Mathematical Modeling - Fourier Series

Presentation by Asimakis Kydros, 3881

The libraries used are listed below:

```
# Always run this cell
import numpy as np
import matplotlib.pyplot as plt
from typing import Callable
from scipy.integrate import quad
```

The Fourier series is a way to approximate any given and adequate function as a linear combination of a series of sines and cosines.

In the general case, the Fourier series of f is written and calculated as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi x}{p}) + b_n sin(\frac{n\pi x}{p})$$

where f is $2p$-periodic and integrable in the range $[-p, p]$. Then, the constants $a_0, a_n, b_n, n \geq 1$ are defined as

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)cos(\frac{n\pi x}{p})dx, n \geq 0$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)sin(\frac{n\pi x}{p})dx, n \geq 1$$

The above procedure is implemented below; the given domain is assumed to be of the format $(-p, p)$, as the *half period* p is calculated as the absolute mean of the bounds.

Note that the function

```
quad(func, a, b)
```

calculates the definite integral

$$\int_a^b func(x)dx$$

and returns a list with a bunch of information, of which we only care about the value (index 0).

After calculating all the constants, the values of the Fourier series of f $\forall x \in (-p, p)$ up to a given depth n (by default the industrial standard 50) are returned in a list, along with the corresponding x-axis.

```
def fourier_series(func: Callable, domain: tuple, depth: int = 50):
    p = (abs(domain[0]) + abs(domain[1])) * 0.5
    a = [1 / np.pi * quad(lambda x: func(x) * np.cos(n * np.pi * x / p), -p, p)[0] for n in range(depth)]
    b = [1 / np.pi * quad(lambda x: func(x) * np.sin(n * np.pi * x / p), -p, p)[0] for n in range(depth)]
    fs = []
    X = np.arange(-p, p, 0.001)
    for x in X:
        fs.append(
            a[0] * 0.5 + sum(a[n] * np.cos(n * np.pi * x / p) + b[n] * np.sin(n * np.pi * x / p) for n in range(1, depth))
        )
    return X, fs
```

Let's test it; suppose the identity function

$$f(x) = x,\ x \in (-\pi, \pi]$$

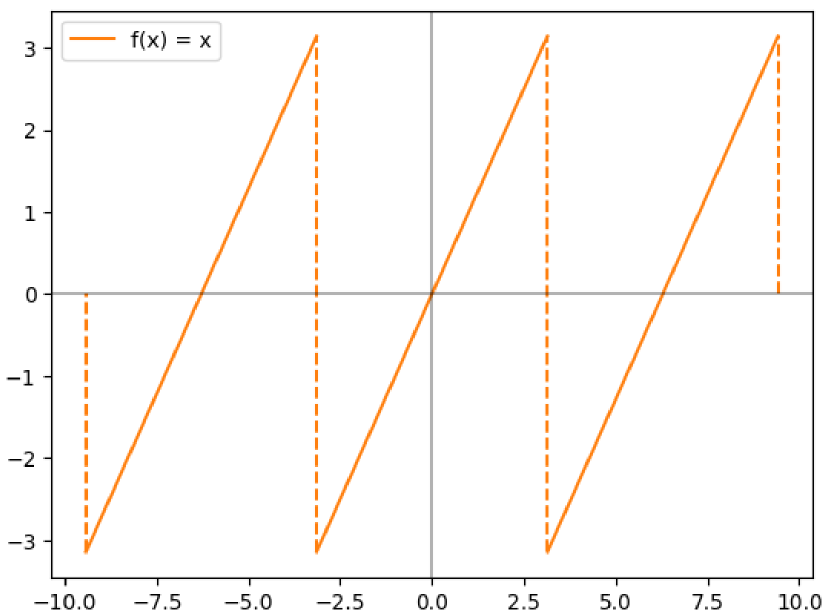
Firstly, we have to extend it in the entirety of $[-\pi, \pi]$. We define the extension of f as

$$\tilde{f}(x) = \begin{cases} f(x), & x \in (-\pi, \pi] \\ f(\pi), & x = -\pi \end{cases} = \begin{cases} x, & x \in (-\pi, \pi] \\ \pi, & x = -\pi \end{cases}$$

Now, we have to consider the 2π-*periodic extension* F of $\tilde{f}$:

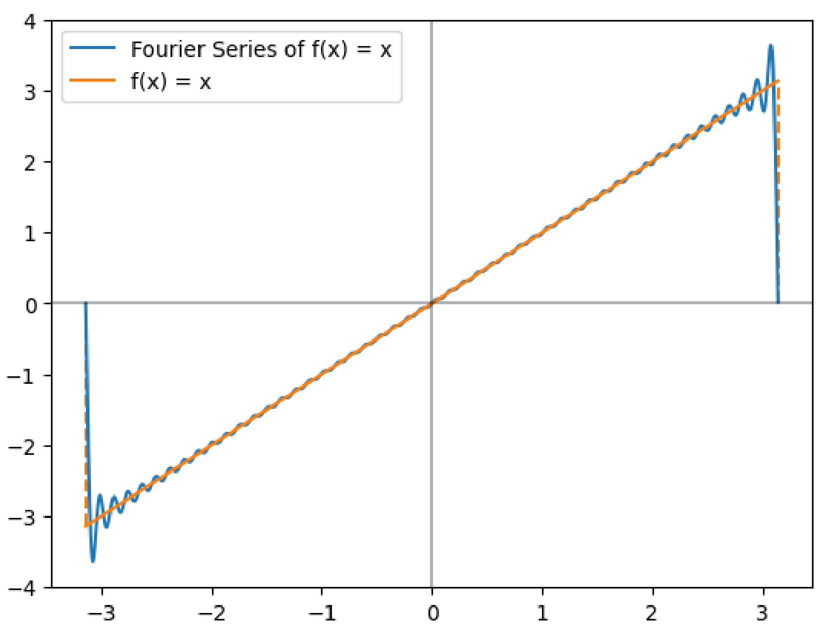
```
x = np.arange(-np.pi, np.pi, 0.001)
plt.plot(x - 2 * np.pi, x, 'C1')
plt.plot(x + 2 * np.pi, x, 'C1')
plt.vlines(-np.pi, -np.pi, np.pi, color='C1', linestyles='dashed')
plt.vlines(np.pi, -np.pi, np.pi, color='C1', linestyles='dashed')
plt.vlines(3 * np.pi, 0, np.pi, color='C1', linestyles='dashed')
plt.vlines(-3 * np.pi, -np.pi, 0, color='C1', linestyles='dashed')
plt.plot(x, x, 'C1', label='f(x) = x')
plt.legend(loc='upper left')
plt.axhline(color='black', alpha=0.3)
```

```
plt.axvline(color='black', alpha=0.3)
plt.show()
```



Now it makes sense to define a Fourier series; since F is periodic in $\mathbb{R}$, we choose to operate in the subset $[-p, p]$, i.e. in f itself:

```
x, y = fourier_series(lambda x: x, (-np.pi, np.pi))
plt.vlines(np.pi, 0, np.pi, color='C1', linestyles='dashed')
plt.vlines(-np.pi, -np.pi, 0, color='C1', linestyles='dashed')
plt.plot(x, y, 'C0', label='Fourier Series of f(x) = x')
plt.plot(x, x, 'C1', label='f(x) = x')
plt.legend(loc='upper left')
plt.axhline(color='black', alpha=0.3)
plt.axvline(color='black', alpha=0.3)
plt.show()
```



We can see that the approximation given by the Fourier series almost perfectly alignes with the actual function in the range $(-\pi, \pi)$. The more it approaches the points of discontinuity $x = -\pi, x = \pi$ though, the error increases.

This behavior is known as the *Gibbs phenomenon*.

Let's see one more example; suppose the function

$$f(x) = \begin{cases} x^2, & 0 < x \leq \pi \\ 0, & -\pi < x \leq 0 \end{cases}$$

in code written as

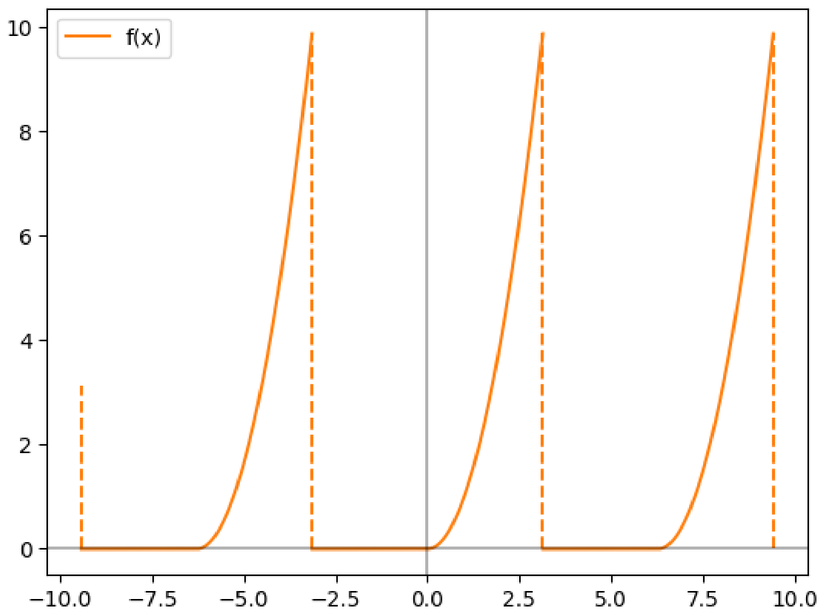
```
def f(x: list[float]) -> float:
    return x * x if 0 < x <= np.pi else 0
```

Extending it into the entirety of $[-\pi, \pi]$ we have

$$\tilde{f}\,(x) = \begin{cases} f(x), & x \in (-\pi, \pi] \\ f(\pi), & x = -\pi \end{cases} = \begin{cases} x^2, & x \in (0, \pi] \vee x = -\pi \\ 0, & x \in (-\pi, 0] \end{cases}$$

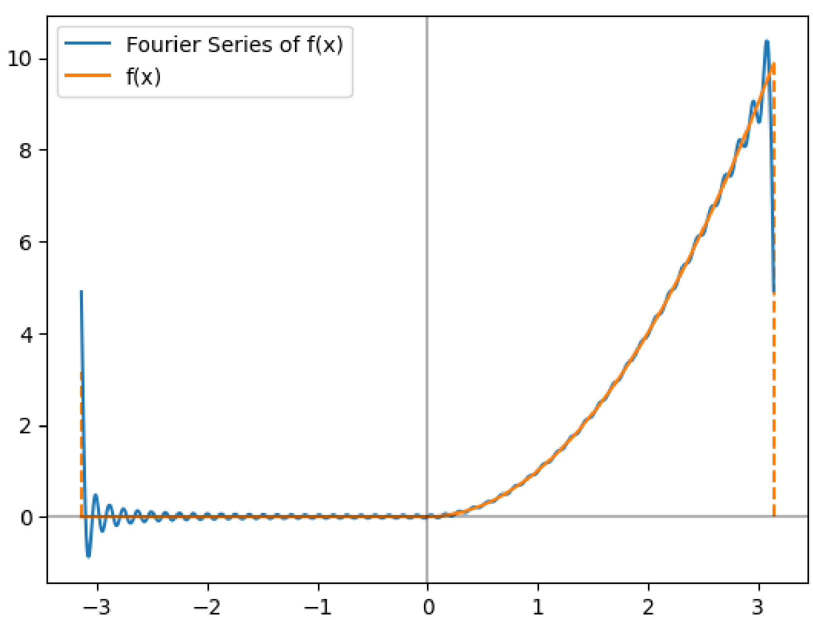
and considering the 2π-periodic extension F of $\tilde{f}$ we have

```
x = np.arange(-np.pi, np.pi, 0.001)
fvals = [f(i) for i in x]
plt.plot(x - 2 * np.pi, fvals, 'C1')
plt.plot(x + 2 * np.pi, fvals, 'C1')
plt.vlines(-np.pi, 0, np.pi * np.pi, color='C1', linestyles='dashed')
plt.vlines(np.pi, 0, np.pi * np.pi, color='C1', linestyles='dashed')
plt.vlines(3 * np.pi, 0, np.pi * np.pi, color='C1', linestyles='dashed')
plt.vlines(-3 * np.pi, 0, np.pi, color='C1', linestyles='dashed')
plt.plot(x, fvals, 'C1', label='f(x)')
plt.legend(loc='upper left')
plt.axhline(color='black', alpha=0.3)
plt.axvline(color='black', alpha=0.3)
plt.show()
```



Again we choose to work in the subset $[-\pi, \pi]$ of the domain of F, therefore in f, and we have

```
x, y = fourier_series(f, (-np.pi, np.pi))
plt.vlines(-np.pi, 0, np.pi, color='C1', linestyles='dashed')
plt.vlines(np.pi, 0, np.pi * np.pi, color='C1', linestyles='dashed')
plt.plot(x, y, 'C0', label='Fourier Series of f(x)')
plt.plot(x, [f(i) for i in x], 'C1', label='f(x)')
plt.legend(loc='upper left')
plt.axhline(color='black', alpha=0.3)
plt.axvline(color='black', alpha=0.3)
plt.show()
```



where we can again notice the amazing accuracy of the Fourier series' approximation of f in $(-\pi, \pi)$, but the points of discontinuity $x = -\pi, x = \pi$ again form the Gibbs phenomenon.

References:

- *Applied Mathematics* by N.L.Tsitsas
- https://docs.scipy.org/doc/scipy/tutorial/integrate.html