1. Disprove the following statement: Let n be an integer. If $\frac{(n+1)(n+2)}{2}$ is odd then $\frac{n^2(n+1)^2}{2}$ is odd.

   To disprove this statement we just need to provide a counterexample: a number n such that $\frac{(n+1)(n+2)}{2}$ is odd and $\frac{n^2(n+1)^2}{2}$ is even. The number $n = 0$ works.

2. Let n be a natural number. Show that $\sum_{i=0}^{n}(3i-2) = \frac{3n^2-n-4}{2}$.

   We argue by induction. In the base step, we check that $\sum_{i=0}^{0}(3i-2) = -2$ equals $\frac{3\cdot 0^2-0-4}{2} = -2$, which it does.

   In the induction step we let k be a natural number and assume that $\sum_{i=0}^{k}(3i-2) = \frac{3k^2-k-4}{2}$. We want to show that the same holds for $k+1$. So we compute:

$$\begin{aligned}\sum_{i=0}^{k+1}(3i-2) = \sum_{i=0}^{k}(3i-2) + 3(k+1) - 2 &= \frac{3k^2-k-4}{2} + 3k + 1 \\ &= \frac{(3k^2-k-4)+(6k+2)}{2} = \frac{3k^2+5k-2}{2} \\ &= \frac{3(k+1)^2-(k+1)-4}{2}\end{aligned}$$

3. Let n be a natural number. Show that $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

   We argue by induction. In the base step we check that $\sum_{i=0}^{0} i^2 = 0$ equals $\frac{0(0+1)(2\cdot 0+1)}{6} = 0$, which it does.

   In the induction step we let k be a natural number and assume that $\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. We want to show that the same holds for $k+1$. So we compute:

$$\begin{aligned}\sum_{i=0}^{k+1} i^2 = \sum_{i=0}^{k} i^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)(2k^2+k)+6(k+1)^2}{6} = \frac{(k+1)(2k^2+k+6k+6)}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}\end{aligned}$$

4. Define a sequence recursively by letting $a_0 = 1$ and $a_n = 2a_{n-1}$ for $n \geq 1$. Find a closed form expression for this sequence and prove that it is correct.

   We write out a few of the initial terms of the sequence: $a_0 = 1, a_1 = 2, a_2 = 4, a_3 = 8, \ldots$. It seems like $a_n = 2^n$ might be the closed form we are looking for. Let us check that it is correct by induction.

In the base step we simply check that $a_0 = 1$ equals $2^0 = 1$. In the induction step we let k be a natural number and assume that $a_k = 2^k$. We need to show that $a_{k+1} = 2^{k+1}$. But by the recursive definition, $a_{k+1} = 2a_k = 2 \cdot 2^k = 2^{k+1}$.

5. Let n be a natural number. Show that $3 \mid n^3 - n$.

   We argue by induction. In the base step we check that $0^3 - 0 = 0$ is divisible by 3, which it is.

   In the induction step we let k be a natural number and assume that $k^3 - k$ is divisible by 3. We need to see that $(k+1)^3 - (k+1)$ is also divisible by 3. We compute

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 + 3k^2 + 2k - 1 = (k^3 - k) + (3k^2 + 3k) \\ &= (k^3 - k) + 3(k^2 + k)\end{aligned}$$

   We know that $k^3 - k$ is divisible by 3 from the inductive hypothesis and $3(k^2 + k)$ is obviously divisible by 3. So their sum $(k+1)^3 - (k+1)$ is also divisible by 3.

   Comment: There is a slicker way of proving this avoiding induction. We can notice that $n^3 - n = n(n+1)(n-1)$ is a product of three consecutive integers. One of the three must be divisible by 3, so the entire product will be as well.