Exam 1: Numerical Methods M. ASISH YADAV

1. A) 6.5.14

Required to prove $(1+\Delta t)^n - e^{n\Delta t} \to 0$ as $\Delta t \to 0$, with $n\Delta t = t$ fixed

let us expand $e^{n\Delta t}$

$$e^{n\Delta t} = 1 + n\Delta t + \frac{1}{2}n^2\Delta t^2 + O(\Delta t^3)$$

let us write taylor series expansion of $(1+\Delta t)^n$

$$(1+\Delta t)^n = 1 + n\Delta t + \frac{1}{2}n(n-1)\Delta t^2 + O(\Delta t^3)$$

Consider the difference $(1+\Delta t)^n - e^{n\Delta t}$

$$\Rightarrow (1+\Delta t)^n - e^{n\Delta t} = -\frac{n}{2}\Delta t^2 + O(\Delta t^3)$$

Given that $\Delta t \to 0$, this means higher powers of Δt will be almost (close to) zero. Hence we can say that

As $\Delta t \to 0$, $(1+\Delta t)^n - e^{n\Delta t} \to 0$ with $n\Delta t = t$ fixed

B) 6.5.16

$$\frac{du}{dt} = -\frac{du}{dx}$$

Downwind approximation:

$$\frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} = -\frac{(u(t,x+h) - u(t,x))}{h}$$

$u(t,x) = e^{i(kx+\omega t)}$

At time $t=0$

$u(0,x) = e^{ikx}$

writing downward approximation for first time step

$$\frac{u(\Delta t, x) - u(0,x)}{\Delta t} = -\frac{(u(0,x+h) - u(0,x))}{h}$$

$$u(\Delta t, x) = u(0,x) - \frac{\Delta t}{h}\left(u(0,x+h) - u(0,x)\right)$$

$$u(\Delta t, x) = e^{ikx} - \frac{\Delta t}{h}\left(e^{ikx}\, e^{ikh} - e^{ikx}\right)$$

$$\Rightarrow u_1 = e^{ikx}\left(1 - \frac{\Delta t}{h}\left(e^{ikh} - 1\right)\right)$$

$$\Rightarrow u_1 = G u_0$$

where $G = 1 - \frac{\Delta t}{h}\left(e^{ikh} - 1\right)$

$$\Rightarrow G = 1 + \frac{\Delta t}{h}\left(1 - e^{ikh}\right)$$

we can see that

$|G| > 1$ ($\because$ Real part is greater than 1 always and imaginary part when considered for magnitude will never yield ~~total~~ lesser than 1)

Also solutions take the form

$u_n = G^n e^{ikx}$ n denotes time steps

c) <u>6.5.16</u>

$$\frac{du}{dt} = \frac{[u(t,x+h) - u(t,x)]}{h}$$

$$u = e^{i\omega t} e^{ikx}$$

Initial condition

$$u(0,x) = e^{ikx}$$

After first time step ~~ie $u_1 = u(\Delta t, x)$~~

$$u_1 = \frac{u(0,x+h) - u(0,x)}{h}$$

$$= \frac{e^{ikx} \cdot e^{ikh} - e^{ikx}}{h}$$

$$u_1 = e^{ikx}\left(\frac{e^{ikh}-1}{h}\right) = a_1 e^{ikx} \left(a_1 = \frac{e^{ikh}-1}{h}\right)$$

$$u_2 = \frac{u_1(0,x+h) - u_1(0,x)}{h}$$

$$= \frac{a_1 e^{ik(x+h)} - a_1 e^{ikx}}{h}$$

$$= a_1 e^{ikx} \left(\frac{e^{ikh} - 1}{h}\right)$$

$$= a_1^2 e^{ikx}$$

$\vdots$

$\Rightarrow \quad u_n = a_1^n e^{ikx}$ $\qquad a_1 = \frac{e^{ikh} - 1}{h}$

$|a_1|$ is a bounded term. e^{ikx} is also a bounded term

$\therefore$ It is stable

D) $u_t = -u_x \quad , \; -\infty < x < \infty$

$$\frac{du}{dt} = -\frac{du}{dx}$$

$$u(x,t) = e^{ikx} e^{i\omega t}$$

$$\frac{du}{dt} = i\omega\, e^{ikx} e^{i\omega t}$$

$$\frac{du}{dx} = ik\, e^{ikx} e^{i\omega t}$$

From given condition

$$i\omega = i.k \quad \Rightarrow \omega = -k$$

$u(x,t) = e^{ikx} e^{-ikt}$

This is a bounded solution

~~So~~ The semidiscretisation

$$\frac{du}{dt} = \frac{u(x+h,t) - u(x,t)}{h}$$

we have proved in previous question that this is a bounded solution and stable for initial condition e^{ikx} always.

Hence it is bounded in both considerations

e) 6.5.19

u_{xx} is replaced by Au

$u_t = u_{xx}$ is replaced by

$$\frac{u_{n+1} - u_n}{\Delta t} = \theta A u_{n+1} + (1-\theta) A u_n$$

$$(I - \Delta t \theta A) u_{n+1} = (I + (1-\theta)\Delta t A) u_n$$

$$u_{n+1} = (I - \Delta t \theta A)^{-1} (I + (1-\theta)\Delta t A) u_n$$

$$G = (I - \Delta t \theta A)^{-1} (I + (1-\theta)\Delta t A)$$

In order to check for stability threshold we need to look at eigen values of G

Let eigen values of A be represented by λ_i

Then eigen values of G i.e. λ_G

$$\lambda_G = \frac{1+(1-\theta)\Delta t \lambda_i}{1-\Delta t \theta \lambda_i}$$

For this to be stable

$$|\lambda_G| \leq 1$$

Given that λ_i ranges from $-\frac{4}{h^2}$ to 0

when $\lambda_i = 0$

$$\lambda_G = 1 \quad \text{(stable)}$$

when $\lambda_i = \frac{-4}{h^2}$

$$-1 \leq \lambda_G \leq 1$$

Case (i): $\lambda_G \leq 1$

$$\Rightarrow \frac{1+(1-\theta)\Delta t \lambda_i}{1-\Delta t \theta \lambda_i} \leq 1$$

$$1+(1-\theta)\Delta t\lambda_i \le 1-\Delta t\theta\lambda_i$$

$$1-\Delta t\theta\lambda_i+\Delta t\lambda_i \le 1-\Delta t\theta\lambda_i$$

$$\Delta t\lambda_i \le 0$$

$$\lambda_i \le 0$$

$$-\frac{4}{h^2} \le 0$$

This case doesn't yield any useful conclusion

Case(ii) : $\lambda_n \ge -1$

$$\frac{1+(1-\theta)\Delta t\lambda_i}{1-\Delta t\theta\lambda_i} \ge -1$$

$$1 \cancel{-\Delta t\theta\lambda_i+\Delta t\lambda_i \ge \Delta t\theta\lambda_i-1}$$

$$\cancel{\Delta t\theta\lambda_i \le 2}$$

$$1+(1-\theta)\Delta t\left(\frac{-4}{h^2}\right) \ge \Delta t\theta\left(\frac{-4}{h^2}\right)-1$$

$$1-\frac{\Delta t}{h^2}\cdot 4+4\theta\frac{\Delta t}{h^2} \ge -\frac{4\Delta t\theta}{h^2}-1$$

$$2 \ge \frac{\Delta t}{h^2}(4-8\theta)$$

$$\frac{\Delta t}{h^2} \le \frac{1}{(2-4\theta)}$$

$\therefore \quad \frac{\Delta t}{h^2} \leq (2-4\theta)^{-1}$

As Δt & h^2 are both positive

$\frac{1}{2-4\theta} \geq 0$

this holds when

$2-4\theta > 0$

$\Rightarrow \quad \theta < \frac{1}{2}$

Stability threshold $\frac{\Delta t}{h^2} \leq (2-4\theta)^{-1}$ for $\theta < \frac{1}{2}$

when $\theta \geq \frac{1}{2}$

~~Kdgb~~ Lets consider $\theta = \frac{1}{2}$

$$|\lambda_G| = \left| \frac{1 + \frac{\Delta t}{2}\left(\frac{-4}{h^2}\right)}{1 - \frac{\Delta t}{2}\left(\frac{-4}{h^2}\right)} \right| = \left| \frac{1 - \frac{2\Delta t}{h^2}}{1 + \frac{2\Delta t}{h^2}} \right| < 1$$

Similarly when $\theta \geq \frac{1}{2}$ $\quad |\lambda_G| < 1$

$\therefore$ It is stable when $\theta \geq \frac{1}{2}$

When $\theta = 1$

$$\lambda_G = \frac{1}{1 - \Delta t \lambda_i} = \frac{1}{1 + \frac{4\Delta t}{h^2}}$$ (Backward Euler)

$|\lambda_G| < 1$

Hence stable