CS5070 Mathematical Structures for Computer Science - Notes 5

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August 14, 2021

Symbolic Logic and Proofs (Chapter 3)

- Given a few mathematical statements or facts, we can draw some conclusions
- We start with some given conditions, the *premises* of our argument, and from these we find a consequence of interest, our *conclusion*
- Logic aims to determine in which cases a conclusion is, or is not, a consequence of a set of premises
- An **argument** is a set of statements, one of which is called the conclusion and the rest of which are called premises
- An argument is said to be **valid** if the conclusion must be true whenever the premises are all true
- It is possible for all the premises to be true and the conclusion to be false

Propositional Logic

- A **proposition** is simply a statement
- Propositional logic studies the ways statements can interact with each other
- Most statements are implications:

$$P \implies Q$$

- To decide the truth value of $(P \implies Q) \vee (Q \implies R)$, either $P \implies Q$ must be true or $Q \implies R$ must be true, or both
- The most direct technique for determining the overall truth value is to use a **truth table**
- See Example 3.1.1 and Example 3.1.2

- A **tautology** is a statement that is always true in the basis of its ligical form
- For any truth values of P and Q, the statements $\neg P \vee Q$ and $P \implies Q$ either both true or both false.

  These statements are **logically equivalent**
- To verify that two statements are logically equivalent, a truth table can be used
- DeMorgan's Law

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

Implication and Negation

- $P \implies Q$ is logically equivalent to $\neg P \vee Q$
- Double negation:

  $\neg\neg P$ is logically equivalent to P
- We can use these laws to simplify statement expressions
- The negation of an implication is not an implication; it is a conjunction

  $\neg(P \implies Q)$ is logically equivalent to $P \wedge \neg Q$

Deductions

- $$\begin{array}{r} P \implies Q \\ P \\ \hline \therefore\ Q \end{array}$$
- This is a **deduction rule**, an argument form that is always valid
- This particular one is known as *modus ponens*
- If $P \implies Q$ is true and P is also true, the Q must be true

More Deduction Rules

- Another example of a rule:

$$\begin{array}{rl} & P \implies Q \\ & \neg P \implies Q \\ \hline \therefore & Q \end{array}$$

  If $P \implies Q$ and $\neg P \implies Q$ are true, then Q is alo true
- I this a valid rule ?

$$\begin{array}{rc} & P \implies R \qquad Q \implies R \\ & R \\ \hline \therefore & P \vee Q \end{array}$$

  This deduction rule is not valid

Another Deduction Rule and Predicate Logic

- $$\frac{P \implies R \qquad Q \implies R \qquad P \vee Q}{\therefore \quad R}$$
- Predicates use variables, such as $P(x)$ and to write statements, quantifiers are used

$$\forall x((P(x) \wedge x > 2) \implies O(x))$$

- The statement:

$$\neg \exists x \forall y (x \geq y)$$

can be written in the equivalent form:

$$\forall x \exists y (y < x)$$

Proofs

Proof by contradiction

- The proof is an argument
- There is a sequence of statements, the last being the *conclusion* that follows from the previous statements
- The conclusion must be true if the premises are true
- See Theorem 3.2.1 in page 214
- When the conclusion is not true, then the premise must be false

Direct Proof

- Direct proofs are especially useful when proving implications
- The general format to prove $P \implies Q$ is:

  Assume P. Explain, explain, ... explain. Therefore Q

- To prove universal statements $\forall x(P(x) \implies Q(x))$
  We assume $P(x)$ is true and deduce $Q(x)$ for *all* x
  We fix x to be an arbitrary element
- See Examples 3.2.2 and 3.2.3 in page 216

Proof By Contrapositive

- An implication $P \implies Q$ is logically equivalent to its contrapositive $\neg Q \implies \neg P$
- The general format to prove $P \implies Q$ is:

  Assume $\neg Q$. Explain, explain, ... explain. Therefore $\neg P$

- As before, if there are variables and quantifiers, we set them to be arbitrary elements of our domain
- See Examples 3.2.4 and 3.2.5 in page 217

Proof By Counter Example

- Since so many statements in mathematics are universal, making their negations existential, we can often prove that a statement is false by providing a counter-example
- See Examples 3.2.10 in page 221

Proof By Cases

- Prove that P is true by proving that $Q \implies P$ and $\neg Q \implies P$ for some statement Q
- Suppose we want to prove P. We know$Q$ that at least one of the statements $Q_1, Q_2 \ldots Q_n$ is true
- If we can show that $Q_1 \implies P$ and $Q_2 \implies P$ and so on up to $Q_n \implies P$, then we conclude P.
- We need to be sure that one of the cases must be true
- See Examples 3.2.11 in page 222