Abelian extensions arising from elliptic curves with complex multiplication

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For this talk, we will focus on elliptic curves E with complex multiplication and fix F to be the minimal field of definition, i.e. $F = \mathbb{Q}(j(E))$.

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- One can show that over $\overline{F}$, we have that $E[N] \cong \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$.
- We will be interested in the **N^{th}-division field of E/F**,

$$F(E[N]) = F(\{x(P), y(P) : P \in E[N]\}),$$

  i.e., the field of definition of the coordinates of points in $E[N]$.

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Let E be an elliptic curve defined over $\mathbb{Q}$. Consider $\mathbb{Q}(E[N])/\mathbb{Q}$, where

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Question

What is $\mathrm{Gal}(\mathbb{Q}(E[N])/\mathbb{Q})$*?*

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Let $E/\mathbb{Q}$ be an elliptic curve and $N \geq 2$.

Definition

Let $\rho_{E,N}$ be the mod N Galois representation attached to E:

$$\rho_{E,N} \colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Aut}(E[N]) \cong \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z}),$$

so we have $G_{E,N} := \operatorname{Gal}(\mathbb{Q}(E[N])/\mathbb{Q}) \cong \operatorname{im}(\rho_{E,N})$.

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Question

Can $\mathrm{Gal}(\mathbb{Q}(E[N])/\mathbb{Q})$ *be abelian, i.e., can* $\mathrm{im}(\rho_{E,N})$ *be abelian?*

What is known?

Theorem (Lozano-Robledo, González-Jiménez, 2015)

Let $E/\mathbb{Q}$ be an elliptic curve. Let $N \geq 2$.

- $\mathbb{Q}(E[N]) = \mathbb{Q}(\zeta_N)$ *only for* $N = 2, 3, 4,$ *or* 5.
- *More generally, if* $\mathbb{Q}(E[N])/\mathbb{Q}$ *is abelian, then* $N = 2, 3, 4, 5, 6,$ *or* 8.

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Can $F(E[N])/F$ be abelian?

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Theorem (H. and Lozano-Robledo, 2023)

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$, $f \geq 1$. Let $N \geq 2$ and let

$$G_{E,N} = \mathrm{Gal}(\mathbb{Q}(j_{K,f}, E[N])/\mathbb{Q}(j_{K,f}))$$

be the Galois group of the N-division field of E.

If $G_{E,N}$ is abelian, then N must equal $2, 3$, or 4. Further, if $G_{E,N}$ is abelian, then it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$ for some $0 \leq k \leq 3$.

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Theorem (H. and Lozano-Robledo, 2023)

(1) *If $j_{K,f} \neq 0, 1728$, then $G_{E,N}$ is abelian if and only if:*

- $\underline{N = 2}$ *and either*
  - $\Delta_K f^2 \equiv 0 \bmod 4$, *or*
  - $\Delta_K \equiv 1 \bmod 8$ *and* $f \equiv 1 \bmod 2$.

  In this case, $G_{E,2} \cong /2$.

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(2) *If* $j_{K,f} = 1728$*, then* $G_{E,N}$ *is abelian if and only if:*

- $\underline{N = 2}$*. In this case,* $G_{E,2} \cong \{0\}$ *or* $/2$ *according to whether* E *is given by* $y^2 = x^3 - dx$ *with* d *a square or a non-square in , respectively.*
- $\underline{N = 4}$ *and* $E/$ *is given by* $y^2 = x^3 + dx$ *with* $d \in \{\pm 1, \pm 4\}$ *or* $d = \pm t^2$ *for some square-free integer* $t \notin \{\pm 1, \pm 2\}$*, in which case* $G_{E,4} \cong (/2)^2$ *or* $(/2)^3$*, resp.*

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- $\underline{N = 2}$ *and* $E/$ *is given by* $y^2 = x^3 + d$ *with* d *a cube in . Then* $G_{E,2} \cong /2$*.*
- $\underline{N = 3}$ *and* $E/$ *is given by* $y^2 = x^3 + d$ *such that* $4d$ *is a cube in . If in addition* d *and* $3d$ *are not squares, then* $G_{E,3} \cong (/2)^2$*, and if* d *or* $3d$ *is a square, then* $G_{E,3} \cong /2$*.*

Example

Theorem (1a): If $N = 2$, then $G_{E,2}$ is abelian if and only if $j_{K,f} \neq 0, 1728$ and $\Delta_K f^2 \equiv 0 \bmod 4$. In this case, $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

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Example (32.1-a1)

$$E/\mathbb{Q}(\sqrt{2}) : y^2 + \sqrt{2}xy = x^3 + x^2 + \left(15\sqrt{2} - 22\right)x + 46\sqrt{2} - 69$$

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One can check that $E(\mathbb{Q}(\sqrt{2}))[2] \cong \mathbb{Z}/2\mathbb{Z}$ is generated by a non-trivial point of two torsion defined over $\mathbb{Q}(\sqrt{2})$, namely

$$P = \left(2\sqrt{2} - \frac{3}{2}, \ \frac{3}{4}\sqrt{2} - 2\right).$$

Sketch of proof

Theorem (H. and Lozano-Robledo, 2023)

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(1) For an elliptic curve $E/\mathbb{Q}(j_{K,f})$ with CM by an arbitrary order $\mathcal{O}_{K,f}$, Lozano-Robledo explicitly describes the groups of $\mathrm{GL}(2, \mathbb{Z}_p)$ that can occur as images of ρ_{E,p^∞}, up to conjugation.

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(2) We understand what subgroups of $\mathcal{N}_{\delta,\phi}(N)$ are images of $\rho_{E,N}$ and we give conditions that will help characterize when a subgroup of $\mathcal{N}_{\delta,\phi}(N)$ is abelian (e.g. the Cartan subgroup is abelian).

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(3) We apply the results from above to all possible images $G_{E,N} = \operatorname{im} \rho_{E,N}$ from (1) and analyze under what circumstances we have that $G_{E,N}$ is abelian.

□

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(3) Let $N \geq 3$, $d \in F$ such that $\sqrt{d} \notin K$, and E^d be the twist of E by d. Then there is an explicitly computable integer $\alpha = \alpha(E^d)$, unique up to some power, such that $F(\sqrt{\alpha}) \subseteq F(E[N])$.

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Therefore, we have that $K(j_{K,f}, \zeta_N, \sqrt{\alpha})$ is an abelian extension contained in $F(E[N])/F$, which is sometimes just $K(j_{K,f}, \zeta_N)$ if $\sqrt{\alpha} \in K(j_{K,f}, \zeta_N)$.

Field diagram

Let $N \geq 3$. Let $G_{E,N} = \mathrm{Gal}(\mathbb{Q}(j_{K,f}, E[N])/\mathbb{Q}(j_{K,f}))$.
Let $G'_{E,N}$ denote the commutator subgroup of $G_{E,N}$.

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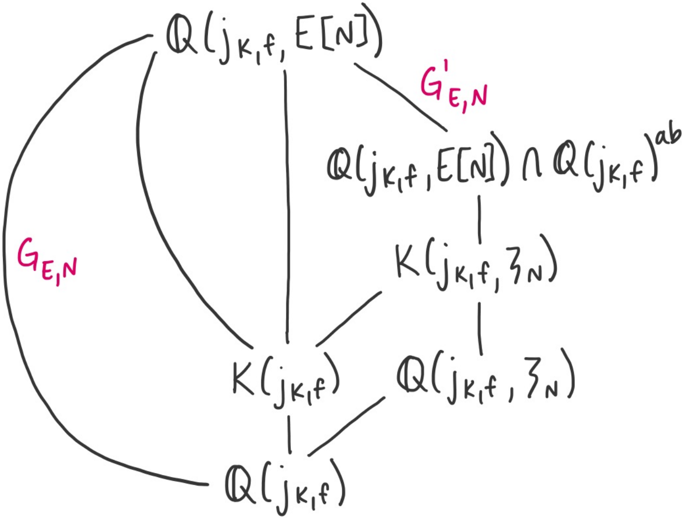

How to find the maximal abelian subextension

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(4) It turns out that $U = L$, so it must be that

$$K(j_{K,f}, \zeta_{p^n}) \quad \text{or} \quad K(j_{K,f}, \zeta_{p^n}, \sqrt{\alpha})$$

is the maximal abelian subextension of $\mathbb{Q}(j_{K,f}, E[p^n])/\mathbb{Q}(j_{K,f})$.

$\square$

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(3) *Let $p = 2$ such that $2 \mid \Delta_K f^2$.*

- *If $\Delta_K f^2 = -12$ or -28, then* $\mathbb{Q}(E[2^n]) \cap \mathbb{Q}^{ab} = K(\zeta_{2^{n+1}})$.
- *If $\Delta_K f^2 = -4, -8,$ or -16, then*

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Example

Example ($p = 7$ and $\Delta_K f^2 = -7$)

Let $E/\mathbb{Q} : y^2 = x^3 - 140x - 784$ (3136.n4), where $j(E) = -3375$.

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The simplest CM curve E' has image

$$G_{E',7^n} = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \left\{ \begin{pmatrix} a & b \\ \delta b & a \end{pmatrix} : a \in (\mathbb{Z}/7\mathbb{Z})^{\times 2}, b \in \mathbb{Z}/7\mathbb{Z} \right\} \right\rangle,$$

which is an index 2 subgroup of $\mathcal{N}_{\delta,0}(7^n)$. Thus, $\mathbb{Q}(E[7^n]) \cap \mathbb{Q}^{\text{ab}} = \mathbb{Q}(\zeta_{7^n})$.

Questions?