1. Let A, B be sets. Show that $A \cup (B \cap A) = A$ and $A \cap (B \cup A) = A$.

   Proof. Let x be an arbitrary element of A. Then clearly $x \in A \cup (B \cap A)$, so $A \subseteq A \cup (B \cap A)$. Conversely if x is an element of $A \cup (B \cap A)$ then either $x \in A$ or $x \in B \cap A$. But in either case we can conclude that $x \in A$, so $A \cup (B \cap A) \subseteq A$.

   For the other equality, suppose $x \in A$. Then also $x \in B \cup A$, so $x \in A \cap (B \cup A)$, showing that $A \subseteq A \cap (B \cup A)$. Conversely, if $x \in A \cap (B \cup A)$, then clearly $x \in A$, which shows that $A \cap (B \cup A) \subseteq A$. □

2. Let A, B be sets. Show that if $A \subseteq B$ then $A \setminus B \subseteq B \setminus A$.

   Proof. Assume that $A \subseteq B$. This means that $A \setminus B = \emptyset$ and clearly $\emptyset \subseteq B \setminus A$. □

3. If A, B are sets, define $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Show that $A \triangle B$ and $A \cap B$ are disjoint.

   Proof. Suppose not. Then $(A \cap B) \cap (A \triangle B)$ is nonempty; fix an element x of this intersection. Since $x \in A \cap B$ we see that x is in both A and B. Since $x \in A \triangle B$ we see that either $x \in A \setminus B$ or $x \in B \setminus A$. In either of the two cases x cannot be in both A and B, contradicting our earlier observation. □

4. If A, B are sets, show that $A \cup B = (A \triangle B) \cup (A \cap B)$.

   Proof. Let $x \in (A \triangle B) \cup (A \cap B)$ be arbitrary. Then either $x \in A \triangle B$ or $x \in A \cap B$. If $x \in A \cap B$ then in particular also $x \in A$ so $x \in A \cup B$. If, on the other hand, $x \in A \triangle B$ then either $x \in A \setminus B$ or $x \in B \setminus A$. In the first case $x \in A$ and in the second $x \in B$, so overall we get $x \in A \cup B$ in either case. This shows that $(A \triangle B) \cup (A \cap B) \subseteq A \cup B$.

   Conversely, suppose $x \in A \cup B$. We distinguish two cases, depending on whether $x \in A \cap B$. If so then clearly also $x \in (A \triangle B) \cup (A \cap B)$. Now consider the case when $x \notin A \cap B$. Since $x \in A \cup B$, x is in at least one of A and B and, since $x \notin A \cap B$, x is not in both of them. We may thus assume, without loss of generality, that $x \in A$ and $x \notin B$. But then $x \in A \setminus B$ and so $x \in A \triangle B$ and $x \in (A \triangle B) \cup (A \cap B)$. This shows that $A \cup B \subseteq (A \triangle B) \cup (A \cap B)$. □

5. If A, B are sets, show that $A \triangle B = \emptyset$ iff $A = B$.

   Proof. Suppose that $A = B$. Then $A \setminus B = B \setminus A = \emptyset$ and this means that $A \triangle B = \emptyset \cup \emptyset = \emptyset$.

   Now suppose that $A \triangle B = \emptyset$. We can conclude that $A \setminus B = \emptyset$, which shows that $A \subseteq B$, and $B \setminus A = \emptyset$, which shows that $B \subseteq A$. Taken together, we get that $A = B$. □

6. Let a, b be integers and $a \neq 0$. Show that if $a \mid b$ then $a^{10} \mid b^{10}$.

Proof. Suppose that $a \mid b$. Then there is an integer k such that $b = ka$. Therefore $b^{10} = k^{10}a^{10}$ which means that $a^{10} \mid b^{10}$. $\square$

7. Let a be an integer. Show that if $6 \mid 5a$ then $6 \mid a$.

   Proof. Suppose that $6 \mid 5a$. It follows that $2 \mid 5a$ and $3 \mid 5a$. Since 2 is prime we get that either $2 \mid 5$ or $2 \mid a$ and similarly for 3. But since 5 is not divisible by either 2 or 3, we can conclude that both 2 and 3 divide a. This means that the prime factorization of a includes both 2 and 3 as factors and we can conclude that a is divisible by 6. $\square$

   Comments: If you want to avoid arguing via prime factorizations, you could prove this by considering the remainder of a when divided by 6. Since a is divisible by both 2 and 3, the remainder must be as well and the only such remainder is 0.

   Please note that it is not true in general that if a number c is divisible by two numbers a, b then c is divisible by the product ab (see problem 9 on the first midterm).

8. Let x, y be integers. Show that x and y have opposite parities iff $(x - y + 3)^3$ is even.

   Proof. x and y have opposite parities iff x and $-y$ have opposite parities iff $x - y$ is odd iff $x - y + 3$ is even iff $(x - y + 3)^3$ is even. $\square$

9. Let x be an integer. Show that $2 \mid (x^4 + 1)$ iff $4 \mid (x^2 - 1)$.

   Proof. Suppose that $2 \mid (x^4 + 1)$. Then x^4 is odd, meaning that x is itself odd. Writing $x = 2k + 1$ we can see that $x^2 = 4k^2 + 4k + 1$ and so $x^2 - 1$ is divisible by 4.

   Conversely, suppose that $4 \mid (x^2 - 1)$. Then x^2 is odd, meaning that x is itself odd. But then x^4 is also odd and so $x^4 + 1$ is even. $\square$

10. Let x be an integer. Show that if $x^2 - 1$ is even then it is divisible by 8. Give a counterexample to disprove the same claim about $x^2 - 5$.

    Proof. Suppose that $x^2 - 1$ is even. Then x^2 is odd so x must be odd and we can write $x = 2k + 1$ for some integer k. Then $x^2 - 1 = 4k^2 + 4k = 4k(k + 1)$. One of k and $k + 1$ is even, so $4k(k + 1)$ is divisible by 8.

    To see that it can happen that $x^2 - 5$ is even but not divisible by 8 just take $x = 5$. $\square$

11. If a, b, c are integers and $c \neq 0$ define $a \equiv b \pmod{c}$ if $5 \mid (a - b)$.

    Show that if $a \equiv b \pmod{c}$ and $d \equiv e \pmod{c}$ then $(a + d) \equiv (b + e) \pmod{c}$.

    Proof. Our assumptions tell us that n divides both $a - b$ and $c - d$, thus there are integers k, l such that $a - b = kn$ and $c - d = ln$. By adding these two equations we get that

    $$a + c - (b + d) = a - b + c - d = kn + ln = (k + l)n$$

    which means that n divides $a + c - (b + d)$ and so $a + c \equiv b + d \pmod{n}$. $\square$

12. Show that there is no integer x such that $x^4 \equiv 2 \pmod 5$.

Proof. Consider the possible remainders of x modulo 5.

If $x \equiv 0 \pmod 5$ then $x^4 \equiv 0 \pmod 5$. If $x \equiv 1 \pmod 5$ then $x^4 \equiv 1 \pmod 5$. If $x \equiv 2 \pmod 5$ then $x^4 \equiv 16 \equiv 1 \pmod 5$. If $x \equiv 3 \pmod 5$ then $x^4 \equiv 81 \equiv 1 \pmod 5$. If $x \equiv 4 \pmod 5$ then $x^4 \equiv 256 \equiv 1 \pmod 5$.

So in no case is $x^4 \equiv 2 \pmod 5$. □

Comments: You may have noticed that $x^4 \equiv 1 \pmod 5$ always holds, except when x is divisible by 5. This is an instance of Fermat's little theorem, which states that if p is prime and x is not divisible by p then $x^{p-1} \equiv 1 \pmod p$. Euler's totient theorem generalizes this to nonprime moduli.

13. Show that two integers x, y have the same parity iff $x \equiv y \pmod 2$.

Proof. x and y have the same parity iff x and $-y$ have the same parity iff $x - y$ is even iff $x \equiv y \pmod 2$. □

14. Let a be an integer and $n \geq 2$ a natural number. Show that if $a \equiv 0 \pmod n$ then $a^2 \equiv 0 \pmod{n^2}$. Give a counterexample to show that if a, b are integers then $a \equiv b \pmod n$ does not imply that $a^2 \equiv b^2 \pmod{n^2}$.

Proof. Suppose that $a \equiv 0 \pmod n$. Then there is an integer k such that $a = kn$. Therefore $a^2 = k^2n^2$, which means that $a^2 \equiv 0 \pmod{n^2}$.

For the second part, consider $n = 3, a = 1, b = 4$. We see that $1 \equiv 4 \pmod 3$ but $1 \not\equiv 16 \pmod 9$. □

15. Let a be an integer. Show that $a^4 \equiv (5 - a)^4 \pmod 5$.

Proof. Clearly $(5 - a) \equiv (-a) \pmod 5$ and $(-a)^4 \equiv a^4 \pmod 5$. Putting these two together yields $(5 - a)^4 \equiv a^4 \pmod 5$. □

16. Show that $\sqrt{6}$ is irrational.

Proof. Suppose not. Then there are integers a, b with $b \neq 0$ such that $\sqrt{6} = \frac{a}{b}$. We may further assume that a and b have no common factors. Squaring both sides of the equality and simplifying gives us $a^2 = 6b^2$. In particular, a^2 (and thus a) is even. Writing $a = 2k$ for some integer k, we obtain $4k^2 = 6b^2$ or equivalently $2k^2 = 3b^2$, which means that $3b^2$ is even. Since 3 is odd, b^2 must be even, meaning that b is even. But this means that a and b have a common factor of 2, contradicting our initial assumption. □

17. Show that $\sqrt{2} + \sqrt{3}$ is irrational.

Proof. Suppose not. Then there are integers a, b with $b \neq 0$ such that $\sqrt{2} + \sqrt{3} = \frac{a}{b}$. Squaring both sides yields $2 + 2\sqrt{6} + 3 = \frac{a^2}{b^2}$ or, equivalently, $\sqrt{6} = \frac{a^2}{2b^2} - \frac{5}{2}$. But this would mean that $\sqrt{6}$ was rational, contradicting the previous problem. $\square$

18. Show that $\log_6 7$ is irrational.

Proof. Suppose not. Then there are integers a, b with $b \neq 0$ such that $\log_6 7 = \frac{a}{b}$; in fact, since $\log_6 7$ is positive, we may take both a and b to be natural numbers. The definition of the logarithm gives $6^{\frac{a}{b}} = 7$. Taking b-th powers we get $6^a = 7^b$. Since both a and b are natural numbers, both sides of these equation are natural numbers. But the left one is even while the right one is odd. Contradiction. $\square$

19. Recall the Fibonacci sequence, defined by $a_0 = 0, a_1 = 1$ and $a_n = a_{n-2} + a_{n-1}$ for $n \geq 2$. Let n be a natural number. Show that $\sum_{k=0}^{n} a_k = a_{n+2} - 1$.

Proof. We proceed by induction. In the base case we check that the equation holds when $n = 0$. And it does, since $\sum_{k=0}^{0} a_k = 0 = 1 - 1 = a_2 - 1$.

In the induction step we assume that $\sum_{k=0}^{n} a_k = a_{n+2} - 1$ and prove that $\sum_{k=0}^{n+1} a_k = a_{n+3} - 1$ as follows:

$$\begin{aligned}\sum_{k=0}^{n+1} a_k &= \sum_{k=0}^{n} a_k + a_{n+1}\\ &= a_{n+2} - 1 + a_{n+1}\\ &= a_{n+3} - 1\end{aligned}$$

where we used the inductive hypothesis to go from the first to the second line and the recursive definition of the sequence to go from the second to the third line. This finishes the induction and the proof. $\square$

20. Define a sequence by $a_1 = -3, a_2 = 0$ and $a_{n+1} = 7a_n - 10a_{n-1}$ for $n \geq 2$. Show that $a_n = 2 \cdot 5^{n-1} - 5 \cdot 2^{n-1}$ for all positive natural n.

Proof. We argue by induction on n. In the base step we check the equality for $n = 1$ and 2. Indeed, $a_1 = -3 = 2 - 5$ and $a_2 = 0 = 2 \cdot 5 - 5 \cdot 2$.

In the induction step we assume that $a_k = 2 \cdot 5^{k-1} - 5 \cdot 2^{k-1}$ for all $1 \leq k \leq n$ and show that the same holds for $k = n + 1$. We get

$$\begin{aligned}a_{n+1} &= 7a_n - 10a_{n-1}\\ &= 7(2 \cdot 5^{n-1} - 5 \cdot 2^{n-1}) - 10(2 \cdot 5^{n-2} - 5 \cdot 2^{n-2})\\ &= 14 \cdot 5^{n-1} - 35 \cdot 2^{n-1} - 4 \cdot 5^{n-1} + 25 \cdot 2^{n-1}\\ &= 10 \cdot 5^{n-1} - 10 \cdot 2^{n-1}\\ &= 2 \cdot 5^n - 5 \cdot 2^n\end{aligned}$$

$\square$

21. Let $0 < x < 1$ be a real number. Show that $(1+x)^n < 1 + 2^n x$ for all natural n.

Proof. Argue by induction. In the base step we check that the inequality holds for $n = 0$. It does, as $(1+x)^0 = 1 < 1 + x = 1 + 2^0 x$.

In the induction step we assume that $(1+x)^n < 1 + 2^n x$ and prove that $(1+x)^{n+1} < 1 + 2^{n+1} x$. Since $x < 1$ we have $(1+x)^n < 2^n$. Multiplying both sides by x yields

$$x(1+x)^n < 2^n x$$

We now recall the inductive hypothesis

$$(1+x)^n < 1 + 2^n x$$

and add these two inequalities. We get

$$(1+x)^n + x(1+x)^n < 1 + 2^n x + 2^n x$$

or equivalently

$$(1+x)^{n+1} < 1 + 2^{n+1} x$$

$\square$

Comments: This is the better, more elegant proof. There is also a more brute force proof using the binomial theorem, if you are familiar with it. There we would avoid induction and argue as follows: by the binomial theorem $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. We can then string together the following

$$\begin{aligned}
(1+x)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \sum_{k=1}^{n} \binom{n}{k} x^k \\
&\leq 1 + \sum_{k=1}^{n} \binom{n}{k} x \\
&= 1 + x \sum_{k=1}^{n} \binom{n}{k} \\
&= 1 + x(2^n - 1) < 1 + 2^n x
\end{aligned}$$

22. Define a sequence by $a_0 = 1, a_1 = 3$ and $a_n = 2a_{n-1} + 8a_{n-2}$ for $n \geq 2$. Show that $a_n \leq 4^n$ for all natural n.

Proof. Argue by induction on n. In the base step we check the inequality for $n = 0$ and 1. Indeed $a_0 = 1 \leq 4^0$ and $a_1 = 3 \leq 4^1$.

In the induction step we assume that $a_k \leq 4^k$ for all $k \leq n$ and show that $a_{n+1} \leq 4^{n+1}$. We get this by

$$a_{n+1} = 2a_n + 8a_{n-1} \leq 2 \cdot 4^n + 8 \cdot 4^{n-1} = 4 \cdot 4^n = 4^{n+1}$$

$\square$