CS5070 Mathematical Structures for Computer Science - Notes 1

Jose Garrido

Department of Computer Science
College of Computing and Software Engineering
Kennesaw State University

jgarrido@kennesaw.edu

August 14, 2021

Basic Definitions

Statement

A declarative sentence that is true or false. It is also known as a **proposition**.

Atomic Statement

A statement that cannot be divided into smaller statements.

Molecular (or Composite) Statement

A statement that is not atomic.

Basic Definitions

Logical Connectives

Are used to define more complicated statements (molecular) out of simpler ones atomic or molecular).

Binary Connectives

Used to connect two statements. These are: *and*, *or*, *xor*.

Unary Connectives

Applied to a single statement. This is *not*; also known as negation.

Propositional Variables

Represent statements. These are denoted by $P, Q, R, S, \ldots$.

Logical Connectives

The logical connectives are: $\wedge$, $\vee$, $\implies$, $\iff$, $\neg$. Examples:

- $P \wedge Q$, known as conjunction
- $P \vee Q$, known as disjunction
- $P \oplus Q$, known as exclusive or
- $P \implies Q$, known as implication (or conditional)
- $P \iff Q$, known as biconditional (if and only if)
- $\neg P$, known as negation

The truth value of a statement is determined by the truth values of its parts

Truth Conditions for Connectives

- $P \wedge Q$ is true when both P and Q are true (logical and)
- $P \vee Q$ is true when P or Q, or both are true, (inclusive or)
- $P \oplus Q$ is true when exactly one P or Q are true. and is false otherwise (exclusive or)
- $P \implies Q$ is true when P is false or Q is true or both are true
- $P \iff Q$ is true when P and Q are both true, or both false
- $\neg P$ is true when P is false

Implications

- An implication or conditional is a molecular (compound) statement of the form:

$$P \implies Q$$

  where P and Q are statements.
- P is the **hypothesis** (or **antecedent**)
- Q is the **conclusion** (or **consequent**)
- An implication is true if P is false or Q is true (or both), and false otherwise
- The only way for $P \implies Q$ to be false is for P to be true and Q to be false

Direct Proofs of Implications

- To proof an implication $P \implies Q$, it is enough to assume P, and from it, deduce Q.
- Explain why Q is true, but you get to assume P is true first
- Prove: If two numbers a and b are even, then their sum is even

DeMorgan's Law

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

Converse and Contrapositive

- The converse of an implication $P \implies Q$, is the implication $Q \implies P$
- The converse is not logically equivalent to the original implication. That is, whether the converse of an implication is true is independent of the truth of the implication
- The contrapositive of an implication $P \implies Q$ is the statement $\neg Q \implies \neg P$
- An implication and its contrapositive are logically equivalent (they are both true or both false)
- Understanding converses and contrapositives can help understand implications and their truth values
- The contrapositive always has the same truth value as its original implication.

Biconditional

- When the implication $P \implies Q$ and the implication $Q \implies P$ are both true, then P and Q are equivalent and is denoted as $P \iff Q$
- The biconditional statement $P \iff Q$ is logically equivalent to $(P \implies Q) \wedge (Q \implies P)$
- You can think of a biconditional (if and only if) statement as having two parts: an implication and a converse
- 'P is becessary for Q' means $Q \implies P$
- 'P is sufficient for Q' means $P \implies Q$
- 'P is necessary and sufficient for Q, then $P \iff Q$

Truth Tables

These tables show the truth value of outcome for every combination of the various truth values of the sentences (propositions)

P	Q	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
t	t	t	t	t	t
t	f	f	t	f	f
f	t	f	t	t	f
f	f	f	f	t	t

Predicates and Quantifiers

- A sentence that contains variables is known as a **predicate**. For example, $P(x)$ and $Q(y)$
- The **existential quantifier** is denoted as $\exists$ and means 'there exists'. For example,

$$\exists x\,(x < 0)$$

  asserts that there is a number less than 0
- The **universal quantifier** is denoted as $\forall$ and means 'for all'. For example,

$$\forall x\,(x \geq 0)$$

  asserts that every number is greater than or equal to 0

Quantifiers and Negation

- $\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$
- $\neg \exists x P(x)$ is equivalent to $\forall x \neg P(x)$
- Example, prove:

$$\exists x \, \forall y (y \geq x)$$

  We could quantify over natural numbers $(0, 1, 2, \ldots)$

- Another use of the universal quantifier:

$$\forall x \, (S(x) \implies R(x))$$

Sets

- A set is an **unordered collection** of objects
- Two sets are equal exactly if they contain the exact same elements
- Notation: set A contains the elements $1, 2, 3$

$$A = \{1, 2, 3\}$$

- Notation: element a is a member of the set containing the elements a, b, c

$$a \in \{a, b, c\}$$

- Notation: element d is not in the set containing elements a, b, c

$$d \notin \{a, b, c\}$$

- Similarly, element 4 is not in set A

$$4 \notin A$$

Set of Natural Numbers

- $\mathbb{N}$ is a set of natural numbers

$$\mathbb{N} = \{0, 1, 2, 3 \ldots\}$$

- Let A be the set of all even natural numbers

$$A = \{0, 2, 4, 6, \ldots\}$$

- A better way to denote this is:

$$A = \{x \in \mathbb{N} : \exists n \in \mathbb{N} \, (x = 2n)\}$$

- Another way

$$A = \{x \in \mathbb{N} \mid x \text{ is even}\}$$

- The set of integers; positive and negative whole numbers is denoted as $\mathbb{Z}$. Using this, we can define the set:

$$A = \{x \in \mathbb{Z} : x^2 \in \mathbb{N}\}$$

Additional Notation

- $\mathbb{N}$ is the set of natural numbers
- $\mathbb{Z}$ is the set of integer numbers
- $\mathbb{R}$ is a set of real numbers
- $\mathbb{Q}$ is the of rational numbers
- The empty set is denoted as $\varnothing$
- The universe set, the set of all elements, is denoted as $\mathcal{U}$
- $\mathcal{P}(A)$ is the power set of any set A
- The cardinality of set A is denoted as $|A|$ and is the number of elements in A

Set Relationships

- Given sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$, clearly:

  $A \neq B$ and $A \subset B$

  In simple words, sets A and B are not equal. Set A is a **subset** of set B because every element of A is also an element of B.

- The **cardinality** of set A is 3 and the cardinality of set B is 4.

$$|A| = 3, \quad |B| = 4$$

- The collection of all subsets of B is a new set, which is a set of sets. This set is known as the **power set** of B and denoted as $\mathcal{P}(B)$
- Given set $A = \{1, 2, 3\}$, the set of all subsets of A is:

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Set Operations

- The **union** of two sets is the operation that combines two sets to get the collection of objects that are in either set.

  $$C = A \cup B$$

  C is a new set, the union of sets A and B.

- The **intersection** of two sets A and B results in another set with the elements that are exactly the elements in both A and B

  $$C = A \cap B$$

- The **complement** of a set A is a set that has the elements not in set A. When set D is the complement of set A, it is denoted as:

  $$D = \bar{A}$$

  Another notation for the complement of A is A'

More Set Operations

- The intersection of set A and complement of set B is denoted:

$$A \cap \bar{B}$$

- Another way to write the previous operation is defined by the **set difference** as:

$$A \cap \bar{B} = A \setminus B$$

- The set **Cartesian product** of sets A and B is defined by:

$$A \times B = \{(a, b) : a \in A \land b \in B\}$$

  This is the set of all ordered pairs (a, b) such that elements a are in set A and elements b are in set B.

Cartesian Product

Given sets A and B, the following set is defined:

$$A \times B = \{(x, y) : x \in A \wedge y \in B\}$$

This is the set of all ordered pairs (x, y) where x is an element of A and y is an element of B. This is the Cartesian product of A and B.

Let $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots, b_n\}$

$A \times B = \{$

$$\begin{array}{lllll}
(a_1, b_1), & (a_1, b_2), & (a_1, b_3), & \ldots & (a_1, b_n), \\
(a_2, b_1), & (a_2, b_2), & (a_2, b_3), & \ldots & (a_2, b_n), \\
(a_3, b_1), & (a_3, b_2), & (a_3, b_3), & \ldots & (a_3, b_n), \\
\vdots & & & & \\
(a_m, b_1), & (a_m, b_2), & (a_m, b_3), & \ldots & (a_m, b_n)
\end{array}$$

$\}$

More Set Operations

- The set **difference** $A \setminus B$ of sets A and B is the set that consists of all those elements in A that are not in set B.
- The **symmetric difference** of two sets A and B, also known as the disjunctive union, is the set of elements which are in either of the sets, but not in their intersection. This operation is denoted as

$$A \oplus B = A \triangle B = (A \setminus B) \cup (B \setminus A)$$

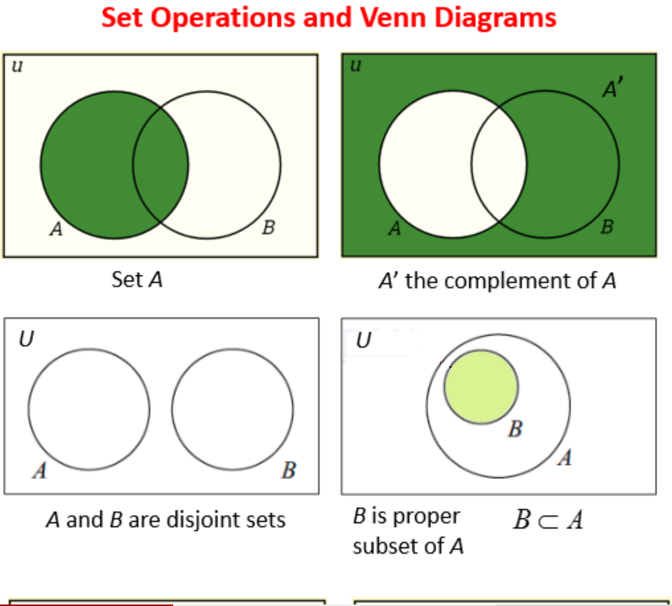
- The main property is that an element in $A \oplus B$:

$$x \in A \triangle B \iff (x \in A \cup B \wedge x \notin A \cap B)$$

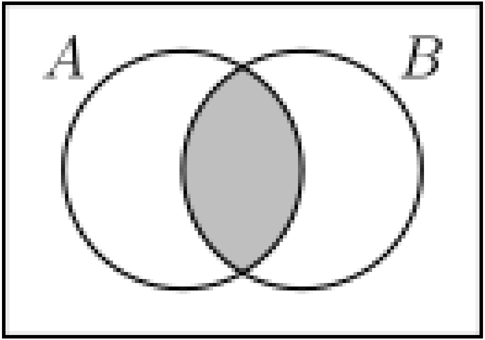
- For example, the symmetric difference of the sets $A = \{1, 2, 3\}$ and $B = \{3, 4\}$ is

$$A \oplus B = \{1, 2, 4\}$$

Venn Diagrams

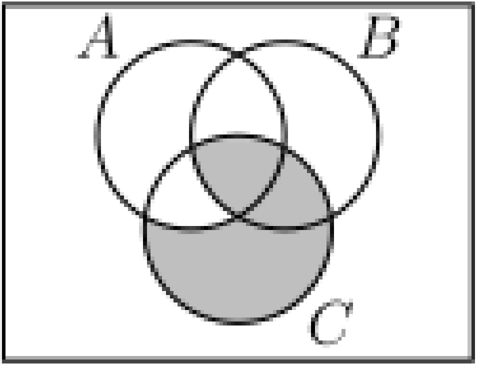


Venn Diagrams: Intersection



Venn Diagrams: More

$(B \cap C) \cup (C \cap \bar{A})$



Questions?