Useful definitions:

- An integer $n$ is *even* if there is an integer $k$ such that $n = 2k$.
- An integer $n$ is *odd* if there is an integer $k$ such that $n = 2k + 1$.
- If $a, b$ are integers and $a \neq 0$ then $a$ *divides* $b$ (written $a \mid b$) if there is an integer $k$ such that $b = ak$. We write $a \nmid b$ if $a$ does not divide $b$.
- A natural number $p > 1$ is *prime* if whenever $p$ divides a product of two integers $p$ divides (at least) one of the two factors. If a natural number $n > 1$ is not prime we call it *composite*.

You may use the following facts without proof or reference:

- Any logical equivalence or equality of sets proved in class or given as homework.
- The axioms for integers given in class.
- The sum of two integers is even iff they have the same parity.
- The product of two integers is even iff at least one of them is even.
- Any reasonable manipulation of inequalities between real numbers, e.g. from $x \leq y$ and $z \leq w$ conclude $x + z \leq y + w$ or from $x \leq y$ and $z \geq 0$ conclude $xz \leq yz$.
- A natural number $p > 1$ is prime iff its only positive factors are 1 and $p$.
- If $p$ is prime then $\sqrt{p}$ is irrational. In particular, $\sqrt{2}$ is irrational.
- (the fundamental theorem of arithmetic) Any natural number $n > 1$ can be written uniquely as a product of (not necessarily distinct) primes. In particular, every natural $n > 1$ has a prime factor.
- (Euclid's theorem) There are infinitely many primes.

If you need to use some other fact you need to either give a proof for it or give a reference to either the lectures or the textbook.

1. Show that $(\sim (P \iff Q)) \implies (P \vee Q)$ is a tautology. [10 pts]

2. Let $A, B$ and $C$ be sets. Show that $(A \setminus B) \setminus C = A \setminus (B \cup C)$. [10 pts]

3. Show that if $n$ is an integer then either $n^2$ is odd or 4 divides $n^2$. [10 pts]

   (Hint: Split into cases depending on the parity of $n$. Alternatively, split into cases depending on the parity of $n^2$ and use a theorem from class.)

4. Show that if $x$ is rational and $y$ is irrational then $xy$ is irrational. [10 pts]

5. Let $n$ be an even integer and $m$ an odd integer.

   (a) Find an example of integers $n, m, a, b, c, d$ such that $an + bm$ is even and $cn + dm$ is odd. [5 pts]

   (b) If $a, b$ are arbitrary integers, show that $an + bm$ is even iff $b$ is even. [5 pts]

6. Show that $\sum_{k=1}^{n}(k \cdot k!) = (n+1)! - 1$ for any integer $n \geq 1$. [10 pts]

7. (a) Show that the sum of two rational numbers is rational. [3 pts]

   (b) Give an example of two irrational numbers whose sum is rational (depending on your example you may want to justify why the sum is rational). [3 pts]

   (c) Given an example of two irrational numbers whose sum is irrational (with a justification why the sum is actually irrational). [4 pts]

8. (extra credit)

(a) Show that if $x, y$ are real numbers and $n \geq 2$ a natural number then [5 pts]

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$$

(Hint: This is easier than it looks and you don't need induction. Just try to compute the right side.)

(b) Use part (a) to show that if $n > 1$ is composite then $2^n - 1$ is composite. [5 pts] (Primes of the form $2^n - 1$ are called *Mersenne primes*. Part (b) shows that the $n$ for any Mersenne prime must be prime itself. There are currently only 48 known Mersenne primes and the largest known prime number is a Mersenne prime.)