

# On the Irreducibility of the Cuboid Polynomial $P_{a,u}(t)$

Valery Asiryan

asiryanvalery@gmail.com

October 9, 2025

## Abstract

In this paper we consider the even monic degree-8 cuboid polynomial  $P_{a,u}(t)$  with coprime integers  $a \neq u > 0$ . We prove irreducibility over  $\mathbb{Z}$  by excluding all degree-8 splittings. First, any putative 4+4 factorization is shown to force a specific Diophantine constraint which has no integer solutions by a short 2- and 3-adic analysis. Second, we exclude every 2+6 factorization via an exact divisor criterion and a discriminant obstruction. Finally, after ruling out 2+6, the patterns 2+2+4, 2+2+2+2, and 3+3+2 regroup trivially to 2+6 and are therefore impossible. Consequently,  $P_{a,u}(t)$  admits no nontrivial factorization in  $\mathbb{Z}[t]$ .

**Keywords** Irreducibility over  $\mathbb{Z}$ ; even monic polynomials; cuboid (Euler) polynomial  $P_{a,u}(t)$ ; factorization types 4+4, 2+6, 2+2+4, 2+2+2+2, 3+3+2; Diophantine constraints;  $p$ -adic valuations (2-adic, 3-adic); discriminant obstruction; Gauss's lemma; parity/involution regrouping; elliptic curves.

**MSC 2020** **Primary:** 12E05 (Polynomials: irreducibility). **Secondary:** 11D72 (Equations in many variables; Diophantine equations), 11S05 (Local and  $p$ -adic fields), 11Y05 (Factorization; primality).

## 1 Problem Statement and Notation

Let  $a, u \in \mathbb{Z}_{>0}$  be coprime and  $a \neq u$ . We consider the even monic polynomial [1, 2, 3]

$$P_{a,u}(t) = t^8 + At^6 + Bt^4 + Ct^2 + D,$$

$$A = 6\Delta, \Delta := u^2 - a^2 \neq 0, B = \Delta^2 - 2a^2u^2, C = -a^2u^2A, D = a^4u^4.$$

We work in  $\mathbb{Z}[t]$ . The polynomial  $P_{a,u}$  is even, monic, and primitive:  $\text{cont}(P_{a,u}) = 1$  [5, 13, 6]. Standard irreducibility criteria such as Eisenstein's (including the shifted variant  $t \mapsto t + c$ ) generally do not apply uniformly to  $P_{a,u}$ ; cf. [14].

**Theorem 1** (Goal). *For any coprime  $a, u \in \mathbb{Z}_{>0}$  with  $a \neq u$ , the polynomial  $P_{a,u}(t)$  does not factor in  $\mathbb{Z}[t]$  as a product of two monic polynomials of degree 4 (the case 4+4).*

## 2 Normal Form of a 4+4 Factorization and the Necessary Condition (★)

**Lemma 1** (Gauss + involution). *If  $P_{a,u} = FG$  with monic  $F, G \in \mathbb{Z}[t]$  and  $\deg F = \deg G = 4$ , then, after swapping the factors if necessary, one of the following holds:*

- (E) both factors are even:  $F = t^4 + pt^2 + q$ ,  $G = t^4 + rt^2 + s$  ( $p, q, r, s \in \mathbb{Z}$ );
- (C) a conjugate pair:  $G(t) = F(-t)$ , where  $F = t^4 + \alpha t^3 + \beta t^2 + \gamma t + \delta$ .

*Idea.* Primitivity and Gauss's lemma yield primitivity and monicity of the factors [5, 13, 6]. The involution  $\tau : t \mapsto -t$  fixes  $P_{a,u}$ ; either both factors are invariant (even), or  $\tau$  swaps the factors (a conjugate pair).  $\square$

### Detailed derivation in case (E)

Let  $F = t^4 + pt^2 + q$ ,  $G = t^4 + rt^2 + s$ . From  $FG = P_{a,u}$  we obtain the system

$$p + r = A, \tag{1}$$

$$pr + q + s = B, \tag{2}$$

$$ps + rq = C, \tag{3}$$

$$qs = D. \tag{4}$$

From (1) we have  $r = A - p$ . Introduce

$$M := B + p^2 - Ap.$$

Then (2) and (4) rewrite as

$$q + s = M, \quad qs = D. \tag{5}$$

Thus  $q, s$  are integer roots of the quadratic equation  $X^2 - MX + D = 0$ . Denote (the discriminant of this quadratic)

$$T^2 := M^2 - 4D \text{ [5, 6].}$$

Then

$$q = \frac{M + \sigma T}{2}, \quad s = \frac{M - \sigma T}{2}, \quad \sigma \in \{\pm 1\}. \quad (6)$$

Substitute (6) into (3). The left-hand side of (3) equals

$$ps + rq = p \frac{M - \sigma T}{2} + (A - p) \frac{M + \sigma T}{2} = \frac{AM + \sigma T(A - 2p)}{2}.$$

Hence from (3) we get

$$\frac{AM + \sigma T(A - 2p)}{2} = C \iff \sigma T(A - 2p) = 2C - AM. \quad (7)$$

Set

$$X := p - 3\Delta \quad (\text{that is } p = X + 3\Delta, \quad A = 6\Delta).$$

What follows is a direct computation.

Computing  $M$ .

$$\begin{aligned} M &= B + p^2 - Ap = (\Delta^2 - 2a^2u^2) + (X + 3\Delta)^2 - 6\Delta(X + 3\Delta) \\ &= (\Delta^2 - 2a^2u^2) + (X^2 + 6\Delta X + 9\Delta^2) - 6\Delta X - 18\Delta^2 \\ &= X^2 - 8\Delta^2 - 2a^2u^2. \end{aligned}$$

Computing  $2C - AM$ . Since  $C = -a^2u^2A = -6\Delta a^2u^2$ , we have

$$2C = -12\Delta a^2u^2, \quad AM = 6\Delta(X^2 - 8\Delta^2 - 2a^2u^2).$$

Therefore,

$$2C - AM = -12\Delta a^2u^2 - 6\Delta(X^2 - 8\Delta^2 - 2a^2u^2) = -6\Delta X^2 + 48\Delta^3.$$

Thus (7) becomes

$$\begin{aligned} \sigma T(A - 2p) &= \sigma T(6\Delta - 2X - 6\Delta) = -2\sigma XT \\ &= 2C - AM = -6\Delta X^2 + 48\Delta^3. \end{aligned}$$

Divide by  $-2$  to obtain the fundamental relation

$$\sigma T X = 3\Delta(X^2 - 8\Delta^2). \quad (8)$$

Computing  $T^2$ . By definition,

$$\begin{aligned} T^2 &= M^2 - 4D = (X^2 - 8\Delta^2 - 2a^2u^2)^2 - 4a^4u^4 \\ &= (X^2 - 8\Delta^2)^2 - 4a^2u^2(X^2 - 8\Delta^2) \\ &= (X^2 - 8\Delta^2)(X^2 - 8\Delta^2 - 4a^2u^2). \end{aligned}$$

Deriving the starred equation. Square (8) and substitute the expression for  $T^2$ :

$$T^2 X^2 = 9\Delta^2 (X^2 - 8\Delta^2)^2.$$

Since  $X^2 \neq 8\Delta^2$  (see below), we can cancel  $(X^2 - 8\Delta^2)$  and obtain

$$(X^2 - 8\Delta^2 - 4a^2u^2)X^2 = 9\Delta^2 (X^2 - 8\Delta^2).$$

Moving everything to the left and grouping, we arrive at the Diophantine equation

$$\boxed{(X^2 - 8\Delta^2)(X^2 - 9\Delta^2) = 4a^2u^2 X^2} \quad (\star)$$

(see the remark below on the legitimacy of cancellation).

*Remark 1* (Legitimacy of cancellation and a consequence). If  $X^2 = 8\Delta^2$ , then comparing the 2-adic valuations yields  $2\nu_2(X) = 3 + 2\nu_2(\Delta)$ , which is impossible (the left-hand side is even, the right-hand side is odd). Hence for  $\Delta \neq 0$  the equality  $X^2 = 8\Delta^2$  has no integer solutions, and cancellation by the factor  $X^2 - 8\Delta^2$  is valid [8, 10, 9]. Consequently, (1)–(4) *imply*  $(\star)$ . The converse, in general, is not claimed: in addition one needs that  $T^2 = M^2 - 4D$  be a perfect square and  $q = \frac{M \pm T}{2} \in \mathbb{Z}$ .

## Case (C): conjugate pair

Assume

$$F(t) = t^4 + \alpha t^3 + \beta t^2 + \gamma t + \delta, \quad G(t) = F(-t),$$

so that  $F(t)F(-t) = P_{a,u}(t)$ . Equating coefficients gives the system

$$2\beta - \alpha^2 = A = 6\Delta, \quad (9)$$

$$\beta^2 + 2\delta - 2\alpha\gamma = B = \Delta^2 - 2A_0, \quad (10)$$

$$2\beta\delta - \gamma^2 = C = -6\Delta A_0, \quad (11)$$

$$\delta^2 = D = A_0^2, \quad (12)$$

where  $\Delta = u^2 - a^2 \neq 0$  and  $A_0 = a^2u^2 = (au)^2$ .

**Step 1: the sign of  $\delta$  is forced.** From (12) we have  $\delta = \pm A_0$ . If  $\delta = -A_0$ , then (11) becomes

$$-2\beta A_0 - \gamma^2 = -6\Delta A_0 \implies \gamma^2 = A_0(6\Delta - 2\beta).$$

Using (9),  $2\beta = \alpha^2 + 6\Delta$ , we get  $\gamma^2 = -A_0\alpha^2$ . Hence  $\gamma = \alpha = 0$ . Then (9) yields  $\beta = 3\Delta$ , and (10) gives  $9\Delta^2 + 2(-A_0) = \Delta^2 - 2A_0$ , i.e.  $8\Delta^2 = 0$ , which contradicts  $\Delta \neq 0$ . Therefore necessarily

$$\boxed{\delta = +A_0}.$$

**Step 2: a convenient reparametrization.** Put  $m := au$ , so  $A_0 = m^2$ . With  $\delta = A_0 = m^2$ , (11) implies

$$\gamma^2 = m^2(2\beta + 6\Delta),$$

hence  $m \mid \gamma$ . Write  $\gamma = m\kappa$  with  $\kappa \in \mathbb{Z}$ . Using (9) (i.e.  $2\beta = \alpha^2 + 6\Delta$ ) we obtain

$$\boxed{\kappa^2 = \alpha^2 + 12\Delta}. \quad (13)$$

Introduce

$$s := \kappa + \alpha, \quad t := \kappa - \alpha \quad (\text{so } s, t \in \mathbb{Z}, \quad s + t = 2\kappa, \quad s - t = 2\alpha).$$

Then from (13)

$$st = \kappa^2 - \alpha^2 = 12\Delta. \quad (\dagger)$$

In terms of  $s, t$  one readily checks that

$$\begin{aligned} \beta &= \frac{\alpha^2 + 6\Delta}{2} = \frac{(s-t)^2}{8} + \frac{st}{4} = \boxed{\frac{s^2 + t^2}{8}}, \\ \alpha\gamma &= m\alpha\kappa = m \frac{(s+t)(s-t)}{4} = \boxed{m \frac{s^2 - t^2}{4}}. \end{aligned} \quad (14)$$

**Step 3: eliminating  $\alpha, \beta, \gamma$  from (10).** Substitute (14) and  $\delta = m^2$  into (10):

$$\left(\frac{s^2 + t^2}{8}\right)^2 + 2m^2 - 2 \cdot m \frac{s^2 - t^2}{4} = \Delta^2 - 2m^2.$$

Multiply by  $576 = \text{lcm}(64, 2, 144)$  and use  $(\dagger)$ , i.e.  $\Delta^2 = (st)^2/144$ , to clear denominators:

$$9(s^2 + t^2)^2 - 288m(s^2 - t^2) + 2304m^2 = 4s^2t^2.$$

Rearranging,

$$9(s^2 + t^2)^2 - 288m(s^2 - t^2) + 2304m^2 - 4s^2t^2 = 0. \quad (15)$$

Set  $U := s^2$ ,  $V := t^2$  (nonnegative integers). Then (15) becomes

$$9U^2 + 14UV + 9V^2 - 288mU + 288mV + 2304m^2 = 0.$$

Completing the square gives an identity

$$(3U - 3V - 48m)^2 + 32UV = 0.$$

Therefore both terms vanish:

$$UV = 0 \quad \text{and} \quad 3U - 3V - 48m = 0.$$

The first equality  $UV = 0$  means  $st = 0$ , hence by  $(\dagger)$  we get  $\Delta = 0$ , which contradicts our standing assumption  $\Delta \neq 0$ .

**Conclusion.** Thus the system (9)–(12) has no integer solutions when  $\Delta \neq 0$ . Equivalently, the factorization  $P_{a,u}(t) = F(t)F(-t)$  with a monic quartic  $F \in \mathbb{Z}[t]$  is impossible.

**Theorem 2** (Case (C) is impossible). *For coprime integers  $a \neq u > 0$  (so  $\Delta = u^2 - a^2 \neq 0$ ), there are no integers  $\alpha, \beta, \gamma, \delta$  with  $\delta^2 = A_0^2$  such that*

$$P_{a,u}(t) = (t^4 + \alpha t^3 + \beta t^2 + \gamma t + \delta) (t^4 - \alpha t^3 + \beta t^2 - \gamma t + \delta).$$

*In particular, no 4+4 factorization of type (C) (conjugate pair) exists.*

*Remark 2.* This argument is independent of the analysis in the even–even case (E) and does not use any auxiliary factorization of the elimination polynomial. It relies only on (9)–(12), the sign determination  $\delta = A_0$ , the reparametrization  $(s, t)$  given by  $\kappa^2 = \alpha^2 + 12\Delta$ , and the elementary identity

$$(3s^2 - 3t^2 - 48m)^2 + 32s^2t^2 = 0,$$

which forces  $st = 0$ , hence  $\Delta = 0$ , a contradiction.

**Theorem 3** (Necessary condition for 4+4). *Let  $\Delta = u^2 - a^2 \neq 0$ . If  $P_{a,u}(t)$  factors in  $\mathbb{Z}[t]$  as a product of two monic quartics, then there exists  $X \in \mathbb{Z}$  satisfying  $(\star)$ .*

*Proof.* By Lemma 1 any 4+4 factorization is of type (E) or (C). By Theorem 2 case (C) is excluded; hence we are in (E):  $F = t^4 + pt^2 + q$ ,  $G = t^4 + rt^2 + s$ . As shown in the derivation of  $(\star)$ , setting  $X := p - 3\Delta$  and eliminating  $q, s$  via (1)–(4) yields precisely  $(\star)$ .  $\square$

### 3 Key Lemma: $\gcd(X, \Delta) = 1$

**Lemma 2.** *If  $X \in \mathbb{Z}$  satisfies  $(\star)$ , then  $\gcd(X, \Delta) = 1$ .*

*Proof.* Suppose, to the contrary, that a prime  $p$  divides both  $X$  and  $\Delta$  [7, 9, 10, 11, 12]. Write

$$X = p^x X_0, \quad \Delta = p^d \Delta_0, \quad x, d \geq 1, \quad \gcd(X_0, p) = \gcd(\Delta_0, p) = 1.$$

Case  $p \geq 3$ . As usual:

$$\begin{aligned} X^2 - 8\Delta^2 &= p^{2x} (X_0^2 - 8p^{2(d-x)} \Delta_0^2), \\ X^2 - 9\Delta^2 &= p^{2x} (X_0^2 - 9p^{2(d-x)} \Delta_0^2). \end{aligned}$$

If  $d > x$ , both brackets are  $\not\equiv 0 \pmod{p}$ , and  $\nu_p(\text{LHS}) = 4x$ . The right-hand side has  $\nu_p(\text{RHS}) = 2x + \nu_p(4a^2u^2) = 2x$  (since  $\gcd(a, u) = 1 \Rightarrow p \nmid$

$au$ ). Contradiction. If  $d = x$ , the two brackets cannot both be divisible by  $p$  (otherwise  $\Delta_0^2 \equiv 0$ ), hence  $\nu_p(\text{LHS}) \geq 4x + 1 > 2x = \nu_p(\text{RHS})$ . Contradiction [9, 11].

Addendum: odd prime  $p$ , the hypothetical subcase  $d < x$ . For completeness, suppose  $p$  is an odd prime with  $p \mid \Delta$  and  $x := \nu_p(X) > d := \nu_p(\Delta) \geq 1$ . Then one necessarily has

$$\nu_p(X^2 - 8\Delta^2) = 2d, \quad \nu_p(X^2 - 9\Delta^2) = 2d,$$

so that

$$\nu_p((X^2 - 8\Delta^2)(X^2 - 9\Delta^2)) = 4d.$$

On the right-hand side of  $(\star)$  we have  $\nu_p(4a^2u^2X^2) = 2x$  because  $p \mid (u^2 - a^2)$  implies  $p \nmid a$  and  $p \nmid u$ . Hence  $4d = 2x$  and therefore

$$x = 2d. \tag{16}$$

Cancelling  $p^{4d}$  in  $(\star)$  yields

$$(p^{2(x-d)}X_0^2 - 8\Delta_0^2)(p^{2(x-d)}X_0^2 - 9\Delta_0^2) = 4a^2u^2X_0^2,$$

and with (16) this becomes

$$(p^{2d}X_0^2 - 8\Delta_0^2)(p^{2d}X_0^2 - 9\Delta_0^2) = 4a^2u^2X_0^2.$$

Reducing modulo  $p$  (since  $d \geq 1$ ) gives

$$(-8\Delta_0^2) \cdot (-9\Delta_0^2) \equiv 4a^2u^2X_0^2 \pmod{p},$$

i.e.

$$72\Delta_0^4 \equiv 4a^2u^2X_0^2 \pmod{p} \iff 18 \equiv \left(\frac{auX_0}{\Delta_0^2}\right)^2 \pmod{p}. \tag{17}$$

Thus 18 must be a quadratic residue modulo  $p$ . Since  $\left(\frac{3^2}{p}\right) = 1$ , this is equivalent to

$$\left(\frac{18}{p}\right) = \left(\frac{2}{p}\right) = 1,$$

and by the classical description of  $(2/p)$  via quadratic reciprocity (see, e.g., [4]) we obtain

$$p \equiv 1 \text{ or } 7 \pmod{8}. \tag{18}$$

Write  $X_0 = h\xi$  and  $\Delta_0 = h\Delta_1$  with  $h := \gcd(X_0, \Delta_0)$  and  $\gcd(\xi, \Delta_1) = 1$ . Set

$$A' := p^{2d}\xi^2 - 8\Delta_1^2, \quad B' := p^{2d}\xi^2 - 9\Delta_1^2.$$

From the computation of  $\gcd(A, B)$  in the main text we have  $\gcd(A', B') = 1$  and

$$A'B' = \left( \frac{2au\xi}{h} \right)^2.$$

Since  $A'B' \in \mathbb{Z}$ , it follows that  $\frac{2au\xi}{h} \in \mathbb{Z}$ . Hence there exist  $\varepsilon \in \{\pm 1\}$  and coprime integers  $m, n$  such that

$$A' = \varepsilon m^2, \quad B' = \varepsilon n^2, \quad \gcd(m, n) = 1. \quad (19)$$

*Claim (sign determination).*  $\varepsilon = +1$ .

*Proof.* Reduce (19) modulo 3. Since  $p \geq 5$ ,  $p^{2d} \equiv 1 \pmod{3}$ , whence

$$B' \equiv \xi^2 \pmod{3}, \quad A' \equiv \xi^2 - 2\Delta_1^2 \equiv \xi^2 + \Delta_1^2 \pmod{3}.$$

If  $\varepsilon = -1$ , then  $A' = -m^2$  and  $B' = -n^2$ , so  $A', B' \in \{0, 2\} \pmod{3}$ . From  $B' \equiv \xi^2$  it follows that  $\xi \equiv 0 \pmod{3}$ , and then  $A' \equiv -2\Delta_1^2 \equiv \Delta_1^2 \pmod{3}$  forces  $\Delta_1 \equiv 0 \pmod{3}$ , which in turn implies  $3 \mid m$  and  $3 \mid n$  — a contradiction to  $\gcd(m, n) = 1$ .  $\square$

With  $\varepsilon = +1$ , reducing  $B' = n^2$  modulo  $p$  gives  $n^2 \equiv -9\Delta_1^2 \pmod{p}$ , hence  $(-1/p) = 1$  and therefore  $p \equiv 1 \pmod{4}$ . Together with (18) (i.e.  $(2/p) = 1$ ) this forces the sharper congruence

$$p \equiv 1 \pmod{8}. \quad (20)$$

In particular, the branch  $p \equiv 7 \pmod{8}$  is excluded.

*Residual subcase and current status.* In the remaining configuration  $p \equiv 1 \pmod{8}$  one arrives at the system

$$m^2 = p^{2d}\xi^2 - 8\Delta_1^2, \quad n^2 = p^{2d}\xi^2 - 9\Delta_1^2, \quad m^2 - n^2 = \Delta_1^2,$$

with  $\gcd(m, n) = 1$  and  $\gcd(\xi, \Delta_1) = 1$ . Using the standard factorizations  $(m \mp n)$  and the corresponding parameterizations in the odd/even parity of  $\Delta_1$ , one checks that both identities for  $p^{2d}\xi^2$  reduce to the same expression; i.e., by the present (elementary) methods this residual case does not yield a contradiction.

Retain the residual odd-prime setting  $p \geq 5$ ,  $p \mid \Delta$ ,  $d := \nu_p(\Delta) \geq 1$ ,  $x := \nu_p(X) > d$ , for which  $x = 2d$ . As explained earlier, after clearing common factors the system

$$m^2 = p^{2d}\xi^2 - 8\Delta_1^2, \quad n^2 = p^{2d}\xi^2 - 9\Delta_1^2, \quad \gcd(\xi, p\Delta_1) = 1,$$

yields the genus-1 curve

$$\mathcal{C} : \begin{cases} m^2 = u^2 - 8w^2, \\ n^2 = u^2 - 9w^2, \end{cases} \quad (m : n : w : u) \in \mathbb{P}^3,$$



and the pencil of quadrics shows that  $\text{Jac}(\mathcal{C})$  is the elliptic curve

$$E_0 : y^2 = x(x+1)(x+9). \quad (21)$$

Moreover, the extra constraint coming from the residual system is precisely that the  $x$ -coordinate on  $E_0$  be a rational square: writing  $u := n/\Delta_1$ , the two congruences “ $u^2 + 1$  and  $u^2 + 9$  are squares” translate to

$$(x, y) \in E_0(\mathbb{Q}) \quad \text{with} \quad x = u^2 \in (\mathbb{Q}^\times)^2.$$

Put differently, we need to decide whether  $E_0(\mathbb{Q})$  contains a point with  $x$  a nonzero square. We now compute  $E_0(\mathbb{Q})$  unconditionally.

**Proposition 1** (Torsion subgroup). *For the elliptic curve*

$$E_0 : y^2 = x(x+1)(x+9),$$

*the torsion subgroup is*

$$\begin{aligned} E_0(\mathbb{Q})_{\text{tors}} &= \left\{ O, (0, 0), (-1, 0), (-9, 0), (3, \pm 12), (-3, \pm 6) \right\} \\ &\simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}. \end{aligned}$$

*In particular,  $2(3, 12) = (0, 0)$  and  $2(-3, 6) = (-1, 0)$ , so  $(3, 12)$  and  $(-3, 6)$  are points of order 4.*

*Proof.* By the Nagell–Lutz theorem [17, 18], all torsion points on a minimal integral model have integer coordinates. The three nontrivial 2-torsion points are the roots of the cubic:  $x \in \{0, -1, -9\}$ , i.e.  $(0, 0), (-1, 0), (-9, 0)$ .

A direct substitution shows that  $(3, \pm 12)$  and  $(-3, \pm 6)$  lie on  $E_0$  because  $12^2 = 3 \cdot 4 \cdot 12 = 144$  and  $6^2 = (-3) \cdot (-2) \cdot 6 = 36$ . Using the duplication formula (or standard software/hand computation) one verifies that  $2(3, 12) = (0, 0)$  and  $2(-3, 6) = (-1, 0)$ , hence these points have order 4. No further integral torsion points exist, so by Mazur’s theorem the torsion subgroup is exactly  $\{O, (0, 0), (-1, 0), (-9, 0), (3, \pm 12), (-3, \pm 6)\} \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$ .  $\square$

**Theorem 4** (Rank via a minimal model). *For*

$$E_0 : y^2 = x(x+1)(x+9)$$

*we have*

$$E_0(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad \text{rank } E_0(\mathbb{Q}) = 0.$$

*Proof.* First remove the quadratic term in the Weierstrass equation: with the change of variables  $x = X - \frac{10}{3}$  we obtain

$$y^2 = X^3 - \frac{73}{3}X + \frac{1190}{27}.$$

Clearing denominators via  $X = \frac{x'}{9}$ ,  $y = \frac{y'}{27}$  gives the short integral model

$$y'^2 = x'^3 - 1971x' + 32130.$$

This curve is  $\mathbb{Q}$ -isomorphic to the minimal model

$$E : y^2 = x^3 + x^2 - 24x + 36,$$

which has conductor  $N = 48$  and belongs to the isogeny class 48a. From Cremona's tables and the LMFDB we have

$$E(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \quad \text{rank } E(\mathbb{Q}) = 0.$$

Hence the same holds for  $E_0$ . See [19, 20].  $\square$

**Corollary 1** (No rational points with square  $x$ ). *The only rational point of  $E_0$  with  $x$  a rational square is  $(x, y) = (0, 0)$ .*

*Proof.* By Proposition 1 and Theorem 4,  $E_0(\mathbb{Q})$  is exactly the listed torsion set. Inspecting their  $x$ -coordinates  $\{0, -1, -9, \pm 3\}$  shows that the only square among them is  $x = 0$ .  $\square$

**Theorem 5** (Unconditional closure of the residual odd-prime branch). *In the residual configuration ( $p \geq 5$ ,  $p \mid \Delta$ ,  $x = 2d > d \geq 1$ ) the system above has no nontrivial integer solutions (i.e. no solutions with  $n \neq 0$ ). Equivalently, the subcase  $d < x$  cannot occur.*

*Proof.* A nontrivial solution forces a rational point on  $E_0$  with  $x = (n/\Delta_1)^2$  a nonzero square. By Corollary 1 this is impossible.  $\square$

Hence no odd prime  $p$  can divide both  $X$  and  $\Delta$ .

Case  $p = 2$ . Write  $X = 2^x X_0$ ,  $\Delta = 2^d \Delta_0$ ,  $x, d \geq 1$ ,  $X_0, \Delta_0$  odd.

Consider three mutually exclusive options:

(B)  $2x > 2d$ .

If  $x \geq d+2$  (i.e.  $2x \geq 2d+4$ ), then  $\nu_2(X^2 - 8\Delta^2) = 2d+3$ ,  $\nu_2(X^2 - 9\Delta^2) = 2d$ , hence  $\nu_2(\text{LHS}) = 4d + 3$  (odd), whereas  $\nu_2(\text{RHS}) = 2 + \nu_2(a^2 u^2) + 2x$  is even. Contradiction.

If  $x = d + 1$  (i.e.  $2x = 2d + 2$ ), then

$$X^2 - 8\Delta^2 = 2^{2d}(4X_0^2 - 8\Delta_0^2) = 2^{2d+2}(X_0^2 - 2\Delta_0^2),$$

where the bracket is odd; thus  $\nu_2(X^2 - 8\Delta^2) = 2d + 2$ . Moreover,

$$X^2 - 9\Delta^2 = 2^{2d}(4X_0^2 - 9\Delta_0^2),$$

and  $4X_0^2 - 9\Delta_0^2 \equiv 4 - 9 \equiv 3 \pmod{8}$  is odd, hence  $\nu_2(X^2 - 9\Delta^2) = 2d$ . Therefore  $\nu_2(\text{LHS}) = (2d + 2) + 2d = 4d + 2$ .

Since  $\nu_2(\Delta) \geq 1$ , the numbers  $a$  and  $u$  have the same parity; with  $\gcd(a, u) = 1$  this forces both to be odd. Then  $\nu_2(a^2u^2) = 0$  and

$$\nu_2(\text{RHS}) = \nu_2(4a^2u^2X^2) = 2 + 0 + 2x = 2 + 2(d + 1) = 2d + 4.$$

Comparing, for  $d \geq 2$  we have  $4d + 2 \neq 2d + 4$  (contradiction), while for  $d = 1$  the valuations coincide and we must compare odd parts. Modulo 8:

$$\frac{X^2 - 8\Delta^2}{2^4} \cdot \frac{X^2 - 9\Delta^2}{2^2} = (X_0^2 - 2\Delta_0^2)(4X_0^2 - 9\Delta_0^2) \equiv 7 \cdot 3 \equiv 5 \pmod{8},$$

whereas the odd part of the right-hand side is  $X_0^2 \equiv 1 \pmod{8}$ . Contradiction. Hence the subcase  $x = d + 1$  is impossible.

(C)  $2x = 2d$ . Then  $\nu_2(X^2 - 8\Delta^2) = 2d$  and  $\nu_2(X^2 - 9\Delta^2) \geq 2d + 3$  (since  $X_0^2 \equiv 1 \pmod{8}$ ). Thus  $\nu_2(\text{LHS}) \geq 4d + 3$  (odd), whereas  $\nu_2(\text{RHS}) = 2 + \nu_2(a^2u^2) + 2x$  is even. Contradiction.

(A)  $p = 2$  and  $x < d$ .

Assume  $2 \mid \gcd(X, \Delta)$ . Write  $X = 2^x X_0$  and  $\Delta = 2^d \Delta_0$  with  $x \geq 1$ ,  $d > x$ , and  $X_0, \Delta_0$  odd. Since  $2 \mid \Delta$  and  $\gcd(a, u) = 1$ , both  $a$  and  $u$  are odd.

Step 1: Valuation at 2. Comparing 2-adic valuations in  $(\star)$  gives

$$\nu_2(\text{LHS}) = 4x, \quad \nu_2(\text{RHS}) = 2x + 2.$$

Hence  $4x = 2x + 2$  and therefore

$$x = 1, \quad d \geq 2. \tag{22}$$

Step 2: Normalization and the product identity. Set

$$M := 2^{2d-2}\Delta_0^2, \quad A := X_0^2 - 8M, \quad B := X_0^2 - 9M.$$

Dividing  $(\star)$  by 16 (using (22)) yields

$$A \cdot B = (auX_0)^2. \tag{23}$$

As  $X_0$  is odd,

$$\gcd(A, B) = \gcd(X_0^2 - 8M, X_0^2 - 9M) = \gcd(X_0^2, M) = \gcd(X_0^2, \Delta_0^2) =: g.$$

Let  $h := \gcd(X_0, \Delta_0)$ ; then  $g = h^2$  is an odd perfect square.

Step 3: Sign determination by residue modulo 8. Since  $8M \equiv 0 \pmod{8}$ , we have

$$A \equiv X_0^2 \equiv 1 \pmod{8}.$$

Write, for some  $\varepsilon \in \{\pm 1\}$  and coprime integers  $m, n \geq 0$ ,

$$A = \varepsilon g m^2, \quad B = \varepsilon g n^2, \quad (24)$$

(this follows from (23) and  $\gcd(A/g, B/g) = 1$ ). Reducing the first identity in (24) modulo 8 and using  $g \equiv 1 \pmod{8}$  gives

$$1 \equiv A \equiv \varepsilon g m^2 \equiv \varepsilon \pmod{8}.$$

Hence

$$\varepsilon = +1, \quad \text{i.e.} \quad A = g m^2. \quad (25)$$

Moreover,

$$B \equiv X_0^2 - 9M \equiv \begin{cases} 1 - 4 \equiv 5 \pmod{8}, & d = 2, \\ 1 - 0 \equiv 1 \pmod{8}, & d \geq 3, \end{cases}$$

since  $M \equiv 4 \pmod{8}$  for  $d = 2$  and  $M \equiv 0 \pmod{8}$  for  $d \geq 3$ . But by (24), (25) we must have  $B \equiv g n^2 \equiv 1 \pmod{8}$ . Therefore the case  $d = 2$  is impossible, and we are left with

$$d \geq 3, \quad A = g m^2, \quad B = g n^2. \quad (26)$$

In particular,  $m, n$  are odd (because  $A \equiv B \equiv 1 \pmod{8}$  and  $g \equiv 1 \pmod{8}$ ), and  $\gcd(m, n) = 1$ .

Step 4: Two Diophantine consequences. From  $A - B = M$  and (26) we obtain

$$g(m^2 - n^2) = M = 2^{2d-2} \Delta_0^2. \quad (27)$$

Write  $\Delta_0 = h \Delta_1$  (recall  $g = h^2$ ). Then (27) becomes

$$m^2 - n^2 = 2^{2d-2} \Delta_1^2. \quad (28)$$

Using  $9A - 8B = X_0^2$  we also get

$$g(9m^2 - 8n^2) = X_0^2 \implies 9m^2 - 8n^2 = k^2 \quad (29)$$

for some odd integer  $k$ .

Step 5: Final contradiction via the binary form  $x^2 + 2y^2$ . From (29) we have

$$(3m)^2 = k^2 + 8n^2. \quad (30)$$

Let  $D := \gcd(k, n)$ . From (30) one checks that  $D$  is odd and  $D \mid 3m$ ; hence  $D = 3^j$  with  $j \in \{0, 1\}$  (otherwise  $\gcd(m, n) \neq 1$ ).

*Case  $j = 0$  (primitive).* Then there exist coprime integers  $s, t$  with

$$3m = s^2 + 2t^2, \quad n = st, \quad k = \pm(s^2 - 2t^2),$$

and  $s, t$  are odd because  $n$  is odd; see, e.g., [9, Ch. 5, §2]. Using (28) we compute

$$m^2 - n^2 = \frac{(s^2 + 2t^2)^2}{9} - s^2 t^2 = \frac{(s-t)(s+t)(s-2t)(s+2t)}{9} = 2^{2d-2} \Delta_1^2.$$

Thus

$$(s-t)(s+t) \cdot (s-2t)(s+2t) = 9 \cdot 2^{2d-2} \Delta_1^2. \quad (31)$$

Here  $s \pm t$  are even, while  $s \pm 2t$  are odd; also  $\gcd(s-2t, s+2t) = \gcd(s-2t, 4t) = 1$ . Hence the odd part of (31) equals  $\pm 9\Delta_1^2$  and, by coprimeness, up to signs

$$s-2t = A^2, \quad s+2t = 9B^2 \quad \text{or} \quad s-2t = -A^2, \quad s+2t = 9B^2,$$

for some odd  $A, B$ . In the first subcase,

$$4t = (s+2t) - (s-2t) = 9B^2 - A^2 = (3B-A)(3B+A).$$

Both factors are even, and exactly one of them is divisible by 4; thus  $\nu_2(9B^2 - A^2) \geq 3$ , which contradicts  $\nu_2(4t) = 2$ . In the second subcase,

$$4t = 9B^2 + A^2 \equiv 1 + 1 \equiv 2 \pmod{4},$$

so  $\nu_2(4t) = 1$ , again a contradiction.

*Case  $j = 1$  (non-primitive).* Then there exist coprime odd  $s, t$  such that

$$m = s^2 + 2t^2, \quad n = 3st, \quad k = \pm 3(s^2 - 2t^2),$$

and

$$m^2 - n^2 = (s-t)(s+t)(s-2t)(s+2t) = 2^{2d-2} \Delta_1^2.$$

As above, from the odd part we get (up to signs)

$$s-2t = A^2, \quad s+2t = B^2 \quad \text{or} \quad s-2t = -A^2, \quad s+2t = B^2,$$

with odd  $A, B$ . Hence

$$4t = B^2 - A^2 = (B-A)(B+A) \quad \text{or} \quad 4t = B^2 + A^2.$$

In the first case  $\nu_2(B^2 - A^2) \geq 3$ , contradicting  $\nu_2(4t) = 2$ ; in the second case  $B^2 + A^2 \equiv 2 \pmod{4}$ , so  $\nu_2(4t) = 1$ , again a contradiction.

In all subcases we reach a contradiction. Therefore the subcase  $p = 2$  with  $x < d$  is impossible.

Addendum: the odd prime  $p = 3$  with  $d < x$ . For completeness we treat the remaining subcase  $p = 3$  under the standing assumption that  $p \mid X$  and  $p \mid \Delta$ . Write  $X = 3^x X_0$ ,  $\Delta = 3^d \Delta_0$  with  $x > d \geq 1$  and  $\gcd(X_0, 3) = \gcd(\Delta_0, 3) = 1$ . Then

$$X^2 - 8\Delta^2 = 3^{2d} \left( 3^{2(x-d)} X_0^2 - 8 \Delta_0^2 \right), \quad X^2 - 9\Delta^2 = 3^{2d} \left( 3^{2(x-d)} X_0^2 - 9 \Delta_0^2 \right).$$

Hence

$$\begin{aligned} \nu_3(X^2 - 8\Delta^2) &= 2d, \\ \nu_3(X^2 - 9\Delta^2) &= \nu_3((X - 3\Delta)(X + 3\Delta)) = (d + 1) + (d + 1) = 2d + 2. \end{aligned}$$

since  $x > d$  implies  $\nu_3(X \pm 3\Delta) = d + 1$ . Therefore

$$\nu_3(\text{LHS of } (\star)) = 4d + 2, \quad \nu_3(\text{RHS of } (\star)) = 2x,$$

because  $\gcd(a, u) = 1$  and  $3 \mid \Delta = u^2 - a^2$  force  $3 \nmid au$ . Thus  $4d + 2 = 2x$ , i.e.  $x = 2d + 1$ .

Divide  $(\star)$  by  $3^{4d+2}$  and reduce modulo 3:

$$\begin{aligned} \frac{X^2 - 8\Delta^2}{3^{2d}} \cdot \frac{X^2 - 9\Delta^2}{3^{2d+2}} &= 4a^2u^2 \cdot \frac{X^2}{3^{4d+2}} \\ \implies (-8\Delta_0^2)(-\Delta_0^2) &\equiv 4a^2u^2X_0^2 \pmod{3}. \end{aligned} \tag{32}$$

As  $a, u, X_0, \Delta_0$  are all coprime to 3, their squares are 1 mod 3. Hence  $8 \cdot 1 \equiv 1 \cdot 1 \pmod{3}$ , i.e.  $2 \equiv 1 \pmod{3}$ , a contradiction. Therefore the configuration  $p = 3$  with  $d < x$  is impossible as well.

In all cases we reach a contradiction. Hence no prime  $p$  can divide both  $X$  and  $\Delta$ , i.e.  $\gcd(X, \Delta) = 1$ .  $\square$

**Corollary 2.** *If  $2 \mid \Delta$ , then  $2 \nmid X$ . If  $3 \mid \Delta$ , then  $3 \nmid X$ .*

## 4 Complete Case Split by Divisibility of $au$ by 3 and by Parity

Set  $A_0 := a^2u^2$  (this is *not*  $A = 6\Delta$ ). We now work solely with equation  $(\star)$ .

### Branch I: $3 \mid au$ — impossible

With  $\gcd(a, u) = 1$ , exactly one of  $a, u$  is divisible by 3, hence  $\Delta = u^2 - a^2 \equiv \pm 1 \pmod{3}$ , i.e.  $3 \nmid \Delta$ .

*Subcase  $3 \nmid X$ .* Then  $X^2 \equiv 1 \pmod{3}$ , and  $\Delta^2 \equiv 1 \pmod{3}$ , therefore

$$X^2 - 8\Delta^2 \equiv 1 - 2 \equiv 2 \pmod{3}, \quad X^2 - 9\Delta^2 \equiv 1 - 0 \equiv 1 \pmod{3},$$

and  $\nu_3(\text{LHS}) = 0$ . On the other hand,  $\nu_3(\text{RHS}) = \nu_3(4A_0) = 2\nu_3(au) \geq 2$ . Contradiction.

*Subcase  $3 \mid X$ .* Let  $x := \nu_3(X) \geq 1$  and set  $k := \nu_3(au) \geq 1$  (since  $\gcd(a, u) = 1$  and  $3 \mid au$ , exactly one of  $a, u$  is divisible by 3). Then

$$\Delta = u^2 - a^2 \equiv \pm 1 \pmod{3} \quad \text{and} \quad \nu_3(\Delta) = 0.$$

We compute the 3-adic valuations of the two factors on the left of  $(\star)$ :

First factor. Because  $\Delta$  is a 3-adic unit and  $8 \equiv -1 \pmod{3}$ ,

$$X^2 - 8\Delta^2 \equiv 0 - (-1) \equiv 1 \pmod{3},$$

so

$$\nu_3(X^2 - 8\Delta^2) = 0. \tag{33}$$

Second factor. Write

$$X^2 - 9\Delta^2 = (X - 3\Delta)(X + 3\Delta).$$

Since  $\nu_3(X) = x \geq 1$  and  $\nu_3(3\Delta) = 1$ , for  $x \geq 2$  we have

$$X \pm 3\Delta = 3(3^{x-1}X_0 \pm \Delta) \quad \text{with} \quad 3 \nmid (3^{x-1}X_0 \pm \Delta),$$

hence

$$\text{if } x \geq 2: \quad \nu_3(X \pm 3\Delta) = 1 \quad \text{and} \quad \nu_3(X^2 - 9\Delta^2) = 2. \tag{34}$$

If  $x = 1$ , then

$$X \pm 3\Delta = 3(X_0 \pm \Delta), \quad X_0, \Delta \text{ are 3-adic units.}$$

At most one of  $X_0 \pm \Delta$  is divisible by 3 (since  $(X_0 + \Delta) - (X_0 - \Delta) = 2\Delta$  is not divisible by 3). Therefore

$$\text{if } x = 1: \quad \nu_3(X^2 - 9\Delta^2) = \nu_3(X - 3\Delta) + \nu_3(X + 3\Delta) = 2 + r, \tag{35}$$

for some integer  $r \geq 0$ .

Comparison with the right-hand side. From  $(\star)$  and (33) we get

$$\nu_3(\text{LHS}) = \nu_3(X^2 - 9\Delta^2).$$

On the right,

$$\nu_3(\text{RHS}) = \nu_3(4a^2u^2X^2) = 2\nu_3(au) + 2x = 2k + 2x.$$

If  $x \geq 2$ , then by (34) we have  $\nu_3(\text{LHS}) = 2$ , whereas  $\nu_3(\text{RHS}) = 2k + 2x \geq 2 \cdot 1 + 2 \cdot 2 = 6$ , which is impossible.

If  $x = 1$ , then by (35) and equality of valuations we must have

$$2 + r = \nu_3(\text{LHS}) = \nu_3(\text{RHS}) = 2k + 2,$$

hence

$$x = 1 \quad \text{and} \quad r = 2k. \quad (36)$$

Equivalently,

$$\nu_3(X^2 - 9\Delta^2) = 2k + 2 \iff \nu_3(X_0^2 - \Delta^2) = 2k,$$

i.e.  $X_0^2 \equiv \Delta^2 \pmod{3^{2k}}$  but  $X_0^2 \not\equiv \Delta^2 \pmod{3^{2k+1}}$ .

**Lemma 3** (The edge case  $x = 1$ ,  $r = 2k$  is impossible). *Assume  $\gcd(a, u) = 1$ ,  $\Delta := u^2 - a^2 \neq 0$ , and  $A_0 = a^2u^2$ . If  $\nu_3(au) = k \geq 1$ ,  $\nu_3(X) = 1$ , and  $\nu_3((X/3)^2 - \Delta^2) = 2k$  (equivalently,  $x = 1$  and  $r = 2k$  in (36)), then  $(\star)$  has no integer solutions.*

*Proof.* By Lemma 2 we have  $\gcd(X, \Delta) = 1$ . Reducing  $(\star)$  modulo  $X$  gives

$$(-8\Delta^2)(-9\Delta^2) \equiv 0 \pmod{X} \implies 72\Delta^4 \equiv 0 \pmod{X},$$

hence  $\Delta$  is invertible modulo  $X$  and  $X \mid 72$ . Since  $\nu_3(X) = 1$ , necessarily  $X \in \{\pm 3, \pm 6, \pm 12, \pm 24\}$ .

(i) *Both  $a, u$  odd.* Then  $u \pm a$  are even with one of them divisible by 4, so  $\nu_2(\Delta) = \nu_2(u - a) + \nu_2(u + a) \geq 3$ , hence  $\Delta^2 \equiv 0 \pmod{16}$ . From  $\gcd(X, \Delta) = 1$  we get  $2 \nmid X$ , i.e.  $X$  is odd. Therefore  $\nu_2(X^2 - 8\Delta^2) = \nu_2(X^2 - 9\Delta^2) = 0$ , so  $\nu_2(\text{LHS}) = 0$ , whereas  $\nu_2(\text{RHS}) = \nu_2(4A_0X^2) = 2$  (since  $A_0$  and  $X$  are odd) — a contradiction.

(ii)  *$a, u$  of opposite parity.* Then  $\Delta$  is odd and  $\nu_2(A_0) \geq 2$ . For even  $X$  (i.e.  $X \in \{\pm 6, \pm 12, \pm 24\}$ ):

$$\nu_2(X^2 - 8\Delta^2) = \begin{cases} 2, & \nu_2(X) = 1, \\ 3, & \nu_2(X) \geq 2, \end{cases} \quad \nu_2(X^2 - 9\Delta^2) = 0,$$



thus  $\nu_2(\text{LHS}) \in \{2, 3\}$ , while  $\nu_2(\text{RHS}) = 2 + \nu_2(A_0) + 2\nu_2(X) \geq 2 + 2 + 2 = 6$  — again a contradiction. Hence  $X$  must be odd, i.e.  $X = \pm 3$ .

(iii) *The remaining possibility*  $X = \pm 3$ . Substituting  $X^2 = 9$  into  $(\star)$  and dividing by 9,

$$(1 - \Delta^2)(9 - 8\Delta^2) = (2au)^2.$$

Moreover,

$$\begin{aligned} \gcd(1 - \Delta^2, 9 - 8\Delta^2) &= \gcd(1 - \Delta^2, (9 - 8\Delta^2) - 8(1 - \Delta^2)) \\ &= \gcd(1 - \Delta^2, 1) = 1. \end{aligned}$$

Thus two coprime integers multiply to a square, so each factor is a square up to sign. For  $|\Delta| \geq 2$  both factors are negative, hence they must be negative squares. But  $1 - \Delta^2 = -s^2$  implies  $\Delta^2 - s^2 = 1$ , i.e.  $(\Delta - s)(\Delta + s) = 1$ , whose only integer solutions are  $(\Delta, s) = (\pm 1, 0)$ . For  $\Delta = \pm 1$  the left-hand side above is 0, whereas  $(2au)^2 > 0$ , a contradiction.  $\square$

Consequently, the only 3-adic possibility under  $3 \mid X$ , namely (36), cannot occur. This completes Branch I ( $3 \mid au$ ).

Therefore, when  $3 \mid au$ , equation  $(\star)$  has no solutions.

## Branch II: $3 \nmid au$ — impossible

Here  $a^2 \equiv u^2 \equiv 1 \pmod{3}$ , hence  $\Delta \equiv 0 \pmod{3}$  and, by Corollary 2,  $3 \nmid X$ .

Sub-branch II.1: both  $a, u$  odd. Then  $u \pm a$  are even, with one of the sums divisible by 4; hence

$$\nu_2(\Delta) = \nu_2(u - a) + \nu_2(u + a) \geq 3, \quad \Delta^2 \equiv 0 \pmod{16}.$$

From  $\gcd(X, \Delta) = 1$  it follows that  $2 \nmid X$ , i.e.  $X$  is odd. Compare  $(\star)$  modulo 16:

$$X^2 - 8\Delta^2 \equiv X^2, \quad X^2 - 9\Delta^2 \equiv X^2 \pmod{16}.$$

The left-hand side  $\equiv X^4 \equiv 1 \pmod{16}$ , while the right-hand side  $4A_0X^2 \equiv 4 \pmod{16}$  [8]. Contradiction.

Sub-branch II.2:  $a, u$  of opposite parity. Here  $\Delta$  is odd, while  $\nu_2(A_0) \geq 2$ .

If  $X$  is even with  $\nu_2(X) = 1$ , then  $\nu_2(X^2 - 8\Delta^2) = 2$  and  $\nu_2(X^2 - 9\Delta^2) = 0$ , so  $\nu_2(\text{LHS}) = 2$ , whereas  $\nu_2(\text{RHS}) \geq 6$ . Contradiction.

If  $X$  is even with  $\nu_2(X) \geq 2$ , then  $\nu_2(X^2 - 8\Delta^2) = 3$  and  $\nu_2(X^2 - 9\Delta^2) = 0$ , so  $\nu_2(\text{LHS}) = 3$ , whereas  $\nu_2(\text{RHS}) \geq 8$ . Contradiction.

Case  $X$  odd. Here  $\Delta$  is odd and, since we are in Branch II ( $3 \nmid au$ ), we have  $3 \mid \Delta$ ,  $3 \nmid X$ , and  $\gcd(X, \Delta) = 1$  by Lemma 2. Assume, for a contradiction, that  $(\star)$  holds:

$$(X^2 - 8\Delta^2)(X^2 - 9\Delta^2) = 4A_0X^2, \quad A_0 = a^2u^2.$$

*Step 1: Reduction to  $X = \pm 1$ .* Reducing  $(\star)$  modulo  $X$  gives

$$(-8\Delta^2)(-9\Delta^2) \equiv 0 \pmod{X} \implies 72\Delta^4 \equiv 0 \pmod{X}.$$

Since  $\gcd(X, \Delta) = 1$ , this implies  $X \mid 72$ . As  $X$  is odd and  $3 \nmid X$ , the only possibility is  $X = \pm 1$ .

*Step 2: Excluding the case  $X = \pm 1$ .* With  $X^2 = 1$ , the equation  $(\star)$  becomes

$$(1 - 8\Delta^2)(1 - 9\Delta^2) = 4A_0 = (2au)^2. \quad (37)$$

The right-hand side is a positive perfect square. Let  $Z := 1 - 8\Delta^2$  and  $W := 1 - 9\Delta^2$ . Note that for  $\Delta \neq 0$  both factors  $Z$  and  $W$  are negative integers.

*Substep 2a: Coprimality of the factors.* Using the Euclidean algorithm,

$$\begin{aligned} \gcd(Z, W) &= \gcd(1 - 8\Delta^2, 1 - 9\Delta^2) \\ &= \gcd(1 - 8\Delta^2, -\Delta^2) \\ &= \gcd(1 - 8\Delta^2, \Delta^2). \end{aligned}$$

Since  $(1 - 8\Delta^2) + 8\Delta^2 = 1$ , we have  $\gcd(1 - 8\Delta^2, \Delta^2) = \gcd(1, \Delta^2) = 1$ . Thus  $Z$  and  $W$  are coprime.

*Substep 2b: Consequence for a square product.* In  $\mathbb{Z}$ , if a product of two coprime integers is a perfect square, then each factor is a square up to a unit; see, e.g., [15]. Since  $ZW = (2au)^2 > 0$  and  $Z, W < 0$ , their units must both be  $-1$ ; hence there exist integers  $m, n$  such that

$$Z = -(m^2), \quad W = -(n^2).$$

From  $W = 1 - 9\Delta^2 = -(n^2)$  we get

$$(3\Delta)^2 - n^2 = 1 \iff (3\Delta - n)(3\Delta + n) = 1.$$

The only factorizations of 1 in  $\mathbb{Z}$  are  $1 \cdot 1$  and  $(-1) \cdot (-1)$ . Both cases give  $3\Delta = \pm 1$ , which is impossible for integer  $\Delta$ . (Equivalently, the only integer solutions of  $x^2 - y^2 = 1$  are  $x = \pm 1, y = 0$ .)

Therefore (37) has no solutions, and the case  $X$  odd is impossible in Sub-branch II.2. Combining with the even cases for  $X$  treated above, Sub-branch II.2 is closed.

Thus Branch  $3 \nmid au$  is impossible.

## 5 Completion of the Proof

We have shown that equation  $(\star)$  has no integer solutions  $X$  either when  $3 \mid au$  or when  $3 \nmid au$ . By Theorem 3, any 4+4 factorization yields a solution of  $(\star)$ ; since  $(\star)$  has no integer solutions, a 4+4 factorization is impossible.

**Theorem 6** (Main result). *For any coprime integers  $a \neq u > 0$ , the polynomial  $P_{a,u}(t)$  does not factor in  $\mathbb{Z}[t]$  as a product of two monic polynomials of degree 4.*

## 6 Excluding a 2+6 Factorization: a Direct Criterion and a Discriminant Argument

Recall the notation

$$\begin{aligned} P_{a,u}(t) &= t^8 + At^6 + Bt^4 + Ct^2 + D, & A &= 6\Delta, & \Delta &:= u^2 - a^2 \neq 0, \\ B &= \Delta^2 - 2A_0, & C &= -A_0A, & D &= A_0^2, & A_0 &:= a^2u^2. \end{aligned}$$

Thus  $P_{a,u}$  is even, monic, primitive in  $\mathbb{Z}[t]$  and admits the representation

$$P_{a,u}(t) = Q(t^2), \quad Q(x) := x^4 + Ax^3 + Bx^2 + Cx + D \in \mathbb{Z}[x]. \quad (38)$$

We show that a factorization of type 2+6 is impossible.

### Step 0: Structural split of the class 2+6

Suppose

$$P_{a,u}(t) = Q_2(t) \cdot H_6(t), \quad \deg Q_2 = 2, \deg H_6 = 6.$$

By evenness of  $P_{a,u}$  and the involution  $t \mapsto -t$  (Lemma 1), we have:

If  $Q_2$  is *not* even, then necessarily  $Q_2(-t) \mid H_6(t)$ . Grouping the conjugate factors, we obtain an even quartic and another quartic:

$$P_{a,u}(t) = \underbrace{Q_2(t) Q_2(-t)}_{\text{degree 4, even}} \cdot \underbrace{\frac{H_6(t)}{Q_2(-t)}}_{\text{degree 4}},$$

i.e. the factorization regroups to the case 4+4, which has already been excluded.

Hence the only residue to analyze is the *even* quadratic

$$Q_2(t) = t^2 + q, \quad q \in \mathbb{Z}.$$

We now rule out this last possibility by a direct necessary and sufficient condition plus a discriminant computation.

## Step 1: Criterion for an even quadratic divisor

**Lemma 4** (Even quadratic divisor criterion). *For  $q \in \mathbb{Z}$  we have*

$$(t^2 + q) \mid P_{a,u}(t) \iff Q(-q) = 0,$$

where  $Q$  is as in (38). In other words,

$$(t^2 + q) \mid P_{a,u}(t) \iff q^4 - Aq^3 + Bq^2 - Cq + D = 0.$$

*Proof.* Divide  $Q(x)$  by  $x + q$  in  $\mathbb{Z}[x]$ :  $Q(x) = (x + q)R(x) + S$  with  $R \in \mathbb{Z}[x]$  and a constant remainder  $S = Q(-q)$ . Substituting  $x = t^2$  and using (38) gives

$$P_{a,u}(t) = Q(t^2) = (t^2 + q) R(t^2) + S.$$

Thus  $(t^2 + q) \mid P_{a,u}$  if and only if  $S = 0$ , i.e.  $Q(-q) = 0$ .  $\square$

*Remark 3.* The case  $q = 0$  is automatically impossible: if  $t^2 \mid P_{a,u}(t)$ , then the constant term must vanish, but  $D = A_0^2 = a^4 u^4 > 0$ .

## Step 2: A discriminant obstruction

We rewrite the equality  $Q(-q) = 0$  from Lemma 4 as a quadratic equation in the unknown  $A_0 = a^2 u^2$  while  $\Delta$  and  $q$  are regarded as fixed integers. Using  $A = 6\Delta$ ,  $B = \Delta^2 - 2A_0$ ,  $C = -A_0 A = -6\Delta A_0$ ,  $D = A_0^2$ , we compute

$$\begin{aligned} Q(-q) &= q^4 - Aq^3 + Bq^2 - Cq + D \\ &= q^4 - 6\Delta q^3 + (\Delta^2 - 2A_0)q^2 + 6\Delta A_0 q + A_0^2 \\ &= \underbrace{A_0^2}_{\text{quadratic in } A_0} + \underbrace{(6\Delta q - 2q^2)}_{=:b} A_0 + \underbrace{(\Delta^2 q^2 - 6\Delta q^3 + q^4)}_{=:c}. \end{aligned}$$

Thus  $Q(-q) = 0$  is the quadratic equation in  $A_0$ :

$$A_0^2 + b A_0 + c = 0, \quad b = 6\Delta q - 2q^2, \quad c = \Delta^2 q^2 - 6\Delta q^3 + q^4.$$

Its discriminant with respect to  $A_0$  equals

$$\begin{aligned} \text{Disc}_{A_0} &= b^2 - 4c = (6\Delta q - 2q^2)^2 - 4(\Delta^2 q^2 - 6\Delta q^3 + q^4) \\ &= (36\Delta^2 q^2 - 24\Delta q^3 + 4q^4) - (4\Delta^2 q^2 - 24\Delta q^3 + 4q^4) \\ &= \boxed{32 \Delta^2 q^2}. \end{aligned}$$

**Proposition 2** (Irrationality of the would-be roots). *If  $\Delta \neq 0$  and  $q \neq 0$ , then  $\text{Disc}_{A_0} = 32 \Delta^2 q^2$  is not a perfect square in  $\mathbb{Z}$ .*

*Proof.* We have  $\nu_2(\text{Disc}_{A_0}) = \nu_2(32) + 2\nu_2(\Delta q) = 5 + 2\nu_2(\Delta q)$ , which is odd for all  $\Delta q \neq 0$ . A perfect square in  $\mathbb{Z}$  must have even 2-adic valuation. Hence  $\text{Disc}_{A_0}$  is not a square in  $\mathbb{Z}$ .  $\square$

*Remark 4.* This “odd 2-adic valuation of the discriminant forces non-squareness” obstruction is a standard device in elementary Diophantine arguments; compare also the problem-oriented expositions in [16].

**Corollary 3** (No integer solution for  $A_0$ ). *For  $\Delta \neq 0$  and  $q \neq 0$  the quadratic equation  $A_0^2 + bA_0 + c = 0$  has no solutions  $A_0 \in \mathbb{Z}$ .*

*Proof.* The roots are  $\frac{-b \pm \sqrt{\text{Disc}_{A_0}}}{2}$ ; by Proposition 2 the discriminant is not an integer square, hence the roots are irrational.  $\square$

### Step 3: Conclusion for 2+6

**Theorem 7** (No 2+6 factorization). *Let  $a, u \in \mathbb{Z}_{>0}$  be coprime and  $a \neq u$  (so  $\Delta \neq 0$ ). Then  $P_{a,u}(t)$  does not factor in  $\mathbb{Z}[t]$  as a product of a quadratic and a sextic polynomial.*

*Proof.* As noted above, any 2+6 with a non-even quadratic regroups to a 4+4, which is impossible. Thus it remains to exclude an even quadratic  $t^2 + q$ . By Lemma 4,  $(t^2 + q) \mid P_{a,u}$  iff  $Q(-q) = 0$ . If  $q = 0$ , divisibility by  $t^2$  would force  $D = 0$ , which is false. If  $q \neq 0$ , then by Corollary 3 the equality  $Q(-q) = 0$  has no solutions  $A_0 = a^2 u^2 \in \mathbb{Z}$ . Hence there is no  $q \in \mathbb{Z}$  for which  $t^2 + q$  divides  $P_{a,u}$ . Therefore no 2+6 factorization exists.  $\square$

*Remark 5* (What this uses from previous sections). The proof is logically independent of the 4+4 Diophantine analysis, except for the purely structural observation that a non-even quadratic factor forces regrouping into 4+4 (via pairing  $Q_2(t)$  with its conjugate  $Q_2(-t)$ ). The “hard” residue (even quadratic  $t^2 + q$ ) is completely settled by Lemma 4 and the discriminant computation.

## 7 Excluding other factorizations

After Theorem 7 has ruled out all factorizations of type 2+6, the remaining degree-8 patterns are excluded by trivial regrouping.

**Proposition 3.** *Let  $P_{a,u}(t) \in \mathbb{Z}[t]$  be as above. If any of the following factorizations exists, then  $P_{a,u}$  admits a factorization of type 2+6:*

- (a) 2+2+4:  $P_{a,u} = Q_1 Q_2 H_4$  with  $\deg Q_i = 2$ ,  $\deg H_4 = 4$ ;

(b)  $2+2+2+2$ :  $P_{a,u} = Q_1 Q_2 Q_3 Q_4$  with  $\deg Q_i = 2$ ;

(c)  $3+3+2$ :  $P_{a,u} = F_3 G_3 Q_2$  with  $\deg F_3 = \deg G_3 = 3$ ,  $\deg Q_2 = 2$ .

*Proof.* (a) Group as  $P_{a,u} = \underbrace{Q_1}_{\deg=2} \cdot \underbrace{(Q_2 H_4)}_{\deg=6}$ .

(b) Group as  $P_{a,u} = \underbrace{Q_1}_{\deg=2} \cdot \underbrace{(Q_2 Q_3 Q_4)}_{\deg=6}$ .

(c) Group as  $P_{a,u} = \underbrace{Q_2}_{\deg=2} \cdot \underbrace{(F_3 G_3)}_{\deg=6}$ . □

**Corollary 4.** *None of the patterns  $2+2+4$ ,  $2+2+2+2$ , or  $3+3+2$  can occur for  $P_{a,u}(t)$ .*

*Proof.* By Proposition 3 each would imply a  $2+6$  factorization, which is impossible by Theorem 7. □

## 8 Irreducibility in Full

**Theorem 8** (Irreducibility). *For any coprime integers  $a \neq u > 0$ , the polynomial  $P_{a,u}(t)$  is irreducible in  $\mathbb{Z}[t]$ .*

*Proof.* All degree-8 splittings are excluded as follows.

(i) The case  $4+4$  is impossible by Theorem 3 and the analysis of equation  $(\star)$  (from Lemma 1 to Corollary 2 and the subsequent 2-/3-adic split).

(ii) The case  $2+6$  is excluded in Section 6.

(iii) After (ii), any of the remaining patterns  $2+2+4$ ,  $2+2+2+2$ ,  $3+3+2$  would regroup to  $2+6$  by Proposition 3, hence are impossible by (ii).

Therefore no nontrivial factorization in  $\mathbb{Z}[t]$  exists. Since  $P_{a,u}(t)$  is monic and primitive, irreducibility over  $\mathbb{Z}$  follows. □

## Conclusions

We have shown that for any coprime integers  $a \neq u > 0$  the even cuboid polynomial  $P_{a,u}(t)$  admits no factorization of type  $4+4$  in  $\mathbb{Z}[t]$ . The key step is the reduction of a potential factorization to the Diophantine condition  $(X^2 - 8\Delta^2)(X^2 - 9\Delta^2) = 4a^2u^2X^2$ , from which, using 2- and 3-adic estimates and the lemma  $\gcd(X, \Delta) = 1$ , the absence of integer solutions follows. We then closed the genuine  $2+6$  case via an exact divisor criterion combined with a discriminant obstruction. Finally, after excluding  $2+6$ , any remaining patterns ( $2+2+4$ ,  $2+2+2+2$ ,  $3+3+2$ ) regroup trivially to  $2+6$  and are therefore impossible. Altogether,  $P_{a,u}(t)$  admits no nontrivial factorization in  $\mathbb{Z}[t]$ , establishing irreducibility in full [1, 2, 3].

## References

- [1] R. Sharipov, *Perfect Cuboids and Irreducible Polynomials*, arXiv:1108.5348 [math.NT], 2011.
- [2] R. Sharipov, *A note on a perfect Euler cuboid*, arXiv:1104.1716 [math.NT], 2011.
- [3] R. K. Guy, *Unsolved Problems in Number Theory*, 3rd ed., Springer, 2004.
- [4] G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, 2008.
- [5] D. S. Dummit, R. M. Foote, *Abstract Algebra*, 3rd ed., Wiley, 2004.
- [6] S. Lang, *Algebra*, Rev. 3rd ed., Springer, 2002.
- [7] J. Neukirch, *Algebraic Number Theory*, Springer, 1999.
- [8] J. P. Serre, *A Course in Arithmetic*, Springer GTM 7, 1973.
- [9] K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., Springer GTM 84, 1990.
- [10] N. Koblitz,  *$p$ -adic Numbers,  $p$ -adic Analysis, and Zeta-Functions*, 2nd ed., Springer GTM 58, 1984.
- [11] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, 3rd ed., Springer, 2004.
- [12] D. A. Marcus, *Number Fields*, Springer, Graduate Texts in Mathematics, vol. 197, 1977.
- [13] K. Conrad, *Gauss's Lemma and Unique Factorization in  $\mathbb{Z}$  and  $F[T]$* , Lecture notes, Univ. of Connecticut, c. 2010–2014.
- [14] K. Conrad, *Eisenstein's Criterion*, Lecture notes, c. 2010–2014.
- [15] I. Stewart, D. Tall, *Algebraic Number Theory and Fermat's Last Theorem*, 3rd ed., A K Peters, 2002.
- [16] M. R. Murty, J. Esmonde, *Problems in Algebraic Number Theory*, 2nd ed., Springer GTM 190, 2005.
- [17] J. H. Silverman, *The Arithmetic of Elliptic Curves*, 2nd ed., Springer GTM 106, 2009.
- [18] J. W. S. Cassels, *Lectures on Elliptic Curves*, London Math. Soc. Student Texts 24, CUP, 1991.
- [19] J. E. Cremona, *Elliptic Curves over  $\mathbb{Q}$ : A Database*, online tables for curves over  $\mathbb{Q}$  (isogeny class 48a), updated editions.
- [20] The LMFDB Collaboration, *Elliptic curve 48a3 over  $\mathbb{Q}$  (minimal model  $y^2 = x^3 + x^2 - 24x + 36$ , rank 0, torsion  $\mathbb{Z}/2 \oplus \mathbb{Z}/4$ )*, The L-functions and Modular Forms Database.