Elliptic curves with complex multiplication and abelian division fields

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For this talk, we will focus on elliptic curves E with complex multiplication and fix F to be the minimal field of definition, i.e. $F = \mathbb{Q}(j(E))$.

Why do we care?

The classification of abelian division fields of elliptic curves over $\mathbb{Q}$ has numerous applications, for example in:

- the classification of torsion subgroups of elliptic curves (Chou, Lozano-Robledo, González-Jiménez),
- classifying isogeny-torsion graphs (Chiloyan, Lozano-Robledo)
- Brauer groups (Várilly-Alvarado, Viray),
- non-monogenic number fields (Smith),
- congruences between elliptic curves (Cremona, Freitas),
- classification of Galois representations (Lozano-Robledo, Rouse, Sutherland, Zureick-Brown).

What is an elliptic curve?

Definition

An *elliptic curve* E defined over a field K (char. $\neq 2, 3$) is an equation of the form

$$y^2 = x^3 + Ax + B, \quad A, B \in K,$$

where $4A^3 + 27B^2 \neq 0$ (for smoothness). More precisely, an elliptic curve defined over a field K is a smooth projective curve of genus 1, with at least one K-rational point.

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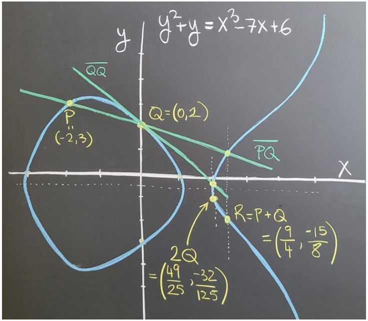
There is a group law (abelian) on the L-rational points of E

$$E(L) = \{(x, y) \in E : x, y \in L\} \cup \mathcal{O},$$

with coordinates in any field $L \supset K$. We call $E(L)$ the *Mordell-Weil group* of E/L.



Mordell-Weil Theorem

Example

Let $E/\mathbb{Q} : y^2 = x^3 + 13x - 34$ (40.a4) be an elliptic curve. Then

$$E(\mathbb{Q}) = \{\mathcal{O}, (7, -20), (2, 0), (7, 20)\} = \langle (7, 20) \rangle \cong \mathbb{Z}/4\mathbb{Z}.$$

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Now consider the same curve E defined over $\mathbb{Q}(i)$. Then

$$E(\mathbb{Q}(i)) = \langle (1 + 2i, -2 - 6i), (-3, -10i) \rangle \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

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Theorem (Mordell-Weil, 1928)

Let F be a number field and let E/F be an elliptic curve. Then $E(F)$ is a finitely generated abelian group. In particular,

$$E(F) \cong E(F)_{tors} \oplus \mathbb{Z}^{R_{E/F}},$$

where $E(F)_{tors}$ is a finite subgroup and $R_{E/F} \geq 0$.

Mordell-Weil groups

Example

(1) $E_1/\mathbb{Q} : y^2 = x^3 + 1$ (36.a4) only has six rational torsion points,

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(2) $E_2/\mathbb{Q} : y^2 = x^3 - 2$ (1728.o3) does not have any rational torsion points (other than $\mathcal{O}$). However, there is a point of infinite order,

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(3) $E_3/\mathbb{Q} : y^2 = x^3 - 1156x$ (18496.j3) has both torsion and infinite order points,

$$E_3(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^2,$$

where the torsion subgroup is generated by $\langle (0, 0), (34, 0) \rangle$, and the free part is generated by $\langle (-2, 48), (-16, 120) \rangle$.

Torsion subgroups

Definition

Let F be a number field and let E/F be an elliptic curve. Let $N \in \mathbb{Z}^+$ and

$$E[N] = \{P \in E(\overline{F}) : [N]P = \mathcal{O}\},$$

be the *N-torsion subgroup* of $E(\overline{F})$.

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We will be interested in the N^{th}-division field of E over F,

$$F(E[N]) = F(\{x(P), y(P) : P \in E[N]\}).$$

Galois groups

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Question

Can $\mathrm{Gal}(\mathbb{Q}(E[N])/\mathbb{Q})$ *be abelian?*

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- Halberstadt, Merel (2001), Merel and Stein (2001), and Rebolledo (2003), show that if p is prime, and $F(E[p]) = \mathbb{Q}(\zeta_p)$, then $p = 2, 3, 5$ or $p > 1000$.

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- When $F = \mathbb{Q}$, Paladino (2010) gives a classification as a two parameter family of all elliptic curves $E/\mathbb{Q}$ with $\mathbb{Q}(E[3]) = \mathbb{Q}(\zeta_3)$.

What is known?

Theorem (Lozano-Robledo, González-Jiménez, 2015)

Let $E/\mathbb{Q}$ be an elliptic curve. Let $N \geq 2$.

- $\mathbb{Q}(E[N]) = \mathbb{Q}(\zeta_N)$ *only for* $N = 2, 3, 4,$ *or* 5.
- *More generally, if* $\mathbb{Q}(E[N])/\mathbb{Q}$ *is abelian, then* $N = 2, 3, 4, 5, 6,$ *or* 8.

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Theorem (Lozano-Robledo, González-Jiménez, 2015)

Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication. Let $N \geq 2$.

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Let E be an elliptic curve defined over a number field F and let $N \geq 2$. Can $F(E[N])/F$ be abelian?

Complex multiplication (CM)

Definition

Let E be an elliptic curve defined over a field F. We say that E has *complex multiplication* (CM) if $\mathrm{End}(E) \supsetneq \mathbb{Z}$.

If E/F has CM, then $\mathrm{End}(E) \cong \mathcal{O}_{K,f}$, where $\mathcal{O}_{K,f}$ is the order in an imaginary quadratic field K with index $f \geq 1$ in $\mathcal{O}_K$, also called the conductor.

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Example

The elliptic curve $E/\mathbb{Q} : y^2 = x^3 + x$ (64.a1) has the endomorphism

$$\phi(x, y) = (-x, iy),$$

where for $(x, y) \in E$, we have $(iy)^2 = (-x)^3 + (-x)$, so $(-x, iy) \in E$.

In this case, $\mathrm{End}(E) \cong \mathbb{Z}[i] = \mathcal{O}_{K,1}$, the maximal order of $K = \mathbb{Q}(i)$.

Notation

- K be an imaginary quadratic field,
- Δ_K is the discriminant of the ring of integers $\mathcal{O}_K$,
- $\mathcal{O}_{K,f}$ be the order of conductor $f \geq 1$ in K, with discriminant $\Delta_K f^2$,
- $j_{K,f}$ is a j-invariant associated to the order $\mathcal{O}_{K,f}$, i.e., $j(\mathbb{C}/\mathcal{O}_{K,f})$.

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Example

Let $E/\mathbb{Q}(\sqrt{2})$ be the elliptic curve given by (32.1-a1),

$$y^2 + \sqrt{2}xy = x^3 + x^2 + \left(15\sqrt{2} - 22\right)x + 46\sqrt{2} - 69,$$

with CM by $\mathcal{O}_{K,4} = \mathbb{Z}[4i]$, where $K = \mathbb{Q}(i)$.
Here, $j_{K,4} = -29071392966\sqrt{2} + 41113158120$, so $\mathbb{Q}(j_{K,4}) = \mathbb{Q}(\sqrt{2})$.

When is $\mathbb{Q}(j_{K,f}, E[N])/\mathbb{Q}(j_{K,f})$ abelian?

Theorem 1 (H. and Lozano-Robledo, 2023)

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$. Let $N \geq 2$ and let

$$G_{E,N} = \mathrm{Gal}(\mathbb{Q}(j_{K,f}, E[N])/\mathbb{Q}(j_{K,f}))$$

be the Galois group of the N-division field of E.

If $G_{E,N}$ is abelian, then N must equal $2, 3$, or 4. Furthermore, if $G_{E,N}$ is abelian, then it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$ for some $0 \leq k \leq 3$.

$G_{E,2} = \mathrm{Gal}(\mathbb{Q}(j_{K,f}, E[2])/\mathbb{Q}(j_{K,f}))$

Theorem 1 (H. and Lozano-Robledo, 2023)

If $N = 2$, then $G_{E,2}$ is abelian if and only if one of the following holds:

(a) $j_{K,f} \neq 0, 1728$ *and either*
- $\Delta_K f^2 \equiv 0 \bmod 4$, *or*
- $\Delta_K \equiv 1 \bmod 8$ *and f is odd.*

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Example ($j_{K,f} \neq 0, 1728$)

- $E_1/\mathbb{Q} : y^2 = x^3 + x^2 - 13x - 21$ (256.a1) has $j_{K,1} = 8000$,

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- $E_2/\mathbb{Q} : y^2 + xy = x^3 - x^2 - 107x + 552$ (49.a2) has $j_{K,1} = -3375$,

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- $E_2/\mathbb{Q} : y^2 + xy = x^3 - x^2 - 107x + 552$ (49.a2) has $j_{K,1} = -3375$, where $K = \mathbb{Q}(\sqrt{-7})$, $\Delta_K = -7 \equiv 1 \bmod 8$, and $f = 1$.

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(a) $j_{K,f} \neq 0, 1728$ *and either*
- $\Delta_K f^2 \equiv 0 \bmod 4$, *or*
- $\Delta_K \equiv 1 \bmod 8$ *and f is odd.*

In this case $G_{E,2} \cong \mathbb{Z}/2\mathbb{Z}$.

Example ($j_{K,f} \neq 0, 1728$)

- $E_1/\mathbb{Q} : y^2 = x^3 + x^2 - 13x - 21$ (256.a1) has $j_{K,1} = 8000$, where $K = \mathbb{Q}(\sqrt{-2})$, $\Delta_K = -8$, and $f = 1$, so $\Delta_K f^2 \equiv 0 \bmod 4$. Therefore, $G_{E_1,2} \cong \mathbb{Z}/2\mathbb{Z}$.
- $E_2/\mathbb{Q} : y^2 + xy = x^3 - x^2 - 107x + 552$ (49.a2) has $j_{K,1} = -3375$, where $K = \mathbb{Q}(\sqrt{-7})$, $\Delta_K = -7 \equiv 1 \bmod 8$, and $f = 1$. Therefore, $G_{E_2,2} \cong \mathbb{Z}/2\mathbb{Z}$.

$G_{E,2} = \mathrm{Gal}(\mathbb{Q}(j_{K,f}, E[2])/\mathbb{Q}(j_{K,f}))$

Example ($j_{K,f} \neq 0, 1728$)

- Let $E_3/\mathbb{Q}(\sqrt{2})$ be given by (32.1-a1)

$$y^2 + \sqrt{2}xy = x^3 + x^2 + \left(15\sqrt{2} - 22\right)x + 46\sqrt{2} - 69.$$

Recall that E_3 has CM by $\mathcal{O}_{K,4} = \mathbb{Z}[4i]$ where $K = \mathbb{Q}(i)$ and

$$j_{K,4} = -29071392966\sqrt{2} + 41113158120.$$

We have $\Delta_K f^2 = -4 \cdot 16 = -64 \equiv 0 \bmod 4$, so $G_{E_3,2} \cong \mathbb{Z}/2\mathbb{Z}$.

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One can check that $E_3(\mathbb{Q}(\sqrt{2}))[2] \cong \mathbb{Z}/2\mathbb{Z}$ is generated by a point of order 2 defined over $\mathbb{Q}(\sqrt{2})$, namely

$$P = \left(2\sqrt{2} - \frac{3}{2},\ \frac{3}{4}\sqrt{2} - 2\right).$$

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Theorem 1 (H. and Lozano-Robledo, 2023)

(b) $j_{K,f} = 1728$, *so* $E/\mathbb{Q}$ *is given by* $y^2 = x^3 - dx$ *with* d *in* $\mathbb{Z}$. *Then*

- *If* d *is a square, then* $G_{E,2}$ *is trivial.*
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Example ($j_{K,f} = 1728$ and $j_{K,f} = 0$)

- $E_4/\mathbb{Q} : y^2 = x^3 - x$ (32.a3) has $j_{K,1} = 1728$ and $d = 1$.

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- $E_6/\mathbb{Q} : y^2 = x^3 + 1$ (36.a4) has $j_{K,1} = 0$ and $d = 1$.

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- $E_6/\mathbb{Q} : y^2 = x^3 + 1$ (36.a4) has $j_{K,1} = 0$ and $d = 1$. Therefore, $G_{E_6,2} \cong \mathbb{Z}/2\mathbb{Z}$.

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If $N = 3$, then $G_{E,3}$ is abelian if and only if $j(E) = 0$ and $E/\mathbb{Q}$ is given by $y^2 = x^3 + d$ such that $4d$ is a cube in $\mathbb{Z}$.

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Example

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- $E_1/\mathbb{Q} : y^2 = x^3 - 4x$ (64.a3) has $j(E_1) = 1728$ and $d = -4$. Therefore, $G_{E_1,4} \cong (\mathbb{Z}/2\mathbb{Z})^2$.
- $E_2/\mathbb{Q} : y^2 = x^3 + 9x$ (576.c4) has $j(E_2) = 1728$ and $d = 3^2$.

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How do we study this?

Let E be an elliptic curve defined over a number field F and let $N \geq 2$.

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Definition

Let $\rho_{E,N}$ be the *mod N Galois representation attached to E*:

$$\rho_{E,N}\colon \ \mathrm{Gal}(F(E[N])/F) \to \mathrm{Aut}(E[N]) \cong \mathrm{GL}(2, \mathbb{Z}/N\mathbb{Z}).$$

So we have $G_{E,N} = \mathrm{Gal}(F(E[N])/F) = \mathrm{im}(\rho_{E,N}) \subseteq \mathrm{GL}(2, \mathbb{Z}/N\mathbb{Z})$.

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So we have $G_{E,N} = \operatorname{Gal}(F(E[N])/F) = \operatorname{im}(\rho_{E,N}) \subseteq \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z})$.

For an elliptic curve with CM, we know that $G_{E,N} \subseteq \operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z})$ is actually contained in something smaller, which is almost abelian.

How do we study this?

Let E be an elliptic curve defined over a number field F and let $N \geq 2$.

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$G_{E,N}$ is contained in the normalizer of Cartan subgroup of $\operatorname{GL}(2, \mathbb{Z}/N\mathbb{Z})$.

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We define the *Cartan subgroup* $\mathcal{C}_{\delta,\phi}(N)$ of $\mathrm{GL}(2, \mathbb{Z}/N\mathbb{Z})$ by

$$\mathcal{C}_{\delta,\phi}(N) = \left\{ \begin{pmatrix} a + b\phi & b \\ \delta b & a \end{pmatrix} : a, b \in \mathbb{Z}/N\mathbb{Z},\ \det \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

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Theorem (Lozano-Robledo, 2021)

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$, let $N \geq 3$, and let $\rho_{E,N}$ be the Galois representation

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3. *The index of $\operatorname{im}(\rho_{E,N}) \subseteq \mathcal{N}_{\delta,\phi}(N)$ coincides with the order of*

$$\operatorname{Gal}(K(j_{K,f}, E[N])/K(j_{K,f}, h(E[N])),$$

for a Weber function h, and it is a divisor of the order of $\mathcal{O}_{K,f}^\times/\mathcal{O}_{K,f,N}^\times$, where $\mathcal{O}_{K,f,N}^\times = \{u \in \mathcal{O}_{K,f}^\times : u \equiv 1 \bmod N\mathcal{O}_{K,f}\}$.

Sketch of proof of Theorem 1

Theorem 1 (H. and Lozano-Robledo, 2023)

Let E/F be an elliptic curve with CM and $F = \mathbb{Q}(j(E))$. Then $F(E[N])/F$ is abelian only for $N = 2$, 3, or 4.

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(3) We apply the results from (2) to all possible $G_{E,p} = \operatorname{im} \rho_{E,p}$ from (1) where $p \mid N$ and analyze under what circumstances $G_{E,N}$ is abelian.

$\square$

Why is $G_{E,N}$ abelian only for $N = 2, 3$, or 4?

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- $p \nmid 2\Delta_K f$ and $j_{K,f} \neq 0$, or $p > 3$ and $j_{K,1} = 0$
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- $p = 2$ and $j_{K,f} \neq 0, 1728$, or $j_{K,f} = 1728$, or $j_{K,f} = 0$, respectively
  - $G_{E,2^n} \cong \mathcal{N}_{\delta,\phi}(2^n)$ or something smaller. $G_{E,2}$ may be abelian.
  - $[\mathcal{N}_{-1,0}(2^n) : G_{E,2^n}] = 1, 2$, or 4. $G_{E,2}$ and $G_{E,4}$ may be abelian.
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Question

What is the maximal abelian extension contained in $F(E[N])/F$?

Field diagram

Let $N \geq 3$. Let $G_{E,N} = \mathrm{Gal}(\mathbb{Q}(j_{K,f}, E[N])/\mathbb{Q}(j_{K,f}))$.
Let $\mathcal{CC}_{E,N}$ denote the commutator subgroup of $G_{E,N}$.

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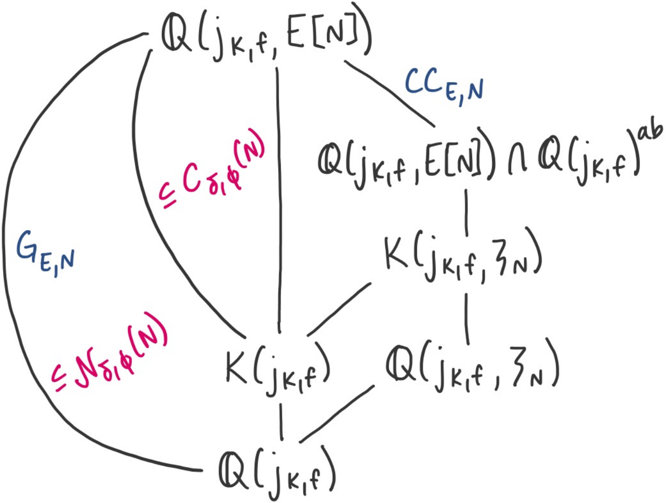

Results for $p = 3$

Theorem 2 (H., 2023)

Let $E/\mathbb{Q}$ be an elliptic curve with $j(E) = 0$. Then for $n \geq 2$,

$[\mathcal{N}_{\delta,0}(3^n) : G_{E,3^n}]$	$\mathbb{Q}(E[3^n]) \cap \mathbb{Q}^{ab}$	$\mathcal{CC}_{E,3^n}$
1	$\mathbb{Q}(\zeta_{3^n}, \sqrt{\alpha})$	$\mathbb{Z}/3^n\mathbb{Z}$ $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3^{n-1}\mathbb{Z}$
2	$\mathbb{Q}(\zeta_{3^n})$	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3^{n-1}\mathbb{Z}$
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where α is a square-free integer, $\alpha \neq -3$. Note, $K = \mathbb{Q}(\sqrt{-3}) \subseteq \mathbb{Q}(\zeta_{3^n})$.

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The Galois groups of the maximal abelian extensions are,

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Results for $p > 3$ prime

Theorem (Daniels, Lozano-Robledo, 2021)

Let $E/\mathbb{Q}$ be an elliptic curve and $p > 2$ a prime. If $\rho_{E,p} \subseteq \mathcal{N}_{\delta,\phi}(p)$, then

$$K_E(p) = \mathbb{Q}(E[p]) \cap \mathbb{Q}^{ab} \subseteq \mathbb{Q}(\zeta_p, \sqrt{d}),$$

for some $d \in \mathbb{Z}$. Thus, $\mathrm{Gal}(K_E(p)/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\times$ or $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})^\times$.

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for some $d \in \mathbb{Z}$. Thus, $\mathrm{Gal}(K_E(p)/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\times$ or $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})^\times$.

Conjecture (H., 2023)

Let $E/\mathbb{Q}(j_{K,f})$ be an elliptic curve with CM by $\mathcal{O}_{K,f}$ for $f \geq 1$. Let $p > 3$ be a prime and let $n \in \mathbb{Z}^+$. Then

$$\mathbb{Q}(j_{K,f}, E[p^n]) \cap \mathbb{Q}(j_{K,f})^{ab} = \begin{cases} K(\zeta_{p^n}), \\ K(\zeta_{p^n}, \sqrt{\alpha}), \end{cases}$$

where $\alpha \in \mathbb{Q}(j_{K,f})$ is square-free such that $\alpha \neq 0, 1$ and $\sqrt{\alpha} \notin K$.

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(4) It turns out that $U = L$, so it must be that $K(\zeta_{p^n})$ or $K(\zeta_{p^n}, \sqrt{\alpha})$ is the maximal abelian subextension of $\mathbb{Q}(j_{K,f}, E[p^n])/\mathbb{Q}(j_{K,f})$.

$\square$

Questions?