CS5070 Mathematical Structures for Computer Science - Notes 4

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Sequences (Chapter 2)

- A sequence is an <u>ordered</u> list of numbers: $\langle a_0, a_1, a_2, a_3, \ldots \rangle$.
- A sequence is a type of function. The sequence $(a_n)_{n \geq 0}$ is a function with domain $\mathbb{N}$ where a_n is the image of the natural number n.
- A <u>closed formula</u> for a sequence $(a_n)_{n \in \mathbb{N}}$ is a formula for a_n using a fixed finite number of operations on n.
- A recursive definition (or inductive definition) for a sequence $(a_n)_{n \in \mathbb{N}}$ consists of a recurrence relation and an initial condition.
- An example of a closed formula is: $a_n = n^2 + 3$
- An example of a recursive definition is:
  $a_n = 2a_{n-1}$ with $a_0 = 27$

Known Sequences

- The **square numbers** sequence $1, 4, 9, 16, 25, \ldots$; this sequence $(s_n)_{n \geq 1}$ has a closed formula

$$s_n = n^2$$

- The **triangular numbers** sequence $1, 3, 6, 10, 15, 21, \ldots$; this sequence $(T_n)_{n \geq 1}$ has closed formula

$$T_n = \frac{n(n+1)}{2}$$

- The **powers of 2** sequence $1, 2, 4, 8, 16, 32, \ldots$; this sequence $(a_n)_{n \geq 0}$ has closed formula

$$a_n = 2^n$$

- The **Fibonacci numbers** sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ is defined recursively by

$$F_n = F_{n-1} + F_{n-2} \quad \text{with } F_1 = F_2 = 1$$

Closed Formula of Sequences

- One method to derive a closed formula for a sequence, is to relate a given sequence to a known sequence. See Example 2.1.4.
- Partial Sums. Given any sequence $(a_n)_{n\in\mathbb{N}}$ we can always form a new sequence $(b_n)_{n\in\mathbb{N}}$ by

$$b_n = a_0 + a_1 + a_2 + \cdots + a_n$$

- Since the terms of (b_n) are the sums of the initial part of the sequence (a_n) ways, I (b_n) is known as the **sequence of partial sums of** (a_n). It is sometimes possible to find a closed formula for (b_n) from the closed formula for (a_n). See page 143.
- To simplify writing the sum, the notation $\sum_{k=1}^{n} a_k$ is used.
- The notation for multiplying the a_k terms is $\prod_{k=1}^{n} a_k$

Arithmetic Sequences

- If the terms of a sequence differ by a constant, the sequence is arithmetic.
- If the initial term (a_0) of the sequence is a and the **common difference** is d,
  Recursive definition: $a_n = a_{n-1} + d$ with $a_0 = a$
  Closed formula: $a_n = a + d \cdot n$
- For example (see Example 2.2.1), for the sequence $2, 5, 8, 11, 14, \ldots$, the common difference $d = 3$ and the recursive definition is $a_n = a_{n-1} + 3$ with $a_0 = 2$.

  The closed formula is $a_n = 2 + 3 \cdot n$

Geometric Sequences

- A sequence like: $2, 6, 18, 54, \ldots$ has a difference between terms that is not constant, however, the **ratio** between successive terms is constant. It is a **geometric** sequence.
- The recursive definition for the geometric sequence with initial term a and common ration r is:

$$a_n = a_{n-1} \cdot r \quad a_0 = a$$

- The closed formula for this sequence is:

$$a_n = a \cdot r^n$$

- The sequence: $3, 6, 12, 24, 48, \ldots$ has the common ration $r = 2$. The recursive definition is: $a_n = 2a_{n-1}$ with $a_0 = 3$. See Example 2.2.2.

  The closed formula is: $a_n = 3 \cdot 2^n$

- The sequence $(T_n)_{n\geq 1}$ that starts $1, 3, 6, 10, 15, \ldots$ is not arithmetic and is not geometric.
- Note that the *differences* between terms form an arithmetic sequence: $2, 3, 4, 5, 6, \ldots$.
- This means that the nth term of the sequence (T_n) is the *sum* of the first n terms of the sequence $1, 2, 3, 4, 5, \ldots$
- (T_n) is the **sequence of partial sums** of the sequence $1, 2, 3, \ldots$

$$T_n = 1 + 2 + 3 + \cdots + n$$

- In general,

$$T_n = \frac{n(n+1)}{2}$$

More Complex Sequences

- The sequence $\langle 1, 5, 14, 30, 55 \ldots \rangle$ is not arithmetic and is not geometric; its sequence of **differences** might be arithmetic or geometric.
- The sequence of differences is: $\langle 4, 9, 16, 25, \ldots \rangle$.
- This sequence is not arithmetic because its differences are not constant
- The second differences is: $\langle 5, 7, 9, \ldots \rangle$ is an arithmetic sequence because the differences are constant with value 2.
- The first sequence had third differences constant.
- This original sequence is known as $\triangle^3-$constant
- In general, a sequence is $\triangle^k-$constant if the kth differences are constant

Finite Differences

- The sequence of differences between terms gives information about the rate of growth of the sequence
- The closed formula for a sequence will be a degree k polynomial if and only if the sequence is $\triangle^k-$constant
- If a sequence is $\triangle^3-$constant, it will have a cubic (degree 3 ploynomial) for its closed formula
- If the closed formula is a degree k polynomial, we just need $k+1$ data points to 'fit' the polynomial to the data.
- See Example 2.3.2 on page 162

Solving Recurrence Relations

- A recurrence relation is a recursive definition without the initial conditions
- For the sequence: $1, 5, 17, 53, 161, 485, \ldots$, the first differences is: $4, 123, 36, 108, \ldots$
- This sequnce is geometric with a comong ratio $r = 3$
- The recurrence relation is $a_n = 3a_{n-1} + 2$, and the initial condition is $a_0 = 1$ To solve this recurrence relation, **iteration** is used
- Starting with the known initial condition, iterate finding the next term until a_n is found. See Example 2.4.4 and Example 2.4.5

Characteristic Roots

- From the recurrence relation $a_n + \alpha a_{n=1} + \beta a_{n-2} = 0$, the characteristic polynomial is

$$x^2 + \alpha x + \beta$$

- The characteristic equation is:

$$x^2 + \alpha x + \beta = 0$$

- The solution to the recurrence relation is:

$$a_n = ar_1^n + br_2^n$$

  Where r_1 and r_2 are distinct roots and a and b are constants determined by the initial conditions
- See Example 2.4.6

Induction

- Mathematical induction is another proof technique
- To prove that $P(n)$ is true for all $n \geq 0$
  1. Base case: prove $P(0)$ is true
  2. Inductive case: Prove $P(k) \rightarrow P(k+1)$ for all $k \geq 0$
- Therefore by the principle of mathematical induction, the statement $P(n)$ is true for all $n \geq 0$
- See Example 2.5.1