Numerical Methods Midterm - 2 M. ASISH YADAV

Homework - 5

1. Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad x>0,\ y>0,\ x+2y<2$$

$u(x,0) = x(2-x)$ $\quad u(0,y)=0 \quad$ $u(2-2y, y) = 0$

Test functions

$T_0(x,y) = x(2-x-2y)$

$T_1(x,y) = xy(2-x-2y)$

$T_2(x,y) = xy^2(2-x-2y)$

a) Raleigh Ritz method :

Energy function $\quad P(u) = \int_\Omega \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) dx\,dy$

Case - 1 : Let

$$u(x,y) = u_0 T_0(x,y) + u_1 T_1(x,y) \qquad (\text{where } u_0 = 1)$$

$$\cancel{\frac{\partial u}{\partial x}} = \cancel{u}\ u(x,y) = \sum_{i=0}^{1} u_i T_i(x,y)$$

$$\frac{\partial u}{\partial x} = \sum_{j=0}^{1} u_j \frac{\partial T_j}{\partial x} \qquad \frac{\partial u}{\partial y} = \sum_{j=0}^{1} u_j \frac{\partial T_j}{\partial y}$$

$$P(u) = \frac{1}{2}\iint_\Omega \left(\left(\sum_{j=0}^{1} u_j \frac{\partial T_j}{\partial x}\right)^2 + \left(\sum_{j=0}^{1} u_j \frac{\partial T_j}{\partial y}\right)^2\right) dx\,dy$$

$$\frac{\partial P}{\partial u_i} = \int_\Omega \left(\left(\sum_{j=0}^{1} u_j \frac{\partial T_j}{\partial x}\right)\frac{\partial T_i}{\partial x} + \left(\sum_{j=0}^{1} u_j \frac{\partial T_j}{\partial y}\right)\frac{\partial T_i}{\partial y}\right) d\Omega$$

To get an approximate solution, we need to find u_1

$$\frac{\partial P}{\partial u_1} = 0$$

$$\Rightarrow \frac{\partial P}{\partial u_1} = \int_\Omega \left(\left(\frac{\partial T_0}{\partial x} + u_1 \frac{\partial T_1}{\partial x} \right) \frac{\partial T_1}{\partial x} + \left(\frac{\partial T_0}{\partial y} + u_1 \frac{\partial T_1}{\partial y} \right) \frac{\partial T_1}{\partial y} \right) d\Omega = 0$$

$$\Rightarrow u_1 \int_\Omega \left(\left(\frac{\partial T_1}{\partial x} \right)^2 + \left(\frac{\partial T_1}{\partial y} \right)^2 \right) d\Omega = - \int_\Omega \left(\frac{\partial T_0}{\partial x} \frac{\partial T_1}{\partial x} + \frac{\partial T_0}{\partial y} \frac{\partial T_1}{\partial y} \right) d\Omega$$

$$\Rightarrow u_1 = \frac{- \int_0^2 \int_1^{x(2-x)} \left(\frac{\partial T_0}{\partial x} \frac{\partial T_1}{\partial x} + \frac{\partial T_0}{\partial y} \frac{\partial T_1}{\partial y} \right) dy \, dx}{\int_0^2 \int_1^{x(2-x)} \left(\left(\left(\frac{\partial T_1}{\partial x} \right)^2 + \left(\frac{\partial T_1}{\partial y} \right)^2 \right) dy \right) dx}$$

$T_0(x,y) = 2x - x^2 - 2xy$ $\qquad$ $T_1(x,y) = 2xy - x^2y - 2xy^2$

$\frac{\partial T_0}{\partial x} = 2 - 2x - 2y$ $\qquad$ $\frac{\partial T_1}{\partial x} = 2y - 2xy - 2y^2$

$\frac{\partial T_0}{\partial y} = -2x$ $\qquad$ $\frac{\partial T_1}{\partial y} = 2x - x^2 - 4xy$

using mathematica to solve for u_1, we get

$$u_1 = \frac{\frac{2354}{315}}{\frac{-18800}{2079}} = -0.8264 \approx -0.83$$

$\therefore u(x,y) = T_0(x,y) - 0.83\, T_1(x,y)$

$u(x,y) = x(2-x-2y)(1-0.83y)$

$\hookrightarrow$ an approximate solution

<u>Case-2:</u>

Let $u(x,y) = u_0 T_0(x,y) + u_1 T_1(x,y) + u_2 T_2(x,y)$ (where $u_0 = 1$)

$$P(u) = \frac{1}{2}\int_{\Omega}\left(\left(\sum_{j=0}^{2} u_j \frac{\partial T_j}{\partial x}\right)^2 + \left(\sum_{j=0}^{2} u_j \frac{\partial T_j}{\partial y}\right)^2\right) dx\, dy$$

$$\frac{\partial P}{\partial u_i} = \int_{\Omega}\left(\left(\sum_{j=0}^{2} u_j \frac{\partial T_j}{\partial x}\right)\frac{\partial T_i}{\partial x} + \left(\sum_{j=0}^{2} u_j \frac{\partial T_j}{\partial y}\right)\frac{\partial T_i}{\partial y}\right) d\Omega$$

To get an approximate solution, we need to find u_1 & u_2

$$\frac{\partial P}{\partial u_1} = 0 \quad \& \quad \frac{\partial P}{\partial u_2} = 0$$

$$\frac{\partial P}{\partial u_i} = \sum_{j=0}^{2} u_j \int_{\Omega}\left(\frac{\partial T_i}{\partial x}\frac{\partial T_j}{\partial x} + \frac{\partial T_i}{\partial y}\frac{\partial T_j}{\partial y}\right) d\Omega = 0$$

As we have $u_0 = 1$, the resulting system of equations can be represented as follows:

$$\begin{bmatrix} \int_{\Omega}\left(\left(\frac{\partial T_1}{\partial x}\right)^2 + \left(\frac{\partial T_1}{\partial y}\right)^2\right) d\Omega & \int_{\Omega}\left(\left(\frac{\partial T_1}{\partial x}\right)\left(\frac{\partial T_2}{\partial x}\right) + \left(\frac{\partial T_1}{\partial y}\right)\left(\frac{\partial T_2}{\partial y}\right)\right) d\Omega \\ \int_{\Omega}\left(\frac{\partial T_1}{\partial x}\frac{\partial T_2}{\partial x} + \frac{\partial T_1}{\partial y}\frac{\partial T_2}{\partial y}\right) d\Omega & \int_{\Omega}\left(\left(\frac{\partial T_2}{\partial x}\right)^2 + \left(\frac{\partial T_2}{\partial y}\right)^2\right) d\Omega \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} -\int_{\Omega}\left(\frac{\partial T_0}{\partial x}\frac{\partial T_1}{\partial x} + \frac{\partial T_0}{\partial y}\frac{\partial T_1}{\partial y}\right) d\Omega \\ -\int_{\Omega}\left(\frac{\partial T_0}{\partial x}\frac{\partial T_2}{\partial x} + \frac{\partial T_0}{\partial y}\frac{\partial T_2}{\partial y}\right) d\Omega \end{Bmatrix}$$

$T_0 = 2x - x^2 - 2xy \qquad T_1(x,y) = 2xy - x^2y - 2xy^2 \qquad T_2(x,y) = 2xy^2 - x^2y^2 - 2xy^3$

$\frac{\partial T_0}{\partial x} = 2 - 2x - 2y \qquad \frac{\partial T_1}{\partial x} = 2y - 2xy - 2y^2 \qquad \frac{\partial T_2}{\partial x} = 2y^2 - 2xy^2 - 2y^3$

$\frac{\partial T_0}{\partial y} = -2x \qquad \frac{\partial T_1}{\partial y} = 2x - x^2 - 4xy \qquad \frac{\partial T_2}{\partial y} = 4xy - 2x^2y - 6xy^2$

using Mathematica, the resulting matrices are:

$$\begin{bmatrix} -9.0428 & -9.9763 \\ -9.9763 & -11.6084 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 7.473 \\ 7.5275 \end{Bmatrix}$$

$$\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} -2.1398 \\ 1.1905 \end{Bmatrix}$$

$$\therefore u(x,y) = T_0(x,y) - 2.14\, T_1(x,y) + 1.19\, T_2(x,y)$$

$$u(x,y) = x(2-x-2y)(1 - 2.14y + 1.19y^2)$$

$\hookrightarrow$ an approximate solution to given equation

b) Galerkin method: Case: 1

Let,

$$u(x,y) = T_0(x,y) + u_1 T_1(x,y)$$

$$R(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial T_0}{\partial x} + u_1 \frac{\partial T_1}{\partial x} \qquad \frac{\partial u}{\partial y} = \frac{\partial T_0}{\partial y} + u_1 \frac{\partial T_1}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 T_0}{\partial x^2} + u_1 \frac{\partial^2 T_1}{\partial x^2} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 T_0}{\partial y^2} + u_1 \frac{\partial^2 T_1}{\partial y^2}$$

$$T_0(x,y) = 2x - x^2 - 2xy \qquad T_1(x,y) = 2xy - x^2 y - 2xy^2$$

$$\frac{\partial T_0}{\partial x} = 2 - 2x - 2y \qquad \frac{\partial T_1}{\partial x} = 2y - 2xy - 2y^2$$

$$\frac{\partial^2 T_0}{\partial x^2} = -2 \qquad \frac{\partial^2 T_1}{\partial x^2} = -2y$$

$$\frac{\partial T_0}{\partial y} = -2x \, ; \, \frac{\partial^2 T_0}{\partial y^2} = 0 \qquad \frac{\partial T_1}{\partial y} = 2x - x^2 - 4xy \, ; \, \frac{\partial^2 T_1}{\partial y^2} = -4x$$

$$R(x,y) = -2 - 2u_1 y - 4u_1 x$$

$$\cancel{\int_0^1 V_1(x,y) R(x,y) = 0} \qquad \int_\Omega V_1(x,y) R(x,y) \, d\Omega = 0$$

$$V_1(x,y) = T_1(x,y) = 2xy - x^2 y - 2xy^2$$

$$V_1(x,y) R(x,y) = (-2 - 2u_1 y - 4u_1 x)(2xy - x^2 y - 2xy^2)$$

$$= -4xy + 2x^2 y + 4xy^2 - 2u_1 (y + 2x)(2xy - x^2 y - 2xy^2)$$

$$= -2(2xy - x^2 y - 2xy^2) - 2u_1 (y + 2x)(2xy - x^2 y - 2xy^2)$$

$$\Rightarrow \int_\Omega v(x)R(x)\,d\Omega = -2\int_\Omega (2xy - x^2y - 2xy^2)\,d\Omega - c_1\int_\Omega (2y+4x)(2xy-x^2y-2xy^2)\,d\Omega = 0$$

$$\Rightarrow \quad c_1 = \frac{-2\int_\Omega (2xy - x^2y - 2xy^2)\,d\Omega}{\int_\Omega (2y+4x)(2xy-x^2y-2xy^2)\,d\Omega}$$

$$\Rightarrow c_1 = \frac{-\frac{36}{35}}{\frac{4162}{945}} = \frac{-1.0286}{4.4042} = -0.2336 \approx -0.23$$

$$\therefore u(x,y) = T_0(x,y) - 0.23\,T_1(x,y)$$

$$\Rightarrow \quad u(x,y) = x(2-x-2y)(1-0.23y)$$

Case 2:

Let,

$$u(x,y) = T_0(x,y) + c_1 T_1(x,y) + c_2 T_2(x,y)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 T_0}{\partial x^2} + c_1\frac{\partial^2 T_1}{\partial x^2} + c_2\frac{\partial^2 T_2}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 T_0}{\partial y^2} + c_1\frac{\partial^2 T_1}{\partial y^2} + c_2\frac{\partial^2 T_2}{\partial y^2}$$

$$R(x,y) = \frac{\partial^2 T_0}{\partial x^2} + \frac{\partial^2 T_0}{\partial y^2} + c_1\left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2}\right) + c_2\left(\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2}\right)$$

$$T_2(x,y) = xy^2(2-x-2y) = 2xy^2 - x^2y^2 - 2xy^3$$

$$\frac{\partial T_2}{\partial x} = 2y^2 - 2xy^2 - 2y^3 \qquad \frac{\partial T_2}{\partial y} = 4xy - 2x^2y - 6xy^2$$

$$\frac{\partial^2 T_2}{\partial x^2} = -2y^2 \qquad \frac{\partial^2 T_2}{\partial y^2} = 4x - 2x^2 - 12xy$$

$$R(x,y) = -2 + u_1(-2y-4x) + u_2(-2y^2+4x-2x^2-12xy)$$

$$\cancel{\int_\Omega V_1 R} \qquad \int_\Omega V_1(x,y)\,R(x,y)\,d\Omega = 0 \qquad \int_\Omega V_2(x,y)\,R(x,y)\,d\Omega = 0$$

$$V_1(x,y) = 2xy - x^2y - 2xy^2$$
$$V_2(x,y) = 2xy^2 - x^2y^2 - 2xy^3$$

$$\int_\Omega V_1(x,y)R(x,y)\,d\Omega = -2\int_\Omega V_1(x,y)\,d\Omega + u_1\int_\Omega(-2y-4x)V_1(x,y)\,d\Omega$$
$$+ u_2\int_\Omega(-2y^2+4x-2x^2-12xy)V_1(x,y)\,d\Omega = 0$$

$$\Rightarrow u_1\int_\Omega(-2y-4x)\,V_1(x,y)\,d\Omega + u_2\int_\Omega(-2y^2+4x-2x^2-12xy)\,V_1(x,y)\,d\Omega = 2\int_\Omega V_1(x,y)\,d\Omega$$

$$\int_\Omega V_2(x,y)\,R(x,y)\,d\Omega = 0$$

$$\Rightarrow u_1\int_\Omega(-2y-4x)\,V_2(x,y)\,d\Omega + u_2\int_\Omega(-2y^2+4x-2x^2-12xy)\,V_2(x,y)\,d\Omega = 2\int_\Omega V_2(x,y)\,d\Omega$$

$$\begin{bmatrix} \int_\Omega(-2y-4x)\,V_1(x,y)\,d\Omega & \int_\Omega(-2y^2+4x-2x^2-12xy)\,V_1(x,y)\,d\Omega \\ \int_\Omega(-2y-4x)\,V_2(x,y)\,d\Omega & \int_\Omega(-2y^2+4x-2x^2+12xy)\,V_2(x,y)\,d\Omega \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 2\int_\Omega V_1(x,y)\,d\Omega \\ 2\int_\Omega V_2(x,y)\,d\Omega \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4.4042 & -8.8695 \\ -3.577 & -7.4395 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1.0286 \\ 0.8402 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -0.1926 \\ -0.0203 \end{Bmatrix} \approx \begin{Bmatrix} -0.19 \\ -0.02 \end{Bmatrix}$$

$$\therefore\ u(x,y) = T_0(x,y) - 0.19\, T_1(x,y) - 0.02\, T_2(x,y)$$

$$\Rightarrow u(x,y) = x(2-x-2y)(1-0.19y-0.02y^2)$$

Homework - 6

(a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on Ω

u on boundary is given (u on C_u)

$$U(x,y) = \sum_{I=1}^{N} U_I \hat{T}_I(x,y)$$

for $(x,y) \in e$ $$U(x,y) = \sum_{I=1}^{N} \sum_{i=1}^{n_{le}} U_I t_{Ig_e(i)} T_i(\xi^1_e(x,y), \xi^2_e(x,y))$$

$$= \sum_{i=1}^{n_{le}} U_{g_e(i)} T_i(\xi_{e}(x,y))$$

$$\oint_{C_u} -v\left(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y\right) ds + \int_S \left[\frac{\partial v}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial U}{\partial y}\right] dx\,dy = 0$$

$$\int_S (\nabla v \cdot \nabla U)\, dx\,dy = \oint_{C_u} v(\nabla U \cdot n)\, ds$$

$$U(x,y) = \sum_{I=1}^{N} U_I \hat{T}_I(x,y) \quad ; \quad V(x,y) = \sum_{I=1}^{N} v_I \hat{T}_I(x,y)$$

$$\Rightarrow \sum_{I=1}^{N} \left(\oint_{C_u} \hat{T}_I (\nabla U \cdot n)\, ds\right) v_I = \sum_{I=1}^{N} \left(\int_S \nabla \hat{T}_I \cdot \nabla U \, dx\,dy\right) v_I$$

stiffness matrix

$$K_{IJ}\Big|_e = \sum_{P=1}^{n_g} \left[\sum_{\alpha=1}^{n_{le}} \sum_{\beta=1}^{n_{le}} \delta_{Ig_e(\alpha)} \sum_{j=1}^{2} \left(\frac{\partial T_\alpha}{\partial x_j}(\xi^P) \frac{\partial T_\beta}{\partial x_j}(\xi^P)\right) \delta_{Jg_e(\beta)}\right] J_e(\xi^P) \omega_P$$

n_g - no. of gauss points — J_e - Determinant of Jacobian

$n_{le} = 4$ ($\because$ 4 noded element) — ω_P - Gaussian weights

In Question - 1

$u_1 = u_4 = u_7 = u_{11} = u_{12} = u_{13} = u_{14} = u_{15} = u_{16} = 0$

$u_2 = 0.64 \quad u_5 = 0.96 \quad u_8 = 0.96$

on solving the problem using C++ code as submitted

$u_3 = -0.0262 \;;\; u_6 = 0.0797 \;;\; u_9 = 0.2331 \;;\; u_{10} = 0.0124$

In Question - 2

$\cancel{u_1 = u_{12} = u_{13} = u_{14} = u_{15} = u_{16} = 0}$

$u_2 = 0.64 \quad u_5 = 0.96 \quad u_8 = 0.96 \quad u_1 = 0 \quad u_{12} = 0$

on solving the problem using C++ code as submitted

$u_3 = -0.0343 \quad u_4 = 0.0069 \quad u_6 = 0.1215 \quad u_7 = 0.0226$

$u_9 = 0.2702 \quad u_{10} = 0.0328 \quad u_{11} = 0.01 \quad u_{13} = -0.1618$

$u_{14} = -0.1875 \quad u_{15} = -0.0035 \quad u_{16} = 0.0011$