Semi-definite relaxations for Optimal Power Flow

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Abstract

The power flow equations regulate the power injections and voltages in an electric power systems. A number of control and optimization problems are based on them, including optimal power flow problem. In this work we are investigating the convex relaxation of that problem using the semi-definite programming.

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1 Introduction to optimal power flow

With the expand of large power transmission systems there is a problem of finding the minimum cost of generators. There are number of limitations to this:

- Different cost of electricity generation, as well as the cost of switching on and off generators;
- Limitation in the maximum output power, as well as in its transmission along the lines (exceeding the border can lead to sagging of the power line, which can have a bad impact on the surrounding space);
- Power generation taking into account losses on branches;
- The voltage consumption must be provided with the least deviation, otherwise it can lead to damage to equipment and machinery;
- It should be possible to redirect the flow in the event of a line failure, failure of transformers, generators, or sudden shutdown of consumers.

In all these cases, the provider can do the least loss of their funds. Optimal power flow (OPF) is the basic tool for investigating these requirement.

Optimal power flow is a optimization problem widely employed in analysis and operation of power system. It is based on power flow equations: the relationship between voltage input and power injections at buses in an electric power system; the OPF problem itself is stated as minimization of operating cost for an electric power system with network-related and engineering constraints. Those equations are nonlinear and pose a non-convex optimization problems; in some cases resulting problem can be NP-hard [1] and contain multiple local solutions.

The history of the field is vast ([2], [3]). There exist a variety of iterative methods, often based on Newton's method [4], that are capable of solving many practical large-scale power flow problems. Convergence of these iterative methods depends on the selection of an appropriate initialization. However, selecting a reasonable initializations is a notoriously challenging task, especially in presence of parameter value spikes (anomalies), which can be caused, for example, by contingencies. Consequently, there has been significant recent interest in alternatives to Newton-based methods: one example of those will be analytic continuation theory from complex analysis. It claimed to be capable of reliably finding a stable solution for any feasible set of power flow equations ([5]). Another proposition is the use of relaxations. Convex relaxations lower bound the objective function, can certify infeasibility, and are even able to solve OPF globally in some cases.

2 OPF problem formulation

There are many multi-node systems. Typical 3-bus scheme is shown on Figure 1. Where in nodes 1 and 2 located generators, and in bus 3 we have demand (customer).

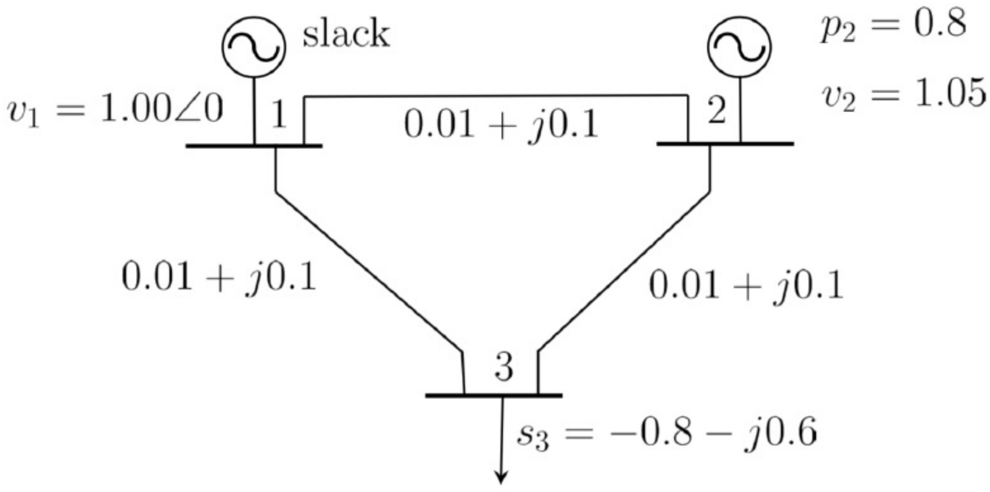

Figure 1: Typical 3-bus scheme.

Let us consider n-bus system. where $\mathbb{N} = \{1, 2, \ldots, n\}$ represents the set of all buses, $\mathbb{G}$ is the set of buses with generators, and $\mathbb{L}$ represents the set of all branches. As known, P_{Dk} is the active load, and Q_{Dk} represent reactive load demand at each bus $k \in \mathbb{N}$. Also let P_{Gk} and Q_{Gk} to be active and reactive generator outputs, where $k \in \mathbb{G}$. For OPF is necessary to consider $V_{d_k} + j \cdot V_{q_k}$ as the voltage phasor at each bus $k \in \mathbb{N}$. Upper and lower limits we can define as superscripts "max" and "min". S_{lm}^{max} represents the maximum apparent power flow accepted on branch $(l,m) \in \mathbb{L}$. Let $Y = G + j \cdot B$ denote the admittance matrix of the system. And let enter the convex quadratic cost function for generator: $f_k(P_{Gk}) = c_{2k} P_{Gk}^2 + c_{1k} P_{Gk} + c_{0k}$.

The classical formulation of the OPF problem is:

$$
\begin{gathered}
\min_{P_G, Q_G, S, V_d, V_q} \sum_{k \in \mathbb{G}} f_k(P_{Gk}) \\
\text{subject to:} \\
P_{G_k}^{min} \le P_{G_k} \le P_{G_k}^{max}, \forall k \in \mathbb{G} \\
Q_{G_k}^{min} \le Q_{G_k} \le Q_{G_k}^{max}, \forall k \in \mathbb{G} \\
(V_k^{min})^2 \le V_{d_k}^2 + V_{q_k}^2 \le (V_k^{max})^2, \forall k \in \mathbb{N} \\
\| S_{lm} \le S_{lm}^{max}, \forall (l,m) \in \mathbb{L} \\
P_{Gk} - P_{Dk} = V_{dk} \sum_{i=1}^{n} (G_{ki} V_{di} - B_{ki} V_{q_i}) + V_{qk} \sum_{i=1}^{n} (B_{ki} V_{di} + G_{ki} V_{q_i}), \forall k \in \mathbb{N} \\
Q_{Gk} - Q_{Dk} = V_{dk} \sum_{i=1}^{n} (-B_{ki} V_{di} - G_{ki} V_{q_i}) + V_{qk} \sum_{i=1}^{n} (G_{ki} V_{di} - B_{ki} V_{q_i}), \forall k \in \mathbb{N}
\end{gathered}
\tag{1}
$$

Constraints enforce engineering limits on active and reactive power injection, voltage magnitudes, and apparent-power line flows. Two last constraints are the power flow equations associated with the transmission network. This problem can be solved with mathematics and programming packages. OPF problem is known now, so let us focus in this particular work on semi-definite programming (SDP).

3 Semi-definite programming in OPF

SDP may solve a wide range of OPF problems, even when those problems do not satisfy any known sufficient conditions ([6]). This fact suggests the potential for developing less strict conditions, which is promising. Infeasibility of a relaxed problem delivers the infeasibility of the original problem (although the reverse is not true). However, SDP has it's shortcomings: there exist practical problems for which the SDP relaxation fails to yield the global solution ([7]). Moreover, SDP solutions naturally may not exactly satisfy the power flow equations. This may be unacceptable for some applications, thus requiring local minima search algorithms, that return a feasible power flow solution at the cost of increased computational difficulty and missing a global solution.

The final scope of relaxation methods is to be complementary to a local search methods to do the heavy lifting in terms of computationally hard operations and later defer to yield solutions which both satisfy power flow equations and have reasonable runtime.

4 Case study

Upper indices correspond to transposed vectors/matrices.

Let voltage coordinates vector be $x = (V_{d,1} \dots V_{d,n}, V_{q,1} \dots V_{q,n})^T$. Denote $W = xx^T$; we assume that it has rank 1. Let e_i be a standart basis in $\mathbb{R}^n$. Denote $Y_i = e_i e^i Y$. Constant matrices used in the SD relaxation are then:

$$Z_i = \frac{1}{2} \begin{bmatrix} Re(Y_i + Y^i) & Im(Y^i - Y_i) \\ Im(Y^i - Y_i) & Re(Y_i + Y^i) \end{bmatrix}$$

$$M_i = \begin{bmatrix} e_i e^i & 0 \\ 0 & e_i e^i \end{bmatrix}$$

Active power injection at bus i equals to $tr(Z_i W)$, where tr is the trace operator. The reactive power injection at bus i equals $tr(\overline{Z_i} W)$, where $\overline{Z}$ is a complex conjugate of Z. Squared voltage magnitude equals to $tr(M_i W)$.

Constant matrices regarding power flow are given similarly. Let b_{lm} be a total shunt susceptance and y_{lm} be a series admittance of the line from bus l to bus m. Denote $Y_{lm} = (\frac{1}{2} b_{lm} + y_{lm}) e_l e^m$. Then, for similarly defined Z_{lm}, active and reactive power flow are given by $tr(Z_{lm} W)$ and $tr(\overline{Z_{lm}} W$ respectively.

Semi-definite relaxation is obtained by replacement of the rank one constraint of W with positive semidefiness: $W \succeq 0$. Let $f(W) = tr(Z_k W) + P_{Dk}$. Then SD relaxation for OPF is given by:

$$
\begin{gathered}
\min_{W} \sum_{k \in \mathbb{G}} \alpha_k \\
\alpha_k = c_{k,2} f^2(W) + c_{k,1} f(W) + c_{k,0} \\
f(W) = tr(Z_k W) - P_{Dk} \\
\text{subject to:} \\
P_{G_k}^{min} - P_{Dk} \le tr(Z_k W) \le P_{G_k}^{max} - P_{Dk}, \forall k \in \mathbb{G} \\
Q_{G_k}^{min} - Q_{Dk} \le tr(\overline{Z_k} W) \le Q_{G_k}^{max} - Q_{Dk}, \forall k \in \mathbb{G} \\
(V_k^{min})^2 \le tr(M_k W) \le (V_k^{max})^2, \forall k \in \mathbb{N} \\
\begin{bmatrix} -(S_{lm}^{max})^2 & tr(Z_{lm} W) & tr(\overline{Z_{lm}} W \\ tr(Z_{lm} W) & -1 & 0 \\ tr(\overline{Z_{lm}} W & 0 & -1 \end{bmatrix} \succeq 0 \\
\begin{bmatrix} \alpha_k - c_{k,1} f(W) - c_{k,0} & -\sqrt{c_{k,2}} f(W) \\ -\sqrt{c_{k,2}} f(W) & 1 \end{bmatrix} \succeq 0 \\
W \succeq 0
\end{gathered}
\tag{2}
$$

This formulation forbids more than a single generator per bus and does not allow parallel lines. A solution has zero duality gap iff $rank(W) \le 2$ and matrix defined by the dual variables of the SD optimization problem has a 2-dim nullspace. In this case a rank one matrix can be obtained, through which a globally optimal voltage values may be inferred.

For the implementation, we choose the 3-bus scheme shown in Figure 2. The transmission capacity of the line from bus-2 to bus-3 is above-bounded by 60 MVA, and the transmission capacities of the other branches are not constrained. So there we can set the limits of 1000 MVA. The generation cost is described in the table 1. The impedances of 3-bus system lines are shown in the table 2.

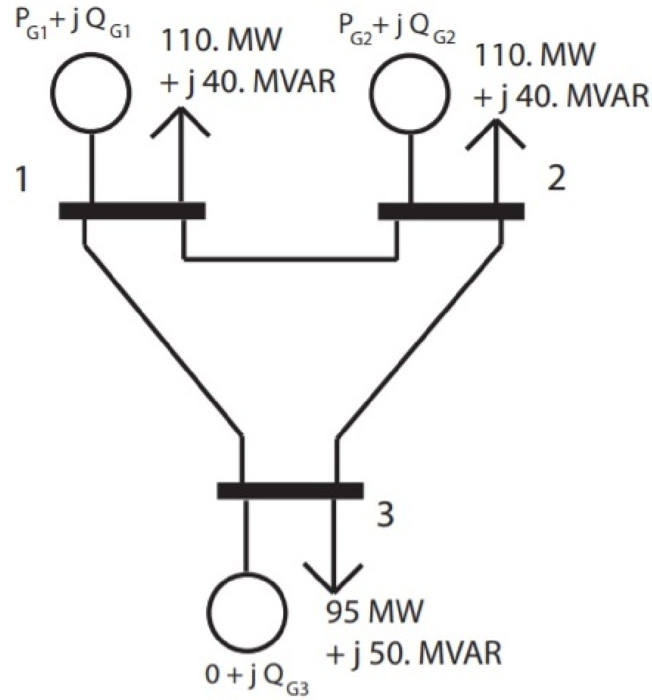

Figure 2: Case study 3-bus scheme.

Table 1: Three-bus system generator cost data

Generator	c_2	c_1	c_0
1	\$ 0.11 per MWh^2	\$ 5 per MWh	\$ 0
2	\$ 0.085 per MWh^2	\$ 1.2 per MWh	\$ 0
3	\$ 0 per MWh^2	\$ 0 per MWh	\$ 0

Table 2: Three-Bus System Network Data

From Bus	To Bus	R	X	b
1	3	0.065	0.620	0.450
3	2	0.025	0.750	0.700
1	2	0.042	0.900	0.300

We implemented this method in Julia language [11], which is widely used for OPF problem solving. For the problem formulation JuMP package [12] was applied, because it is the most popular solver for optimization problems in Julia. We tried solvers SCS and Mosek as they are designed for solving non-linear programs and can deal with SDP problems. The method itself was sequentially implemented to a program. Also for checking the solution we used Convex package [13], which can solve SDP problems as well.

As a result we have infeasible status in Convex package, while JuMP refused to solve the program with a quadratic objective function and conic constraints. Probably we should try the latest JuMP version. We are also going to realize a code in Python and Matlab and try to rewrite the problem according to the article [10].

5 Conclusion

In order to provide an optimal operating point for the electric power system, the OPF problem looks for the optimal solution for a program with total generation cost as a typical objective and engineering inequality and network quality constraints. The non-linear power flow equations impose the non-convexity on the OPF problem making it NP-hard. The non-convexity of the OPF problem has raised interests among scientists since its first introduction. Among many proposed techniques one has become promising - semidefinite programing (SDP).

The nonlinearity of the power flow equations leads to the nonconvexity of the OPF problem, with the problem being NP-hard. The methods of solution have become a constant topic for research. Suggested many methods of solution: from a DC OPF to second order cone programming. However, it is worth considering that traditional methods can be guaranteed to get a point of local optimum and also depend on the starting point.

The use of convex optimization did not miss this type of task by the party. In particular, a semi-definite relaxation of the power flow equation was applied to the OPF problem. For this case, the use of techniques in convex optimization allowed us to find the global minimum. This is possible under the condition of "dense" relaxation - satisfaction of the rank condition to obtain a solution with zero duality gap.

Although quite a large number of tasks gives the trivial solution, there is a nonzero cases of the duality gap. This makes it possible to find the lower bounds of the optimal target point, even if it is not possible to obtain physically significant solutions in the OPF problem by the SDP relaxation method. There are enough cases when the task was set very strongly, what leads to only zero result. And sometimes it is possible to find a solution for a system of 3 nodes. When for systems with 4 buses already most solutions can be zero. Also SDP method needs more computational power, than simple DC-OPF problem.

Our case became one of such examples when constraints are given very rigidly, and the solution obtained by our method leads to a zero result, that is, an infeasible solution. The formulation of the problem was formed in 2 packages that support SDP. Different solvers were used, but still this led to the absence of a solution. But it is possible to formulate the problem in other constraints, and we want to try it.

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