The second and last Take-Home exam for 12/24-703 consists of Homeworks 5 and 6 attached.

Homework 5

12/24-703 Numerical Methods in Engineering

1. Consider Laplace's equation in a triangular domain with Dirichlet boundary conditions:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad x > 0, y > 0, x + 2y < 2, \\
&u(x,0) = x(2 - x) \\
&u(0, y) = 0 \\
&u\left(2 - 2y, y\right) = 0.
\end{aligned} \tag{1}
$$

Consider the trial and test functions

$$
\begin{aligned}
T_0\left(x, y\right) &= x\left(2 - x - 2y\right) \\
T_1\left(x, y\right) &= xy\left(2 - x - 2y\right) \\
T_2\left(x, y\right) &= xy^2\left(2 - x - 2y\right),
\end{aligned}
$$

and note that an approximation of the form

$$
U\left(x, y\right) = T_0\left(x, y\right) + \sum_{i=1}^{1 or 2} U_i T_i\left(x, y\right) \tag{2}
$$

satisfies the boundary conditions for all possible values of U_2 and U_3.

a) Formulate an appropriate energy function for the problem and use the Ritz method to obtain the approximate solutions to (1) using first one and then two of the trial functions given.

b) Use the Galerkin method with first one test/trial function T_1 and then two test/trial functions T_1, T_2 to obtain approximate solutions to (1) of the form (2).

Homework 6

12/24-703 Numerical Methods in Engineering

Problem 1. Consider Laplace's equation in a triangular domain:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad for \quad x > 0, y > 0, x + 2y < 2 \tag{1}$$

Consider the following Dirichlet boundary conditions:

$$\begin{aligned} u(x,0) &= x(2-x) \\ u(0,y) &= 0 \\ u(2-2y,y) &= 0. \end{aligned} \tag{2}$$

Consider the finite element mesh of the domain in Figure 1, consisting of quadrilateral elements. Here, each element shows the element number in the square box at its center and

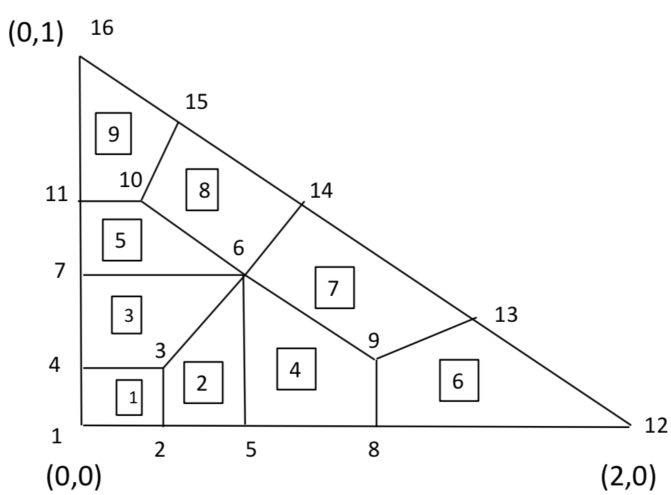

Figure 1: Mesh

the global node number of the nodes associated with the element.

Table 1 shows the global coordinates of the global nodes used in the mesh. Table 2 shows the global nodes associated with each element, which is called the connectivity matrix. Use the isoparametric master element as shown in Figure 2. The isoparametric shape functions

Node number	x-coord	y-coord
1	0	0
2	0.4	0
3	0.4	0.2
4	0	0.2
5	0.8	0
6	0.8	0.4
7	0	0.4
8	1.2	0
9	1.2	0.2
10	0.4	0.6
11	0	0.6
12	2	0
13	1.5	0.25
14	1	0.5
15	0.5	0.75
16	0	1

Table 1: Nodal coordinates.

Element number	Node 1	Node 2	Node 3	Node 4
1	1	2	3	4
2	2	5	6	3
3	4	3	6	7
4	5	8	9	6
5	7	6	10	11
6	8	12	13	9
7	9	13	14	6
8	6	14	15	10
9	11	10	15	16

Table 2: Element connectivity.

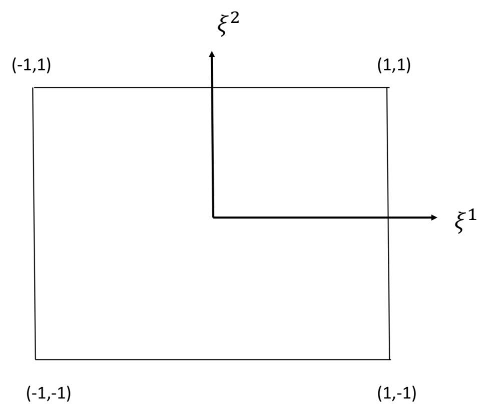

Figure 2: Master element

corresponding to this master element are

$$\begin{aligned}
T_1 &= \frac{1}{4}(1-\xi^1)(1-\xi^2) \\
T_2 &= \frac{1}{4}(1+\xi^1)(1-\xi^2) \\
T_3 &= \frac{1}{4}(1+\xi^1)(1+\xi^2) \\
T_4 &= \frac{1}{4}(1-\xi^1)(1+\xi^2).
\end{aligned} \tag{3}$$

The formula for Gauss integration in two-dimensions with 2×2 integration points is

$$I = \int_{-1}^{1} \int_{-1}^{1} f(\xi^1, \xi^2) d\xi^1 d\xi^2 \approx \sum_{i=1}^{2} \sum_{j=1}^{2} H_i H_j f(\xi_i^1, \xi_j^2). \tag{4}$$

where H_i and H_j are the weights and (ξ_i^1, ξ_j^2) are the integration points. The weights and integration points for 2×2 integration are given in Table 3. We have $H_1 H_1 = \omega_1$, $H_1 H_2 = \omega_2$, $H_2 H_1 = \omega_3$ and $H_2 H_2 = \omega_4$, where ω_J are the weights corresponding to integration points J (following the notation used in the class). Please write your own solver to solve the resulting system of equations using Gauss elimination.

i	H_i	ξ_i^1, ξ_i^2
1	1.0000	-0.57735
2	1.0000	0.57735

Table 3: Gauss quadrature.

Problem 2.

Solve the Laplace equation (1) but with the boundary conditions

$$\begin{aligned} & u(x, 0) = x(2 - x) \\ & \nabla u \cdot \mathbf{n} = 0 \quad on \quad x = 0 \\ & \nabla u \cdot \mathbf{n} = 0 \quad on \quad x + 2y = 2 \end{aligned} \tag{5}$$

where $\mathbf{n}$ is the outward directed boundary normal.

Use the same mesh and its details and the same quadrature rule as provided in the previous problem. Use the solver you wrote for the previous problem to solve the resulting system of equations.