1. Let a, b be integers. Prove that $a + b$ is even if and only if a and b have the same parity (i.e. they are both even or both odd).

   (Hint: depending on how you approach this problem, you may need to do two cases for each of the two directions. The cases will be very similar to one another.)

   ($\Longleftarrow$): Let a and b be integers with the same parity. We consider two cases:

   Case 1: If a and b are even then there are integers k and l such that $a = 2k$ and $b = 2l$. Then $a + b = 2k + 2l = 2(k + l)$ is even, by definition.

   Case 2: If a and b are odd then there are integers k and l such that $a = 2k + 1$ and $b = 2l + 1$. Then $a + b = 2k + 1 + 2l + 1 = 2(k + l + 1)$ is even, by definition.

   ($\Longrightarrow$): We argue by contrapositive. So let a and b be integers with opposite parity. We consider two cases:

   Case 1: If a is even and b is odd then there are integers k and l such that $a = 2k$ and $b = 2l + 1$. Then $a + b = 2k + 2l + 1 = 2(k + l) + 1$ is odd, by definition.

   Case 2: If a is odd and b is even then there are integers k and l such that $a = 2k+1$ and $b = 2l$. Then $a + b = 2k + 1 + 2l = 2(k + l) + 1$ is odd, by definition.

2. Let a, b be integers. Prove that ab is odd if and only if both a and b are odd.

   ($\Longleftarrow$): Let a and b be odd integers. Then there are integers k and l such that $a = 2k + 1$ and $b = 2l + 1$. So $ab = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$ is odd.

   ($\Longrightarrow$): We argue by contrapositive. So let a and b be integers, at least one of which is even. We consider two cases:

   Case 1: If a is even then there is an integer k such that $a = 2k$. Then $ab = 2kb$ is even.

   Case 2: If b is even then there is an integer k such that $b = 2k$. Then $ab = 2ka$ is even.

Here is a new property of integers that we can prove things about.

Definition. If a and b are integers and $a \neq 0$, we say that a *divides* b (in symbols $a \mid b$; note that the line is straight, not slanted) if there is an integer k such that $b = ak$. We also say that b is divisible by a or that a is a factor or a divisor of b. We write $a \nmid b$ to say that a does not divide b.

For example, saying that an integer is even (according to the definition from class) means exactly that 2 divides it (according to this definition). The divisibility relation allows us to talk about more things than just evenness and oddness.

3. Let a, b, c be integers and $a \neq 0$ and $b \neq 0$. Show that if $a \mid b$ and $b \mid c$ then $a \mid c$.

   Suppose that $a \mid b$ and $b \mid c$. By definition there are integers k and l such that $b = ak$ and $c = bl$. Combining these two equations, we get $c = (ak)l = a(kl)$. By definition, this means that $a \mid c$.

4. Let a be an integer. Show that if 2 divides a^2 then 4 divides a^2.

   (Hint: use a theorem from class.)

   Suppose that a is an integer and 2 divides a^2, i.e. a^2 is even. By a theorem from class, it follows that a itself is even, so we can write $a = 2k$ for some integer k. Then $a^2 = (2k)^2 = 4k^2$, so $4 \mid a^2$.

5. Let a and b be integers and $a \neq 0$.

   (a) Show that $a \mid a$.

   Let a be a nonzero integer. To see that $a \mid a$, we need to find an integer k such that $a = ak$. Clearly $k = 1$ works.

   (b) Assume that a and b are positive. Show that if $a \mid b$ and $b \mid a$ then $a = b$.

   Let a, b be integers such that $a \mid b$ and $b \mid a$. This means that there are integers k, l such that $b = ak$ and $a = bl$. Combining these two equalities, we get $b = blk$. Since $b \neq 0$, we can cancel b and get $lk = 1$. Since k and l are integers, we conclude that either $k = l = 1$ or $k = l = -1$. The second option is impossible, since then $a = -b$, which contradicts our assumption that a and b positive. So we must have $k = l = 1$, which means that $a = b$.