1. Prove that if A, B are sets and $A \cap B = A$ then $A \subseteq B$.

   Assume that $A \cap B = A$. Let x be an arbitrary element of A. By our assumption that $A = A \cap B$, it follows that $x \in A \cap B$, so $x \in B$. This proves that $A \subseteq B$.

   Comment: A slicker way to prove this is to recall that we already proved on a previous homework that $A \cap B \subseteq B$. Since $A = A \cap B$, it immediately follows that $A \subseteq B$.

2. Suppose A, B are sets and $A \subseteq B$. Prove that $A \cup B = B$.

   (Hint: at some point in the proof it will be useful to split into cases.)

   Assume that $A \subseteq B$. We need to show that $A \cup B \subseteq B$ and $B \subseteq A \cup B$. We proved the second one of these in class, so let us focus on the first one. Let x be an arbitrary element of $A \cup B$. Then either $x \in A$ or $x \in B$.

   Case 1: If $x \in B$, we are done.

   Case 2: If $x \in A$ then, since we are assuming that $A \subseteq B$, it follows that $x \in B$.

   In both cases we concluded that $x \in B$, proving that $A \cup B \subseteq B$ and this finishes the proof that $A \cup B = B$.

3. Prove that the sum of two odd integers is even.

   Let a and b be odd integers. By definition there are integers k, l such that $a = 2k + 1$ and $b = 2l + 1$. Then we can write

   $$a + b = 2k + 1 + 2l + 1 = 2(k + l) + 2 = 2(k + l + 1)$$

   Since $k + l + 1$ is an integer, $a + b$ is even by definition.

4. Let x and y be integers. Show that if x is even or $y = 0$ then xy is even.

   Assume that x is even or that $y = 0$. We split into cases:

   Case 1: Suppose that x is even. Then $x = 2k$ for some integer k. But then $xy = 2ky$ and, since ky is an integer, xy is even by definition.

   Case 2: Suppose that $y = 0$. Then $xy = 0$ is an even number.

   In both of the cases we showed that xy is even, finishing the proof.

5. Let a and b be integers. Show that $(a + b)^2 = a^2 + b^2$ if and only if at least one of a and b is 0.

   We are proving a biconditional statement, so we need to show two directions:

   $(\Longrightarrow)$: Suppose that $(a + b)^2 = a^2 + b^2$. Expanding, this means that

   $$a^2 + 2ab + b^2 = a^2 + b^2$$

so $2ab = 0$ or, even more simply, $ab = 0$. But for the product to be 0, at least one of a and b must be 0.

$(\Longleftarrow)$: Suppose that either $a = 0$ or $b = 0$. We split into cases:

Case 1: If $a = 0$ then $a^2 + b^2 = 0^2 + b^2 = b^2$ and $(a + b)^2 = (0 + b)^2 = b^2$ are equal, as required.

Case 2: If $b = 0$ then arguing as in case 1 shows that $(a + b)^2 = a^2 + b^2$.

In both cases we got the desired equality, finishing the proof.

Comment: Alternatively, we could have noticed in the proof of the $(\Longrightarrow)$ direction that all of the equation manipulations we did were reversible. So in fact that part shows that $(a + b)^2 = a^2 + b^2$ is *equivalent* to saying that $ab = 0$ which is equivalent to saying that at least one of a and b is 0. If we notice this then the $(\Longleftarrow)$ part of the proof becomes unnecessary.

6. (extra credit) Let x be an integer. Prove that if 2^{2x} is an odd integer then 2^{-2x} is an odd integer.

   Assume that x is an integer and 2^{2x} is an odd integer. The only way this can happen is if $x = 0$; otherwise, if $x \geq 1$ then 2^{2x} is a product of a number of 2s, which is obviously even, and if $x \leq -1$ then 2^{2x} is a negative power of 2, which won't be an integer. So, from our assumption we can conclude that $x = 0$. But then $2^{-2x} = 2^0 = 1$ is also an odd integer, as required.