1. Show that $(P \implies Q) \vee (Q \implies P)$ is a tautology. [5 pts]

   To show that the statement is a tautology we compute its truth table and verify that only T appears in its column.

P	Q	$P \implies Q$	$Q \implies P$	$(P \implies Q) \vee (Q \implies P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

2. Show, using the logical equivalences given in class (or as Theorems 2.17 & 2.18 in the textbook), that the statements $P \implies (Q \implies R)$ and $(P \wedge Q) \implies R$ are logically equivalent. Do not use truth tables. [5 pts]

   Let's start with one of the statements and produce a string of equivalent statements, ending at the other statement.

$$
\begin{aligned}
P \implies (Q \implies R) &\equiv P \implies (\sim Q \vee R) \equiv \sim P \vee (\sim Q \vee R) \\
&\equiv (\sim P \vee \sim Q) \vee R \equiv \sim (P \wedge Q) \vee R \\
&\equiv (P \wedge Q) \implies R
\end{aligned}
$$

   Comments: Instead of starting with one statement and ending up with the other, we could also have worked with each side separately and met up somewhere in the middle. This is often easier.

3. (a) Show that $P \implies Q$ and $\sim Q \implies \sim P$ are logically equivalent. [5 pts]
   (These two implications are *contrapositives* of one another).

   (b) Show that $P \implies Q$ and $Q \implies P$ are not logically equivalent.
   (These two implications are *converses* of one another).

   (a) We can show this using a truth table or alternatively by observing that

$$P \implies Q \equiv \sim P \vee Q \equiv \sim P \vee \sim\sim Q \equiv \sim Q \implies \sim P$$

   (b) We show that the two statements are not logically equivalent by building their truth tables and observing that their respective columns do not match.

P	Q	$P \implies Q$	$Q \implies P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Comments: A few people attempted to prove that converse implications are not logically equivalent by applying certain logical equivalence rules. But if you think about it, to show logical inequivalence via these rules, you would need to show that there is *no* string of logical equivalences starting with one implication and ending with the converse. There isn't a clear way of doing this.

4. Consider the statement [5 pts]

$$P(n) : 2n^2 + 11 \text{ is prime}$$

(a) Is the statement $\forall n \in \{0, 1, 2, 3, 4, 5\} \colon P(n)$ true or false? Justify your answer.

(b) Is the statement $\sim \exists n \in \mathbb{N} \colon P(n)$ true or false? Justify your answer.

(c) Is the statement $\forall n \in \mathbb{N} \colon P(n)$ true or false? Justify your answer.
(You may want to find a list of primes online.)

(a) The statement says that $2n^2 + 11$ is prime if n is any natural number between 0 and 5. We can just check this by just plugging in these values for n . It turns out we get primes in all six cases, so the original statement is true.

(b) The statement says that there is no natural number n for which $2n^2 + 11$ would be prime. But in part (a) we found several n which make this expression prime. Therefore the statement is false.

(c) The statement says that $2n^2 + 11$ is prime for any natural number n. It is not immediately clear whether this is true or false, so to get a better idea we should try out a few possible values for n. It turns out that taking $n = 11$ makes $2n^2 + 11$ composite. Therefore the original statement is false.

Interesting, if irrelevant facts: The formula $2n^2 + 11$ gives primes for integers n between 0 and 10. By changing the formula we can get longer stretches of primes; the most well known is probably $n^2 + n + 41$, due to Euler, which gives primes for the first 40 natural numbers. We could play around with this and try to get polynomials which give more and more primes, but unfortunately a theorem of Goldbach says that no polynomial with integer coefficients can have prime values for all natural inputs n. Nevertheless, a fancy result in mathematical logic called the MRDP theorem (due to Matiyasevich, Robinson, Davis and Putnam) says that there is a polynomial with integer coefficients (necessarily in many variables) such that the first coordinates of its roots are exactly the primes.

5. Using the logical equivalences from class we can see that any statement is [5 pts] logically equivalent to another statement which only involves negations, conjunctions and disjunctions.

Argue that in fact any statement is logically equivalent to one which only involves negations and conjunctions.

(Hint: you need to show that the statement $P \vee Q$ is logically equivalent to a statement involving only negations and conjunctions).

Following the hint we observe that

$$P \vee Q \equiv \sim\sim P \vee \sim\sim Q \equiv \sim (\sim P \wedge \sim Q)$$

and so $P \vee Q$ is equivalent to a statement involving only negations and conjunctions.

Comment: As far as finishing the argument goes, recall that we can unravel biconditionals into implications and conjunctions via

$$P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$$

and implications into disjunctions and negations via

$$P \implies Q \equiv \sim P \vee Q$$

It now follows that we can translate any statement (built up from negations, conjunctions, disjunctions, implications and biconditionals) into a logically equivalent statement only involving negations and conjunctions.

6. (extra credit) Consider the following truth table [5 pts]

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

The connective $\uparrow$ is called the *Sheffer stroke* or *NAND*.

Show that any statement is logically equivalent to one which only involves Sheffer strokes.

(Hint: following problem 5, you need to show that the statements $\sim P$ and $P \wedge Q$ are logically equivalent to statements involving only Sheffer strokes).

We spend some time playing around with truth tables and eventually come up with the following:

P	Q	$P \uparrow Q$	$\sim P$	$P \uparrow P$	$P \wedge Q$	$(P \uparrow Q) \uparrow (P \uparrow Q)$
T	T	F	F	F	T	T
T	F	T	F	F	F	F
F	T	T	T	T	F	F
F	F	T	T	T	F	F

The truth table shows that $\sim P \equiv P \uparrow P$ and $P \wedge Q \equiv (P \uparrow Q) \uparrow (P \uparrow Q)$.