CS5070 Mathematical Structures for Computer Science - Notes 3

José Garrido

Department of Computer Science
College of Computing and Software Engineering
Kennesaw State University

jgarrido@kennesaw.edu

August 14, 2021

Counting (Chapter 1)

Counting Introduction

Consider this rather simple counting problem: there are 14 varieties of donuts, and 16 types of hot dogs. If you want either a donut or a dog, how many options do you have? This isn't too hard, just add 14 and 16.

Additive Principle

The additive principle states that if event A can occur in m ways, and event B can occur in n **disjoint** ways, then $|A \cup B|$ is $m + n$.

Disjoint Events

Consider that events A and B be disjoint: i.e., that there is no way for event A and event B to both happen at the same time. Thus, $A \cap B = \emptyset$

More on Counting

Multiplicative Principle

The multiplicative principle states that if event A can occur in m ways, and each possibility for A allows for exactly n ways for event B, then the event 'A and B' can occur in $m \cdot n$ ways. Events A and B occur independently.

The College of Computing offers in the fall semester seven sections of Programming I and nine sections of Lab Programming I. Students can take any combination of a section of the lecture class and a section of the lab section. How many different ways can a student register these?
Answer: $7 \cdot 9$.

Counting with Sets

Using Sets

Instead of event A and event B, we use set A and a set B. The sets will contain all the different ways the event can happen.

Additive Principle with Sets

Given two sets A and B, if $A \cap B = \emptyset$, that is, if there is no element common to both A and B, then

$$|A \cup B| = |A| + |B|$$

Multiplicative Principle with Sets

The set **Cartesian product** of sets A and B is defined by:

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

This is the set of all ordered pairs (a, b) such that elements a are in set A and elements b are in set B.
Example of a Cartesian product:

$$\{1, 2\} \times \{3, 4, 5\} = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$$

Cartesian Product Counting (2)

$$A \times B = \{$$

$$\begin{array}{lllll}
(a_1, b_1), & (a_1, b_2), & (a_1, b_3), & \dots & (a_1, b_n), \\
(a_2, b_1), & (a_2, b_2), & (a_2, b_3), & \dots & (a_2, b_n), \\
(a_3, b_1), & (a_3, b_2), & (a_3, b_3), & \dots & (a_3, b_n), \\
\vdots & & & & \\
(a_m, b_1), & (a_m, b_2), & (a_m, b_3), & \dots & (a_m, b_n)
\end{array}$$

There are m rows and n pairs, for a total of $m \cdot b$ pairs

Multiplicative Principle with Sets

We can consider that $A \times B$ is really the union of m disjoint sets. Each of those sets has n elements. The total is $n + n + n + \cdots + n = m \cdot n$.

The Multiplicative Principle
Given two sets A and B,

$$|A \times B| = |A| \cdot |B|$$

This can be extended to any number of sets

The cardinality of a union of two sets A and B, is:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Cardinality of a Union of 3 Sets

For any finite sets A, B, and C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

This process is called the *Principle of Inclusion/Exclusion* (PIE)

Example 1.1.8

$$f = \begin{array}{c|ccccccc} \textit{Subject} & A & B & C & AB & AC & BC & ABC \\ \hline \textit{Failed} & 12 & 5 & 8 & 2 & 6 & 3 & 1 \end{array}$$

How many students failed at least one subject?
$|A| = 12$, $|B| = 5$, $|C| = 8$, $|A \cap B| = 2$, $|A \cap C| = 6$, $|B \cap C| = 3$, and $|A \cap B \cap C| = 1$

$|A \cup B \cup C| = 15$

Subsets

The powerset of a set is a set of all subsets of a set
For set $A = \{a, b, c\}$, the powerset of A is:

$$\mathcal{P}(A) = \{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

The number of elements in set A is $k = |A|$ and the value is 3.

The number of subsets of set A is $n = 2^k$, and this value is 8.

How many subsets of A have two elements? We know that 3 subsets have cardinality 2.

Decomposition Principle

- Many larger problems are solved by partitioning a large problem into smaller problems (sub-problems)
- Each sub-problem is solved independently of other sub-problems
- The solution of the all sub-problems is combined to get the final result of the original problem. In general, the additive principle is applied in this step.

Bit Strings

- An n-bit string is a bit string of length n
- The **weight** of a bit string is the number of 1's in it
- $\mathbf{B}^n$ is the set of all n-bit strings
- $\mathbf{B}^n_k$ is the set of all n-bit strings of weight k
- The elements of of the set $\mathbf{B}^3_2$ are the strings 011, 101, and 110.
- How many strings have length 5? The answer is 2^5, which is 32.
- How many of those have weight 3?

$$|\mathbf{B}^5_3| = |\mathbf{B}^4_2| + |\mathbf{B}^4_3|$$

$$|\mathbf{B}^4_2| = |\mathbf{B}^3_1| + |\mathbf{B}^3_2| \text{ and } |\mathbf{B}^4_3| = |\mathbf{B}^3_2| + |\mathbf{B}^3_3|$$

- With $A = \{1, 2, 3, 4, 5\}$, the number of 3-element subsets of A is the same as counting the number of 5-bit strings of weight 3.

Binomial Coefficients

For each integer $n \geq 0$ and integer k with $0 \leq k \leq n$

$$\binom{n}{k}$$

- $\binom{n}{k} = |\mathbf{B}_k^n|$, the number of n-bit strings of weight k
- $\binom{n}{k}$ is the number of subsets of a set of size n each with cardinality k
- $\binom{n}{k}$ is the coefficient $x^k y^{n-k}$ in the expansion of $(x+y)^n$
- $\binom{n}{k}$ is the number of ways to select k objects from a total of n objects

Applying Binomial Coefficients

- How many subsets of $\{1, 2, 3, 4, 5\}$ contain exactly 3 elements?
  Answer: $\binom{5}{3}$
- How many bit strings have length 5 and weight 3? Answer: $\binom{5}{3}$
- Recurrence Relation for $\binom{n}{k}$

$$|\mathbf{B}_k^n| = |\mathbf{B}_{k-1}^{n-1}| + |\mathbf{B}_k^{n-1}|$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Permutations

- A permutation is a (possible) **ordered** arrangement of objects. For example, there are 6 permutations of the letters a, b, c:

$$abc, acb, bac, bca, cab, cba$$

- This problem determines how many ordered arrangements of distinct objects can be counted.
- The permutation of n elements is:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

- How many permutations exist of k objects choosing those objects from a larger collection of n objects? This is denoted $P(n, k)$ also called k-**permutation of** n **elements** with $1 \leq k \leq n$

$$P(n, k) = \frac{n!}{(n-k)!} = n(n-1)(n-2) \cdots (n-(k-1))$$

Combinations

- Combinations count the **unordered** selections of objects.
- Combinations are the number of ways k items can be grouped from a total of n items: $C(n,k)$

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k(k-1)(k-2)\cdots 1}$$

- Example. How many possible different teams of three students can be formed in a class section of 27 students?

$$\binom{27}{3} = \frac{27!}{3!24!} = \frac{27\cdot 26\cdot 25}{3\cdot 2\cdot 1} = \frac{17,550}{6} = 2,925$$

- To count permutations, use $P(n,k)$
- To count combinations, use $C(n,k)$ or $\binom{n}{k}$

Counting with Repetitions

- The selections of objects allow repeated objects
- Permutations with repetitions can be computed using the multiplicative principle
- The number of $k-$permutations of a set of n objects with repetitions is computed by n^k
- How many words of length 5 can be constructed with 26 upper case letters? Solution is 26^5

Combinations with Repetitions

- Combinations with repetitions can be computed using the expression of the binomial coefficient
- The number of $k-$combinations from a set of n objects with repetitions is computed by:

$$C(n+k-1,k)=C(n+k-1,n-1)$$

- A tray has four different types of cookies. How many ways can six cookies be selected?

$$C(9,6)=C(9,3)=84$$