12 755 – Finite Elements in Mechanics I

Assignment 3

Due Oct 11th in class – Late assignments receive no credit

Discussions with other students about concepts and overall approaches to solving the assignments is permitted. However, copying another students work, or sharing or viewing code, are not permitted. The final submission must reflect your own work and understanding.
Do not submit your code.

Problem 1: Nonuniform 1D bar

Consider the governing equation for the axial deformation $u(x)$ of a 1D bar subject to a given axial loading $p(x)$:

$$\frac{d}{dx}\left(E(x)A(x)\frac{d}{dx}u(x)\right) + p(x) = 0 \tag{1}$$

subject to these boundary conditions:

$$u(L) = 0, \quad \left.\frac{du}{dx}\right|_{x=0} = \frac{1}{6} \tag{2}$$

Use $L = 1, E(x) = 1, A(x) = 1, p(x) = x$.

a) Using ξ as the natural coordinate, write down the linear shape functions $\left(\hat{N}_i(\xi)\right)$ for the master element.

b) Approximate the displacement $u(\xi) = \sum_{i=1}^{2} \hat{u}_i \hat{N}_i(\xi)$. For the Galerkin approach, use the weight functions $\hat{N}_i(\xi)$. Use the weak form (HW2 Problem 2a) to obtain the element stiffness matrix K_e and the element load vector f_e.

c) Assemble the K_e's and f_e's to obtain the global stiffness matrix K and global load vector f. Solve $K\hat{u} = f$ for $N = 4, 10, 100, 200, 500$ elements and plot these on the same graph.

d) Plot the stress $\sigma = E\dfrac{du}{dx}$ in the bar for the results from (c). Notice that the displacements u is continuous while the stress σ is discontinuous. Discuss the trends with increasing N, and especially with respect to boundary conditions.

Problem 2: 1D bar with Quadratic element

Consider the axial deformation $u(x)$ of a 1D bar subject to a given axial loading $p(x)$ and subject to fixed end boundary conditions.

$$\frac{d}{dx}\left(E(x)A(x)\frac{d}{dx}u(x)\right) + p(x) = 0 \tag{3}$$

$$u(0) = u(L) = 0 \tag{4}$$

Use $L = 0.7, p(x) = x$,

$$E(x)A(x) = \begin{cases} 1 & 0 \le x \le 0.3 \\ 4 & 0.3 \le x \le 0.7 \end{cases} \tag{5}$$

a) Write down the shape function for quadratic elements in terms of the natural coordinates ξ.

b) Write down the element stiffness matrix and load vector.

c) Assemble the element stiffness matrix and element load vector to obtain the global system of equations. Solve for u and plot on the same graph for $N = 4, 10, 100, 500$ elements. Observe that the solution converges with h-refinement.

d) Plot the stress in the bar for the above values of N. Note that stress is piecewise linear in this problem but it is still discontinuous across the elements. Discuss this.

Problem 3: 1D bar on a foundation

Consider the axial deformation $u(x)$ of a 1D bar bonded to a foundation along the length.

$$\frac{d}{dx}\left(E(x)A(x)\frac{d}{dx}u(x)\right) + Cu(x) + p(x) = 0 \tag{6}$$

The boundary conditions are

$$u(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=1} = 1 \tag{7}$$

Use $L = 1, E(x) = 1, A(x) = 1, C = 1, p(x) = 0$. The term $Cu(x)$ is the force applied by the foundation.

a) Write down the weak form of the above equation and implement the boundary conditions.

b) Using quadratic shape functions, solve for the displacement u. Plot u for $N = 10, 100, 500$ elements.

d) Obtain the analytical solution and plot against the Finite Element solution for 500 elements.