* If the Question is like $\int \frac{dx}{\sqrt{(x-\alpha)\,(x-\beta)}}$ then

we will consider ~~x-α~~ either $x-\alpha = z^2$ or $x-\beta = z^2$.

Example:

$$\int \frac{dx}{\sqrt{(x-4)\,(x-7)}}$$

Suppose,

$x-4 = z^2$

or, $dx = 2z\,dz$

And, $x-7 = x-4-3$

$= z^2-3$

$$= \int \frac{2z\,dz}{\sqrt{z^2(z^2-3)}}$$

$$= \int \frac{2z\,dz}{z\sqrt{z^2-3}}$$

$$= 2\int \frac{dz}{\sqrt{z^2-3}}$$

$$= 2\ln\left(z+\sqrt{z^2-3}\right) + c$$ [(1x) নং formula]

$$= 2\ln\left(\sqrt{x-4}+\sqrt{x-4-3}\right) + c$$

$$= 2\ln\left(\sqrt{x-4}+\sqrt{x-7}\right) + c$$

* If the question is like $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$ then we will consider $cx+d = z^2$.

<u>Example:</u>

$$\int \frac{dx}{(x+2)\sqrt{x+3}}$$

Suppose,

$x+3 = z^2$

$\Rightarrow dx = 2z\,dz$

$x+2 = x+3-1$

$= z^2-1$

$$= \int \frac{2z\,dz}{(z^2-1)\sqrt{z^2}}$$

$$= 2\int \frac{dz}{z^2-1}$$

$$= 2 \times \frac{1}{2} \ln\left|\frac{z-1}{z+1}\right| + c$$

$$= \ln\left|\frac{\sqrt{x+3}-1}{\sqrt{x+3}+1}\right| + c$$

<u>Ans</u>

* If the question is like $\displaystyle\int \frac{dx}{(ax^2+bx+c)\sqrt{gx+h}}$ then

we will consider $gx+h = z^2$.

Example: $\displaystyle\int \frac{dx}{(x^2-16)\sqrt{x+1}}$

Suppose,

$x+1 = z^2$

$\Rightarrow dx = 2z\,dz$

$x^2-16 = (x-4)(x+4)$

$= (x+1-5)(x+1+3)$

$= (z^2-5)(z^2+3)$

$$= \int \frac{2z\,dz}{(z^2-5)(z^2+3)\,z}$$

$$= 2\int \frac{dz}{z^4-2z^2-15}$$

$$= 2\int \frac{dz}{(z^2-1)^2-16}$$

$$= 2 \times \frac{1}{8} \ln\left|\frac{z^2-1-4}{z^2-1+4}\right| + c$$

$$= \frac{1}{4}\ln\left|\frac{x+1-1-4}{x+1-1+4}\right| + c$$

$$= \frac{1}{4}\ln \cancel{x+1}\left|\frac{x-4}{x+4}\right| + c$$

Ans

* If the question is like $\int \frac{Px+q}{\sqrt{ax^2+bx+c}}\,dx$ then

we will consider $ax^2+bx+c = z$.

Example: $\int \frac{2x+7}{\sqrt{x^2-2x+8}}\,dx$

Suppose,

$x^2-2x+8 = z$

$\Rightarrow (2x-2)\,dx = dz$

~~$2x+7 = 2x$~~

$= \int \frac{2x-2+9}{\sqrt{x^2-2x+8}}\,dx$

$= \int \frac{2x-2}{\sqrt{x^2-2x+8}}\,dx + \int \frac{9}{\sqrt{x^2-2x+8}}\,dx$

$= \int \frac{dz}{\sqrt{z}} + \int \frac{9}{\sqrt{(x-1)^2+7}}\,dx$

~~$= 2\sqrt{z} + 9\ln(x+1)+\sqrt{x=}$~~

$= 2\sqrt{z} + 9\ln\left\{(x-1)+\sqrt{(x-1)^2+7}\right\} + c$

$= 2\sqrt{x^2-2x+8} + 9\ln\left(x-1+\sqrt{x^2-2x+8}\right) + c$

Ans

* If the question is like $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$ then first we will consider $x = \frac{1}{u}$ then solving that equation we will get another one. In that equation we will consider $p+qu^2 = z^2$.

Example:

$$\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$$

Suppose,

$$x = \frac{1}{u}$$

$$\Rightarrow dx = -\frac{1}{u^2}\,du$$

$$= \int \frac{-\frac{1}{u^2}\,du}{\left(1+\frac{1}{u^2}\right)\sqrt{1-\frac{1}{u^2}}}$$

$$= \int \frac{-\frac{1}{u^2}\,du}{\frac{1}{u^2}(u^2+1)\times u\sqrt{u^2-1}}$$

$$= \int \frac{-u\,du}{(u^2+1)\sqrt{u^2-1}}$$

Suppose,

$$u^2-1 = z^2$$

$$\Rightarrow 2u\,du = 2z\,dz$$

$$\cancel{\Rightarrow u\,du = \frac{dz}{2}}$$

$$\Rightarrow u\,du = z\,dz$$

$$\cancel{= \int \frac{-\frac{dz}{2}}{(z+2)\sqrt{z}}}$$

$$= \int \frac{-z\,dz}{(z^2+2)\times z}$$

$$= -1 \int \frac{dz}{z^2+2}$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \frac{z}{\sqrt{2}} + c$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{v^2-1}}{\sqrt{2}} + c$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{\frac{1}{x^2}-1}}{\sqrt{2}} + c$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{1-x^2}}{x\sqrt{2}} + c \quad \underline{\text{Ans}}$$

* If the question is like $\int \frac{dx}{(px+q)\sqrt[n]{ax^2+bx+c}}$ then we will consider $px+q = \frac{1}{z}$.

<u>Example:</u>

$$\int \frac{dx}{(x-1)\sqrt{x^2+1}}$$

Suppose,

$$x-1 = \frac{1}{z}$$

$$\Rightarrow dx = -\frac{1}{z^2}\,dz$$

$$x = \frac{1}{z}+1 = \frac{1+z}{z}$$

$$= \int \frac{-\frac{1}{z^2}\,dz}{\frac{1}{z}\sqrt{\frac{1}{z^2}+\frac{2+2z}{z}}}$$

$$= \int \frac{-\frac{1}{z^2}\,dz}{\frac{1}{z^2}\sqrt{1+2z+2z^2}}$$

$$= -1\int \frac{dz}{\sqrt{1+2z+2z^2}}$$

$$= -1\int \frac{dz}{\sqrt{1+2(z^2+z)}}$$

$$= -1\int \frac{dz}{\sqrt{1+2\left(z+\frac{1}{2}\right)^2-\frac{1}{2}}}$$

$$= -1\int \frac{dz}{\sqrt{2\left\{\left(z+\frac{1}{2}\right)^2-\frac{1}{4}\right\}}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\left(z+\frac{1}{2}\right)^2-\frac{1}{4}}$$

$$= -\frac{1}{\sqrt{2}} \times \frac{1}{2\times\frac{1}{2}} \ln\left|\frac{z+\frac{1}{2}-\frac{1}{2}}{z+\frac{1}{2}+\frac{1}{2}}\right| + c$$

$$= -\frac{1}{\sqrt{2}} \ln\left|\frac{\frac{1}{x-1}}{\frac{1}{x-1}+\cancel{x}1}\right| + c$$

$$= -\frac{1}{\sqrt{2}} \ln\left|\frac{1}{x}\right| + c \quad \text{Ans.}$$

Exercise: (i) $\displaystyle\int \frac{x(\tan^{-1}x)^2}{(1+x^2)\sqrt{1+x^2}}\,dx$

Suppose, $\tan^{-1}x = z$

$\Rightarrow \dfrac{1}{1+x^2}\,dx = dz$

$$\Rightarrow \int \frac{\tan z \cdot z^2\,dz}{\sqrt{1+\tan^2 z}}$$

$$\Rightarrow \int \sin z \cdot z^2\,dz$$

$$\cancel{\Rightarrow z}\int$$

$$\Rightarrow z^2\int \sin z\,dz - \int\left\{\frac{d}{dx}z^2\int \sin z\,dz\right\}dz$$

$$\Rightarrow z^2(-\cos z) - \int\{2z(-\cos z)\}\,dz$$

$$\Rightarrow -z^2\cos z + \int 2z\cos z\,dz$$

$$\Rightarrow -z^2\cos z + 2z\int \cos z\,dz - \int\left\{\frac{d}{dx}z\int \cos z\,dz\right\}dz$$

$$\Rightarrow -z^2\cos z + 2z\sin z - \int \sin z\,dz$$

$$\Rightarrow 2z\sin z - z^2\cos z + \cos z + c$$

$$\Rightarrow 2\tan^{-1}x \sin(\tan^{-1}x) - (\tan^{-1}x)^2 \cos(\tan^{-1}x) + \cos(\tan^{-1}x) + C$$

Ans

(11) $\int \frac{x^2 - x + 1}{x^2 + x + 1} dx$

$$\Rightarrow \int \frac{x^2 + x + 1 - 2x}{x^2 + x + 1} dx$$

$$\Rightarrow \int 1 - \frac{2x}{x^2 + x + 1} dx$$

$$\Rightarrow \int 1 \, dx - \int \frac{2x}{x^2 + x + 1} dx$$

Suppose,

$x^2 + x + 1 = \cancel{dx}\, z$

$(2x + 1) dx = dz$

$$\Rightarrow x - \int \frac{2x+1}{x^2 + x + 1} dx - \int \frac{1}{x^2 + x + 1} dx$$

$$\Rightarrow x - \int \frac{1}{z} dz - \int \frac{1}{x^2 + x + \frac{1}{4} + 1 - \frac{1}{4}} dx$$

$$\Rightarrow x - \ln z - \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\, dx$$

$$\Rightarrow x - \ln z - \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$

$$\Rightarrow x - \ln(x^2+x+1) - \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c$$

(iii) $\int \sqrt{\tan x} + \sqrt{\cot x}\, dx$

$$= \int \frac{\tan x + 1}{\sqrt{\tan x}}$$

Suppose,

$\tan x = z^2$

$\Rightarrow \sec^2 x\, dx = 2z\, dz$

$\Rightarrow dx = \frac{2z}{\sec^2 x} dz$

$= \frac{2z}{1+\tan^2 x} dz$

$= \frac{2z}{1+z^4}$

$$= \int \frac{z^2+1}{z} \times \frac{2z}{1+z^4}\, dz$$

$$= 2\int \frac{z^2+1}{z^4+1}\, dz$$

$$= 2\int \frac{1+z^2}{z^2 + \frac{1}{z^2}}\, dz$$

$$= 2\int \frac{1+\frac{1}{z^2}}{\left(z-\frac{1}{z}\right)^2+2}\,dz$$

Suppose,

$z - \frac{1}{z} = u$

$\left(1+\frac{1}{z^2}\right)dz = du$

$$= 2\int \frac{du}{u^2+2}$$

$$= 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{2} + C$$

$$= \sqrt{2}\tan^{-1}\frac{z-\frac{1}{z}}{\sqrt{2}} + C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2}\sqrt{\tan x}}\right) + C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + C$$

(iv) $$\int \frac{x^3-2x^2+x-7}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int \frac{x(x^2-2x+1)-7}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int \frac{x(x^2-2x+3)-2x-2-5}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int x\sqrt{x^2+2x+3} - \frac{2x+2}{\sqrt{x^2+2x+3}} - \frac{5}{\sqrt{x^2+2x+3}}\,dx$$

$$= \frac{1}{2}\int (2x+2-2)\sqrt{x^2+2x+3}\,dx - 2\sqrt{x^2+2x+3}$$
$$- 5\int \frac{dx}{\sqrt{(x+1)^2+(\sqrt{2})^2}} + c$$

$$= \frac{1}{2}\int (2x+2)\sqrt{x^2+2x+3}\,dx - 2\int\sqrt{x^2+2x+3}\,dx - 2\sqrt{x^2+2x+3}$$
$$- 5\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}\right) + c$$

$$= 3\sqrt{(x^2+2x+3)^3} - 2\int\sqrt{(x+1)^2+(\sqrt{2})^2}\,dx - 2\sqrt{x^2+2x+3}$$
$$- \frac{5}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + c$$

$$= 3\sqrt{(x^2+2x+3)^3} - 2x\frac{(x+1)\sqrt{(x+1)^2+(\sqrt{2})^2}}{2} + \cancel{\ln x + \sqrt{}}$$
$$2\ln\left\{(x+1)+\sqrt{(x+1)^2+(\sqrt{2})^2}\right\} \cancel{+2\sqrt{2}\sqrt{}} - 2\sqrt{x^2+2x+3}$$
$$- \frac{5}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + c$$

Ans

(v) $\displaystyle\int \frac{dx}{\sin x + \cos x}$

$$= \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2} - \tan^2\frac{x}{2} + 1}\, dx$$

Suppose,

$\tan\frac{x}{2} = z$

$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\, dx = dz$

$\Rightarrow \sec^2\frac{x}{2}\, dx = 2dz$

$$= 2\int \frac{dz}{2z + 1 - z^2}$$

$$= 2\int \frac{dz}{1 - (z^2 - 2z)}$$

$$= 2\int \frac{dz}{1 - (z^2 - 2z + 1) + 1}$$

$$= 2\int \frac{dz}{2 - (z-1)^2}$$

$$= 2 \times \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2} + z - 1}{\sqrt{2} - z + 1} \right| + c$$

$$= \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2} + \tan\frac{x}{2} - 1}{\sqrt{2} - \tan\frac{x}{2} + 1} \right| + c$$

(vi) $$\int \frac{\sin x}{\sqrt{a^2 \cos^2 x + b^2 \sin x}} \, dx$$

Suppose,

$\cos x = z$

$\Rightarrow -\sin x \, dx = dz$

$$= \int \frac{\sin x}{\sqrt{a^2 \cos^2 x + b^2 - b^2 \cos^2 x}} \, dx$$

$$= \int \frac{\sin x}{\sqrt{(a^2 - b^2)\cos^2 x + b^2}} \, dx$$

$$= -1 \int \frac{dz}{\sqrt{(a^2 - b^2) z^2 + b^2}}$$

$$= -1 \int \frac{dz}{\sqrt{z^2 + \frac{b^2}{a^2 - b^2}}}$$

$$= -1 \ln \left| z + \sqrt{z^2 + \frac{b^2}{a^2-b^2}} \right| + c$$

$$= -\ln \left(\cos x + \sqrt{\cos^2 x + \frac{b^2}{a^2-b^2}} \right) + c$$

(vii) $$\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x} \, dx$$

$$2\sin x + 3\cos x = l(3\sin x + 4\cos x) + m(3\cos x - 4\sin x)$$

$$= \cancel{3l + 4m} \; (3l - 4m)\sin x + (4l + 3m)\cos x$$

equating the coefficient of $\sin x$ and $\cos x$;

$3l - 4m = 2 \qquad \therefore l = \frac{18}{25}$

$4l + 3m = 3 \qquad \therefore m = \frac{1}{25}$

Now,
$$\int \frac{\frac{18}{25}(3\sin x + 4\cos x) + \frac{1}{25}(3\cos x - 4\sin x)}{3\sin x + 4\cos x} \, dx$$

$$= \int \frac{18}{25} + \frac{1}{25} \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x} dx$$

suppose,

$3\sin x + 4\cos x = z$

$\Rightarrow (3\cos x - 4\sin x) dx = dz$

$$= \frac{18}{25} x + \int \frac{1}{25} \frac{1}{z} dz$$

$$= \frac{18}{25} x + \frac{1}{25} \ln(3\sin x + 4\cos x) + c$$

(viii) $$\int_0^\pi \frac{dx}{1 - 2a\cos x + a^2}$$

$$= \int_0^\pi \frac{dx}{(1+a^2) - 2a\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$= \int_0^\pi \frac{1+\tan^2\frac{x}{2}}{(1+a^2)(1+\tan^2\frac{x}{2}) - 2a(1-\tan^2\frac{x}{2})} dx$$

$$= \int_0^\pi \frac{\sec^2\frac{x}{2}}{(1+a^2) + (1+a^2)\tan^2\frac{x}{2} - 2a + 2a\tan^2\frac{x}{2}} dx$$

$$= \int_0^{\pi} \frac{\sec^2 \frac{x}{2}}{(1-a)^2 + (1+a)^2 \tan^2 \frac{x}{2}} \, dx$$

Suppose,

$\tan \frac{x}{2} = z$

$\Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} \, dx = dz$

$\Rightarrow \sec^2 \frac{x}{2} \, dx = 2 \, dz$

$$= 2 \int_0^{\infty} \frac{dz}{\left(\frac{1-a}{1+a}\right)^2 + z^2}$$

$$= \frac{2}{(1+a)^2} \times \frac{1+a}{(1-a)} \left[\tan^{-1} \frac{z(1+a)}{1-a} \right]_0^{\infty}$$

$$= \frac{2}{1-a^2} \frac{\pi}{2}$$

$$= \frac{\pi}{1-a^2} \quad \underline{\text{Ans}}$$

(ix) $$\int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

Suppose,

$\tan x = z$

$\Rightarrow \sec^2 x \, dx = dz$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} \, dx$$

$$= \frac{1}{b^2} \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\left(\frac{a}{b}\right)^2 + \tan^2 x} \, dx$$

$$= \frac{1}{b^2} \int_0^{\infty} \frac{dz}{\left(\frac{a}{b}\right)^2 + z^2}$$

$$= \frac{1}{b^2} \times \frac{b}{a} \left[\tan^{-1} \frac{za}{b}\right]_0^{\infty}$$

$$= \frac{\pi}{2ab} \quad \underline{\text{Ans}}$$

(x) $$\int_0^{\frac{\pi}{2}} \frac{dx}{4 + 5 \sin x}$$

Suppose,

$\tan \frac{x}{2} = z$

$\Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} \, dx = dz$

$\Rightarrow \sec^2 \frac{x}{2} \, dx = 2 dz$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{4 + 5\left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 \frac{x}{2}}{4 + 4 \tan^2 \frac{x}{2} + 10 \tan \frac{x}{2}} \, dx$$

$$= 2 \int_0^{1} \frac{dz}{4 + 4z^2 + 10z}$$

$$= 2\int_0^1 \frac{dz}{(2z)^2 + 2\cdot\frac{5}{2}\cdot 2z + \frac{25}{4} - \frac{25}{4} + 4}$$

$$= 2\int_0^1 \frac{dz}{\left(2z + \frac{5}{2}\right)^2 - \frac{9}{4}}$$

$$= \frac{1}{8}\int_0^1 \frac{dz}{\left(z + \frac{5}{2}\right)^2 - \frac{9}{16}}$$

$$= \frac{1}{8} \times \frac{1}{2 \times \frac{3}{4}} \left[\ln \frac{z + \frac{5}{2} - \frac{3}{4}}{z + \frac{5}{2} + \frac{3}{4}}\right]_0^1$$

$$= \frac{1}{12}\left(\ln \frac{1 + \frac{5}{2} - \frac{3}{4}}{1 + \frac{5}{2} + \frac{3}{4}} - \ln \frac{\frac{5}{2} - \frac{3}{4}}{\frac{5}{2} + \frac{3}{4}}\right)$$

$$= \frac{1}{12}\left(\ln \frac{11}{17} - \ln \frac{7}{13}\right)$$

$$= \frac{1}{12} \ln\left(\frac{\frac{11}{17}}{\frac{7}{13}}\right)$$

$$= \frac{1}{12} \ln \frac{143}{119}$$

Ans

(xi) $$\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^4 x \, dx}{(a^2+b^2\tan^2 x)^2}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x(1+\tan^2 x) \, dx}{(a^2+b^2\tan^2 x)^2}$$

suppose,

$b\tan x = a\tan\theta$

$\Rightarrow b\sec^2 x \cdot dx$

$= a\sec^2\theta \, d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{\cancel{\sec^2 x}\left(1+\frac{a^2}{b^2}\tan^2\theta\right)\frac{a}{b}\sec^2\theta}{(a^2+a^2\tan^2\theta)^2}$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} (b^2+a^2\tan^2\theta)\cos^2\theta \, d\theta$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} (b^2\cos^2\theta + a^2\sin^2\theta) \, d\theta$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}} (b^2 - b^2\sin^2\theta + a^2\sin^2\theta) \, d\theta$$

$$= \frac{1}{a^3b^3} \int_0^{\frac{\pi}{2}} \left\{ b^2 + \sin^2\theta \, (a^2 - b^2) \right\} d\theta$$

$$= \frac{a^2-b^2}{a^3b^3} \int_0^{\frac{\pi}{2}} \left(\frac{b^2}{a^2-b^2} + \frac{1}{2} \, 2\sin^2\theta \right) d\theta$$

$$= \frac{a^2-b^2}{a^3b^3} \int_0^{\frac{\pi}{2}} \left(\frac{b^2}{a^2-b^2} + \frac{1}{2} - \frac{1}{2}\cos 2\theta \right) d\theta$$

$$= \frac{a^2-b^2}{a^3b^3} \left[\frac{b^2}{a^2-b^2}\theta + \frac{\theta}{2} - \frac{1}{4}\sin 2\theta \right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2-b^2}{a^3b^3} \left(\frac{b^2}{a^2-b^2} \frac{\pi}{2} + \frac{\pi}{4} \right)$$

$$= \frac{1}{a^3b} \frac{\pi}{2} + \frac{a^2-b^2}{a^3b^3} \frac{\pi}{4}$$

Ans

Properties of Definate Integrats:

(i) $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(z)\,dz$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$[a < c < b]$

(iv) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(v) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$

(vi) $\int_0^{2a} f(x)\,dx = \int_0^a f(x) + \int_0^a f(2a-x)\,dx$

(vii) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$

[if $f(2a-x) = f(x)$]

Exercise:

① $\int_0^{\frac{\pi}{2}} \log \sin x \, dx = I$ --- (i)

$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right)$ [using $\int_x^a f(x)\,dx = \int_0^a f(a-x)\,dx$]

$= \int_0^{\frac{\pi}{2}} \log \cos x \, dx$ --- (ii)

Adding (i) and (ii) $\Rightarrow$

$$I + I = \int_0^{\frac{\pi}{2}} \left[\log(\sin x) + \log(\cos x)\right] dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log(\sin x \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \frac{1}{2}(2 \sin x \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \frac{1}{2} \sin 2x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \frac{1}{2} \, dx + \int_0^{\frac{\pi}{2}} \cancel{\sin 2x} \log \sin 2x \, dx$$

$$= \log\frac{1}{2}\int_0^{\frac{\pi}{2}} dx + \int_0^{\frac{\pi}{2}} \log\sin 2x \, dx$$

$$= \frac{\pi}{2}\log\frac{1}{2} + \frac{1}{2}\int_0^{\pi} \log\sin z \, dz \quad [2x = z]$$

$$= \frac{\pi}{2}\log\frac{1}{2} + \int_0^{\frac{\pi}{2}} \log\sin z \, dz \quad \left[\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx\right]$$

$$= \frac{\pi}{2}\log\frac{1}{2} + \int_0^{\frac{\pi}{2}} \log\sin x \, dx \quad \left[\because \int_a^{b} f(x)\,dx = \int_b^{a} f(z)\,dz\right]$$

$$= \frac{\pi}{2}\log\frac{1}{2} + I$$

$$\therefore 2I = \frac{\pi}{2}\log\frac{1}{2} + I$$

$$\therefore I = \frac{\pi}{2}\log\frac{1}{2}$$

Ans

(ii) $$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = I$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx \quad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = I$$

$$I + I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} 1\,dx$$

$$2I = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4}$$

Ans

Walli's formulla:

If n be a positive integer then,

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} ; & \text{when } n \text{ is even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 ; & \text{when } n \text{ is odd} \end{cases}$$

let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$$= \int_0^{\frac{\pi}{2}} \sin^n \left(\frac{\pi}{2} - x\right) dx \quad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \cos^n x\, dx$$

Now, $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \sin x\, dx.$$

$$= \left[-\sin^{n-1}x \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -(n-1)\sin^{n-2}x \cos^2 x \, dx$$

[$\int uv\,dx$ - [illegible]]

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x \cos^2 x \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x \,(1-\sin^2 x)\, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x - \sin^n x \, dx$$

$$\therefore I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n + nI_n - I_n = (n-1) I_{n-2}$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$$

Changing n by $n-2$, $n-4$, $n-6$, ... etc successively,

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$$I_{n-6} = \frac{n-7}{n-6} I_{n-8}$$

$\cdots$

when n is even,

$$I_6 = \frac{5}{6} I_4$$

$$I_4 = \frac{3}{4} I_2$$

$$I_2 = \frac{1}{2} I_0$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

$$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$= \frac{\pi}{2}$$

$$\therefore I_2 = \frac{1}{2} \times \frac{\pi}{2}$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

when n is odd,

$$I_7 = \frac{6}{7} I_5$$

$$I_5 = \frac{4}{5} I_3$$

$$I_3 = \frac{2}{3} I_1$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin^1 x \, dx$$

$$= \left[-\cos x\right]_0^{\frac{\pi}{2}}$$

$$= \left[-\cos \frac{\pi}{2} + \cos 0\right]$$

$$= 1$$

$$I_3 = \frac{2}{3} \times 1$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

Question: Obtain reduction ~~formular~~ formula for $\int_0^{\frac{\pi}{4}} \tan^n x \, dx$ and hence find $\int_0^{\frac{\pi}{4}} \tan^6 x \, dx$.

Solve: Let,

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \left(\sec^2 x - 1\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x - \tan^{n-2} x \, dx$$

suppose,

$\tan x = z$

$\Rightarrow \sec^2 x \, dx = dz$

$$= \int_0^1 z^{n-2} \, dz - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

$$= \left[\frac{z^{n-1}}{n-1}\right]_0^1 - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

In the similar way,

$$I_{n-2} = \frac{1}{n-3} - I_{n-4}$$

$$I_{n-4} = \frac{1}{n-5} - I_{n-6}$$

$$I_n = \frac{1}{n-1} - \frac{1}{n-3} + \frac{1}{n-5} - \frac{1}{n-7} + \cdots$$

Question: Obtain reduction formula for $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$.

Solve: Let,

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \, (\sin^m x \cos x) \, dx$$

$$= \left[\cos^{n-1} x \, \frac{\sin^{m+1} x}{m+1} \right]_0^{\frac{\pi}{2}} - \int (n-1) \cos^{n-2} x \, (-\sin x) \frac{\sin^{m+1} x}{m+1} \, dx$$

$$= 0 + \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^{m+2} x \cos^{n-2} x \, dx$$

$$I_{m,n} = \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^m x \sin^2 x \cos^{n-2} x \, dx$$

$$= \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^m x \left(1 - \cos^2 x\right) \cos^{n-2} x \, dx$$

$$= \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^m x \cos^{n-2} x \cancel{- \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin m} \, dx$$

$$- \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$$

$$I_{m,n} = \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m-1} I_{m,n}$$

$$\Rightarrow I_{m,n}\left(1 + \frac{n-1}{m+1}\right) = \frac{n-1}{m+1} I_{m,n-2}$$

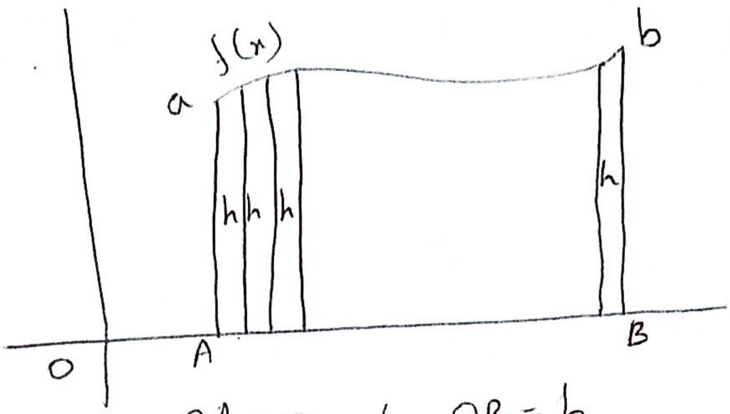
$$\Rightarrow I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$$

$$I_{m,n-2} = \frac{n-3}{m+n-2} I_{m,n-4}$$

$$I_{m,n-4} = \frac{n-5}{m+n-4} I_{m,n-6}$$

$$\therefore I_{m,n} = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdot \frac{n-7}{m+n-6} \cdots$$

Integration as the limit of sum:



$OA = a$; $OB = b$

$AB = OB - OA$

$\Rightarrow nh = b - a$ where $h \to 0$

$\Rightarrow n = \frac{b-a}{h}$ where $n \to \alpha$

$$\text{Area} = \int_a^b f(x)\,dx = \lim_{h \to 0} hf(a) + hf(a+h) + hf(a+2h) + \dots + hf\{a+(n-1)h\}$$

$$= \lim_{h \to 0} h\{f(a) + f(a+h) + f(a+2h) + \dots\}$$

$$= \lim_{h \to 0} h \sum_{r=0}^{n-1} f(a + rh)$$

$$= \lim_{n \to \alpha} \frac{b-a}{n} \sum_{r=0}^{n-1} f\left(a + r \cdot \frac{b-a}{n}\right)$$

if $a=0$ and $b=1$, then,

$$\int_0^1 f(x)\, dx = \lim_{n \to \alpha} \frac{1}{n} \sum_{r=0}^{n-1} f\left(a + \frac{r}{n}\right)$$

Question:

Evalvate

$$\lim_{n \to \alpha} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{1}{8n}\right]$$

Solve:

Given,

$$\lim_{n \to \alpha} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{1}{8n}\right]$$

$$= \lim_{n \to \alpha} \left[\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{n^2}{(n+n)^3}\right]$$

$$= \lim_{n \to \alpha} \sum_{r=0}^{n} \frac{n^2}{(n+r)^3}$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{n^2}{n^3\left(1+\frac{r}{n}\right)^3}$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{n} \cdot \frac{1}{\left(1+\frac{r}{n}\right)^3}$$

$$= \int_0^1 \frac{1}{(1+x)^3}\, dx$$

$$= \left[-\frac{1}{2(1+x)^2} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{8}$$

$$= \frac{3}{8}$$ Ans

Beta function: The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$; $m, n > 0$ is known as first Eulerian integral. The above function defined for positive value of m and n ($m>0$, $n>0$) is known as Beta function.

It is denoted by $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$.

Gamma function: The integral $\int_0^\alpha e^{-x} x^{n-1} dx$ is called the second Eulerian integral. The function defined for positive value of n $(n>0)$ is known as gamma function. It is denoted by

$$\Gamma n = \int_0^\alpha e^{-x} x^{n-1} dx.$$

(i) $\Gamma n+1 = n!$

(ii) $\Gamma n+1 = n \Gamma n-1$

(iii) $B(m,n) = \dfrac{\Gamma n \, \Gamma m}{\Gamma m+n}$ [Relation between γ and B function]

(iv) $\Gamma 1 = 1$

(v) $\Gamma \frac{1}{2} = \sqrt{\pi}$

(vi) $\Gamma n \, \Gamma n-1 = \dfrac{\pi}{\sin n\pi}$

Differenent form of gamma function:

~~Whe~~ We have, $\cancel{d}\ \Gamma n = \int_0^{\infty} e^{-x}\, x^{n-1}\, dx$ --- (1)

(i) put $x = \lambda y$ in (1) =)

$dx = \lambda\, dy$

$$\Gamma n = \int_0^{\infty} e^{-\lambda y} (\lambda y)^{n-1} \lambda\, dy$$

$$= \lambda^n \int_0^{\infty} e^{\lambda y} (y)^{n-1}\, dy$$

$$\therefore \frac{\Gamma n}{\lambda^n} = \int_0^{\infty} e^{-\lambda x}\, x^{n-1}\, dx$$

(ii) Put $x = y^2$ in (1) =)

$dx = 2y\, dy$

$$\Gamma n = \int_0^{\infty} e^{-y^2} y^{2n-2} \cdot 2y\, dy$$

$$= 2\int_0^{\infty} e^{-y^2} y^{2n-1}\, dy$$

$$\Gamma n = 2\int_0^{\infty} e^{-x^2} x^{2n-1}\, dx$$

(iii) Put $x = -(m+1)\log y$ in (1) $\Rightarrow$

$dx = -(m+1)\frac{1}{y}\, dy$

x	0	∞
y	1	0

$$\Gamma n = \int_1^0 e^{(m+1)\log y} \{-(m+1)\log y\}^{n-1} \{-(m+1)\tfrac{1}{y}\}\, dy$$

$$= \int_0^1 e^{\log y^{m+1}} (m+1)^{n-1+1} (-\log y)^{n-1} \cdot \frac{1}{y}\, dy$$

$$= (m+1)^n \int_0^1 y^{m+1} \left(\log \frac{1}{y}\right)^{n-1} \frac{1}{y}\, dy$$

$$\Gamma n = (m+1)^n \int_0^1 y^m \left(\log \frac{1}{y}\right)^{n-1} dy$$

$$\Gamma n = (m+1)^n \int_0^1 x^m \left(\log \frac{1}{x}\right)^{n-1} dx$$

(iv) $\cancel{\Gamma n = n\int_0^{\infty} e^{-x^n}\, dx}$

(IV) $\Gamma n = \frac{1}{n} \int_0^\infty e^{-x^{\frac{1}{n}}} dx$

(V) $\Gamma n = \int_0^1 \left(\log \left(\frac{1}{x} \right) \right)^{n-1} dx$

Different form of Beta function:

We have $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

(i) Put $x = \frac{y}{1+y}$; $dx = \frac{(1+y)\,dy - y\,dy}{(1+y)^2}$

$\rightarrow y = \frac{x}{1-x}$ $\Rightarrow dx = \frac{dy}{(1+y)^2}$

$$\beta(m,n) = \int_0^\infty \left(\frac{y}{1+y} \right)^{m-1} \left(1 - \frac{y}{y+1} \right)^{n-1} \frac{1}{(1+y)^2} dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+1}} dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+1}} dy$$

$$dx = 2\sin\theta\cos\theta\, d\theta$$

(ii) Put $x = \sin^2\theta$;

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\, (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$$

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$$

$$\frac{\beta(m,n)}{2} = \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$$

We know,

$$\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}$$

$$\therefore \frac{\Gamma m\, \Gamma n}{2\Gamma m+n} = \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta.$$

~~Put~~ (iii) Put $x = \frac{y}{a}$: $dx = \frac{1}{a} dy$

$$\beta(m,n) = \int_0^a \frac{1}{a^{m+n-1}} \cdot y^{m-1} (a-y)^{m-n} \, dy$$

~~*Syme~~ * Symmetrie property : $\beta(m,n) = \beta(n,m)$

Relation between Gamma and Beta function:

We have, $\Gamma m = \int_0^\infty e^{-\lambda} \lambda^{m-1} \, d\lambda$ ---- (i)

$$\frac{\Gamma n}{\lambda^n} = \int_0^\infty e^{-\lambda x} x^{n-1} \, dx$$

$$\Gamma n = \int_0^\infty e^{-\lambda x} \lambda^n x^{n-1} \, dx \quad \text{--- (ii)}$$

multiply (i) and (ii) $\Rightarrow$

$$\Gamma m \, \Gamma n = \int_0^\infty \left(\int_0^\infty e^{-\lambda x} e^{-\lambda} \lambda^{m-1} \lambda^n \, d\lambda \right) x^{n-1} \, dx$$

$$= \int_0^\infty \left(\int_0^\infty e^{-\lambda(1+x)} \lambda^{m+n-1} \, d\lambda \right) x^{n-1} \, dx$$

$$= \int_0^{\alpha} \frac{\Gamma m+n}{(1+x)^{m+n}} x^{n-1} \, dx$$

$$\Rightarrow \frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \int_0^{\alpha} \frac{x^{n-1}}{(1+x)^{m+n}} \, dx = \beta(m, n)$$

$$\therefore \beta(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

<u>Question:</u> Evaluate $\int_0^{\alpha} e^{-x^2} \, dx$

<u>Solution:</u>

$$\int_0^{\alpha} e^{-x^2} \, dx$$

$$= \int_0^{\alpha} e^{-x^{\frac{1}{2}}} \, dx$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} \int_0^{\alpha} e^{-x^{\frac{1}{2}}} \, dx$$

$$= \frac{1}{2} \Gamma \frac{1}{2}$$

$$= \frac{\sqrt{\pi}}{2}$$

<u>Ans</u>

Alternative Solution:

$$\int_0^\infty e^{-x^2} dx$$

Put $x^2 = z$

$2x\,dx = dz$

$dx = \frac{1}{2} \cdot \frac{1}{x} dz$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{z}} dz$

$$= \int_0^\infty e^{-z} \cdot \frac{1}{2} z^{-\frac{1}{2}} dz$$

$$= \frac{1}{2} \int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz$$

$$= \frac{1}{2} \times \Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2} \text{ Ans}$$

Question: find the value of $\Gamma\frac{1}{2}$.

Solve: We know,

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^0\theta \cos^0\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1} = 2\int_0^{\frac{\pi}{2}} d\theta$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi$$

$$\therefore \Gamma\frac{1}{2} = \sqrt{\pi} \text{ Ans}$$

Question: Show that $\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m \Gamma n}{\Gamma m+n}$

and hence show that $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}$

Solution: We have,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

Put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta \, d\theta$

$$B(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta \; 2\cos\theta\sin\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma m \Gamma n}{\Gamma m+n} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m \Gamma n}{2\Gamma m+n}$$

Put $2m-1 = p$; $2n-1 = q$

$\Rightarrow m = \frac{p+1}{2}$, $n = \frac{q+1}{2}$

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \, \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \; \Gamma\frac{q+1}{2}}{2\,\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}$$

$$= \frac{\Gamma\frac{p+1}{2} \; \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

Ans

Question: Evaluate $\int_0^1 x^3 (1-x)^4 \, dx$

Solve: Given,

$$\int_0^1 x^3 (1-x)^4 \, dx$$

$$= \int_0^1 x^{4-1} \left(1-x\right)^{5-1} dx$$

$$= B(4, 5)$$

$$= \frac{\Gamma 4 \; \Gamma 5}{\Gamma 4+5} = \frac{\Gamma 4 \; \Gamma 5}{\Gamma 9}$$

$$= \frac{3! \times 4!}{8!}$$

$$= \frac{1}{280}$$

Ans

Question: Evaluate $\displaystyle\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

Solve: Given,

$$I = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= I_1 + I_2 \qquad \left[\begin{array}{l} \because\ I_1 = \displaystyle\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \\ I_2 = \displaystyle\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \end{array}\right]$$

$$I_2 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \quad ; \quad \text{Put } x = \frac{1}{y}$$

$$dx = -\frac{1}{y^2}\,dy$$

$$= \int_\infty^1 \frac{\left(\frac{1}{y}\right)^{n-1}}{\left(1+\frac{1}{y}\right)^{m+n}}\left(-\frac{1}{y^2}\right)dy$$

$$= \int_1^\infty \frac{y^{m+n}}{y^{n-1}(y+1)^{m+n}} \cdot \frac{1}{y^2}\,dy$$

$$= \int_1^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} \, dy$$

$$I_2 = \int_1^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} \, dy$$

$$= \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$

$$I = I_1 + I_2$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} \, dx + \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$

$$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$

$$= B(m, n)$$

$$= \frac{\Gamma m \; \Gamma n}{\Gamma m+n}$$

<u>Ans</u>

Jacobians: If $u_1, u_2, \dots, u_n$ are f^n of n variables $x_1, x_2, \dots, x_n$ then the determinent

$$\begin{vmatrix} \frac{du_1}{dx_1} & \frac{du_1}{dx_2} & \cdots & \frac{du_1}{dx_n} \\ \frac{du_2}{dx_1} & \frac{du_2}{dx_2} & \cdots & \frac{du_2}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{du_n}{dx_1} & \frac{du_n}{dx_2} & \cdots & \frac{du_n}{dx_n} \end{vmatrix}$$

is called the jacobians of $u_1, u_2, \dots, u_n$ with respect to ~~x_1, x~~ $x_1, x_2, \dots, x_n$. It is denoted by $\frac{d(u_1, u_2, \dots, u_n)}{d(x_1, x_2, \dots, x_n)}$ or $J(u_1, u_2, \dots, u_n)$.

~~* If~~ * If $J = 0$, then $u_1, u_2, \dots, u_n$ are dependent on each other.

Question: If $u = x^2 + y^2 + z^2$; $v = x + y + z$; $w = xy + yz + zx$, show that the jacobian $\frac{d(u,v,w)}{d(x,y,z)}$ vanish and find the relation between u, v, w.

Solve: Given that,

$$u = x^2 + y^2 + z^2 \; ; \quad v = x + y + z \; ; \quad w = xy + yz + zx$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 1 \qquad \frac{dw}{dx} = y + z$$

$$\frac{du}{dy} = 2y \qquad \frac{dv}{dy} = 1 \qquad \frac{dw}{dy} = x + z$$

$$\frac{du}{dz} = 2z \qquad \cancel{dz}\;\frac{dv}{dz} = 1 \qquad \frac{dw}{dz} = y + x$$

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{vmatrix} = \begin{vmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 0$$

Ans

$V = x+y+z$

$V^2 = (x+y+z)^2$

$= (x^2+y^2+z^2+2xy+2yz+2xz)$

$= U + 2w$

$\therefore V^2 = U + 2w$

Ans

Question: show that the function $\cancel{u = x+y+z}$, $u = x+y-z$, $v = x-y+z$ and $w = x^2+y^2+z^2-2yz$ are not independent of one another and also show that $u^2+v^2 = 2w$.

Solve: Given,

$u = x+y-z$; $v = x-y+z$; $w = x^2+y^2+z^2-2yz$

$\cancel{u = x+y+z}$

$\frac{du}{dx} = 1$ $\frac{dv}{dx} = 1$ $\frac{dw}{dx} = 2x$

$$\cancel{dx}\ \frac{du}{dy} = 1 \qquad \frac{dv}{dy} = -1 \qquad \frac{dw}{dy} = 2y - 2z$$

$$\frac{du}{dz} = -1 \qquad \frac{dv}{dz} = 1 \qquad \frac{dw}{dz} = 2z - 2y$$

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2x \\ 1 & -1 & 2y-2z \\ -1 & 1 & 2z-2y \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 2x \\ 0 & -1 & 2y-2z \\ 0 & 1 & 2z-2y \end{vmatrix}$$

$$= 2 \cdot 2(-z+y-y+z)$$

$$= 2 \cdot 2 \cdot 0$$

$$= 0$$

~~Here~~ Here, $J(u,v,w) = 0$, so they are dependent on each other.

Now,

$$u^2+v^2 = (x+y-z)^2 + (x-y+z)^2$$

$$= (x+y)^2 + z^2 - 2(x+y)z + (x-y)^2 + z^2 + \overset{2(x-y)z}{\cancel{2(x+y)z}}$$

$$= x^2 + \cancel{2xy} + y^2 + z^2 - \cancel{2xz} - 2yz + x^2 - \cancel{2xy} + y^2 + z^2$$
$$\cancel{+2xz} \cancel{+2yz} + \cancel{2xz} - 2yz$$

$$= 2x^2 + 2y^2 + 2z^2 - 4yz$$

$$= 2(x^2 + y^2 + z^2 - 2yz)$$

$$= 2w$$

$$\therefore u^2 + v^2 = 2w$$

Ans

* Differentiation under integral sign:

General form: If $f(x,t)$ is a continuous and continuously differentiable function and the limits of integration $a(x)$ and $b(x)$ are continuously differentiable function of x

then,

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f\{x, b(x)\}.b'(x) - f\{x, a(x)\}\,a'(x) + \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)\,dt.$$

When $a(x)$ and $b(x)$ are constant then

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{d}{dx} f(x,t)\,dt.$$

This is known as Leibnitz's rule.

~~Question: Evaluate~~

Question: Evaluate $\displaystyle\int_0^{\alpha} \frac{\tan^{-1}(ax)}{x(1+x^2)}\, dx$

Solve: Let,

$$I = \int_0^{\alpha} \frac{\tan^{-1}(ax)}{x(1+x^2)}\, dx$$

$$\Rightarrow \frac{dI}{da} = \frac{d}{da}\left\{ \int_0^{\alpha} \frac{\tan^{-1}(ax)}{x(1+x^2)}\, dx \right\}$$

$$= \int_0^{\alpha} \frac{x}{(1+a^2x^2)\, x\,(1+x^2)}\, dx$$

$$= \int_0^{\alpha} \frac{1}{(1+x^2)(1+a^2x^2)}\, dx$$

$$= \frac{a^2}{1-a^2} \int_0^{\alpha} \left[\frac{1}{a^2(1+x^2)} - \frac{1}{1+a^2x^2} \right] dx$$

$$= \frac{a^2}{1-a^2} \left[\frac{1}{a^2} \int_0^{\alpha} \frac{1}{1+x^2}\, dx - \int_0^{\alpha} \frac{1}{1+a^2x^2}\, dx \right]$$

$$= \frac{1}{1-a^2}\left[\tan^{-1}x\right]_0^{\alpha} - \frac{a^2}{1-a^2}\int_0^{\alpha} \frac{1}{a^2\left(x^2+\frac{1}{a^2}\right)}\,dx$$

$$= \frac{1}{1-a^2}\left[\tan^{-1}\alpha - \tan^{-1}0\right] - \frac{a}{1-a^2}\left[\tan^{-1}ax\right]_0^{\alpha}$$

$$= \frac{1}{1-a^2}\frac{\pi}{2} - \frac{a}{1-a^2}\frac{\pi}{2}$$

$$= \frac{\pi}{2}\frac{1}{1-a^2}(1-a)$$

$$= \frac{\pi}{2}\frac{1}{1+a}$$

$$\Rightarrow dI = \frac{\pi}{2(1+a)}\,da$$

Now, integrating,

$$\Rightarrow \int dI = \int \frac{\pi}{2(1+a)}\,da$$

$$\Rightarrow I = \frac{\pi}{2}\int \frac{1}{1+a}\,da$$

$$= \frac{\pi}{2}\log(1+a) + c$$

when $a=0$ then $I=0$

$\therefore 0 = 0 + c$

$\Rightarrow c = 0$

$$\therefore I = \frac{\pi}{2} \log(1+a)$$

Ans

Jacobian of implicit function:

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\dfrac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_1, x_2, \ldots, x_n)}}{\dfrac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)}}$$

Question: If $u = x(1-r^2)^{-\frac{1}{2}}$, $v = y(1-r^2)^{-\frac{1}{2}}$, $w = z(1-r^2)^{-\frac{1}{2}}$, where $r^2 = x^2+y^2+z^2$ then find the value of $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

solve: Given that,

$$u = x(1-r^2)^{-\frac{1}{2}}$$

$$\Rightarrow u^2 = x^2(1-r^2)^{-1}$$

$$\Rightarrow x^2 = u^2(1-r^2)$$

$$\Rightarrow x^2 - u^2(1-x^2-y^2-z^2) = 0$$

similarly,

$$y^2 - v^2(1-x^2-y^2-z^2) = 0$$

$$z^2 - w^2(1-x^2-y^2-z^2) = 0$$

Let, $F_1 = x^2 - u^2(1-x^2-y^2-z^2)$

$F_2 = y^2 - v^2(1-x^2-y^2-z^2)$

$F_3 = z^2 - w^2(1-x^2-y^2-z^2)$

$$\frac{\delta f_1}{\delta x} = 2x + 2xu^2 \quad ; \quad \frac{\delta f_1}{\delta y} = 2yu^2 \quad , \quad \frac{\delta f_1}{\delta z} = 2zu^2$$

$$= 2x(1+u^2)$$

$$\frac{\delta f_2}{\delta x} = 2xv^2 \quad ; \quad \frac{\delta f_2}{\delta y} = 2y(1+v^2) \quad ; \quad \frac{\delta f_2}{\delta z} = 2zv^2$$

$$\frac{\delta f_3}{\delta x} = 2xw^2 \quad ; \quad \frac{\delta f_3}{\delta y} = 2yw^2 \quad ; \quad \frac{\delta f_3}{\delta z} = 2z(1+w^2)$$

$$\therefore \frac{\delta(f_1, f_2, f_3)}{\delta(x, y, z)} = \begin{vmatrix} \frac{\delta f_1}{\delta x} & \frac{\delta f_1}{\delta y} & \frac{\delta f_1}{\delta z} \\ \frac{\delta f_2}{\delta x} & \frac{\delta f_2}{\delta y} & \frac{\delta f_2}{\delta z} \\ \frac{\delta f_3}{\delta x} & \frac{\delta f_3}{\delta y} & \frac{\delta f_3}{\delta z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x(1+u^2) & 2yu^2 & 2zu^2 \\ 2xv^2 & 2y(1+v^2) & 2zv^2 \\ 2xw^2 & 2yw^2 & 2z(1+w^2) \end{vmatrix}$$

$$= 8xyz \begin{vmatrix} 1+u^2 & u^2 & u^2 \\ v^2 & 1+v^2 & v^2 \\ w^2 & w^2 & 1+w^2 \end{vmatrix}$$

$$= \frac{8xyz}{1-x^2-y^2-z^2}$$

$$\frac{\partial f_1}{\partial u} = -2u(1-x^2-y^2-z^2) \qquad \frac{\partial f_1}{\partial v} = 0 \; ; \; \frac{\partial f_1}{\partial w} = 0$$

$$= -2u(1-r^2)$$

$$\frac{\partial f_2}{\partial u} = 0 \; ; \; \frac{\partial f_2}{\partial v} = -2v(1-r^2) \; ; \; \frac{\partial f_2}{\partial w} = 0$$

$$\frac{\partial f_3}{\partial u} = 0 \; ; \; \frac{\partial f_3}{\partial v} = 0 \; ; \; \frac{\partial f_3}{\partial w} = -2w(1-r^2)$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} -2u(1-r^2) & 0 & 0 \\ 0 & -2v(1-r^2) & 0 \\ 0 & 0 & -2w(1-r^2) \end{vmatrix}$$

$$= -8uvw(1-r^2)^3$$

$$= -8xyz(1-r^2)^{\frac{3}{2}}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{8xyz(1-r^2)^{-1}}{-8xyz(1-r^2)^{\frac{3}{2}}}$$

$$= (1-r^2)^{-\frac{5}{2}} \quad \text{Ans}$$

* Let x be the length and y be the ~~breadth~~ width of a rectangle, then

perimeter, $s = 2(x+y) = \text{Constant} = 2K$ (say)

$$\Rightarrow x+y = K$$

$$\Rightarrow y = K-x \quad \text{---} \quad (i)$$

Area of the rectangle is $A = xy$

$$= x(K-x)$$

$$= xK - x^2$$

$$\Rightarrow \frac{dA}{dx} = K - 2x \text{ - - - (ii)}$$

For minima and maxima, $\frac{dA}{dx} = 0$

$$\Rightarrow K - 2x = 0$$

$$\Rightarrow x = \frac{K}{2}$$

From equation (ii),

$\frac{d^2A}{dx^2} = -2$, which is negative for all values of x.

From equation (i),

$$y = K - \frac{K}{2}$$

$$= \frac{K}{2}$$

$\therefore x = y$, that is the rectangular converts to square.