Some notes on tensor notation and Cartesian Tensors.

Tensor notation is extremely convenient A 01
we only introduce the absolute basics here.

The components of a vector may be indicated by an index
So $[\underset{\sim}{V}]_i = V_i$; i.e. the i^{th} component of vector V in a particular coordinate system is given as V_i.

In any equation involving indices, there must be "indicial homogenity".
i.e. $V_i + U_j = W_k$ is total nonsense!
When indices are repeated it denotes a summation.
Consider a Cartesian coordinate system. The dot product of two vectors is given by $V_i U_i = \sum_i V_i U_i = V_1 U_1 + V_2 U_2 + V_3 U_3$.
This is called a contraction. Note that the result is a scalar.
$V_i U_i = V_k U_k$. Thus the summed over index is really a "dummy index".
We now introduce two very important tensors
1. the Kronecker delta $\delta_{ij} = 1$ when $i = j$
$= 0$ otherwise

2. the alternator or Levi-Civita symbol ε_{ijk} is defined as
$\varepsilon_{ijk} = 1$ when $i \neq j \neq k$ & i, j, k are a cyclic permutation
$= 0$ ~~~~ when $i = j$; $i = k$ or $j = k$
$\cancel{\varepsilon_{123} =}$ $= -1$ when $i \neq j \neq k$ & i, j, k are not a cyclic permutation

$\varepsilon_{123} = 1$, $\varepsilon_{213} = -1$, $\varepsilon_{112} = 0$
$\varepsilon_{231} = 1$, $\cancel{\varepsilon_{213} = 1}$ $\varepsilon_{132} = -1$

The cross product of 2 vectors is given by
$$[\underset{\sim}{u} \times \underset{\sim}{v}]_i = \varepsilon_{ijk} u_j v_k$$

Thus $\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = a_i \varepsilon_{ijk} b_j c_k$.
Note that in 3D $\delta_{ii} = 3$; $\delta_{ij} \varepsilon_{ijk} = 0$
Another useful property is $\boxed{\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}}$

The curl of a vector field may be written in tensor notation as

$$\cancel{\nabla \times v} \quad (\text{curl } \underset{\sim}{v})_i = \varepsilon_{ijk}\, \partial_j v_k .$$

by ∂_j we mean $\dfrac{\partial (\ )}{\partial x_j}$ where x_j is the j^{th} coordinate

Thus $\text{div}(\underset{\sim}{v}) = \partial_j v_j$

<u>Exercises</u>:

1. Prove that $\delta_{ij}\, \varepsilon_{ijk} = 0$
2. Find the expression for curl (curl ($\underset{\sim}{v}$))
3. What is div curl ($\underset{\sim}{v}$)? Prove the result with tensor notation.
4. P.T. $v_j \partial_j v_i = \partial_i (\frac{1}{2} v_j v_j) + \varepsilon_{ijk}\, w_j v_k$

   where $w_j = \varepsilon_{jlm}\, \partial_l v_m$

   what do the terms on LHS & RHS mean in Vector notation?
5. What is $\varepsilon_{ijk}\, \varepsilon_{ilm}$?

<u>Note</u>: when using indicial notation make sure that no term has the same index repeated more than twice. For example $v_i u_i w_i$ is nonsensical and confusing.

The above has only exposed you to the tip of the iceberg. There is a lot more that tensor calculus has to offer. The above has only exposed you to the basics of Cartesian tensors and should suffice for the purposes of this course.

There are plenty of good references on the subject. <u>Hobson, Riley & Bence</u> has a good chapter. <u>Karamcheti</u> ~~has a~~ Krishnamurthy has a book on the topic. <u>Barry Spain's</u> Tensor Calculus is a nice book though advanced. <u>I S. Sokolnikoff's</u> book on the subject is a classic.

The vortex blob method.

Consider an incompressible fluid with a constant kinematic viscosity $\nu = \mu/\rho$. The fluid is Newtonian. A03

The governing equations for this are

$$\operatorname{div} \vec{V} = 0 \quad ; \text{ or } ; \quad \partial_i v_i = 0 \;\rightarrow \text{mass cons.}$$

&

$$\partial_t v_i + v_j \partial_j v_i = -\frac{1}{\rho}\partial_i p + \nu \partial_j \partial_j v_i \;\rightarrow \text{mom.-cons.}$$

These represent 4 P.D.E's. In this case the temperature equation is decoupled from the mass & momentum equations.

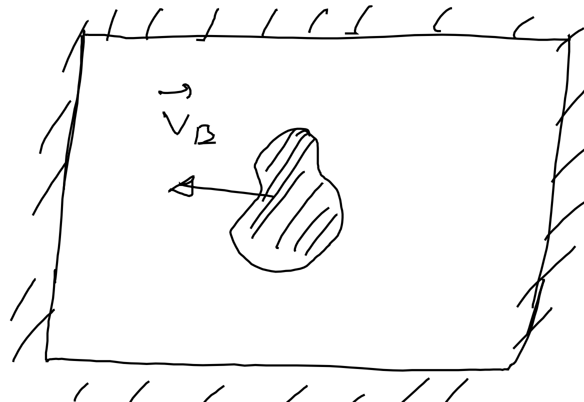
We have 4 unknowns (v_i, p) & 4 equations.

In addition we may specify the B.C.'s to be

$$v_i t_i = V_{Bt}$$

$$v_i n_i = V_{Bn}$$

where t_i, n_i are components of the tangent & normal on a solid body. V_{Bt} & V_{Bn} are the tangential & normal components of the body velocity. Additionally we may specify the B.C. at infinity. Note that we have implicitly assumed that we are solving for the case where a body is immersed in a fluid of infinite extent. This can easily be extended to handle cases like shown below, where the body is immersed in a fluid which is itself placed in a stationary or moving vessel.



Regardless, it is clear that the PDE's above along with the supplied B.C's + an initial condition specifying the unknowns at $t=0$ say are sufficient to find a unique solution to the problem.

This is called a primitive variable formulation.

The vortex method utilizes a velocity vorticity formulation of the governing PDE's. This enables for an elegant Lagrangian method of solution. To do this consider the curl of the momentum eqn.

$$\text{curl}\left(\partial_t v_i + v_j \partial_j v_i = -\frac{1}{\rho}\partial_i p + \nu \partial_j \partial_j v_i \right)$$

This leaves us with

$$\partial_t \omega_i + \text{curl}(v_j \partial_j v_i) = \nu \partial_j \partial_j \omega_i \longrightarrow (1)$$

where $\omega_j = \varepsilon_{jik} \partial_i v_k$.

To simplify the second term on the LHS consider the identity

$$v_j \partial_j v_i = \partial_i\left(\tfrac{1}{2} v_j v_j\right) + \varepsilon_{ijk} \omega_j v_k .$$

[You should be able to show that this is true easily $, to do this expand ω_j from its definition and use the ~~definition~~ relation

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{lk}$$ and expand]

Taking the curl of the above we have.

$$\text{curl}(v_j \partial_j v_i) = \varepsilon_{ijk} \partial_j (\varepsilon_{klm} \omega_l v_m)$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(v_m \partial_j \omega_l + \omega_l \partial_j v_m)$$

$$= v_j \partial_j \omega_i + \underbrace{\omega_i \partial_j v_j}_{\substack{= 0 \text{ (mass cons)} \\ \partial_j v_j = 0}} - \underbrace{v_i \partial_j \omega_j}_{= 0 \text{ (div curl } v \equiv 0)} - \omega_j \partial_j v_i$$

$$= v_j \partial_j \omega_i - \omega_j \partial_j v_i$$

$\therefore$ (1) becomes $\partial_t \omega_i + v_j \partial_j \omega_i = \omega_j \partial_j v_i + \nu \partial_j \partial_j \omega_i$

$$\Rightarrow \boxed{\frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega}\cdot\text{grad})\underset{\sim}{v} + \nu \nabla^2 \underset{\sim}{\omega}} \longrightarrow (2)$$

This is the N.S. eqn in vorticity ~~form~~ - velocity form.

If we look at equation ②, the LHS represents the material derivative of vorticity. [Note: please read up on any elementary text on vorticity and what it physically means etc. ~~If~~ Any good fluid mechanics text will do.]

The second term on the RHS represents the molecular diffusion of the vorticity.

The first term on the RHS represents the vorticity stretching term.

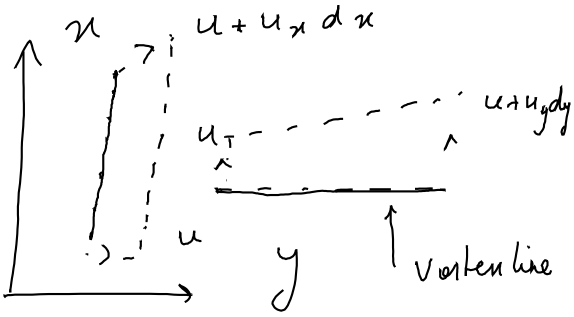
To see why this is called "vortex stretching" consider the 'x' component of the PDE ②.

$$\frac{D\omega_x}{Dt} = \omega_x \partial_x u + \omega_y \partial_y u + \omega_z \partial_z u + \nu \nabla^2 \omega_x$$

Look at the first term on the R.H.S. If $\partial_x u$ is +ve along ~~ω_x~~ the direction of x with ω_x itself +ve then clearly along 'x' if we look at a vortex line, [a vortex line is one tangent to which the vorticity lies at all points of the line, it is similar to a stream line.] then that vortex line will be streched by the velocity gradient along ~~that~~ the 'x' direction. Imagine what would happen to a material line drawn on the fluid along 'x' with a gradient ~~along x~~ of 'u' along 'x'; clearly the line will be streched.

Similarly $\omega_y \partial_y u$ represents an elongation of the vortex lines along 'y' ~~along the~~ elongated along the 'x' direction.

This is illustrated (somewhat!) in the figure shown on the right.



Thus the equation ② governs the change of vorticity in time.

The B.C's remain the same; No penetration on solid walls and no slip on solid walls.

The initial condition on V is now expressed as an initially specified ω distribution.

Note that the pressure has vanished. Thus if we solve for the vorticity alone, the problem has been solved.

Now, given the new PDE we must solve it. A 06.

Let us first simplify it.

Consider a 2D flow. In such a flow, ω_z alone exists. ~~$\left(v \cdot \frac{\partial \omega_z}{\partial z}\right)$~~

Both ω_x & ω_y are not present. ~~Similarly~~ By ω_x we mean the component of ω along 'x'. The component of V along 'z' is negligible and gradients of V along z are insignificant therefore

$$\omega_j \partial_j v_i = 0 \quad \left[\text{since } \omega_z \partial_z u \text{ or } \omega_z \partial_z V = 0\right]$$

thus the stretching term vanishes. By the way, this is both the blessing and curse of 2D flow simulations!

Thus <u>in 2D</u> we have

$$\frac{D\omega}{Dt} = \nu \nabla^2 \omega \quad \left[\text{where } \omega \text{ is the component of } \omega \text{ along '}z\text{' axis}\right]$$

Thus in 2D the change of vorticity in the flow is only due to viscosity.

Now consider an inviscid flow [or a fluid with negligible viscosity].

we get (in 2D)

$$\frac{D\omega}{Dt} = 0$$

This is an extremely simple equation and is the starting point for simulations with the vortex method. The equation says that in 2D, incompressible, inviscid (ideal!) fluids that the vorticity is a <u>material property</u>. I.e. the vorticity of the fluid flows along with the fluid. This is called the "<u>advection equation</u>" since the equation governs the advection of the vorticity.

Let us now see how this equation can be solved.

The given problem is, given ~~$\omega(\vec{r}, t=0)$~~ $\omega(\vec{r}, t=0)$ solve $\frac{D\omega}{Dt} = 0$. such that there is no penetration on solid walls.

[Note: we cannot also specify a no-slip B.C. here!]

AD7

To further simplify consider the case when there is no solid body. That is, there is just an infinite mass of fluid. We'll get back to boundaries later.

We are given an initial distribution of w at $t=0$ and we must now "advect" this vorticity in time according to $\frac{Dw}{Dt}=0$.

A very simple way of doing this is to break up the $w(\vec{r},0)$ distribution into many small particles labelled by the index 'i' say to carry the vorticity in time. Let the position of each of these particles be given as $\vec{r}_i(t)$. Let each particle carry some initial vorticity times the area it occupies, say $w_i \cdot \Delta x \Delta y$ or $w_i h^2$ (for constant $\Delta x, \Delta y$). Then clearly $\frac{Dw}{Dt}=0$ implies that the vorticity carried by these particles does not change in time.

Clearly the particles must move with the local velocity V if this is to be true. Therefore

$$\frac{d\vec{r}_i}{dt} = \vec{V}_i$$

, where V_i is the velocity of the flow at the i^{th} particle's position.

Thus, if each particle moves according to $\frac{d\vec{r}_i}{dt}=\vec{V}_i$ and continues to carry the original vorticity w_i, the PDE has been solved! Therefore the solution of the PDE has been reduced to solving a system of ODE's given an initial condition.

The only thing we dont yet know to find is given ~~an initial~~ a $w(\vec{r},t)$ say, how do we find $\vec{V}_i$ at each particle. If we know this, clearly we can easily solve our PDE using the ODE.

Clearly $\vec{V}$ & w are related as $\text{curl}(\vec{V})\cdot\hat{k}=w$ where $\hat{k}$ is the unit vector along 'z' axis. We can use this to find $\vec{V}$ given w.

To do this we first introduce the streamfunction.

Recall from elementary fluid mechanics that [A08]

$$(u, v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) = \left(\frac{\partial}{\partial y}, \frac{-\partial}{\partial x}\right)\psi = (\partial_y, -\partial_x)\psi \quad (3)$$

Thus it is easy to see that

$$\nabla^2 \psi = -\omega.$$

Given ω we can find ψ and from (3) above we can solve for $\vec{v}$ (or (u,v)).

We may use a Green's function to solve the above PDE to find ψ ~~given~~ given ω. Consider a linear differential operator $\mathcal{L}$ (like the ∇^2 operator) if we have $\mathcal{L}\psi = f$ then the Greens function of $\mathcal{L}$ can be used to solve for ψ. The Greens function is defined as that function G such that $\mathcal{L}\, G(x - x_0) = \delta(x - x_0)$ [in 1D]

replace x, x_0 by suitable equivalents in multiple dimensions.

Given this consider $\mathcal{L}\psi = f$. Now convolve this with G to get

$$G * \mathcal{L}\psi = G * f$$

The LHS is $\int G(x - x')\, \mathcal{L}\psi(x')\, dx' = \int G(x')\, \mathcal{L}\psi(x - x')\, dx'$

It can be shown that $\frac{d}{dx}(f * g) = \frac{df}{dx} * g = f * \frac{dg}{dx}$ and likewise for partial differentiation. Thus

$$\int G(x')\left(\mathcal{L}\psi(x - x')\right) dx' = \int \left(\mathcal{L}\, G(x - x')\right) \psi(x')\, dx'$$

From the definition of the Greens function we have $\mathcal{L}G(x - x') = \delta(x - x')$

$$\therefore \int \delta(x - x')\, \psi(x')\, dx' = G * f.$$

$$\Rightarrow \boxed{\psi(x) = G * f.}$$

This can be extended to 2D and our ∇^2 operator in 2 & 3 dimensions. Thus if we know the Greens function for the $-\nabla^2$ operator we can "invert" the $\nabla^2\psi = -\omega$ equation and find $\underset{\sim}{V}$ from ω through ψ.

Consider the 2D case. $\nabla^2 G(\underset{\sim}{r}, \underset{\sim}{r_0}) = \delta(r - r_0)$

Since $\delta(\bar{r} - \bar{r}_0)$ is spherically symmetric we expect a symmetric solution where $G = G(r)$ & not a fn of θ.

Consider $\nabla^2 G = -\delta(\underset{\sim}{r} - \underset{\sim}{r_0})$

Integrate over the volume on both sides with ~~the~~ volume containing the point $\underset{\sim}{r_0}$. We get

$$\int \nabla^2 G(\underset{\sim}{r}, r_0)\, dV = -\int \delta(\underset{\sim}{r} - \underset{\sim}{r_0})\, dV$$
$$= -1$$

From Gauss divergence theorem for the LHS we have $\oint \underset{\sim}{\nabla} G(\underset{\sim}{r}, r_0) \cdot \hat{n}\, ds$

Taking a circular contour and assuming radial symmetry we get

$$\oint \underset{\sim}{\nabla} G(r, r_0) \cdot \hat{n}\, ds = \int_0^{2\pi} \frac{\partial G}{\partial r} |\underset{\sim}{r} - \underset{\sim}{r_0}|\, d\theta = 2\pi \frac{\partial G}{\partial r} |\underset{\sim}{r} - \underset{\sim}{r_0}|$$

$$\therefore \frac{\partial G}{\partial r} = -\frac{1}{2\pi |\underset{\sim}{r} - \underset{\sim}{r_0}|}$$

Note: this is all for 2D.

$$\therefore G(r, r_0) = -\frac{1}{2\pi} \ln |\underset{\sim}{r} - \underset{\sim}{r_0}|$$

Thus the stream function corresponding to the Dirac delta ~~is given~~ placed at the origin is $G(r) = -\frac{1}{2\pi} \log r$.

Therefore, for an arbitrary vorticity distribution we have

$$\psi = G * \omega = -\iint \frac{1}{2\pi} \ln |\underset{\sim}{r} - \underset{\sim}{r'}|\, \omega(r')\, dx'\, dy'$$

~~The~~ Using this ψ we may calculate $\underset{\sim}{V}$ from equation (3) [Pg. A08]

Let us step back for a moment and write the Greens function in complex co-ordinates for convenience.

The complex potential Φ is defined as $\Phi = \phi + i\psi$

where ϕ is the velocity potential and ψ the stream function.

the complex velocity $V(z) = u - iv = \frac{d\Phi}{dz} = \frac{\partial \Phi}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$

In complex analysis, an analytic function is one whose derivative is the same along all directions. Thus $\frac{d\Phi}{dz} = \frac{\partial \Phi}{\partial x} = -i\frac{\partial \Phi}{\partial y}$ $[\because z = x + iy]$

Consider a point vortex in 2D in polar co-ordinates at origin. [A 10]
we know that $u_r = 0$ and $u_\theta = \frac{\Gamma}{2\pi r}$ (anti-clockwise +ve)

Thus to find the potential, we have

$$\frac{\partial \phi}{\partial r} = 0 \Rightarrow \phi = f(\theta) \; ; \; \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r} \Rightarrow \phi = \frac{\Gamma}{2\pi}\theta + \text{Const}$$

w.l.g. we can say $\boxed{\phi = \frac{\Gamma}{2\pi}\theta.}$

To find ψ we have

$$\frac{\partial \psi}{\partial r} = -\frac{\Gamma}{2\pi r} \quad \text{and} \quad \frac{\partial \psi}{\partial \theta} = 0 \Rightarrow \psi = f(r)$$

$$\therefore \psi = -\frac{\Gamma}{2\pi} \ln r$$

$$\therefore \Phi = \frac{\Gamma}{2\pi}(\theta - i \ln r)$$

Now $z = re^{i\theta}$

$\Rightarrow \ln z = \ln r + i\theta$

therefore $\Phi = \frac{-i\Gamma}{2\pi} \ln z$ is the complex potential of a point vortex at the origin.

This is very convenient since

$$u - iv = \frac{d\Phi}{dz} = \frac{-i\Gamma}{2\pi z}$$

For a point vortex at point z_0 we clearly have

$$\boxed{u - iv = \frac{-i\Gamma}{2\pi (z - z_0)}}$$

Thus we can use complex notation to simplify our notation in 2D.

Anyway, now that we have found G in $\hat{r}$ we can clearly see that $G(z) = -\frac{1}{2\pi} \ln |z|$

Let us find (u, v) from equation (3)

$$(u, v) = \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)\psi = \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right) \iint G(z - z')\, \omega(z')\, dx'dy'$$

since $G = \frac{-1}{2\pi} \ln r$

$r = \sqrt{x^2 + y^2}$

$$\frac{\partial G}{\partial y} = \frac{\partial G}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial G}{\partial \theta}\frac{\partial \theta}{\partial y} = 0$$

$$= \frac{-1}{2\pi r} \cdot \frac{y}{r}$$

Similarly $\frac{\partial G}{\partial x} = \frac{-x}{2\pi r^2}$

$\therefore \quad (u, v) = \frac{-1}{2\pi} \iint \underbrace{\left(\frac{y-y'}{|r-r'|^2}, \frac{x-x'}{|r-r'|^2} \right)}_{K(x,y) \text{ or } K(z)} \omega(x', y')\, dx' dy'$

We may simplify and write this as

$(u, v) = \iint K(z - z')\, \omega(z')\, dx' dy'$

where $K(z) = -\frac{1}{2\pi r^2}(y - x) = \left(\frac{-i}{2\pi (z)} \right)^*$

where $(\ )^*$ indicates complex conjugation, $z^* = x - iy$.

Now, we may discretize the above integral into particles carrying the vorticity & each particle being a <u>point vortex</u>.
$K(z)$ is called the <u>Cauchy velocity kernel</u>. Thus we have

$\omega(z) = \sum_i \omega(z_i)\, \Delta x \Delta y\, \delta(z - z_i)$;

$\omega(z_i)\, \Delta x \Delta y$ is nothing but the circulation of that element of vorticity

therefore $\boxed{\omega(z) = \sum_i \Gamma_i\, \delta(z - z_i).}$

The velocity of this is

$\boxed{V(z) = \sum_i \Gamma_i\, K(z - z_i)}$

Now, given this it is easy to see that we can integrate $\frac{D\omega}{Dt} = 0$.
Since for each vortex we have $V(z_i) = \sum_{j,\, i \neq j} \Gamma_j\, K(z_i - z_j)$.

We must enforce $i \neq j$ since $K(z)$ is singular when $z = 0$.
Thus we can solve the ODE $\frac{dz_i}{dt} = V_i$ and solve $\frac{D\omega}{Dt} = 0$.

There is however a problem. $w(z)$ is really a continuous & perhaps differentiable function and we have represented it or rather discretized it into Dirac delta pulses. This leads to the singularity in the vorticity & velocity fields. This situation may be improved by using an approximate Dirac delta function $f_\delta(z)$ which tends to a Dirac delta as the parameter δ tends to 0. i.e. $\lim_{\delta \to 0} f_\delta(z) \to \delta(z)$

δ is called the core radius or smoothing radius.

Thus given the ~~core radius~~ function f_δ, how are we to calculate w & v?

Clearly $w(z) = \sum_j \Gamma_j f_\delta(z - z_j)$

But what about $v(z)$?

Since $\nabla^2 \psi = -w$ and ∇^2 is linear we have

$\nabla^2 \psi_\delta = -f_\delta$

or $\nabla^2 G_\delta = -f_\delta$

It is obvious that the solution is $G_\delta = G * f_\delta$.

Thus we can find the "Greens function" for the f_δ & thereby obtain $v_\delta(z)$ by differentiating this.

To do this we first assume that f_δ is symmetric radially. Thus G_δ is symmetric (G is symmetric) & we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} G_\delta\right) = -f_\delta(r)$$

$$\therefore \frac{\partial}{\partial r} G_\delta = -\frac{1}{r}\int_0^r r' f_\delta(r')\, dr'$$

Now, to get K_δ we have $K_\delta = \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right) G_\delta = \left(\frac{y}{r}, \frac{-x}{r}\right)\frac{\partial G_\delta}{\partial r}$

$$\therefore K_\delta = -(y, -x)\frac{1}{2\pi r^2}\underbrace{2\pi \int_0^r r' f_\delta(r')\, dr'}_{k(r)} = K(z)\, k(r)$$

There are different types of blobs based on the f_δ chosen. Here are a few examples

Chorin Blob :	$f_\delta(r) = \frac{1}{2\pi r \delta}$ $r < \delta$; 0 $r \geq \delta$	$k(r) = r/\delta$ $r < \delta$; 1 $r \geq \delta$
Saffman Blob	$f_\delta(r) = \frac{1}{\pi \delta^2}$ $r < \delta$; 0 $r > \delta$	$k(r) = (r/\delta)^2$ $r < \delta$; 1 $r \geq \delta$
Krasny	$f_\delta(r) = \frac{\delta^2}{\pi(r^2+\delta^2)^2}$	$k(r) = \frac{r^2}{r^2+\delta^2}$
Beale Majda 2nd order	$f_\delta(r) = \frac{e^{-r^2/\delta^2}}{\pi \delta^2}$	$k(r) = 1 - e^{-r^2/\delta^2}$

It is relatively straightforward to plot the resulting velocity as a function of r for each kind of blob.

Now, given an ω distribution we may write an approximate $\omega(z)$ as

$$\omega(z) = \sum_j \Gamma_j f_\delta(z - z_j)$$

The velocity $V_\delta(z) = \sum_j \Gamma_j k_\delta(z - z_j)$.

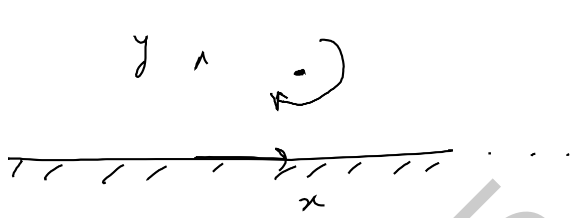
Given this we may solve the ODE $\frac{dz_i}{dt} = V_\delta(z_i)$ and solve the Euler equations in incompressible flow.

Boundary conditions

Thus far we have assumed that there is an infinite mass of fluid with no boundaries, solid or otherwise. The only B.C. was that $\lim_{r\to\infty} V(r) \to 0$ or to a constant in the case of a free-stream.

Now let us consider the case when we have a solid boundary. In general we can come up with complicated geometries but for simplicity we first consider the case of external flow past a body.

Consider first the case of flow in the upper half plane. Take a point vortex placed in this domain.



Clearly at $y = 0$ the streamfunction ψ must be ~~zero~~ a constant.
Without any special treatment a point vortex in this region would not satisfy the boundary condition.
To solve this we may add a mirror point vortex with opposite sign. i.e. if z is the position & Γ the strength we add another point vortex at z^* with $-\Gamma$ strength. This would make the $y=0$ line a constant (zero) streamline, thus solving the B.C.
Therefore the Greens function in this case has changed from that of the free-space Greens function.
This approach works ~~for~~ and can be made to work for more complex geometries by using conformal transformations.
However this method is hard to implement for truly generic B.C's.
To handle those we may need to use a technique like the panel method.
To see how this works note that if we use the free-space Greens function we have

$$V_w(z) = \sum_j \Gamma_j K(z - z_j) \quad (\text{as } K_g(z - z_j))$$

i.e. this is the velocity due to the vorticity in free space.
This doesn't satisfy the B.C. We may add a potential velocity to the velocity s.t. w doesn't change & thus

$$V(z) = V_w(z) + \nabla\phi$$

since $\text{curl}(\text{grad}\,\phi) = 0$ w is unaffected.
Now this $V(z)$ must satisfy mass conservation & therefore

$$\text{div}(V) = 0 \Rightarrow \text{div}(V_w) + \nabla^2\phi = 0$$

$\text{div}(V_w) = 0$ by construction (prove this!)

$\therefore$ we must have that $\nabla^2\phi = 0$

In addition we have the B.C. that $\vec{V}\cdot\hat{e}_n = \vec{V}_B\cdot\hat{e}_n$ [$\vec{V}_B$ is body velocity & $\hat{e}_n$ is normal vector on body]

Therefore we basically have to find a solution for $\nabla^2 \phi = 0$ such that $\vec{V} \cdot \hat{e}_n = \vec{V}_B \cdot \hat{e}_n$

$$\Rightarrow \quad \vec{V}_w(z) \cdot \hat{e}_n + \frac{\partial \phi}{\partial n} = \vec{V}_B \cdot \hat{e}_n$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial n} = \vec{V}_B \cdot \hat{e}_n - \vec{V}_w \cdot \hat{e}_n$$

This is basically the solution to the Laplace equation with Neumann type B.C.

The Laplace equation is linear so we may assemble a set of solutions ϕ_1, ϕ_2 etc. such that $\phi_1 + \phi_2 + \dots + \phi_n$ ~~solve~~ satisfy the B.C.

To do this we may use a panel method.

The basic idea for the panel method is to

* discretize the geometry into linear or curved elements called panels.
* distribute some singularity on the surface of the panels. This may be in the form of sources, doublets or vorticity.
* Pick a control point on each panel.
* Satisfy the B.C. on the control points.
* Use the B.C condition to setup a system of equations to solve for the unknown strengths.

As a concrete example, take each panel to be flat (linear). Place a point vortex or source of strength Γ_i or q_i on each. The panels are numbered 'i'. Now on each control point z_{c_i} say we solve for the B.C like so. Take the case of a point vortex placed at z_{c_i}

$$V_{panels} = \sum_j \frac{-i\Gamma_j}{2\pi (z - z_j)} \quad \therefore \quad V_{panel}(z_k) = \sum_{\substack{j \\ j \neq k}} \frac{-i\Gamma_j}{2\pi (z_k - z_j)}$$

To solve B.C we have

$$V_{panel}(z_k) \cdot \hat{e}_{n_k} = \vec{V}_B(z_k) \cdot \hat{e}_{n_k} - V_w(z_k) \cdot \hat{e}_{n_k}$$

where $\hat{e}_{n_k}$ is $\hat{e}_n$ at z_k & $\vec{V}_B(z_k)$ is similarly velocity at z_k.

$$\therefore \left(\sum_{\substack{j \\ j \neq k}} \frac{-i\Gamma_j}{2\pi(z_k - z_j)} \right) \cdot \hat{en}_k = \vec{V}_B(z_k)\cdot \hat{en}_k - \vec{V}_\infty(z_k)\cdot \hat{en}_k$$

for all panels $k = 1, \ldots N$.

This is clearly a system of linear equations in Γ_j. We may write this in matrix form as.

$$\left(A_{kj}\,\Gamma_j\right)\cdot \hat{e}_k = \left(\vec{V}_B - \vec{V}_\infty(z_k)\right)\cdot \hat{e}_k$$

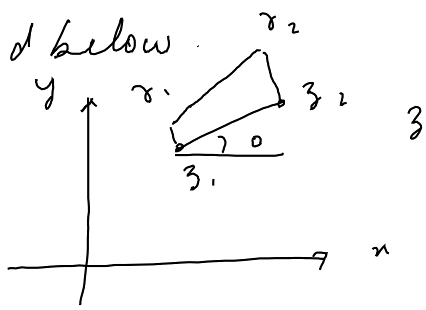
This system may be easily (or not!) solved, to obtain the Γ_j's.

Once Γ_j is known, the problem is solved.

The above represents a ~~lumped~~ lumped mass approach that is inaccurate.

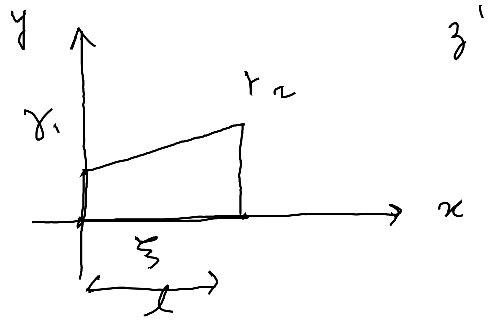
We quickly formulate a linear panel method below.

Consider figure on the right. What is $V(z)$ due to this panel?



We rotate the coordinates to a local coordinate system z'

Clearly, $z' = (z - z_1)e^{-i\theta}$; $V(z) = V(z')\,e^{i\theta}$



$$V(z') = \frac{-i}{2\pi}\int_0^l \frac{\gamma(\xi)\,d\xi}{(z' - \xi)} \quad ; \quad \gamma(\xi) = \gamma_1 + \frac{(\gamma_2 - \gamma_1)\,\xi}{l}$$

Integrating the above we can get

$$V(z') = \frac{-i}{2\pi}\left\{ \gamma_1\left[\left(\frac{z'}{l} - 1\right)\ln\left(\frac{z' - l}{z'}\right) + 1\right] - \gamma_2\left[\frac{z'}{l}\ln\left(\frac{z' - l}{z'}\right) + 1\right]\right\}$$

Thus using $V(z')$ & $V(z) = V(z')\,e^{i\theta}$ we can find the effect of a linear vortex panel rather easily.

Using the panel method we can now solve the Euler equation in 2D for an ideal fluid in the presence of arbitrary boundaries that are solid.