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Grungy Times: The Euler-Lagrange Equations

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Abstract

The Euler-Lagrange equations are ewwler. Here's what we can do to make them...kewwler.

I. INTRODUCTION

Euler-Lagrange forms an integral part of our petrochemical production pipeline.

For systems with one degree of freedom, the *action integral* called $J(y)$ can be expressed as:

$$J(y) = \int_0^T F(t, y, \dot{y})\ dt \tag{1}$$

II. FALLING BODIES

Any body falling sufficiently close to the surface of the Earth is subject to an approximately constant gravitational force $F = mg$, assuming a perfectly spherical Earth, and that the object's height above the surface of the Earth is much less than its radius. Its trajectory is parameterized by a function $y(t)$ which expresses its height relative to the surface of the Earth. We concern ourselves here with a trajectory going from $t = 0, y_0 = y(0)$ to $t = T, y = y(T)$.

The kinetic energy of a falling body is simply $T = \frac{1}{2}m\dot{y}^2$. Its gravitational potential energy is the integral of the gravitational force from its origin to its destination; assuming the surface of the Earth as the origin ($y = 0$), $V = mgy$.

We have defined an isolated system; as such, in the confines of the system, there can be no change in the net amount of energy. The energy can be represented $E_{tot} = T - V$, which expands to $E_{tot} = \frac{1}{2}m\dot{y}(t)^2 - mgy(t)$. This function $E_{tot}(t, y, \dot{y})$ is what we need to conserve.

Thus, we can define the action integral J as follows:

$$J = \int_0^T \frac{1}{2}m\dot{y}(t)^2 - mgy(t)\ dt$$

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By the Euler-Lagrange constraint, then,

$$\frac{\partial}{\partial y}\left[E_{tot}(t, y, \dot{y})\right] - \frac{\partial^2}{\partial t \partial y}\left[E_{tot}(t, y, \dot{y})\right] = 0$$

We can evaluate these partial derivatives, of course, and this evaluation gives us the equation:

$$m\ddot{y}(t) = -mg$$
$$\Longrightarrow \ddot{y}(t) = -g$$

This can now be integrated with respect to time.

$$\iint \ddot{y}(t)\ dt\ dt = \iint -g\ dt\ dt$$
$$\Longrightarrow y(t) = \iint -g\ dt\ dt$$
$$\Longrightarrow y(t) = \int -gt + C_1\ dt$$
$$\Longrightarrow y(t) = -\frac{1}{2}gt^2 + C_1 t + C_2$$

The constants C_1 and C_2 are really just the initial velocity and position of the object, respectively. This function describes a parabolic trajectory, which we know to be true.

III. THE BRACHISTOCHRONE PROBLEM

The brachistochrone curve is simply the curve along which a frictionless particle will slide in the minimum time. Brachistochrone trajectories are especially interesting in the context of space travel, since they provide the most efficient trajectories with which to traverse long distances.

The problem is, specifically, to find the curve $y(x)$ between points $y(a) = A$ and $y(b) = B$, along which a particle will slide in the minimum time.

Here, we will trace out Johann Bernoulli's solution to the brachistochrone problem, and justify all the steps he took.

We begin with the fundamental theorem of calculus, which shows that:

$$T = \int_0^t dt$$

Next, we proceed to a chain rule equality. Though this is perhaps unpalatable to most mathematicians, it is trivial to show through the chain rule that $\frac{dt}{ds}ds = dt$:

$$\implies \int_0^T dt = \int_0^L \frac{dt}{ds} ds$$

However, there is an important simplification we can make here. Velocity is defined as $v = \frac{ds}{dt}$, and a simple inversion shows that $\frac{dt}{ds} = \frac{1}{v}$:

$$\int_0^L \frac{dt}{ds} ds = \int_0^L \frac{1}{v}\, ds$$

Let us assume a coordinate system in which the particle begins at the origin, and the positive y direction points upwards, against the direction of gravity. We can now find the velocity of the particle as a function of its position, by using the principle of energy conservation.

Additionally, let $h = 0$ be the initial position of the particle. We know that the gravitational potential energy is simply $V_g = mgh$, and that the kinetic energy in the absence of rotation or friction is $T = \frac{1}{2}mv^2$. Thus,

$$\begin{aligned} &mgh = -mgy + \frac{1}{2}mv^2 \\ &\implies mgy = \frac{1}{2}mv^2 \\ &\implies v = \sqrt{2gy} \end{aligned}$$

Now, we can substitute this back into our original integral,

$$\int_0^L \frac{1}{v}\, ds = \int_0^L \frac{1}{\sqrt{2gy}}\, ds$$

However, it's important to remember that here, ds is actually just $\sqrt{dx^2 + dy^2}$. This can then be algebraically simplified to $dx\sqrt{1 - \left(\frac{dy}{dx}\right)^2}$. Substituting this into our integral,

$$\int_0^L \frac{1}{\sqrt{2gy}}\, ds = \sqrt{\frac{1}{2g}} \int_a^b \sqrt{\frac{1 + y'^2}{y}}\, dx$$

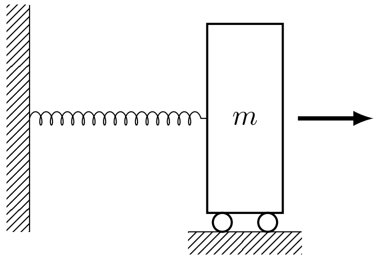

FIG. 1. A single-spring, single-mass system.

This is simply integrating over the x domain of the path, as opposed to the y domain.

We can now solve for the Euler-Lagrange equations of $\sqrt{\frac{1}{2g}}\int_a^b \sqrt{\frac{1+y'^2}{y}}\ dx$:

$$F_y = -\sqrt{\frac{1+(y')^2}{8g}}y^{-\frac{3}{2}}$$

$$F_{y'} = \frac{y}{\sqrt{2gy}\sqrt{1+(y')^2}} \quad F_{y'x} = \frac{\frac{1}{2}y'^2\left[y\left(1+y'^2\right)\right]^{-1/2}\left(2y''y+y'^2+1\right)+y''(\sqrt{y}\sqrt{1+y'^2})}{\sqrt{2gy\left(1+y'^2\right)}}$$

We know that $F_{y'x} = F_y$. Therefore,

$$\sqrt{\frac{1+y'^2}{8g}}y^{-\frac{3}{2}} - \frac{\frac{1}{2}y'^2\left[y\left(1+y'^2\right)\right]^{-1/2}\left(2y''y+y'^2+1\right)+y''(\sqrt{y}\sqrt{1+y'^2})}{\sqrt{2gy\left(1+y'^2\right)}} = 0$$

After some tedious simplification, which is redacted for clarity,

$2y''y' + y'^2 + 1 = 0$ $\square$

IV. THE SPRING-MASS SYSTEM

Spring-mass systems are very common in physics.

V. GENERALISING THE LAGRANGIAN: DEGREES OF FREEDOM

VI. THE TWO MASS, THREE SPRING SYSTEM

VII. THE RAY EQUATION; FERMAT'S PRINCIPLE

The velocity of an object in 2D can be broken down into its basis component, and the line element can be written interm of velocity.

$$v = \sqrt{x'^2 + y'^2} \qquad ds = vdt = \sqrt{x'^2 + y'^2}dt$$

Then the total time taken can be calculated by

$$T = \int_P \frac{ds}{c(x,y)} = \int_P \frac{\sqrt{x'^2 + y'^2}dt}{c(x,y)} = \int_P F(t,x,x',y,y')dt$$

$$\text{where } F(t,x,x',y,y') = \frac{\sqrt{x'^2 + y'^2}}{c(x,y)}$$

To minimize time, the variation need to be stationary at zero pertubation. n_x, n_y are small pertubations to the system.

$$\begin{aligned}
\delta T = & \left.\frac{dT(t, x+\epsilon n_x, y+\epsilon n_y, x'+\epsilon n_x', y'+\epsilon n_y')}{d\epsilon}\right|_{\epsilon=0} = 0 \\
= & \int \left(\frac{\partial F}{\partial x} n_x + \frac{\partial F}{\partial x'} n_x' + \frac{\partial F}{\partial y} n_y + \frac{\partial F}{\partial y'} n_y' \right) \\
= & \int \left(\frac{\partial F}{\partial x} n_x + \frac{\partial F}{\partial x'} \frac{dn_x}{dt} + \frac{\partial F}{\partial y} n_y + \frac{\partial F}{\partial y'} \frac{dn_y}{dt} \right) \\
= & \int \left(\frac{\partial F}{\partial x} n_x - n_x \frac{d}{dt}\frac{\partial F}{\partial x'} + \frac{\partial F}{\partial y} n_y - n_y \frac{d}{dt}\frac{\partial F}{\partial y'} \right) \\
= & \int \left(\left[\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x'} \right] n_x + \left[\frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial y'} \right] n_y \right) dt = 0
\end{aligned}$$

Which can be summarized as a pair of lagrange equations

$$\frac{\partial F}{\partial x} = \frac{d}{dt}\frac{\partial F}{\partial x'} \qquad 1)$$

$$\frac{\partial F}{\partial y} = \frac{d}{dt}\frac{\partial F}{\partial y'} \qquad 2)$$

Carry out the derivation for equation 1(i.e. the x curve).

$$\frac{\partial F}{\partial x} = \sqrt{x'^2 + y'^2}\frac{\partial}{\partial x}\frac{1}{c(x,y)} = -\frac{\sqrt{x'^2 + y'^2}}{c(x,y)^2}\frac{\partial c(x,y)}{\partial x}$$

use the fact that $x' = \frac{dx}{dt}$

$$\frac{d}{dt}\frac{\partial F}{\partial x'} = \frac{d}{dt}\frac{\partial}{\partial x'}\frac{\sqrt{x'^2 + y'^2}}{c(x,y)} = \frac{d}{dt}\frac{x'}{c(x,y)\sqrt{x'^2 + y'^2}} = \frac{d}{dt}\frac{dx}{c(x,y)\sqrt{x'^2 + y'^2}dt}$$

We can convert the independent variable form t to s by multiplying both side of equation 1 by $\frac{dt}{ds} = (x'^+ y'^2)^{\frac{1}{2}}$:

$$\frac{\partial F}{\partial x}\frac{dt}{ds} = -(x'^+ y'^2)^{\frac{1}{2}}\frac{\sqrt{x'^2 + y'^2}}{c(x,y)^2}\frac{\partial c(x,y)}{\partial x} = -\frac{1}{c(x,y)^2}\frac{\partial c(x,y)}{\partial x}$$

Using chain rule and the fact that $ds = (x'^+ y'^2)^{\frac{1}{2}}dt$:

$$\frac{d}{dt}\frac{\partial F}{\partial x'}\frac{dt}{ds} = \frac{dt}{ds}\frac{d}{dt}\frac{dx}{c(x,y)\sqrt{x'^2 + y'^2}dt} = \frac{d}{ds}\frac{dx}{c(x,y)ds}$$

. The exact same procedure can be carry out on equation 2, which together would yield the following system:

$$\frac{d}{ds}\frac{dx}{c(x,y)ds} = -\frac{1}{c(x,y)^2}\frac{\partial c(x,y)}{\partial x}$$

$$\frac{d}{ds}\frac{dy}{c(x,y)ds} = -\frac{1}{c(x,y)^2}\frac{\partial c(x,y)}{\partial y}$$

In fact, the system can be generalized into

$$\frac{d}{ds}\frac{1}{c(\vec{r})}\frac{d\vec{r}}{ds} = -\frac{1}{c(\vec{r})^2}\nabla c(\vec{r})$$

Where $\vec{r}$ is the position vector of the object and ∇ is the gradient, and the expressioin describe a n-equation system component wise.
Taking $\vec{r} = (x, y)$ would yield the result derived above.