# Design of an Electromagnetic Railgun

1   Calculations

1.1   Force on the Projectile

The first step is to calculate the force exerted on the projectile as a function of the position of the projectile and the current running through the system (assuming the projectile is within the barrel, i.e. $0 < x < L$). We begin by computing the magnitude of the magnetic field generated by one rail (the effective length of the rail is the position of the projectile):
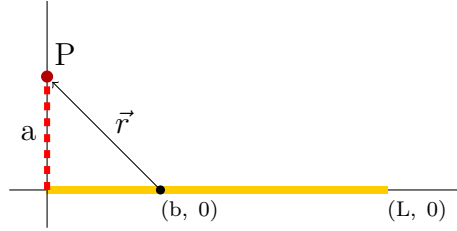
Given that point P is located at the coordinates $(0, a)$ (as the projectile is always at the end of the portion of the rail which current is running in):

$$d\vec{B} = \frac{\mu_0 I d\vec{s} \times \hat{r}}{4\pi \|\vec{r}\|^2} = \frac{\mu_0 I}{4\pi} \cdot \frac{d\vec{s} \times \hat{r}}{\|\vec{r}\|^2} = \frac{\mu_0 I}{4\pi} \cdot \frac{d\vec{s} \times \left(\frac{\vec{r}}{\|\vec{r}\|}\right)}{\|\vec{r}\|^2} = \frac{\mu_0 I}{4\pi} \cdot \frac{d\vec{s} \times \vec{r}}{\|\vec{r}\|^3} \tag{1}$$

We begin by solving for the infinitesimal portion of the rails:

$$d\vec{s} = db\hat{\boldsymbol{i}} \tag{2}$$

We then solve for the distance and direction to the point where the magnetic field is being observed:

$$\vec{r} = (0 - b)\hat{\boldsymbol{i}} + (a - 0)\hat{\boldsymbol{j}} = -b\hat{\boldsymbol{i}} + a\hat{\boldsymbol{j}} \tag{3}$$

$$\|\vec{r}\| = \sqrt{(-b)^2 + a^2} = \sqrt{b^2 + a^2} \tag{4}$$

$$\|\vec{r}\|^3 = (b^2 + a^2)^{3/2} \tag{5}$$

We then write and simplify the expression for $d\vec{B}$:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \cdot \frac{d\vec{s} \times \vec{r}}{\|\vec{r}\|^3} = \frac{\mu_0 I}{4\pi} \cdot \frac{db\hat{\boldsymbol{i}} \times (-b\hat{\boldsymbol{i}} + a\hat{\boldsymbol{j}})}{(b^2 + a^2)^{3/2}} \tag{6}$$

$$db\hat{\boldsymbol{i}} \times (-b\hat{\boldsymbol{i}} + a\hat{\boldsymbol{j}}) = adb\hat{\boldsymbol{k}} \tag{7}$$

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \cdot \frac{adb\hat{\boldsymbol{k}}}{(x^2 + a^2)^{3/2}} \tag{8}$$

We take the norm of $d\vec{B}$ to find the magnitude of the field:

$$B = \frac{\mu_0 I}{4\pi} \cdot \frac{adb}{(x^2 + a^2)^{3/2}} \tag{9}$$

Finally, we integrate along the rail to find B at a given distance a from the rail, where x is the effective length of the rails:

$$B(a) = \frac{\mu_0 I a}{4\pi} \int_0^x \frac{db}{(b^2 + a^2)^{3/2}} = \frac{\mu_0 I x}{4\pi a \sqrt{x^2 + a^2}} \tag{10}$$

Now that we have computed the magnitude of the magnetic field of the rails, we proceed by calculating the force exerted on the projectile. By the right-hand rule, both rails produce a magnetic field such that the force applied on the projectile is forward. Therefore, we can use the Lorentz force law (for a wire) on the projectile (given that d is the separation between the rails and w is the width of the rails):

$$F = I \int B ds = I \int_{(w/2)}^{(d+w/2)} B(a) da \tag{11}$$

$$F = I \int_{(w/2)}^{(d-w/2)} \frac{\mu_0 I x}{4\pi a \sqrt{x^2 + a^2}} da = \frac{\mu_0 I^2 x}{4\pi} \int_{(w/2)}^{(d-w/2)} \frac{da}{a \sqrt{x^2 + a^2}} \tag{12}$$

Integrating yields the final expression for force on the projectile when it is a given distance through the barrel (where $x = 0$ is the start of the barrel):

$$F(I, x) = \frac{\mu_0 I^2}{\log(d - w/2) - \log(x\sqrt{(d - w/2)^2 + x^2} + x^2) + \log(w/2) - \log(x\sqrt{(w/2)^2 + x^2} + x^2)}{4\pi} \tag{13}$$

1.2 Current in Rails and Projectile

It follows that the next step is to calculate the amount of current running through the projectile. We begin by calculating the total capacitance of the capacitor bank:

$$C = N \cdot C_{\text{each}} \tag{14}$$

Next, we find the resistance of the circuit as a function of time, temperature (which itself is a function of distance along the rails), and effective rail length (position of the projectile).

$$R_{\text{rail}} = \frac{\int \rho \, dl}{A} \tag{15}$$

$$\rho(l) = \rho_0(1 + \alpha(T(l) - 20)) = \rho_0(1 + \alpha T(l) - 20\alpha) = \rho_0(1 - 20\alpha) + \rho_0 \alpha \cdot T(l) \tag{16}$$

Given the width of the rail is w and the thickness of the rail is k:

$$R_\text{rail} = \frac{\int \rho(l)dl}{A} = \frac{\rho_0(1 - 20\alpha)\int dl + \rho_0\alpha\int T(l)dl}{wk} \tag{17}$$

$$R_\text{rail} = \frac{\rho_0 l(1 - 20\alpha) + \rho_0\alpha(\int T(l)dl)}{wk} \tag{18}$$

Assuming the resistance of the projectile is negligible, we find that the total resistance is (given that R_circuit is the additional resistance of the capacitor circuit and wiring):

$$R_\text{total} = 2R_\text{rail} + R_\text{circuit} = 2\frac{\rho_0 l(1 - 20\alpha) + \rho_0\alpha(\int T(l)dl)}{wk} + R_\text{circuit} \tag{19}$$

Now, we can use the capacitor discharge formulas to find the amount of current running through the circuit:

$$V = \frac{Q}{C} \tag{20}$$

$$\frac{Q}{C} = IR \tag{21}$$

We write this in terms of time and solve for the rate of change of charge:

$$\frac{Q}{C} = \frac{dQ}{dt}R \tag{22}$$

$$I = \frac{dQ}{dt} = \frac{Q}{R_\text{total}C} \tag{23}$$

In addition, we can find the initial charge as a function of voltage and capacitance:

$$Q = \frac{V}{C} \tag{24}$$

Because resistance is constantly changing, we will use Euler's method to approximate the current and charge over time.

1.3 Temperature Increase of Rails

By the joule-heating effect, we know that the heat q added to a resistor can be found by the following formula:

$$q = Pt = I^2 Rt \tag{25}$$

Therefore, we can calculate the amount of heat added to a system in a given amount of time:

$$q = Pt = \left(\frac{Q}{R_\text{rail}C}\right)^2 R_\text{rail}t = \frac{Q^2 t}{C^2 R_{rail}} \tag{26}$$

Next, we need to calculate the change in temperature due to the added heat:

$$q = mc\Delta T \tag{27}$$

$$\Delta T = \frac{q}{mc} \tag{28}$$

The specific heat c will depend on the metal used, and we can calculate the volume and mass as a function of the density p of the metal and the effective length x of the rails:

$$V = xwk \tag{29}$$

$$m = pV = pxwk \tag{30}$$

$$\Delta T = \frac{q}{pxwkc} \tag{31}$$

$$\Delta T = \frac{\frac{Q^2 t}{C^2 R_{total}}}{pxwkc} = \frac{Q^2 t}{pxwkc \cdot C^2 R_{rail}} \tag{32}$$

We must note that the increase in temperature only applies between the start of the rail and the point x meters into the rail. Therefore, it must be calculated point-by-point.

2 Results

2.1 Implementation

We implement a simulation using equations (13) (18) (19) (23) (24) (32). Code for the simulation can be seen: