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(1) (a) A single qubit is in the form $a|0\rangle + b|0\rangle$

and $a|0\rangle + b|1\rangle = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$.

where $|a|^2 + |b|^2 = 1$.

Example:

(1) $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle) + \frac{1}{\sqrt{2}}(|1\rangle \otimes |1\rangle)$

$= \frac{1}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right\} + \frac{1}{\sqrt{2}}\left\{\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}\right\}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$= \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}$.

(2) $|10\rangle = |1\rangle \otimes |0\rangle$

$= \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

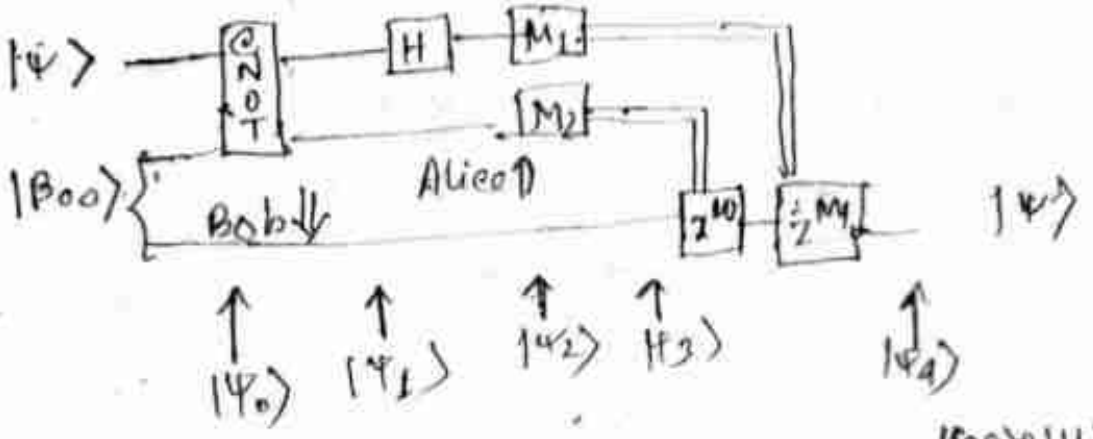
(1) (b) (i) $\begin{pmatrix}1&0&0&0&\cdots&0\\0&1&0&0&\cdots&0\\ \vdots & & & & & \\0&0&0&0&\cdots&1\end{pmatrix}_{2^n \times 2^n}$.

for input $\begin{pmatrix}a_1\\a_2\\\vdots\\a_{2^n}\end{pmatrix}$, the output will be $\begin{pmatrix}a_1\\a_2\\\vdots\\a_{2^n}\end{pmatrix}$

(ii) $\begin{pmatrix}0&0&\cdots&1\\0&0&\cdots&1&0\\1&0&0&0\end{pmatrix}$. for input $\begin{pmatrix}a_1\\a_2\\\vdots\\a_{2^n}\end{pmatrix}$ and output

$= \begin{pmatrix}a_{2^n}\\\vdots\\a_1\end{pmatrix}$

② Teleportation:



Here, $|\psi_0\rangle = |\psi\rangle|B_{00}\rangle = (\alpha|0\rangle + \beta|1\rangle)\frac{(|00\rangle+|11\rangle)}{\sqrt{2}}$

and $|\psi_1\rangle = \alpha|0\rangle\frac{(|00\rangle+|11\rangle)}{\sqrt{2}} + \beta|1\rangle\frac{(|10\rangle+|01\rangle)}{\sqrt{2}}$

$$|\psi_2\rangle = \alpha\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right) + \beta\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{|10\rangle+|01\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2}|00\rangle(\alpha|0\rangle+\beta|1\rangle) \quad \text{(Nothing to do)}$$
$$+\frac{1}{2}|01\rangle(\beta|0\rangle+\alpha|1\rangle) \quad \text{(Apply x gate)}$$
$$+\frac{1}{2}|10\rangle(\alpha|0\rangle-\beta|1\rangle) \quad \text{(Apply z gate)}$$
$$-\frac{1}{2}|11\rangle(\beta|0\rangle-\alpha|1\rangle) \quad \text{(Apply both X and Z gates)}$$

Now, (i) if Bob get 00, then he will do nothing.

(ii) if Bob get 01 then he will apply X-gate

(iii) if Bob get 10 then Bob apply Z-gate

(iv) if Bob get 11, then Bob will apply X and Z both. then back $|\psi\rangle$.

(3) (a)

Let us consider a boolean function ~~$f:\{0,1\}^n$~~
$f:\{0,1\}^n \rightarrow \{0,1\}$. Deutsch-Jozsa algorithm finds the function is balanced or constant
~~The~~ function is ~~balanced~~ constant if it gives 0's or all 1's and balanced if it gives half inputs 0's and half inputs are 1's.

(b) Deutch-Jozsa Algorithm:

At-first two quantum registers, the first one is an n-bit qubit quantum register with all the qubits are intialized to $|0\rangle$. and the 2nd one is a 1 qubit register, intialized to $|1\rangle$.

$|\psi_0\rangle = |0\rangle^{\otimes n} |1\rangle$.

Then apply Hadamard gate to each qubit.

$$|\psi_1\rangle = \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \{0,1\}^n} |x\rangle (|0\rangle - |1\rangle).$$

Apply the quantum oracle $U_f : |x\rangle|y\rangle \rightarrow |x\rangle|y \oplus f(x)\rangle$ on $|\psi_1\rangle$:

$$|\psi_2\rangle = \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \{0,1\}^n} |x\rangle (|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle)$$

$$\Rightarrow |\psi_2\rangle = \frac{1}{\sqrt{2^{n+1}}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle (|0\rangle - |1\rangle)$$

Ignore the last qubit from the 2nd register and apply Hadamard gate to all n qubits from the last register

$$H^{\otimes n} \left(\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \left[\frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \right]$$

$$= \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} \left[\sum_{x \in \{0,1\}^n} (-1)^{f(x)} (-1)^{x \cdot y} \right] |y\rangle.$$

Then measure all the n qubits from the first register:

if $f(x)$ is constant then check that the above prob is 1 & if $f(x)$ is balanced the probability is 0.

Hence after measurement if we get $|0\rangle^{\otimes n}$ we can conclude that $f(x)$ is constant function then $f(x)$ is balanced.

(6) (A) Correctness of Algorithm:

If f is constant: Then there is no effect on input qubits. It states before and after querying the oracle are same. ~~Since H-gate is self inverse~~

$$H^{\otimes n}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = \frac{1}{\sqrt{2^n}}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \xrightarrow{\text{apply } U_f} H^{\otimes n}\frac{1}{\sqrt{2^n}}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$$

In Case if f is blanced:

The input register would be an equal superposition of all states. then for exactly half of these states,

$U_f \frac{1}{\sqrt{2^n}}[1, \cdots -1]^t = \frac{1}{\sqrt{2^n}}\cdot[-1, -1 \cdots 1]^t$, so the Kickback phase add negative phase in the first part, after applying Hadamard gate the gate we receiver is orthogonal to $|00\cdots0\rangle$. Hence, that cannot measure in the all 0-state

③ ⓐ In deterministic classical algorithm the worst case we might have to check $2^{n-1}+1$ queries so is required in the worst case. and in probabistic classical algo a constant is many quires can generate the answer with prob. $\leq \frac{1}{2^k}$ But Deutsch-Jozsa algo can conclude the result deterministically with just a single query.

⑧ (a)

Given boolean function

$f(x_1, x_2, x_3) = x_1 x_2 x_3$

Now, $|\psi_0\rangle = |000\rangle \otimes |1\rangle$

Then $|\psi_1\rangle = \frac{1}{2\sqrt{2}} \{ |000\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle \} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\Rightarrow |\psi_2\rangle = \frac{1}{2\sqrt{2}} \{ (-1)^0 |000\rangle + (-1)^0 |001\rangle + (-1)^0 |010\rangle + (-1)^0 |011\rangle + (-1)^0 |100\rangle + (-1)^0 |101\rangle + (-1)^0 |110\rangle + (-1) |111\rangle \} \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$

$= \frac{1}{2\sqrt{2}} \{ |000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle - |111\rangle \} \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$

Now we ignore the last q-bit form the 2nd register

$|\psi_2\rangle = \frac{1}{2\sqrt{2}} \{ |000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle - |111\rangle \}$

Now, $|\psi_3\rangle = H^{\otimes 3} [|\psi_2\rangle]$.

Finally we calculate the prob. of getting $|0\rangle^{\otimes n}$

is $\frac{1}{2^n} \left[\sum_{x \in \{0,1\}^3} (-1)^{f(x)} \right]^2$

Here $f(1,1,1) = 1$ for all other cases of outputs 0. Hence the probability of getting $|0\rangle^{\otimes n}$ is

$\frac{1}{2^6} \left[(-1)^0 + (-1)^0 + (-1)^0 + (-1)^0 + (-1)^0 + (-1)^0 + (-1)^0 + (-1)^0 \right]^2$

$= \frac{1}{2^6} \{ 1+1+1+1+1+1+1-1 \}^2$

$= \frac{1}{36} \cdot 6^2 = \frac{36}{64} = 0.5625$

Hence for DJ algo. if f is assumed to be either balance or constant. The given output 1 in one case and 0 in others. Hence f is neither balanced nor constant. So ~~according to the given algo~~, we cannot get any information

(4) (a)

Given a function $f: \{0,1\}^n \to \{0,1\}$.
The goal is to find $x \in \{0,1\}^n$, such that $f(x)=1$, or to conclude that no such x exists i.e $f=0$, a constant function.

Let, $A = \{x \in \{0,1\}^n : f(x)=1\}$

$B = \{x \in \{0,1\}^n : f(x)=0\}$

Also let $|A|=a$, $|B|=b$, then $N=2^n$, $a+b=N$.

Begin with a state: $|\psi_0\rangle = |0\rangle^{\otimes n}$

Matrix Representation $|\psi_0\rangle = \begin{pmatrix}1\\0\end{pmatrix}^{\otimes n}$

Apply the Hadamard gate to each of these qubits $|\psi_1\rangle = (H|0\rangle)^{\otimes n}$

$$|\psi_1\rangle = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)^{\otimes n} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}^{\otimes n}$$

$$= \frac{1}{\sqrt{2^n}} \cdot \sum_{x \in \{0,1\}^n} |x\rangle = \frac{1}{\sqrt{N}} \cdot \sum_{x \in \{0,1\}^n} |x\rangle$$

consider the states: $|A\rangle = \frac{1}{\sqrt{a}} \sum_{x \in A} |x\rangle$ & $|B\rangle = \frac{1}{\sqrt{b}} \sum_{x \in B} |x\rangle$

Note that $|A\rangle$ & $|B\rangle$ are orthogonal. consdine the space spanned by $|A\rangle$ & $|B\rangle$.

Then $|\psi_1\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle = \frac{1}{\sqrt{N}} \left(\sum_{x \in A} |x\rangle + \sum_{x \in B} |x\rangle\right)$

$$= \frac{1}{\sqrt{N}} \left(\sqrt{a} \times \frac{1}{\sqrt{a}} \sum_{x \in A} |x\rangle + \sqrt{b} \times \frac{1}{\sqrt{b}} \sum_{x \in B} |x\rangle\right)$$

$$\Rightarrow |\psi_1\rangle = \sqrt{\frac{a}{N}}\,|A\rangle + \sqrt{\frac{b}{N}}\,|B\rangle$$

$$|\psi_1\rangle = \sqrt{\frac{a}{N}}\,|A\rangle + \sqrt{\frac{b}{N}}\,|B\rangle$$

Assuming $\sqrt{\frac{a}{N}} = \sin\theta$ & $\sqrt{\frac{b}{N}} = \cos\theta$ geometrically we can think of $|\psi_1\rangle$ is making an angle θ with the state $|B\rangle$. This implies $\theta = \sin^{-1}\sqrt{\frac{a}{N}}$.

$Z_f|x\rangle = (-1)^{f(x)}|x\rangle$, then $Z_0|x\rangle = \begin{cases} -|x\rangle & \text{if } x = 0^n \\ |x\rangle & \text{if } x \neq 0^n \end{cases}$

Matrix Represent, $Z_0 = I - 2|0^n\rangle\langle 0^n|$

for $n=2$, $Z_0 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}(1000)$

$= \begin{pmatrix} -1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$

Define the unitary operator $G = -H^{\otimes n} Z_0 H^{\otimes n} Z_f$

$H^{\otimes n} Z_0 H^{\otimes n} = H^{\otimes n}(I - 2|0^n\rangle\langle 0^n|)H^{\otimes n}$

$= (H^{\otimes n} - 2H^{\otimes n}|0^n\rangle\langle 0^n|)H^{\otimes n}$

$= I - 2H^{\otimes n}|0^n\rangle\langle 0^n|H^{\otimes n}$

$= I - 2|\psi_1\rangle\langle\psi_1|$

Thus $G = (I - 2|\psi_1\rangle\langle\psi_1|)(-Z_f)$

Same derivation for $n=2$

$H^{\otimes n} Z_0 H^{\otimes n}$

$= \frac{1}{2}\begin{pmatrix} 1&1&1&1 \\ 1&-1&1&-1 \\ 1&1&-1&-1 \\ 1&-1&-1&1 \end{pmatrix}\begin{pmatrix} -1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1&1&1&1 \\ 1&-1&1&-1 \\ 1&1&-1&-1 \\ 1&-1&-1&1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1&-1&-1&-1 \\ -1&1&-1&-1 \\ -1&-1&1&-1 \\ -1&-1&-1&1 \end{pmatrix}$

similar $I - 2|\psi_1\rangle\langle\psi_1|$

$= \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} - 2\cdot\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\frac{1}{2}(1\ 1\ 1\ 1)$

$= \frac{1}{2}\begin{pmatrix} 1&-1&-1&-1 \\ -1&1&-1&-1 \\ -1&-1&1&-1 \\ -1&-1&-1&1 \end{pmatrix}$

Hence $H^{\otimes n} Z_0 H^{\otimes n} = I - 2|\psi_1\rangle\langle\psi_1|$

that.

Note that $G|A\rangle = [I - 2|\psi_1\rangle\langle\psi_1|](-I)|A\rangle$

$= [I - 2|\psi_1\rangle\langle\psi_1|]\cdot|A\rangle$

$= |A\rangle - 2\langle\psi_1|A\rangle|\psi_1\rangle$

$= |A\rangle - 2\sqrt{\frac{a}{N}}\left(\sqrt{\frac{a}{N}}|A\rangle + \sqrt{\frac{b}{N}}|B\rangle\right)$

$= \left(1 - \frac{2a}{N}\right)|A\rangle - \frac{2\sqrt{ab}}{N}|B\rangle$

Similarly $G|B\rangle = \frac{2\sqrt{ab}}{N}|A\rangle - \left(1 - \frac{2b}{N}\right)|B\rangle$

Thus G can be considered as a matrix

$$\begin{pmatrix} -\left(1-\frac{2b}{N}\right) & -\frac{2\sqrt{ab}}{N} \\ \frac{2\sqrt{ab}}{N} & \left(1-\frac{2b}{N}\right) \end{pmatrix}$$

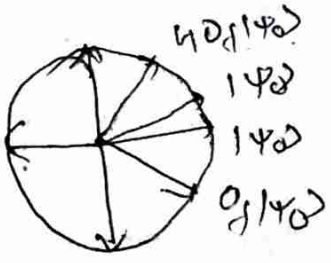
Now using $N = a+b$, we get:

$$\begin{pmatrix} \frac{b-a}{N} & -\frac{2\sqrt{ab}}{N} \\ \frac{2\sqrt{ab}}{N} & \frac{b-a}{N} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{b}{N}} & -\sqrt{\frac{a}{N}} \\ \sqrt{\frac{a}{N}} & \sqrt{\frac{b}{N}} \end{pmatrix}^2$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^2$$

Thus G can be considered as a rotation matrix which on application to a state increase it angle by 2θ. & our goal is to increase the prob. of getting $|A\rangle$ i.e $\sin\theta \approx 1$

Each application of G_0 of G_0, amplifies the angle from θ to 3θ.



After k applications of amplitude amplification operator G_0 of the resulting state is of the following form $|\psi_k\rangle = \sin(2k+1)\theta|x_0\rangle + \cos(2k+1)\theta|\phi_0\rangle$

Prob. of observing x_0 from $|\psi_k\rangle$ is

$\sin^2(2k+1)\theta$

calculating the number of iteration $\theta \to \theta$

the sucess prob. is $\sin^2(2k+1)\theta$

To make the sucess prob. $\frac{1}{2}$ we need

$\sin^2(2k+1)\theta = \frac{1}{2}$

$\Rightarrow (2k+1)\theta = \arcsin\frac{1}{\sqrt{2}}$

$\Rightarrow (2k+1)\theta = \frac{\pi}{4} \Rightarrow k \approx \frac{\pi}{8\theta}$

$= \frac{\pi}{8}\sqrt{2^n}$

prepare the intial state.

$$|\psi_0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

Apply $G = -H^{\otimes n} Z_0 H^{\otimes n} Z_f$

$= (1 - 2|\psi_1\rangle\langle\psi_1|)(-Z_f)$ on the state $|\psi_0\rangle$ for approx $\frac{\pi\sqrt{N}}{4}$ many times

Measure the state in computation basis.

check $|x'\rangle \xrightarrow{Z_f} (-1)^{f(x')}|x'\rangle = -|x'\rangle$ or not.

If yes then we got the correct result

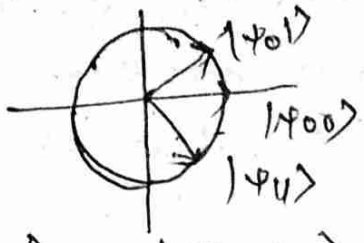
otherwise incorrect

④ (b) Grover algorithm performs a search over an unordered set of $N = 2^n$ items to find an unique element that satisfy some condition. while the best classical algo for a search requires $O(N)$ searchs over unordered data times quantum computers is only $O(\sqrt{N})$ of operation an quadratic speed up.

(5) (a) Alice randomly guarntees two strings of bits $x, y \in \{0,1\}^n$

Define $|\psi_{00}\rangle = |0\rangle$, $|\psi_{10}\rangle = |1\rangle$, $|\psi_{01}\rangle = |+\rangle$, & $|\psi_{11}\rangle = |-\rangle$

we have these 4 states as



Alice prepares m qbits in the state $|\psi_{x,y}\rangle = |\psi_{x_1 y_1}\rangle \cdots |\psi_{x_m y_m}\rangle$.

and sends these m q-bits over quantum channel to Bob. Bob recives m q-bits, although they may not longer is a stable $|\psi_{xy}\rangle$ because eve may have tampered with them or possibly the channel is noisy.

Bob randomly choices $y' \in \{0,1\}^m$ & measures each q-bit reached from Alice to follows.

- If $y_i' = 0$, Bob measures qubit i.
- If $y_i' =$ Bob performs a Hadamard transform to q bit i' then measures it with respect to the standard basis

Let $x' \in \{0,1\}^m$ be the string corresponding to the results of Bobs measurements. The important thing to note at this point is that if $y_i = y_i'$ for some i & there was no noise or eavesdropping then it is certain $x_i = x_i'$

Finally, Alice & Bob publicly compare y and y' They discard bits x_i & x_i' for which $y_i \neq y_i'$ The reaming bits of x & x' represent a 'secret private' key that will go into the next stange of the protocol

2nd Stage of protocol:

Alice & Bob now need to estimate how much Eve might know about x and x'. They do this by some of bits x and x'.

comparing these bits publicly they can estimate the error rate with high accuracy and if it is too large they about.. the maximum error rate can be tolerated is about 11%. If they have acceptable error rate Alice & Bob will have two strings x and x' that agree in a high percentange of positions with high prob. They have some bound on the amount of information Eve posses about the given strings.