Math 156.01 — Spring 2017 Midterm exam

Name: EMPLID:

Please write your solutions in an organized and systematic manner.
Please show all work for partial credit. I may give no credit for solutions with no supporting work.
The maximum number of points is shown next to each problem.

Useful definitions:

- An integer n is *even* if there is an integer k such that $n = 2k$.
- An integer n is *odd* if there is an integer k such that $n = 2k + 1$.
- If a, b are integers and $a \neq 0$ then a *divides* b (written $a \mid b$) if there is an integer k such that $b = ak$. We write $a \nmid b$ if a does not divide b.

You may use the following facts without proof or reference:

- Any logical equivalence or equality of sets proved in class or given as homework.
- The axioms for integers given in class.
- The sum of two integers is even iff they have the same parity.
- The product of two integers is even iff at least one of them is even.

If you need to use some other fact you need to either give a proof for it or give a reference to either the lectures or the textbook.

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1. Show that $(\sim (P \iff Q)) \implies (P \vee Q)$ is a tautology. [10 pts]

2. Let A, B and C be sets. Show that $(A \setminus B) \setminus C = A \setminus (B \cup C)$. [10 pts]

3. Show that if n is an integer then either n^2 is odd or 4 divides n^2. [10 pts]

   (Hint: Split into cases depending on the parity of n. Alternatively, split into cases depending on the parity of n^2 and use a theorem from class.)

4. Find an example of a subset A of $\mathbb{N}$ such that $\{1, 2, 3\} \subseteq A$, $(7, 2) \in A \times A$ and $|\mathcal{P}(A)| = 16$. Is there more than one example? [10 pts]

5. Let n be an even integer and m an odd integer.

   (a) Find an example of integers n, m, a, b, c, d such that $an + bm$ is even and $cn + dm$ is odd. [5 pts]

   (b) If a, b are arbitrary integers, show that $an + bm$ is even iff b is even. [5 pts]

6. Let a, b and c be positive natural numbers.

   (a) Show that if $a|b$ and $a|c$ then $a|bc$. [5 pts]

   (b) Give an example where $a|c$ and $b|c$ but $ab \nmid c$. [5 pts]

7. Show that there is no triple of distinct real numbers a, b, c for which the numbers ab, bc and ac would be equal. [10 pts]

   (Hint: Argue by contradiction and consider cases depending on whether one of a, b, c is 0 or not.)

8. Let A and B be sets.

(a) Show that if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ then $A \subseteq B$. [5 pts]

(b) Show that if $\mathcal{P}(A) = \mathcal{P}(B)$ then $A = B$. [5 pts]
(You may use part (a) here even if you didn't prove it.)

9. Let x and y be integers. Show that x and y are both even iff xy and $x+y$ are both even. [10 pts]

10. Show that $\sum_{i=0}^{n}(i \cdot i!) = (n+1)! - 1$ for any natural number n. [10 pts]
(Recall that $0! = 1, 1! = 1, 2! = 2 \cdot 1 = 2, 3! = 3 \cdot 2 \cdot 1 = 6$, etc.)