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1 A Cauchy–Schwarz Integral

Claim:

$$\left(\int_a^b f(x)dx\right)^4 \leq (b-a)^2 \int_a^b e^x f^2(x)dx \int \frac{f^2(x)}{e^x}dx$$

Proof: This is simply the integral case of the C.S. Inequality (of which we already are aware in discrete form).

It states

$$\int_a^b f^2(x)dx \int_a^b g^2(x)dx \geq \left(\int_a^b f(x)g(x)dx\right)^2$$

Here both

$$e^x f^2(x), \frac{f^2(x)}{e^x} \geq 0$$

So,

$$\int_a^b (\sqrt{e^x} f(x))^2 dx \int_a^b (\frac{f(x)}{\sqrt{e^x}})^2 dx \geq \left(\int_a^b f^2(x)dx\right)^2$$

$$\Rightarrow (b-a)^2 \int_a^b e^x f^2(x)dx \int \frac{f^2(x)}{e^x}dx \geq (b-a)^2 \left(\int_a^b f^2(x)dx\right)^2$$

Now,

$$\int_a^b 1^2 dx \int_a^b f^2(x)dx \geq \left(\int_a^b f(x)dx\right)^2$$

Here,

$$\int_a^b 1^2 dx \int_a^b f^2(x)dx \geq 0$$

,

$$\Rightarrow \left(\int_a^b 1dx \int_a^b f^2(x)dx\right)^2 = (b-a)^2 \left(\int_a^b f^2(x)dx\right)^2 \geq \left(\int_a^b f(x)dx\right)^4$$

$$\Rightarrow (b-a)^2 \int_a^b e^x f^2(x)dx \int \frac{f^2(x)}{e^x}dx \geq (b-a)^2 \left(\int_a^b f^2(x)dx \right)^2 \geq \left(\int_a^b f(x)dx \right)^4$$

which was what we needed to prove.

QED