\mathrm{Stab}\,\mathcal{C} = \mathrm{Stab}_0\,\mathcal{C}$. The final statement follows from the fact that every $\tau \in \mathrm{Stab}_0\,\mathcal{C}$ is, up to rotation, in the braid group orbit of a standard stability condition. $\square$

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