Math 156.01 — Spring 2017 Homework set #2

Name:

Please write your solutions in an organized and systematic manner; use scratch paper to solve the problems first and then write up a neat solution with the relevant work shown.

1. If $A = \{1, 2\}$, compute $A \times \mathcal{P}(A)$. [5 pts]

2. For $n \in \mathbb{N}$ let $A_n = (\frac{1}{n+1}, 3 - \frac{1}{n+1})$ (an open interval). Compute $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$. [5 pts]

3. (a) Consider the statement scheme [5 pts]

$$P(x) : x(x-1) = 6$$

where x has domain $\mathbb{N}$. Find those x for which $P(x)$ is true.

(b) Consider the statement scheme

$$Q(n) : n \text{ and } n+2 \text{ are prime}$$

where n has domain $\mathbb{N}$. Find five values of n which make $Q(n)$ true.[1]

[1] Pairs $n, n+2$ such that both of them are prime are called *twin primes*. It is unknown whether there are infinitely many twin primes. In 2013 Yitang Zhang made significant progress by proving that there are infinitely many pairs of primes differing by at most 70,000,000. The discovery earned him many prizes, including a MacArthur grant.

We saw the truth tables for the logical connectives in class. We can now apply these one after another to create truth tables for more complicated statements made up from more than just one connective. For example, suppose we wanted to create the truth table for the statement $P \wedge (P \implies Q)$. We make a column for each of the component statements and build the entire thing by combining two columns at a time, like so:

P	Q	$P \implies Q$	$P \wedge (P \implies Q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

Given two statements R and S, we say that they are *logically equivalent* (written $R \equiv S$) if their columns in the truth table are the same, meaning that given the same truth values for the basic statements involved, R and S have the same truth values. For example, the truth table we have above shows that $P \wedge (P \implies Q)$ is logically equivalent to $P \wedge Q$. Intuitively, logically equivalent statements are "the same", they say the exact same thing. This is useful, because to prove a statement it suffices to prove a logically equivalent statement and this is often easier.
A statement is a *tautology* if its column in the truth table is all Ts (i.e. if it is logically equivalent to the statement T). A statement is a *contradiction* if its column in the truth table is all Fs (i.e. if it is logically equivalent to the statement F).

4. (a) Show that $P \implies Q$ is logically equivalent to $Q \vee \sim P$. [5 pts]

   (b) Show that $(P \implies Q) \vee (Q \implies P)$ is a tautology.

5. (a) Show that $P \implies Q$ and $\sim Q \implies \sim P$ are logically equivalent. [5 pts]
(These two implications are *contrapositives* of one another).

(b) Show that $P \implies Q$ and $Q \implies P$ are not logically equivalent.
(These two implications are *converses* of one another).

6. (extra credit) Consider the following truth table [5 pts]

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

The connective $\uparrow$ is called the *Sheffer stroke* or *NAND*.

Find two statements using only Sheffer strokes (and no other logical connectives) that are equivalent to $\sim P$ and $P \wedge Q$, respectively.