12 755 – Finite Elements in Mechanics I

Assignment 4

Due Oct 25th in class – Late assignments receive no credit

Discussions with other students about concepts and overall approaches to solving the assignments is permitted. However, copying another students work, or sharing or viewing code, are not permitted. The final submission must reflect your own work and understanding.
Do not submit your code.

Problem I: Weak Form and Gauss-divergence theorem

a) **Weak formulation of the heat transfer equation in 3D.**

Consider a body Ω. The boundary $\partial\Omega$ of the region Ω is divided into two disjoint regions Γ_n and Γ_T such that $\partial\Omega = \Gamma_n \bigcup \Gamma_T$. The corresponding strong form for the heat equation in 3D is given by

$$\begin{aligned} &\operatorname{div}(\boldsymbol{K}\nabla T) + Q = 0 \text{ in } \Omega \\ &\hat{\boldsymbol{n}} \cdot \boldsymbol{q} = h(x,y,z) \text{ on } \Gamma_n \\ &T = g(x,y,z) \text{ on } \Gamma_T, \end{aligned} \tag{1}$$

where $\boldsymbol{K}(x,y,z)$ is the 3×3 thermal conductivity matrix, $Q(x,y,z)$ is the heat source term, $T(x,y,z)$ is the temperature, $\boldsymbol{q}$ is the heat flux, $\hat{\boldsymbol{n}}$ is the unit outward normal to the boundary $\partial\Omega$, and $h(x,y,z)$ and $g(x,y,z)$ are given functions.

Derive the weak form of the equation.

b) **Gauss Divergence theorem in 2D.**

Consider a 2D body Ω with boundary $\partial\Omega$ and unit outward normal $\hat{\boldsymbol{n}}$. Consider a vector-valued function $\boldsymbol{f}(x,y) = (\phi(x,y), \psi(x,y))$ defined on Ω. Show that

$$\int_\Omega \operatorname{div} \boldsymbol{f} \, \mathrm{d}A =: \int_\Omega \left(\frac{\partial}{\partial x}\phi(x,y) + \frac{\partial}{\partial y}\psi(x,y) \right) \mathrm{d}A = \int_{\partial\Omega} \hat{\boldsymbol{n}} \cdot \boldsymbol{f} \, \mathrm{d}S \tag{2}$$

In class you have shown the first part, i.e.,

$$\int_\Omega \frac{\partial}{\partial x}\phi(x,y) \, \mathrm{d}A = \int_{\partial\Omega} \phi(x,y) \, \mathrm{d}y, \tag{3}$$

Proceeding in an analogous way, show that

$$\int_\Omega \frac{\partial}{\partial y}\psi(x,y) \, \mathrm{d}A = -\int_{\partial\Omega} \psi(x,y) \, \mathrm{d}x \tag{4}$$

Problem II: Shape functions for 2D elements

Find and plot all the shape functions for the following elements. Show all necessary steps.

a) **8 node rectangular element ("serendipity elements")**
Consider a rectangular element defined by the nodes below:

Node 1 $(x,y) = (-1,-1)$

Node 2 $(x, y) = (1, -1)$

Node 3 $(x, y) = (1, 1)$

Node 4 $(x, y) = (-1, 1)$

Node 5 $(x, y) = (0, -1)$

Node 6 $(x, y) = (1, 0)$

Node 7 $(x, y) = (0, 1)$

Node 8 $(x, y) = (-1, 0)$

b) **9 node rectangular element**
Consider a rectangular element defined by 9 nodes. Nodes 1 through 8 are the same as that for the serendipity element, and the location of Node 9 is $(x, y) = (0, 0)$.

Problem III: Heat Transfer in 2D

Solve the 2D heat transfer problem on the rectangular body Ω with corners $(0, 0)$, $(0, 2)$, $(2, 0)$, $(2, 2)$ subject to boundary conditions:

$$T(0, y) = 100,\ T(2, y) = 200,\ T(x, 0) = T(x, 2) = 50(2 - x) + 100x \tag{5}$$

$Q = 0$ and $\boldsymbol{K}$ is the conductivity matrix.

a) Write down the weak form.

b) Write down the shape functions for a 4-node rectangular element (Q4).

c) For each of the following cases, (1) discretize Ω into 4 Q4 elements, (2) write down the corresponding element stiffness matrices (K_e) and element load vectors (f_e), (3) assemble (K_e) and (f_e) into the global stiffness matrix (K) and global load vector (f), (4) solve for T, and (5) plot the scalar temperature field with a contour plot, and overlay the vector heat flux field.

1) $\boldsymbol{K}$ is isotropic and homogeneous, with $K_{xx} = K_{yy} = 10, K_{xy} = K_{yx} = 0$.

2) $\boldsymbol{K}$ is diagonal and homogeneous, for both the following cases:
i) $K_{xx} = 1, K_{yy} = 10, K_{xy} = K_{yx} = 0$
ii) $K_{xx} = 1, K_{yy} = 0.1, K_{xy} = K_{yx} = 0$
Discuss the similarity/difference of solutions in i), ii).

3) Of the two choices below for $\boldsymbol{K}$, only one is positive definite:
i) $\boldsymbol{K} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$
ii) $\boldsymbol{K} = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$
Choose the appropriate one and solve.