PHYS 3926: Project 3

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1 Introduction

The purpose of this simulation was to determine the mass-radius relation of white dwarf stars. In order to achieve this, a set of coupled ordinary differential equations (ODE's) which govern the internal density structure of white dwarf stars was solved numerically using Python. Furthermore, solutions yielded by Runge-Kutta Method of order 4 (RK45) and 3 (RK23) were compared. The theoretically determined mass-radius relation was then compared to observational data from *Tremblay et al. (2017)*. From the theoretical relation, the Chandrasekhar limiting mass - the maximum allowed mass of a white dwarf star - was estimated and found to be in accordance with results from *Kippenhahn and Weigert (1990)*. The coupled ODEs which govern the internal density are:

$$\frac{d\rho'}{dr'} = -K_1 \frac{m'\rho'}{\gamma(x) r'^2} \tag{1}$$

$$\frac{dm'}{dr'} = K_2 r'^2 \rho' \tag{2}$$

where

$$\gamma(x) = \frac{x^2}{3(1+x^2)^{1/2}} \tag{3}$$

The primed variables were modified into dimensionless quantities and are defined as:

$$r' = \frac{r}{R_0} \tag{4}$$

$$m' = \frac{m}{M_0} \tag{5}$$

$$\rho' = \frac{\rho}{\rho_0} \tag{6}$$

where $R_0 = 7.72 \times 10^8/\mu_e$cm, $M_0 = 5.67 \times 10^{33}/\mu_e^2$g, and $\rho_0 = 9.74 \times 10^5 \mu_e$gcm^{-3}. Here, μ_e is the number of nucleons per electron (taken as 2 for the purposes of this investigation). The quantities listed were chosen specifically in order to force the coefficients K_1 and K_2 to be unity.

2 Tasks

2.1 Task 1 and 2

The goal of Task 1 was to determine the radius and mass of model white dwarfs. Firstly, the ODE's were defined as:

```
def odes(r, y):
    x = y[0]**(1./3)
    gamma = (x**2)/(3*np.sqrt((1+x**2)))

    # ODEs
```

```
    drho_dr = -(y[1]*y[0])/(gamma*r**2)
    dm_dr = (r**2)*y[0]

    return [drho_dr, dm_dr]
```

Function `odes()` was created to be used with `scipy.integrate.solve_ivp()`. The variables used in the function are the primed quantities. Next, the function `whiteDwarf(rho_c, method)` was created in order to handle the solutions of the ODE's. The parameter `rho_c` takes an array of values which govern the family of solutions. Parameter `method` allows the user to chose the method of integration used by `solve_ivp`. The function utilizes a for loop to iterate over all the values of `rho_c` as follows:

```
for i in range(len(rho_c)):
    rho_temp = rho_c[i]
    rSpan = [3e-14, 10] # Integration range
    initCond = [rho_temp, 0]
    # Set terminal condition
    rho_f = lambda r, y: y[0] - 5.13e-17 # Density very close to zero
    rho_f.terminal = True
    # Solve IVP
    sol = scint.solve_ivp(odes, t_span=rSpan, y0=initCond, method=method, events=rho_f)

    # Unit Conversion before adding to lists to be returned
    rhoValues = sol.y[0] * rho0
    mValues = sol.y[1] * M0
    rValues = sol.t * R0
    masses.append(mValues[-1])
    densities.append(rhoValues[-1])
    radii.append(rValues[-1])
```

Firstly, the loop chooses an appropriate integration range. The integration range could not begin at $r = 0$ since it would cause a zero division error. The average radius of a white dwarf star is 7000km, therefore a relative radius of 1×10^{-5}cm seems comparable to zero. This value was converted to the primed variable holding a value of 3×10^{-14} using equation (4). The upper limit of integration was set very large in order for the `events` functionality of `solve_ivp` to handle termination of the integration. The integration should terminate when density falls to zero, therefore it was determined that a density of $\rho = 1 \times 10^{-10}$gcm^{-3} was small enough; the value was then converted in accordance to equation (6). `rho_f` determined the termination condition. The loop then converts the value into their respective un-primed quantities. The function returns three arrays, each containing the final values for radius, mass and density, respectively, for each `rho_c` value.

Ten evenly spaced values of ρ_c in range 10^{-1} to 2.5×10^6 were determined through the use of the `logspace()`. Next, the solutions for these values were determine and plotted.

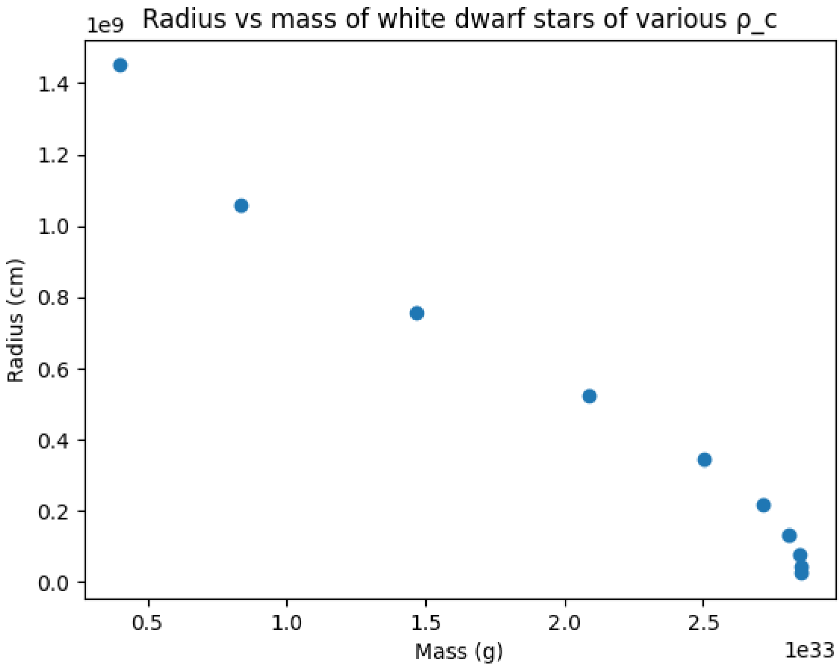

Figure 1: Represents the mass-radius relation of white dwarf stars using various ρ_c values.

The plot seems to be approaching a mass limit which we know to be the Chandrasekhar limit. *Kippenhahn and Weigert (1990)* determined the limit to be $M_{Ch} = 5.836/\mu_e^2$ in units of solar mass. Converting this to grams, it is $M_{Ch} = 2.902 \times 10^{33}$g. From the plot, the limit can be estimated to be approximately 2.9×10^{33}g. The limit determined by the code can be seen to agree with that of *Kippenhahn and Weigert (1990)*.

2.2 Task 3

The purpose of task 3 was to compare two integration methods used in `solve_ivp`. The following code was implemented:

```
solutions1 = whiteDwarf(ran[3:6])
solutions2 = whiteDwarf(ran[3:6], 'RK23')
```

Firstly, 2 sets of 3 solutions were determined. `solutions1` was found using RK45 and `solutions2` used RK23 (the default `method` parameter in `whiteDwarf()` is RK45). The following graph presenting the differences was produced.

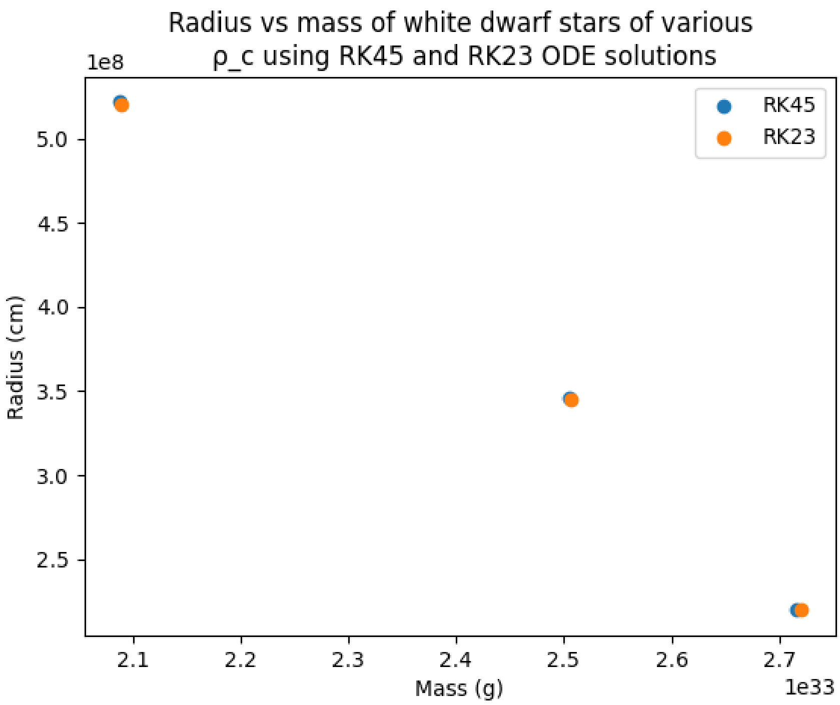

Figure 2: Compares solutions of ODE's using RK23 and RK45

As one can observe, the graph appears to show minimal differences in the solutions. However, upon further analysis, the numerical differences are as follows.

Table 1: Difference in values of mass and radius from RK45 and RK23.

Methods	**Mass (g)**			**Radius (cm)**		
RK45	2.087×10^{33}	2.505×10^{33}	2.716×10^{33}	5.218×10^{8}	3.464×10^{8}	2.204×10^{8}
RK23	2.089×10^{33}	2.507×10^{33}	2.721×10^{33}	5.209×10^{8}	3.454×10^{8}	2.197×10^{8}
Difference in methods	2.074×10^{30}	1.962×10^{30}	4.574×10^{30}	1.013×10^{6}	9.474×10^{5}	6.541×10^{5}

Although figure 2 represents the differences as minimal, inspection of table 1 shows that the difference in masses are on the order of 10^{30}cm and the differences in radii are on the order of 10^{6}g. While graphically, these differences seem inadmissible, numerically they are very large.

2.3 Task 4

The objective of this task was to plot the white dwarf masses and radii obtained from binary systems measured by the Gaia satellite from *Tremblay et al. (2017)*. The csv file containing the data was read by the code and organized into a 2-dimensional numpy array. The mass data was converted to grams and the radius data to cm before adding them to the array. The Data, along with the error bars, was then plotted against the theoretically determined mass-radius relation obtained in section 2.1.

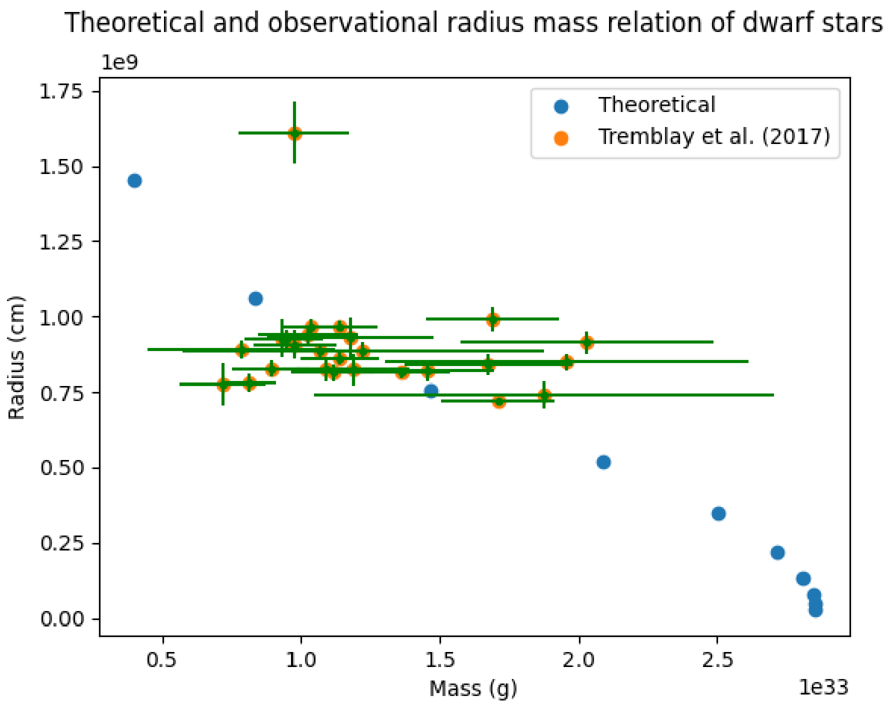

Figure 3: Theoretical mass-radius relation of white dwarf stars compared to observational data from *Tremblay et al. (2017)*.

As one can observe, there appears to be little correlation between the observational and theoretical data. The observational data also large uncertainties in mass which partially accounts for its inaccuracy in that respect. However, the uncertainties in radius are relatively small. There are several points for which the error range lies outside of the path of the mass-radius relation. Should one exclude the obvious outlying point at $R \approx 1.62 \times 10^9$cm, much of the data still does not adhere to the trend. Overall, the data seems to have low accuracy with moderate precision when compared to the theoretical data.

3 Conclusion

The goal of this investigation was to investigate the mass-radius relation of white-dwarf stars, determine the Chandrasekhar limit, and compare observational data to the theoretically determined relation. The mass-radius relation was determined by implementing a set of coupled ODE's which govern the internal density structure of white dwarf stars and subsequently using `solve_ivp` to obtain solutions for various initial densities. The mass-radius relation was found to be in accordance with the expected outcome. The relation trended asymptotically towards the Chandrasekhar mass as given by *Kippenhahn and Weigert (1990)*. Next, the differences between RK45 and RK23 methods for solving coupled ODE's found to produce fairly large discrepancies in solutions which appear minimal when graphed. Furthermore, the observational data from *Tremblay et al. (2017)* seemed to be in minor agreement with the theoretical relation. Even with the uncertainties accounted for, the trend did not pass through several data points which points towards a low accuracy despite its moderate precision.

References

[1] Kippenhahn, Rudolf, and Alfred Weigert. Stellar Structure and Evolution. Springer, 1990.

[2] Monthly Notices of the Royal Astronomical Society, Volume 465, Issue 3, March 2017, Pages 2849–2861, https://doi.org/10.1093/mnras/stw2854