Linearization basics

- Linearization is Taylor expanding a function about a point and ignoring the higher order terms
- We do the expansion about a fixed point
    - quesiton - can we expand about any point and still inearize a system?

Taylor expansion

Tayloe expand $\dot{x} = f(x, u)$ to approximate $f(x + \Delta x, u + \Delta u)$
with $\Delta x = x - x_0$, $\Delta u = u - u_0$

$\dot{x} = \frac{d}{dt}(x_0 + \Delta x) = \frac{d}{dt}\Delta x$, OR $f(x, u) = \frac{d}{dt}\Delta x$

$$f(x,u)\Big|_{x_0,u_0} = f(x_0, u_0) + \frac{\partial f}{\partial x}\Big|_{x_0,u_0} . (x - x_0) + \frac{\partial f}{\partial u}\Big|_{x_0,u_0} . (u - u_0) +$$

$$\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_{x_0,u_0} . (x - x_0)^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial u^2}\Big|_{x_0,u_0} . (u - u_0)^2 + \ \dots$$

Since linearization is to be valid in a small zone around the *fixed points*

$$(x - x_0)^n \approx (u - u_0)^n \approx 0 \ \ \forall \, n \geq 2$$

$$f(x_0, u_0) = 0$$

Hence we can say

$$\dot{\Delta x}\Big|_{x_0,u_0} = \frac{\partial f}{\partial x}\Big|_{x_0,u_0} . (x - x_0) + \frac{\partial f}{\partial u}\Big|_{x_0,u_0} . (u - u_0)$$

$$\dot{\Delta x} = \frac{Df}{Dx}\Delta x + \frac{Df}{Du}\Delta u$$

$$\dot{\Delta x} = A\Delta x + B\Delta u$$

It is customary (abuse of notation) to drop the Δ. Hence, $\dot{x} = Ax + Bu$

D is the Jacobian matrix if x and u are vectors (system of equations)

$$\frac{Df}{Dx} = \begin{bmatrix} \frac{f_1}{x_1}, \frac{f_1}{x_2} \\ \frac{f_2}{x_1}, \frac{f_2}{x_2} \end{bmatrix}, \ \frac{Df}{Du} = \begin{bmatrix} \frac{f_1}{u_1}, \frac{f_1}{xu_2} \\ \frac{f_2}{u_1}, \frac{f_2}{u_2} \end{bmatrix}$$

Example

propeller mechanical model

$$J\dot{\omega} + b(\omega)\omega^2 = T_{aero}$$

$$\dot{\omega} = -\frac{b(\omega)\omega^2}{J} + \frac{T_{aero}}{J}$$

$J = properller\ inertia\ (kg.\, m^2)$
$b(w) = aero\ damping\ coefficient\ (Nm.\, s^2)$
$\omega = speed\ of\ the\ propeller(rad/s) \leftarrow\ x$ (state)
$T_{aero} = aero\ resistive\ torque\ (Nm) \leftarrow\ u$ (input)