1. Disprove the following statement: Let n be an integer. If $\frac{(n+1)(n+2)}{2}$ is odd then $\frac{n^2(n+1)^2}{2}$ is odd. [5 pts]

   This is a universal statement (it asserts something is true for all n), so to disprove it we just need to find a single counterexample. Let's take $n = 0$. Then $\frac{(n+1)(n+2)}{2} = 1$ is odd but $\frac{n^2(n+1)^2}{2} = 0$ is even. Thus $n = 0$ is a counterexample to the implication being true.

2. Suppose that x is a rational number and y is an irrational number. Show that $x + y$ is irrational. [5 pts]

   Fix a rational number x and an irrational number y. We can write x as a fraction $x = \frac{a}{b}$ where both a and b are integers and $b \neq 0$. Now assume toward a contradiction that $x + y$ is rational. Then we can write $x + y$ as a fraction $x + y = \frac{c}{d}$ for some integers c, d with $d \neq 0$. But now $y = \frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{bd}$ is written as a fraction of integers, contradicting our assumption that it was irrational.

3. Show that there are a rational number a and an irrational number b such that a^b is irrational. [5 pts]

   (Hint: We proved in class that $\sqrt{2}$ is irrational and this easily implies that $\frac{1}{2\sqrt{2}}$ is also irrational. Now consider the number $2^{\frac{1}{2\sqrt{2}}}$. Depending on whether this is rational or irrational find a and b.)

   Following the hint let us consider two cases, depending on whether $2^{\frac{1}{2\sqrt{2}}}$ is rational or irrational.

   Case 1: If $2^{\frac{1}{2\sqrt{2}}}$ is irrational then we can just take $a = 2$ and $b = \frac{1}{2\sqrt{2}}$ to satisfy the requirements of the problem.

   Case 2: If $2^{\frac{1}{2\sqrt{2}}}$ is rational we can take $a = 2^{\frac{1}{2\sqrt{2}}}$ and $b = \sqrt{2}$. This solves the problem since $a^b = 2^{\frac{1}{2\sqrt{2}}\sqrt{2}} = 2^{\frac{1}{1}} = \sqrt{2}$ is irrational.

   Comment: Here is a solution to this problem which is much more elegant than my own. It goes as follows:

   Fix irrational x, y such that x^y is rational; as mentioned above the existence of such x and y was proved in class. For the problem, take $a = x^y$ and $b = \frac{1}{y}$. Since y was irrational b is also irrational and a is rational by our choice of x and y. But this solves the problem, since $a^b = (x^y)^{\frac{1}{y}} = x$ is irrational.

4. Let n be a natural number. Show that $\sum_{k=0}^{n}(3k-2) = \frac{3n^2-n-4}{2}$. [5 pts]

   We prove this by induction on n.

   In the base step we check that the equality holds when $n = 0$. And it does, since $\sum_{k=0}^{0}(3k-2) = -2$ and $\frac{3(0)^2-0-4}{2} = -2$.

In the induction step we make the inductive hypothesis that $\sum_{k=0}^{n}(3k-2) = \frac{3n^2-n-4}{2}$ and try to prove that $\sum_{k=0}^{n+1}(3k-2) = \frac{3(n+1)^2-(n+1)-4}{2}$. We can proceed as follows

$$\begin{aligned}\sum_{k=0}^{n+1}(3k-2) &= \sum_{k=0}^{n}(3k-2) + 3(n+1) - 2 \\ &= \frac{3n^2-n-4}{2} + 3(n+1) - 2 = \frac{3n^2-n-4+6n+6-4}{2} \\ &= \frac{3n^2+5n-2}{2} = \frac{3(n+1)^2-(n+1)-4}{2}\end{aligned}$$

where we used the inductive hypothesis to go from the first to the second line. This finishes the induction and the proof.

5. Show that $n! > 2^n$ for any natural number $n \geq 4$. [5 pts]

We prove this by induction on $n \geq 4$.

In the base step we check that the inequality holds when $n = 4$. And it does, since $4! = 24$ and $2^4 = 16$.

In the induction step we make the inductive hypothesis that $n! > 2^n$ and try to prove that $(n+1)! > 2^{n+1}$. Now then,

$$\begin{aligned}(n+1)! &= (n+1)n! \\ &> (n+1)2^n \\ &> 2 \cdot 2^n = 2^{n+1}\end{aligned}$$

where we used the inductive hypothesis to go from the first to the second line and the fact that $n+1 > 2$ to go from the second to the third line. This finishes the induction and the proof. □

Comments: It is interesting to note that the induction step would have gone through even for $n \geq 1$. Nevertheless, we need $n \geq 4$ to do the base step (and to make the statement actually true).

6. (extra credit) Let $x > -1$ be a real number and n a natural number. Prove Bernoulli's inequality: [5 pts]

$$(1+x)^n \geq 1 + nx$$

(Hint: Argue by induction on n.)

We prove this by induction on n.

In the base step we check that the inequality holds when $n = 0$. And it does, since $(1+x)^0 = 1$ (it is *vital* here that $x \neq -1$ since the expression 0^0 is problematic) and $1 + 0x = 1$.

In the induction step we make the inductive hypothesis that $(1+x)^n \geq 1+nx$ and try to prove that $(1+x)^{n+1} \geq 1+(n+1)x$. The following works:

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n(1+x) \\ &\geq (1+nx)(1+x) = 1+x+nx+nx^2 = 1+(n+1)x+nx^2 \\ &\geq 1+(n+1)x\end{aligned}$$

where we used the inductive hypothesis (and the fact that $1+x \geq 0$) to go from the first to the second line and the fact that $nx^2 \geq 0$ (since $n \geq 0$ and $x^2 \geq 0$) to go from the second to the third line. This finishes the induction and the proof.