1. Let A, B be sets. Show that $A \cup (B \cap A) = A$ and $A \cap (B \cup A) = A$.

   Proof. Let x be an arbitrary element of A. Then clearly $x \in A \cup (B \cap A)$, so $A \subseteq A \cup (B \cap A)$. Conversely if x is an element of $A \cup (B \cap A)$ then either $x \in A$ or $x \in B \cap A$. But in either case we can conclude that $x \in A$, so $A \cup (B \cap A) \subseteq A$.

   For the other equality, suppose $x \in A$. Then also $x \in B \cup A$, so $x \in A \cap (B \cup A)$, showing that $A \subseteq A \cap (B \cup A)$. Conversely, if $x \in A \cap (B \cup A)$, then clearly $x \in A$, which shows that $A \cap (B \cup A) \subseteq A$. $\square$

2. Let A, B be sets. Show that if $A \subseteq B$ then $A \setminus B \subseteq B \setminus A$.

   Proof. Assume that $A \subseteq B$. This means that $A \setminus B = \emptyset$ and clearly $\emptyset \subseteq B \setminus A$. $\square$

3. If A, B are sets, define $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

   Show that $A \triangle B$ and $A \cap B$ are disjoint.

   Proof. Suppose not. Then $(A \cap B) \cap (A \triangle B)$ is nonempty; fix an element x of this intersection. Since $x \in A \cap B$ we see that x is in both A and B. Since $x \in A \triangle B$ we see that either $x \in A \setminus B$ or $x \in B \setminus A$. In either of the two cases x cannot be in both A and B, contradicting our earlier observation. $\square$

4. If A, B are sets, show that $A \cup B = (A \triangle B) \cup (A \cap B)$.

   Proof. Let $x \in (A \triangle B) \cup (A \cap B)$ be arbitrary. Then either $x \in A \triangle B$ or $x \in A \cap B$. If $x \in A \cap B$ then in particular also $x \in A$ so $x \in A \cup B$. If, on the other hand, $x \in A \triangle B$ then either $x \in A \setminus B$ or $x \in B \setminus A$. In the first case $x \in A$ and in the second $x \in B$, so overall we get $x \in A \cup B$ in either case. This shows that $(A \triangle B) \cup (A \cap B) \subseteq A \cup B$.

   Conversely, suppose $x \in A \cup B$. We distinguish two cases, depending on whether $x \in A \cap B$. If so then clearly also $x \in (A \triangle B) \cup (A \cap B)$. Now consider the case when $x \notin A \cap B$. Since $x \in A \cup B$, x is in at least one of A and B and, since $x \notin A \cap B$, x is not in both of them. We may thus assume, without loss of generality, that $x \in A$ and $x \notin B$. But then $x \in A \setminus B$ and so $x \in A \triangle B$ and $x \in (A \triangle B) \cup (A \cap B)$. This shows that $A \cup B \subseteq (A \triangle B) \cup (A \cap B)$. $\square$

5. If A, B are sets, show that $A \triangle B = \emptyset$ iff $A = B$.

   Proof. Suppose that $A = B$. Then $A \setminus B = B \setminus A = \emptyset$ and this means that $A \triangle B = \emptyset \cup \emptyset = \emptyset$.

   Now suppose that $A \triangle B = \emptyset$. We can conclude that $A \setminus B = \emptyset$, which shows that $A \subseteq B$, and $B \setminus A = \emptyset$, which shows that $B \subseteq A$. Taken together, we get that $A = B$. $\square$

6. Let a, b be integers and $a \neq 0$. Show that if $a \mid b$ then $a^{10} \mid b^{10}$.

Proof. Suppose that $a \mid b$. Then there is an integer k such that $b = ka$. Therefore $b^{10} = k^{10}a^{10}$ which means that $a^{10} \mid b^{10}$. $\square$

7. Let a be an integer. Show that if $6 \mid 5a$ then $6 \mid a$.

   Proof. Suppose that $6 \mid 5a$. It follows that $2 \mid 5a$ and $3 \mid 5a$. Since 2 is prime we get that either $2 \mid 5$ or $2 \mid a$ and similarly for 3. But since 5 is not divisible by either 2 or 3, we can conclude that both 2 and 3 divide a. This means that the prime factorization of a includes both 2 and 3 as factors and we can conclude that a is divisible by 6. $\square$

8. Let x, y be integers. Show that x and y have opposite parities iff $(x - y + 3)^3$ is even.

   Proof. x and y have opposite parities iff x and $-y$ have opposite parities iff $x - y$ is odd iff $x - y + 3$ is even iff $(x - y + 3)^3$ is even. $\square$

9. Let x be an integer. Show that $2 \mid (x^4 + 1)$ iff $4 \mid (x^2 - 1)$.

   Proof. Suppose that $2 \mid (x^4 + 1)$. Then x^4 is odd, meaning that x is itself odd. Writing $x = 2k + 1$ we can see that $x^2 = 4k^2 + 4k + 1$ and so $x^2 - 1$ is divisible by 4.

   Conversely, suppose that $4 \mid (x^2 - 1)$. Then x^2 is odd, meaning that x is itself odd. But then x^4 is also odd and so $x^4 + 1$ is even. $\square$

10. Let x be an integer. Show that if $x^2 - 1$ is even then it is divisible by 8. Give a counterexample to disprove the same claim about $x^2 - 5$.

    Proof. Suppose that $x^2 - 1$ is even. Then x^2 is odd so x must be odd and we can write $x = 2k + 1$ for some integer k. Then $x^2 - 1 = 4k^2 + 4k = 4k(k + 1)$. One of k and $k + 1$ is even, so $4k(k + 1)$ is divisible by 8.

    To see that it can happen that $x^2 - 5$ is even but not divisible by 8 just take $x = 5$. $\square$

11. If a, b, c are integers and $c \neq 0$ define $a \equiv b \pmod{c}$ if $c \mid (a - b)$.

    Show that if $a \equiv b \pmod{c}$ and $d \equiv e \pmod{c}$ then $(a + d) \equiv (b + e) \pmod{c}$.

    Proof. Our assumptions tell us that c divides both $a - b$ and $d - e$, thus there are integers k, l such that $a - b = kc$ and $d - e = lc$. By adding these two equations we get that

    $$a + d - (b + e) = a - b + d - e = kc + lc = (k + l)c$$

    which means that c divides $a + d - (b + e)$ and so $a + d \equiv b + e \pmod{c}$. $\square$

12. Show that there is no integer x such that $x^4 \equiv 2 \pmod{5}$.

Proof. Consider the possible remainders of x modulo 5.

If $x \equiv 0 \pmod 5$ then $x^4 \equiv 0 \pmod 5$. If $x \equiv 1 \pmod 5$ then $x^4 \equiv 1 \pmod 5$. If $x \equiv 2 \pmod 5$ then $x^4 \equiv 16 \equiv 1 \pmod 5$. If $x \equiv 3 \pmod 5$ then $x^4 \equiv 81 \equiv 1 \pmod 5$. If $x \equiv 4 \pmod 5$ then $x^4 \equiv 256 \equiv 1 \pmod 5$.

So in no case is $x^4 \equiv 2 \pmod 5$. □

Comments: Alternatively you can solve this by splitting into cases depending on what the remainder of x is when divided by 5, writing $x = 5k$ or $x = 5k + 1$ or $x = 5k + 2$, etc., and then computing what x^4 is.

13. Show that two integers x, y have the same parity iff $x \equiv y \pmod 2$.

Proof. x and y have the same parity iff x and $-y$ have the same parity iff $x - y$ is even iff $x \equiv y \pmod 2$. □

14. Let a be an integer and $n \geq 2$ a natural number. Show that if $a \equiv 0 \pmod n$ then $a^2 \equiv 0 \pmod{n^2}$. Give a counterexample (i.e. values for a, b, n) to show that if a, b are integers then $a \equiv b \pmod n$ does not imply that $a^2 \equiv b^2 \pmod{n^2}$.

Proof. Suppose that $a \equiv 0 \pmod n$. Then there is an integer k such that $a = kn$. Therefore $a^2 = k^2n^2$, which means that $a^2 \equiv 0 \pmod{n^2}$.

For the second part, consider $n = 3, a = 1, b = 4$. We see that $1 \equiv 4 \pmod 3$ but $1 \not\equiv 16 \pmod 9$. □

15. Let a be an integer. Show that $a^4 \equiv (5 - a)^4 \pmod 5$.

Proof. Clearly $(5 - a) \equiv (-a) \pmod 5$ and $(-a)^4 \equiv a^4 \pmod 5$. Putting these two together yields $(5 - a)^4 \equiv a^4 \pmod 5$. □

Comments: Alternatively, you can solve this just by writing out what it means for $a^4 \equiv (5 - a)^4 \pmod 5$ to hold.

16. Show that $\sqrt{2} + \sqrt{3}$ is irrational.

Proof. Suppose not. Then there are integers a, b with $b \neq 0$ such that $\sqrt{2} + \sqrt{3} = \frac{a}{b}$. Squaring both sides yields $2 + 2\sqrt{6} + 3 = \frac{a^2}{b^2}$ or, equivalently, $\sqrt{6} = \frac{a^2}{2b^2} - \frac{5}{2}$. But this would mean that $\sqrt{6}$ was rational, contradicting a problem from a homework. □

17. Show that $\log_6 7$ is irrational.

Proof. Suppose not. Then there are integers a, b with $b \neq 0$ such that $\log_6 7 = \frac{a}{b}$; in fact, since $\log_6 7$ is positive, we may take both a and b to be natural numbers. The definition of the logarithm gives $6^{\frac{a}{b}} = 7$. Taking b-th powers we get $6^a = 7^b$. Since both a and b are natural numbers, both sides of these equation are natural numbers. But the left one is even while the right one is odd. Contradiction. □

18. Let a_n be the terms of the Fibonacci sequence (i.e. $a_0 = 0, a_1 = 1$, $a_n = a_{n-2} + a_{n-1}$). Prove Cassini's identity: $a_{n-1}a_{n+1} - a_n^2 = (-1)^n$.

Proof. We argue by induction. In the base step we check that the identity holds when $n = 1$. Plugging in we get $0 \cdot 1 - 1^2 = (-1)^1$, which is true.

In the induction step we assume that we have $a_{k-1}a_{k+1} - a_k^2 = (-1)^k$ for some natural number k. We want to see that the same holds for $k+1$. We compute

$$\begin{aligned} a_{(k+1)-1}a_{(k+1)+1} - a_{k+1}^2 &= a_k(a_k + a_{k+1}) - a_{k+1}^2 = a_k^2 + a_k a_{k+1} - a_{k+1}^2 \\ &= a_k^2 + a_k a_{k+1} - a_{k+1}a_{k+1} \\ &= a_k^2 + a_k a_{k+1} - (a_{k-1} + a_k)a_{k+1} \\ &= a_k^2 + a_k a_{k+1} - a_{k-1}a_{k+1} - a_k a_{k+1} \\ &= a_k^2 - a_{k-1}a_{k+1} = -(-1)^k = (-1)^{k+1} \end{aligned}$$

where we used the induction hypothesis to get from the next-to-last line to the last one. □

19. Define a sequence by $a_1 = -3, a_2 = 0$ and $a_{n+1} = 7a_n - 10a_{n-1}$ for $n \geq 2$. Show that $a_n = 2 \cdot 5^{n-1} - 5 \cdot 2^{n-1}$ for all positive natural n.

Proof. We argue by induction on n. In the base step we check the equality for $n = 1$ and 2. Indeed, $a_1 = -3 = 2 - 5$ and $a_2 = 0 = 2 \cdot 5 - 5 \cdot 2$.

In the induction step we assume that $a_k = 2 \cdot 5^{k-1} - 5 \cdot 2^{k-1}$ for all $1 \leq k \leq n$ and show that the same holds for $k = n+1$. We get

$$\begin{aligned} a_{n+1} &= 7a_n - 10a_{n-1} \\ &= 7(2 \cdot 5^{n-1} - 5 \cdot 2^{n-1}) - 10(2 \cdot 5^{n-2} - 5 \cdot 2^{n-2}) \\ &= 14 \cdot 5^{n-1} - 35 \cdot 2^{n-1} - 4 \cdot 5^{n-1} + 25 \cdot 2^{n-1} \\ &= 10 \cdot 5^{n-1} - 10 \cdot 2^{n-1} \\ &= 2 \cdot 5^n - 5 \cdot 2^n \end{aligned}$$

□

20. Let $x > -1$ be a real number and n a natural number. Prove Bernoulli's inequality: $(1+x)^n \geq 1 + nx$.

Proof. We argue by induction on n. In the base step we need to show that the inequality holds when $n = 0$. Plugging in, we get $(1+x)^0 = 1 \geq 1 + 0 \cdot x$, which is true (we also need to know that $1 + x \neq 0$).

In the induction step we assume that $(1+x)^k \geq 1 + kx$ holds, for some natural number k, and we want to prove the same about $k+1$. We can compute

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k(1+x) \geq (1+kx)(1+x) \\ &= 1 + x + kx + kx^2 \\ &= 1 + (k+1)x + kx^2 \geq 1 + (k+1)x \end{aligned}$$

where we used the induction hypothesis in the first line. In the last line we used the fact that $kx^2 \geq 0$, since k is a natural number and x^2 is never negative, regardless of what x is. $\square$

21. Let $0 < x < 1$ be a real number. Show that $(1+x)^n < 1 + 2^n x$ for all natural n.

Proof. Argue by induction. In the base step we check that the inequality holds for $n = 0$. It does, as $(1+x)^0 = 1 < 1 + x = 1 + 2^0 x$.

In the induction step we assume that $(1+x)^n < 1 + 2^n x$ and prove that $(1+x)^{n+1} < 1 + 2^{n+1} x$. Since $x < 1$ we have $(1+x)^n < 2^n$. Multiplying both sides by x yields

$$x(1+x)^n < 2^n x$$

We now recall the inductive hypothesis

$$(1+x)^n < 1 + 2^n x$$

and add these two inequalities. We get

$$(1+x)^n + x(1+x)^n < 1 + 2^n x + 2^n x$$

or equivalently

$$(1+x)^{n+1} < 1 + 2^{n+1} x$$

$\square$

22. Define a sequence by $a_0 = 1, a_1 = 3$ and $a_n = 2a_{n-1} + 8a_{n-2}$ for $n \geq 2$. Show that $a_n \leq 4^n$ for all natural n.

Proof. Argue by induction on n. In the base step we check the inequality for $n = 0$ and 1. Indeed $a_0 = 1 \leq 4^0$ and $a_1 = 3 \leq 4^1$.

In the induction step we assume that $a_k \leq 4^k$ for all $k \leq n$ and show that $a_{n+1} \leq 4^{n+1}$. We get this by

$$a_{n+1} = 2a_n + 8a_{n-1} \leq 2 \cdot 4^n + 8 \cdot 4^{n-1} = 4 \cdot 4^n = 4^{n+1}$$ $\square$