Solution to AMM Problem 12432

Alex Jian

University of Michigan, Ann Arbor

1 Problem

Suppose that k and n are integers with $n \geq 2$ and $1 \leq k \leq n$. What is the average value of $\sum_{i=1}^{\pi(k)} \pi(i)$ over all permutations π of $\{1, \dots, n\}$?

2 Solution

Let $S = \sum_{i=1}^{\pi(k)} \pi(i)$. The average value in question is the expected value, $E[S]$ $= E[\sum_{i=1}^{\pi(k)} \pi(i)]$, taken over all permutations π of $\{1, \dots, n\}$. The first problem is that $\pi(k)$, the number of terms in the summation, varies with π. To deal with this, we consider $E[S]$ for fixed values of $\pi(k)$, then sum over all possible values of $\pi(k)$ using the law of total expectation:

$$E[S] = \sum_{j=1}^{n} P(\pi(k) = j) \cdot E[S|\pi(k) = j]. \tag{1}$$

All permutations are equally likely, so $\pi(k)$ is uniformly distributed on $\{1, \dots, n\}$, and for $1 \leq j \leq n$, $P(\pi(k) = j) = \frac{1}{n}$. For the expectation part, by linearity,

$$E[S|\pi(k) = j] = E[\sum_{i=1}^{\pi(k)} \pi(i)|\pi(k) = j] = \sum_{i=1}^{j} E[\pi(i)|\pi(k) = j]. \tag{2}$$

Let $1 \leq i \leq \pi(k)$. We want to find $E[\pi(i)|\pi(k) = j]$, which we can do by conditioning on i. If $i \neq k$, then $\pi(i) \neq j$. There are no other constraints, and all permutations π satisfying $\pi(k) = j$ are still equally likely, so under these conditions $\pi(i)$ is uniformly distributed on $\{1, \dots, j-1, j+1, \dots, n\}$:

$$P(\pi(i) = m|\pi(k) = j,\, i \neq k) = \frac{1}{n-1},\ m \in \{1, \dots, j-1, j+1, \dots n\}, \tag{3}$$

$$E[\pi(i)|\pi(k) = j,\, i \neq k] = \frac{(1 + \cdots + n) - j}{n-1} = \frac{\frac{n(n+1)}{2} - j}{n-1}. \tag{4}$$

If $i = k$, $\pi(i) = \pi(k) = j$. From (2), we have

$$E[S|\pi(k) = j] = \sum_{i=1}^{j} E[\pi(i)|\pi(k) = j].$$

The value of this summation depends on the summation bound j: If $j < k$, then for $1 \leq i \leq j$, $i \neq k$, so using (4), we get

$$E[S|\pi(k) = j, j < k] = \sum_{i=1}^{j} E[\pi(i)|\pi(k) = j, i \neq k]$$

$$= \sum_{i=1}^{j} \frac{1}{n-1}(\frac{n(n+1)}{2} - j) = \frac{j}{n-1}(\frac{n(n+1)}{2} - j).$$

If $j \geq k$, then the summation includes the kth index, so we can write:

$$E[S|\pi(k) = j, j \geq k] = j + \sum_{i=1, i \neq k}^{j} E[\pi(i)|\pi(k) = j, i \neq k]$$

$$= j + \frac{j-1}{n-1}(\frac{n(n+1)}{2} - j).$$

since $E[\pi(i)] = j$ when $i = k$. Breaking down the summation in (1) and plugging in $P(\pi(k) = j) = \frac{1}{n}$, we get

$$E[S] = \sum_{j=1}^{n} P(\pi(k) = j) \cdot E[S|\pi(k) = j]$$

$$= \frac{1}{n} \cdot (\sum_{j=1}^{k-1} E[S|\pi(k) = j, j < k] + \sum_{j=k}^{n} E[S|\pi(k) = j, j \geq k])$$

$$= \frac{1}{n} \cdot (\sum_{j=1}^{k-1} \frac{j}{n-1}(\frac{n(n+1)}{2} - j) + \sum_{j=k}^{n}(j + \frac{j-1}{n-1}(\frac{n(n+1)}{2} - j))).$$

At this point we have written $E[S]$ as a summation involving the two constants n and k as well as the summation index j. The power of j in this summation is at most 2. As a result, we can use the following summation identities to write this expression in terms of n and k:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \tag{5}$$

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}. \tag{6}$$

Doing this and simplifying gives the final answer:

$$E[S] = \frac{3n^3 + 2n^2 - 9n - 6k^2 + 12k + 6kn - 8}{12(n-1)}. \tag{7}$$