02/01/23

# Differential Calculus

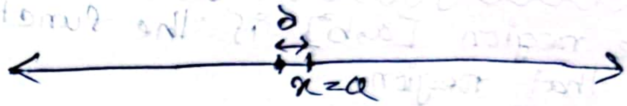
## Limit Continuity and Differentiability

**# Limit:** A function $f(x)$ is said to have a limit $l$ and $x$ tends to $a$, provided given any positive quantity $\epsilon$ (however small) we can determine another positive quantity $\delta$ (depending on $\epsilon$) such that $|f(x)-l| < \epsilon$ for all values of $x$ satisfying $0 < |x-a| \leq \delta$ but $x \neq a$ and we write it as $\lim_{x\to a} f(x) = l$.

For L.H.L, $0 < a-x \leq \delta$

$\Rightarrow a-\delta \leq x$

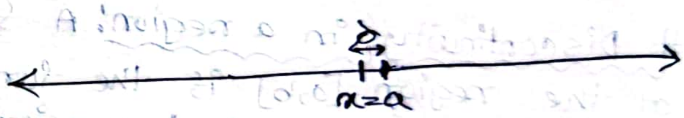
$\Rightarrow \lim_{x\to a^-} f(x) = l$



For R.H.L, $0 < x-a \leq \delta$

$\Rightarrow x \leq a+\delta$

$\Rightarrow \lim_{x\to a^+} f(x) = l$



When the value of R.H.L and L.H.L are equal, the function has a limit.

Suppose, L.H.L = R.H.L = $m$, then the limit of the function is $m$.

# What is the distinction between $\lim_{x\to a} f(x)$ and $f(a)$?

CS CamScanner

04/01/23

# Differential Calculus

**# Continuity at $x=a$:** A function $f(x)$ is said to be continious at $x=a$ if $\lim_{x\to a} f(x)$ exists, is finite and is equal to $f(a)$.

It can be written as: $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

$$\text{or, } \lim_{h\to 0^-} f(a+h) = \lim_{h\to 0^+} f(a+h) = f(a)$$

i.e., L.H.L = R.H.L = F.V (Functional value)

**# continuity in a region:** A function $f(x)$ is said to be continious at the region $[a,b]$ if the function is defined for all values of $x$ at that region.

**# Discontinuity in a region:** A function $f(x)$ is said to be discontinious at the region $[a,b]$ if the function is undefined for one or more values at that region.

**# Differentiability at $x=a$:** A function $f(x)$ is said to be differentiable at $x=a$ if, $\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$

i.e, L.H.D = R.H.D

It can be written as $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

L.H.D → Left hand derivative

**# Differentiability in a region:** A function $f(x)$ is said to be differentiable at a region if the function's derivatives (L.H.D = R.H.D) are defined in that region.

⊛ Every differentiable function is continious.

CamScanner

# Prob: Show that every finite differentiable function is continious, but the converse is not true in general.

Soln: suppose the function $f(x)$ is differentiable at $x=a$

i.e; $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$; which is finite

Now, $f(a+h)-f(a) = \frac{f(a+h)-f(a)}{h} \cdot h$

Taking $\lim_{h\to 0}$ on both sides we get,

$$\lim_{h\to 0}\left[f(a+h)-f(a)\right] = \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h} \times h\right]$$

$$\Rightarrow \lim_{h\to 0}\left[f(a+h)-f(a)\right] = \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right] \times \lim_{h\to 0} h$$

$$\Rightarrow \lim_{h\to 0} f(a+h) - f(a) = f'(a) \times 0$$

$$\Rightarrow \lim_{h\to 0} f(a+h) = f(a)$$

i.e; Limiting value = functional value

So, by definition, the function is continious.

Consider the function $f(x)=|x|$

i.e, $f(x) = x$, when $x>0$
$= -x$, when $x<0$
$= 0$, when $x=0$

At $x=0$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} (-x) = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} x = 0$$

And, $f(0)=0$

$\therefore$ L.H.L = R.H.L = F.V

So, the function is continious at $x=0$

Now, at $x=0$, $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{(h-0)}{-h} = -1$

For, $f(0-h)$
Lo range is $x<0$
$\therefore f(x) = -x$
$\therefore f(-h) = h$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(h-0)}{h} = 1$$

$\therefore$ L.H.D $\neq$ R.H.D at $x=0$

So, $f(x)$ is not differentiable at $x=0$

CS CamScanner

Thus; every continious function is not differentiable.

CS CamScanner

Prob:-1

H.W Given, $f(x) = 1$ for $x < 0$

$= 1 + \sin x$ for $0 \le x < \pi/2$

$= 2 + (x - \pi/2)^2$. for $\pi/2 \le x$

Discuss the differentiability of $f(x)$ at $x=0$ and $\pi/2$.

$$\left[ L.H.D = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \text{ ; at } x=0 \right]$$

H.W

Prob:2 Check the continuity and differentiability of the function at $x=0$.

$f(x) = x\cos(1/x)$ ; $x \neq 0$

$= 0$ ; $x = 0$

Soln: We have $|\cos(1/x)| \le 1$

Now, $\lim_{x \to 0} f(x) = \lim_{x \to 0} x\cos 1/x = \lim_{x \to 0} x \times \lim_{x \to 0} \cos(1/x)$

$= 0 \times$ a finite value lies between -1 to 1.

$= 0$

and $f(0) = 0$

$\therefore$ limiting value = functional value

So, the function is continious at $x=0$.

Now, $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{h\cos(1/h) - 0}{h}$

$= \lim_{h \to 0} \cos(\frac{1}{h})$, which doesn't exist because the value lies between -1 to 1

So, $f'(0)$ does not exist at $x=0$.

**Prob:** $\lim_{x\to 3} \frac{x^2-9}{x-3}$

$= \lim_{x\to 3} \frac{2x}{1}$

$= 2\times 3 = 6$

**Prob:** Find the values of a and b such that the function;

$f(x) = x+\sqrt{2}\, a\sin x, \quad 0\le x< \pi/4$

$= 2x\cot x + b, \quad \pi/4 \le x < \pi/2$

$= a\cos 2x - b\sin x, \quad \pi/2 \le x \le \pi$

is continious for all values of $x$ in the interval $[0, \pi]$.

**Prob:** $f(x) = -2\sin x, \quad -\pi \le x \le \pi/2$

$= a\sin x + b, \quad -\pi/2 < x < \pi/2$

$= \cos x, \quad \pi/2 \le x \le \pi$

Find the values of a and b if $f(x)$ is continious for all values of $x$ in the interval $[-\pi, \pi]$.

# Implicit function: If any variable (say) y cannot be expressed in another variable (say) x, then the function including x and y is called implicit function. This can be written as:

$$f(x,y) = 0$$

# Explicit function: If any variable (say) y can be expressed in another variable (say) x, then the function including x and y is called explicit function. This can be written as:

$$y = f(x)$$

# Parametric Equation: If the variables of any equation can be represented by another parameter, then the equation is called parametric equation.

Suppose, $y = \tan^{-1}\frac{2t}{1-t^2}$, $x = \sin^{-1}\frac{2t}{1+t^2}$

Put, $t = \tan\theta$

then, $y = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} = \tan^{-1}\tan 2\theta = 2\theta$ ; $\therefore \frac{dy}{d\theta} = 2$

CamScanner

$\Rightarrow x = \sin^{-1} \sin 2\theta$

$= 2\theta$

$\therefore \frac{dx}{d\theta} = 2$

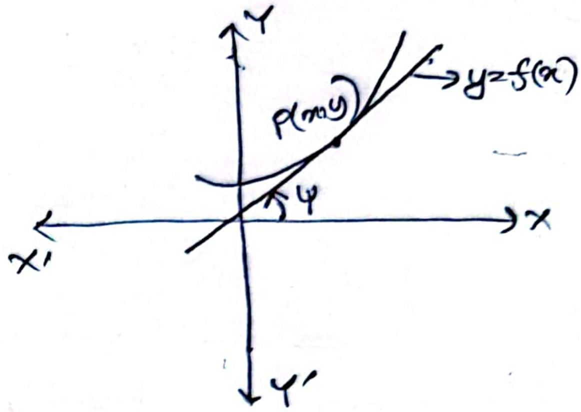
$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 2 \times \frac{1}{2} = 1$

MATH 1107

## Differential Calculus

*** Significance of derivatives:**

1) $\frac{dy}{dx}$ = The rate of change of $y$ with respect to $x$
2) $\frac{dy}{dx} = \tan\psi$




The derivative of $f(x)$ at $(x,y)$ is equal to the trigonometrical ~~angle of the tangent~~ which the tangent makes with the positive direction of X-axis. It is denoted by $\frac{dy}{dx} = \tan\psi$; where $\psi$ is the angle made with the positive direction of x-axis.

**# Find the derivative of $x^y + (\sin y)^{\cos x} = x+y$**

$\Rightarrow \quad x^y + (\sin y)^{\cos x} = x+y$

$\Rightarrow \frac{d}{dx}(x^y) + \frac{d}{dx}\{(\sin y)^{\cos x}\} = 1 + \frac{dy}{dx}$

$\Rightarrow \cancel{x^y\left(\ln x \cdot \frac{dy}{dx} + \frac{y}{x}\right) + (\sin y)^{\cos x}}$

$\Rightarrow x^y\left(\ln x \frac{dy}{dx} + \frac{y}{x}\right) + (\sin y)^{\cos x}\{-\sin x \cdot \ln(\sin x) + \cot x \cdot \cos x) = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(x^y \cdot \ln x - 1) =$

$p = x^y$

$q = (\sin y)^{\cos x}$

$\frac{dp}{dx} \rightarrow$

$\ln(p) = y \ln x$

$\Rightarrow \frac{1}{p} \cdot \frac{dp}{dx} = \frac{dy}{dx}\ln x + \frac{y}{x}$

$\Rightarrow \frac{dp}{dx} = x^y \cdot \ln x \cdot \frac{dy}{dx} + y/x$

$\ln q = \cos x \cdot \ln(\sin y)$

$\Rightarrow \frac{dq}{dx} = q\ \{-\sin x \cdot \ln(\sin y) + \frac{\cos x}{\sin y} \cdot \cos y \cdot \frac{dy}{dx}$

$\Rightarrow x^y\left(\ln x \frac{dy}{dx} + \frac{y}{x}\right) + (\sin y)^{\cos x}\{-\sin x \cdot \ln(\sin y) + \cot y \cdot \cos x \cdot \frac{dy}{dx}\} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}\{x^y \cdot \ln x + (\sin y)^{\cos x} \cdot \cot y \cdot \cos x - 1\} = 1 - y \cdot x^{y-1} + (\sin y)^{\cos x} \cdot \sin x \cdot \ln(\sin y)$

$$\Rightarrow \frac{dy}{dx} = \frac{(\sin y)^{\cos x} \cdot \sin x \cdot \ln(\sin y) + 1 - y \cdot x^{y-1}}{x^y \cdot \ln x + (\sin y)^{\cos x} \cdot \cot y \cdot \cos x - 1}$$

Or,

$$x^y + (\sin y)^{\cos x} = x+y$$

$$\Rightarrow e^{y\ln x} + e^{\cos x \ln(\sin y)} = x+y$$

Differenting both side with respect to $x$,

$$\Rightarrow e^{y\ln x}\cdot\left(\frac{y}{x} + \ln x\cdot\frac{dy}{dx}\right) + e^{\cos x \ln(\sin y)}\left\{\frac{\cos x}{\sin y}\cdot\cos y\cdot\frac{dy}{dx} + \ln(\sin y)\sin x\right\}$$

$$= 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left\{e^{y\ln x}\cdot\ln x + e^{\cos x\cdot\ln(\sin y)}\cdot\cos x\cdot\cot y - 1\right\}$$

$$= \left\{1 - e^{y\ln x}\cdot\frac{y}{x} + e^{\cos x\cdot\ln(\sin y)}\cdot\ln(\sin y)\sin x\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - y\cdot x^{y-1} + (\sin y)^{\cos x}\cdot\ln(\sin y)\cdot\sin x}{x^y\cdot\ln x + (\sin y)^{\cos x}\cdot\cos x\cdot\cot y - 1}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2i}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

23/01/2023

$\boxed{\text{MATH 1107}}$

# Differential calculus

## Successive differentiations:

- $y = x^4$, find $y_4, y_5$.

$y_1 = 4x^3$

$y_2 = 4 \times 3x^2$

$y_3 = 4 \times 3 \times 2 \times x$

$y_4 = 4 \times 3 \times 2 \times 1 = 4!$

$y_5 = 0$

- $y = \sin(ax+b)$, $\cos(ax+b)$, $\log(ax+b)$, $\frac{1}{x^2+a^2}$, $\tan^{-1} x/a$, $\frac{x}{x^2+a^2}$, $e^{ax}\sin bx$, $e^{ax}\cos bx$, $\sin 2x \sin 3x \sin 4x$ etc.

- $y = \cos(ax+b)$

$\therefore y_1 = -a\sin(ax+b) = a\cos\{\pi/2 + (ax+b)\}$

$y_2 = a^2 \cos\{2 \cdot \pi/2 + (ax+b)\}$

— — — — — — — —

— — — — — — — —

$y_n = a^n \cos\{n\pi/2 + (ax+b)\}$

$\cancel{y_n = (-1)^n a^n \cos\{n\pi/2 + (ax+b)\}}$

- $y = \log(ax+b)$

$y_1 = \frac{a}{(ax+b)} = a(ax+b)^{-1}$

$y_2 = (-1)\, a^2 (ax+b)^{-2}$

$y_3 = (-1)(-2) \cdot a^3 (ax+b)^{-3}$

$y_4 = (-1)(-2)(-3)\, a^4 (ax+b)^{-4}$

$= (-1)^{4-1} \cdot (4-1)!\, a^4 (ax+b)^{-4}$

— — — — — — — — —

— — — — — — — —

CS CamScanner

$$y_n = (-1)^{n-1}.(n-1)!\, a^n (ax+b)^{-n}$$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\sin^2 x = 1 - \cos 2x$

$2\cos^2 x = 1 + \cos 2x$

$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

de moivre's theorem

- $$y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)}$$

$$= \frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

$$= \frac{1}{2ia}\left[(x-ia)^{-1} - (x+ia)^{-1}\right]$$

$$\therefore y_n = \frac{1}{2ia}(-1)^n.n! \left[(x-ia)^{-(n+1)} - (x+ia)^{-(n+1)}\right] \quad \text{—— (1)}$$

let, $a = r\sin\theta$

$x = r\cos\theta$

therefore, $r = \sqrt{x^2+a^2}$ , $\theta = \tan^{-1}\frac{a}{x}$

$$\therefore (x-ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta - i\sin\theta)^{-(n+1)} = r^{-(n+1)}\{\cos(n+1)\theta + i\sin(n+1)\theta\}$$

de moivre's theorem

$$\therefore (x+ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta + i\sin\theta)^{-(n+1)} = r^{-(n+1)}\{\cos(n+1)\theta - i\sin(n+1)\theta\}$$

$$\therefore y_n = \frac{(-1)^n.n!}{2ia}.r^{-(n+1)}\left[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta + i\sin(n+1)\theta\right]$$

[from (1)]

$$\Rightarrow y_n = \frac{(-1)^n.n!}{2ia} r^{-(n+1)} . 2i\sin(n+1)\theta$$

$$\Rightarrow y_n = \frac{(-1)^n}{a}.n!.r^{-(n+1)}.\sin(n+1)\theta \quad ; \text{where } r = \sqrt{x^2+a^2},\ \theta = \tan^{-1}\frac{a}{x}$$

$$\Rightarrow y_n = \frac{(-1)^n}{x}.n!\,(a^2+x^2)^{-\frac{(n+1)}{2}}.\sin\{(n+1)\tan^{-1} a/x\}$$

Or,

$$y_n = \frac{(-1)^n}{x}.n!\left(\frac{a}{\sin\theta}\right)^{-(n+1)}.\sin(n+1)\theta$$

$$\Rightarrow y_n = \frac{(-1)^n.n!}{a^{n+2}}.(\sin\theta)^{n+1}.\sin(n+1)\theta$$

MATH 1107

## Differential Calculus

# $y = e^{ax}\sin bx$, $e^{ax}\cos bx$

$y = \sin^2 x\cos^3 x$, $\sin^3 x\cdot\cos^3 x$, $\sin^4 x\cos^4 x$

$y = \sin x\cdot\cos 2x\cdot\sin 3x$

$y = \tan^{-1}\frac{x}{a}$, $\frac{1}{x^2+a^2}$, $\frac{x}{x^2+a^2}$, $\frac{x^2}{(x-a)(x-b)}$, $\frac{1}{(x+a)^2+b^2}$

# $y = x^{2n}$, show that $y_n = 2^n\{1\cdot3\cdot5\cdots(2n-1)\}\cdot x^n$

# $u = \sin ax + \cos bx$, show that, $u_n = a^n\{1+(-1)^n\sin 2ax\}^{1/2}$

# Statement of Leibnitz theorem:

If $u$ and $v$ are two functions of $x$, then the $n$th derivative of their product is.

$$(uv)_n = u_n v + \cancel{u_{n}}\, {}^nc_1\cdot u_{n-1}\cdot v_1 + {}^nc_2\cdot u_{n-2}\cdot v_2 + \cdots + {}^nc_r\cdot u_{n-r}\cdot v_r + \cdots + u v_n$$

where the suffixes in $u$ and $v$ ~~are~~ denote the orders of differentiations w.r.to $x$.

Prob-1: If $y = \sin^{-1}x \rightarrow \boxed{y_{n+2}, y_{n+1}, y_n \text{ relations}} \rightarrow (y_n)_0$

2. $y = \sin(m\sin^{-1}x)$
3. $x = \sin\left(\frac{1}{m}\log y\right)$
4. $y^{1/m} + y^{-1/m} = 2x$
5. $y = e^{\tan^{-1}x}$
6. $y = \tan^{-1}x \longrightarrow (y_n)_0$
7. $y = (\sin^{-1}x)^2$
8. $y = a\cos(\log x) + b\sin(\log x)$
9. $y = (x^2-1)^n$

(12) $u = \sin ax + \cos ax$, show that, $u_n = a^n\{1+(-1)^n \sin 2ax\}^{1/2}$

$\Rightarrow u_n = a^n \cdot \sin\{n\pi/2 + ax\} + a^n \cdot \cos\{n\pi/2 + ax\}$

$\Rightarrow u_n = a^n\{\sin(n\pi/2 + ax) + \cos(n\pi/2 + ax)\}$

$\Rightarrow u_n^2 = a^{2n} \quad \{1 + \sin(n\pi + 2ax)\}$

$\Rightarrow u_n^2 = a^{2n} \quad \{1 + (-1)^n \sin 2ax\}$

$\therefore u_n = a^n\{1 + (-1)^n \sin 2ax\}^{1/2}$

$\sin(n\pi+\theta)$:

$\sin(\pi+\theta) = (-1)\sin\theta$

$\sin(2\pi+\theta) = (-1)^2\sin\theta$

$\sin(3\pi+\theta) = (-1)^3\sin\theta$

$\sin(4\pi+\theta) = (-1)^4\sin\theta$

$\therefore \sin(n\pi+\theta) = (-1)^n \sin\theta$

(13) $y = x^{2n}$, show that, $y_n = 2^n\{1.3.5 --- (2n-1)\}x^n$

$\Rightarrow y = x^{2n}$

$y_1 = 2n x^{2n-1}$

$y_2 = 2n(2n-1)x^{2n-1}$

— — — —

— — — —

$y_n = \{2n(2n-1)\cdot(2n-2) - --- (2n-n+1)\}x^{2n-n}$

$= \{2n(2n-1)\cdot(2n-2) \cdot --- \cdot (n+1)\}\cdot x^n$

$= \dfrac{2n(2n-1)(2n-2) \cdot --- (n+1) \cdot n(n-1) \cdot --- 3.2.1 \cdot x^n}{n!}$

$= \dfrac{2^n \cdot n! \{(2n-1)(2n-3) \cdot ---- 5.3.1\} \cdot x^n}{n!}$

$= 2^n \cdot \{1.3.5 ---- (2n-1)\} \cdot x^n$

(14) $y = e^{ax}\sin bx$, find $y_n$.

$\Rightarrow y_1 = e^{ax}(a\sin bx + b\cos bx)$

let, $a = r\cos\theta$ and $b = r\sin\theta$

then, $r = \sqrt{a^2+b^2}$ ; $\theta = \tan^{-1}(b/a)$

$\Rightarrow y_1 = e^{ax}(r\cos\theta \sin bx + r\sin\theta\cos bx)$

$= r \cdot e^{ax} \cdot \sin(bx+\theta)$

$y_2 = r \cdot e^{ax} \{ a\sin(bx+\theta) + b\cos(bx+\theta) \}$

$= r \cdot e^{ax} \{ r\cos\theta \sin(bx+\theta) + r\sin\theta \cos(bx+\theta) \}$

$= r^2 \cdot e^{ax} \sin(bx+\theta+\theta)$

$= r^2 \cdot e^{ax} \cdot \sin(bx+2\theta)$

- - - - - - - - - - - - - - -

$\therefore y_n = r^n \cdot e^{ax} \cdot \sin(bx+n\theta)$

$= (a^2+b^2)^{n/2} \cdot e^{ax} \cdot \sin\{ bx + n\tan^{-1}(b/a) \}$

$\boxed{4}$ $y^{1/m} + y^{-1/m} = 2x$

$\Rightarrow \frac{1}{m} y^{\frac{1}{m}-1} \cdot y_1 - \frac{1}{m} y^{-\frac{1}{m}-1} \cdot y_1 = 2$

$\Rightarrow y_1 \left( \frac{y^{1/m}}{my} - \frac{y^{-1/m}}{my} \right) = 2$

$\Rightarrow y_1 (y^{1/m} - y^{-1/m}) = 2my$

$\Rightarrow y_1 \sqrt{[(y^{1/m}+y^{-1/m})^2 - 4]} = 2my$

$\Rightarrow y_1 \sqrt{4x^2-4} = 2my$

$\Rightarrow 2y_1 \sqrt{x^2-1} = 2my$

$\Rightarrow y_1\sqrt{x^2-1} = my$ —— (i)

$\Rightarrow y_1^2 (x^2-1) = m^2y^2$

$\Rightarrow 2y_1y_2(x^2-1) + 2x \cdot y_1^2 = m^2 \cdot 2y \cdot y_1$

$\Rightarrow y_2(x^2-1) + xy_1 = m^2 y$ —— (ii)

Differentiating eq$^n$ (ii) n times by libnitz theorem, we get,

$\Rightarrow y_{n+2}(x^2-1) + {}^nC_1 \cdot y_{n+1} \cdot 2x + {}^nC_2 \cdot y_n \cdot 2 + y_{n+1} \cdot x + {}^nC_1 \cdot y_n \cdot 1 = m^2 y_n$

$\Rightarrow y_{n+2} \cdot (x^2-1) + 2nx \cdot y_{n+1} + \frac{2n(n-1)}{2} \cdot y_n + x \cdot y_{n+1} + ny_n = m^2 y_n$

$\Rightarrow (x^2-1)y_{n+2} + (2nx+x) \cdot y_{n+1} + (n^2 - n + n - m^2) y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + x(2n+1) \cdot y_{n+1} + (n^2-m^2) y_n = 0$

CS CamScanner

⑨ $y = (x^2-1)^n$ ;

$\Rightarrow y_1 = n(x^2-1)^{n-1} \cdot 2x = \frac{n(x^2-1)^n}{(x^2-1)} \cdot 2x$

~~$\Rightarrow y_1(1-x$~~

$\Rightarrow y_1(x^2-1) = 2nxy$

② $y = \sin(m\sin^{-1}x)$ —— ①

$\Rightarrow y_1 = \cos(m\sin^{-1}x) \cdot \frac{m}{\sqrt{1-x^2}}$ [Differentiating both side w.r.to $x$]

$\Rightarrow (1-x^2)y_1^2 = m^2\{1-\sin(m\sin^{-1}x)^2\}$

$\Rightarrow (1-x^2)y_1^2 = m^2(1-y^2)$.

$\Rightarrow 2y_1 \cdot y_2(1-x^2) + y_1^2(-2x) = m^2(-2y \cdot y_1)$ [Differentiating both side w.r.to $x$]

$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0$ —— ⓘⓘ

Differentiating eqn ⓘⓘ $n$ times by libnitz theorem, we get,

$y_{n+2} \cdot (1-x^2) + {}^nc_1 \cdot y_{n+1} \cdot (-2x) + {}^nc_2 \cdot y_n \cdot (-2) - y_{n+1} \cdot x - {}^nc_1 \cdot y_n \cdot 1 + m^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2nx \cdot y_{n+1} - n(n-1) \cdot y_n - x \cdot y_{n+1} - ny_n + m^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - x(2n+1)y_{n+1} + (m^2-n^2)y_n = 0$

⑨ $y = (x^2-1)^n$ —— ①

$\Rightarrow y_1 = n(x^2-1)^{n-1} \cdot 2x = \frac{n(x^2-1)^n}{(x^2-1)} \cdot 2x$

$\Rightarrow y_1(x^2-1) = 2nxy$

$\Rightarrow y_2(x^2-1) + y_1(2x) = 2ny + 2nxy_1$

$\Rightarrow (x^2-1)y_2 + 2x(1-n)y_1 - 2ny = 0$ —— ⓘⓘ

Differentiating eqn ⓘⓘ $n$ times by libnitz theorem, we get,

$y_{n+2} \cdot (x^2-1) + {}^nc_1 \cdot y_{n+1} \cdot 2x + {}^nc_2 \cdot y_n \cdot 2 + y_{n+1} \cdot (2x+2nx) + {}^nc_1 \cdot y_n \cdot (2+2n)$
$- 2n \cdot y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + 2nx \cdot y_{n+1} + n(n-1) \cdot y_n + (2x+2nx) \cdot y_{n+1} + (2n+2n^2) \cdot y_n - 2n \cdot y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + (4nx+2x) \cdot y_{n+1} + (n^2-n+2n+2n^2-2n) \cdot y_n = 0$

$\Rightarrow (x^2-1) \cdot y_{n+2} + (4nx+2x) \cdot y_{n+1} + (3n^2-n) \cdot y_n = 0$

CamScanner

(3) $x = \sin\left(\frac{1}{m}\log y\right)$

$\Rightarrow \sin^{-1}x = \frac{1}{m}\log y$

$\Rightarrow \log y = m\sin^{-1}x$

$\Rightarrow y = e^{m\sin^{-1}x}$

$\Rightarrow y_1 = e^{m\sin^{-1}x} \cdot \frac{m}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)y_1^2 = m^2 (e^{m\sin^{-1}x})^2$

$\Rightarrow (1-x^2)y_1^2 = m^2y^2$

$\Rightarrow 2y_1y_2(1-x^2) - 2xy_1^2 = 2m^2 \cdot y \cdot y_1$

$\Rightarrow (1-x^2)y_2 - xy_1 = m^2y$ ——— (1)

Differentiating eqn (1) n times by libnitz theorem, we get,

$y_{n+2}(1-x^2) + nc_1 \cdot y_{n+1} \cdot (-2x) + nc_2 \cdot y_n \cdot (-2) - y_{n+1} \cdot x - nc_1 \cdot y_n = m^2y_n$

$\Rightarrow (1-x^2)\, y_{n+2} - 2nx \cdot y_{n+1} + (-n^2+n) \cdot y_n - x \cdot y_{n+1} - ny_n - m^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x \cdot y_{n+1} - (n^2+m^2)\, y_n = 0$

(5) $y = e^{\tan^{-1}x}$

$\Rightarrow y_1 = e^{\tan^{-1}x} \frac{1}{1+x^2}$

$\Rightarrow (1+x^2)y_1 = y$

$\Rightarrow y_2(1+x^2) + 2xy_1 = y_1$

$\Rightarrow (1+x^2)\, y_2 + (2x-1)\, y_1 = 0$ ——— (1)

Differentiating eqn (1) n times by libnitz theorem,

$y_{n+2}(1+x^2) + nc_1 \cdot y_{n+1} \cdot 2x + nc_2 \cdot y_n \cdot 2 + y_{n+1} \cdot (2x-1) + nc_1 \cdot y_n \cdot 2 = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2nx \cdot y_{n+1} + (n^2-n)\, y_n + (2x-1)\, y_{n+1} + 2n \cdot y_n = 0$

$\Rightarrow (1+x^2)\, y_{n+2} + (2nx+2x-1)\, y_{n+1} + (n^2+n)\, y_n = 0$

(6) $y = \tan^{-1}x$

$\Rightarrow y_1 = \frac{1}{1+x^2}$

$\Rightarrow (1+x^2)y_1 = 1$

$\Rightarrow (1+x^2)\, y_2 + 2xy_1 = 0$ ——— (1)

Differentiating eqn (1) n times by libnitz theorem,

$y_{n+2}(1+x^2) + nc_1 \cdot y_{n+1} \cdot 2x + nc_2 \cdot y_n \cdot 2 + 2 \cdot y_{n+1} \cdot x + 2 \cdot nc_1 \cdot y_n \cdot 1 = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2nx \cdot y_{n+1} + (n^2-n)\, y_n + 2x \cdot y_{n+1} + 2n \cdot y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + (2nx+2x)\, y_{n+1} + (n^2+n)\, y_n = 0$

CS CamScanner

⑦ $y = (\sin^{-1}x)^2$

$\Rightarrow y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)y_1^2 = 4(\sin^{-1}x)^2$

$\Rightarrow (1-x^2)y_1^2 = 4y$

$\Rightarrow 2y_1y_2(1-x^2) + y_1^2(-2x) = 4y_1$

$\Rightarrow (1-x^2)y_2 - xy_1 - 2 = 0$ —①

Differentiating eq$^n$ ① n-times by leibnitz theorem,

$y_{n+2}(1-x^2) + {}^nC_1 \cdot y_{n+1} \cdot (-2x) + {}^nC_2 \cdot y_n(-2) - y_{n+1} \cdot x - {}^nC_1 \cdot y_n \cdot 1 = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2nx \cdot y_{n+1} + (-n^2+n)y_n - x \cdot y_{n+1} - ny_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x \cdot y_{n+1} - n^2y_n = 0$

⑧ $y = a\cos(\log x) + b\sin(\log x)$

$\Rightarrow y_1 = -\frac{a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x)$

$\Rightarrow xy_1 = -a\sin(\log x) + b\cos(\log x)$

$\Rightarrow xy_2 + y_1 = -\frac{a}{x}\cos(\log x) - \frac{b}{x}\sin(\log x)$

$\Rightarrow x^2y_2 + y_1x = -\{a\cos(\log x) + b\sin(\log x)\}$

$\Rightarrow x^2y_2 + xy_1 = -y$

$\Rightarrow x^2y_2 + xy_1 + y = 0$ —①

Differentiating eq$^n$ ① n times by leibnitz theorem,

$y_{n+2} \cdot x^2 + {}^nC_1 \cdot y_{n+1} \cdot 2x + {}^nC_2 \cdot y_n \cdot 2 + y_{n+1} \cdot x + {}^nC_1 \cdot y_n \cdot 1 + y_n = 0$

$\Rightarrow y_{n+2} \cdot x^2 + 2nx \cdot y_{n+1} + (n^2-n)y_n + x \cdot y_{n+1} + ny_n + y_n = 0$

$\Rightarrow x^2 \cdot y_{n+2} + (2n+1)x \cdot y_{n+1} + (n^2-n+n+1)y_n = 0$

$\Rightarrow x^2 \cdot y_{n+2} + (2n+1)x \cdot y_{n+1} + (n^2+1)y_n = 0$

⑨ $y = \sin^{-1}x$

$\Rightarrow y_1 = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)y_1^2 = 1$

$\Rightarrow 2y_1 \cdot y_2(1-x^2) - 2x \cdot y_1^2 = 0$

$\Rightarrow (1-x^2)y_2 - xy_1 = 0$ —①

Differentiating eq$^n$ ① 'n times by leibnitz theorem,

$y_{n+2}(1-x^2) + {}^nC_1 \cdot y_{n+1} \cdot (-2x) + {}^nC_2 \cdot y_n(-2) - y_{n+1} \cdot x - {}^nC_1 \cdot y_n \cdot 1 = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2nx \cdot y_{n+1} + (-n^2+n)y_n - x \cdot y_{n+1} - ny_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x \cdot y_{n+1} - n^2y_n = 0$

CamScanner

⊛ $y = e^{ax}\cdot\cos bx$ ; find $y_n$.

$\Rightarrow y_1 = ae^{ax}\cdot\cos bx + e^{ax}(-b\sin bx)$

$\Rightarrow y_1 = e^{ax}(a\cos bx - b\sin bx)$

let, $a = r\cos\theta$ and $b = r\sin\theta$

then, $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$

$\therefore y_1 = e^{ax}(r\cos\theta\cos bx - r\sin\theta\sin bx)$

$= e^{ax}\cdot r(\cos\theta\cos bx - \sin\theta\cdot\sin bx)$

$= e^{ax}\cdot r\cos(bx+\theta)$

$\Rightarrow y_2 = r\cdot e^{ax}\{a\cos(bx+\theta) - b\sin(bx+\theta)\}$

$= r^2 e^{ax}\{\cos\theta\cos(bx+\theta) - \sin\theta\sin(bx+\theta)\}$

$= r^2 e^{ax}\cdot\cos(bx+2\theta)$

--- --- --- --- ---

--- --- --- --- ---

$\therefore y_n = r^n\cdot e^{ax}\cdot\cos(bx+n\theta)$

$= (a^2+b^2)^{n/2}\cdot e^{ax}\cdot\cos\{bx + n\tan^{-1}(b/a)\}$

⊛ $y = \sin^2 x\cos^2 x$ ; find $y_n$.

$\Rightarrow y = \frac{1}{4}(2\sin x\cos x)^2$

$\Rightarrow y = \frac{1}{4}\sin^2 2x$

$\Rightarrow y = \frac{1}{8}(1-\cos 4x) = -\frac{1}{8} - \frac{1}{8}\cos 4x$

$\Rightarrow y_1 = -\frac{1}{8}\cdot 4^1\cdot\cos(\pi/2+4x)$

$\Rightarrow y_2 = -\frac{1}{8}\cdot 4^2\cdot\cos(2\cdot\pi/2+4x)$

$\Rightarrow y_3 = -\frac{1}{8}\cdot 4^3\cdot\cos(3\cdot\pi/2+4x)$

--- --- --- --- ---

--- --- --- --- ---

$\therefore y_n = -\frac{1}{8}\cdot 4^n\cdot\cos(n\pi/2+4x)$

⊛ $y = \sin^2 x\cos^3 x = \frac{1}{4}\cos x(2\sin x\cos x)^2$

$\Rightarrow y = \frac{1}{4}\cos x\cdot\sin^2 2x$

$\Rightarrow y = \frac{1}{8}\cos x(1-\cos 4x)$

$\Rightarrow y = \frac{1}{8}\cos x - \frac{1}{8}\cos x\cos 4x$

$\Rightarrow y = \frac{1}{8}\cos x - \frac{1}{16}(\cos 5x + \cos 3x)$

$\Rightarrow y = \frac{1}{8}\cos x - \frac{1}{16}\cos 5x - \frac{1}{16}\cos 3x$

$\Rightarrow y_1 = -\frac{1}{8}\sin x + \frac{5}{16}\sin 5x + \frac{3}{16}\sin 3x$

$= \frac{1}{8}\cos(\pi/2+x) - \frac{1}{16}\cdot 5^1\cdot\cos(\pi/2+5x) - \frac{1}{16}\cdot 3^1\cdot\cos(\pi/2+3x)$

CS CamScanner

$\Rightarrow y_2 = \frac{1}{8}\cdot\cos(2\cdot\pi/2+x) - \frac{1}{16}\cdot 5^2\cos(2\cdot\pi/2+5x) - \frac{1}{16}\cdot 3^2\cdot\cos(2\cdot\pi/2+3x)$

- - - - - - - - - - - - - - - - - - - -

- - - - - - - - - - - - - - - - - - - -

$\therefore y_n = \frac{1}{8}\cos(n\pi/2+x) - \frac{1}{16}\cdot 5^n\cdot\cos(n\pi/2+5x) - \frac{1}{16}\cdot 3^n\cdot\cos(n\pi/2+3x)$

(*) $y = \sin^4 x\cos^4 x = (\sin x\cos x)^4 = \frac{1}{16}(2\sin x\cos x)^4$

$\Rightarrow y = \frac{1}{16}(\sin^2 2x)^2$

$\Rightarrow y = \frac{1}{16\times 4}(1-\cos 4x)^2$

$\Rightarrow y = \frac{1}{64}(1+\cos^2 4x - 2\cos 4x)$

$\Rightarrow y = \frac{1}{64} - \frac{1}{32}\cos 4x + \frac{1}{64\times 2}(1+\cos 8x)$

$\Rightarrow y = \frac{1}{64} - \frac{1}{32}\cos 4x + \frac{1}{128} + \frac{1}{128}\cos 8x$

$\therefore y_n = \frac{1}{128}\cdot 8^n\cdot\cos(n\pi/2+8x) - \frac{1}{32}\cdot 4^n\cdot\cos(n\pi/2+4x)$

(*) $y = \sin x\cdot\cos 2x\cdot\sin 3x$

$\Rightarrow y = \frac{1}{2}(2\sin x\cdot\cos 2x)\cdot\sin 3x = \frac{1}{2}(\sin 3x - \sin x)\sin 3x$

$\Rightarrow y = \frac{1}{4}(2\sin^2 3x - 2\sin 3x\sin x)$

$\Rightarrow y = \frac{1}{4}(1-\cos 6x - \cos 2x + \cos 4x)$

$\Rightarrow y = \frac{1}{4} - \frac{1}{4}\cos 6x - \frac{1}{4}\cos 2x + \frac{1}{4}\cos 4x$

$\therefore y_n = -\frac{1}{4}\cdot 6^n\cdot\cos(n\pi/2+6x) - \frac{1}{4}\cdot 2^n\cdot\cos(n\pi/2+2x) + \frac{1}{4}\cdot 4^n\cdot\cos(n\pi/2+4x)$

(*) $y = \tan^{-1} x/a$

$\Rightarrow y_1 = \frac{1}{1+x^2/a^2} = \frac{a}{a^2+x^2} = \frac{a}{(x-ia)(x+ia)}$

$\Rightarrow y_1 = \frac{1}{2i}\left[\frac{1}{(x-ia)} - \frac{1}{(x+ia)}\right] = \frac{1}{2i}\left[(x-ia)^{-1} - (x+ia)^{-1}\right]\cdot(1-1)!\,(-1)^{1-1}$

$\Rightarrow y_2 = \frac{1}{2i}(-1)\left[(x-ia)^{-2} - (x+ia)^{-2}\right] = \frac{(-1)^{2-1}\cdot(2-1)!}{2i}\left[(x-ia)^{-2} - (x+ia)^{-2}\right]$

$\Rightarrow y_3 = \frac{1}{2i}(-1)(-2)\left[(x-ia)^{-3} - (x+ia)^{-3}\right] = \frac{(-1)^{3-1}\cdot(3-1)!}{2i}\left[(x-ia)^{-3} - (x+ia)^{-3}\right]$

- - - - - - - - - - - - - - - - - - - -

- - - - - - - - - - - - - - - - - - - -

$\therefore y_n = \frac{(-1)^{n-1}\cdot(n-1)!}{2i}\left[(x-ia)^{-n} - (x+ia)^{-n}\right]$

let, $x = r\cos\theta$, $a = r\sin\theta$

then, $r = \sqrt{a^2+x^2}$ and, $\theta = \tan^{-1}(a/x)$

CamScanner

$\therefore (x-ia)^{-n} = r^{-n}(\cos\theta - i\sin\theta)^{-n} = r^{-n}\{\cos(n\theta) + i\sin(n\theta)\}$ [De moivre's theorem]

& $(x+ia)^{-n} = r^{-n}(\cos\theta + i\sin\theta)^{-n} = r^{-n}\{\cos(n\theta) - i\sin(n\theta)\}$ [ " ]

$\therefore y_n = \frac{(-1)^{n-1}\cdot(n-1)!}{2i} r^{-n}\{\cos(n\theta) + i\sin(n\theta) - \cos(n\theta) + i\sin(n\theta)\}$

$= (-1)^{n-1}\cdot(n-1)!\cdot r^{-n}\cdot\sin(n\theta)$

$= (-1)^{n-1}\cdot(n-1)!\cdot\left(\frac{a}{\sin\theta}\right)^{-n}\cdot\sin(n\theta)$ ; where $\theta = \tan^{-1}(a/x)$

(*) $y = \frac{x}{x^2+a^2}$ ; find $y_n$.

$\Rightarrow y = \frac{x}{(x-ia)(x+ia)}$

let, $\frac{x}{(x-ia)(x+ia)} = \frac{A}{(x-ia)} + \frac{B}{(x+ia)}$ —— (i)

$\Rightarrow x = A(x+ia) + B(x-ia)$

Eq

$\Rightarrow x = (A+B)x + ia(A-B)$

equating co-efficient from both sides,

$A+B = 1$ —— (ii) $\quad A-B = 0$

$\Rightarrow A = B$ —— (iii)

From (ii) & (iii), we get, $A = B = \frac{1}{2}$

Putting the value of A & B in eq$^n$ (i),

$\frac{x}{(x-ia)(x+ia)} = \frac{1}{2}\left[\frac{1}{(x-ia)} + \frac{1}{(x+ia)}\right] = \frac{1}{2}\left[(x-ia)^{-1} + (x+ia)^{-1}\right]$

$\therefore y = \frac{1}{2}\left[(x-ia)^{-1} + (x+ia)^{-1}\right]$

$\Rightarrow y_1 = \frac{1}{2}(-1)\left[(x-ia)^{-2} + (x+ia)^{-2}\right] = \frac{1}{2}(-1)^1\cdot 1!\left[(x-ia)^{-(1+1)} + (x+ia)^{-(1+1)}\right]$

$\Rightarrow y_2 = \frac{1}{2}(-1)(-2)\left[(x-ia)^{-3} + (x+ia)^{-3}\right] = \frac{1}{2}(-1)^2\cdot 2!\left[(x-ia)^{-(2+1)} + (x+ia)^{-(2+1)}\right]$

- - - - - - - - - - - - - - - - - - - -

- - - - - - - - - - - - - - - - - - - -

$\therefore y_n = \frac{1}{2}(-1)^n\cdot n!\left[(x-ia)^{-(n+1)} + (x+ia)^{-(n+1)}\right]$

let, $x = r\cos\theta$ and $a = r\sin\theta$

then, $r = \sqrt{a^2+x^2}$ and $\theta = \tan^{-1}(a/x)$

$\therefore (x-ia)^{-(n+1)} = r^{-(n+1)}\cdot(\cos\theta - i\sin\theta)^{-(n+1)}$

$= r^{-(n+1)}\cdot[\cos(n+1)\theta + i\sin(n+1)\theta]$ [De, moivre's theorem]

$(x+ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta + i\sin\theta)^{-(n+1)}$

$= r^{-(n+1)}[\cos(n+1)\theta - i\sin(n+1)\theta]$ [ " ]

CamScanner

$\therefore y_n = \frac{1}{2}(-1)^n \cdot n! \cdot r^{-(n+1)} \cdot 2\cos(n+1)\theta$

$= (-1)^n \cdot n! \cdot \left(\frac{x}{\cos\theta}\right)^{-(n+1)} \cdot \cos(n+1)\theta$ ; where $\theta = \tan^{-1}(a/x)$

(*) $y = \frac{1}{(x+a)^2+b^2} = \frac{1}{(x+a)^2 - (bi)^2} = \frac{1}{(x+a+ib)(x+a-ib)}$

$\Rightarrow y = \frac{1}{2ib}\left[\frac{1}{(x+a-ib)} - \frac{1}{(x+a+ib)}\right] = \frac{1}{2ib}\left[(x+a-ib)^{-1} - (x+a+ib)^{-1}\right]$

$\Rightarrow y_1 = \frac{1}{2ib}(-1)\left[(x+a-ib)^{-2} - (x+a+ib)^{-2}\right] = \frac{(-1)^1 \cdot 1!}{2ib}\left[(x+a-ib)^{-(1+1)} - (x+a+ib)^{-(1+1)}\right]$

$\Rightarrow y_2 = \frac{1}{2ib}(-1)(-2)\left[(x+a-ib)^{-3} - (x+a+ib)^{-3}\right] = \frac{(-1)^2 \cdot 2!}{2ib}\left[(x+a-ib)^{-(2+1)} - (x+a+ib)^{-(2+1)}\right]$

- - - - - - - - - - - - - - - - - - - -

- - - - - - - - - - - - - - - - - - - -

$\therefore y_n = \frac{(-1)^n \cdot n!}{2ib}\left[(x+a-ib)^{-(n+1)} - (x+a+ib)^{-(n+1)}\right]$

let, $x+a = r\cos\theta$ and $b = r\sin\theta$

then, $r = \sqrt{(x+a)^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{x+a}\right)$

$\therefore (x+a-ib)^{-(n+1)} = r^{-(n+1)}(\cos\theta - i\sin\theta)^{-(n+1)}$

$= r^{-(n+1)}\{\cos(n+1)\theta + i\sin(n+1)\theta\}$ [De moivre's theorem]

& $(x+a+ib)^{-(n+1)} = r^{-(n+1)}\{\cos(n+1)\theta - i\sin(n+1)\theta\}$ [ " ]

$\therefore y_n = \frac{(-1)^n \cdot n!}{2ib} r^{-(n+1)} \cdot 2i\sin(n+1)\theta$

$= \frac{(-1)^n \cdot n!}{b}\left(\frac{b}{\sin\theta}\right)^{-(n+1)} \cdot \sin(n+1)\theta$ ; where $\theta = \tan^{-1}\left(\frac{b}{a+x}\right)$

(*) $y = \frac{x^2}{(x-a)(x-b)}$ ; find $n$th derivative of $y$.

let, $\frac{x^2}{(x-a)(x-b)} \equiv 1 + \frac{A}{x-a} + \frac{B}{x-b}$

$\Rightarrow x^2 \equiv x^2 - (a+b)x + ab + A(x-b) + B(x-a)$

$\Rightarrow x^2 \equiv x^2 + x(A+B-a-b) + ab - Ab - Ba$

equating co-efficient from both side,

$A+B-a-b = 0$

$\Rightarrow A+B = a+b$

$\Rightarrow B = a+b-A$

$ab - Ab - Ba = 0$

$\Rightarrow ab - Ab - (a+b-A)a = 0$

$\Rightarrow Ab = ab - a^2 - ab + Aa$

$\Rightarrow A(b-a) = -a^2$

$\Rightarrow A = -\frac{a^2}{\ldots}$

$\therefore B = a+b+\frac{a^2}{b-a}$

$= ab-a^2+b^2-ab+a^2$

$= \frac{b^2}{b-a}$

$$\therefore y = 1 + \frac{\left(-\frac{a^2}{b-a}\right)}{(x-a)} + \frac{b^2}{x-b}$$

$$\Rightarrow y = 1 + \left(-\frac{a^2}{b-a}\right)(x-a)^{-1} + b^2(x-b)^{-1}$$

$$\Rightarrow y_1 = \left(-\frac{a^2}{b-a}\right)(-1)(x-a)^{-2} + b^2(-1)(x-b)^{-2}$$

$$\Rightarrow y_2 = \left(-\frac{a^2}{b-a}\right)(-1)^2 \cdot 2!\,(x-a)^{-(2+1)} + b^2 \cdot (-1)^2 \cdot 2!\,(x-b)^{-(2+1)}$$

-- -- -- -- -- -- -- --

-- -- -- -- -- --

$$\therefore y_n = \left(-\frac{a^2}{b-a}\right)(-1)^n \cdot n!\,(x-a)^{-(n+1)} + b^2 \cdot (-1)^n \cdot n!\,(x-b)^{-(n+1)}$$

MATH 1107

# Differential Calculus

## Expansion of Function

# $y = \tan^{-1}x \longrightarrow$ (i) $(1+x^2)y_1 = 0$ ; find $(y_n)_0$

(ii) $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

$\Rightarrow y_1 = \dfrac{1}{1+x^2}$

$\Rightarrow (1+x^2)y_1 = 1$ —— (i)

Differentiating n times by Leibnitz theorem,

$y_{n+1}\cdot(1+x^2) + {}^nC_1\cdot y_n\cdot 2x + {}^nC_2\cdot y_{n-1}\cdot 2 = 0$

$\Rightarrow (1+x^2)y_{n+1} + 2nx\cdot y_n + (n^2-n)y_{n-1} = 0$

$\Rightarrow (1+x^2)y_{n+1} + 2nx\cdot y_n + n(n-1)y_{n-1} = 0$ —— (ii)

Put $x=0$ in eq$^n$ (ii), we get,

$(y_{n+1})_0 + n(n-1)(y_{n-1})_0 = 0$

$\therefore (y_{n+1})_0 = -n(n-1)(y_{n-1})_0$ —— (3)

From eq$^n$ (i),

$(y_1)_0 = 1$

Put $n = 1, 2, 3, ---$ in eq$^n$ (3) we get,

$(y_2)_0 = -1(1-1)(y_0)_0$

$\Rightarrow (y_2)_0 = 0$

Similarly, $(y_4)_0 = -3(3-1)(y_2)_0$

$\Rightarrow (y_4)_0 = 0$

$\therefore (y_6)_0 = 0$

$---$

$(y_n)_0 = 0$ ; when $n$ is even

Again, $(y_3)_0 = -2(2-1)\cdot(y_1)_0 = -2\cdot 1$

~~$(y_5)_0 = -5\cdot 4(y_3)_0 = (-1)^2 5\cdot 4\cdot 3\cdot 1\cdot 2$~~

$(y_5)_0 = -4\cdot 3\cdot(y_3)_0 = (-1)^2\cdot 4\cdot 3\cdot 2\cdot 1 = (-1)^{\frac{5-1}{2}}\cdot 4! = (-1)^{\frac{5-1}{2}}(5-1)!$

$(y_7)_0 = -6\cdot 5\cdot(y_5)_0 = (-1)^3\cdot 6! = (-1)^{\frac{7-1}{2}}\cdot 6! = (-1)^{\frac{7-1}{2}}\cdot(7-1)!$

$-------$

$\therefore (y_n)_0 = (-1)^{\frac{n-1}{2}}\cdot(n-1)!$ ; when $n$ is odd.

# Statement of Rolle's theorem:

If (i) $f(x)$ is continious in the closed interval $[a,b]$

(ii) $f'(x)$ exists in the open interval $(a,b)$

and (iii) $f(a) = f(b)$

then there exists one value of $x$ (say $c$) lies between a and b $(a<c<b)$ such that $f'(c)=0$

**Problem:** Verify Rolle's theorem for the function $f(x) = x^2 - 5x + 6$ ; $1 \le x \le 4$

**Solⁿ:** we observe that,

$f(x)$ is continious in the closed interval $[1,4]$

And, $f'(x) = 2x-5$ ; which is exists in the open interval $(1,4)$

Now, $f(1) = 2$

$f(4) = 2$

$\therefore f(1) = f(4)$

Since, $f'(x) = 0$

$\Rightarrow 2x - 5 = 0$

$\Rightarrow x = \frac{5}{2} = 2.5$

$\therefore 1 < 2.5 < 4$

So, Rolle's theorem is verified for the function.

# Statement of Mean Value theorem:

If (i) $f(x)$ is continious in the closed interval $[a,b]$

(ii) $f'(x)$ exists in the open interval $(a,b)$

Then there exists at least one value of $x$ (say $c$) lies between a and b $(a<c<b)$ such that, $f(b) - f(a) = (b-a) f'(c)$

$$\Rightarrow f(b) = f(a) + (b-a) f'(c) \quad \text{—— (i)}$$

since $c$ lies between a and b, so, $c$ can be written as

$c = a + \theta(b-a)$ ; where $0<\theta<1$

Put, $b - a = h$

$\Rightarrow b = a+h$

From eqⁿ (i),

$$f(a+h) = f(a) + hf'(a+\theta h) \quad \text{—— (ii)}, \; 0<\theta<1$$

Put $a = x$ in eqⁿ (ii), we get,

$$f(x+h) = f(x) + hf'(x+\theta h), \; 0<\theta<1 \quad \text{—— (iii)}$$

06-02-2023

MATH 1107

# Differential Calculus

## # Taylor's theorem:

If (i) $f^{n-1}(x)$ exists in the ~~th~~ closed interval $[a,b]$

(ii) $f^{n}(x)$ exists in the open interval $(a,b)$

then there exists at least one value of $x$ (say $c$) lies between $a$ and $b$ $(a<c<b)$ such that,

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^n}{n!} f^n(c) \quad \text{--- (i)}$$

Note: Since $c$ lies between $a$ and $b$, so, $c$ can be written as

$$c = a + \theta(b-a), \; 0<\theta<1$$

and $b = a+h \Rightarrow b-a = h$

From eqn (i),

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a+\theta h), \; 0<\theta<1 \quad \text{--- (ii)}$$

Also, write $x$ for $a$,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots + \underbrace{\frac{h^n}{n!} f^n(x+\theta h)}_{R_n \text{ (remainder) (lagrange's form)}}, \; 0<\theta<1 \quad \text{--- (iii)}$$

Note: If $R_n \to 0$ when $n \to \infty$, then the taylor's series in finite form extended to infinity is valid. Then the series becomes:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

This series is called Taylor's series in infinite form.

## # Maclaurin's theorem:

Putting $x=0$ and $h=x$ in Taylor's series we get,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \underbrace{\frac{x^n}{n!} f^n(\theta x)}_{R_n \text{ (remainder) (lagrange's form)}} \quad \text{--- (i)} \; ; 0<\theta<1$$

Note: If $R_n \to 0$ when $n \to \infty$, then the maclaurin's series in finite form extended to infinity is valid. Then the series becomes:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

The series is called Maclaurin's series in infinite form.

CamScanner

# Prob: Find the value of $\sin 32°$ and $\sin 48°$

Solⁿ: we know the taylor's series is:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots \text{———} (1)$$

put $x = 30°$ and $h = 2° = \frac{2\pi}{180}$ radian

Let, $f(x) = \sin x$ $\therefore f(30°) = \sin 30° = \frac{1}{2}$

$f(x+h) = \sin(x+h)$ $\therefore f(32°) = \sin 32°$

$f'(x) = \cos x$ $\therefore f'(30°) = \cos 30° = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x$ $\therefore f''(30°) = -\sin 30° = -\frac{1}{2}$

— — — — — — — — —

— — — — — — — — —

$\therefore$ from (1) we get,

$$\sin 32° = \frac{1}{2} + \frac{2\pi}{180} \times \frac{\sqrt{3}}{2} + \frac{\left(\frac{2\pi}{180}\right)^2}{2!} \times \left(-\frac{1}{2}\right)$$

$$= \frac{1}{2} + \frac{\pi}{90} \times \frac{\sqrt{3}}{2} - \frac{\pi^2}{90^2 \times 4}$$

# Prob: Expand $\sin x$ or $\cos x$ in powers of $(x - \frac{\pi}{2})$ or $x$.

Solⁿ: we have the taylor's series is:

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots \text{———} (1)$$

Put $b = x$, $a = \frac{\pi}{2}$, then

$$f(x) = f(\tfrac{\pi}{2}) + (x - \tfrac{\pi}{2}) f'(\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^2}{2!} f''(\tfrac{\pi}{2}) + \cdots \text{———} (ii)$$

here $f(x) = \sin x$

$\Rightarrow f(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$\therefore f'(x) = \cos x$

$\therefore f'(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$f''(x) = -\sin x$

$\therefore f''(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

— — — — — — — — —

— — — — — — — — —

$\therefore$ from (ii) we get,

$$\sin x = 1 + (x - \tfrac{\pi}{2}) \times 0 + \frac{(x-\frac{\pi}{2})^2}{2!} \times (-1) + \cdots$$

$$= 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \cdots$$

CamScanner

Prob: Expand $e^x$ in powers of $x$.
/ $\sin x$ "
/ $\cos x$ "
/ $\log(1+x)$ "

1) Expand $e^x$ in powers of $x$.

$\Rightarrow$ We know the maclaurin's series is:

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \text{------}$ —— ①

here, $f(x) = e^x$

$\therefore f(0) = e^0 = 1$

$f'(x) = e^x$

$\therefore f'(0) = e^0 = 1$

$f''(x) = e^x$

$\therefore f''(0) = e^0 = 1$

— — — — —

— — — — — —

$\therefore$ from ① we get,

$e^x = 1 + x \times 1 + \frac{x^2}{2!} \times 1 + \text{---}$

$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \text{----}$

2) Expand $\sin x$ in powers of $x$.

$\Rightarrow$ we know the maclaurin's series is:

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \text{------}$ —— ①

Here, $f(x) = \sin x$

$\therefore f(0) = \sin 0^\circ = 0$

$f'(x) = \cos x$

$\therefore f'(0) = \cos 0^\circ = 1$

$f''(x) = -\sin x$

$\therefore f''(0) = -\sin 0^\circ = 0$

$f'''(x) = -\cos x$

$\therefore f'''(0) = -\cos 0^\circ = -1$

— — — — — —

— — — — — —

$\therefore$ from ① we get,

$\sin x = 0 + x \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times (-1) + \text{---}$

$\Rightarrow \sin x = x - \frac{x^3}{3!} + \text{-----}$

CS CamScanner

3) Expand $\cos x$ in powers of $x$.

$\Rightarrow$ We know the maclaurin's series is:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \frac{x^5}{5!} f'''''(0) + \ldots \quad (1)$$

Here, $f(x) = \cos x$

$\therefore f(0) = 1$

$f'(x) = -\sin x$

$\therefore f'(0) = 0$

$f''(x) = -\cos x$

$\therefore f''(0) = -1$

$f'''(x) = \sin x$

$\therefore f'''(0) = 0$

$f''''(x) = \cos x$

$\therefore f''''(0) = 1$

$f'''''(x) = -\sin x$

$\therefore f'''''(0) = 0$

— — — ———

— — — ——

$\therefore$ From (1) we get,

$$\cos x = 1 + x \times 0 + \frac{x^2}{2!} \times (-1) + \frac{x^3}{3!} \times 0 + \frac{x^4}{4!} \times 1 + \frac{x^5}{5!} \times 0.$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

4) Expand $\log(1+x)$ in powers of $x$

$\Rightarrow$ We know the maclaurin's series is:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \ldots \quad (1)$$

Here, $f(x) = \log(1+x)$

$\therefore f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x}$

$\therefore f'(0) = \frac{1}{1} = 1$

$f''(x) = -\frac{1}{(1+x)^2}$

$\therefore f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$

$\therefore f'''(0) = \frac{2}{1} = 2$

CamScanner

$$f''''(x) = -\frac{6}{(1+x)^4}$$

$$\therefore f''''(0) = -\frac{6}{1} = -6$$

$\therefore$ From (1) we get,

$$\log(1+x) = 0 + x \times 1 + \frac{x^2}{2!} \times (-1) + \frac{x^3}{3!} \times 2 + \frac{x^4}{4!} \times (-6) + \text{---} + \text{---}$$

$$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{---} \cdots$$

MATH 1107

13-02-23

# Differential Calculus

# Prob: Verify mean value theorem for the function

i) $f(x) = x(x-1)(x-2)$ ; $0 \le x \le 4$

ii) $f(x) = |x|$, $-1 \le x \le 1$

iii) $f(x) = |x|$, $0 \le x \le 1$

# Prob:

# Expand $\sin x, \cos x, e^{ax}\sin bx, e^{ax}\cos bx$ etc in powers of $x$

# Differentiation of known series:

Prob: Assuming expansion of $\sin x$, find the series of $\cos x$

# Method of undetermined co-efficients:

Prob: Expand $\log(1+x)$ or $\log(1-x)$ in powers of $x$

# Prob: Find the value of $\theta$ in the mean value theorem ~~f(x+h)~~

$f(x+h) = f(x) + hf'(x+\theta h)$ ; where $f(x) = \log x$

# Prob: i) Expand $e^x$ in powers of $(x-1)$

ii) Expand $2x^3+3x^2+5$ in powers of $(x-2)$

Expand $e^{ax}\sin bx$ in powers of $x$.

$\Rightarrow$ Here, $f(x) = e^{ax}\sin bx$

$f^n(x) = r^n e^{ax}\sin(bx + n\tan^{-1} b/a)$ ; $r = \sqrt{a^2+b^2}$

$f^n(\theta x) = r^n e^{a\theta x}\sin(b\theta x + n\tan^{-1} b/a)$

We know the maclaurin's series is;

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^n}{n!}f^n(\theta x) \quad \text{——— (1)}$$

Here, $|R_n| = \left|\frac{x^n}{n!}f^n(\theta x)\right|$

$$= \left|\frac{(rx)^n}{n!}e^{a\theta x}\right| \left|\sin(b\theta x + n\tan^{-1} b/a)\right| \le \left|\frac{(rx)^n}{n!}e^{a\theta x}\right|$$

as $|\sin(\;)| \le 1$,

which tends to 0 as $n$ tends to $\infty$ because

$$\lim_{n\to\infty} R_n \le \lim_{n\to\infty} \frac{(rx)^n}{n!}e^{a\theta x} = 0$$

∴ Finite form extended to infinity is valid.

Rule:

1) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

2) $\lim_{n\to\infty} x^n = 0 ; -1 \le x \le 1$

# Assuming expansion of $\sin x$, find the series of $\cos x$.

⟹ We know that,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{--- (i)}$$

$$\therefore \cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \cdots$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

# Expand $\log(1+x)$ or $\log(1-x)$ in powers of $x$

⟹ let,

$$\log(1+x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad \text{--- (i)}$$

$$\Rightarrow \frac{1}{1+x} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots$$

$$\Rightarrow (1+x)(a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots) = 1 \quad \text{--- (ii)}$$

Equating the co-efficient of $x^n$ on both sides we get,

$$(n+1) a_{n+1} + n a_n = 0$$

$$\Rightarrow a_{n+1} = -\frac{n}{n+1} a_n \quad \text{--- (iii)}$$

Put $x = 0$ in (i) & (ii),

$$\Rightarrow 0 = a_0$$

$$a_1 = 1$$

$$a_2 = -\tfrac{1}{2} a_1 = -\tfrac{1}{2}$$

$$a_3 = -\tfrac{2}{3} a_2 = \tfrac{1}{3}$$

$$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

CamScanner

MATH 1107

27-02-2023

# Indeterminent form

**Definition:**

The limit of $\phi(x)/\psi(x)$ as $x \to a$ is, in general, equal to the ~~quot~~ quotient of the limiting values of the numerator and denominator, but when this two limits are both zero, that rule is no longer applicable, since the limit takes the form $0/0$ which is meaningless. This type of forms which are meaningless, is called indeterminant form. There are 7 types of indeterminant forms which are:

$0/0, \infty/\infty, 0\times\infty, \infty-\infty, 0^0, \infty^0, 1^{\pm\infty}$

**# Statement of L' Hospital's Rule:**

If $\phi(x), \psi(x)$ as also their derivatives $\phi'(x), \psi'(x)$ are continious at $x=a$. and if $\phi(a) = \psi(a) = 0$ (ie. $\lim_{x\to a} \phi(x) = \lim_{x\to a} \psi(x) = 0$), then,

$$\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\psi'(x)} = \frac{\phi'(a)}{\psi'(a)}, \text{ provided } \psi'(a) \neq 0$$

**# Problem:**

1. $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$

2. $\lim_{x\to 0} \dfrac{\log(x^2)}{\log(\cot^2 x)}$

3. $\lim_{x\to 1} \left\{ \dfrac{1}{x^2-1} - \dfrac{2}{x^4-1} \right\}$

4. $\lim_{x\to\infty} 2^x \sin\dfrac{a}{2^x}$

5. $\lim_{x\to 0} (\cos x)^{\cot^2 x}$

6. $\lim_{x\to 0} \left(\dfrac{\tan x}{x}\right)^{1/x}$

7. $\lim_{x\to 0} \log_{\tan^2 x}(\tan^2 x)$

CamScanner

MATH 1107

02-03-23

Prob:

i. $\lim_{x\to 0} (\cos x)^{\cot^2 x}$

ii. $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$, $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$.

iii. $\lim_{x\to 0} \log_{\tan^2 x}(\tan^2 2x)$

iv. $\lim_{x\to \pi/2} (\sin x)^{\tan x}$

v. If $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$ is finite, find the value of a and the value of limit.

vi. Find a, b such that $\lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1$

vii. Determine the value of a, b, c so that $\frac{ae^x - b\cos x + ce^{-x}}{x\sin x} \to 2$ as $x\to 0$

Soln: iii) we have, $\log_a x = \log_a e \times \log_e x = \frac{\log x}{\log a}$

So, $\lim_{x\to 0} \log_{\tan^2 x}(\tan^2 2x) = \lim_{x\to 0} \frac{\log(\tan^2 2x)}{\log(\tan^2 x)}$

$= \lim_{x\to 0} \frac{\cancel{2}\log(\tan 2x)}{\cancel{2}\log(\tan x)}$, $\frac{\infty}{\infty}$ form

$= \lim_{x\to 0} \frac{\frac{1}{\tan 2x} \cdot 2\sec^2 2x}{\frac{1}{\tan x} \cdot \sec^2 x}$

$= \lim_{x\to 0} \frac{2\sin 2x}{\sin 4x}$, $\frac{0}{0}$ form

$= \lim_{x\to 0} \frac{4\cos 2x}{4\cos 4x}$

$= 1$ (Ans)

ii) $\underline{\text{Soln:}}$

Let, $y = \left(\frac{\tan x}{x}\right)^{1/x^2}$

$\Rightarrow \log y = \frac{1}{x^2} \log\left(\frac{\tan x}{x}\right)$

$\Rightarrow \log y = \dfrac{\log\left(\frac{\tan x}{x}\right)}{x^2}$

$\Rightarrow \lim\limits_{x\to 0} \log y = \lim\limits_{x\to 0} \dfrac{\log\left(\frac{\tan x}{x}\right)}{x^2}$, $0/0$ form

$= \lim\limits_{x\to 0} \dfrac{\frac{x}{\tan x}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)}{2x}$

$= \lim\limits_{x\to 0} \dfrac{x\sec^2 x - \tan x}{2x^2 \tan x}$, $0/0$ form

$= \lim\limits_{x\to 0} \dfrac{\sec^2 x - 2\sec^2 x \tan x \cdot x - \sec^2 x}{2x^2\sec^2 x + 4x\tan x}$,

$= \lim\limits_{x\to 0} \dfrac{-2x(1+\tan^2 x)\tan x}{2x^2\sec^2 x + 4x\tan x}$

$= \lim\limits_{x\to 0} \dfrac{x\tan x + x\tan^3 x}{x^2\sec^2 x + 2x\tan x}$, $0/0$ form

$= \lim\limits_{x\to 0} \dfrac{x\sec^2 x + \tan x + 3\tan x\sec^2 x \cdot x + \tan^3 x}{2x\sec^2 x + 2\sec^2 x\tan x \cdot x^2 + 2\tan x + 2x\sec^2 x}$

$\therefore \lim\limits_{x\to 0} \log y = \lim\limits_{x\to 0} \dfrac{x\sec^2 x + \tan x + 3x(\tan x + \tan^3 x) + \tan^3 x}{\cancel{4x\tan^2}\ 4x\sec^2 x + 2x^2(\tan x + \tan^3 x) + 2\tan x}$, $0/0$ form

$= \lim\limits_{x\to 0} \dfrac{2x\sec^2 x\tan x + \sec^2 x + \sec^2 x + 3(\tan x + \tan^3 x) + 3x(\sec^2 x + 3\tan^2 x\sec^2 x) + 3\tan^2 x \cdot \sec^2 x}{4x \times 2\sec^2 x \cdot \tan x + 4\sec^2 x + 4x(\tan x + \tan^3 x) + 2x^2(\sec^2 x + 3\tan^2 x \cdot \sec^2 x) + 2\sec^2 x}$

$\Rightarrow \lim\limits_{x\to 0} \log y = \dfrac{2}{6} = 1/3$

$\Rightarrow \lim\limits_{x\to 0} y = e^{1/3}$

CamScanner

v) $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$, $\frac{0}{0}$ form

$= \lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2}$

Since $\lim_{x\to 0} 3x^2 = 0$, so for the limiting value the form should be $\frac{0}{0}$

Thus, $\lim_{x\to 0} (2\cos 2x + a\cos x) = 0$

$\Rightarrow 2 + a = 0$

$\Rightarrow a = -2$

$\therefore \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2}$, $\frac{0}{0}$ form

$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x}$, $\frac{0}{0}$ form

$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6}$

$= \frac{-8+2}{6} = \frac{-6}{6} = -1$

vi) $\lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3}$, $\frac{0}{0}$ form

$= \lim_{x\to 0} \frac{x(-a\sin x) + (1+a\cos x) - b\cos x}{3x^2}$

$= \lim_{x\to 0} \frac{-ax\sin x + 1 + (a-b)\cos x}{3x^2}$ ——— ①

For finite limit, the form should be $\frac{0}{0}$

So, $\lim_{x\to 0} \{-ax\sin x + (a-b)\cos x + 1\} = 0$ as $\lim_{x\to 0} 3x^2 = 0$

$\Rightarrow a - b + 1 = 0$

$\Rightarrow b = 1 + a$

From ① we get,

$\lim_{x\to 0} \frac{-ax\sin x + 1 - \cos x}{3x^2}$, $\frac{0}{0}$ form

$= \lim_{x\to 0} \frac{-ax\cos x - a\sin x + \sin x}{6x}$, $\frac{0}{0}$ form

CS CamScanner

$$= \lim_{x \to 0} \frac{ax\sin x - a\cos x - a\cos x + \cos x}{6}$$

$$= \frac{-2a+1}{6}$$

$\therefore \frac{-2a+1}{6} = 1$ (Given)

$\Rightarrow -2a+1 = 6$

$\Rightarrow -2a = 5$

$\Rightarrow a = -5/2$ (Ans)

13/03/23

# Partial Differentiation

**Def$^n$:** The result of differentiating $u=f(x,y)$ with respect to $x$ treating $y$ as a constant is called the partial derivative of $u$ with respect to $x$. It can be written one of the form $\frac{\partial u}{\partial x}$ or $u_x$ or $\frac{\partial f}{\partial x}$, $f_x$ etc.

**# Homogeneous function:** A function $f(x,y)$ is said to be homogeneous of degree $n$ if it can be written one of the form $f(\lambda x,\lambda y)=\lambda^n f(x,y)$ or $x^n\varphi(y/x)$ or $y^n\varphi(x/y)$.

**Ex:** $f(x,y)=x^3y+x^2y^2$

$$\Rightarrow f(\lambda x,\lambda y)=\lambda^4(x^3y+x^2y^2)=\lambda^4 f(x,y)$$

$$f(x,y)=x^3y+x^2y^2$$

$$=x^4\left\{y/x+(y/x)^2\right\}$$

$$=x^4\varphi(y/x)$$ ; this is a homogenous function of degree 4.

$$* \quad \frac{x^3+y^3}{x+y}=\frac{x^3\{1+(y/x)^3\}}{x(1+y/x)}=x^2\frac{\{1+(y/x)^3\}}{\{1+(y/x)\}}=x^2\varphi(y/x)$$

**# Euler's theorem:** If $f(x,y)$ is a homogenous function of degree $n$ having continious partial derivatives, thus

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$$

**# Converse of Euler's theorem:** If $f(x,y,z)$ admits of continious partial derivatives and satisfies the relation $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x,y,z)$ then $f(x,y,z)$ is a homogenous function of degree $n$.

**Prob: 1.** If $u=\tan^{-1}\frac{x^3+y^3}{x+y}$, then $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\sin 2u$

**Prob: 2.** If $u=2\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\cot u/2=0$

**Prob: 3** If $u=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0$

**Prob: 4** If $u=\operatorname{cosec}^{-1}\sqrt{\frac{\sqrt{x}+\sqrt{y}}{x^{1/3}+y^{1/3}}}$, then,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right)$$

CamScanner

Solution:

Prob-1: Given,

$$u = \tan^{-1}\frac{x^3+y^3}{x+y}$$

$$\Rightarrow \tan u = \frac{x^3+y^3}{x+y} = \frac{x^3\{1+(y/x)^3\}}{x\{1+(y/x)\}}$$

$$\Rightarrow \tan u = x^2\,\phi(y/x)$$

$\therefore \tan u$ is a homogeneous function of degree 2.

Let, $v = \tan u$, so by euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v$$

$$\Rightarrow x\sec^2 u\cdot\frac{\partial u}{\partial x} + y\sec^2 u\cdot\frac{\partial u}{\partial y} = 2\tan u$$

Dividing both side by $\sec^2 u$,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}\times\cos^2 u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \quad \text{(Proved)}$$

Prob-2: Given,

$$u = 2\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \cos\frac{u}{2} = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \cos\frac{u}{2} = \frac{x\{1+(y/x)\}}{x^{1/2}\{1+(y/x)^{1/2}\}}$$

$$\Rightarrow \cos\frac{u}{2} = x^{1/2}\,\phi(y/x)$$

$\therefore \cos\frac{u}{2}$ is a homogeneous function of degree $\frac{1}{2}$.

Let, $v = \cos\frac{u}{2}$, so by euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{1}{2}v$$

$$\Rightarrow x\left(-\sin\frac{u}{2}\right)\cdot\frac{1}{2}\cdot\frac{\partial u}{\partial x} + y\left(-\sin\frac{u}{2}\right)\cdot\frac{1}{2}\cdot\frac{\partial u}{\partial y} = \frac{1}{2}\cos\frac{u}{2}$$

Dividing both side by $\frac{1}{2}\sin\frac{u}{2}$,

$$-x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = \cot\frac{u}{2}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \cot\frac{u}{2} = 0 \quad \text{(Proved)}$$

CS CamScanner

Prob-3: Given,

$$u = \cos^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \cos u = \frac{x\{1+(y/x)\}}{x^{1/2}\{1+(y/x)^{1/2}\}}$$

$$\Rightarrow \cos u = x^{1/2}\phi(y/x)$$

$\therefore \cos u$ is a homogeneous function of degree $1/2$

Let, $v = \cos u$, so by euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{1}{2}v$$

$$\Rightarrow x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

Dividing both side by $\sin u$,

$$\Rightarrow -x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = \frac{1}{2}\cot u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0 \text{ (Proved)}$$

Prob-4: Given,

$$u = \operatorname{cosec}^{-1}\sqrt{\frac{\sqrt{x}+\sqrt{y}}{x^{1/3}+y^{1/3}}}$$

$$\Rightarrow \operatorname{cosec}^2 u = \frac{x^{1/2}\{1+(y/x)^{1/2}\}}{x^{1/3}\{1+(y/x)^{1/3}\}}$$

$$\Rightarrow \operatorname{cosec}^2 u = x^{1/6}\phi(y/x)$$

$\therefore \operatorname{cosec}^2 u$ is a homogeneous function of degree $1/6$.

Let, $v = \operatorname{cosec}^2 u$, so by euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{1}{6}v$$

$$\Rightarrow x\, 2\operatorname{cosec} u\,(-)\operatorname{cosec} u \cdot \cot u \cdot \frac{\partial u}{\partial x} + y \cdot 2\operatorname{cosec} u \cdot (-)\operatorname{cosec} u \cdot \cot u\,\frac{\partial u}{\partial y} = \frac{1}{6}\operatorname{cosec}^2 u$$

Dividing both side by $2\operatorname{cosec}^2 u \cot u$,

$$\Rightarrow -x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = \frac{1}{12}\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u \quad \text{—— (1)}$$

Partially differentiating eqn (1) with respect to $x$, and $y$ respectively we get,

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{12}\sec^2 u \cdot \frac{\partial u}{\partial x} \quad \text{—— (ii)}$$

CS CamScanner

And,

$$x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = -\frac{1}{12}\sec^2 u\,\frac{\partial u}{\partial y} \quad \text{—— (iii)}$$

Now, multiplying $x$ with eqn (ii) and multiplying with eqn (iii) we get,

$$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + yx\frac{\partial^2 u}{\partial x\partial y} = -\frac{1}{12}\sec^2 u\cdot x\frac{\partial u}{\partial x} \quad \text{—— (iv)}$$

$$xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\sec^2 u\cdot y\frac{\partial u}{\partial y} \quad \text{—— (v)}$$

Now, from (iv) + (v) we get,

$$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} + xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) - \frac{1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u + \frac{1}{12}\sec^2 u\left(\frac{1}{12}\tan u\right) \quad [\text{from (i)}]$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left(1 + \frac{\sec^2 u}{12}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left(\frac{12 + \sec^2 u}{12}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left(\frac{12 + 1 + \tan^2 u}{12}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$$

(Proved)

CamScanner

16/03/23

## Differential Calculus

**Prob:**

If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$, then $U_x^2+U_y^2+U_z^2=2(xU_x+yU_y+zU_z)$

**Solution:**

Given,

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1 \quad \text{--- (1)}$$

Partially differentiating eq$^n$ (1) with respect to $x, y, z$ respectively, we get,

$$\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u)^2}U_x-\frac{y^2}{(b^2+u)^2}\cdot U_x-\frac{z^2}{(c^2+u)^2}U_x=0$$

$$\Rightarrow \frac{2x}{a^2+u}-U_x\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]=0$$

$$\Rightarrow U_x=\frac{2x}{(a^2+u)F}, \text{ where } F=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

Similarly, we get,

$$U_y=\frac{2y}{(b^2+u)F}$$

$$U_z=\frac{2z}{(c^2+u)F}$$

$$\therefore U_x^2+U_y^2+U_z^2=\frac{4x^2}{(a^2+u)^2F^2}+\frac{4y^2}{(b^2+u)^2F^2}+\frac{4z^2}{(c^2+u)^2F^2}$$

$$\Rightarrow U_x^2+U_y^2+U_z^2=\frac{4}{F^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$\Rightarrow U_x^2+U_y^2+U_z^2=\frac{4}{F^2}\cdot F=\frac{4}{F}$$

Again,

$$2(xU_x+yU_y+zU_z)=\frac{4x^2}{(a^2+u)F}+\frac{4y^2}{(b^2+u)F}+\frac{4z^2}{(c^2+u)F}$$

$$\Rightarrow 2(xU_x+yU_y+zU_z)=\frac{4}{F}\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]=\frac{4}{F}\cdot 1=\frac{4}{F}$$

$$\therefore U_x^2+U_y^2+U_z^2=2(xU_x+yU_y+zU_z)$$

(Proved)

CS CamScanner

# Prob:
If $U = F(y-z, z-x, x-y)$ then find $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}$

Solution:

Let,

$p = y-z$ , $q = z-x$ , $r = x-y$

$\Rightarrow \frac{\partial p}{\partial x} = 0$ , $\frac{\partial q}{\partial x} = -1$ , $\frac{\partial r}{\partial x} = 1$

$\Rightarrow \frac{\partial p}{\partial y} = 1$ , $\frac{\partial q}{\partial y} = 0$ , $\frac{\partial r}{\partial y} = -1$

$\Rightarrow \frac{\partial p}{\partial z} = -1$ , $\frac{\partial q}{\partial z} = 1$ , $\frac{\partial r}{\partial z} = 0$

$\therefore U = F(p, q, r)$ —— (i)

Now,

$$\frac{\partial U}{\partial x} = \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial F}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial F}{\partial p} \cdot 0 + \frac{\partial F}{\partial q} \cdot (-1) + \frac{\partial F}{\partial r} (1)$$

$$\Rightarrow \frac{\partial U}{\partial x} = -\frac{\partial F}{\partial q} + \frac{\partial F}{\partial r} \text{ —— (ii)}$$

Again,

$$\frac{\partial U}{\partial y} = \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial F}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$\Rightarrow \frac{\partial U}{\partial y} = \frac{\partial F}{\partial p} (1) + \frac{\partial F}{\partial q} (0) + \frac{\partial F}{\partial r} (-1)$$

$$\Rightarrow \frac{\partial U}{\partial y} = \frac{\partial F}{\partial p} - \frac{\partial F}{\partial r} \text{ —— (iii)}$$

Also,

$$\frac{\partial U}{\partial z} = \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial F}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$\Rightarrow \frac{\partial U}{\partial z} = \frac{\partial F}{\partial p} (-1) + \frac{\partial F}{\partial q} (1) + \frac{\partial F}{\partial r} \cdot (0)$$

$$\Rightarrow \frac{\partial U}{\partial z} = -\frac{\partial F}{\partial p} + \frac{\partial F}{\partial q} \text{ —— (iv)}$$

From, (ii)+(iii)+(iv) we get,

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = -\frac{\partial F}{\partial q} + \frac{\partial F}{\partial r} + \frac{\partial F}{\partial p} - \frac{\partial F}{\partial r} - \frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}$$

$$= 0$$

(Ans)

CS CamScanner

# Prob:

If $u = x^n F(y/x, z/x)$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Solution:

Let, $p = y/x$ and $q = z/x$

$\therefore \frac{\partial p}{\partial x} = -\frac{y}{x^2}$ ; $\frac{\partial q}{\partial x} = -\frac{z}{x^2}$

$\frac{\partial p}{\partial y} = 1/x$ ; $\frac{\partial q}{\partial y} = 0$

$\frac{\partial p}{\partial z} = 0$ ; $\frac{\partial q}{\partial z} = \frac{1}{x}$

$\therefore U = x^n F(p,q)$ —— (i)

Now, partially differentiating eqn (i) with respect to $x, y, z$ respectively we get,

$\frac{\partial U}{\partial x} = nx^{n-1} F(p,q) + x^n \left[\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x}\right]$ [multiplying $x$ with both side]

$\Rightarrow x\frac{\partial U}{\partial x} = nx^n F(p,q) + x\cdot x^n \left[\frac{\partial p}{\partial x}\cdot\frac{\partial f}{\partial p} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x}\right]$

$\Rightarrow x\frac{\partial U}{\partial x} = nx^n F(p,q) + x^n\cdot x\left[\frac{\partial f}{\partial p}\left(-\frac{y}{x^2}\right) + \frac{\partial f}{\partial q}\cdot\left(-\frac{z}{x^2}\right)\right]$

$\Rightarrow x\frac{\partial U}{\partial x} = nx^n F(p,q) - x^n \left[\frac{\partial f}{\partial p}\cdot y/x + \frac{\partial f}{\partial q}\cdot z/x\right]$ —— (ii)

Again,

$\frac{\partial U}{\partial y} = x^n\left[\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y}\right]$

$\Rightarrow \frac{\partial U}{\partial y} = x^n\left[\frac{\partial f}{\partial p}\cdot 1/x + \frac{\partial f}{\partial q}\cdot(0)\right]$

$\Rightarrow \frac{\partial U}{\partial y} = x^n \frac{\partial f}{\partial p}\cdot 1/x$ [Multiplying $y$ with both side]

$\Rightarrow y\frac{\partial U}{\partial y} = x^n \frac{\partial f}{\partial p}\cdot y/x$ —— (iii)

And,

$\frac{\partial U}{\partial z} = x^n\left[\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial z}\right]$

$\Rightarrow \frac{\partial U}{\partial z} = x^n\left[\frac{\partial f}{\partial p}\cdot(0) + \frac{\partial f}{\partial q}\cdot 1/x\right]$

$\Rightarrow \frac{\partial U}{\partial z} = x^n \frac{\partial f}{\partial q}\cdot 1/x$ [Multiplying $z$ with both side]

$\Rightarrow z\frac{\partial U}{\partial z} = x^n\cdot\frac{\partial f}{\partial q}\cdot z/x$ —— (iv)

CS CamScanner

From (ii) + (iii) + (iv) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= nx^n f(p,q) - x^n \frac{\partial f}{\partial p}\cdot \frac{y}{x} - x^n \frac{\partial f}{\partial q}\cdot \frac{z}{x} + x^n \frac{\partial f}{\partial p}\cdot \frac{y}{x} + x^n \frac{\partial f}{\partial q}\cdot \frac{z}{x}$$

$$= nx^n F(p,q)$$

$$= nx^n F(y/x, z/x)$$

$$= nU$$ (Ans)

# Prob:

If $U = \log(x^3+y^3+z^3-3xyz)$, Prove, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

Solution:

Given, $U = \log(x^3+y^3+z^3-3xyz)$ —— (i)

Now, partially differentiating eqn (i) with respect to $x, y, z$ respectively we get,

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \times \frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} \text{ —— (ii)}$$

Again, Similarly,

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} \text{ —— (iii)}$$

and, $$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz} \text{ —— (iv)}$$

From (ii) + (iii) + (iv) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2+3y^2+3z^2-3xy-3yz-3zx}{x^3+y^3+z^3-3xyz}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$ (Proved)

# Prob:

If $U = f(x+ay) + \phi(x-ay)$ then, prove that, $\frac{\partial^2 U}{\partial y^2} = a^2 \frac{\partial^2 U}{\partial x^2}$

Solution:

Given,

$$U = f(x+ay) + \phi(x-ay)$$

$$\Rightarrow \frac{\partial U}{\partial x} = f'(x+ay) + \phi'(x-ay)$$

$$\Rightarrow \frac{\partial^2 U}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \quad \text{(i)}$$

Now,

$$\frac{\partial U}{\partial y} = a f'(x+ay) + (-a)\phi'(x-ay)$$

$$\Rightarrow \frac{\partial^2 U}{\partial y^2} = a^2 f''(x+ay) + (-a)(-a)\phi''(x-ay)$$

$$\Rightarrow \frac{\partial^2 U}{\partial y^2} = a^2 \{f''(x+ay) + \phi''(x-ay)\}$$

$$\Rightarrow \frac{\partial^2 U}{\partial y^2} = a^2 \frac{\partial^2 U}{\partial x^2} \quad [\text{from (i)}]$$

(Proved)

# Prob:

If $U = \log(x^2+y^2+z^2)$, then prove that, $x\frac{\partial^2 U}{\partial x^2} = y\frac{\partial^2 U}{\partial y^2} = z\frac{\partial^2 U}{\partial z^2}$

Solution:

Given, $U = \log(x^2+y^2+z^2) \quad \text{(i)}$

Now partially differentiating eqn (i) with respect to $x, y, z$ respectively. we get,

$$\frac{\partial U}{\partial x} = \frac{2x}{(x^2+y^2+z^2)} \quad \text{(ii)}$$

$$\frac{\partial U}{\partial y} = \frac{2y}{(x^2+y^2+z^2)} \quad \text{(iii)}$$

$$\frac{\partial U}{\partial z} = \frac{2z}{(x^2+y^2+z^2)} \quad \text{(iv)}$$

Now partially differentiating eqn (ii), (iii) & (iv) with respect to $x, y$ and $z$ respectively we get,

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left\{\frac{2x}{x^2+y^2+z^2}\right\} =$$

CamScanner

MATH 1107 03/04/23

# Differential Calculus

## Maxima & Minima

If $y=f(x)$ and we put value $\frac{dy}{dx}=0$ then get values for $x$; those values are called critical values.
for determining maxima and minima, we first need to find $\frac{d^2y}{dx^2}$ then put those critical values in $\frac{d^2y}{dx^2}$. If we get positive value for any critical value that critical value is called minima. and vice versa.
If we don't get minima or maxima for a critical value, then we need to differentiate that function until we get maxima or minima.
Suppose, $f'(x)=0=f''(x)=f'^2(x)=f^{(n-1)}x=0$ and $f^n(x)\neq 0$ and if $n$ is odd, then neither maxima nor minima, These values are called point of inflexion. If $n$ is even and the value is positive then minima and vice versa.

* criticals points are also called sattle points.
* for $f(x,y)$, suppose,

$$r=\frac{d^2f}{dx^2}, \quad s=\frac{d^2f}{dx\,dy}, \quad t=\frac{d^2f}{dy^2}$$

If $rt-s^2 \geqslant 0$, then the function has a minima or maxima
If $rt-s^2<0$, then the function has neither minima nor maxima
If both $r,t>0$, then the function has minima.
If both $r,t<0$, then the function has maxima.

## # Problem:

1) Examine the maxima and minima of $\sin x(1+\cos x)$
2) Examine the maxima and minima of $f(x)=5x^6-18x^5+15x^4-10$
3) Examine the maxima and minima of $f(x)=\left(\frac{1}{\sqrt[n]{x}}\right)^{1/x}$

$$\text{or, } f(x)=x^{1/x}$$
$$\text{or, } f(x)=\frac{x}{\log x}$$
$$\text{or, } f(x)=\left(\frac{1}{x}\right)^x$$

4) Examine the maxima and minima of. $U=\frac{4}{x}+\frac{36}{y}$ and $x+y=2$
5) If $\frac{x}{2}+\frac{y}{3}=1$, find the maximum value of $xy$ and minimum value of $x^2+y^2$
6) If $xy=4$, find the maximum or minimum $xy$ and $4x+9y$.
7) Show that, $f(x,y)=x^3+3x^2+4xy+y^2$ attains a minimum at $(2/3,-4/3)$
8) Find the extreme or turning value of $f(x,y)=2x^2-xy+2y^2-20x$.

CamScanner

3) Solution:

1) Let, $y = \sin x(1+\cos x)$

$\Rightarrow \frac{dy}{dx} = \cos x(1+\cos x) + \sin x(-\sin x)$

$\Rightarrow \frac{dy}{dx} = \cos^2 x + \cos x - \sin^2 x$

$\Rightarrow \frac{dy}{dx} = 2\cos^2 x - 1 + \cos x$ —— (i)

$\Rightarrow \frac{dy}{dx} = 2\cos^2 x + 2\cos x - \cos x - 1$

$\Rightarrow \frac{dy}{dx} = 2\cos x(\cos x + 1) - 1(\cos x + 1)$

$\Rightarrow \frac{dy}{dx} = (2\cos x - 1)(\cos x + 1)$

For maxima or minima, $\frac{dy}{dx} = 0$

i.e, either $2\cos x - 1 = 0$ or, $\cos x + 1 = 0$

$\Rightarrow \cos x = \frac{1}{2}$ $\Rightarrow \cos x = -1$

$\Rightarrow x = \frac{\pi}{3}$ $\Rightarrow x = \pi$

Now, $\frac{d^2y}{dx^2} = -4\sin 2x - \sin x$ —— (ii)

Putting $x = \frac{\pi}{3}$ in eqn (ii),

$-4\sin\left(\frac{2\pi}{3}\right) - \sin\frac{\pi}{3} = -4 \times \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{5\sqrt{3}}{2} < 0$

$\therefore$ for $x = \frac{\pi}{3}$, the function has a maximum value.

Now, putting $x = \pi$ in eqn (i),

$\frac{d^2y}{dx^2} = 0$

So, at $x = \pi$, there is no maxima and minima.

Again,

$\frac{d^3y}{dx^3} = -8\cos 2x - \cos x$ —— (iii)

Putting $x = \pi$ in eqn (iii) we get,

$-8 \times \cos 2\pi - \cos\pi = -8 + 1 = -7 < 0$

So, at $x = \pi$, there is no minima or maxima. It is called the point of inflexion. The point of inflexion is at $x = \pi$.

CS CamScanner

2) Let, $y = 5x^6 - 18x^5 + 15x^4 - 10$

$\therefore \frac{dy}{dx} = 30x^5 - 90x^4 + 60x^3$ —— (i)

$\Rightarrow \frac{dy}{dx} = 30x^3 (x^2 - 3x + 2)$

for maxima and minima,

$$\frac{dy}{dx} = 0$$

$\therefore 30x^3 (x^2 - 3x + 2) = 0$

$\Rightarrow 30x^3 (x-1)(x-2) = 0$

$\Rightarrow x = 0, 0, 0, 1, 2$

Now, $\frac{d^2y}{dx^2} = 30(5x^4 - 12x^3 + 6x^2)$ —— (ii)

Now, put $x=1$ in eqn (ii),

$$\frac{d^2y}{dx^2} = 30(5-12+6) = -30 < 0$$

$\therefore$ at $x=1$, the function has a maximum value

again, put $x=2$ in eqn (ii),

$$\frac{d^2y}{dx^2} = 30(80-96+24) = 240 > 0$$

$\therefore$ at $x=2$, the function has a minimum value

Now, put $x=0$ in eqn (ii),

$$\frac{d^2y}{dx^2} = 0$$

So, ~~at~~ Now. we have to examine higher order derivatives,

$$\frac{d^3y}{dx^2} = 120(5x^3 - 9x^2 + 3x)$$

for, $x=0$, $\frac{d^3y}{dx^2} = 0$

Again, $\frac{d^4y}{dx^4} = 360(5x^2 - 6x + 1)$

for, $x=0$, $\frac{d^4y}{dx^4} = 360 > 0$

Since, even order derivative is positive for $x=0$.
So, for $x=0$, the function has minimum value.

CamScanner

[3] Let, $f(x) = x^{1/x}$

$\therefore \log f(x) = \frac{1}{x}\log x$ —— (i)

Diffentiating eqn (i) with respect to $x$,

$$\frac{1}{f(x)} f'(x) = \frac{1}{x}\cdot\frac{1}{x} - \frac{1}{x^2}\log x = \frac{1}{x^2}(1-\log x)$$

$$\therefore f'(x) = f(x)\left\{\frac{1}{x^2}(1-\log x)\right\} \text{ —— (ii)}$$

For, maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow f(x)\left\{\frac{1}{x^2}(1-\log x)\right\} = 0$$

Here, $\frac{1}{x^2}(1-\log x) = 0$

$\Rightarrow \log x = 1$

$\Rightarrow \log x = \log e$

$\Rightarrow x = e$

Now, differentiating eqn (ii) with respect to $x$,

$$f''(x) = f'(x)\left\{\frac{1}{x^2}(1-\log x)\right\} + f(x)\left\{-\frac{3}{x^3} + \frac{2}{x^3}\log x\right\} \text{ —— (iii)}$$

put $x = e$ in eqn (iii),

$$f''(e) = f'(e)\left\{\frac{1}{e^2}(1-\log e)\right\} + f(e)\left\{\frac{-3}{e^3} + \frac{2}{e^3}\log e\right\}$$

$$= 0 + e^{1/e}\left\{-\frac{1}{e^3}\right\}$$

$$= -\frac{e^{1/e}}{e^3} < 0$$

$\therefore$ The function has maximum value at $x = e$.

---

Let, $f(x) = \frac{x}{\log x}$

$$\therefore f'(x) = \frac{\log x - 1}{(\log x)^2} \text{ —— (i)}$$

For maxima or minima,

$$f'(x) = 0$$

$$\Rightarrow \frac{\log x - 1}{(\log x)^2} = 0$$

$\Rightarrow \log x - 1 = 0$

$\Rightarrow \log x = 1$

$\Rightarrow \log x = \log e$

$\Rightarrow x = e$

Now, differentiating eq$^n$ (i) with respect to $x$,

$$f''(x) = \frac{(\log x)^2 \frac{1}{x} - (\log x - 1)' \cdot 2\log x \cdot \frac{1}{x}}{(\log x)^4} \quad \text{——— (ii)}$$

put $x=e$ in eq$^n$ (ii),

$$f''(e) = \frac{1 \times \frac{1}{e} - 0}{(1)^4} = \frac{1}{e} > 0$$

$\therefore$ The function has minimum value at $x=e$.

---

Let, $f(x) = \left(\frac{1}{x}\right)^x$

~~$\therefore f(x) =$~~

$\therefore \log f(x) = x(\log \frac{1}{x}) = x(\log 1 - \log x)$

$\Rightarrow \log f(x) = -x\log x$ ——— (i) $[\because \log 1 = 0]$

Now Differentiating eq$^n$ (i) with respect to $x$,

$$\frac{1}{f(x)} f'(x) = -\log x - 1$$

$$\Rightarrow f'(x) = f(x)(-\log x - 1) \quad \text{——— (ii)}$$

For maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow f(x)(-\log x - 1) = 0$$

$$\Rightarrow -\log x - 1 = 0$$

$$\Rightarrow \log x^{-1} = 1$$

$$\Rightarrow \log \frac{1}{x} = \log e$$

$$\Rightarrow x = \frac{1}{e}$$

Now, differentiating eq$^n$ (ii) with respect to $x$,

$$\frac{1}{f(x)} f''(x) - \frac{f'(x)}{\{f(x)\}^2} = -\frac{1}{x}$$

$$\Rightarrow f''(x) = f(x) \frac{f'(x)}{\{f(x)\}^2} - \frac{f(x)}{x}$$

$$\Rightarrow f''(x) = \frac{f'(x)}{f(x)} - \frac{f(x)}{x} \quad \text{——— (iii)}$$

put $x = \frac{1}{e}$ in eq$^n$ (iii),

$$f''(\tfrac{1}{e}) = -e^{1+\frac{1}{e}} < 0$$

$\therefore$ The function has a maximum value at $x = \frac{1}{e}$.

---

Let, $y = \left(\frac{1}{\sqrt[n]{x}}\right)^x$

$\Rightarrow \log y = \frac{1}{x} \log \frac{1}{\sqrt[n]{x}}$

$\Rightarrow \log y = \frac{1}{x}\left[\log 1 - \frac{1}{n}\log x\right]$

$\Rightarrow \log y = -\frac{1}{nx}\log x$ ——— (i)

Now, differentiating eq$^n$ (i) with respect to $x$, we get,

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = -\frac{1}{nx}\cdot\frac{1}{x} + \frac{1}{nx^2}\log x$ ——— (ii)

$\Rightarrow \frac{dy}{dx} = \frac{y}{nx^2}(\log x - 1)$ ——— (iii)

For maxima or minima, $\frac{dy}{dx} = 0$

$\Rightarrow \log x - 1 = 0$

$\Rightarrow \log x = 1 = \log e$

$\Rightarrow x = e$

Now, differentiating eq$^n$ (iii) with respect to $x$, we get,

$-\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2} = \frac{nx^2\cdot 1/x - 2nx(\log x - 1)}{n^2x^4}$

$\Rightarrow -\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2} = \frac{nx(1 - 2\log x + 2)}{n^2x^4}$

$\Rightarrow -\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2} = \frac{3 - 2\log x}{nx^3}$ ——— (iv)

put $x = e$ in eq$^n$ (iv) we get,

$\frac{1}{y}\cdot\frac{d^2y}{dx^2} = \frac{3-2}{ne^3} = \frac{1}{ne^3} > 0$

So, for $x = e$, the function has a minimum value and the minimum value is $\left(\frac{1}{\sqrt[n]{e}}\right)^{1/e}$.

4) Given, $x + y = 2$

$\therefore y = 2 - x$ ——— (i)

Let, $p = \frac{4}{x} + \frac{36}{y}$

$\Rightarrow p = \frac{4}{x} + \frac{36}{2-x}$

$\Rightarrow p = \frac{8 - 4x + 36x}{2x - x^2}$

$\Rightarrow p = \frac{8(1+4x)}{2x - x^2}$

$\therefore \frac{dp}{dx} = \frac{8\times 4\times(2x - x^2) - 8(1+4x)(2-2x)}{(2x - x^2)^2}$

$$\Rightarrow \frac{dp}{dx} = \frac{32(2x-x^2) - 16(1+4x)(1-x)}{(2x-x^2)^2}$$

$$\Rightarrow \frac{dp}{dx} = \frac{64x-32x^2 - 16(1-x+4x-4x^2)}{(2x-x^2)^2}$$

$$\Rightarrow \frac{dp}{dx} = \frac{64x-32x^2-48x-16+64x^2}{(2x-x^2)^2}$$

$$\Rightarrow \frac{dp}{dx} = \frac{32x^2+16x-16}{(2x-x^2)^2} \quad \text{—— (ii)}$$

For maxima and minima, $\frac{dp}{dx} = 0$

$\therefore 16(2x^2+x-1) = 0$

$\Rightarrow 2x^2+2x-x-1 = 0$

$\Rightarrow 2x(x+1) - 1(x+1) = 0$

$\Rightarrow (2x-1)(x+1) = 0$

$\Rightarrow x = \frac{1}{2}, -1$

Now, differentiating eqn (ii) with respect to $x$, we get,

$$\frac{d^2p}{dx^2} = \frac{(2x-x^2)^2(64x+16) - (32x^2+16x-16)\,2(2x-x^2)(2-2x)}{(2x-x^2)^4} \quad \text{—— (iii)}$$

put $x=-1$ in eqn (iii) we get,

$\frac{d^2p}{dx^2} = -\frac{16}{3} < 0$

$\therefore$ for $x=-1$, the function has a maximum value and the maximum value is 8.

put $x=\frac{1}{2}$ in eqn (iii) we get,

$\frac{d^2p}{dx^2} = \frac{256}{3} > 0$

So, for $x=\frac{1}{2}$, the function has a minimum value and the minimum value is 32.

5) Given,

$$\frac{x}{2}+\frac{y}{3}=1$$

$$\Rightarrow 3x+2y=6$$

$$\Rightarrow y=\frac{6-3x}{2} \quad \text{—— (i)}$$

Let, $p=xy=x\left(\frac{6-3x}{2}\right) = \frac{6x-3x^2}{2} = \frac{3}{2}(2x-x^2)$

$\therefore \frac{dp}{dx}=\frac{3}{2}(2-2x)$ —— (ii)

For maxima or minima, $\frac{dp}{dx}=0$

$$\Rightarrow \frac{3}{2}(2-2x)=0$$

$$\Rightarrow 2x=2$$

$$\Rightarrow x=1$$

Now, Differentiating eqn (ii) with respect to $x$, we get,

$$\frac{d^2p}{dx^2}=\frac{3}{2}(-2)=-3<0$$

$\therefore$ The function has maximum value at $x=1$ and the maximum value is $3/2$.

Now, let, $q=x^2+y^2 = x^2+\left(\frac{6-3x}{2}\right)^2$

$$\Rightarrow q = x^2+\frac{36+9x^2-36x}{4}$$

$$\Rightarrow q = \frac{13x^2-36x+36}{4}$$

Now, $\frac{dq}{dx}=\frac{1}{4}(26x-36)$ —— (iii)

For maxima or minima, $\frac{dq}{dx}=0$

$$\therefore 26x-36=0$$

$$\Rightarrow x=\frac{36}{26}=\frac{18}{13}$$

Now, differentiating eqn (iii) with respect to $x$, we get,

$$\frac{d^2q}{dx^2}=\frac{1}{4}\times 26 = 13/2 > 0$$

$\therefore$ the function has minimum value at $x=\frac{18}{13}$ and the minimum value is $\frac{36}{13}$.

CS CamScanner

6) Given,

$yx = 4$

$\therefore y = 4/x$ —— (i)

Let, $4x + 9y = p \Rightarrow p = 4x + \frac{36}{x}$

$\therefore \frac{dp}{dx} = 4 - \frac{36}{x^2}$ —— (ii)

For maxima or minima,

$$\frac{dp}{dx} = 0$$

$$\Rightarrow 4 - \frac{36}{x^2} = 0$$

$$\Rightarrow x^2 = \frac{36}{4} = 9$$

$$\Rightarrow x = \pm 3$$

Now, differentiating eqn (ii) with respect to $x$,

$$\frac{d^2p}{dx^2} = 0 + \frac{72}{x^3} = \frac{72}{x^3} \text{ —— (iii)}$$

put $x = 3$ in eqn (iii) we get, $\frac{d^2p}{dx^2} = 8/3 > 0$

So, the function has a minimum value at $x = 3$ and the minimum value is 24.

put $x = -3$ in eqn (iii) we get, $\frac{d^2p}{dx^2} = -8/3 < 0$

$\therefore$ for $x = -3$, the function has a maximum value and the maximum value is 0.

CS CamScanner

7) Given, $f(x,y) = x^3+3x^2+4xy+y^2$ —— (I)

Partial:

Differentiating eqn (I) w.r.t $x$,

$$f_x = 3x^2+6x+4y \quad \text{—— (II)}$$

Again, differentiating eqn (II) w.r.t $x$,

$$f_{xx} = 6x+6 \quad \text{—— (III)}$$

Now, differentiating eqn (I) w.r.t $y$,

$$f_y = 2y+4x \quad \text{—— (IV)}$$

Again differentiating eqn (IV) w.r.t $y$,

$$f_{yy} = 2 \quad \text{—— (V)}$$

Now, differentiating eqn (I) w.r.t $xy$,

$$f_{xy} = 4 \quad \text{—— (VI)}$$

For maxima or minima,

$f_x = 0$ and $f_y = 0$

$\therefore 3x^2+6x+4y = 0$ —— (VIII)

$\therefore 4x+2y = 0$ —— (VII)

$\Rightarrow y = -2x$

Now, from (VIII), we get,

$$-8x+6x+3x^2=0$$

$$\Rightarrow x(3x-2)=0$$

$$\Rightarrow x = 0, \tfrac{2}{3}$$

From (VII), when $x=0$, $y=0$

when $x=\tfrac{2}{3}$, $y=-\tfrac{4}{3}$

Now, at $\left(\tfrac{2}{3}, -\tfrac{4}{3}\right)$, $f_{xx} = 10$

$f_{yy} = 2$

$f_{xy} = 4$

So, $f_{xx} \times f_{yy} - (f_{xy})^2 = 10\times 2 - 4^2 = 4 > 0$

Here, $f_{xx} = 10 > 0$

So, the function attains a minimum at the point $\left(\tfrac{2}{3}, -\tfrac{4}{3}\right)$.

(Proved)

CamScanner

8) Given,

$$f(x,y) = 2x^2 - xy + 2y^2 - 20x$$

$$\therefore f_x = 4x - y - 20$$

$$f_y = 4y - x$$

$$f_{xx} = 4$$

$$f_{yy} = 4$$

$$f_{xy} = -1$$

For, extremum of $f(x,y)$, $f_x = 0$ and $f_y = 0$

$\therefore 4x - y - 20 = 0$ and, $4y - x = 0$ — (ii)

$\Rightarrow 4x - y = 20$ — (i)

From (i) and (ii) we get,

$$x = \frac{16}{3}, y = 4/3$$

Now, at $(16/3, 4/3)$,

$$f_{xx} \times f_{yy} - (f_{xy})^2 = 16 - 1 = 15 > 0$$

and $f_{xx} = 4 > 0$

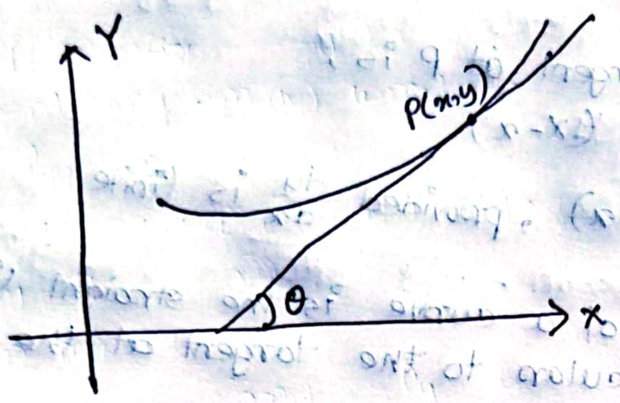
$\therefore f(x,y)$ has a minimum value at $(16/3, 4/3)$ and the minimum value is $:= 2\times\frac{256}{9} - \frac{64}{9} + \frac{32}{9} - 20\times 16/3 = -\frac{160}{3}$

CS CamScanner

# Differential Calculus

## Tangent & Normal

The tangent at P to a given curve is defined as the limiting position of the secant PQ as the point Q approaches P along the curve.




Here, $\frac{dy}{dx} = \tan\theta$

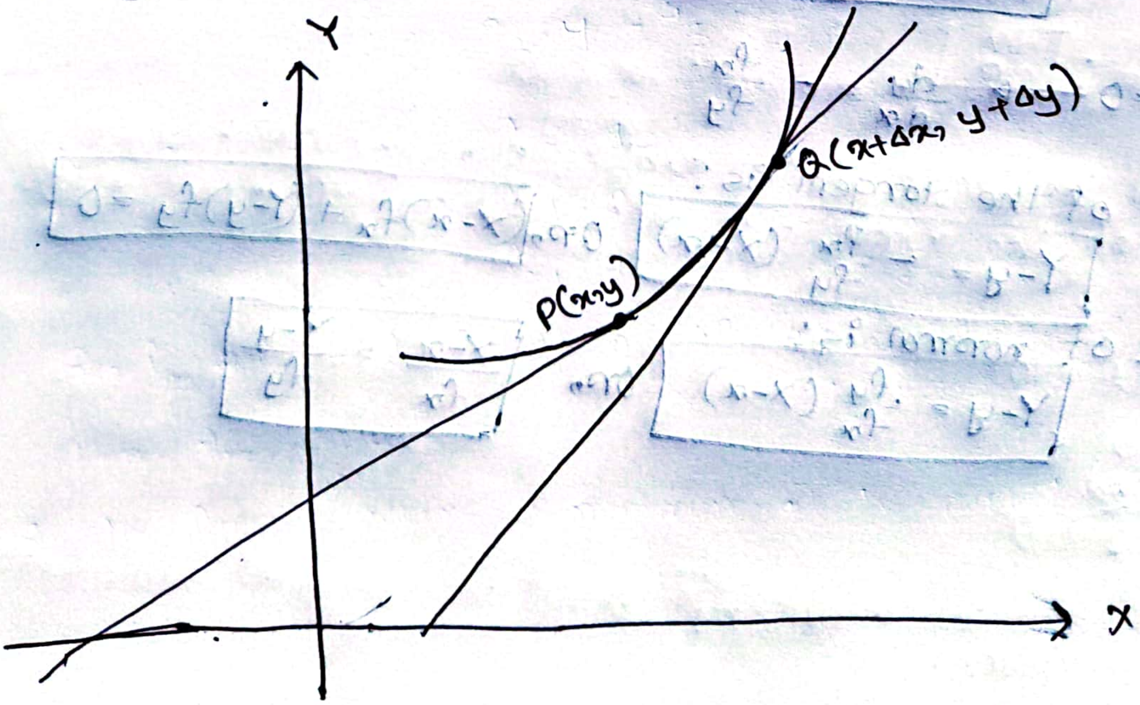
if $\theta = 0$, then, $\frac{dy}{dx} = \tan\theta = 0$

$\boxed{y = f(x)\ ,\ f(x,y) = 0}$

* find the point where tangent is parallel or perpendicular to x-axis.

$\frac{dy}{dx} = \infty$ when, $\theta = \pi/2$

$\Rightarrow \frac{dx}{dy} = 0$ perpendicular



Let, P be a point ~~on~~ $(x,y)$ on the curve, and Q a neighbouring point $(x+\Delta x, y+\Delta y)$ on the curve. The equation of the secant PQ is. ($X, Y$ denoting current co-ordinates)

$$\frac{X-x}{x-x-\Delta x} = \frac{Y-y}{y-y-\Delta y}$$

$$\Rightarrow \frac{X-x}{-\Delta x} = \frac{Y-y}{-\Delta y}$$

$$\Rightarrow Y-y = \frac{\Delta y}{\Delta x}(X-x)$$

$\therefore$ The equation of the tangent at P is,

$$Y-y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}(X-x)$$

$$\Rightarrow Y-y = \frac{dy}{dx}(X-x)$$, provided $\frac{dy}{dx}$ is finite

# The normal at any point of a curve is the straight line through that point drawn perpendicular to the tangent at the point.

For $y=f(x) \rightarrow$

The equation of tangent is:

$$\boxed{Y-y = \frac{dy}{dx}(X-x)}$$

The equation of normal is:

$$\boxed{Y-y = -\frac{dx}{dy}(X-x)}$$

For $f(x,y)=0 \longrightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$

Equation of the tangent is:

$$\boxed{Y-y = -\frac{f_x}{f_y}(X-x)}$$ or, $$\boxed{(X-x)f_x + (Y-y)f_y = 0}$$

Equation of normal is:

$$\boxed{Y-y = \frac{f_y}{f_x}(X-x)}$$ or, $$\boxed{\frac{X-x}{f_x} = \frac{Y-y}{f_y}}$$

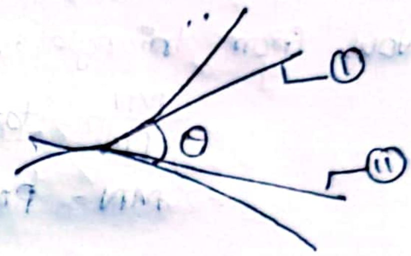
CamScanner

## Angle between two curves:

The angle between two curves is the angle between the tangents to the two curves at the common point of intersection.

Suppose, $f(x,y)=0$ and $\phi(x,y)=0$ be two curves which intersects at point $(x,y)$. Then the tangents of the two curves are:

$$(X-x)f_x+(Y-y)f_y=0 \text{ ——— (i)}$$
$$(X-x)\phi_x+(Y-y)\phi_y=0 \text{ ——— (ii)}$$



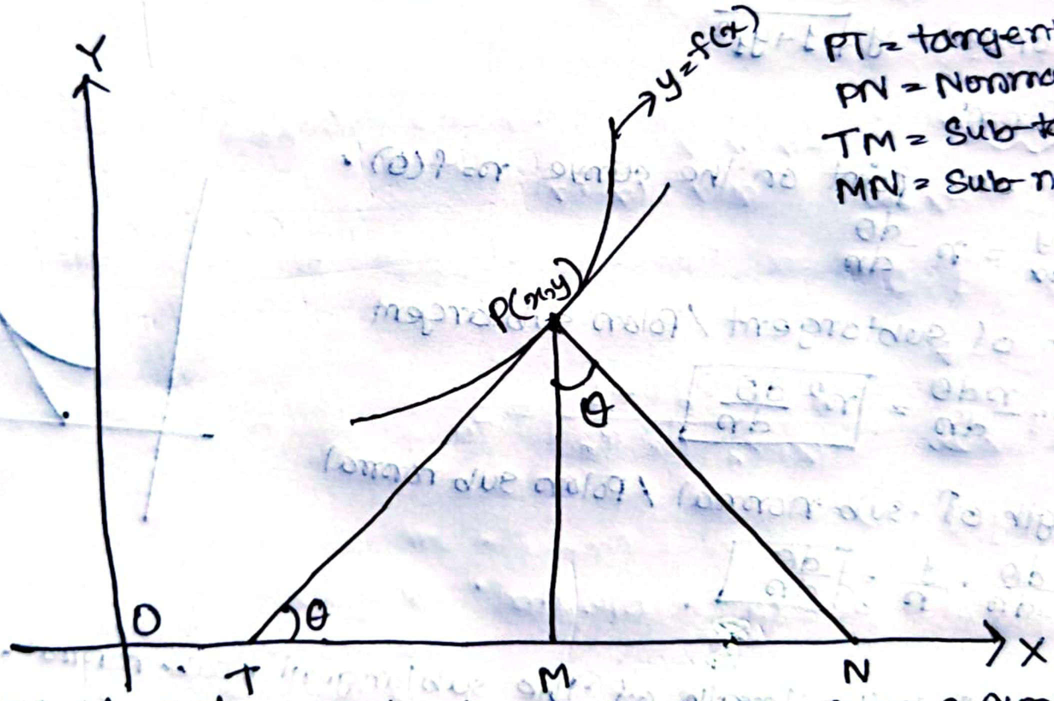

If $\theta$ be the angle between the tangents, then,

$$\tan\theta = \frac{f_x\phi_y \sim f_y\phi_x}{f_x\phi_x+f_y\phi_y}$$

(i) If $\theta=0$, then $f_x\phi_y=f_y\phi_x$

(ii) If $\theta=\pi/2$, then $f_x\phi_x+f_y\phi_y=0$

## (i) Sub-tangent and Sub-normal:




PT = tangent
PN = Normal
TM = Sub-tangent
MN = Sub-normal

Let the tangent and normal at any point $P(x,y)$ on a curve meet the x-axis is T and N respectively and let PM be drawn perpendicular to OX.

Then, TM is called the subtangent and MN is called the sub-normal at $P(x,y)$.

In the right-angled triangle $\Delta PTM$,

$$\frac{TM}{PM}=\cot\theta$$

$$\Rightarrow TM = PM\cot\theta = y\cot\theta = y\frac{dx}{dy} \quad \left[\because \frac{dy}{dx}=\tan\theta\right]$$

$$\therefore \text{Sub-tangent} = \frac{y}{y_1}$$

Again, $\frac{PT}{PM} = \operatorname{cosec}\theta$

$\Rightarrow PT = PM \operatorname{cosec}\theta = y\sqrt{1+\cot^2\theta} = y\sqrt{1+\left(\frac{dx}{dy}\right)^2} = y\sqrt{1+y_1^2}$

$\therefore$ tangent $= \frac{y\sqrt{1+y_1^2}}{y_1}$

Now, from triangle, $\Delta PNM$,

$\frac{MN}{PM} = \tan\theta$

$\Rightarrow MN = PM\tan\theta = y\tan\theta = y\cdot\frac{dy}{dx} = y\cdot y_1$

$\therefore$ Sub-normal $= y\cdot y_1$

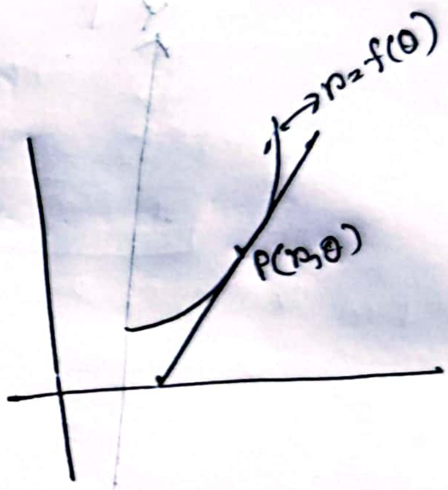
Again, $\frac{PN}{PM} = \sec\theta$

$\Rightarrow PN = PM\sec\theta = y\sec\theta = y\sqrt{1+\tan^2\theta} = y\sqrt{1+y_1^2}$

$\therefore$ Normal $= y\sqrt{1+y_1^2}$

## Polar form:

If $P$ be any point on the curve $r = f(\theta)$, then, $\frac{dy}{dx} = r\frac{d\theta}{dr}$




$\therefore$ Length of subtangent / Polar subtangent

$= r\cdot\frac{r\,d\theta}{dr} = \boxed{r^2\frac{d\theta}{dr}}$

And, Length of sub-normal / Polar sub-normal

$= r\frac{d\theta}{dr}\cdot\frac{1}{r} = \boxed{\frac{d\theta}{dr}}$

# Problem: Find the length of the subtangent, sub-normal, tangent and normal of the curves, $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$ at $\theta$.

# Problem: (i) $r = e^{\theta}$

(ii) $r = a(1-\cos\theta)$

(iii) $r = ae$

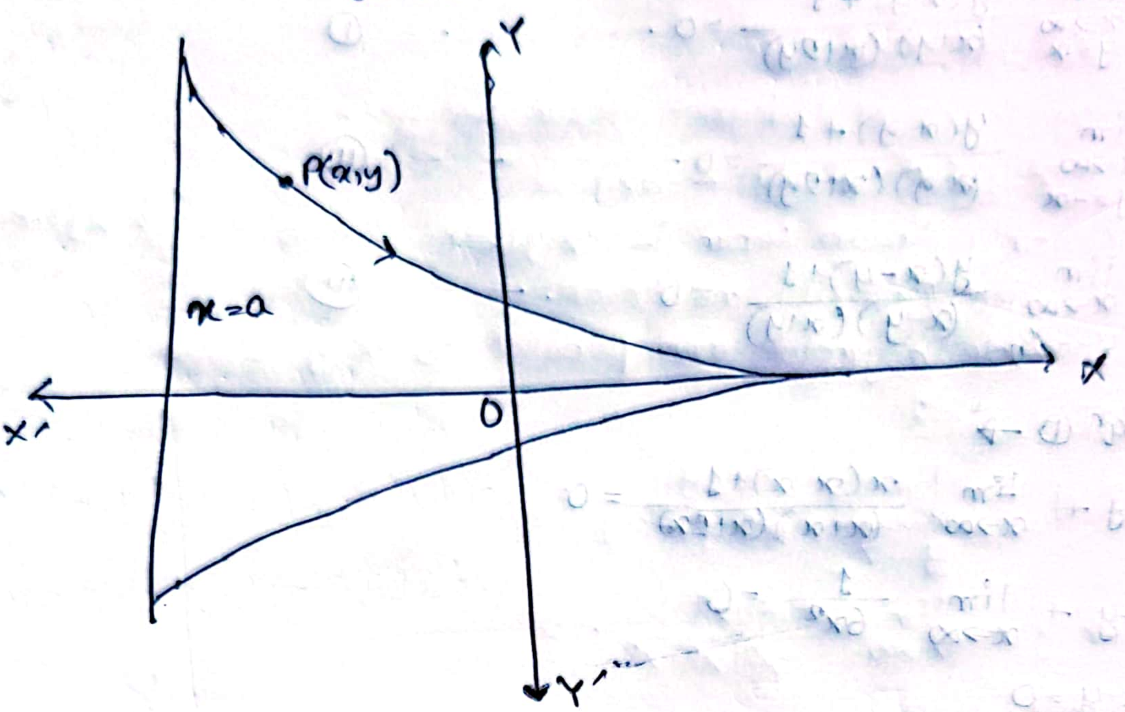
(iv) $r = \frac{2a}{1-\cos\theta}$

# Problem: Show that for the curve $by^2 = (x+a)^3$ the square of subtangent varies as the subnormal.

CS CamScanner

MATH 1107

## Asymptotes




If the curve P is infinite then the distances ~~th~~ of the line tends to zero(0).
When extending finite region, a straight line exists from the origin and the distance of P(x,y) gradually deminishes and the distance will be almost zero that curve is called asymptotes.

$$f(x) = \frac{x+a}{x-a}$$

If $x = a$, then, $f(x) = 0$

$\therefore$ $x = a$ is the asymptotes of the curve.

**#Problem-1:** Find the asymptotes of the curve:

$$x^3 - 2y^3 + xy(2x-y) + y(x-y) + 1 = 0$$

**Solution:**

Highest power indicates how many asymptotes can be possible. We will factorize the highest power term. Then how many linear factor (linear factor means power 1) we will get, the asymptotes will be those much. Now,

We have,

$$x^3 - 2y^3 + xy(2x-y) + y(x-y) + 1 = 0$$

Factorizing the highest degree term, we get,

$$(x-y)(x+y)(x+2y) + y(x-y) + 1 = 0$$

Here, the highest power is 3, so we will find three asymptotes.

CamScanner

The possible asymptotes are parallel to $x+y=0$; $x-y=0$; $x+2y=0$ and the asymptotes are respectively:

$$x-y+\lim_{\substack{x\to\infty \\ y=x}} \frac{y(x-y)+1}{(x+y)(x+2y)} = 0 \quad \text{--- (i)}$$

$$x+y+\lim_{\substack{x\to\infty \\ y=-x}} \frac{y(x-y)+1}{(x-y)(x+2y)} = 0 \quad \text{--- (ii)}$$

$$x+2y+\lim_{\substack{x\to\infty \\ y=-x/2}} \frac{y(x-y)+1}{(x-y)(x+y)} = 0 \quad \text{--- (iii)}$$

From eq$^n$ (i) →

$$x-y+\lim_{x\to\infty} \frac{x(x-x)+1}{(x+x)(x+2x)} = 0$$

$$\Rightarrow x-y+\lim_{x\to\infty} \frac{1}{6x^2} = 0$$

$$\Rightarrow x-y=0$$

From eq$^n$ (ii) →

$$x+y+\lim_{x\to\infty} \frac{-x(x+x)+1}{(x+x)(x-2x)} = 0$$

$$\Rightarrow x+y+1=0$$

From eq$^n$ (iii) →

$$x+2y+\lim_{x\to\infty} \frac{-x/2\,(x+x/2)+1}{(x+x/2)(x-x/2)} = 1$$

$$\Rightarrow x+2y-1=0$$

∴ The asymptotes of that curve are:

$$\left.\begin{array}{l} x-y=0 \\ x+y+1=0 \\ x+2y-1=0 \end{array}\right\} \text{(Ans)}$$

# Problem-2: Find the asymptotes of the curve,
$$2x(y-5)^2 = 3(y-2)(x-1)^2$$

Solution: we have,

$$2x(y-5)^2 = 3(y-2)(x-1)^2$$

Factorizing the highest degree term, we get,

$$xy(2y-3x) + 2x(3x-7y) + 38x - 3y + 6 = 0$$

In this curve, the highest power is 3, so the possible three asymptotes are parallel to $x=0$; $y=0$; $2y-3x=0$.

The asymptote parallel to $x=0$ that is to the Y-axis, is (equating the to zero the co-efficient of highest degree terms of y) $2x=0$ that is $x=0$, the Y-axis itself.

The asymptote parallel to $y=0$ that is to the X-axis, is (in a similar manner),

$$-3y+6=0$$
$$\Rightarrow y=2$$

The assymptote parallel to $2y-3x=0$ is,

$$2y-3x + \lim_{\substack{x\to\infty \\ y=\frac{3}{2}x}} \frac{2x(3x-7y)+38x-3y+6}{xy} = 0$$

$$\Rightarrow 2y-3x + \lim_{x\to\infty} \frac{2x(3x-\frac{21}{2}x)+38x-\frac{9}{2}x+6}{\frac{3}{2}x^2} = 0$$

$$\Rightarrow 2y-3x + \lim_{x\to\infty} \frac{2x^2\left(\frac{6-21}{2}\right)}{\frac{3}{2}x^2} + \lim_{x\to\infty} \frac{38x-\frac{9}{2}x+6}{\frac{3}{2}x^2} = 0$$

$$\Rightarrow 2y-3x + \left(-\frac{15}{1}\times\frac{2}{3}\right) = 0$$

$$\Rightarrow 2y-3x-10=0$$

$\therefore$ The asymptotes of the curves are:
$$\left.\begin{array}{l} x=0 \\ y=2 \\ 2y-3x-10=0 \end{array}\right\} \text{(Ans)}$$

CamScanner

# Problem-3: Find the asymptotes of the curve : $x^3+y^3=3axy$

**Solution:**

We have,

$$x^3+y^3=3axy$$

factorizing the highest degree term we get,

$$(x+y)(x^2+y^2-xy)-3axy=0$$

Here the only linear term is $(x+y)$. The factor $(x^2+y^2-xy)$ has 2 as its highest power. So, it is not linear.

So, the possible asymptote is parallel to $x+y=0$

The asymptote is :

$$x+y-\lim_{\substack{x\to\infty\\ y=-x}}\frac{3axy}{x^2+y^2-xy}=0$$

$$\Rightarrow x+y-\lim_{x\to\infty}\frac{-3ax^2}{x^2+x^2+x^2}=0$$

$$\Rightarrow x+y+a=0$$

$\therefore$ The asymptote of the curve is $x+y+a=0$ (Ans)

**Note:**

If one factor exists several times, the asymptote's will be several times.

**Example:**

$$(x+y)^2(x+2y)+y(x-y)+1=0$$

Here asymptote will be 3 because $(x+y)$ has been repeated two times, factorizing, we get,

$$(x+y)^2(x+2y)+(x+y)(x-y)+bx=0$$

* $n$ numbers of asymptotes means $n$ number of straight line.

In previous example, the asymptote for $(x+y)^2=0$ is:

$$(x+y)^2+(x+y)\lim_{\substack{x\to\infty\\ y=-x}}\frac{x-y}{x+2y}+\lim_{\substack{x\to\infty\\ y=-x}}\frac{6x}{x+2y}=0$$