Useful definitions:

- An integer n is *even* if there is an integer k such that $n = 2k$.
- An integer n is *odd* if there is an integer k such that $n = 2k + 1$.
- If a, b are integers and $a \neq 0$ then a *divides* b (written $a \mid b$) if there is an integer k such that $b = ak$. We write $a \nmid b$ if a does not divide b.

You may use the following facts without proof or reference:

- Any logical equivalence or equality of sets proved in class or given as homework.
- The axioms for integers given in class.
- The sum of two integers is even iff they have the same parity.
- The product of two integers is even iff at least one of them is even.

If you need to use some other fact you need to either give a proof for it or give a reference to either the lectures or the textbook.

1. Show that $(\sim (P \iff Q)) \implies (P \vee Q)$ is a tautology. [10 pts]

   We build a truth table for the statement:

P	Q	$P \vee Q$	$P \iff Q$	$\sim (P \iff Q)$	$(\sim (P \iff Q)) \implies (P \vee Q)$
T	T	T	T	F	T
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	T	F	T

   Since the final column in the truth table consists of only Ts, the statement is a tautology.

2. Let A, B and C be sets. Show that $(A \setminus B) \setminus C = A \setminus (B \cup C)$. [10 pts]

   Suppose $x \in (A \setminus B) \setminus C$. Then, by definition, $x \in A \setminus B$ and $x \notin C$. Again, by definition, $x \in A$ and $x \notin B$. Since $x \notin B$ and $x \notin C$, we conclude that $x \notin B \cup C$. So, by definition, $x \in A \setminus (B \cup C)$. This shows that $(A \setminus B) \setminus C \subseteq A \setminus (B \cup C)$.

   Now suppose that $x \in A \setminus (B \cup C)$. Then, by definition, $x \in A$ and $x \notin B \cup C$, which means that $x \notin B$ and $x \notin C$. Again, by definition, $x \in A \setminus B$ and also $x \in (A \setminus B) \setminus C$. This shows $A \setminus (B \cup C) \subseteq (A \setminus B) \setminus C$.

   Altogether, we have shown that $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

3. Show that if n is an integer then either n^2 is odd or 4 divides n^2. [10 pts]

   (Hint: Split into cases depending on the parity of n. Alternatively, split into cases depending on the parity of n^2 and use a theorem from class.)

   Let n be an integer. We consider two cases:

   Case 1: If n is odd then n^2 is odd, by a known fact.

   Case 2: If n is even, we can write $n = 2k$ for some integer k. Then $n^2 = 4k^2$ is divisible by 4.

   In either case, n^2 is odd or divisible by 4.

4. Find an example of a subset A of $\mathbb{N}$ such that $\{1, 2, 3\} \subseteq A$, $(7, 2) \in A \times A$ and $|\mathcal{P}(A)| = 16$. Is there more than one example? [10 pts]

   Since $|\mathcal{P}(A)| = 2^{|A|}$ we known that A has exactly 4 elements. Three of those are $1, 2$ and 3, as given. We also must have $(7, 2) \in A \times A$ which means that $7 \in A$. All of these requirements together tell us that $A = \{1, 2, 3, 7\}$, and the reasoning given shows that this is the only solution.

5. Let n be an even integer and m an odd integer.

   (a) Find an example of integers n, m, a, b, c, d such that $an + bm$ is even and $cn + dm$ is odd. [5 pts]

   We can simply take $n = a = b = c = 0$ and $m = d = 1$. Then $an + bm = 0$ is even and $cn + dm = 1$ is odd.

   (b) If a, b are arbitrary integers, show that $an + bm$ is even iff b is even. [5 pts]

   Since n is even, so is an. Therefore $an + bm$ is even iff bm is. But since m is odd, bm is even iff b is.

6. Let a, b and c be positive natural numbers.

   (a) Show that if $a|b$ and $a|c$ then $a|bc$. [5 pts]

   Assume that $a \mid b$. Then there is an integer k such that $b = ak$. So then $bc = a(ck)$ is also clearly divisible by a.

   (b) Give an example where $a|c$ and $b|c$ but $ab \nmid c$. [5 pts]

   We can take $a = 2$, $b = 4$, and $c = 4$.

7. Show that there is no triple of distinct real numbers a, b, c for which the numbers ab, bc and ac would be equal. [10 pts]

   (Hint: Argue by contradiction and consider cases depending on whether one of a, b, c is 0 or not.)

   Assume, toward a contradiction, that there are distinct real numbers a, b, c such that ab, bc, and ac are all equal. Now consider two cases:

   Case 1: If $a = 0$ then $ab = 0$, so $bc = 0$ as well. But then one of b and c must equal 0, contradicting the fact that a, b, c were assumed to be distinct.

Case 2: If $a \neq 0$ then we can cancel a from $ab = ac$ to conclude that $b = c$, contradicting the assumption that a, b, c were distinct.

8. Let A and B be sets.

   (a) Show that if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ then $A \subseteq B$. [5 pts]

   Assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ and let x be an arbitrary element of A. Then $\{x\}$ is a subset of A, so $\{x\} \in \mathcal{P}(A)$. By our assumption we get $\{x\} \in \mathcal{P}(B)$, so $x \in B$. This shows that $A \subseteq B$.

   Comment: There is a slicker way to argue: since $A \subseteq A$ we get $A \in \mathcal{P}(A)$ and so $A \in \mathcal{P}(B)$. But this means that A is a subset of B.

   (b) Show that if $\mathcal{P}(A) = \mathcal{P}(B)$ then $A = B$. [5 pts]

   (You may use part (a) here even if you didn't prove it.)

   We are assuming that $\mathcal{P}(A) = \mathcal{P}(B)$ or, in other words, that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ and $\mathcal{P}(B) \subseteq \mathcal{P}(A)$. By part (a) the first one of these implies that $A \subseteq B$ and the second one that $B \subseteq A$. Altogether, we get that $A = B$.

9. Let x and y be integers. Show that x and y are both even iff xy and $x + y$ are both even. [10 pts]

   By known facts, $x + y$ is even iff x and y have the same parity and xy is even iff at least one of x and y is even. Therefore both xy and $x + y$ are even iff at least one of x and y is even and they have the same parity. But this happens exactly when both x and y are even.

10. Show that $\sum_{i=0}^{n}(i \cdot i!) = (n + 1)! - 1$ for any natural number n. [10 pts]

    (Recall that $0! = 1, 1! = 1, 2! = 2 \cdot 1 = 2, 3! = 3 \cdot 2 \cdot 1 = 6$, etc.)

    Argue by induction. In the base step we check that $\sum_{i=0}^{0} i \cdot i^2 = 0$ equals $(0+1)!-1 = 0$, which it does.

    In the induction step we let k be a natural number and assume that $\sum_{i=0}^{k}(i \cdot i!) = (k + 1)! - 1$. We need to show that the same holds for $k + 1$. We thus calculate

$$\begin{aligned}
\sum_{i=0}^{k+1}(i \cdot i!) &= \sum_{i=0}^{k}(i \cdot i!) + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)! \\
&= (k+1)!(1 + k + 1) - 1 = (k+2)! - 1 \\
&= ((k+1)+1)! - 1
\end{aligned}$$