Abelian Extensions Arising from Elliptic Curves with Complex Multiplication

Asimina S. Hamakiotes

University of Connecticut

April 1, 2025
(This is NOT a joke.)

Motivation

Hilbert's 12$^{\text{th}}$ Problem

What are all of the abelian extensions of a number field K?

Motivation

Hilbert's 12^{th} Problem

What are all of the abelian extensions of a number field K?

Let's begin with $K = \mathbb{Q}$.

Motivation

Hilbert's 12^{th} Problem

What are all of the abelian extensions of a number field K?

Let's begin with $K = \mathbb{Q}$.

Example: The extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is abelian:

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \{\sqrt{2} \mapsto \sqrt{2},\ \sqrt{2} \mapsto -\sqrt{2}\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Motivation

Hilbert's 12^{th} Problem

What are all of the abelian extensions of a number field K?

Let's begin with $K = \mathbb{Q}$.

Example: The extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is abelian:

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \{\sqrt{2} \mapsto \sqrt{2},\ \sqrt{2} \mapsto -\sqrt{2}\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Example: Consider $\mathbb{Q}(\alpha)$, where α is a root of $f(x) = x^3 - 3x - 1$. The 3 distinct roots are

$$\alpha, \quad -\alpha^2 + 2, \quad \alpha^2 - \alpha - 2.$$

The extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is abelian:

$$\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q}) = \{\alpha \mapsto \alpha,\ \alpha \mapsto -\alpha^2 + 2,\ \alpha \mapsto \alpha^2 - \alpha - 2\} \cong \mathbb{Z}/3\mathbb{Z}.$$

Cyclotomic fields

Definition

A *cyclotomic field* is an extension of $\mathbb{Q}$ generated by a root of unity: it is $\mathbb{Q}(\zeta_n)$, where ζ_n is a root of $f(x) = x^n - 1$ with order n.

Cyclotomic fields

Definition

A *cyclotomic field* is an extension of $\mathbb{Q}$ generated by a root of unity: it is $\mathbb{Q}(\zeta_n)$, where ζ_n is a root of $f(x) = x^n - 1$ with order n.

The two abelian extensions we saw are in cyclotomic fields:

- $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\zeta_8)$,
- $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\zeta_9)$, where α is a root of $f(x) = x^3 - 3x - 1$.

Cyclotomic fields

Definition

A *cyclotomic field* is an extension of $\mathbb{Q}$ generated by a root of unity: it is $\mathbb{Q}(\zeta_n)$, where ζ_n is a root of $f(x) = x^n - 1$ with order n.

The two abelian extensions we saw are in cyclotomic fields:

- $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\zeta_8)$,
- $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\zeta_9)$, where α is a root of $f(x) = x^3 - 3x - 1$.

Cyclotomic extensions are abelian extensions of $\mathbb{Q}$:

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times,$$

by $\sigma \mapsto a \bmod n$, where $\sigma(\zeta_n) = \zeta_n^a$.

Cyclotomic fields

Definition

A *cyclotomic field* is an extension of $\mathbb{Q}$ generated by a root of unity: it is $\mathbb{Q}(\zeta_n)$, where ζ_n is a root of $f(x) = x^n - 1$ with order n.

The two abelian extensions we saw are in cyclotomic fields:

- $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\zeta_8)$,
- $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\zeta_9)$, where α is a root of $f(x) = x^3 - 3x - 1$.

Cyclotomic extensions are abelian extensions of $\mathbb{Q}$:

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times,$$

by $\sigma \mapsto a \bmod n$, where $\sigma(\zeta_n) = \zeta_n^a$.

All subfields of abelian extensions of $\mathbb{Q}$ are abelian, because quotient groups of abelian groups are abelian.

Kronecker-Weber Theorem

Theorem (Kronecker-Weber Theorem)

The abelian extensions of $\mathbb{Q}$ *are precisely the subfields of cyclotomic fields.*

Kronecker-Weber Theorem

Theorem (Kronecker-Weber Theorem)

The abelian extensions of $\mathbb{Q}$ are precisely the subfields of cyclotomic fields.

Let $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(\zeta_n)$. Then $F/\mathbb{Q}$ is Galois and $\mathrm{Gal}(F/\mathbb{Q})$ is abelian.

Kronecker-Weber Theorem

Theorem (Kronecker-Weber Theorem)

The abelian extensions of $\mathbb{Q}$ are precisely the subfields of cyclotomic fields.

Let $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(\zeta_n)$. Then $F/\mathbb{Q}$ is Galois and $\mathrm{Gal}(F/\mathbb{Q})$ is abelian.

Question: What are abelian extensions of number fields besides $\mathbb{Q}$?

Kronecker-Weber Theorem

Theorem (Kronecker-Weber Theorem)

The abelian extensions of $\mathbb{Q}$ are precisely the subfields of cyclotomic fields.

Let $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(\zeta_n)$. Then $F/\mathbb{Q}$ is Galois and $\mathrm{Gal}(F/\mathbb{Q})$ is abelian.

Question: What are abelian extensions of number fields besides $\mathbb{Q}$?

We can construct abelian extensions of $\mathbb{Q}(i)$ using an elliptic curve!

What is an elliptic curve?

Definition

An *elliptic curve* E defined over a field K is a smooth projective curve of genus 1, with at least one K-rational point.

What is an elliptic curve?

Definition

An *elliptic curve* E defined over a field K is a smooth projective curve of genus 1, with at least one K-rational point.

More concretely, an *elliptic curve* E defined over a field K is given by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in K,$$

plus a "point at infinity" $\mathcal{O}$.

What is an elliptic curve?

Definition

An *elliptic curve* E defined over a field K is a smooth projective curve of genus 1, with at least one K-rational point.

More concretely, an *elliptic curve* E defined over a field K is given by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in K,$$

plus a "point at infinity" $\mathcal{O}$.

When $\mathrm{char}(K) \neq 2, 3$, it may be simplified to a short Weierstrass equation

$$y^2 = x^3 + Ax + B, \quad A, B \in K,$$

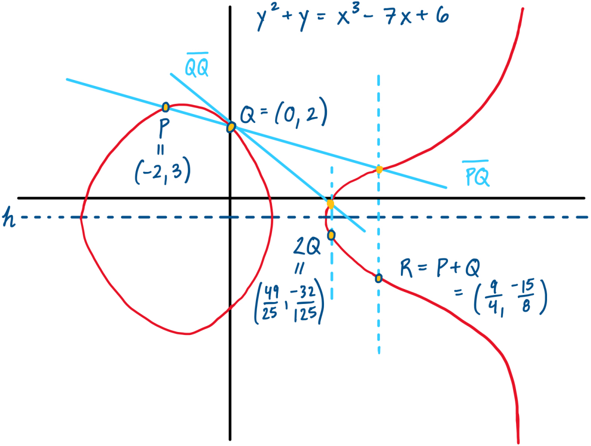
where $4A^3 + 27B^2 \neq 0$ (for smoothness).

Elliptic curve group law

There is a group law (abelian) on the L-rational points of E

$$E(L) = \{(x, y) \in E : x, y \in L\} \cup \mathcal{O},$$

with coordinates in any field $L \supset K$. We call $E(L)$ the *Mordell-Weil group* of E/L.



Torsion points on an elliptic curve

Theorem (Mordell-Weil Theorem)

Let F be a number field and let E/F be an elliptic curve. Then $E(F)$ is a finitely generated abelian group: $E(F) \cong E(F)_{tors} \oplus \mathbb{Z}^r$, where $r \in \mathbb{Z}_{\geq 0}$.

Torsion points on an elliptic curve

Theorem (Mordell-Weil Theorem)

Let F be a number field and let E/F be an elliptic curve. Then $E(F)$ is a finitely generated abelian group: $E(F) \cong E(F)_{tors} \oplus \mathbb{Z}^r$, where $r \in \mathbb{Z}_{\geq 0}$.

Let $N \geq 1$. The point $P \in E(F)$ is an *N-torsion point* if

$$\underbrace{P + P + \cdots + P}_{N \text{ times}} = [N]P = \mathcal{O},$$

where $\mathcal{O}$ is the "point at infinity".

Torsion points on an elliptic curve

Theorem (Mordell-Weil Theorem)

Let F be a number field and let E/F be an elliptic curve. Then $E(F)$ is a finitely generated abelian group: $E(F) \cong E(F)_{tors} \oplus \mathbb{Z}^r$, where $r \in \mathbb{Z}_{\geq 0}$.

Let $N \geq 1$. The point $P \in E(F)$ is an *N-torsion point* if

$$\underbrace{P + P + \cdots + P}_{N \text{ times}} = [N]P = \mathcal{O},$$

where $\mathcal{O}$ is the "point at infinity".

Definition

Let $N \geq 1$ be an integer and let

$$E[N] = \{P \in E(\overline{F}) : [N]P = \mathcal{O}\}.$$

This is called the *N-torsion subgroup of* $E(\overline{F})$.

Division field of an elliptic curve

Let F be a number field and let E/F be an elliptic curve.

Definition

The N^{th}-*division field of* E/F is

$$F(E[N]) = F(\{x(P), y(P) : P \in E[N]\}).$$

This is the field of definition of the coordinates of all points in $E[N]$.

Division field of an elliptic curve

Let F be a number field and let E/F be an elliptic curve.

Definition

The N^{th}-*division field of* E/F is

$$F(E[N]) = F(\{x(P), y(P) : P \in E[N]\}).$$

This is the field of definition of the coordinates of all points in $E[N]$.

Example

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 + x$. We have

$$\begin{aligned} E[2] &= \{\mathcal{O}, (0,0), (i,0), (-i,0)\}, \\ \mathbb{Q}(E[2]) &= \mathbb{Q}(0, i, -i) = \mathbb{Q}(i). \end{aligned}$$

Division field of an elliptic curve

Let F be a number field and let E/F be an elliptic curve.

Definition

The N^{th}-*division field of* E/F is

$$F(E[N]) = F(\{x(P), y(P) : P \in E[N]\}).$$

This is the field of definition of the coordinates of all points in $E[N]$.

Example

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 + x$. We have

$$\begin{aligned} E[2] &= \{\mathcal{O}, (0,0), (i,0), (-i,0)\}, \\ \mathbb{Q}(E[2]) &= \mathbb{Q}(0, i, -i) = \mathbb{Q}(i). \end{aligned}$$

Note that, similar to $\mathbb{Q}(\zeta_N)/\mathbb{Q}$, the extension $F(E[N])/F$ is Galois.

Abelian extensions of $\mathbb{Q}(i)$

Theorem

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 + x$. For each integer $N \geq 1$, let

$$K_N = \mathbb{Q}(i)(E[N]).$$

Then K_N is a Galois extension of $\mathbb{Q}(i)$, and $\mathrm{Gal}(K_N/\mathbb{Q}(i))$ is abelian.

Abelian extensions of $\mathbb{Q}(i)$

Theorem

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 + x$. For each integer $N \geq 1$, let

$$K_N = \mathbb{Q}(i)(E[N]).$$

Then K_N is a Galois extension of $\mathbb{Q}(i)$, and $\mathrm{Gal}(K_N/\mathbb{Q}(i))$ is abelian.

For E above, $\mathbb{Q}(E[N])/\mathbb{Q}$ is Galois and $\mathbb{Q}(i)/\mathbb{Q}$ is Galois, so $K_N/\mathbb{Q}$ is Galois. But $K_N/\mathbb{Q}$ may not be abelian!

Abelian extensions of $\mathbb{Q}(i)$

Example

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 + x$. The 4-torsion points of E are

$$E[4] = \{\mathcal{O}, (0,0), (1, \pm\sqrt{2}), (-1, \pm i\sqrt{2}), (\pm i, 0), \text{and 8 other points}\}$$

and $\mathbb{Q}(E[4]) = \mathbb{Q}(i, \sqrt{1+\sqrt{2}})$.

One can check that $\mathrm{Gal}(\mathbb{Q}(E[4])/\mathbb{Q}(i)) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Note that $\mathrm{Gal}(\mathbb{Q}(E[4])/\mathbb{Q}) \cong D_4$.

Abelian extensions of $\mathbb{Q}(i)$

Example

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 + x$. The 4-torsion points of E are

$$E[4] = \{\mathcal{O}, (0,0), (1, \pm\sqrt{2}), (-1, \pm i\sqrt{2}), (\pm i, 0), \text{and 8 other points}\}$$

and $\mathbb{Q}(E[4]) = \mathbb{Q}(i, \sqrt{1+\sqrt{2}})$.

One can check that $\mathrm{Gal}(\mathbb{Q}(E[4])/\mathbb{Q}(i)) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Note that $\mathrm{Gal}(\mathbb{Q}(E[4])/\mathbb{Q}) \cong D_4$.

Question: What are the abelian extensions of $\mathbb{Q}(i)$?

Abelian extensions of $\mathbb{Q}(i)$

Example

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 + x$. The 4-torsion points of E are

$$E[4] = \{\mathcal{O}, (0,0), (1, \pm\sqrt{2}), (-1, \pm i\sqrt{2}), (\pm i, 0), \text{and 8 other points}\}$$

and $\mathbb{Q}(E[4]) = \mathbb{Q}(i, \sqrt{1+\sqrt{2}})$.

One can check that $\mathrm{Gal}(\mathbb{Q}(E[4])/\mathbb{Q}(i)) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Note that $\mathrm{Gal}(\mathbb{Q}(E[4])/\mathbb{Q}) \cong D_4$.

Question: What are the abelian extensions of $\mathbb{Q}(i)$?

Theorem (Takagi)

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 + x$. The abelian extensions of $\mathbb{Q}(i)$ are precisely the subfields of K_N for some $N \geq 1$.

Complex multiplication (CM)

Definition

Let E be an elliptic curve defined over a field F of characteristic 0. We say that E has *complex multiplication* (CM) if $\mathrm{End}(E) \supsetneq \mathbb{Z}$.

Complex multiplication (CM)

Definition

Let E be an elliptic curve defined over a field F of characteristic 0. We say that E has *complex multiplication* (CM) if $\mathrm{End}(E) \supsetneq \mathbb{Z}$.

If E/F has CM, then $\mathrm{End}(E) \cong \mathcal{O}_{K,f}$, where $\mathcal{O}_{K,f}$ is the order in an imaginary quadratic field K with index $f \geq 1$ in $\mathcal{O}_K$.

Complex multiplication (CM)

Definition

Let E be an elliptic curve defined over a field F of characteristic 0. We say that E has *complex multiplication* (CM) if $\mathrm{End}(E) \supsetneq \mathbb{Z}$.

If E/F has CM, then $\mathrm{End}(E) \cong \mathcal{O}_{K,f}$, where $\mathcal{O}_{K,f}$ is the order in an imaginary quadratic field K with index $f \geq 1$ in $\mathcal{O}_K$.

Example

The elliptic curve $E/\mathbb{Q} : y^2 = x^3 + x$ (64.a1) has the endomorphism

$$\phi(x, y) = (-x, iy),$$

where for $(x, y) \in E$, we have $(iy)^2 = (-x)^3 + (-x)$, so $(-x, iy) \in E$.

In this case, $\mathrm{End}(E) \cong \mathbb{Z}[i] = \mathcal{O}_{K,1}$, the maximal order of $K = \mathbb{Q}(i)$.

Abelian extensions arising from elliptic curves with CM

Theorem

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Let E be an elliptic curve with $\mathrm{End}(E) \cong \mathcal{O}_K$ and j-invariant $j(E)$. Then

Abelian extensions arising from elliptic curves with CM

Theorem

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Let E be an elliptic curve with $\mathrm{End}(E) \cong \mathcal{O}_K$ and j-invariant $j(E)$. Then

(1) *$K(j(E))$ is the Hilbert class field of K,*

Abelian extensions arising from elliptic curves with CM

Theorem

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Let E be an elliptic curve with $\mathrm{End}(E) \cong \mathcal{O}_K$ and j-invariant $j(E)$. Then

(1) $K(j(E))$ *is the Hilbert class field of* K,

(2) $K(j(E), E_{tors})$ *is an abelian extension of* $K(j(E))$,

Abelian extensions arising from elliptic curves with CM

Theorem

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Let E be an elliptic curve with $\mathrm{End}(E) \cong \mathcal{O}_K$ and j-invariant $j(E)$. Then

(1) $K(j(E))$ *is the Hilbert class field of* K*,*

(2) $K(j(E), E_{tors})$ *is an abelian extension of* $K(j(E))$*,*

(3) $K(j(E), x(E_{tors}))$ *is the maximal abelian extension of* K *when* $j(E) \neq 0, 1728$ *(there is an alternate expression when* $j(E) = 0, 1728$*).*

Goal

Question: When can division fields be abelian over $\mathbb{Q}$ or other fields?

Goal

Question: When can division fields be abelian over $\mathbb{Q}$ or other fields?

Let F be a number field, E/F be an elliptic curve, and $N \geq 2$.

Goal

Question: When can division fields be abelian over $\mathbb{Q}$ or other fields?

Let F be a number field, E/F be an elliptic curve, and $N \geq 2$.

- When is the N^{th}-division field $F(E[N])$ abelian over F?

Goal

Question: When can division fields be abelian over $\mathbb{Q}$ or other fields?

Let F be a number field, E/F be an elliptic curve, and $N \geq 2$.

- When is the N^{th}-division field $F(E[N])$ abelian over F?
- If $F(E[N])/F$ is not abelian, then what is the maximal abelian extension of F contained in $F(E[N])$?

What is known?

Theorem (Lozano-Robledo, González-Jiménez, 2015)

Let $E/\mathbb{Q}$ be an elliptic curve. Let $N \geq 2$.

- $\mathbb{Q}(E[N]) = \mathbb{Q}(\zeta_N)$ *only for* $N = 2, 3, 4,$ *or* 5.
- *More generally, if* $\mathbb{Q}(E[N])/\mathbb{Q}$ *is abelian, then* $N = 2, 3, 4, 5, 6,$ *or* 8.

What is known?

Theorem (Lozano-Robledo, González-Jiménez, 2015)

Let $E/\mathbb{Q}$ be an elliptic curve. Let $N \geq 2$.

- $\mathbb{Q}(E[N]) = \mathbb{Q}(\zeta_N)$ *only for* $N = 2, 3, 4,$ *or* 5.
- *More generally, if* $\mathbb{Q}(E[N])/\mathbb{Q}$ *is abelian, then* $N = 2, 3, 4, 5, 6,$ *or* 8.

Theorem (Lozano-Robledo, González-Jiménez, 2015)

Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication. Let $N \geq 2$.

- $\mathbb{Q}(E[N]) = \mathbb{Q}(\zeta_N)$ *only for* $N = 2$ *or* 3.
- *More generally, if* $\mathbb{Q}(E[N])/\mathbb{Q}$ *is abelian, then* $N = 2, 3,$ *or* 4.

What is known?

Theorem (Lozano-Robledo, González-Jiménez, 2015)

Let $E/\mathbb{Q}$ be an elliptic curve. Let $N \geq 2$.

- $\mathbb{Q}(E[N]) = \mathbb{Q}(\zeta_N)$ *only for* $N = 2, 3, 4,$ *or* 5.
- *More generally, if* $\mathbb{Q}(E[N])/\mathbb{Q}$ *is abelian, then* $N = 2, 3, 4, 5, 6,$ *or* 8.

Theorem (Lozano-Robledo, González-Jiménez, 2015)

Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication. Let $N \geq 2$.

- $\mathbb{Q}(E[N]) = \mathbb{Q}(\zeta_N)$ *only for* $N = 2$ *or* 3.
- *More generally, if* $\mathbb{Q}(E[N])/\mathbb{Q}$ *is abelian, then* $N = 2, 3,$ *or* 4.

Let E be an elliptic curve defined over a number field F and let $N \geq 2$.

Notation

The following notation will be used throughout the rest of the talk.

Notation

The following notation will be used throughout the rest of the talk.

- K is an imaginary quadratic field,

Notation

The following notation will be used throughout the rest of the talk.

- K is an imaginary quadratic field,
- Δ_K is the discriminant of the ring of integers $\mathcal{O}_K$,

Notation

The following notation will be used throughout the rest of the talk.

- K is an imaginary quadratic field,
- Δ_K is the discriminant of the ring of integers $\mathcal{O}_K$,
- $\mathcal{O}_{K,f}$ is the order of conductor $f \geq 1$ in K, with discriminant $\Delta_K f^2$,

Notation

The following notation will be used throughout the rest of the talk.

- K is an imaginary quadratic field,
- Δ_K is the discriminant of the ring of integers $\mathcal{O}_K$,
- $\mathcal{O}_{K,f}$ is the order of conductor $f \geq 1$ in K, with discriminant $\Delta_K f^2$,
- $j_{K,f}$ is the j-invariant associated to the order $\mathcal{O}_{K,f}$, i.e., $j(\mathbb{C}/\mathcal{O}_{K,f})$.

Notation

The following notation will be used throughout the rest of the talk.

- K is an imaginary quadratic field,
- Δ_K is the discriminant of the ring of integers $\mathcal{O}_K$,
- $\mathcal{O}_{K,f}$ is the order of conductor $f \geq 1$ in K, with discriminant $\Delta_K f^2$,
- $j_{K,f}$ is the j-invariant associated to the order $\mathcal{O}_{K,f}$, i.e., $j(\mathbb{C}/\mathcal{O}_{K,f})$.

Notation

The following notation will be used throughout the rest of the talk.

- K is an imaginary quadratic field,
- Δ_K is the discriminant of the ring of integers $\mathcal{O}_K$,
- $\mathcal{O}_{K,f}$ is the order of conductor $f \geq 1$ in K, with discriminant $\Delta_K f^2$,
- $j_{K,f}$ is the j-invariant associated to the order $\mathcal{O}_{K,f}$, i.e., $j(\mathbb{C}/\mathcal{O}_{K,f})$.

$E/\mathbb{Q}(j_{K,f})$ is an elliptic curve with CM by $\mathcal{O}_{K,f}$, and a minimal field of definition for E is $\mathbb{Q}(j_{K,f})$.

Notation

The following notation will be used throughout the rest of the talk.

- K is an imaginary quadratic field,
- Δ_K is the discriminant of the ring of integers $\mathcal{O}_K$,
- $\mathcal{O}_{K,f}$ is the order of conductor $f \geq 1$ in K, with discriminant $\Delta_K f^2$,
- $j_{K,f}$ is the j-invariant associated to the order $\mathcal{O}_{K,f}$, i.e., $j(\mathbb{C}/\mathcal{O}_{K,f})$.

$E/\mathbb{Q}(j_{K,f})$ is an elliptic curve with CM by $\mathcal{O}_{K,f}$, and a minimal field of definition for E is $\mathbb{Q}(j_{K,f})$.

Example

Let $E/\mathbb{Q}(\sqrt{2})$ be the elliptic curve given by (32.1-a1),

$$y^2 + \sqrt{2}xy = x^3 + x^2 + \left(15\sqrt{2} - 22\right)x + 46\sqrt{2} - 69,$$

with CM by $\mathcal{O}_{K,4} = \mathbb{Z}[4i]$, where $K = \mathbb{Q}(i)$.
Here, $j_{K,4} = -29071392966\sqrt{2} + 41113158120$, so $\mathbb{Q}(j_{K,4}) = \mathbb{Q}(\sqrt{2})$.

Abelian division fields

Theorem (H. and Lozano-Robledo, 2023)

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$, $f \geq 1$. Let $N \geq 2$ and let

$$G_{E,N} = \mathrm{Gal}(\mathbb{Q}(j_{K,f}, E[N])/\mathbb{Q}(j_{K,f}))$$

be the Galois group of the N^{th}-division field of $E/\mathbb{Q}(j_{K,f})$.

If $G_{E,N}$ is abelian, then N must equal $2, 3$, or 4. Further, if $G_{E,N}$ is abelian, then it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$ for some $0 \leq k \leq 3$.

Abelian division fields

Theorem (H. and Lozano-Robledo, 2023)

(1) *If* $j_{K,f} \neq 0, 1728$*, then* $G_{E,N}$ *is abelian if and only if:*

- $\underline{N = 2}$ *and either*
  - $\Delta_K f^2 \equiv 0 \bmod 4$*, or*
  - $\Delta_K \equiv 1 \bmod 8$ *and* $f \equiv 1 \bmod 2$*.*

  In this case, $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$*.*

Abelian division fields

Theorem (H. and Lozano-Robledo, 2023)

(1) *If $j_{K,f} \neq 0, 1728$, then $G_{E,N}$ is abelian if and only if:*

- $\underline{N = 2}$ *and either*
  - $\Delta_K f^2 \equiv 0 \bmod 4$, *or*
  - $\Delta_K \equiv 1 \bmod 8$ *and* $f \equiv 1 \bmod 2$.

  In this case, $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

(2) *If $j_{K,f} = 1728$, then $G_{E,N}$ is abelian if and only if:*

- $\underline{N = 2}$. *In this case,* $G_{E,2} \cong \{0\}$ *or* $\mathbb{Z}/2\mathbb{Z}$ *according to whether E is given by* $y^2 = x^3 dx$ *with d a square or a non-square in* $\mathbb{Z}$, *respectively.*
- $\underline{N = 4}$ *and $E/\mathbb{Q}$ is given by* $y^2 = x^3 + dx$ *with* $d \in \{\pm 1, \pm 4\}$ *or* $d = \pm t^2$ *for some square-free integer* $t \notin \{\pm 1, \pm 2\}$, *in which case* $G_{E,4} \cong (\mathbb{Z}/2\mathbb{Z})^2$ *or* $(\mathbb{Z}/2\mathbb{Z})^3$, *resp.*

Abelian division fields

Theorem (H. and Lozano-Robledo, 2023)

(1) *If* $j_{K,f} \neq 0, 1728$, *then* $G_{E,N}$ *is abelian if and only if:*

- $\underline{N = 2}$ *and either*
  - $\Delta_K f^2 \equiv 0 \bmod 4$, *or*
  - $\Delta_K \equiv 1 \bmod 8$ *and* $f \equiv 1 \bmod 2$.

  In this case, $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

(2) *If* $j_{K,f} = 1728$, *then* $G_{E,N}$ *is abelian if and only if:*

- $\underline{N = 2}$. *In this case,* $G_{E,2} \cong \{0\}$ *or* $\mathbb{Z}/2\mathbb{Z}$ *according to whether* E *is given by* $y^2 = x^3 dx$ *with* d *a square or a non-square in* $\mathbb{Z}$, *respectively.*
- $\underline{N = 4}$ *and* $E/\mathbb{Q}$ *is given by* $y^2 = x^3 + dx$ *with* $d \in \{\pm 1, \pm 4\}$ *or* $d = \pm t^2$ *for some square-free integer* $t \notin \{\pm 1, \pm 2\}$, *in which case* $G_{E,4} \cong (\mathbb{Z}/2\mathbb{Z})^2$ *or* $(\mathbb{Z}/2\mathbb{Z})^3$, *resp.*

(3) *If* $j_{K,f} = 0$, *then* $G_{E,N}$ *is abelian if and only if:*

- $\underline{N = 2}$ *and* $E/\mathbb{Q}$ *is given by* $y^2 = x^3 + d$ *with* d *a cube in* $\mathbb{Z}$. *Then* $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.
- $\underline{N = 3}$ *and* $E/\mathbb{Q}$ *is given by* $y^2 = x^3 + d$ *such that* $4d$ *is a cube in* $\mathbb{Z}$. *If in addition* d *and* $3d$ *are not squares, then* $G_{E,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$, *and if* d *or* $3d$ *is a square, then* $G_{E,3} \cong \mathbb{Z}/2\mathbb{Z}$.

Example

Theorem (H. and Lozano-Robledo, 2023)

If $j_{K,f} \neq 0, 1728$, then $G_{E,2}$ is abelian if and only if $\Delta_K f^2 \equiv 0 \bmod 4$. In this case, $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

Example

Theorem (H. and Lozano-Robledo, 2023)

If $j_{K,f} \neq 0, 1728$, then $G_{E,2}$ is abelian if and only if $\Delta_K f^2 \equiv 0 \bmod 4$. In this case, $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

Example: Let $K = \mathbb{Q}(i)$ and $E/\mathbb{Q}(\sqrt{2})$ be the elliptic curve (32.1-a1)

$$E/\mathbb{Q}(\sqrt{2}) : y^2 + \sqrt{2}xy = x^3 + x^2 + \left(15\sqrt{2} - 22\right) x + 46\sqrt{2} - 69,$$

Example

Theorem (H. and Lozano-Robledo, 2023)

If $j_{K,f} \neq 0, 1728$, then $G_{E,2}$ is abelian if and only if $\Delta_K f^2 \equiv 0 \bmod 4$. In this case, $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

Example: Let $K = \mathbb{Q}(i)$ and $E/\mathbb{Q}(\sqrt{2})$ be the elliptic curve (32.1-a1)

$$E/\mathbb{Q}(\sqrt{2}) : y^2 + \sqrt{2}xy = x^3 + x^2 + \left(15\sqrt{2} - 22\right) x + 46\sqrt{2} - 69,$$

where $j(E) = -29071392966\sqrt{2} + 41113158120$.

Example

Theorem (H. and Lozano-Robledo, 2023)

If $j_{K,f} \neq 0, 1728$, then $G_{E,2}$ is abelian if and only if $\Delta_K f^2 \equiv 0 \bmod 4$. In this case, $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

Example: Let $K = \mathbb{Q}(i)$ and $E/\mathbb{Q}(\sqrt{2})$ be the elliptic curve (32.1-a1)

$$E/\mathbb{Q}(\sqrt{2}) : y^2 + \sqrt{2}xy = x^3 + x^2 + \left(15\sqrt{2} - 22\right) x + 46\sqrt{2} - 69,$$

where $j(E) = -29071392966\sqrt{2} + 41113158120$.

Note that E has CM by $\mathcal{O}_{K,4} = \mathbb{Z}[\sqrt{-16}]$.

Example

Theorem (H. and Lozano-Robledo, 2023)

If $j_{K,f} \neq 0, 1728$, then $G_{E,2}$ is abelian if and only if $\Delta_K f^2 \equiv 0 \bmod 4$. In this case, $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

Example: Let $K = \mathbb{Q}(i)$ and $E/\mathbb{Q}(\sqrt{2})$ be the elliptic curve (32.1-a1)

$$E/\mathbb{Q}(\sqrt{2}) : y^2 + \sqrt{2}xy = x^3 + x^2 + \left(15\sqrt{2} - 22\right) x + 46\sqrt{2} - 69,$$

where $j(E) = -29071392966\sqrt{2} + 41113158120$.

Note that E has CM by $\mathcal{O}_{K,4} = \mathbb{Z}[\sqrt{-16}]$.

Here $\Delta_K f^2 = -4 \cdot 16 = -64 \equiv 0 \bmod 4$, so $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

Example

Theorem (H. and Lozano-Robledo, 2023)

If $j_{K,f} \neq 0, 1728$, then $G_{E,2}$ is abelian if and only if $\Delta_K f^2 \equiv 0 \bmod 4$. In this case, $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

Example: Let $K = \mathbb{Q}(i)$ and $E/\mathbb{Q}(\sqrt{2})$ be the elliptic curve (32.1-a1)

$$E/\mathbb{Q}(\sqrt{2}) : y^2 + \sqrt{2}xy = x^3 + x^2 + \left(15\sqrt{2} - 22\right) x + 46\sqrt{2} - 69,$$

where $j(E) = -29071392966\sqrt{2} + 41113158120$.

Note that E has CM by $\mathcal{O}_{K,4} = \mathbb{Z}[\sqrt{-16}]$.

Here $\Delta_K f^2 = -4 \cdot 16 = -64 \equiv 0 \bmod 4$, so $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

One can check that $E(\mathbb{Q}(\sqrt{2}))[2] \cong \mathbb{Z}/2\mathbb{Z}$ is generated by a non-trivial 2-torsion point defined over $\mathbb{Q}(\sqrt{2})$, namely

$$P = \left(2\sqrt{2} - \frac{3}{2}, \ \frac{3}{4}\sqrt{2} - 2\right).$$

Example

Theorem 1 (H. and Lozano-Robledo, 2023)

If $j_{K,f} = 0$, then $G_{E,3}$ is abelian if and only if $N = 3$ and $E/\mathbb{Q}$ is given by $y^2 = x^3 + d$ such that $4d$ is a cube in $\mathbb{Z}$.

Example

Theorem 1 (H. and Lozano-Robledo, 2023)

If $j_{K,f} = 0$, then $G_{E,3}$ is abelian if and only if $N = 3$ and $E/\mathbb{Q}$ is given by $y^2 = x^3 + d$ such that $4d$ is a cube in $\mathbb{Z}$.

- *If d and $-3d$ are not squares, then $G_{E,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$.*

Example

Theorem 1 (H. and Lozano-Robledo, 2023)

If $j_{K,f} = 0$, then $G_{E,3}$ is abelian if and only if $N = 3$ and $E/\mathbb{Q}$ is given by $y^2 = x^3 + d$ such that $4d$ is a cube in $\mathbb{Z}$.

- *If d and $-3d$ are not squares, then $G_{E,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$.*
- *If d or $-3d$ is a square, then $G_{E,3} \cong \mathbb{Z}/2\mathbb{Z}$.*

Example

Theorem 1 (H. and Lozano-Robledo, 2023)

If $j_{K,f} = 0$, then $G_{E,3}$ is abelian if and only if $N = 3$ and $E/\mathbb{Q}$ is given by $y^2 = x^3 + d$ such that $4d$ is a cube in $\mathbb{Z}$.

- *If d and $-3d$ are not squares, then $G_{E,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$.*
- *If d or $-3d$ is a square, then $G_{E,3} \cong \mathbb{Z}/2\mathbb{Z}$.*

Example

- $E_1/\mathbb{Q} : y^2 = x^3 + 2$ (1728.n4) has $j_{K,1} = 0$ and $d = 2$.

Example

Theorem 1 (H. and Lozano-Robledo, 2023)

If $j_{K,f} = 0$, then $G_{E,3}$ is abelian if and only if $N = 3$ and $E/\mathbb{Q}$ is given by $y^2 = x^3 + d$ such that $4d$ is a cube in $\mathbb{Z}$.

- *If d and $-3d$ are not squares, then $G_{E,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$.*
- *If d or $-3d$ is a square, then $G_{E,3} \cong \mathbb{Z}/2\mathbb{Z}$.*

Example

- $E_1/\mathbb{Q} : y^2 = x^3 + 2$ (1728.n4) has $j_{K,1} = 0$ and $d = 2$.
  Here $4d = 8$ is a cube in $\mathbb{Z}$, but d and $-3d$ are not squares in $\mathbb{Z}$.

Example

Theorem 1 (H. and Lozano-Robledo, 2023)

If $j_{K,f} = 0$, then $G_{E,3}$ is abelian if and only if $N = 3$ and $E/\mathbb{Q}$ is given by $y^2 = x^3 + d$ such that $4d$ is a cube in $\mathbb{Z}$.

- *If d and $-3d$ are not squares, then $G_{E,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$.*
- *If d or $-3d$ is a square, then $G_{E,3} \cong \mathbb{Z}/2\mathbb{Z}$.*

Example

- $E_1/\mathbb{Q} : y^2 = x^3 + 2$ (1728.n4) has $j_{K,1} = 0$ and $d = 2$.
  Here $4d = 8$ is a cube in $\mathbb{Z}$, but d and $-3d$ are not squares in $\mathbb{Z}$.
  Therefore, $G_{E_1,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$.
- $E_2/\mathbb{Q} : y^2 = x^3 + 16$ (27.a4) has $j_{K,1} = 0$ and $d = 16$.

Example

Theorem 1 (H. and Lozano-Robledo, 2023)

If $j_{K,f} = 0$, then $G_{E,3}$ is abelian if and only if $N = 3$ and $E/\mathbb{Q}$ is given by $y^2 = x^3 + d$ such that $4d$ is a cube in $\mathbb{Z}$.

- *If d and $-3d$ are not squares, then $G_{E,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$.*
- *If d or $-3d$ is a square, then $G_{E,3} \cong \mathbb{Z}/2\mathbb{Z}$.*

Example

- $E_1/\mathbb{Q} : y^2 = x^3 + 2$ (1728.n4) has $j_{K,1} = 0$ and $d = 2$.
  Here $4d = 8$ is a cube in $\mathbb{Z}$, but d and $-3d$ are not squares in $\mathbb{Z}$.
  Therefore, $G_{E_1,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$.
- $E_2/\mathbb{Q} : y^2 = x^3 + 16$ (27.a4) has $j_{K,1} = 0$ and $d = 16$.
  Here $4d = 4^3$ is a cube in $\mathbb{Z}$, and d is a square in $\mathbb{Z}$.

Example

Theorem 1 (H. and Lozano-Robledo, 2023)

If $j_{K,f} = 0$, then $G_{E,3}$ is abelian if and only if $N = 3$ and $E/\mathbb{Q}$ is given by $y^2 = x^3 + d$ such that $4d$ is a cube in $\mathbb{Z}$.

- *If d and $-3d$ are not squares, then $G_{E,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$.*
- *If d or $-3d$ is a square, then $G_{E,3} \cong \mathbb{Z}/2\mathbb{Z}$.*

Example

- $E_1/\mathbb{Q} : y^2 = x^3 + 2$ (1728.n4) has $j_{K,1} = 0$ and $d = 2$.
  Here $4d = 8$ is a cube in $\mathbb{Z}$, but d and $-3d$ are not squares in $\mathbb{Z}$.
  Therefore, $G_{E_1,3} \cong (\mathbb{Z}/2\mathbb{Z})^2$.
- $E_2/\mathbb{Q} : y^2 = x^3 + 16$ (27.a4) has $j_{K,1} = 0$ and $d = 16$.
  Here $4d = 4^3$ is a cube in $\mathbb{Z}$, and d is a square in $\mathbb{Z}$.
  Therefore, $G_{E_2,3} \cong \mathbb{Z}/2\mathbb{Z}$.

How do we study this?

Let E be an elliptic curve defined over a number field F and let $N \geq 2$.

Definition

Let $\rho_{E,N}$ be the *mod N Galois representation attached to E*:

$$\rho_{E,N} \colon \operatorname{Gal}(F(E[N])/F) \hookrightarrow \operatorname{Aut}(E[N]) \cong \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z}).$$

So we have $G_{E,N} = \operatorname{Gal}(F(E[N])/F) = \operatorname{im}(\rho_{E,N}) \subseteq \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z})$.

How do we study this?

Let E be an elliptic curve defined over a number field F and let $N \geq 2$.

Definition

Let $\rho_{E,N}$ be the *mod N Galois representation attached to E*:

$$\rho_{E,N} \colon \operatorname{Gal}(F(E[N])/F) \hookrightarrow \operatorname{Aut}(E[N]) \cong \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z}).$$

So we have $G_{E,N} = \operatorname{Gal}(F(E[N])/F) = \operatorname{im}(\rho_{E,N}) \subseteq \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z})$.

For an elliptic curve with CM, we know that $G_{E,N} \subseteq \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z})$ is actually contained in something smaller, which is almost abelian.

How do we study this?

Let E be an elliptic curve defined over a number field F and let $N \geq 2$.

Definition

Let $\rho_{E,N}$ be the *mod N Galois representation attached to E*:

$$\rho_{E,N} \colon \operatorname{Gal}(F(E[N])/F) \hookrightarrow \operatorname{Aut}(E[N]) \cong \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z}).$$

So we have $G_{E,N} = \operatorname{Gal}(F(E[N])/F) = \operatorname{im}(\rho_{E,N}) \subseteq \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z})$.

For an elliptic curve with CM, we know that $G_{E,N} \subseteq \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z})$ is actually contained in something smaller, which is almost abelian.

$G_{E,N}$ is contained in the normalizer of Cartan subgroup of $\operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z})$ and that has an index 2 abelian subgroup.

How do we study this?

Definition

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$ and let $N \geq 3$.

How do we study this?

Definition

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$ and let $N \geq 3$.

We define associated constants δ and ϕ as follows:

- If $\Delta_K f^2 \equiv 0 \bmod 4$, or N is odd, let $\delta = \Delta_K f^2/4$, and $\phi = 0$.

How do we study this?

Definition

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$ and let $N \geq 3$.

We define associated constants δ and ϕ as follows:

- If $\Delta_K f^2 \equiv 0 \bmod 4$, or N is odd, let $\delta = \Delta_K f^2/4$, and $\phi = 0$.
- If $\Delta_K f^2 \equiv 1 \bmod 4$, and N is even, let $\delta = \frac{(\Delta_K - 1)}{4} f^2$, let $\phi = f$.

How do we study this?

Definition

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$ and let $N \geq 3$.

We define associated constants δ and ϕ as follows:

- If $\Delta_K f^2 \equiv 0 \bmod 4$, or N is odd, let $\delta = \Delta_K f^2/4$, and $\phi = 0$.
- If $\Delta_K f^2 \equiv 1 \bmod 4$, and N is even, let $\delta = \frac{(\Delta_K - 1)}{4} f^2$, let $\phi = f$.

We define the *Cartan subgroup* $\mathcal{C}_{\delta,\phi}(N)$ of $\mathrm{GL}(2, \mathbb{Z}/N\mathbb{Z})$ by

$$\mathcal{C}_{\delta,\phi}(N) = \left\{ \begin{pmatrix} a + b\phi & b \\ \delta b & a \end{pmatrix} : a, b \in \mathbb{Z}/N\mathbb{Z},\ \det \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

How do we study this?

Definition

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$ and let $N \geq 3$.

We define associated constants δ and ϕ as follows:

- If $\Delta_K f^2 \equiv 0 \bmod 4$, or N is odd, let $\delta = \Delta_K f^2/4$, and $\phi = 0$.
- If $\Delta_K f^2 \equiv 1 \bmod 4$, and N is even, let $\delta = \frac{(\Delta_K - 1)}{4} f^2$, let $\phi = f$.

We define the *Cartan subgroup* $\mathcal{C}_{\delta,\phi}(N)$ of $\mathrm{GL}(2, \mathbb{Z}/N\mathbb{Z})$ by

$$\mathcal{C}_{\delta,\phi}(N) = \left\{ \begin{pmatrix} a + b\phi & b \\ \delta b & a \end{pmatrix} : a, b \in \mathbb{Z}/N\mathbb{Z},\ \det \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

The *normalizer of Cartan subroup* $\mathcal{N}_{\delta,\phi}(N)$ of $\mathrm{GL}(2, \mathbb{Z}/N\mathbb{Z})$ is

$$\mathcal{N}_{\delta,\phi}(N) = \left\langle \mathcal{C}_{\delta,\phi}(N), \begin{pmatrix} -1 & 0 \\ \phi & 1 \end{pmatrix} \right\rangle.$$

How do we study this?

Theorem (Lozano-Robledo, 2021)

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$, let $N \geq 3$, and let $\rho_{E,N}$ be the Galois representation

$$\rho_{E,N} \colon \mathrm{Gal}(\overline{\mathbb{Q}(j_{K,f})}/\mathbb{Q}(j_{K,f})) \to \mathrm{Aut}(E[N]) \cong \mathrm{GL}(2, \mathbb{Z}/N\mathbb{Z}).$$

Then

How do we study this?

Theorem (Lozano-Robledo, 2021)

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$, let $N \geq 3$, and let $\rho_{E,N}$ be the Galois representation

$$\rho_{E,N} \colon \operatorname{Gal}(\overline{\mathbb{Q}(j_{K,f})}/\mathbb{Q}(j_{K,f})) \to \operatorname{Aut}(E[N]) \cong \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z}).$$

Then

1. *There is a $\mathbb{Z}/N\mathbb{Z}$-basis of $E[N]$ such that $\operatorname{im}(\rho_{E,N}) \subseteq \mathcal{N}_{\delta,\phi}(N)$,*

How do we study this?

Theorem (Lozano-Robledo, 2021)

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$, let $N \geq 3$, and let $\rho_{E,N}$ be the Galois representation

$$\rho_{E,N}\colon \operatorname{Gal}(\overline{\mathbb{Q}(j_{K,f})}/\mathbb{Q}(j_{K,f})) \to \operatorname{Aut}(E[N]) \cong \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z}).$$

Then

1. *There is a $\mathbb{Z}/N\mathbb{Z}$-basis of $E[N]$ such that $\operatorname{im}(\rho_{E,N}) \subseteq \mathcal{N}_{\delta,\phi}(N)$,*
2. *$\mathcal{C}_{\delta,\phi}(N) \cong (\mathcal{O}_{K,f}/N\mathcal{O}_{K,f})^\times$ is a subgroup of index 2 in $\mathcal{N}_{\delta,\phi}(N)$,*

How do we study this?

Theorem (Lozano-Robledo, 2021)

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$, let $N \geq 3$, and let $\rho_{E,N}$ be the Galois representation

$$\rho_{E,N} \colon \operatorname{Gal}(\overline{\mathbb{Q}(j_{K,f})}/\mathbb{Q}(j_{K,f})) \to \operatorname{Aut}(E[N]) \cong \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z}).$$

Then

1. *There is a $\mathbb{Z}/N\mathbb{Z}$-basis of $E[N]$ such that $\operatorname{im}(\rho_{E,N}) \subseteq \mathcal{N}_{\delta,\phi}(N)$,*
2. *$\mathcal{C}_{\delta,\phi}(N) \cong (\mathcal{O}_{K,f}/N\mathcal{O}_{K,f})^{\times}$ is a subgroup of index 2 in $\mathcal{N}_{\delta,\phi}(N)$,*
3. *The index of $\operatorname{im}(\rho_{E,N}) \subseteq \mathcal{N}_{\delta,\phi}(N)$ is a divisor of $2, 4,$ or 6.*

Sketch of proof

Theorem (H. and Lozano-Robledo, 2023)

Let E/F be an elliptic curve with CM and $F = \mathbb{Q}(j(E))$, then $F(E[N])/F$ is only abelian for $N = 2$, 3, or 4.

Sketch of proof:

Sketch of proof

Theorem (H. and Lozano-Robledo, 2023)

Let E/F be an elliptic curve with CM and $F = \mathbb{Q}(j(E))$, then $F(E[N])/F$ is only abelian for $N = 2$, 3, or 4.

Sketch of proof:

(1) For an elliptic curve $E/\mathbb{Q}(j_{K,f})$ with CM by an arbitrary order $\mathcal{O}_{K,f}$, Lozano-Robledo explicitly describes the subgroups of $\mathrm{GL}(2, \mathbb{Z}_p)$ that can occur as images of ρ_{E,p^∞}, up to conjugation.

Sketch of proof

Theorem (H. and Lozano-Robledo, 2023)

Let E/F be an elliptic curve with CM and $F = \mathbb{Q}(j(E))$, then $F(E[N])/F$ is only abelian for $N = 2$, 3, or 4.

Sketch of proof:

(1) For an elliptic curve $E/\mathbb{Q}(j_{K,f})$ with CM by an arbitrary order $\mathcal{O}_{K,f}$, Lozano-Robledo explicitly describes the subgroups of $\mathrm{GL}(2, \mathbb{Z}_p)$ that can occur as images of ρ_{E,p^∞}, up to conjugation.

(2) We understand what subgroups of $\mathcal{N}_{\delta,\phi}(N)$ are images of $\rho_{E,N}$ and we give conditions that will help characterize when a subgroup of $\mathcal{N}_{\delta,\phi}(N)$ is abelian (e.g. the Cartan subgroup is abelian).

Sketch of proof

Theorem (H. and Lozano-Robledo, 2023)

Let E/F be an elliptic curve with CM and $F = \mathbb{Q}(j(E))$, then $F(E[N])/F$ is only abelian for $N = 2$, 3, or 4.

Sketch of proof:

(1) For an elliptic curve $E/\mathbb{Q}(j_{K,f})$ with CM by an arbitrary order $\mathcal{O}_{K,f}$, Lozano-Robledo explicitly describes the subgroups of $\mathrm{GL}(2, \mathbb{Z}_p)$ that can occur as images of ρ_{E,p^∞}, up to conjugation.

(2) We understand what subgroups of $\mathcal{N}_{\delta,\phi}(N)$ are images of $\rho_{E,N}$ and we give conditions that will help characterize when a subgroup of $\mathcal{N}_{\delta,\phi}(N)$ is abelian (e.g. the Cartan subgroup is abelian).

(3) We apply the above results to all possible images $G_{E,N} = \operatorname{im} \rho_{E,N}$ from (1) and analyze under what circumstances $G_{E,N}$ is abelian.

□

Conditions for determining if $G_{E,N}$ is abelian

Let $\varepsilon \in \{\pm 1\}$ and let

$$c_1 = \begin{pmatrix} -1 & 0 \\ \phi & 1 \end{pmatrix}, \; c_\varepsilon = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \text{ and } \; c_{\delta,\phi}(a,b) = \begin{pmatrix} a+b\phi & b \\ \delta b & a \end{pmatrix}.$$

Lemma (H. and Lozano-Robledo, 2023)

Let $N \geq 2$ and let $G \subseteq \mathcal{N}_{\delta,\phi}(N)$ be a subgroup. If $c_1, c_{\delta,\phi}(a,b) \in G$, for some $a, b \in \mathbb{Z}/N\mathbb{Z}$, such that the two matrices commute, then

$$b\phi \equiv 0 \bmod N \quad \textit{and} \quad 2b \equiv 0 \bmod N. \tag{1}$$

Moreover, if $\phi = 0$, and if $c_\varepsilon, c_{\delta,0}(a,b) \in G$ for some $\varepsilon \in \{\pm 1\}$, such that the two matrices commute, then the same conditions as (1) hold.

Conditions for determining in $G_{E,N}$ is abelian

Let $\varepsilon \in \{\pm 1\}$ and let

$$c_1 = \begin{pmatrix} -1 & 0 \\ \phi & 1 \end{pmatrix}, \; c_\varepsilon = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \text{ and } \; c_{\delta,\phi}(a,b) = \begin{pmatrix} a+b\phi & b \\ \delta b & a \end{pmatrix}.$$

Corollary (H. and Lozano-Robledo, 2023)

Let $N \geq 3$ and let $G \subseteq \mathcal{N}_{\delta,\phi}(N)$ be a subgroup. If $c_1 \in G$ (or $\phi = 0$ and $c_\varepsilon \in G$) and $c_{\delta,\phi}(a,b) \in G$ with $b \in (\mathbb{Z}/N\mathbb{Z})^\times$, then G is non-abelian.

Conditions for determining in $G_{E,N}$ is abelian

Let $\varepsilon \in \{\pm 1\}$ and let

$$c_1 = \begin{pmatrix} -1 & 0 \\ \phi & 1 \end{pmatrix}, \; c_\varepsilon = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \text{ and } \; c_{\delta,\phi}(a,b) = \begin{pmatrix} a+b\phi & b \\ \delta b & a \end{pmatrix}.$$

Corollary (H. and Lozano-Robledo, 2023)

Let $N \geq 3$ and let $G \subseteq \mathcal{N}_{\delta,\phi}(N)$ be a subgroup. If $c_1 \in G$ (or $\phi = 0$ and $c_\varepsilon \in G$) and $c_{\delta,\phi}(a,b) \in G$ with $b \in (\mathbb{Z}/N\mathbb{Z})^\times$, then G is non-abelian.

Proof: Assume that $G \subseteq \mathcal{N}_{\delta,\phi}(N)$ is abelian. Then c_1 (or c_ε if $\phi = 0$) and $c_{\delta,\phi}(a,b)$ commute, so by the previous lemma, we have

$$b\phi \equiv 0 \bmod N \quad \text{and} \quad 2b \equiv 0 \bmod N.$$

If $N \geq 3$ and $b \in (\mathbb{Z}/N\mathbb{Z})^\times$, then $2b \equiv 0 \bmod N \implies 2 \equiv 0 \bmod N$. Therefore, G cannot be abelian.

$\square$

Example of proving that $G_{E,N}$ is not abelian

Corollary (H. and Lozano-Robledo, 2023)

Let $N \geq 3$ and let $G \subseteq \mathcal{N}_{\delta,\phi}(N)$ be a subgroup. If $c_1 \in G$ (or $\phi = 0$ and $c_\varepsilon \in G$) and $c_{\delta,\phi}(a,b) \in G$ with $b \in (\mathbb{Z}/N\mathbb{Z})^\times$, then G is non-abelian.

Example

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$. Let p be an odd prime dividing $\Delta_K f^2$, and $j_{K,f} \neq 0, 1728$. For $\varepsilon \in \{\pm 1\}$, consider the image

$$G_{E,p} = \left\langle \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \left\{ \begin{pmatrix} a & b \\ \delta b & a \end{pmatrix} : a \in (\mathbb{Z}/p\mathbb{Z})^{\times 2},\ b \in \mathbb{Z}/p\mathbb{Z} \right\} \right\rangle .$$

Observe that $c_{\delta,0}(1,1) = \begin{pmatrix} 1 & 1 \\ \delta & 1 \end{pmatrix} \in G_{E,p}$ and $b = 1 \in (\mathbb{Z}/p\mathbb{Z})^\times$.

Therefore, $G_{E,p}$ is not abelian, and hence, G_{E,p^n} is not abelian.

What if $F(E[N])/F$ is not abelian?

Let E/F be an elliptic curve with CM by $\mathcal{O}_{K,f}$ and $F = \mathbb{Q}(j_{K,f})$.

We have seen that $F(E[N])/F$ need not be abelian.

What if $F(E[N])/F$ is not abelian?

Let E/F be an elliptic curve with CM by $\mathcal{O}_{K,f}$ and $F = \mathbb{Q}(j_{K,f})$.

We have seen that $F(E[N])/F$ need not be abelian.

Example

$E/\mathbb{Q} : y^2 = x^3 - 2x$ (256.b1) has $j(E) = 1728$. Observe that

What if $F(E[N])/F$ is not abelian?

Let E/F be an elliptic curve with CM by $\mathcal{O}_{K,f}$ and $F = \mathbb{Q}(j_{K,f})$.
We have seen that $F(E[N])/F$ need not be abelian.

Example

$E/\mathbb{Q} : y^2 = x^3 - 2x$ (256.b1) has $j(E) = 1728$. Observe that

- $\mathbb{Q}(E[5])/\mathbb{Q}$ is not abelian, $G_{E,5} \cong (\mathbb{Z}/4\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}$:

$$G_{E,5} = \left\langle \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq \mathrm{GL}(2, \mathbb{Z}/5\mathbb{Z}).$$

What if $F(E[N])/F$ is not abelian?

Let E/F be an elliptic curve with CM by $\mathcal{O}_{K,f}$ and $F = \mathbb{Q}(j_{K,f})$.
We have seen that $F(E[N])/F$ need not be abelian.

Example

$E/\mathbb{Q} : y^2 = x^3 - 2x$ (256.b1) has $j(E) = 1728$. Observe that

- $\mathbb{Q}(E[5])/\mathbb{Q}$ is not abelian, $G_{E,5} \cong (\mathbb{Z}/4\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}$:

$$G_{E,5} = \left\langle \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq \mathrm{GL}(2, \mathbb{Z}/5\mathbb{Z}).$$

- $\mathbb{Q}(E[4])/\mathbb{Q}$ is not abelian, $G_{E,4} \cong D_4$:

$$G_{E,4} = \left\langle \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq \mathrm{GL}(2, \mathbb{Z}/4\mathbb{Z}).$$

What if $F(E[N])/F$ is not abelian?

Let E/F be an elliptic curve with CM by $\mathcal{O}_{K,f}$ and $F = \mathbb{Q}(j_{K,f})$.
We have seen that $F(E[N])/F$ need not be abelian.

Example

$E/\mathbb{Q} : y^2 = x^3 - 2x$ (256.b1) has $j(E) = 1728$. Observe that

- $\mathbb{Q}(E[5])/\mathbb{Q}$ is not abelian, $G_{E,5} \cong (\mathbb{Z}/4\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}$:

$$G_{E,5} = \left\langle \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq \mathrm{GL}(2, \mathbb{Z}/5\mathbb{Z}).$$

- $\mathbb{Q}(E[4])/\mathbb{Q}$ is not abelian, $G_{E,4} \cong D_4$:

$$G_{E,4} = \left\langle \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq \mathrm{GL}(2, \mathbb{Z}/4\mathbb{Z}).$$

Question

What is the maximal abelian extension contained in $F(E[N])/F$?

Abelian extensions contained in $F(E[N])/F$

Let E/F be an elliptic curve with CM by $\mathcal{O}_{K,f}$, where $F = \mathbb{Q}(j_{K,f})$.

Abelian extensions contained in $F(E[N])/F$

Let E/F be an elliptic curve with CM by $\mathcal{O}_{K,f}$, where $F = \mathbb{Q}(j_{K,f})$.

(1) Let $N \geq 2$. By the existence of the Weil-pairing, $F(\zeta_N) \subseteq F(E[N])$.

Abelian extensions contained in $F(E[N])/F$

Let E/F be an elliptic curve with CM by $\mathcal{O}_{K,f}$, where $F = \mathbb{Q}(j_{K,f})$.

(1) Let $N \geq 2$. By the existence of the Weil-pairing, $F(\zeta_N) \subseteq F(E[N])$.

(2) Let $N \geq 3$. Then, $K \subseteq F(E[N])$.

Abelian extensions contained in $F(E[N])/F$

Let E/F be an elliptic curve with CM by $\mathcal{O}_{K,f}$, where $F = \mathbb{Q}(j_{K,f})$.

(1) Let $N \geq 2$. By the existence of the Weil-pairing, $F(\zeta_N) \subseteq F(E[N])$.

(2) Let $N \geq 3$. Then, $K \subseteq F(E[N])$.

(3) Let $N \geq 3$, $d \in F$ such that $\sqrt{d} \notin K$, and E^d be the twist of E by d. Then there is an explicitly computable integer $\alpha = \alpha(E^d)$ such that $F(\sqrt{\alpha}) \subseteq F(E[N])$, with α unique up to squares when $j_{K,f} \neq 0, 1728$, 4^{th}-powers when $j_{K,f} = 1728$, and 6^{th}-powers when $j_{K,f} = 0$.

Abelian extensions contained in $F(E[N])/F$

Let E/F be an elliptic curve with CM by $\mathcal{O}_{K,f}$, where $F = \mathbb{Q}(j_{K,f})$.

(1) Let $N \geq 2$. By the existence of the Weil-pairing, $F(\zeta_N) \subseteq F(E[N])$.

(2) Let $N \geq 3$. Then, $K \subseteq F(E[N])$.

(3) Let $N \geq 3$, $d \in F$ such that $\sqrt{d} \notin K$, and E^d be the twist of E by d. Then there is an explicitly computable integer $\alpha = \alpha(E^d)$ such that $F(\sqrt{\alpha}) \subseteq F(E[N])$, with α unique up to squares when $j_{K,f} \neq 0, 1728$, 4^{th}-powers when $j_{K,f} = 1728$, and 6^{th}-powers when $j_{K,f} = 0$.

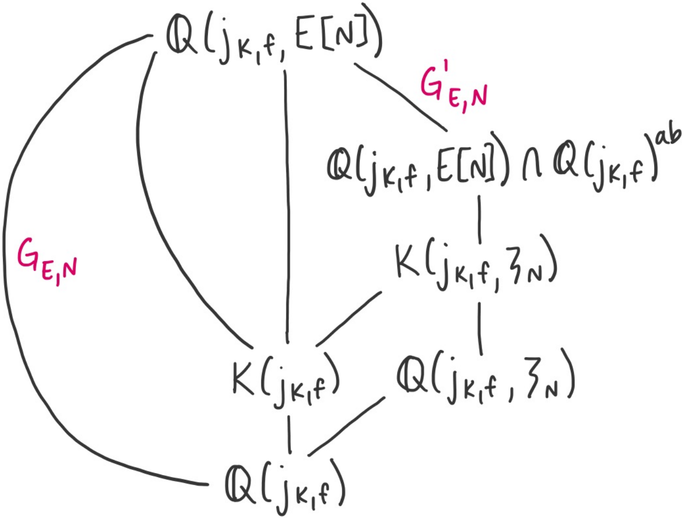
Therefore, we have that $K(j_{K,f}, \zeta_N, \sqrt{\alpha})$ is an abelian extension contained in $F(E[N])/F$, which is sometimes just $K(j_{K,f}, \zeta_N)$ if $\sqrt{\alpha} \in K(j_{K,f}, \zeta_N)$.

Field diagram

Let $N \geq 3$. Let $G_{E,N} = \mathrm{Gal}(\mathbb{Q}(j_{K,f}, E[N])/\mathbb{Q}(j_{K,f}))$.
Let $G'_{E,N}$ denote the commutator subgroup of $G_{E,N}$.

Field diagram

Let $N \geq 3$. Let $G_{E,N} = \mathrm{Gal}(\mathbb{Q}(j_{K,f}, E[N])/\mathbb{Q}(j_{K,f}))$.
Let $G'_{E,N}$ denote the commutator subgroup of $G_{E,N}$.



How to find the max abelian subextension of $F(E[N])/F$

Sketch of proof:

How to find the max abelian subextension of $F(E[N])/F$

Sketch of proof:

(1) We have explicit matrix group descriptions for G_{E,p^n}, so we can compute their commutator subgroups G'_{E,p^n} explicitly.

How to find the max abelian subextension of $F(E[N])/F$

Sketch of proof:

(1) We have explicit matrix group descriptions for G_{E,p^n}, so we can compute their commutator subgroups G'_{E,p^n} explicitly.

(2) We know $K(j_{K,f}, \zeta_{p^n})$ or $K(j_{K,f}, \zeta_{p^n}, \sqrt{\alpha})$ is an abelian extension of K and can use that to find an upper bound for the size of G'_{E,p^n}.

How to find the max abelian subextension of $F(E[N])/F$

Sketch of proof:

(1) We have explicit matrix group descriptions for G_{E,p^n}, so we can compute their commutator subgroups G'_{E,p^n} explicitly.

(2) We know $K(j_{K,f}, \zeta_{p^n})$ or $K(j_{K,f}, \zeta_{p^n}, \sqrt{\alpha})$ is an abelian extension of K and can use that to find an upper bound for the size of G'_{E,p^n}.

(3) We can use the surjective reduction map $\pi : G'_{E,p^{n+1}} \to G'_{E,p^n}$ to get a lower bound for the size of G'_{E,p^n}.

How to find the max abelian subextension of $F(E[N])/F$

Sketch of proof:

(1) We have explicit matrix group descriptions for G_{E,p^n}, so we can compute their commutator subgroups G'_{E,p^n} explicitly.

(2) We know $K(j_{K,f},\zeta_{p^n})$ or $K(j_{K,f},\zeta_{p^n},\sqrt{\alpha})$ is an abelian extension of K and can use that to find an upper bound for the size of G'_{E,p^n}.

(3) We can use the surjective reduction map $\pi : G'_{E,p^{n+1}} \to G'_{E,p^n}$ to get a lower bound for the size of G'_{E,p^n}.

(4) It turns out that the upper and lower bounds agree, so it must be that

$$K(j_{K,f},\zeta_{p^n}) \quad \text{or} \quad K(j_{K,f},\zeta_{p^n},\sqrt{\alpha})$$

is the maximal abelian subextension of $\mathbb{Q}(j_{K,f},E[p^n])/\mathbb{Q}(j_{K,f})$. $\square$

Maximal abelian subextensions

Theorem (H., 2024)

Let $E/\mathbb{Q}$ be an elliptic curve with CM by an order $\mathcal{O}_{K,f}$ of K, $f \geq 1$.

Maximal abelian subextensions

Theorem (H., 2024)

Let $E/\mathbb{Q}$ be an elliptic curve with CM by an order $\mathcal{O}_{K,f}$ of K, $f \geq 1$.

(1) *If p is any prime such that $p \nmid \Delta_K f^2$, then*

$$\mathbb{Q}(E[p^n]) \cap \mathbb{Q}^{ab} = K(\zeta_{p^n}).$$

Maximal abelian subextensions

Theorem (H., 2024)

Let $E/\mathbb{Q}$ be an elliptic curve with CM by an order $\mathcal{O}_{K,f}$ of K, $f \geq 1$.

(1) *If p is any prime such that $p \nmid \Delta_K f^2$, then*

$$\mathbb{Q}(E[p^n]) \cap \mathbb{Q}^{ab} = K(\zeta_{p^n}).$$

(2) *If $p > 2$ is prime such that $p \mid \Delta_K f^2$, then*

$$\mathbb{Q}(E[p^n]) \cap \mathbb{Q}^{ab} = \begin{cases} \mathbb{Q}(\zeta_{p^n}) & \text{if } E \text{ is a "simplest model"}, \\ \mathbb{Q}(\zeta_{p^n}, \sqrt{\alpha}) & \text{if } E \text{ is a twist of previous case.} \end{cases}$$

Maximal abelian subextensions

Theorem (H., 2024)

Let $E/\mathbb{Q}$ be an elliptic curve with CM by an order $\mathcal{O}_{K,f}$ of K, $f \geq 1$.

(1) *If p is any prime such that $p \nmid \Delta_K f^2$, then*

$$\mathbb{Q}(E[p^n]) \cap \mathbb{Q}^{ab} = K(\zeta_{p^n}).$$

(2) *If $p > 2$ is prime such that $p \mid \Delta_K f^2$, then*

$$\mathbb{Q}(E[p^n]) \cap \mathbb{Q}^{ab} = \begin{cases} \mathbb{Q}(\zeta_{p^n}) & \text{if } E \text{ is a "simplest model"}, \\ \mathbb{Q}(\zeta_{p^n}, \sqrt{\alpha}) & \text{if } E \text{ is a twist of previous case.} \end{cases}$$

(3) *Let $p = 2$ and $2 \mid \Delta_K f^2$.*

- *If $\Delta_K f^2 = -12$ or -28, then* $\mathbb{Q}(E[2^n]) \cap \mathbb{Q}^{ab} = K(\zeta_{2^{n+1}})$.
- *If $\Delta_K f^2 = -4, -8,$ or -16, then*

$$\mathbb{Q}(E[2^n]) \cap \mathbb{Q}^{ab} = \begin{cases} \mathbb{Q}(\zeta_{2^{n+1}}) & \text{if } E \text{ is a "simplest model"}, \\ \mathbb{Q}(\zeta_{2^{n+1}}, \sqrt{\alpha}) & \text{if } E \text{ is a twist of previous case.} \end{cases}$$

"Simplest model"

Definition

A *simplest model* at a prime p is an elliptic curve $E/\mathbb{Q}(j_{K,f})$ such that $[\mathcal{N}_{\delta,\phi}(p^n) : G_{E,p^n}] = 2, 4$ or 6 (depending on $j_{K,f}$).

j-invariant	Δ_K	f	Elliptic curve $E_{\Delta_K,f}$
0	-3	1	$y^2 = x^3 + 16$
$2^4 3^3 5^3$		2	$y^2 = x^3 - 15x + 22$
$-2^{15} 3 \cdot 5^3$		3	$y^2 = x^3 - 480x + 4048$
$2^6 3^3$	-4	1	$y^2 = x^3 + x$
$2^3 3^3 11^3$		2	$y^2 = x^3 - 11x + 14$
$-3^3 5^3$	-7	1	$y^2 = x^3 - 1715x + 33614$
$3^3 5^3 17^3$		2	$y^2 = x^3 - 29155x + 1915998$
$2^6 5^3$	-8	1	$y^2 = x^3 - 4320x + 96768$
-2^{15}	-11	1	$y^2 = x^3 - 9504x + 365904$
$-2^{15} 3^3$	-19	1	$y^2 = x^3 - 608x + 5776$
$-2^{18} 3^3 5^3$	-43	1	$y^2 = x^3 - 13760x + 621264$
$-2^{15} 3^3 5^3 11^3$	-67	1	$y^2 = x^3 - 117920x + 15585808$
$-2^{18} 3^3 5^3 23^3 29^3$	-163	1	$y^2 = x^3 - 34790720x + 78984748304$

TABLE 1. CM elliptic curves over $\mathbb{Q}$

Example

Example ($p = 7$ and $\Delta_K f^2 = -7$)

Let $E/\mathbb{Q} : y^2 = x^3 - 140x - 784$ (3136.n4), where $j(E) = -3375$.

Example

Example ($p = 7$ and $\Delta_K f^2 = -7$)

Let $E/\mathbb{Q} : y^2 = x^3 - 140x - 784$ (3136.n4), where $j(E) = -3375$.

Here $K = \mathbb{Q}(\sqrt{-7})$.

Example

Example ($p = 7$ and $\Delta_K f^2 = -7$)

Let $E/\mathbb{Q} : y^2 = x^3 - 140x - 784$ (3136.n4), where $j(E) = -3375$.

Here $K = \mathbb{Q}(\sqrt{-7})$.

In this case, $G_{E,7^n} = \mathcal{N}_{\delta,0}(7^n)$, where $\delta = -7/4$.

Example

Example ($p = 7$ and $\Delta_K f^2 = -7$)

Let $E/\mathbb{Q} : y^2 = x^3 - 140x - 784$ (3136.n4), where $j(E) = -3375$.

Here $K = \mathbb{Q}(\sqrt{-7})$.

In this case, $G_{E,7^n} = \mathcal{N}_{\delta,0}(7^n)$, where $\delta = -7/4$.

E is a quadratic twist of $E'/\mathbb{Q} : y^2 = x^3 - 1715x + 33614$ (49.a2) by -14.

Example

Example ($p = 7$ and $\Delta_K f^2 = -7$)

Let $E/\mathbb{Q} : y^2 = x^3 - 140x - 784$ (3136.n4), where $j(E) = -3375$.

Here $K = \mathbb{Q}(\sqrt{-7})$.

In this case, $G_{E,7^n} = \mathcal{N}_{\delta,0}(7^n)$, where $\delta = -7/4$.

E is a quadratic twist of $E'/\mathbb{Q} : y^2 = x^3 - 1715x + 33614$ (49.a2) by -14.

Thus, $\mathbb{Q}(E[7^n]) \cap \mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\zeta_{7^n}, \sqrt{-14})$.

Example

Example ($p = 7$ and $\Delta_K f^2 = -7$)

Let $E/\mathbb{Q} : y^2 = x^3 - 140x - 784$ (3136.n4), where $j(E) = -3375$.

Here $K = \mathbb{Q}(\sqrt{-7})$.

In this case, $G_{E,7^n} = \mathcal{N}_{\delta,0}(7^n)$, where $\delta = -7/4$.

E is a quadratic twist of $E'/\mathbb{Q} : y^2 = x^3 - 1715x + 33614$ (49.a2) by -14.

Thus, $\mathbb{Q}(E[7^n]) \cap \mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\zeta_{7^n}, \sqrt{-14})$.

The simplest CM curve E' has image

$$G_{E',7^n} = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \left\{ \begin{pmatrix} a & b \\ \delta b & a \end{pmatrix} : a \in (\mathbb{Z}/7\mathbb{Z})^{\times 2}, b \in \mathbb{Z}/7\mathbb{Z} \right\} \right\rangle,$$

which is an index 2 subgroup of $\mathcal{N}_{\delta,0}(7^n)$. Thus, $\mathbb{Q}(E[7^n]) \cap \mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\zeta_{7^n})$.

Questions?

Abelian extensions of $\mathbb{Q}(i)$

Theorem

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = x^3 + x$. For each integer $N \geq 1$, let $K_N = \mathbb{Q}(i)(E[N])$. Then $\operatorname{Gal}(K_N/\mathbb{Q}(i))$ is abelian.

__**Proof:**__ Let $P = (x, y) \in E[N]$ and let A be the action by i on $E[N]$, so

$$A \cdot (x, y) = \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix} \cdot (x, y) = (-x, iy).$$

Let $\sigma \in \operatorname{Gal}(K_N/\mathbb{Q}(i))$ such that $M_\sigma \cdot (x, y) = (\sigma(x), \sigma(y))$.
One can show that $\sigma(A \cdot P) = A \cdot (\sigma(P))$, i.e., that M_σ commutes with A.
Note that the set of all matrices that commute with A are

$$\left\{ \begin{pmatrix} -a & 0 \\ 0 & id \end{pmatrix} : a, d \in \mathbb{Z}/N\mathbb{Z},\ ad \neq 0 \right\},$$

which is an abelian group, and $\{M_\sigma : \sigma \in \operatorname{Gal}(K_N/\mathbb{Q}(i))\}$ is contained in there. Thus, $\operatorname{Gal}(K_N/\mathbb{Q}(i))$ is abelian. $\square$