1. Suppose that x is a rational number and y an irrational number. Show that $x+y$ is irrational.

   Assume towards a contradiction that $x+y$ is rational. By definition, that means that there are integers a, b with $b \neq 0$ such that $x+y=\frac{a}{b}$. Furthermore, since x is rational, there are integers c, d with $d \neq 0$ such that $x=\frac{c}{d}$. But then

   $$y=(x+y)-x=\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

   which contradicts the fact that y is irrational.

2. Show that if x and y are positive real numbers then $\sqrt{x+y} \neq \sqrt{x}+\sqrt{y}$.

   Assume toward a contradiction that $\sqrt{x+y}=\sqrt{x}+\sqrt{y}$. Squaring both sides we get $x+y=x+2\sqrt{x}\sqrt{y}+y$. Cancelling from both sides we get $2\sqrt{x}\sqrt{y}=0$. Squaring again we get $4xy=0$, and the only way this can happen is if at least one of x or y is 0. But this contradicts our initial assumption that x and y are positive.

3. Prove the following statements (two thirds of them are easy):

   (a) There is an irrational number a and a rational number b such that a^b is irrational.

   Let $a=\sqrt{2}$ and $b=1$. Then $a^b=\sqrt{2}$ is irrational.

   (b) There are rational numbers a and b such that a^b is irrational.

   Let $a=2$ and $b=\frac{1}{2}$. Then $a^b=\sqrt{2}$ is irrational.

   (c) There is a rational number a and an irrational number b such that a^b is irrational.

   Consider two cases:

   Case 1: If $2^{\frac{1}{2\sqrt{2}}}$ is irrational we can take $a=2$ and $b=\frac{1}{2\sqrt{2}}$. Then $a^b=2^{\frac{1}{2\sqrt{2}}}$ is irrational, by assumption.

   Case 2: If $2^{\frac{1}{2\sqrt{2}}}$ is rational we can take $a=2^{\frac{1}{2\sqrt{2}}}$ and $b=\sqrt{2}$. Then $a^b=\sqrt{2}$ is irrational.

   Comment: A clever solution for (c) is to remember that we proved in class that there are irrational numbers x, y such that x^y is rational. To solve (c) we can then take $a=x^y$ and $b=\frac{1}{y}$.

4. Show that $\sqrt{3}$ is irrational.

   (Hint: there are a few different ways to prove this. Depending on which strategy you pick, you may need to, along the way, prove that if a^2 is divisible by 3 then a is as well.)

We first prove a lemma: if a is an integer and a^2 is divisible by 3 then a is divisible by 3.

We prove this by contrapositive. Suppose that a is not divisible by 3. We can then consider two cases:

Case 1: If $a = 3k + 1$ for some integer k then $a^2 = 9k^2 + 6k + 1$ is not divisible by 3.

Case 2: If $a = 3k + 2$ for some integer k then $a^2 = 9k^2 + 12k + 4$ is not divisible by 3.

In either case, a^2 is not divisible by 3. This concludes the proof of the lemma.

Returning to the main problem, assume toward a contradiction that $\sqrt{3}$ is rational and we can write $\sqrt{3} = \frac{a}{b}$ for some integers a, b, with $b \neq 0$. We may also assume that a and b have no common factors. Squaring both sides of the equation we get $3 = \frac{a^2}{b^2}$ or, simplified, $a^2 = 3b^2$. So a^2 is divisible by 3, and by the lemma a is divisible by 3 as well. We can thus write $a = 3c$ for some integer c. But then $(3c)^2 = 3b^2$; we can simplify this to get $3c^2 = b^2$. This means that b^2 is divisible by 3, and, by the lemma, so is b. But then both a and b are divisible by 3, contradicting our assumption that a and b have no common factors. Therefore $\sqrt{3}$ must be irrational.

5. Follow these steps to prove that $\sqrt{6}$ is irrational:

   (a) Let a be an integer. Show that if $3 \mid a^2$ then $3 \mid a$ (i.e. prove the hint from problem 4).

   See problem 4.

   (b) Let a be an integer. Show that if $2 \mid a$ and $3 \mid a$ then $6 \mid a$.

   Suppose that $2 \mid a$ and $3 \mid a$. By definition we can find integers k and l such that $a = 2k$ and $a = 3l$. In other words $2k = 3l$. So $3l$ is even and, by a problem on homework 5, l must be even. We can write $l = 2t$. Plugging back in, we get $a = 3(2t) = 6t$, so a is divisible by 6.

   (c) Prove that $\sqrt{6}$ is irrational, using parts (a) and (b).

   Assume towards a contradiction that $\sqrt{6}$ is rational and can be written as $\sqrt{6} = \frac{a}{b}$ for some integers a, b with $b \neq 0$. Furthermore, we may assume that a and b have no common factors. Squaring both sides and simplifying we get $a^2 = 6b^2$. This means that a^2 is divisible by both 2 and 3. By part (a) and a theorem from class, we can conclude that a is divisible by both 2 and 3. By part (b), a must be divisible by 6, so we can write $a = 6c$ for some integer c. Plugging back in and simplifying we get $6c^2 = b^2$. Doing the same argument again, this time for b, we see that b is divisible by 6. But this contradicts our assumption that a and b have no common factors. Therefore $\sqrt{6}$ must be irrational.