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Mathematical Sciences Institute

EXAMINATION: Semester 2 — Mid-Semester, 2016

MATH1013 — Advanced Mathematics and Applications 1

Book A — Calculus

Exam Duration: 120 minutes.
Reading Time: 15 minutes.

Materials Permitted In The Exam Venue:

- One A4 page with hand written notes on both sides.
  (This A4 page is to cover both Algebra and Calculus.)
- Unmarked English-to-foreign-language dictionary (no approval from MSI required).
- **No electronic aids are permitted e.g. laptops, phones, calculators.**

Materials To Be Supplied To Students:

- Scribble Paper.

Instructions To Students:

- Answer the Calculus questions in Book A, and the Algebra questions in Book B, in the spaces provided.
- The Algebra and Calculus sections are worth a total of 50 points each, with the value of each question as shown. It is recommended that you spend equal time on the Calculus and the Algebra papers.
- A good strategy is not to spend too much time on any question. Read them through first and attack them in the order that allows you to make the most progress.
- *You must justify your answers. Please be neat.*

Q1	Q2
10	10

Total / 20

Question 1 **10 pts**

(a) Using the definition of $\sinh x$ in terms of exponentials, find the indefinite integral

$$\int \sinh x \, dx \, . \qquad \textit{2 pts}$$

Write your solution here

(b) Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{(1+x^2)(\tan^{-1} x)^2} \, dx$. *4 pts*

Write your solution here

Answer *Make the substitution* $u = \tan^{-1} x$. *Then* $du = \frac{1}{1+x^2} \, dx$, *so*

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{1}{(1+x^2)(\tan^{-1} x)^2} \, dx &= \int_{\pi/4}^{\pi/3} \frac{du}{u^2} \\
&= \left[-u^{-1}\right]_{\pi/4}^{\pi/3} \\
&= -\frac{3}{\pi} + \frac{4}{\pi} \\
&= \frac{1}{\pi} \, .
\end{aligned}$$

Grading Scheme:

- 1 point for trying a substitution
- 1 point for correctly performing the change of variables
- 1 point for computing the resulting integral
- 1 point for the right answer!

(c) Evaluate $\displaystyle\int_1^2 (\ln x)^2\,dx$. *4 pts*

Write your solution here

Answer

Let $U_1 = (\ln x)^2$ *and* $dV_1 = dx$. *Then* $dU_1 = \frac{2}{x}\ln x\,dx$ *and* $V_1 = x$, *so integration by parts gives*

$$\int_1^2 (\ln x)^2\,dx = \Big[x(\ln x)^2\Big]_1^2 - \int_1^2 x\tfrac{2}{x}\ln x\,dx = 2(\ln 2)^2 - 2\int_1^2 \ln x\,dx\,.$$

Now let $U_2 = \ln x$ *and* $dV_2 = dx$. *Then* $dU_2 = \frac{1}{x}\,dx$ *and* $V_2 = x$, *so integration by parts gives*

$$\begin{aligned}\int_1^2 \ln x\,dx &= \Big[x\ln x\Big]_1^2 - \int_1^2 x\tfrac{1}{x}\,dx\\ &= 2\ln 2 - \int_1^2 dx\\ &= 2\ln 2 - 1\,.\end{aligned}$$

Hence

$$\int_1^2 (\ln x)^2\,dx = 2(\ln 2)^2 - 4\ln 2 + 2\,.$$

Question 2 **10 pts**

Suppose $\alpha > 1$, $K \geq 0$, and that $f : \mathbb{R} \to \mathbb{R}$ has the property that, for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq K|x - y|^{\alpha} .$$

Prove that f is constant.

Hints: for this question you may use the facts that

- *for all $r > 0$, the mapping $t \mapsto t^r$ is continuous on $[0, \infty)$, and*
- *if $g : \mathbb{R}\backslash\{c\} \to \mathbb{R}$, then $\lim_{x\to c} g(x) = 0$ if and only if $\lim_{x\to c} |g(x)| = 0$.*

Given $c \in \mathbb{R}$, consider $h : \mathbb{R}\backslash\{c\} \to \mathbb{R}$ defined by $h(x) = \frac{f(x)-f(c)}{x-c}$.

Finally, please note that $\alpha > 1$, NOT $\alpha > 0$.

Write your solution here

Extra space for previous question