Kennesaw State University

College of Computing and Software Engineering

Department of Computer Science

CS 5070, Mathematical Structures for Computer Science, Section W01

Assignment #4

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07/1/2025

Problem Statement

Problem of this document is to solve the following problem set for assignment #4.

Module 4 Assignment #4 (Sequences)

1. Problem 2.1.10 (page 145)
2. Problem 2.2.3 (page 156)
3. Problem 2.3.8 (page 165)
4. Problem 2.4.7 (page 176)

5. Problem 2.5.7 (page 188)

Summary / Purpose

Purpose of this document is to provide solutions to the problem set outlined in the Problem Statement section.

Solutions

1. Problem 2.1.10 (page 145) – Show that $a_n = 2^n - 5^n$ is also a solution to the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$. What would the initial conditions need to be for this to be the closed formula for the sequence?

   A closed formula for a sequence $(a_n)_{n\epsilon \boldsymbol{N}}$ is a formula for a_n using a fixed finite number of operations on n. A recursive relation is an equation relating a term of the sequence to previous terms (terms with smaller index) and an initial condition.

   For $a_n = 2^n - 5^n$:
   If n = 1; $a_1 = 2^1 - 5^1 = -3$
   If n = 2; $a_2 = 2^2 - 5^2 = -21$
   If n = 3; $a_3 = 2^3 - 5^3 = -117$

   Then, $a_n = 7a_{n-1} - 10a_{n-2} = a_3 = 7(a_2) - 10(a_1) = 7(-21) - 10(-3) = -117$

   Therefore, $2^n - 5^n$ is a solution to the reoccurrence relation $7a_{n-1} - 10a_{n-2}$.

   For it to be a closed formula, we need to find a_0. To do so, we can find substitute in for a_2 in the reoccurrence relation.

$$a_n = 7a_{n-1} - 10a_{n-2} = a_2 = 7(a_1) - 10(a_0) =$$

$$7(-3) - 10(a_0) = -21$$

$$-21 - 10a_0 = -21$$

$$a_0 = 0$$

Therefore our initial condition is when $a_0 = 0$.

2. Problem 2.2.3 (page 156) – Consider the sum 4 + 11 + 18 + 25 + … + 249
   a. How many terms (summands) are in the sum?

      The summation here is an arithmetic sequence because each summand differs by a constant, with that constant being 7, with initial term being 4. Therefore, we can find the number of terms by substituting in:

$$\begin{aligned} 4 + 7n &= 249 \\ 7n &= 245 \\ n &= 35 \end{aligned}$$

      But note that n here is at the a_1 term, 4 is a_0 term so in total there are 36 terms. Therefore, there are 36 terms in the summation.

   b. Compute the sum using a technique discussed in this section.

      Add the first and last terms of the sequence, and then reverse. Let's call the sum *S*. See:

      S = 4 + 11 + 18 + 25 + … + 242 + 249

      S = 249 + 242 + 235 + 228 + … + 11 + 4

      2*S* = 253 + 253 + 253 + 253 + … + 253 + 253

      We know there are 35 terms in the summation. So:

      2*S* = 36*253 = 9108

      S = 4554

      Therefore, the sequence is equal to 4554.

3. Problem 2.3.8 (page 165) – Suppose $a_n = n^2 + 3n + 4$. Find a closed formula for the sequence of differences by computing $a_n - a_{n-1}$.

   For $a_n = n^2 + 3n + 4$:

If n = 0; $a_0 = 0^2 + 3(0) + 4$ = 4
If n = 1; $a_1 = 1^2 + 3(1) + 4 = 8$
If n = 2; $a_2 = 2^2 + 3(2) + 4 = 14$
If n = 3; $a_3 = 3^2 + 3(3) + 4$ = 22
If n = 4; $a_4 = 4^2 + 3(4) + 4$ = 32

We have the sequence of 4, 8, 14, 22, 32. The sequence of differences is: 4, 6, 8, 10. Therefore, a closed formula for the sequence of differences is 2n + 2.

We can also substitute:

$$a_n = n^2 + 3n + 4 \; and \; a_{n-1} = (n-1)^2 + 3(n-1) + 4:$$

So:

$$\begin{aligned} a_n - a_{n-1} &= n^2 + 3n + 4 - [(n-1)^2 + 3(n-1) + 4] \\ &= n^2 + 3n + 4 - (n^2 + n + 2) \\ &= n^2 - n^2 + (3n - n) + (4 - 2) \\ &= 2n + 2 \end{aligned}$$

4. Problem 2.4.7 (page 176) – Solve the reoccurrence relation $a_n = 3a_{n-1} + 10a_{n-2}$ with initial terms $a_0 = 4$ and $a_1 = 1$.

   A reoccurrence relation is a recursive definition without the initial conditions. The characteristic equation is of the form $r^2 - 3r - 10 = 0$. Let's factor this to get the quadratic equation:

$$(r - 5)(r + 2) = 0$$

   The roots are then $r_1 = 5$ and $r_2 = -2$

   The general solution is of the following form:

$$a_n = c_1 r_1^n + c_2 r_2^n$$

   Substitute the roots in to get:

$$a_n = c_1 5^n + c_2 (-2)^n$$

   Now let's use the initial conditions to find c_1 and c_2.

$$a_0 = 4, 4 = c_1 5^0 + c_2(-2)^n$$
$$4 = c_1 + c_2$$
$$a_1 = 1, 1 = c_1 5^1 + c_2(-2)^1$$

$$1 = 5c_1 - 2c_2$$

Solving,

$$8 = 2c_1 + 2c_2$$

$$9 = 7c_1$$

$$c_1 = \frac{9}{7}$$

Plugging into the first equation,

$$4 = \frac{9}{7} + c_2$$

So,

$$c_2 = 4 - \frac{9}{7} = \frac{28 - 9}{7} = \frac{19}{7}$$

Substituting into the general equation, we get our solution to the reoccurrence relation:

$$a_n = \frac{9}{7} * 5^n + \frac{19}{7}(-2)^n$$

5. Problem 2.5.7 (page 188) – Prove by mathematical induction, that $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$, where F_n is the *n*th Fibonacci number ($F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$).

The Fibonacci sequence is a series of numbers where each number is the sum of the two preceding ones, typically starting with 0 and 1. So, the sequence begins 0, 1, 1, 2, 3, 5, 8, 13, and so on.

The base case (n = 0) is: $F_0 = F_2 - 1$ => $F_0 = 1 - 1 = 0$, which is true for n = 0.

Now using induction,

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$$

Let's determine this by solving for F_{n+1}:

$$F_0 + F_1 + F_2 + \cdots + F_n + F_{n+1} = F_{n+3} - 1$$

Solving the right-hand side:

$$F_{n+2} - 1 + F_{n+1} \Rightarrow$$

$$F_{n+2} + F_{n+1} - 1 \Rightarrow$$

$$F_{n+3} = F_{n+2} + F_{n+1}$$

Therefore,

$$F_0 + F_1 \dots . F_n + F_{n+1} = F_{n+3} - 1$$

Using this, we can determine that the sequence

$$F_0 + F_1 + F_2 + \cdots F_{n-1} + F_n = F_{n+2} - 1$$

Is true for all non-negative integers n.

References

[1] Garrido, J. (2021, August 14). *CS5070 Mathematical Structures for Computer Science - Notes 4* [Slide show; Powerpoint]. D2L. https://kennesaw.view.usg.edu/d2l/le/content/3550928/viewContent/55786717/View?ou=3550928

[2] Kennesaw State University, College of Computing and Software Engineering, Department of Computer Science, Mathematical Structures for Computer Science. (n.d.). CS5070 Assignment 4. In *https://kennesaw.view.usg.edu/d2l/le/content/3550928/viewContent/55786683/View*

[3] Levin, O. (2016). *Discrete mathematics: An Open Introduction*.