CS5070 Mathematical Structures for Computer Science - Notes 3b

José Garrido

Department of Computer Science
College of Computing and Software Engineering
Kennesaw State University

jgarrido@kennesaw.edu

August 14, 2021

Permutations with Repetitions

- Permutations with repetitions can be computed using the multiplicative principle
- Example: How many strings of length r can be formed from the uppercase letters? Solution: there are 26 uppercase letters in English, and because each letter can be used repeatedly, there are 26^r strings of uppercase letters of length r.
- The number of $r-$permutations of a set of n objects with repetition allowed is n^r

Combinations with Repetitions

There are $C(n+r-1,r) = C(n+r-1,n-1)$ $r-$combinations from a set with n elements when repetition of elements is allowed

- Example: A cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?
- Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. This is $C(4-6-1,6) = C(9,6)$

  $$C(9,6) = C(9,3) = 84$$

  There are 84 different ways to choose the six cookies

Permutations with Indistinguishable Objects

- How many different strings can be made by reordering the letters of the word SUCCESS?
- Some of the letters in SUCCESS are the same. The word contains three Ss, two Cs, one U, and one E.

  To determine the number of of different strings that can be made by reordering the letters, first note that the three Ss can be placed among the seven positions in $C(7,3)$ different ways, leaving four positions free.

  Then the two Cs can be placed in $C(4,2)$ ways, leaving two free positions.

  The U can be placed in $C(2,1)$ ways. Hence E can be placed in $C(1,1)$ way.
- The number of different strings that can be made is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = 420$$

Theorem

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, $\ldots$, and n_k indistinguishable objects of type k is:

$$C(n, n_1)C(n - n_1, n_2)\cdots C(n - n_1 - \cdots - n_{k-1}, n_k)$$

Which is equal to:

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Binomial Coefficients

For each integer $n \geq 0$ and integer k with $0 \leq k \leq n$

$$\binom{n}{k}$$

- $\binom{n}{k} = |\mathbf{B}_k^n|$, the number of n-bit strings of weight k
- $\binom{n}{k}$ is the number of subsets of a set of size n each with cardinality k
- $\binom{n}{k}$ is the coefficient $x^k y^{n-k}$ in the expansion of $(x+y)^n$
- $\binom{n}{k}$ is the number of ways to select k objects from a total of n objects

Applying Binomial Coefficients

- How many subsets of $\{1, 2, 3, 4, 5\}$ contain exactly 3 elements? Answer: $\binom{5}{3}$
- How many bit strings have length 5 and weight 3? Answer: $\binom{5}{3}$
- Recurrence Relation for $\binom{n}{k}$

$$|\mathbf{B}_k^n| = |\mathbf{B}_{k-1}^{n-1}| + |\mathbf{B}_k^{n-1}|$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Permutations

- A permutation is a (possible) **ordered** arrangement of objects. For example, there are 6 permutations of the letters a, b, c:

$$abc, acb, bac, bca, cab, cba$$

- This problem determines how many ordered arrangements of distinct objects can be counted.
- The permutation of n elements is:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

- How many permutations exist of k objects choosing those objects from a larger collection of n objects? This is denoted $P(n, k)$ also called k-**permutation of** n **elements** with $1 \leq k \leq n$

$$P(n, k) = \frac{n!}{(n-k)!} = n(n-1)(n-2) \cdots (n-(k-1))$$

Combinations

- Combinations count the **unordered** selections of objects.
- Combinations are the number of ways k items can be grouped from a total of n items: $C(n, k)$

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k(k-1)(k-2)\cdots 1}$$

- Example. How many possible different teams of three students can be formed in a class section of 27 students?

$$\binom{27}{3} = \frac{27!}{3!24!} = \frac{27 \cdot 26 \cdot 25}{3 \cdot 2 \cdot 1} = \frac{17,550}{6} = 2,925$$

- To count permutations, use $P(n, k)$
- To count combinations, use $C(n, k)$ or $\binom{n}{k}$

Counting with Repetitions

- The selections of objects allow repeated objects
- Permutations with repetitions can be computed using the multiplicative principle
- The number of $k-$permutations of a set of n objects with repetitions is computed by n^k
- How many words of length 5 can be constructed with 26 upper case letters? Solution is 26^5

Combinations with Repetitions

- Combinations with repetitions can be computed using the expression of the binomial coefficient
- The number of $k-$combinations from a set of n objects with repetitions is computed by:

$$C(n+k-1,k) = C(n+k-1,n-1)$$

- A tray has four different types of cookies. How many ways can six cookies be selected?

$$C(9,6) = C(9,3) = 84$$