1. If $A = \{1, 2\}$, compute $A \times \mathcal{P}(A)$.

   We first write out the power set $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. The elements of $A \times \mathcal{P}(A)$ are ordered pairs, the first component coming from A and the second from $\mathcal{P}(A)$. So we get

   $$A \times \mathcal{P}(A) = \{(1, \emptyset), (1, \{1\}), (1, \{2\}), (1, \{1, 2\}), (2, \emptyset), (2, \{1\}), (2, \{2\}), (2, \{1, 2\})\}$$

   Comment: It may have been confusing to have pairs where one component was a number and the other a set of numbers. The Cartesian product doesn't care, it just pairs up things from the first set with things from the second set.

2. For $n \in \mathbb{N}$ let $A_n = (\frac{1}{n+1}, 3 - \frac{1}{n+1})$ (an open interval). Compute $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$.

   Let's write out some of the sets A_n. We get

   $$A_0 = (1, 2), \quad A_1 = \left(\frac{1}{2}, \frac{5}{2}\right), \quad A_2 = \left(\frac{1}{3}, \frac{8}{3}\right), \quad A_3 = \left(\frac{1}{4}, \frac{11}{4}\right), \ldots$$

   We notice that these intervals are nested, they keep getting bigger. In particular, they all contain the first set $A_0 = (1, 2)$. This is enough to say that their intersection, what they all have in common, is exactly this set, so $\bigcap_{n\in\mathbb{N}} A_n = (1, 2)$.

   The union of these sets contains all the points that are in at least one of these sets. Looking at them, the sets never reach outside the interval $(0, 3)$, so their union also won't. On the other hand, any number in $(0, 3)$ gets into one of the sets A_n. If $x \in (0, 3)$ we can find an n such that $\frac{1}{n+1} < x < 3 - \frac{1}{n+1}$ (since the two endpoints converge to 0 and 3, respectively). But then $x \in A_n$. It follows that the union is just $\bigcup_{n\in\mathbb{N}} A_n = (0, 3)$.

3. (a) Consider the statement scheme

   $$P(x) : x(x - 1) = 6$$

   where x has domain $\mathbb{N}$. Find those x for which $P(x)$ is true.

   We just need to solve the equation. Using some algebra we find that the solutions are $x = 3$ and $x = -2$. But we are only interested in the solutions within the domain of the variable, which is $\mathbb{N}$. So the only valid x is 3.

   (b) Consider the statement scheme

   $$Q(n) : n \text{ and } n + 2 \text{ are prime}$$

   where n has domain $\mathbb{N}$. Find five values of n which make $Q(n)$ true.

This is quite easy to do with a list of primes. We can take

n	$n+2$
3	5
5	7
11	13
17	19
29	31

We saw the truth tables for the logical connectives in class. We can now apply these one after another to create truth tables for more complicated statements made up from more than just one connective. For example, suppose we wanted to create the truth table for the statement $P \wedge (P \implies Q)$. We make a column for each of the component statements and build the entire thing by combining two columns at a time, like so:

P	Q	$P \implies Q$	$P \wedge (P \implies Q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

Given two statements R and S, we say that they are *logically equivalent* (written $R \equiv S$) if their columns in the truth table are the same, meaning that given the same truth values for the basic statements involved, R and S have the same truth values. For example, the truth table we have above shows that $P \wedge (P \implies Q)$ is logically equivalent to $P \wedge Q$. Intuitively, logically equivalent statements are "the same", they say the exact same thing. This is useful, because to prove a statement it suffices to prove a logically equivalent statement and this is often easier.
A statement is a *tautology* if its column in the truth table is all Ts (i.e. if it is logically equivalent to the statement T). A statement is a *contradiction* if its column in the the truth table is all Fs (i.e. if it is logically equivalent to the statement F).

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4. (a) Show that $P \implies Q$ is logically equivalent to $Q \vee \sim P$.

We just need to compare their columns in a truth table.

P	Q	$\sim P$	$Q \vee \sim P$	$P \implies Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since their two columns are the same, $P \implies Q$ and $Q \vee \sim P$ are logically equivalent.

(b) Show that $(P \implies Q) \vee (Q \implies P)$ is a tautology.

We just check that only Ts appear in the statement's column in the truth table.

P	Q	$P \implies Q$	$Q \implies P$	$(P \implies Q) \vee (Q \implies P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

5. (a) Show that $P \implies Q$ and $\sim Q \implies \sim P$ are logically equivalent.

(These two implications are *contrapositives* of one another).

We compare the columns in the truth table and see that the two statements are logically equivalent.

P	Q	$\sim P$	$\sim Q$	$P \implies Q$	$\sim Q \implies \sim P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

(b) Show that $P \implies Q$ and $Q \implies P$ are not logically equivalent.

(These two implications are *converses* of one another).

We show that the two statements are not logically equivalent by building their truth tables and observing that their respective columns do not match.

P	Q	$P \implies Q$	$Q \implies P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

6. (extra credit) Consider the following truth table

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

The connective $\uparrow$ is called the *Sheffer stroke* or *NAND*.

Find two statements using only Sheffer strokes (and no other logical connectives) that are equivalent to $\sim P$ and $P \wedge Q$, respectively.

We spend some time playing around with truth tables and eventually come up with the following:

P	Q	$P \uparrow Q$	$\sim P$	$P \uparrow P$	$P \wedge Q$	$(P \uparrow Q) \uparrow (P \uparrow Q)$
T	T	F	F	F	T	T
T	F	T	F	F	F	F
F	T	T	T	T	F	F
F	F	T	T	T	F	F

The truth table shows that $\sim P \equiv P \uparrow P$ and $P \wedge Q \equiv (P \uparrow Q) \uparrow (P \uparrow Q)$.

Comment: The way you come up with this is really up to you. For me, it's a combination of just playing around and the realization that $P \uparrow Q$ is false exactly when both P and Q are true. This already suggests a solution for $\sim P$ and getting one for $P \wedge Q$ from here is not that bad.