Linearization basics

- Linearization is Taylor expanding a function about a point and ignoring the higher order terms
- We do the expansion about a fixed point
    - quesiton - can we expand about any point and still inearize a system?

Taylor expansion

Taylor expand $\dot{x} = f(x, u)$ to approximate $f(x + \Delta x, u + \Delta u)$
with $\Delta x = x - x_0$, $\Delta u = u - u_0$

$\dot{x} = \frac{d}{dt}(x_0 + \Delta x) = \frac{d}{dt}\Delta x$, OR $f(x, u) = \frac{d}{dt}\Delta x$

$$f(x,u)\Big|_{x_0,u_0} = f(x_0, u_0) + \frac{\partial f}{\partial x}\Big|_{x_0,u_0} .\,(x - x_0) + \frac{\partial f}{\partial u}\Big|_{x_0,u_0} .\,(u - u_0) +$$

$$\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_{x_0,u_0} .\,(x - x_0)^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial u^2}\Big|_{x_0,u_0} .\,(u - u_0)^2 + \ \ldots$$

Since linearization is to be valid in a small zone around the *fixed points*

$$(x - x_0)^n \approx (u - u_0)^n \approx 0 \ \ \forall\, n \geq 2$$

$$f(x_0, u_0) = 0$$

Hence we can say

$$\dot{\Delta x}\Big|_{x_0,u_0} = \frac{\partial f}{\partial x}\Big|_{x_0,u_0} .\,(x - x_0) + \frac{\partial f}{\partial u}\Big|_{x_0,u_0} .\,(u - u_0)$$

$$\dot{\Delta x} = \frac{Df}{Dx}\Delta x + \frac{Df}{Du}\Delta u$$

$$\dot{\Delta x} = A\Delta x + B\Delta u$$

It is customary (abuse of notation) to drop the Δ. Hence, $\dot{x} = Ax + Bu$

D is the Jacobian matrix if x and u are vectors (system of equations)

$$\frac{Df}{Dx} = \begin{bmatrix} \frac{f_1}{x_1}, \frac{f_1}{x_2} \\ \frac{f_2}{x_1}, \frac{f_2}{x_2} \end{bmatrix}, \ \frac{Df}{Du} = \begin{bmatrix} \frac{f_1}{u_1}, \frac{f_1}{xu_2} \\ \frac{f_2}{u_1}, \frac{f_2}{u_2} \end{bmatrix}$$

Example

Single state system - propeller mechanical model

$$J\dot{\omega} + b(\omega)\omega^2 = T_{aero}$$

$$\dot{\omega} = -\frac{b(\omega)\omega^2}{J} + \frac{T_{aero}}{J}$$

$J = properller\ inertia\ (kg.m^2)$
$b(w) = aero\ damping\ coefficient\ (Nm.s^2)$
$\omega = speed\ of\ the\ propeller (rad/s) \leftarrow\ x$ (state)
$T_{aero} = aero\ resistive\ torque\ (Nm) \leftarrow\ u$ (input)

Linearizing

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial w} = \frac{-2b(w)}{J}\bigg|_{b_0, T_{aero_0}} = \frac{-2b_0\omega_0}{J} = A$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial T_{aero}} = \frac{1}{J}\bigg|_{b_0, T_{aero_0}} = \frac{1}{J} = B$$

$$\dot{\Delta} x = A\Delta x + B\Delta u$$

Simulating

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A = -2 * dQdOm2_Nms2 * fixedPoints.speed_radps / inertia_kgm2;
B = 1 / inertia_kgm2;
C = 1;
D = 0;
linSys = ss(A, B, C, D);
linSys = ss(Amatrix, Bmatrix, Cmatrix, Dmatrix);

[speed_radps, time_s] = lsim(linSys, (input.torque_Nm - fixedPoints.torque_Nm), time_s,
0); % lsim(sys, u, t, x0)
speed_radps = speed_radps + fixedPoints.speed_radps;
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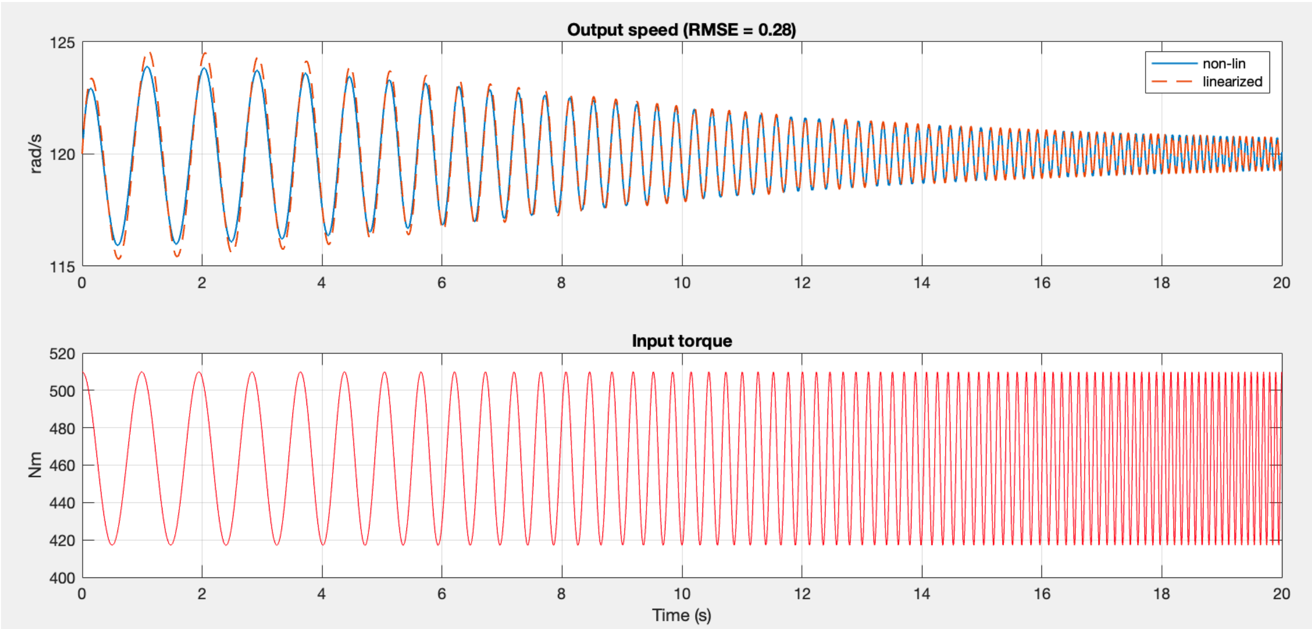
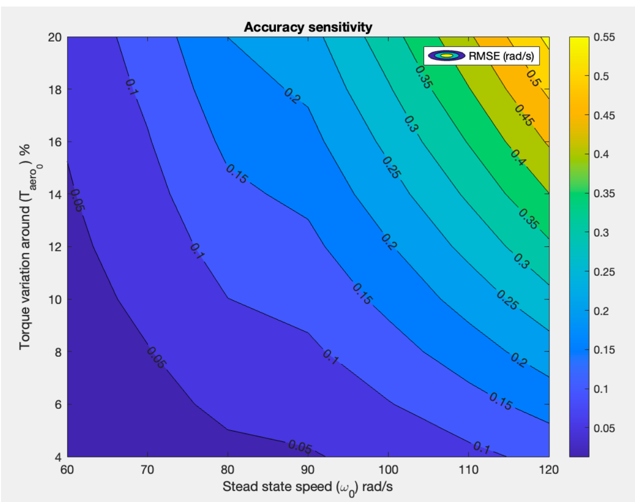
- Input $u =$ to $linSys =$ actual torque - fixedPoint torque
- Output = y from $linSys$ + fixedPoint speed



A sample time series comparison plot



Accuracy sensitivity as linearization fixed point and input variation increase

Two-state system - propeller thermo-mechanical model

Will do in a new branch.