1. Consider the following sets [5 pts]

$$\begin{aligned} A &= \{-1, 1, 2\} \\ B &= \emptyset \\ C &= \{x \in \mathbb{Z}; x^4 - 5x^2 + 4 = 0\} \\ D &= \{x \in \mathbb{N}; \cos(x) = 0\} \end{aligned}$$

The sets A and B are already given in a simple way. We can simplify C and D as well by listing their elements.

The set C consists of integer zeros of the polynomial $x^4 - 5x^2 + 4 = 0$. A calculation shows that these are ± 1 and ± 2, so $C = \{1, -1, 2, -2\}$.

The set D consists of natural number zeros of the function cos. There are no such zeros, so $D = \emptyset$.

(a) Which of these four sets are equal?
We have $B = D$, since both are empty. All other pairs are distinct.

(b) What is the cardinality of C?
We listed the four elements of C above, so $|C| = 4$.

(c) Find two sets among these such that one will be a proper subset of the other.
There are many possible solutions here. For example $B \subset A$ or $B \subset C$, since both A and C are nonempty (but not $B \subset D$ as $B = D$). Another possibility is $A \subset C$ (since every element of A is an element of C and $A \neq C$).

(d) Find $A \cup C$ and $A \cap C$.
Going by the definitions of union and intersection we get $A \cup C = \{1, -1, 2, -2\}$ and $A \cap C = \{1, -1, 2\}$.

(e) Find $A \cap B$.
We see that A and B have no elements in common (because B has no elements at all) and so, going by the definition of intersection, $A \cap B = \emptyset$.

2. Give an example of three sets A, B and C such that $B \neq C$ but $B \setminus A = C \setminus A$. [5 pts]

There are literally infinitely many possible solutions here. For example, we could take

$$A = \{2, 3\}, \quad B = \{1, 2\}, \quad C = \{1, 3\}$$

Then obviously $B \neq C$ but $B \setminus A = \{1\} = C \setminus A$.

Comments: A clever solutions is to let A be any nonempty set and take $B = \emptyset$ and $C = A$.

3. Give an example of four different subsets A, B, C and D of $\{1, 2, 3, 4\}$ such that all 6 intersections of two of them (i.e. $A \cap B$, $A \cap C$, $A \cap D$, $B \cap C$, etc.) will be distinct. [5 pts]

   Again, there are several possible solutions here. For example, we can take

   $$A = \{1, 2\}, \quad B = \{1, 2, 3\}, \quad C = \{2, 3, 4\}, \quad D = \{1, 3, 4\}$$

   The pairwise intersections are then as follows:

   $$\begin{array}{lll} A \cap B = \{1, 2\} & A \cap C = \{2\} & A \cap D = \{1\} \\ B \cap C = \{2, 3\} & B \cap D = \{1, 3\} & \\ C \cap D = \{3, 4\} & & \end{array}$$

   and they are all distinct.

   Comments: If you think hard there is a strategy of finding combinations of sets like these, but for these small examples it is usually better to try a few times and see what works.

4. Find an example of two infinite subsets A_1 and A_2 of $\mathbb{N}$, satisfying $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = \mathbb{N}$. [5 pts]

   One possibility here is to let A_1 be the set of odd numbers and A_2 the set of even numbers. Another is to take A_1 to be the set of prime numbers and A_2 the set of composite numbers.

5. For $A = \{1, 2\}$ and $B = \{4\}$, determine $\mathcal{P}(A \times B)$. [5pts]

   We first compute $A \times B$. This consists of ordered pairs whose fist component comes from A and whose second component comes from B, so we get

   $$A \times B = \{(1, 4), (2, 4)\}$$

   It now remains for us to compute the power set

   $$\mathcal{P}(A \times B) = \{\emptyset, \{(1, 4)\}, \{(2, 4)\}, \{(1, 4), (2, 4)\}\}$$

6. (extra credit) [5 pts]

   (a) Find an example of three infinite subsets A_1, A_2 and A_3 of $\mathbb{N}$, such that $A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$ and $A_1 \cup A_2 \cup A_3 = \mathbb{N}$.

   There are (at least) two approaches here. We can start fresh and try to imitate problem 4. This way we arrive at, for example,

   $$\begin{aligned} A_1 &= \{n \in \mathbb{N}; n \text{ is divisible by } 3\} \\ A_2 &= \{n \in \mathbb{N}; n \text{ gives remainder 1 when divided by } 3\} \\ A_3 &= \{n \in \mathbb{N}; n \text{ gives remainder 2 when divided by } 3\} \end{aligned}$$

A different approach is to take a pair of sets we got in problem 4 and break up one of them further. For example, we can take the pair odd numbers/even numbers and get

$$
\begin{aligned}
A_1 &= \{n \in \mathbb{N}; n \text{ is odd}\} \\
A_2 &= \{n \in \mathbb{N}; n \text{ is even but not divisible by } 4\} \\
A_3 &= \{n \in \mathbb{N}; n \text{ is divisible by } 4\}
\end{aligned}
$$

Or we could take the pair prime numbers/composite numbers and get

$$
\begin{aligned}
A_1 &= \{n \in \mathbb{N}; n \text{ is prime}\} \\
A_2 &= \{n \in \mathbb{N}; n \text{ is composite and even}\} \\
A_3 &= \{n \in \mathbb{N}; n \text{ is composite and odd}\}
\end{aligned}
$$

(b) Find an example of a family $\{A_n\}_{n\in\mathbb{N}}$ satisfying the following conditions:

- each A_n is an infinite subset of $\mathbb{N}$;
- $A_n \cap A_m = \emptyset$ for any two distinct indices n and m;
- $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{N}$.

Let A_n consist of those numbers that are divisible by 2^n but not by 2^{n+1}. For example, A_0 consists of all odd natural numbers and A_1 consists of those even natural numbers which are not divisible by 4. These sets are easily seen to satisfy the requirements.

Comment: The idea here is to try doing the second approach for part (a) over and over again, i.e. split $\mathbb{N}$ into two sets, split one of the parts again, then split one of those parts and so on.

To start with, let A_1 be the set of odd numbers and B_1 the set of even numbers. We decide we won't touch A_1 ever again and it becomes the first member of the family we are building.

We split B_1, the set of even numbers, into two again: A_2, the set of even numbers not divisible by 4, and B_2, the set of numbers divisible by 4. We leave A_2 alone and it becomes the second member of our family.

We split B_2, the set of numbers divisible by 4, into two: A_3, the set of numbers divisible by 4 but not by 8, and B_3, the set of numbers divisible by 8. We leave A_3 alone and put it in our family.

We keep doing this and notice a pattern.