Algorithm complexity analysis

Algoritmos e Estruturas de Dados,

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Algorithm

- **Algorithm**

  set of precise instructions for solving a problem

algorithm $\neq$ program

Algorithm analysis:

- *Correctness*: prove the algorithm is correct
- *Efficiency*: determine the resources requested by the algorithm (time, space)
  - Compare the resource requested by different algorithms that solve the same problem: a more efficient algorithm requires less resources
  - Predict the increasing of required resources as the input size increases

Complexity

- Algorithm space complexity:

  memory space needed to execute

  S(n) – memory space required depending on input size (*n*)

- Algorithm time complexity:

  time it takes to execute

  T(n) - execution time depending on input size (*n*)

Complexity ↑ versus Eficiência ↓

Sometimes, complexity is calculated for the “*best case*” (not too useful), the *“worst case"* (more useful) and the *“average case"* (equally useful)

Complexity

- In general, we are not so much interested in the time and space complexity for small inputs

- What is important is the **growth** of the complexity functions
  - The growth of time and space complexity with increasing input size *n* is a suitable measure for the comparison of algorithms.

- Evaluate growth rate
  - As a function of various terms, growth is determined by the fastest growing term (*dominant term*)
  - Constant coefficients influence the initial progress

*Dominant term

Supose you use n^3 to estimate $n^3+350n^2 + n$

- for $n = 10\ 000$
  - real value = 1 0003 5000 010 000
  - estimated value = 1 000 000 000 000
  - error = 0.35% (not significant)

- for high values of n
  - the dominant term is indicative of the behavior of the algorithm

- for small values of n
  - The dominant term is not necessarily indicative of the behavior, but usually programs run so quickly that it doesn´t matter

The growth of functions

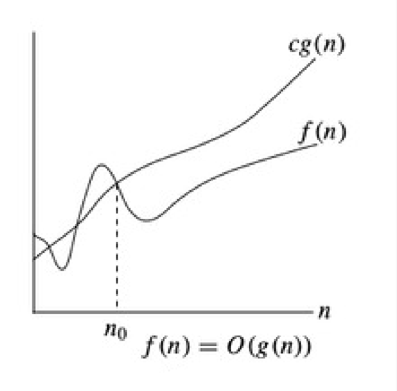
The growth of functions is usually described using the big-O notation

- **Definition**

  $$f(n) = O(g(n))$$

  if there are positive constants c and n_0 such that

  $$f(n) \leq cg(n), \text{ for all } n>n_0$$



- The idea behind the *big-O* notation is to establish an upper boundary for the growth of a function $f(n)$ for large n
  - This boundary is specified by a function $g(n)$ that is usually much simpler than $f(n)$
  - We accept the constant c in the requirement $f(n) \leq cg(n)$ whenever $n > n_0$, because c does not grow with n
  - We are only interested in large n, so it is OK if $f(n) > cg(n)$ for $x \leq n_0$

The growth of functions

Example: $f(n) = n^2 + 2n + 1$

- for $n > 1$:

$$n^2 + 2n + 1 \le n^2 + 2n^2 + n^2$$
$$\Rightarrow n^2 + 2n + 1 \le 4n^2$$

- therefore, for $c=4$ and $n_0=1$: $f(n) \le cn^2$, whenever $n>n_0$

$$\Rightarrow f(n) = O(n^2)$$

Question: if $f(n)$ is $O(n^2)$, is it also $O(n^3)$?

- yes; n^3 grows faster than n^2, so n^3 grows also faster than $f(n)$
- therefore, we always have to find the smallest simple function $g(n)$ for which $f(n)$ is $O(g(n))$

The growth of functions

<u>more examples:</u>

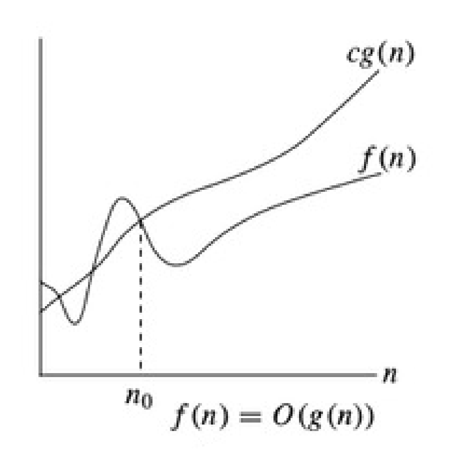
- $c_k\, n^k + c_{k-1}\, n^{k-1} + \ldots + c_0 = O(n^k)$ (c_i - constants)
- $\log_2 n = O(\log n)$ (changing the base is to multiply by a constant)
- $4 = O(1)$ (use 1 for constant order)

Big-O notation

Notation for functions growth

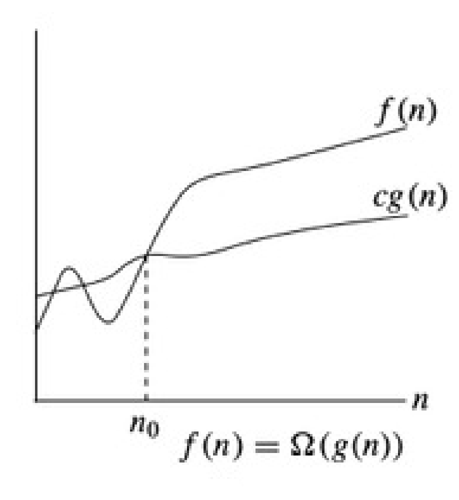
- $f(n) = O(g(n))$

  if there are positive constants c and n_0 such that $f(n) \leq cg(n)$, for $n \geq n_0$



- $f(n) = \Omega(g(n))$

  if there are positive constants c and n_0 such that $f(n) \geq cg(n)$, for $n \geq n_0$

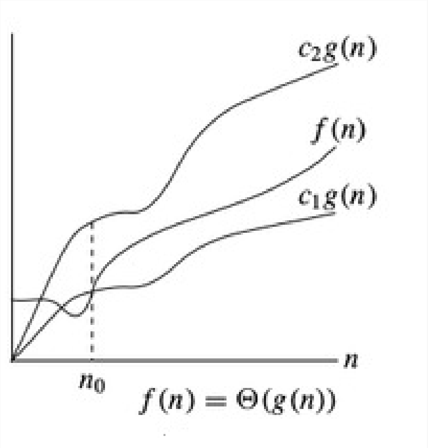


Big-O notation

Notation for functions growth

- $f(n) = \Theta(g(n))$

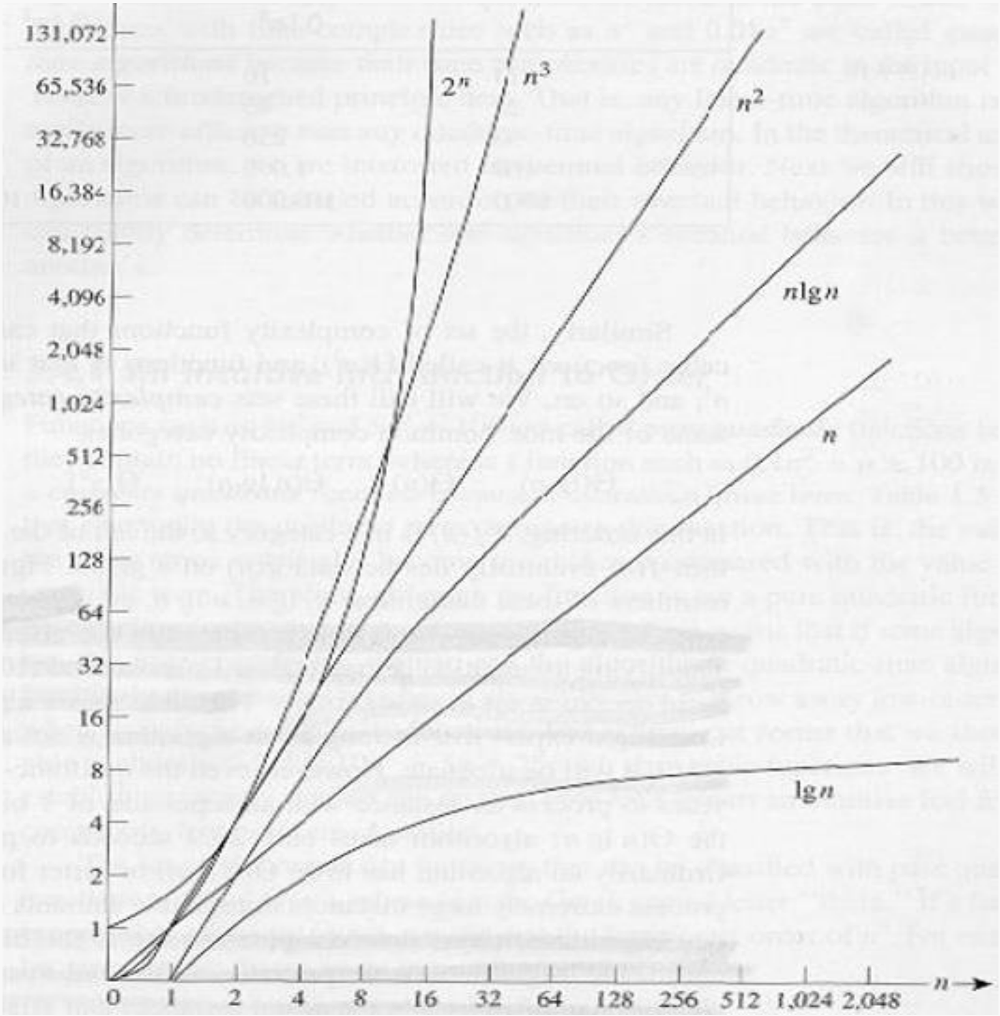
  if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$



- $f(n) = o(g(n))$

  if there are positive constants c and n_0 such that $f(n) < cg(n)$, for $n \geq n_0$

Most common orders of growth



Case study: maximum subsequence

- Problem
  - Given a set of integer values (positive and/or negative) $a_1, a_2, \ldots, a_n$, determine the highest subsequence sum
  - The largest sum subsequence is zero if all values are negative

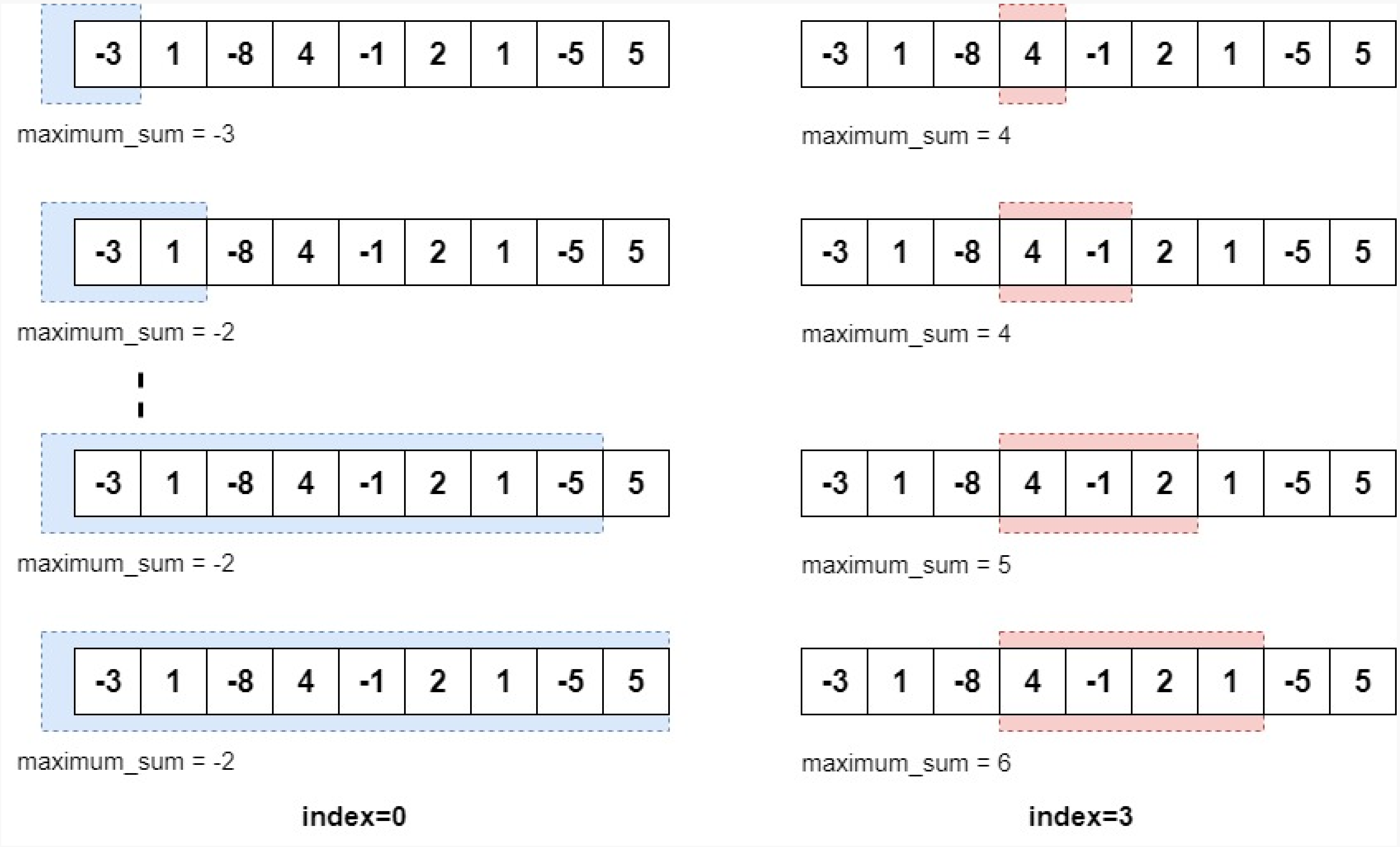
Examples

-2, <u>11, -4, 13</u>, -4, 2

1, -3, <u>4, -2, -1, 6</u>

-3, 1, -8, <u>4, -1, 2, 1</u>, -5, 5

Maximum subsequence – algorithm 1



Maximum subsequence – algorithm 1

```
template <class Comparable>
Comparable maxSubSum1(const vector<Comparable> &a)
{
    Comparable maxSum = 0;
    for (int i = 0 ; i < a.size() ; i++)
        for (int j = i; j < a.size(); j++)
        {
            Comparable thisSum = 0;
            for (int k = i; k <= j; k++)
                thisSum += a[k];
            if (thisSum > maxSum)
                maxSum = thisSum;
        }
    return maxSum;
}
```

Maximum subsequence – algorithm 1

- Space complexity

  $S(n) = O(1)$, does not depend on the input size

- Time complexity
  - cycle of *n* iterations within another cycle of *n* iterations within another cycle of *n* iterations → $T(n) = O(n^3)$
  - value estimated by excess, some cycles have less than *n* iterations

- How to improve time complexity
  - remove a cycle
  - inner cycle is not necessary
  - *thisSum* for next *j* can be easily calculated from the old value of *thisSum*

Maximum subsequence – algorithm 2

```
template <class Comparable>
Comparable maxSubSum2(const vector<Comparable> &a)
{
     Comparable maxSum = 0;
     for (int i = 0 ; i < a.size() ; i++)
      {
        Comparable thisSum = 0;
        for (int j = i; j < a.size(); j++)
        {
           thisSum += a[j];
           if (thisSum > maxSum)
               maxSum = thisSum;
        }
      }
      return maxSum;
}
```

Maximum subsequence – algorithm 2

- Time complexity
  - cycle of n iterations within another cycle of n iterations → $T(n) = O(n^2)$
  - value estimated by excess, some cycles have less than n iterations

- Is it possible to improve?
  - linear algorithm is better: execution time is proportional to input size (hard to do better)
  - if a_{ij} is a subsequence with negative cost, a_{iq} with $q>j$ is not the maximum subsequence

Maximum subsequence – algorithm 3

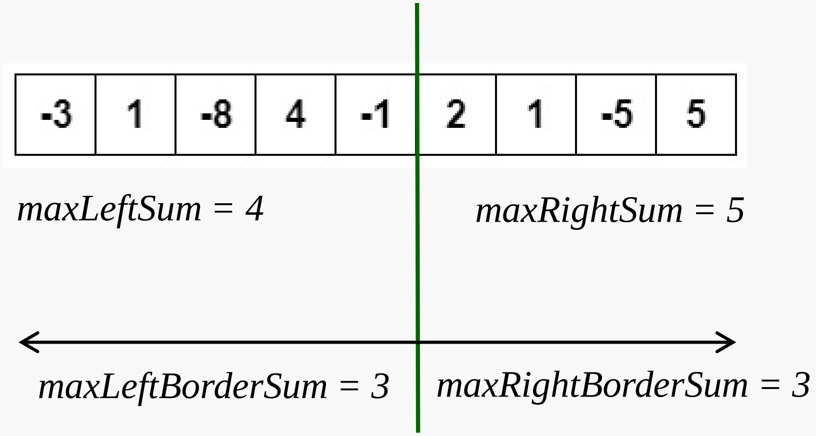
```
template <class Comparable>
Comparable maxSubSum3(const vector<Comparable> &a)
{
    Comparable thisSum = 0; Comparable maxSum = 0;
    for (int j=0 ; j < a.size() ; j++)
    {
       thisSum += a[j];
       if (thisSum > maxSum)
           maxSum = thisSum;
       else if (thisSum < 0)
           thisSum = 0;
    }
    return maxSum;
}
```

Time complexity

$$T(n) = O(n)$$

Maximum subsequence – algorithm 4

- Divide and conquer
  - divide the sequence in half
  - the maximum subsequence is:
    a) in the first half
    b) in the second half
    c) starts in the first half, goes to the last element of it, continues with the first element of the second half, and ends with an element of it
  - calculates the three hypotheses and determines the maximum



a) and b) : recursively calculated

c) : calculated with two sequential cycles

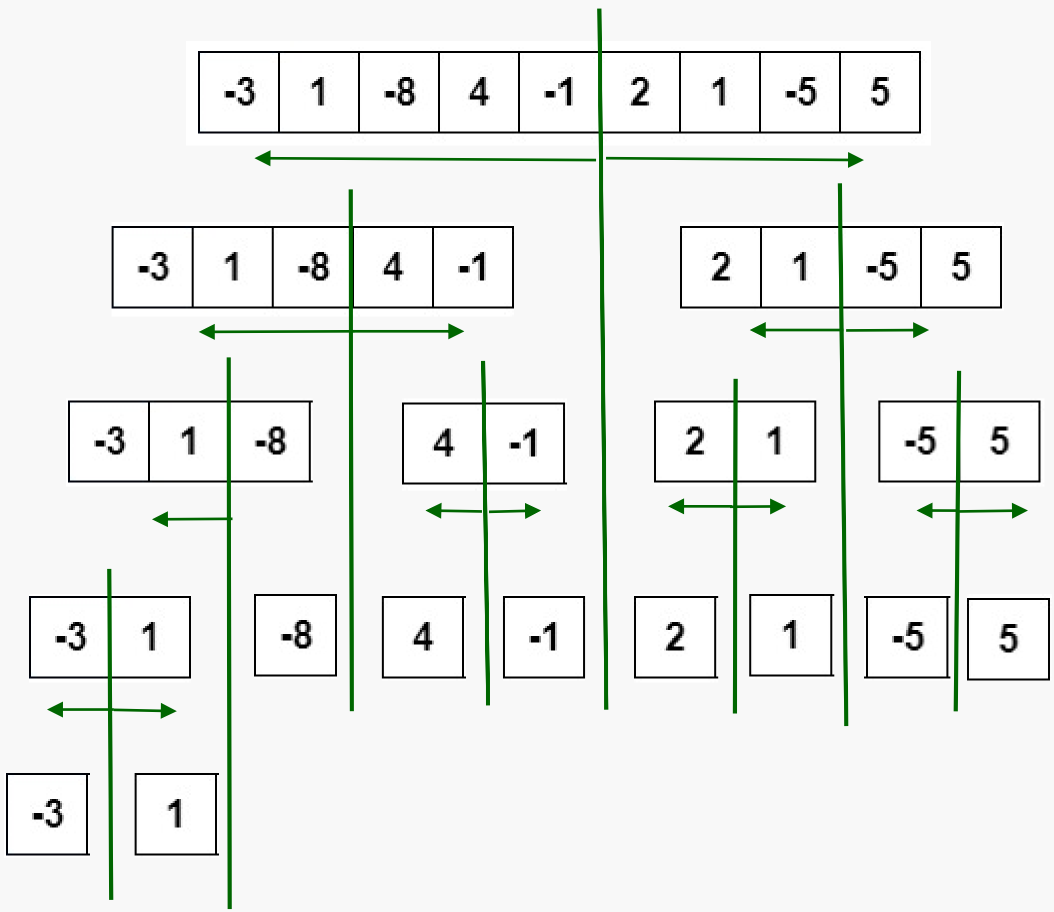
Maximum subsequence – algorithm 4

```
template <class Comparable>
Comparable maxSubSum(const vector<Comparable> &a, int left,
                                                  int right)
{
   Comparable maxLeftBorderSum = 0, maxRightBorderSum = 0
   Comparable leftBorderSum = 0, rightBorderSum = 0;
   int center = (left + right ) / 2;

   if (left == right)
       return ( a[left] > 0 ? a[left] : 0 )

   Comparable maxLeftSum = maxSubSum (a, left, center);
   Comparable maxRightSum = maxSubSum (a, center + 1, right);
```

Maximum subsequence – algorithm 4

```
   for (int i = center ; i >= left ; i--)
   {
      leftBorderSum += a[i];
      if (leftBorderSum > maxLeftBorderSum)
         maxLeftBorderSum = leftBorderSum;
   }
   for (int j = center +1 ; j <= right ; j++)
   {
      rightBorderSum += a[j];
      if (rightBorderSum > maxRightBorderSum)
         maxRightBorderSum = rightBorderSum;
   }

   return max3( maxLeftSum, maxRightSum,
                  maxLeftBorderSum + maxRightBorderSum);
}
```

Maximum subsequence – algorithm 4



Maximum subsequence – algorithm 4

- Time complexity

Let $T(n)$ = execution time for input size n

(remember that constants don't matter)

$$\begin{cases} T(1) = 1 \\ T(n) = 2 \times T(n/2) + n \end{cases}$$

- two recursive calls, each of input size $n/2$; execution time of each recursive call is $T(n/2)$
- execution time of c) is n

recurrence relation

$$T(n/2) = 2 \times T(n/4) + n/2$$

$$T(n/4) = 2 \times T(n/8) + n/4$$

…

$$T(n) = 2 \times 2 \times T(n/4) + 2 \; * n/2 + n$$

$$T(n) = 2 \times 2 \times 2 \times T(n/8) + 2 \times 2 \times n/4 + 2 \times n/2 + \; n$$

…

$$T(n) = 2^k \times T\left(n/2^k\right) + k \times n$$

Maximum subsequence – algorithm 4

$$T(n) = 2^k \times T\left(n/2^k\right) + k \times n$$

- we know that $T(1) = 1$
- $\left(n/2^k\right) = 1 \quad \rightarrow \quad k = \log_2 n$

$$T(n) = n \times 1 + \log_2 n \times n = O(n \times \log n)$$

- Space complexity

$S(n) = O(log\ n)$

Tower of Hanoi problem

Tower of Hanoi is a puzzle invented in 1883

- consists of three rods and multiple disks
- initially, all the disks are placed on one rod, one over the other in ascending order of size

- the objective is to move the stack of disks from the initial rod to another rod, following these rules:
  - A disk cannot be placed on top of a smaller disk
  - No disk can be placed on top of the smaller disk

Time complexity?

Space complexity?