Mass spring damper (unforced $\rightarrow F_{ext} = 0$)

1. Equations of motion

The equation of motion for the unforced mass-spring-damper system:

$$m\ddot{x} + c\dot{x} + kx = F_{ext} = 0$$

State variables:

- $x_1 = x$ (position)
- $x_2 = \dot{x}$ (velocity)

First-order form:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{cases}$$

State-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

---

2. State-space equations in ω_n and ζ form

Using the natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}}$$

and the damping ratio:

$$\zeta = \frac{c}{2\sqrt{mk}}$$

we rewrite the system equation:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

which gives the state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, in terms of ω_n and ζ:

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$$

---

3. Time constant of a second order system

The time constant τ describes how quickly the system's response decays. For a **first-order system**, the time constant is the time it takes for the response to decay to $1/e$ of its initial value.

For a **second-order system**, the dominant time constant is related to the real part of the eigenvalues of A. The characteristic equation is:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

with roots:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- If $\zeta < 1$ (underdamped), the real part is $-\zeta\omega_n$, so the time constant is:

$$\tau = \frac{1}{\zeta\omega_n}$$

- If $\zeta \geq 1$ (overdamped or critically damped), the dominant pole determines τ, but for practical purposes, we still use:

$$\tau \approx \frac{1}{\zeta\omega_n}$$

  But note the farthest pole is $-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

Intuition behind time constant

- A **larger** τ means a slower response.
- A **smaller** τ means a faster response.
- In **underdamped** systems, τ governs how fast the oscillations decay.
- In **overdamped** systems, τ determines how quickly the system settles.