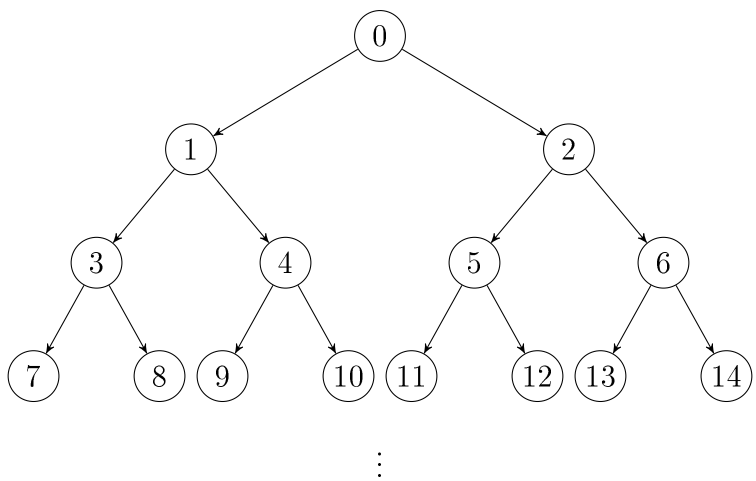
Honour and glory problem #1

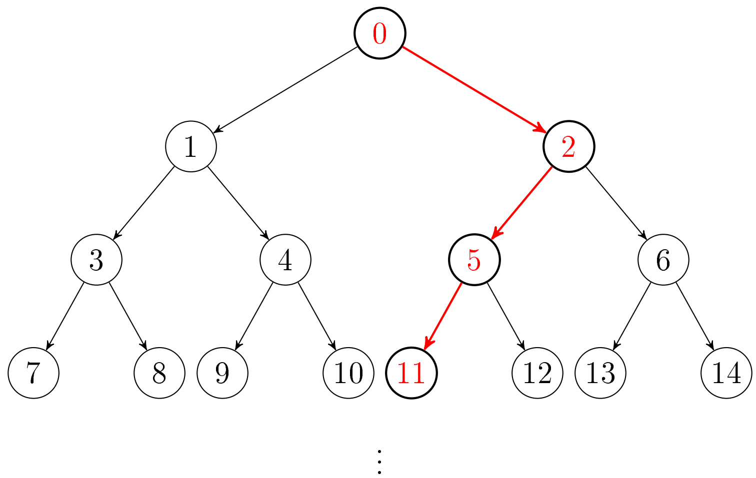
Find an example of a family $\{A_i\}_{i\in I}$ where I is the set of infinite sequences of 0s and 1s (e.g. $111\cdots\in I$ and $010101\cdots\in I$ but $0123\ldots\notin I$) satisfying the following conditions:

- each A_i is an infinite subset of $\mathbb{N}$;
- $A_i \cap A_j$ is finite for any two distinct indices i and j;
- $\bigcup_{i\in I} A_i = \mathbb{N}$.

The difference between this problem and 6 on HW1 is only in the different indexing; instead of having a set in the family for each natural number n we have one for each infinite sequence of 0's and 1's.
There doesn't seem to be a good way of doing this if we think of the natural numbers as a line of points, so let's organize ourselves differently. We will arrange the natural numbers in a structure called a full binary tree:



So we have an arrangement of nodes, labelled with natural numbers, and each one sprouts two edges going to two further nodes. Notice that every node gets a label and every natural number is the label of some node.
Now consider a given infinite sequence of 0's and 1's (let's call these things binary sequences from now on); to be concrete let's take the sequence $10011\ldots$ (here the dots don't represent any particular pattern but only say that the sequence continues in some way). We can interpret this sequence as a list of instructions on how to move through the tree above. We start at the node labelled 0 and look at the first digit of the sequence: if it is a 0 we move left and down a level in the tree and if it is a 1 we move right and down a level in the tree. After this we are at another node and we look at the second digit of the sequence and move again in the same way. With this interpretation every binary sequence determines a path through the tree; the sequence we had before gives the path that starts like this:



For a binary sequence i we let A_i be the set of nodes (or rather their labels) which are hit by the path determined by i. In our example the set would contain $0, 2, 5, 11$ and whatever the nodes the path hits after 11 are.
We have now defined a family $\{A_i\}_{i \in I}$. Does it satisfy the requirements?

- Clearly each A_i is an infinite set of natural numbers, since the path keeps going forever and hits nodes with larger and larger labels.
- If i and j are two distinct binary sequences then the paths they determine must split at some point and after that last node they have in common they never meet again. Thinking about this, the intersection $A_i \cap A_j$ consists precisely of those nodes (or their labels) which belong to both the paths determined by i and j. Since the paths split at some finite level, the intersection must be finite as well.
- It is clear that any element of $\bigcup_{i \in I} A_i$ is a natural number (since we only ever put natural numbers into the A_i). So to see that the union actually equals $\mathbb{N}$ what we need to check is that every natural number n gets into this union. But this basically means that for each node (labelled by n) there needs to be a path through the tree which hits that node and that is easy: we can go from the 0 node to the node n and after that we can just always go left.