1. Let x and y be integers. Show that if x is even or $y = 0$ then xy is even. [5 pts]

   Let x and y be arbitrary integers and assume that x is even or $y = 0$. We split into cases:

   Case 1: If x is even then xy is even, since we proved in class that the product of two integers is even iff at least one of them is even.

   Case 2: If $y = 0$ then $xy = 0$ is even (because $0 = 2 \cdot 0$).

   In both of the cases we proved that xy was even, which concludes the proof.

2. Let a and b be integers. Show that $(a + b)^2 = a^2 + b^2$ if and only if at least one of a and b is 0. [5 pts]

   Fix two arbitrary integers a and b. We are proving a biconditional, so we prove each direction separately.

   ($\Longrightarrow$):

   Assume that $(a + b)^2 = a^2 + b^2$. By multiplying out the left side we get

   $$a^2 + 2ab + b^2 = a^2 + b^2$$

   which yields $2ab = 0$. But the only way this product can be zero is if one of the factors (i.e. one of a or b) is zero, and this is what we wanted to show.

   ($\Longleftarrow$):

   Assume that one of a and b is 0. We may assume without loss of generality that $a = 0$. But then

   $$(a + b)^2 = (0 + b)^2 = b^2 = a^2 + b^2$$

   which is what we wanted to prove.

   Comments: In the proof of the $\Longleftarrow$ direction we would in principle have to consider two cases, depending on which of a and b is zero. But with a tiny bit of foresight we can see that the two cases would be completely symmetric (basically the same calculation up to renaming the variables), so we can get away with just considering one.

3. Let a and b be nonzero integers. [5 pts]

   (a) Show that $a \mid a$.

   (b) Assume additionally that a and b are positive. Show that if $a \mid b$ and $b \mid a$ then $a = b$.

   (a) Let a be a nonzero integer. To show that $a \mid a$ we need to find an integer k such that $a = ak$. After much thought we get the idea that taking $k = 1$ might work, and indeed $a \cdot 1 = a$ and thus $a \mid a$.

   (b) Let a and b be positive integers and assume $a \mid b$ and $b \mid a$. By definition there are integers k and l such that $b = ak$ and $a = bl$. Substituting the second of these equalities into the first one gets us $b = blk$ and therefore, since b is not zero, $lk = 1$. Since l and k are positive integers, the only way their product can be 1 is if $k = l = 1$. But then $a = bl = b$, which was what we wanted to show.

4. Let a, b, c be integers and $a \neq 0$. Show that if $a \nmid bc$ then $a \nmid b$ and $a \nmid c$. [5 pts]

   Pick arbitrary nonzero integers a, b and c. We will prove the statement via contrapositive, so assume that $a \mid b$ or $a \mid c$. We can assume without loss of generality that $a \mid b$. By definition there is then an integer k such that $b = ak$. But then $bc = a(kc)$, which shows that $a \mid bc$, which we wanted to prove.

   Comments: A proof by contrapositive suggests itself here because there does not seem to be any good way of using an assumption of nondivisibility. Saying that $a \nmid bc$ means that there is no k such that $bc = ak$. This is not much of an operative assumption; saying that there is no way of doing something doesn't give me much to work with. On the other hand, an assumption of divisibility has immediate tangible consequences (I get a k and an equation).

   Again, technically we would have had to consider two cases depending on whether $a \mid b$ or $a \mid c$. But, as before, the two cases are symmetric and proceed in the same way so we omit one of them.

   Something that happened far more often than I expected was that people computed the contrapositive wrong: instead of assuming $a \mid b$ or $a \mid c$ they assumed $a \mid b$ *and* $a \mid c$.

5. A positive natural number p is *prime* if $p \mid ab$ implies $p \mid a$ or $p \mid b$ for any integers a, b.

   Show that 2 is prime. [5 pts]

   (Hint: This is really just an application of a theorem from class.)

   Before we start we can notice, by comparing definitions, that a positive natural x is even iff $2 \mid x$. We will use this observation in the proof.

   We need to show that 2 satisfies the given definition of being prime. So let a and b be positive natural numbers and assume that $2 \mid ab$. By our observation above this means that ab is even. By a result proved in class this means that one of a and b is even or, equivalently, that $2 \mid a$ or $2 \mid b$. This concludes the proof.

6. (extra credit) Let n be an integer. Show that $2 \mid (n^4 - 3)$ iff $4 \mid (n^2 + 3)$. [5 pts]

Fix an integer x. We show the two directions of the biconditional separately.

($\Longrightarrow$):

Assume that $2 \mid (n^4 - 3)$. In other words $n^4 - 3$ is even. It follows by a theorem from class that $n^4 - 3 + 3 = n^4$ is odd and by another theorem from class that n is odd. Thus we can find an integer k such that $n = 2k + 1$. But then $n^2 + 3 = (2k+1)^2 + 3 = 4k^2 + 4k + 4$ is clearly divisible by 4.

($\Longleftarrow$):

Assume that $4 \mid (n^2 + 3)$. In particular this means that $n^2 + 3$ is even and, by a theorem from class, that n^2 is odd. Another theorem from class tells us that $(n^2)^2 = n^4$ is also odd and from there we can conclude that $n^4 - 3$ is even.

Comment: There are plenty of approaches to this problem, depending how far you deconstruct the hypothesis and at which point you use existing theorems.