Kennesaw State University

College of Computing and Software Engineering

Department of Computer Science

CS 5070, Mathematical Structures for Computer Science, Section W01

Assignment #6

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Problem Statement

Problem of this document is to solve the following problem set for assignment #6.

Assignment #6 (Chapter 4)

1. Problem 1: 4.2.4 (page 255)
2. Problem 2: 4.3.4 (page 265)
3. Problem 3: Prob 23 page 293

Summary / Purpose

Purpose of this document is to provide solutions to the problem set outlined in the Problem Statement section.

Solutions

1. Problem #1, page 255 (4.2.4): Suppose you have a graph with *v* vertices and *e* edges that satisfies $v = e + 1$. Must the graph be a tree? Prove your answer.

   A graph is an ordered pair $G = (V, E)$ consisting of a nonempty set *V* (called the vertices) and a set *E* (called the edge) of two-element subsets of *V.* This one in particular satisfies the condition $v = e + 1$. We know that a tree is a connected graph with no cycles. To be connected is to mean that we can get from any vertex to any other vertex following some path of edges.

   Suppose we have 4 vertices and 3 edges, we will satisfy the condition $v = e + 1$ since v is 4, and e is 3; i.e 4 = 3 + 1. Our graph could look like this:

   $$G = \{a, b, c, d\}, \{\{a, b\}, \{b, c\}, \{c, d\}\}$$

   This is a straight line, and meets the conditions. However, what if the graph looked like this where *v* = 5, and *e* = 4.

   $$G = \{a, b, c, d, e\}, \{\{a, b\}, \{a, c\}, \{c, d\}, \{a, d\}\}$$

   This is not connected because you can't go from b to e (an isolated vertex), hence it's not a tree. Therefore, with the conditions above, the graph must not technically be a tree.

2. Problem #2, p. 265 (4.3.4): Is it possible for a graph with 10 vertices and edges to be a connected planar graph? Explain.

A graph is planar when none of the edges cross. A connected graph is when we can get from any vertex to any other vertex following some path of edges. Euler's formula for planar graphs is the following:

For any connected planar graph with *v* vertices, *e* edges, and *f* faces, we have $v - e + f = 2$. Let's set *v* to 10 and *e* to 10 in this formula.

$$10 - 10 + f = 2 => f = 2$$

These two faces means one region is bounded and another is the outer region. A n example of such a graph. A graph with 2 vertices and 1 edge is not enough to consider. Suppose the following:

$$v \geq 3 \text{ and thus, } e \leq 3v - 6 \text{ since e = v – 1}$$

Plug in 10 for *v*. We get that $e \leq 3(10) - 6$ = 24. This is larger than our constraint of *e* = 10, the graph is planar. How do we make such a graph connected? A simple circle with vertices on the circumferences with edges connecting them will make sure the graph is connected. Therefore, we can conclude that such a graph can possible.

3. Problem #3, p. 293 (#23): Let *G* be a connected graph with *v* vertices and *e* edges. Use mathematical induction to prove that if G contains exactly one cycle (among other edges and vertices), then *v* = *e*. Note: this is asking you to prove a special case of Euler's formula for planar graphs, so do not use that formula in your proof.

We know that *G* is connected, and it has *v* vertices and *e* edges. Suppose that it has one cycle, then the number of vertices equals the number of edges, *v* = *e*. A connected graph is when we can get from any vertex to any other vertex following some path of edges and a cycle is when we start and stop at the same vertex. A tree is a connected graph with no cycles that also has *e* = *v* – 1.

For our base case, we assume that a graph with v = 1 and e = 1 cannot contain a cycle. A graph with v = 2 and e = 2 can't be a cycle either, so let's say that v = 3 and e =3. This forms a triangle shape, so we can get a cycle.

Our inductive hypothesis is that any connected graph with *v* = *k* holds for all *v* = k +1 and *e* = k + 1. Let a graph *G* be a connected graph with v = k + 1 and exactly 1 cycle. If we remove a single edge from the graph, we'll break the cycle but still have a connected graph; whch can also be a called a tree. If it's a tree, we know that *e* = *v* - 1.

So then:

$$e - 1 = (k + 1) - 1 = k$$

Thus,

$$e - 1 = k \rightarrow e = k + 1$$

Since $v = k + 1$,

$$e = v$$

Therefore, we have shown that a connected graph with *v* vertices and *e* edges, and exactly one cycle, we can say that *v* = *e*.

References

[1] Garrido, J. (2021, August 14). *CS5070 Mathematical Structures for Computer Science - Notes 6* [Slide show; Powerpoint]. D2L.

[2] Kennesaw State University, College of Computing and Software Engineering, Department of Computer Science, Mathematical Structures for Computer Science. (n.d.). CS5070 Assignment 6.

[3] Levin, O. (2016). *Discrete mathematics: An Open Introduction*.