1. Prove that the sum of two odd integers is even. [5 pts]

   Let x and y be arbitrary odd integers. We can write $x = 2k+1$ and $y = 2l+1$ where k and l are some integers. Then

   $$x + y = 2k + 1 + 2l + 1 = 2(k + l + 1)$$

   and, since $k + l + 1$ is an integer, this shows that $x + y$ is even.

2. Let A and B be sets. Prove that if $A \subseteq B$ then $A \cap B = A$. [5 pts]

   (Hint: Remember, to prove two sets X, Y are equal you need to prove $X \subseteq Y$ and $Y \subseteq X$).

   Let A, B be sets and assume $A \subseteq B$. We will show that $A \cap B = A$ and this amounts to showing that $A \cap B \subseteq A$ and $A \subseteq A \cap B$.

   To see that $A \cap B \subseteq A$ we take an arbitrary $x \in A \cap B$. Then, in particular, $x \in A$. Thus $A \cap B \subseteq A$.

   Conversely, to see that $A \subseteq A \cap B$, take an arbitrary $x \in A$. Since we assumed that $A \subseteq B$ it follows that $x \in B$ and therefore $x \in A \cap B$. Thus $A \subseteq A \cap B$.

3. Let x be an integer. Prove that if 2^{2x} is an odd integer then 2^{-2x} is an odd integer. [5 pts]

   Consider what happens with the expression 2^{2x} as x varies. If x is negative then 2^{2x} is a noninteger rational number. If x is positive then 2^{2x} is a positive power of 2 and is thus even (because we can write $2^{2x} = 2 \cdot 2^{2x-1}$ and 2^{2x-1} is an integer if $x > 0$). The only case when 2^{2x} is odd is when $x = 0$ and then $2^{2x} = 2^0 = 1$.

   Now, for the proof: let x be an arbitrary integer and assume that 2^{2x} is an odd integer. By the argument above this means that x must be 0. But then $2^{-2x} = 2^0 = 1$ is also an odd integer.

4. Here is a definition:

   A subset A of $\mathbb{N}$ is *bounded* if there is a natural number n such that $A \subseteq \{0, 1, \ldots, n\}$.

   Prove that the union of two bounded sets is bounded. [5 pts]

   Let A and B be finite subsets of $\mathbb{N}$. Thus there are natural numbers n, m such that $A \subseteq \{0, 1, \ldots, n\}$ and $B \subseteq \{0, 1, \ldots, m\}$.

   Let p be any number larger than both n and m (for example, we could take $p = \max(n, m)$). To see that $A \cup B$ is finite we will show that $A \cup B \subseteq \{0, 1, \ldots, p\}$. So take $x \in A \cup B$. Then either $x \in A$ or $x \in B$. If $x \in A$ then $x \in \{0, 1, \ldots, n\}$ and so $x \in \{0, 1, \ldots, p\}$ by our choice of p. Alternatively, if $x \in B$ then $x \in \{0, 1, \ldots, m\}$ and so again $x \in \{0, 1, \ldots, p\}$ by the choice of p. All in all, this shows that $A \cup B \subseteq \{0, 1, \ldots, p\}$.

5. Let A, B be sets contained in some universal set. Show that if $A \subseteq B$ then $B^c \subseteq A^c$.

[5 pts]

Start by assuming that $A \subseteq B$. To see that $B^c \subseteq A^c$, let us take an arbitrary $x \in B^c$. Then x is not an element of B. Since we assumed that $A \subseteq B$ we can conclude that x is also not an element of A (since otherwise it would be an element of B). So we get $x \in A^c$.

6. (extra credit) [5 pts]

Let $A = \{0, 1, 2, 3, 4, 5\}$. Prove that if x is an element of every subset of A then $x = 1$.

Here is one version of the proof (proceeding directly): Assume that x is an element of every subset of A. In particular, $\{1\}$ is a subset of A so $x \in \{1\}$. It follows that $x = 1$.

Another version of the proof: We can show that the implication holds vacuously. There cannot be an x which is an element of every subset of A. If there were then in particular it would have to be an element of $\emptyset$, but this is clearly nonsense.