Pade approximation (PA)

General description

- At it's core the PA is a fucntion approximation.
- Any fucntion can be approximated as a rational (ratio of) polynominal function.
    - The numerator and denominator can be of different order.
- In the dynamical-systems world, it is commonly used to approximate a time delay.
- A linear ODE can be represented as a Transfer Function.

Mathematically

A system $G(s)$ with a time delay can be represented as

$$G(s) = e^{-st} \frac{1}{\tau s + 1}$$

PA approximates e^{-st} part and the TF becomes, e.g.,

$$G(s) \approx \frac{1 - 0.5s}{1 + 0.5s} . \frac{1}{\tau s + 1}$$

More generally the Pade approximation is of the form

$$\frac{\sum_{j=0 \to m} a_j x^j}{1 + \sum_{k=1 \to n} b_k x_k}$$

This is equated to the first $m + n$ terms of the Taylor exapnsion of e^{-s} which is given by

$$e^{-s} = \sum_{n=o \to \infty} \frac{(-s)^n}{n!}$$

- We typically ignore a higher order coefficient matching for some reason, and that somehow produces better approximations had we considered it.
- We use similar order for numerator and denominator, otherwise the gain of the TF also gets affected (steep fall off at high frequencies)

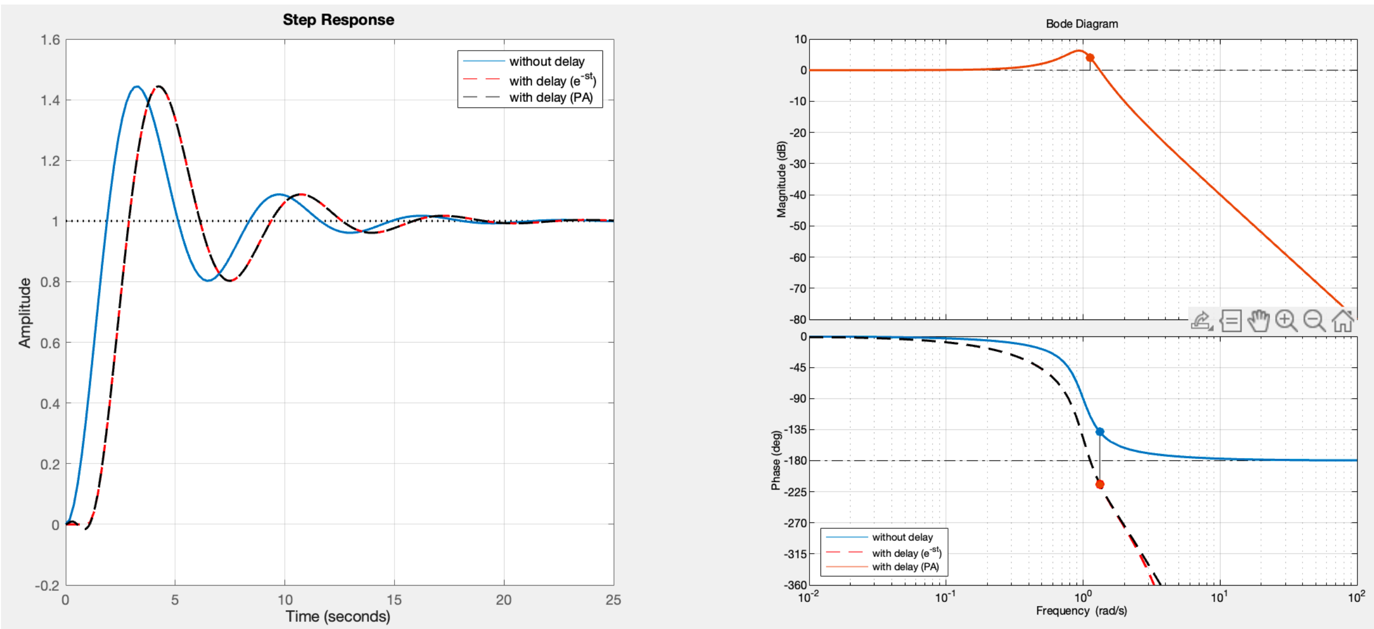
Example

$$TF = \frac{1}{s^2 + 0.5s + 1}$$

$$TF_{delay} = e^{-st}\frac{1}{s^2 + 0.5s + 1}$$

$$TF_{pade(order2)} = \frac{s^2 - 6s + 12}{s^2 + 6s + 12}$$

$$TF_{delayPA} = \frac{s^2 - 6s + 12}{s^4 + 6.5s^2 + 16s^2 + 12s + 12}$$



We can see that $TF_{delayPA}$ (black dotted) is approximating TF_{delay} (red dotted) by matching the phase margin at the cross-over frequency (frequency at -3dB gain)

Note - PA approximates the delay, it does NOT remove the delay meaning it does NOT alleviate the lesser phase margin that comes with delay. It only returns a TF with finite poles so that controller design can be done to increase the phase margin.

Why time delay matters?

- Short answer - it reduces phase margins
- Stability margins are given by i) gain margin, ii) phase margin
    - gain margin - how much (dB) can the system gain increase before system becomes unstable
    - phase margin - how much phase shift can happen before system becomes unstable. It is measured in degrees as $180^0 - phase$ at cross-over frequency (0dB gain frequency)
- Systems with delays mathematically have ∞ poles (or states). So controllers like LQR can not be used.
    - this is why pade approximation works as it converters a ∞ pole system to a finite one (depending on the order of PA)

How to choose the order of PA

- Short answer - PA should match the phase at cut-off frequency (-3dB gain) of the system **with delay**
- Increase the order till that happens

Other usage of PA

Other ways of approximating delays

Asides

1. Number of poles of a system with time delay is ∞

Link, and resources

1. [Brian Douglas, Matlab Techtalk]
2. [pade from Matlab]