PARTIAL DIFFERENTIAL EQUATIONS

One of the main goals of the theory of PDEs is to express the unknown function of several independent variables from an identity where this function appears together with its partial derivatives. Mostly, t denotes the time variable, x,y,z,... stand for the spatial(dimensional) variables.

The general PDE for an unknown function $u = u(x, y, z, t)$ can be written as

$$F\left(x, y, z, t, u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, u_{xz}, u_{xt}, \dots\right) = 0$$

where $(x, y, z) \in \Omega \subset \mathbb{R}^3, t \in I, \Omega$ is a given domain in $\mathbb{R}^3$ and $I \subset \mathbb{R}$ is a time interval.

Problems from mathematics and physics

Differential equations describe many different physical systems, ranging form gravitation to fluid mechanics. They are difficult to study, they usually have individual equation, which needs to be analyzed as a separate problem.

A fundamental question for any PDE is the ***existence and uniqueness*** of a solution for given boundary conditions. For nonlinear equations these questions are in general hard to answer.

Heat equation

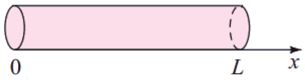

FIGURE 12.3.1 Temperatures in a rod of length L

The equation describing the heat equation problem is

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \qquad k > 0$$

- x is a point along the rod
- t is time;
- k is a physical constant related to the material;
- $u(x,t)$ represents temperature at any point at any time.

So here is the equation describing the problem and the problem is defined also by its boundary and initial values or conditions. For example

- temperature at a point is fixed (mostly at the beginning/end of the rod)
- distribution of temperature in the body at time $t = 0$ is given;
- flow at boundary is not allowed (the end is insulated): $\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$

Laplace's equation

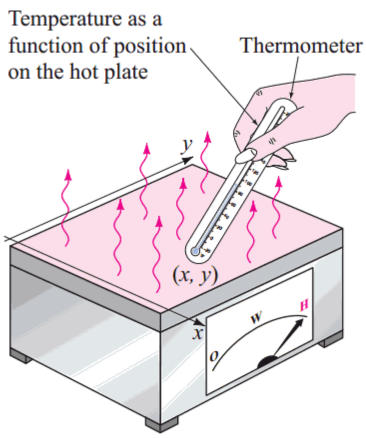

FIGURE 12.2.3 Steady-state temperatures in a rectangular plate

Laplace's equation is useful for solving many physical problems such as electrostatic, gravitational or velocity in fluid mechanics.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(x, y)$ of temperature distribution at any point (x, y) according to boundary conditions.

Partial differential equations can be classified from various points of view.

- ❖
  - If time t is one of the independent variables of the function, then we can speak about ***evolution equations***.
  - If it is not the case (the equation contains only spatial independent variables), we can speak about ***stationary equations***.
- ❖ The highest order of the derivative of the unknown function in the equation determines the order of the equation.
  - If the order is 1, then it is first order, other-wise it is high order PDEs.

- ❖ If all the coefficients of *u* are constants, then it will be PDE with constant coefficients; otherwise it will be PDE with variable coefficients.

❖

- If the equation consists only of a linear combination of u and its derivatives (for example, it does not contain products as uu_x , $u_x u_{yy}, etc$ ...), then it is called a ***linear equation***. Otherwise, it is a ***nonlinear equation***.

A linear equation can be written symbolically by means of a linear differential operator L ,

i.e., the operator with the property

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$$

where α, β are real constants and u, v are real functions.

❖

- The equation

$$L(u) = 0$$

is called ***homogeneous*** ,

- the equation

$$L(u) = f$$

where f is a given function, is called ***nonhomogeneous*** . The function f represents the "right-hand side" of the equation.

Examples: Classify the following equations.

1. The transport equation in one spatial variable

$$u_t + u_x = 0$$

2. The Laplace equation in three spatial variables

$$u_{xx} + u_{yy} + u_{zz} = 0$$

3. The Poisson equation in two spatial variables

$$u_{xx} + u_{yy} = f \quad , \quad f = f(x, y)$$

4. The wave equation with interaction in one spatial variable

$$u_{tt} - u_{xx} + u^3 = 0$$

The interaction is represented by the term u^3 .

5. The diffusion equation in one spatial variable

$$u_t - u_{xx} = f$$

6. The equation of the vibrating beam

$$u_{tt} + u_{xxxx} = 0$$

7. The Schrödinger equation

$$u_t - iu_{xx} = 0$$

8. The equation of a disperse wave

$$u_t - uu_x + u_{xxx} = 0$$

SOLUTION of PDEs

A function u is called a ***solution*** of a partial differential equation if, when substituted (together with its partial derivatives) into the equation, the latter becomes an identity.

It means that the function u must have all derivatives appearing in the equation.

If k is the order of the given partial differential equation, then by its solution we understand a function of the class C^k satisfying the equation at each point. In such a case, we can speak about the classical solution of a PDE. If we solve, for example, the diffusion equation in one spatial variable, that is,

$$u_t = ku_{xx}$$

which is of the second order, then its classical solution will be a C^2-function, i.e., a function having continuous partial derivatives up to the second order at all points (x, t) considered.

Thus, we require the existence and continuity of derivatives $u_{tt}\ and\ u_{xt}$ that ***do not*** occur in the equation at all.

Example 1. Let us try to find a function $u = u(x, y)$ satisfying the equation

$$u_{xx} = 0$$

This problem can be solved by direct integration of the equation

We have obtained a solution for arbitrary functions $f(y)$ and $g(y)$. However, if we want to talk about the classical solution, the functions $f(y)$ and $g(y)$ must be ***twice continuously differentiable.***

Example 2. Let us find for a function $u = u(x, y, z)$ satisfying the equation

$$u_{yy} + u = 0$$

This equation is similar with the ODE for the unknown function $v = v(t)$

when the general solution is a function

with arbitrary constants. The solution of the PDE will be

where $f(x, z)$ and $g(x, z)$ are arbitrary twice continuously differentiable functions.

Example 3. Let us search for a function $u = u(x, y)$ satisfying the equation

$$u_{xy} = 0$$

If we integrate with respect to y , we find

When we integrate with respect to x, then

where $F' = f$.

Consequently, the solution of any partial differential equation depends on arbitrary functions.

Example 4. Let us look for a function $u = u(x, y)$ satisfying the equation

$$(u_{xx})^2 + (u_{yy})^2 = 0$$

From the equation, we can write that

Since the left-hand side is linear in *x* and the right-hand side is linear in *y* , we can say that f_1, f_2, g_1, g_2 are linear functions. So,

$$u(x, y) = axy + bx + cy + d$$

where a, b, c, d are arbitrary real numbers. The solution depends on four arbitrary constants instead of two arbitrary function of one variable!

AUXILIARY CONDITIONS (BOUNDARY AND INITIAL CONDITIONS)

PDE does not provide sufficient information to enable us to determine its solution uniquely. For the unique determination of a solution, we need further information.

Initial Conditions: Condition of the function to be solved for at a particular time.

For example, for the function $u(x,t)$

$u(x,t_0) = f(x)$ or $\left.\frac{\partial u}{\partial t}\right|_{t=t_0} = g(x)$ are initial conditions.

- The number of initial conditions are the highest derivative of u with respect to t.

Ex. $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$, $u(x,0) = x$

Here, highest derivative of u with respect to *t is* 1. So, we need 1 initial condition.

Boundary Conditions: Condition of the function to be solved for at a particular special positions, generally on the edges of the domain where the PDE solves.

We have three types of boundary conditions:

- **Dirichlet Boundary Condition**

Function *u* is specified at the boundary.

Ex. If $u(x,t)$, then $u(0,t) = f_1(t)$ is a Dirichlet boundary condition

If $u(x,y,t)$, then $u(0,y,t) = f_2(y,t)$ is a Dirichlet boundary condition

- **Neumann Boundary Condition**

Spatial derivative of the function *u* is specified at the boundary.

Ex. If $u(x,t)$, then $u_x(0,t) = g_1(t)$ is a Neumann boundary condition

If $u(x,y,t)$, then $u_x(0,y,t) = g_2(y,t)$ is a Neumann boundary condition

- **Robin (Newton) Boundary Condition**

A linear combination of the derivative of the function and the function itself is specified at the boundary.

Ex. If $u(x,t)$, then $u_x(0,t) + ku(0,t) = h(t)$ is a Robin boundary condition

- The needed boundary initial condition number is the sum of the orders of highest partial derivative in each spatial variable.

  Ex.

$$u_{tt} + u_{xxxx} = 0$$

For the solution, we need 2 initial and 4 boundary conditions.

CONSTRUCT OF A PDE

Let us consider the geometric surfaces of two parameters $f(x,y,z,a,b) = 0$ where a, b are arbitrary parameters.
To determine the PDE, we have to take derivative of *f* with respect to *x* and *y*.