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Advanced Interest Rate Models Project

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I. Excercise 1- Cap/Floor pricing

In order to pricing options we had to have appropriate forward rates and discount rate. Using market data we interpolated yield curves. We used function given in the excel spreadsheet during lectures, namely: *Interpolate*.

The next point was to create functions to correctly price options. Let's remind the analytical solution.

Consider shifted lognormal Black formula for standard caplet / floorlet at time $t < T_{i-1} < T_i$:

$$\begin{aligned} cf_i(t, T_{i-1}, T_i, K, \omega) &= NP_d(t, T_i)\tau_x(T_{i-1}, T_i)E_t^{Q_d^{T_i}}\{Max[\omega[L_x(T_{i-1}, T_i) - K], 0]\} = \\ &= NP_d(t, T_i)\tau_x(T_{i-1}, T_i)Black[F_{x,i}(t), K, \lambda_{x,i}, \nu_{x,i}(t, T_{i-1}), \omega], \end{aligned} \tag{1}$$

where T_{i-1} - start accrual time; T_i - end accrual time; K - strike; ω - indicator (for caplet $\omega = +1$, for floorlet $\omega = -1$); $\tau_x(T_{i-1}, T_i)$ is an accrual time period; $F_{x,i}(t)$ - underlying forward rate (EURIBOR), which will be fixed in advance (two business days) before the beginning of the accrual period; $\lambda_{x,i}$ - shift; N - nominal of the contract; $P_d(t; T_i)$ - price of the zero coupon bond at time t which is discounted with OIS rate; $\nu_{x,i}(t, T_{i-1})$ - implied forward variance, which is:

$$\nu_{x,i}(t, T_{i-1}) = \sigma_x(t, T_{i-1})^2\tau_x(t, T_{i-1}). \tag{2}$$

We call $\sigma_x(t, T_{i-1})$ shifted lognormal implied forward volatility.

For the cap / floor contract with time vector $\mathbf{T} = \{T_0, T_1, \ldots, T_n\}$ then we obtain:

$$\begin{aligned} CF(t, \mathbf{T}, K, \omega) &= \sum_{i=1}^{n} cf_i(t, T_{i-1}, T_i, K, \omega) = \\ &= N\sum_{i=1}^{n} P_d(t, T_i)\tau_x(T_{i-1}, T_i)Black[F_{x,i}(t), K, \lambda_{x,i}, \nu_{x,i,n}(t, T_{i-1}, T_n), \omega], \end{aligned} \tag{3}$$

where $\nu_{x,i,n}(t, T_{i-1}, T_n)$ - implied term variance, which is:

$$\nu_{x,i,n}(t, T_{i-1}, T_n) = \sigma_x(t, T_n)^2\tau_x(t, T_{i-1}), \tag{4}$$

where $\sigma_x(t, T_n)$ is shifted lognormal implied term volatility, and it is quoted in the table by broker.

We created function *OptionPrice* in Visual Basic.

```
Public Function OptionPrice( _
    maturity As Variant, _
    forward_vec As Variant, _
    discount_vec As Variant, _
    year_fraction_c As Variant, _
    year_fraction_v As Variant, _
    impl_vol As Variant, _
    lambda As Variant, _
    strike As Variant, _
    nominal As Variant, _
    cp As Variant) As Variant

    Dim price, counter, y_frac_sum As Variant
    Dim black_price, forward As Variant
    counter = maturity * 2 ' number of caplets/florlets
    strike = strike + lambda ' shift a strike
    price = 0
    For i = 2 To counter
    ' calculate implied variance from implied volatility
       variance = impl_vol ^ 2 * year_fraction_v(i)
       forward = forward_vec(i) + lambda ' shift forward

        ' calculate price of single caplet/floorlet using given formula

          black_price = Black(forward, strike, variance, cp)

        ' add calculated price to sum of the whole option
        price = price + year_fraction_c(i) * discount_vec(i) * black_price

    Next i
    OptionPrice = nominal * price

End Function
```

The arguments are:

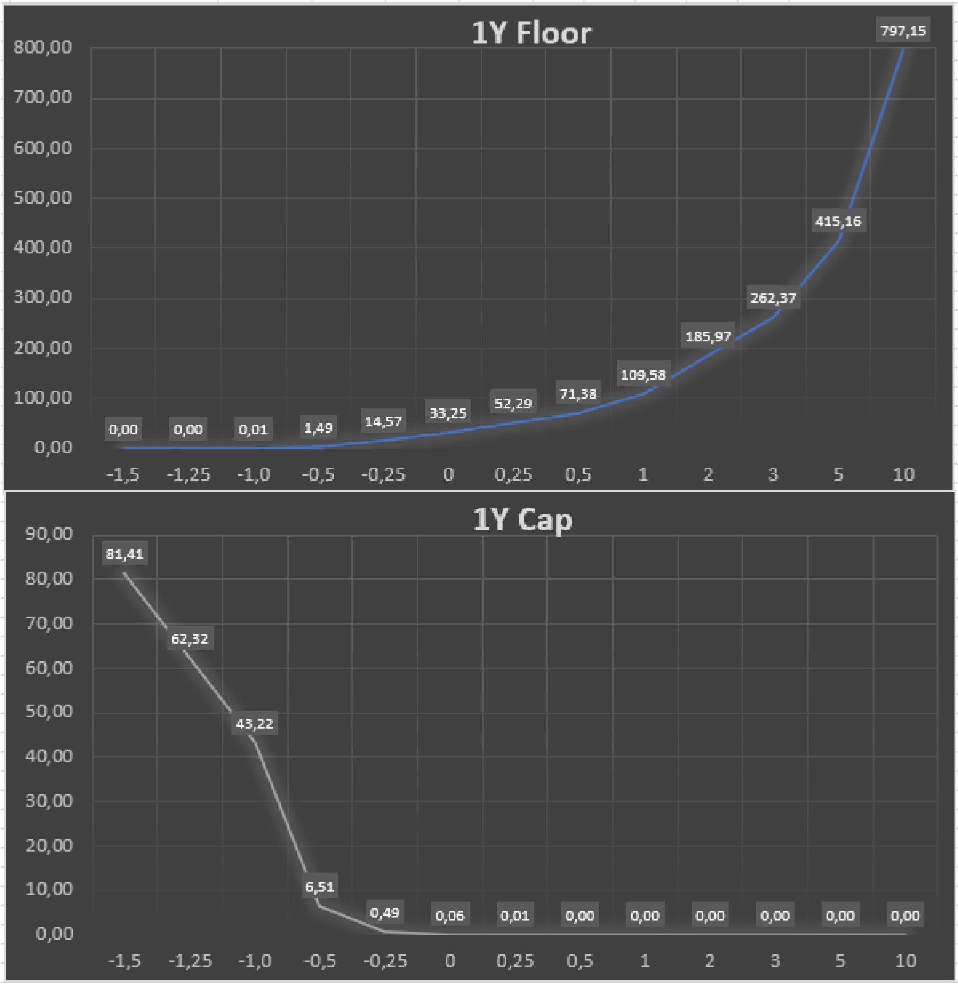
- *maturity*- years to maturity assuming that we are dealing with 6m forward rates. In case of 3m rates we have to multiply number of years by 2,
- *forward_vec*- vector of forward rates,
- *discount_vec*- vector of discount factors, it is longer than vector of forward rates because the first forward rate is no longer consider in option,
- *year_fraction_c*- vector of coupon year fractions,
- *year_fraction_v*- vector of accumulated coupon year fractions,
- *impl_vol*- implied volatlity,
- *lambda*- shift of rate,
- *strike*- strike of the option,
- *nominal*- nominal of the option,
- *cp*- $+/-$ 1 if we calculate cap/floor option.

The results are in *Caps&Floors Pricing* sheet.

16:10	31.paź.19	ICAP	UK69580	VCAP5											
EUR	Floors	-	Premium	Mids	(Eonia	disc)									
	STK	ATM	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	4,91	0,00	0,00	0,01	1,49	14,57	33,25	52,29	71,38	109,58	185,97	262,37	415,16	797,15
18M	-0,4	10,15	0,00	0,01	0,06	4,16	25,73	56,27	87,83	119,54	183,03	310,06	437,09	691,15	1326,30
2Y	-0,4	15,99	0,02	0,06	0,24	7,57	36,85	78,85	122,92	167,39	256,62	435,24	613,88	971,16	1864,36
3Y	-0,3	31,17	0,13	0,32	0,91	10,12	39,81	92,45	152,77	215,44	342,68	598,80	855,23	1368,18	2650,62
4Y	-0,2	71,95	0,93	1,88	3,87	21,11	61,26	128,97	208,86	293,88	469,59	827,56	1187,25	1907,54	3708,90
5Y	-0,2	98,76	2,76	5,02	9,00	36,10	84,97	166,80	263,26	367,95	588,87	1045,42	1506,91	2432,95	4750,34
6Y	-0,2	129,06	6,01	10,00	17,14	54,21	112,62	206,02	317,69	440,30	702,50	1253,94	1816,41	2948,15	5783,54
7Y	-0,1	200,08	10,40	16,62	26,59	75,66	142,28	245,78	371,26	510,18	809,87	1452,54	2113,88	3450,10	6803,18
8Y	0	284,54	14,99	23,85	37,07	97,30	170,97	286,19	424,10	577,68	913,10	1640,66	2398,43	3936,94	7806,16
9Y	0	326,82	20,20	31,55	48,97	119,51	203,16	328,23	477,69	644,79	1014,14	1825,24	2676,66	4416,10	8803,02
10Y	0,04	394,73	27,91	42,02	62,91	144,53	236,70	370,84	533,16	712,38	1113,17	2001,84	2944,96	4879,89	9774,92
12Y	0,16	574,70	47,84	68,12	98,00	202,32	309,08	462,42	643,80	847,24	1305,98	2343,71	3459,12	5777,41	11678,75
15Y	0,29	853,64	90,17	121,22	161,82	300,05	428,13	605,82	818,71	1057,02	1592,81	2839,03	4203,09	7079,81	14463,58
20Y	0,41	1343,73	190,82	242,46	309,46	505,70	677,49	897,19	1157,16	1453,79	2127,15	3695,52	5457,85	9222,74	18984,11
25Y	0,45	1859,13	322,01	400,99	500,25	775,08	987,21	1253,20	1574,37	1938,63	2738,93	4648,87	6795,39	11413,68	23477,30
30Y	0,44	2366,93	484,33	596,86	727,66	1084,30	1345,14	1679,65	2052,87	2473,97	3420,36	5660,60	8195,25	13642,59	27956,70

16:10	31.paź.19	ICAP	UK69580	VCAP4											
EUR	Caps	-	Premium	Mids	(Eonia	disc)									
	STK	ATM	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	2,29	81,41	62,32	43,22	6,51	0,49	0,06	0,01	0,00	0,00	0,00	0,00	0,00	0,00
18M	-0,4	4,96	134,55	102,80	71,09	11,68	1,49	0,27	0,08	0,03	0,01	0,00	0,00	0,00	0,00
2Y	-0,4	9,49	190,02	145,40	100,92	18,93	3,55	0,89	0,30	0,12	0,03	0,00	0,00	0,00	0,00
3Y	-0,3	22,39	299,13	235,20	171,66	52,63	18,20	6,72	2,92	1,46	0,46	0,09	0,03	0,01	0,00
4Y	-0,2	38,06	435,43	346,30	258,22	95,31	45,39	23,02	12,84	7,79	3,35	1,02	0,43	0,13	0,02
5Y	-0,2	76,92	583,55	469,92	358,00	153,32	86,30	52,24	32,81	21,60	10,74	3,72	1,65	0,54	0,10
6Y	-0,2	132,36	746,82	608,98	474,29	227,70	144,29	95,86	65,70	46,48	25,02	9,15	4,31	1,43	0,26
7Y	-0,1	175,43	925,27	763,72	605,92	319,44	218,29	154,02	111,72	82,87	47,01	18,60	8,84	2,88	0,52
8Y	0	226,79	1119,36	934,54	754,07	426,93	306,91	228,44	172,66	132,55	80,60	33,41	16,42	5,43	0,91
9Y	0	315,57	1327,46	1119,05	916,73	547,76	411,66	316,98	246,69	194,04	123,89	55,99	28,41	9,85	1,77
10Y	0,04	401,12	1546,17	1314,85	1090,30	681,06	527,80	416,50	333,38	267,17	177,10	84,03	45,43	16,89	3,26
12Y	0,16	572,75	2014,43	1738,24	1471,66	983,04	793,33	650,21	535,12	442,10	307,90	159,77	89,31	35,88	7,89
15Y	0,29	859,43	2759,98	2418,96	2087,49	1481,58	1237,59	1043,22	884,03	750,28	541,93	299,88	175,66	75,83	18,23
20Y	0,41	1356,04	3977,79	3535,37	3108,30	2316,40	1994,13	1719,77	1485,67	1288,23	973,46	565,56	351,62	163,98	44,02
25Y	0,45	1861,30	5102,61	4568,97	4055,61	3105,20	2704,71	2358,08	2066,63	1818,27	1393,33	852,79	548,83	266,17	77,38
30Y	0,44	2385,72	6157,47	5541,34	4943,49	3842,82	3375,00	2980,86	2625,43	2317,88	1806,97	1132,59	752,63	370,74	111,79

When strike increases, the price increases. The longer maturity time, the higher price is. In the interval of given strikes, the price ranges are bigger for floor options than cap ones.



We can compare obtained prices with those quoted in the market. We present the difference between these values in percentage point.

	STK	ATM	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	63,72%	-	-	-	-25,44%	-	-	-	-	-	-	-	-	-
18M	-0,4	44,95%	-	-	-	3,91%	-	-	-	-	-	-	-	-	-
2Y	-0,4	33,27%	-	-	-	8,12%	-	-	-	-	-	-	-	-	-
3Y	-0,3	19,89%	-	-	-9,37%	1,17%	-	-	-	-	-	-	-	-	-
4Y	-0,2	41,07%	-6,56%	-5,96%	-3,25%	0,50%	-	-	-	-	-	-	-	-	-
5Y	-0,2	16,19%	-8,04%	0,50%	-0,03%	0,27%	-0,04%	-	-	-	-	-	-	-	-
6Y	-0,2	-0,72%	0,16%	0,03%	0,80%	0,39%	0,56%	-	-	-	-	-	-	-	-
7Y	-0,1	8,74%	3,99%	3,89%	2,29%	2,24%	0,91%	-	-	-	-	-	-	-	-
8Y	0	15,20%	-0,08%	3,71%	0,18%	1,36%	-0,02%	-	-	-	-	-	-	-	-
9Y	0	3,10%	1,02%	1,76%	2,03%	1,28%	1,07%	-	-	-	-	-	-	-	-
10Y	0,04	-0,32%	3,37%	2,49%	1,47%	0,37%	0,72%	0,50%	-	-	-	-	-	-	-
12Y	0,16	1,00%	1,79%	1,67%	2,08%	1,16%	1,00%	0,75%	-	-	-	-	-	-	-
15Y	0,29	0,31%	2,47%	2,73%	1,14%	1,03%	0,97%	0,63%	0,58%	-	-	-	-	-	-
20Y	0,41	0,13%	1,50%	1,87%	1,80%	0,54%	1,42%	0,81%	0,36%	-	-	-	-	-	-
25Y	0,45	0,66%	1,58%	1,77%	2,09%	1,32%	1,15%	0,34%	0,41%	-	-	-	-	-	-
30Y	0,44	0,21%	1,33%	1,85%	1,49%	0,77%	0,68%	1,00%	0,58%	-	-	-	-	-	-

	STK	ATM	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	-23,69%	-	-	-	-	-50,90%	-	-	-	-	-	-	-	-
18M	-0,4	-29,09%	-	-	-	-	49,15%	-	-	-	-	-	-	-	-
2Y	-0,4	-20,89%	-	-	-	-	-11,19%	-10,62%	-	-	-	-	-	-	-
3Y	-0,3	-13,90%	-	-	-	-	1,10%	-4,04%	-2,69%	46,33%	-	-	-	-	-
4Y	-0,2	-25,37%	-	-	-	-	0,86%	0,09%	-1,19%	-2,64%	11,75%	2,38%	-	-	-
5Y	-0,2	-9,51%	-	-	-	-	-	2,43%	2,53%	2,86%	7,39%	-6,99%	-17,66%	-45,93%	-
6Y	-0,2	1,81%	-	-	-	-	-	1,98%	2,65%	3,30%	4,27%	1,72%	7,73%	43,03%	-
7Y	-0,1	-4,66%	-	-	-	-	-	1,33%	1,56%	2,31%	2,20%	3,32%	10,56%	-3,89%	-48,44%
8Y	0	-8,18%	-	-	-	-	-	1,53%	1,56%	1,96%	3,33%	4,39%	2,64%	8,63%	-9,38%
9Y	0	-0,45%	-	-	-	-	-	1,60%	1,52%	1,59%	2,39%	3,68%	5,24%	9,43%	-11,44%
10Y	0,04	1,29%	-	-	-	-	-	-	1,64%	1,59%	1,78%	2,48%	3,25%	5,55%	8,67%
12Y	0,16	0,66%	-	-	-	-	-	-	1,16%	0,94%	1,28%	2,42%	2,66%	5,53%	12,77%
15Y	0,29	0,99%	-	-	-	-	-	-	-	1,25%	1,11%	2,35%	2,13%	3,88%	7,22%
20Y	0,41	1,05%	-	-	-	-	-	-	-	0,96%	1,51%	1,17%	1,62%	3,78%	4,82%
25Y	0,45	0,77%	-	-	-	-	-	-	-	1,24%	0,89%	1,52%	2,01%	3,17%	6,00%
30Y	0,44	1,00%	-	-	-	-	-	-	-	1,00%	0,89%	1,30%	2,82%	2,42%	5,46%

The divergences are bigger for cap options than floor one, but not for ATM options.

Option ATM have larger divergences than others. Also options with short maturities have bigger differences than for longer periods. It applies to both cap and floor options.

Differences between market floor prices and those from model are more visible for negative strikes. However for cap options it works in opposite way- the bigger positive strike, the bigger divergences between market and model prices.

Why there are divergences?

1) The cumulative distribution function does not have analytical solution, so one needs to implement numerical method. Depends on the methodology used, there might be differences.
2) The LN dynamics is not assumed to be a good proxy for the true underlying dynamics. The Black formula is actually used as a model based parametric form useful to map option prices into LN implied volatilities.

II. Excercise 2- Cap/Floor volatility

Implied volatility is the market's forecast of a likely movement in a security's price. It is a metric used by investors to estimate future fluctuations (volatility) of a security's price based on certain predictive factors. Professional traders often use implied volatility as a metric of option value instead of quoted price on the market. There is one to one relationship between option price and implied volatility. Implied volatility can be determined by using an option pricing model. We can substitute quoted price of the option on the right hand side of pricing formula and solve equation for volatility as a parameter,

$$N\sum_{i=1}^{n} P_d(t,T_i)\tau_x(T_{i-1},T_i)Black[F_{x,i}(t),K,\lambda_{x,i},\nu_{x,i,n}(t,T_{i-1},T_n),\omega] = \text{known value}. \tag{5}$$

Unfortunately, we can not solve it analytically, only numerically. The method used is **Newton–Raphson** which uses the derivative of the option price with respect to the volatility (the vega) in the calculation. The general formula can be written as,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{6}$$

At the beginning of this excercise, we implemented function to calculate vega (look part III for the analytical formula).

```
Public Function vegaOption( _
    maturity As Variant, _
    forward_vec As Variant, _
    discount_vec As Variant, _
    year_fraction_c As Variant, _
    year_fraction_v As Variant, _
    impl_vol As Variant, _
    lambda As Variant, _
    strike As Variant, _
    nominal As Variant) As Variant

    Dim vega, counter, y_frac_sum As Variant
    Dim vega_single, forward As Variant
    Dim d1d2 As Variant
    counter = maturity * 2 ' number of caplets/florlets
    strike = strike + lambda ' shift a strike
    vega = 0

    For i = 2 To counter
        forward = forward_vec(i) + lambda ' shift forward
        variance = impl_vol ^ 2 * year_fraction_v(i) ' implied variance
        d1d2 = BlackD1D2(forward, strike, variance)
        vega_single = discount_vec(i) * year_fraction_c(i) * Sqr(year_fraction_v(i)) * forward * Exp(-0.5 * d1d2(0) * d1d2(0))
        vega = vega + vega_single
    Next i

    vegaOption = nominal * vega / Sqr(3.1415926 * 2)
End Function
```

The next step was to implement Newton-Raphson method.

```
Public Function OptionImplied( _
    mkt_price As Variant, _
    maturity As Variant, _
    forward_vec As Variant, _
    discount_vec As Variant, _
    year_fraction_c As Variant, _
    year_fraction_v As Variant, _
    lambda As Variant, _
    strike As Variant, _
    nominal As Variant, _
    cp, _
    error As Variant) As Variant


    Dim volatility, dv, price_error As Variant
    Dim vegaOption, price, OptionPrice, variance As Variant
    Dim vega, counter, y_frac_sum As Variant
    volatility = 0.2 ' start with artificial volatility
    dv = error + 2


    strike = strike + lambda ' shift a strike
    While Abs(dv) > error
        counter = maturity * 2 ' number of caplets/florlets
        vega = 0
        price = 0
        For i = 2 To counter
            Dim vega_single, forward As Variant
            forward = forward_vec(i) + lambda ' shift forward
            variance = volatility ^ 2 * year_fraction_v(i) ' implied variance

            Dim d1d2 As Variant
            d1d2 = BlackD1D2(forward, strike, variance)
             'calculate vega for single caplet/floorlet
            vega_single = discount_vec(i) * year_fraction_c(i) * Sqr(year_fraction_v(i)) * forward * Exp(-0.5 * d1d2(0) * d1c
            vega = vega + vega_single

            Dim black_price As Variant
            black_price = Black(forward, strike, variance, cp) 'calculate price of single caplet/florlet
            price = price + year_fraction_c(i) * discount_vec(i) * black_price
        Next i

        vegaOption = nominal * vega / Sqr(3.1415926 * 2)
        OptionPrice = nominal * price

        price_error = OptionPrice - mkt_price
        dv = (price_error / vegaOption)
        volatility = volatility - dv

    Wend

    OptionImplied = volatility
    End Function
```

New arguments are:

- *mkt_price*- market price of the option,
- *error*- error of the estimation.

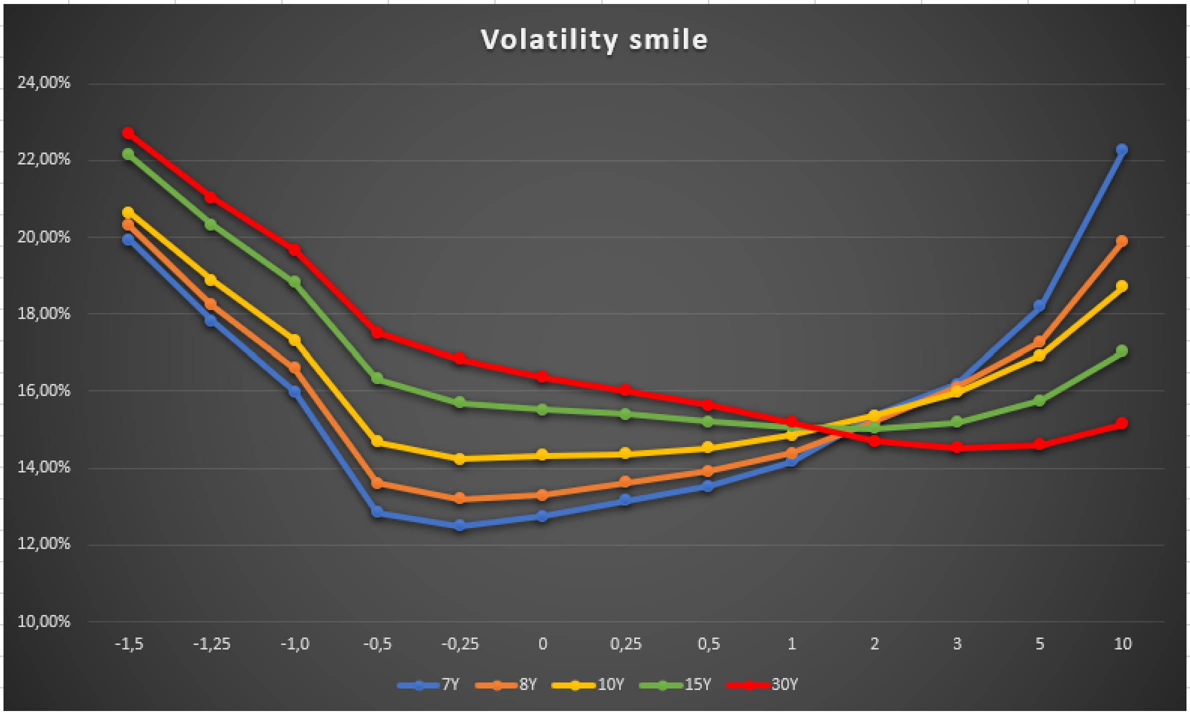
The results are in *Implied volatilities* sheet.

				Shifted	Black	Volatilities		Shift:	3,0%						
	STK	**ATM**	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	2,33%				7,32%	9,31%								
18M	-0,4	3,49%				6,44%	6,62%								
2Y	-0,4	4,63%				6,65%	7,97%	8,93%							
3Y	-0,3	6,29%				8,15%	8,04%	9,04%	9,87%	9,92%					
4Y	-0,2	5,02%	17,78%	15,47%	13,30%	9,58%	9,43%	10,20%	11,05%	11,89%	12,90%	15,55%			
5Y	-0,2	8,79%	18,88%	16,38%	14,30%	10,88%	10,60%	11,12%	11,76%	12,37%	13,44%	15,91%	17,70%	21,07%	
6Y	-0,2	11,86%	19,49%	17,30%	15,37%	11,98%	11,64%	12,01%	12,51%	13,00%	13,91%	15,54%	16,69%	18,08%	
7Y	-0,1	11,46%	19,93%	17,83%	15,99%	12,85%	12,50%	12,75%	13,16%	13,53%	14,18%	15,37%	16,18%	18,20%	22,26%
8Y	0	11,10%	20,30%	18,23%	16,59%	13,60%	13,20%	13,29%	13,63%	13,93%	14,39%	15,21%	16,11%	17,28%	19,90%
9Y	0	13,44%	20,35%	18,51%	16,89%	14,11%	13,69%	13,75%	14,01%	14,24%	14,62%	15,32%	15,90%	16,94%	19,47%
10Y	0,04	14,46%	20,64%	18,87%	17,32%	14,67%	14,22%	14,32%	14,38%	14,52%	14,85%	15,36%	15,96%	16,93%	18,71%
12Y	0,16	14,90%	21,40%	19,61%	18,07%	15,51%	15,00%	14,99%	14,91%	14,98%	15,07%	15,34%	15,67%	16,41%	17,90%
15Y	0,29	15,39%	22,15%	20,33%	18,82%	16,32%	15,70%	15,52%	15,40%	15,21%	15,08%	15,03%	15,18%	15,75%	17,02%
20Y	0,41	15,65%	22,79%	20,96%	19,46%	17,05%	16,35%	16,00%	15,74%	15,44%	15,11%	14,81%	14,80%	15,15%	16,17%
25Y	0,45	15,68%	22,77%	21,05%	19,62%	17,37%	16,67%	16,26%	15,94%	15,59%	15,19%	14,78%	14,68%	14,87%	15,64%
30Y	0,44	15,81%	22,68%	21,03%	19,66%	17,52%	16,82%	16,37%	16,01%	15,63%	15,19%	14,69%	14,53%	14,60%	15,15%

We can compare obtained volatilities with those quoted in the market. We present the difference between these values in percentage point.

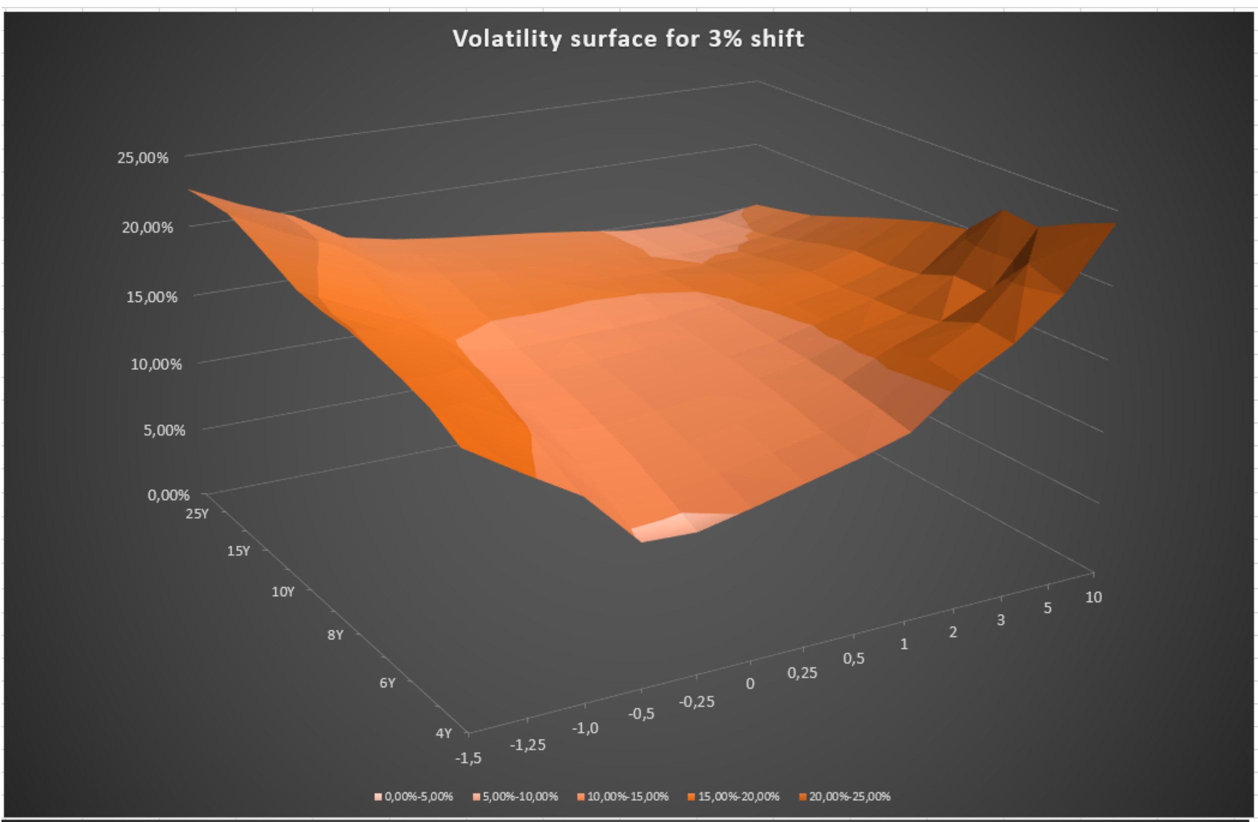
	STK	**ATM**	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	-62,91%				18,05%	24,17%								
18M	-0,4	-46,82%				-2,43%	-11,72%								
2Y	-0,4	-33,37%				-5,03%	4,93%	2,61%							
3Y	-0,3	-20,60%				-0,56%	-0,80%	1,62%	0,73%	-7,31%					
4Y	-0,2	-46,38%	1,04%	1,13%	0,75%	-0,25%	-0,79%	-0,05%	0,46%	0,80%	-2,31%	-0,35%			
5Y	-0,2	-17,35%	1,51%	-0,10%	0,01%	-0,14%	0,04%	-1,59%	-1,20%	-1,06%	-1,91%	1,36%	2,93%	7,48%	
6Y	-0,2	0,78%	-0,03%	-0,01%	-0,22%	-0,21%	-0,53%	-1,55%	-1,52%	-1,49%	-1,37%	-0,37%	-1,26%	-4,35%	
7Y	-0,1	-9,66%	-0,83%	-0,94%	-0,67%	-1,17%	-0,79%	-1,19%	-1,06%	-1,24%	-0,85%	-0,83%	-1,92%	0,57%	7,00%
8Y	0	-17,15%	0,02%	-0,93%	-0,06%	-0,71%	0,01%	-1,54%	-1,21%	-1,22%	-1,47%	-1,24%	-0,57%	-1,28%	1,04%
9Y	0	-3,51%	-0,23%	-0,46%	-0,63%	-0,66%	-0,82%	-1,77%	-1,30%	-1,11%	-1,19%	-1,18%	-1,25%	-1,54%	1,40%
10Y	0,04	0,37%	-0,78%	-0,67%	-0,47%	-0,19%	-0,53%	-0,53%	-1,53%	-1,21%	-0,99%	-0,88%	-0,86%	-1,02%	-1,01%
12Y	0,16	-1,16%	-0,45%	-0,48%	-0,69%	-0,59%	-0,69%	-0,71%	-1,23%	-0,83%	-0,83%	-1,00%	-0,82%	-1,16%	-1,63%
15Y	0,29	-0,36%	-0,68%	-0,84%	-0,40%	-0,52%	-0,63%	-0,54%	-0,64%	-1,25%	-0,82%	-1,12%	-0,75%	-0,93%	-1,06%
20Y	0,41	-0,15%	-0,48%	-0,66%	-0,70%	-0,28%	-0,90%	-0,64%	-0,36%	-1,03%	-1,22%	-0,62%	-0,64%	-1,00%	-0,79%
25Y	0,45	-0,73%	-0,57%	-0,69%	-0,90%	-0,73%	-0,75%	-0,27%	-0,39%	-1,33%	-0,74%	-0,84%	-0,83%	-0,88%	-1,03%
30Y	0,44	-0,23%	-0,52%	-0,79%	-0,70%	-0,46%	-0,47%	-0,81%	-0,56%	-1,06%	-0,74%	-0,73%	-1,17%	-0,70%	-0,97%

The biggest differences are between options at the money. The smaller time to maturity, there bigger is the difference. Our calculations underestimate the implied volatility of ATM options. Conversely, the implied volatility of options with strike -0.5 % and -0.25 % and one year to maturity is bigger than in the market. However, except for a couple of volatilities, values are within the range of +/- 2 %. Options with a shorter time to maturity (except those ATM) are usually overestimated. The longer time to maturity, the volatility tends to be underestimated.



The more an option is ITM or OTM, the greater its implied volatility becomes. Implied volatility tends to be lowest with ATM options. Markets have significant skew, so to hedge against it, market participants buys OTM options. The volatility smile's existence shows that ITM and OTM options tend to be more in demand than ATM options. Demand drives prices, which affects implied volatility.

Our calculations are able to recreate the state of the market. The graphs indeed create smile's. Additionally, we can observe patterns. The longer time to maturity, the higher implied volatility at the lowest possible strike and lower volatility at the highest strike.



A. Problematic inversion

Calculation of implied volatility is probably the most frequently executed numerical task in practical financial mathematics. Nonetheless, it is not always an easy task.

- The obvious problem in a world with negative rates is choosing appropriate shift λ. If it is too low we might not obtain result at all because log-normal distributions assumes that values are strictly positive. However, with high value of λ we decrease implied volatility.
- One of the element of pricing function is the cumulative normal distribution ($\Phi(z)$) function which can not be compute analytically. When its argument z is significantly positive, as is the situation with options deeply in the money, $\Phi(z)$ become indistinguishable from 1.
- The unpleasant feature of the Black option formula is the fact that it is convex for low volatilities and concave for higher volatilities. This means that, given an arbitrary initial guess, a Newton iteration may, if the initial guess is too low and in the convex domain, be fast forwarded to very high volatilities, and not rarely to levels where the numerical implementation of our (normalised or conventional) Black function no longer distinguishes the result from the limit value for high volatilities. When the latter happens, the iteration ceases and fails. Conversely, when the arbitrary initial guess is too high and in the concave domain, the first step may attempt to propel volatility to the negative domain.

III. Excercise 3- Cap/Floor risk

A. Pricing formulas

We are interested in delta and vega sensitivities to forward and discount curves and implied term bolatilities, which are by definition:

$$\begin{aligned}
\Delta_{x,i}^{CF} &= \frac{\partial CF(t, \mathbf{T}, K, \omega)}{\partial F_{x,i}(t)}, \\
\Delta_{d,i}^{CF} &= \frac{\partial CF(t, \mathbf{T}, K, \omega)}{\partial Z_{d,i}(t)}, \\
V_{x}^{CF} &= \frac{\partial CF(t, \mathbf{T}, K, \omega)}{\partial \sigma_x(t, T_n)}.
\end{aligned} \tag{7}$$

Then we are going to calculate parallel deltas:

$$\begin{aligned}
\Delta_{x}^{CF} &= \sum_{i=1}^{n} \frac{\partial CF(t, \mathbf{T}, K, \omega)}{\partial F_{x,i}(t)}, \\
\Delta_{d}^{CF} &= \sum_{i=1}^{n} \frac{\partial CF(t, \mathbf{T}, K, \omega)}{\partial Z_{d,i}(t)}.
\end{aligned} \tag{8}$$

Then Δ_x^{CF}, Δ_d^{CF} and V we are going to compare with numerically calculated ones. We use forward rates instead of zero rates for Δ_x^{CF} because the underlying rate for the contract is forward rate.

B. Analytical calculation of greeks

1) The Black formula:

$$Black[F_{x,i}(t), K, \lambda_{x,i}, \nu_x(t; T_{i-1}), \omega] = \omega[(F_{x,i}(t) + \lambda_{x,i})\Phi(\omega d_i^{+}) - (K + \lambda_{x,i})\Phi(\omega d_i^{-})], \tag{9}$$

where

$$\begin{aligned}
d_i^{+} &= \frac{\ln \frac{F_{x,i}(t) + \lambda_{x,i}}{K + \lambda_{x,i}} + \frac{1}{2}\nu_x(t; T_{i-1})}{\sqrt{\nu_x(t; T_{i-1})}}, \\
d_i^{-} &= d_i^{+} - \sqrt{\nu_x(t; T_{i-1})}, \\
\Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy.
\end{aligned} \tag{10}$$

2) Delta to forward rates: We will start with pillar delta for a single caplet / floorlet. Using (1):

$$\begin{aligned}
&\forall j \neq i : \Delta_{x,j}^{cf_i} = 0, \\
&\Delta_{x,i}^{cf_i} = \frac{cf_i(t, T_{i-1}, T_i, K, \omega)}{\partial F_{x,i}(t)} = \\
&= NP_d(t, T_i)\tau_x(T_{i-1}, T_i)\omega[\Phi(\omega d_i^+) + (F_{x,i}(t) + \lambda_{x,i})\frac{\partial \Phi(\omega d_i^+)}{\partial F_{x,i}} - (K + \lambda_{x,i})\frac{\partial \Phi(\omega d_i^-)}{\partial F_{x,i}}].
\end{aligned} \tag{11}$$

Notice that:

$$\begin{aligned}
&\frac{\partial d_i^+}{\partial F_{x,i}} = \frac{\partial d_i^-}{\partial F_{x,i}} = \frac{1}{(F_{x,i}(t) + \lambda_{x,i})\sqrt{\nu_x(t; T_{i-1})}}, \\
&\frac{\partial \Phi(\omega d_i^+)}{\partial F_{x,i}} = \frac{\omega e^{-\frac{1}{2}(\omega d_i^+)^2}}{\sqrt{2\pi\nu_x(t; T_{i-1})}(F_{x,i}(t) + \lambda_{x,i})}, \\
&\frac{\partial \Phi(\omega d_i^-)}{\partial F_{x,i}} = \frac{\omega e^{-\frac{1}{2}(\omega d_i^-)^2}}{\sqrt{2\pi\nu_x(t; T_{i-1})}(F_{x,i}(t) + \lambda_{x,i})}.
\end{aligned} \tag{12}$$

Since $\omega^2 = 1$:

$$\Delta_{x,i}^{cf_i} = NP_d(t, T_i)\tau_x(T_{i-1}, T_i)[\omega\Phi(\omega d_i^+) + \frac{(F_{x,i}(t) + \lambda_{x,i})e^{-\frac{1}{2}(d_i^+)^2} - (K + \lambda_{x,i})e^{-\frac{1}{2}(d_i^-)^2}}{\sqrt{2\pi\nu_x(t; T_{i-1})}(F_{x,i}(t) + \lambda_{x,i})}]. \tag{13}$$

We will proove that $(F_{x,i}(t) + \lambda_{x,i})e^{-\frac{1}{2}(d_i^+)^2} - (K + \lambda_{x,i})e^{-\frac{1}{2}(d_i^-)^2} = 0$:

$$\begin{aligned}
&(F_{x,i}(t) + \lambda_{x,i})e^{-\frac{1}{2}(d_i^+)^2} - (K + \lambda_{x,i})e^{-\frac{1}{2}(d_i^-)^2} = e^{-\frac{1}{2}(d_i^-)^2}[(F_{x,i}(t) + \lambda_{x,i})e^{-\frac{1}{2}[(d_i^+)^2 - (d_i^-)^2]} - (K + \lambda_{x,i})], \\
&(d_i^+)^2 - (d_i^-)^2 = \frac{4\ln\frac{F_{x,i}(t) + \lambda_{x,i}}{K + \lambda_{x,i}}\frac{1}{2}\nu_x(t; T_{i-1})}{\nu_x(t; T_{i-1})} = 2\ln\frac{F_{x,i}(t) + \lambda_{x,i}}{K + \lambda_{x,i}}, \\
&e^{-\frac{1}{2}[(d_i^+)^2 - (d_i^-)^2]} = e^{-\ln\frac{F_{x,i}(t) + \lambda_{x,i}}{K + \lambda_{x,i}}} = \frac{K + \lambda_{x,i}}{F_{x,i}(t) + \lambda_{x,i}}, \\
&(F_{x,i}(t) + \lambda_{x,i})e^{-\frac{1}{2}[(d_i^+)^2 - (d_i^-)^2]} - (K + \lambda_{x,i}) = 0.
\end{aligned} \tag{14}$$

Then $\Delta_{x,i}^{cf_i} = N\omega P_d(t, T_i)\tau_x(T_{i-1}, T_i)\Phi(\omega d_i^+)$.

Since cap / floor has one caplet (floorlet) for every forward rate, and since the sensitivity of caplet (floorlet) is non-zero only for the corresponding forward rate, using (3) the total sensitivity of the cap (floor) to the single forward rate will be the same as for the corresponding caplet (floorlet):

$$\Delta_{x,i}^{CF} = \Delta_{x,i}^{cf_i} = N\omega P_d(t, T_i)\tau_x(T_{i-1}, T_i)\Phi(\omega d_i^+). \tag{15}$$

Then cap (floor) sensitivity to the parralel shift of the forward curve can be then presented as:

$$\Delta_x^{CF} = N\omega\sum_{i=1}^{n} P_d(t, T_i)\tau_x(T_{i-1}, T_i)\Phi(\omega d_i^+), \tag{16}$$

which was implemented in Excel as following:

```
Public Function DeltaForward( _
    maturity As Variant, _
    forward_vec As Variant, _
    discount_vec As Variant, _
    year_fraction_c As Variant, _
    year_fraction_v As Variant, _
    impl_vol As Variant, _
    lambda As Variant, _
    strike As Variant, _
    nominal As Variant, _
    cp As Variant) As Variant

    Dim delta, counter As Variant

    counter = maturity * 2
    strike = strike + lambda
    delta = 0

    For i = 2 To counter
        Dim forward, d1d2, variance As Variant

        forward = forward_vec(i) + lambda
        variance = impl_vol ^ 2 * year_fraction_v(i)

        d1d2 = BlackD1D2(forward, strike, variance)

        delta = delta + discount_vec(i) * year_fraction_c(i) * NormsDist_Wilmott(cp * d1d2(0))

    Next i

    DeltaForward = delta * nominal * cp
End Function
```

3) Delta to discount rates: In general forward rates that are built inside the formula can depend on discount rates. In particular before the crisis EURIBOR and Eonia OIS had a constant spread and then the sensitivity of the Black formula to the discount rate should have been calculated using general formula with Jacobian matrices where $dZ_x = dZ_d$. However nowadays the connection between underlying rate and the discount rate is much more complicated. We will consider them as independent rates which is a simplifying (though not very realistic) assumption.

We again start with pillar delta for a single caplet / floorlet:

$$\begin{aligned} &\forall j \neq i : \Delta_{d,j}^{cf_i} = 0, \\ &\Delta_{d,i}^{cf_i} = \frac{\partial cf_i(t, T_{i-1}, T_i, K, \omega)}{\partial Z_{d,i}(t)} = \\ &= N \frac{\partial P_d(t, T_i)}{\partial Z_{d,i}(t)} \tau_x(T_{i-1}, T_i) Black[F_{x,i}(t), K, \lambda_{x,i}, \nu_{x,i}(t, T_{i-1}), \omega]. \end{aligned} \tag{17}$$

Since $P_d(t, T_i) = e^{-Z_{d,i}(t)\tau(t,T_i)}$, we obtain $\frac{\partial P_d(t,T_i)}{\partial Z_{d,i}(t)} = -\tau(t, T_i) P_d(t, T_i)$. Then:

$$\begin{aligned} &\Delta_{d,i}^{cf_i} = -N\tau(t, T_i) P_d(t, T_i) \tau_x(T_{i-1}, T_i) Black[F_{x,i}(t), K, \lambda_{x,i}, \nu_{x,i}(t, T_{i-1}), \omega] = \\ &= -\tau(t, T_i) cf_i(t, T_{i-1}, T_i, K, \omega). \end{aligned} \tag{18}$$

Since again the sensitivity of caplet (floorlet) is non-zero only for the corresponding discount rate, the total sensitivity of the cap (floor) to the single forward rate will be:

$$\Delta_{d,i}^{CF} = \Delta_{d,i}^{cf_i} = -\tau(t, T_i) cf_i(t, T_{i-1}, T_i, K, \omega). \tag{19}$$

Then cap (floor) sensitivity to the parralel shift of the discount curve can be then presented as:

$$\Delta_d^{CF} = -\sum_{i=1}^{n} \tau(t, T_i) cf_i(t, T_{i-1}, T_i, K, \omega), \tag{20}$$

which was implemented in Excel as following:

```
Public Function DeltaDisc( _
    maturity As Variant, _
    forward_vec As Variant, _
    discount_yf As Variant, _
    discount_rate As Variant, _
    year_fraction_c As Variant, _
    year_fraction_v As Variant, _
    impl_vol As Variant, _
    lambda As Variant, _
    strike As Variant, _
    nominal As Variant, _
    cp As Variant) As Variant

    Dim delta, counter As Variant
    counter = maturity * 2 ' number of caplets/floorlets
    strike = strike + lambda ' shift a strike
    delta = 0
    For i = 2 To counter

      Dim black_price, forward As Variant
      forward = forward_vec(i) + lambda ' shift forward
      variance = impl_vol ^ 2 * year_fraction_v(i) ' calculate implied variance from implied volatility
      discount_vec = Exp(-discount_yf(i) * discount_rate(i))

       ' calculate price of single caplet/florlet using given formula
       black_price = Black(forward, strike, variance, cp)

       delta = delta + discount_yf(i) * year_fraction_c(i) * discount_vec * black_price

    Next i

    DeltaDisc = nominal * delta * (-1)
End Function
```

4) Vegas to forward and term volatility: For single caplet (floorlet) vega $V_x^{cf_i}$should be calculated as a derivative with respect to the corresponding forward volatility $\sigma_x(t, T_{i-1})$ for each $cf_i(t, T_{i-1}, T_i, K, \omega)$.

In case of cap (floor) we will calculate vega $V_x^{CF}$to the term volatility (common $\sigma_x(t, T_n)$ for all pillars).

First we set:

$$\begin{aligned} &\tau_{x,i} \equiv \tau_x(t; T_{i-1}), \\ &\sigma_{x,i} \equiv \sigma_x(t; T_{i-1}), \\ &\nu_{x,i} \equiv \nu_x(t; T_{i-1}) = \sigma_{x,i}^2 \tau_{x,i}. \end{aligned} \tag{21}$$

For single carpet (floorlet) we obtain:

$$\begin{aligned} V_x^{cf_i} &= \frac{\partial cf_i(t, T_{i-1}, T_i, K, \omega)}{\partial \sigma_{x,i}} = \\ &= N\omega^2 P_d(t, T_i)\tau_x(T_{i-1}, T_i)\frac{1}{\sqrt{2\pi}}[(F_{x,i}(t) + \lambda_{x,i})e^{-\frac{1}{2}(\omega d_i^+)^2}\frac{\partial d_i^+}{\partial \sigma_{x,i}} - (K + \lambda_{x,i})e^{-\frac{1}{2}(\omega d_i^-)^2}\frac{\partial d_i^-}{\partial \sigma_{x,i}}]. \end{aligned} \tag{22}$$

Then

$$\begin{aligned} d_i^+ &= \frac{\ln \frac{F_{x,i}(t)+\lambda_{x,i}}{K+\lambda_{x,i}} + \frac{1}{2}\sigma_{x,i}^2 \cdot \tau_{x,i}}{\sigma_{x,i}\sqrt{\tau_{x,i}}}, d_i^- = d_i^+ - \sigma_{x,i}\sqrt{\tau_{x,i}}, \\ \frac{\partial d_i^+}{\partial \sigma_{x,i}} &= -\frac{\ln \frac{F_{x,i}(t)+\lambda_{x,i}}{K+\lambda_{x,i}}}{\sigma_{x,i}^2\sqrt{\tau_{x,i}}} + \frac{1}{2}\sqrt{\tau_{x,i}}, \\ \frac{\partial d_i^-}{\partial \sigma_{x,i}} &= -\frac{\ln \frac{F_{x,i}(t)+\lambda_{x,i}}{K+\lambda_{x,i}}}{\sigma_{x,i}^2\sqrt{\tau_{x,i}}} - \frac{1}{2}\sqrt{\tau_{x,i}}. \end{aligned} \tag{23}$$

Since $\omega^2 = 1$, from (22), (23) we obtain:

$$\begin{aligned} V_x^{cf_i} = \frac{NP_d(t, T_i)\tau_x(T_{i-1}, T_i)}{\sqrt{2\pi}} \times \{ &\frac{\ln \frac{F_{x,i}(t)+\lambda_{x,i}}{K+\lambda_{x,i}}}{\sigma_{x,i}^2\sqrt{\tau_{x,i}}}[(F_{x,i}(t) + \lambda_{x,i})e^{-\frac{1}{2}(d_i^+)^2} - (K + \lambda_{x,i})e^{-\frac{1}{2}(\omega d_i^-)^2}] + \\ &\frac{1}{2}\sqrt{\tau_{x,i}}[(F_{x,i}(t) + \lambda_{x,i})e^{-\frac{1}{2}(d_i^+)^2} + (K + \lambda_{x,i})e^{-\frac{1}{2}(d_i^-)^2}]\}. \end{aligned} \tag{24}$$

From (14) we have the final expression:

$$V_x^{cf_i} = \frac{NP_d(t, T_i)\tau_x(T_{i-1}, T_i)}{\sqrt{2\pi}}\sqrt{\tau_{x,i}}(F_{x,i}(t) + \lambda_{x,i})e^{-\frac{1}{2}(d_i^+)^2}. \tag{25}$$

For cap (floor) denote $\sigma_x = \sigma_x(t, T_n)$ the calculations are completely the same.

$$V_x^{CF} = \frac{\partial CF(t,\mathbf{T},K,\omega)}{\partial \sigma_x} = \sum_{i=1}^{n} NP_d(t,T_i)\tau_x(T_{i-1},T_i)\frac{1}{\sqrt{2\pi}}\sqrt{\tau_{x,i}}(F_{x,i}(t)+\lambda_{x,i})e^{-\frac{1}{2}(\tilde{d}_i^+)^2},$$
$$\tilde{d}_i^+ = \frac{\ln\frac{F_{x,i}(t)+\lambda_{x,i}}{K+\lambda_{x,i}} + \frac{1}{2}\sigma_x^2\tau_{x,i}}{\sigma_x\sqrt{\tau_{x,i}}}. \tag{26}$$

Implementation of the function in Excel has been shown earlier.

C. Numerical

To compare numerical and analytical results we implemented functions to calculate all greeks numerically:

Delta forward:

```
Public Function NumericalDeltaForward( _
    maturity As Variant, _
    forward_vec As Variant, _
    discount_vec As Variant, _
    year_fraction_c As Variant, _
    year_fraction_v As Variant, _
    impl_vol As Variant, _
    lambda As Variant, _
    strike As Variant, _
    nominal As Variant, _
    cp As Variant, _
    shock As Variant) As Variant

    Dim priceh, pricel, counter As Variant
    counter = maturity * 2 ' number of caplets/florlets
    strike = strike + lambda ' shift a strike
    pricel = 0
    priceh = 0
    For i = 2 To counter

       Dim black_price, forward As Variant
       forward = forward_vec(i) + lambda + shock ' shift forward
       variance = impl_vol ^ 2 * year_fraction_v(i) ' calculate implied variance from implied volatility
        ' calculate price of single caplet/florlet using given formula
        black_price = Black(forward, strike, variance, cp)
        ' add calculated price to sum of the whole option
        priceh = priceh + year_fraction_c(i) * discount_vec(i) * black_price
    Next i
    For i = 2 To counter
       forward = forward_vec(i) + lambda - shock ' shift forward
       variance = impl_vol ^ 2 * year_fraction_v(i)  ' calculate implied variance from implied volatility
        ' calculate price of single caplet/florlet using given formula
        black_price = Black(forward, strike, variance, cp)
        ' add calculated price to sum of the whole option
        pricel = pricel + year_fraction_c(i) * discount_vec(i) * black_price
    Next i
    NumericalDeltaForward = nominal * (priceh - pricel) / 2 / shock
End Function
```

Delta discount:

```
Public Function NumericalDeltaDisc( _
   maturity As Variant, _
   forward_vec As Variant, _
   discount_yf As Variant, _
   discount_rate As Variant, _
   year_fraction_c As Variant, _
   year_fraction_v As Variant, _
   impl_vol As Variant, _
   lambda As Variant, _
   strike As Variant, _
   nominal As Variant, _
   cp As Variant, _
   shock As Variant) As Variant

   Dim priceh, pricel, counter As Variant
   Dim black_price, forward As Variant
   Dim discount_vech, discount_vecl As Variant
   counter = maturity * 2 ' number of caplets/florlets, we exclude the first rate
   strike = strike + lambda ' shift a strike
   pricel = 0
   priceh = 0
   y_frac_sum = 0
   For i = 2 To counter
      discount_vech = Exp(-discount_yf(i) * (discount_rate(i) + shock))
      discount_vecl = Exp(-discount_yf(i) * (discount_rate(i) - shock))
      forward = forward_vec(i) + lambda ' shift forward
      'Debug.Print year_fraction(i)
      variance = impl_vol ^ 2 * year_fraction_v(i)  ' calculate implied variance from implied volatility
       ' calculate price of single caplet/florlet using given formula
       black_price = Black(forward, strike, variance, cp)
       ' add calculated price to sum of the whole option
       priceh = priceh + year_fraction_c(i) * discount_vech * black_price
       pricel = pricel + year_fraction_c(i) * discount_vecl * black_price
   Next i
   NumericalDeltaDisc = nominal * (priceh - pricel) / (2 * shock)

End Function
```

Vegas were calculated in the spreadshit.

Comparing results shown pretty good accuracy which is getting better with decreasing "shock" parameter. For deltas shock was chosen to be 1 bps and 0.1 bps, while for volatility it was 100 bps and 10 bps. The table illustrates the differences between numerical results with smaller shock and analytical solution.

Delta forward:

Heat map

% differences between market and our prices

Parallel deltas (forward curve)
Floors
Analytical / numerical

	STK	ATM	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	0,00%	0,19%	0,08%	0,01%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
18M	-0,4	0,00%	0,06%	0,01%	-0,01%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
2Y	-0,4	0,00%	0,02%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
3Y	-0,3	0,00%	0,00%	-0,01%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
4Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
5Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
6Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
7Y	-0,1	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
8Y	0	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
9Y	0	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
10Y	0,04	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
12Y	0,16	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
15Y	0,29	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
20Y	0,41	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
25Y	0,45	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
30Y	0,44	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%

Parallel deltas (forward curve)
Caps
Analytical / numerical

	STK	ATM	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%	0,00%	0,02%	0,11%	0,25%	0,27%	0,30%	0,35%
18M	-0,4	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,04%	0,16%	0,28%	0,32%	0,37%
2Y	-0,4	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,01%	0,08%	0,17%	0,34%	0,38%
3Y	-0,3	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,01%	0,03%	0,11%	0,30%
4Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,02%	0,10%
5Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,05%
6Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,03%
7Y	-0,1	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,01%
8Y	0	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,01%
9Y	0	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
10Y	0,04	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
12Y	0,16	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
15Y	0,29	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
20Y	0,41	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
25Y	0,45	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
30Y	0,44	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%

Delta discount:

Parallel deltas (discount curve)
Floors
Analytical / numerical

	STK	ATM	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
18M	-0,4	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
2Y	-0,4	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
3Y	-0,3	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
4Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
5Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
6Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
7Y	-0,1	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
8Y	0	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
9Y	0	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
10Y	0,04	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
12Y	0,16	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
15Y	0,29	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
20Y	0,41	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
25Y	0,45	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
30Y	0,44	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%

Parallel deltas (discount curve)
Cap
Analytical / numerical

	STK	ATM	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
18M	-0,4	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
2Y	-0,4	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
3Y	-0,3	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
4Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
5Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
6Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
7Y	-0,1	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
8Y	0	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
9Y	0	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
10Y	0,04	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
12Y	0,16	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
15Y	0,29	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
20Y	0,41	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
25Y	0,45	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
30Y	0,44	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%

Vega:

Vegas
Caps / floors
Analytical / numerical

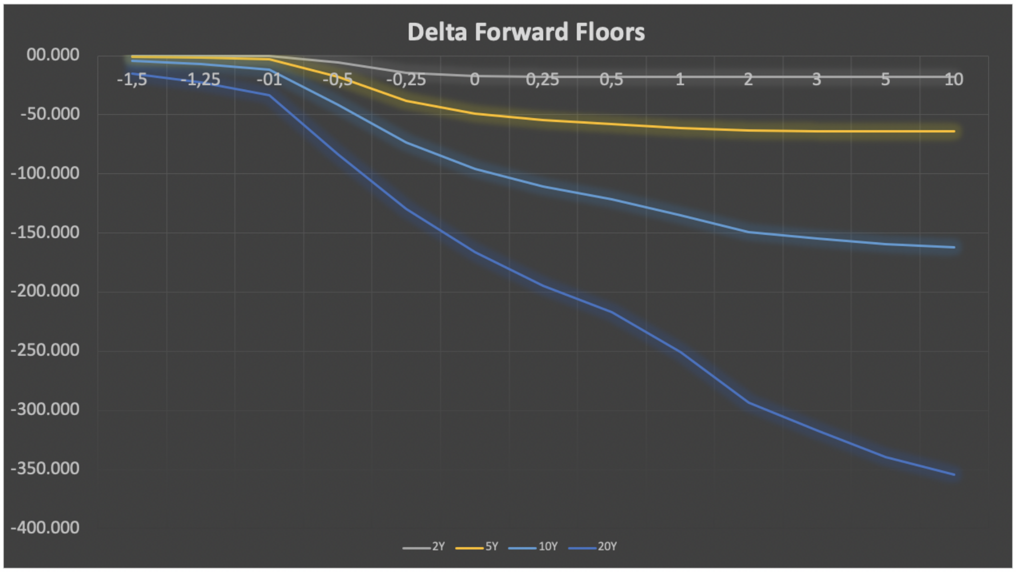
	STK	ATM	-1,5	-1,25	-1,0	-0,5	-0,25	0	0,25	0,5	1	2	3	5	10
1Y	-0,4	0,00%	0,05%	-0,03%	-0,05%	0,00%	0,01%	-0,02%	-0,04%	-0,03%	0,01%	0,11%	0,16%	0,21%	0,27%
18M	-0,4	0,00%	-0,01%	-0,03%	-0,03%	0,00%	0,01%	-0,01%	-0,02%	-0,03%	-0,03%	0,04%	0,13%	0,20%	0,27%
2Y	-0,4	0,00%	-0,02%	-0,02%	-0,01%	0,00%	0,00%	0,00%	-0,01%	-0,02%	-0,03%	0,00%	0,06%	0,17%	0,26%
3Y	-0,3	0,00%	-0,01%	-0,01%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%	-0,02%	0,00%	0,05%	0,18%
4Y	-0,2	0,00%	-0,01%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%	-0,01%	0,00%	0,05%
5Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%	-0,01%	0,02%
6Y	-0,2	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%	0,01%
7Y	-0,1	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%	0,00%
8Y	0	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%	-0,01%
9Y	0	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%
10Y	0,04	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%
12Y	0,16	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%
15Y	0,29	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	-0,01%
20Y	0,41	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
25Y	0,45	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
30Y	0,44	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%

Some small differences for greeks (those which are themselves very small numbers 0.0001) during numerical calculation can be explained by rounding error in Excel (since we are using finite differences).

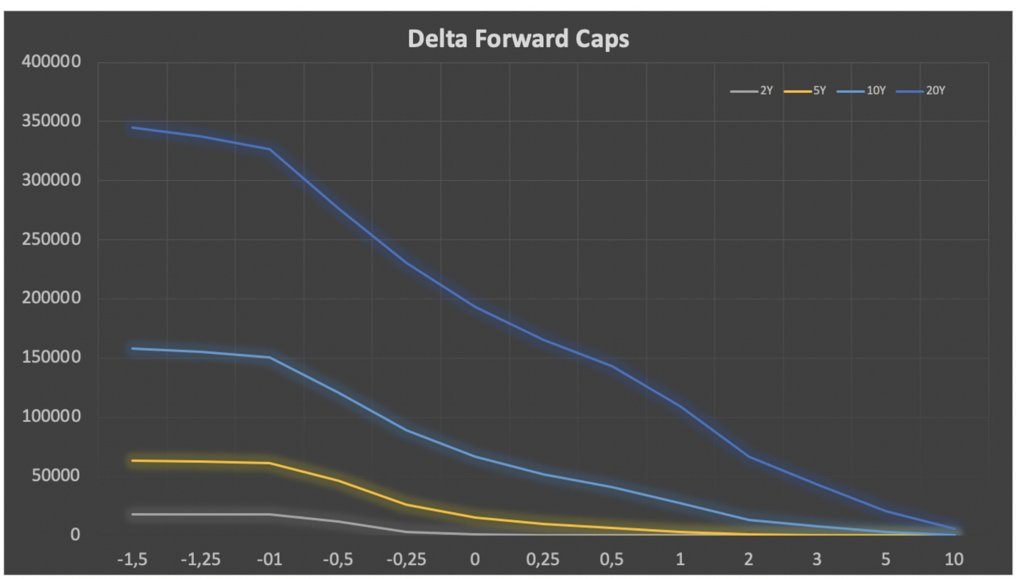
D. Greek plots

We ploted greeks for 2Y, 5Y, 10Y, 20Y caps and floors. The first quite intuitive observation is that caps / floors with larger maturities have bigger sensitivities, which logically corresponds to less definite payoffs further in the future.

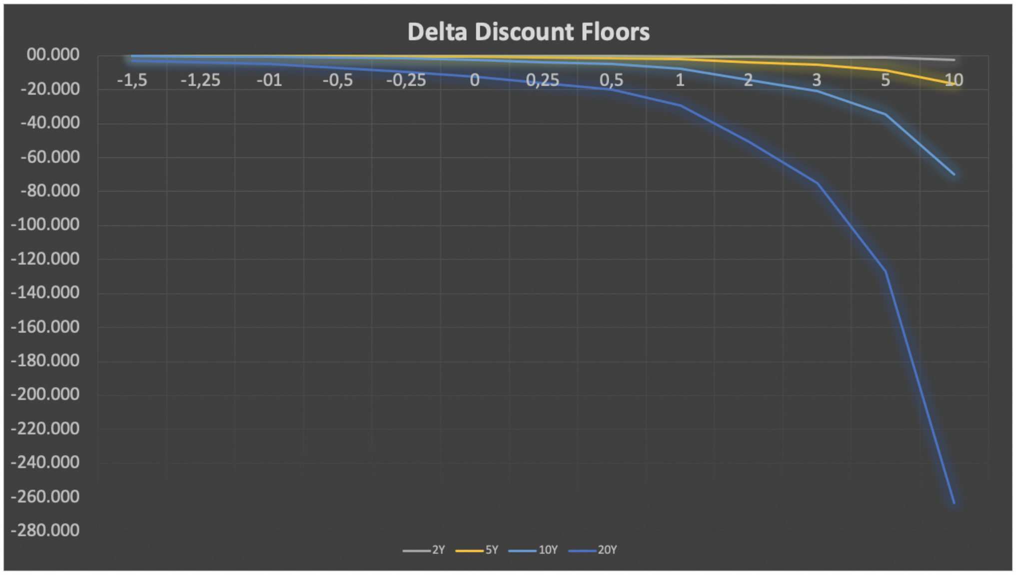
Deltas forward for floors depend on forward rates negatively (as logically follows from intuition) and the derivative delta is as big (in absolute value) as deep the floor is in the money (since in the money floors tend to be similar to simple swaps and reflect underlying rates):

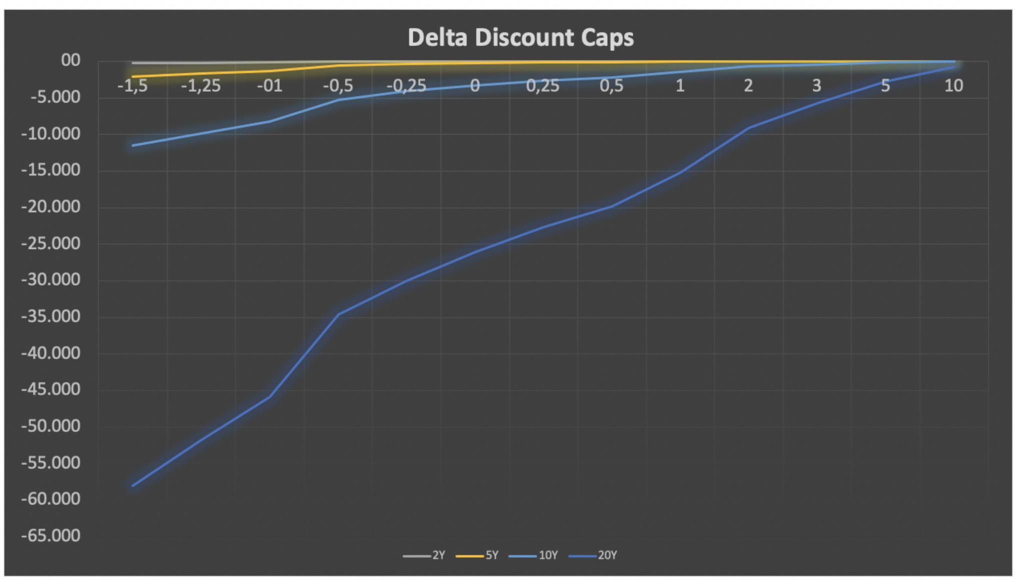


For caps the trend is similar while sensitivities are always positive.

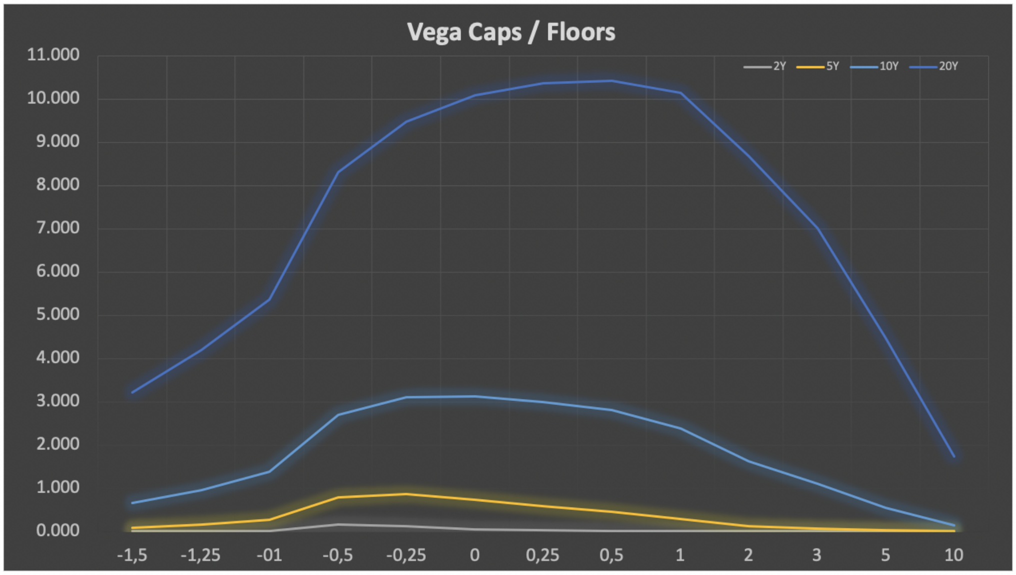


Delta to the discount curve will be bigger (in absolute value) for the contracts with bigger payoff further in the future, so for higher strikes floors and lower strikes caps. Sign will be always negative since bigger discount rate decreases the present value of future payoffs.





Vegas for caps and floors are the same, are always positive (since bigger volatility implies bigger price) and logically bigger in the area close to ATM. Caps/floors deeply out of money cost little and then their price change is smaller while those deeply in the money tend to reflect forward rates and less depend on volatility.


Vega Caps / Floors
2Y
5Y
10Y
20Y
11.000
10.000
9.000
8.000
7.000
6.000
5.000
4.000
3.000
2.000
1.000
0.000
-1,5
-1,25
-01
-0,5
-0,25
0
0,25
0,5
1
2
3
5
10