[Proof by Induction]

(NOTES)

1. Consider the sequence $b_0 = 3$ and for $n > 0$, $b_n = b_{n-1} + n$.
Prove for all $n \in \mathbb{N}_0$,
$$b_n = \frac{n^2+n+6}{2}.$$

Proof:

(a) Base Case: Let $n=0$. $b_0 = 3$ and $\frac{6}{2} = 3$.

(b) Inductive Step:

- Hypothesis: Assume the statement is true when $n = k$. Therefore $b_k = \frac{k^2+k+6}{2}$.
- We want to show $b_{k+1} = \frac{(k+1)^2+(k+1)+6}{2}$.
- Consider b_{k+1}:

$$b_{k+1} = b_k + (k+1)$$
$$= \frac{k^2+k+6}{2} + (k+1) \quad \text{[by hypothesis]}$$
$$= \frac{k^2+k+6}{2} + \frac{2k+2}{2}$$
$$= \frac{k^2+3k+8}{2}$$
$$= \frac{(k+1)^2+(k+1)+6}{2}$$

Suppose, for sake of contradiction, ~~...~~
$\exists x \in [0, \frac{\pi}{2}]$ ....

$x \in A$, then $x \in B$
$x \in B$, then $x \in A$

Thus we have shown through induction that $b_n = \frac{n^2+n+6}{2}$ for all $n \in \mathbb{N}_0$.

Strong induction:

- Hypothesis: Assume the statement is ~~...~~ true for $n = 0, 1, \cdots, m$ | $n \in \{0, 1, \cdots, m\}$.

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1B)

i] False: because negative x + negative x is less than x.

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ii] False: because $1 \in \mathbb{N}$ and $1^2 = 1$.

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iii] True: $m = -3$, $n = 1$ $\Rightarrow$ $7 - 6 = 1$

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iv] False: For $\exists x \in \mathbb{Z}$, have $y = x - 1$.

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v] True: for $\forall y \in \mathbb{Z}$, have $x = y - 1$.

1c) i] • Base Case: Let $n = 0$. Then $0 = \frac{0(0+1)}{2}$ is true, meaning the statement holds for base case.

• Inductive Hypothesis: Assume the statement is true when $n = K$. Therefore assume

$$0 + 1 + \cdots + K = \frac{K(K+1)}{2}.$$

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ii] • Base case: . Sorry

Let $n = 1$. Then $a_1 = 1$ and $1^2 = 1$. Therefore statement is true for case 1.

Let $n = 2$. Then $a_2 = 4$ and $2^2 = 4$. Therefore statement is true for case 2.

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ii cont] • Inductive Hypothesis: Assume the statement is true when $n \in \{1, 2, \cdots, m\}$. Meaning for all $n \in \{1, 2, \cdots, m\}$, $a_n = n^2$.

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2b) Let $2x+5 \geq 1$.

Then $2x \geq -4$ and $x \geq -2$.

Now consider $x^3 + 2x^2$. We can factor x^3+2x^2 as

$$x^3 + 2x^2 = x^2(x+2).$$

Observe that $x^2(x+2)$ is only negative when $x < -2$. Therefore, because $x \geq -2$, $x^3+2x^2 \geq 0$. Therefore, by proof of contrapositive:

If $x^3+2x^2 < 0$, then $x < -2$, meaning $2x+5 < 1$.

2c) Assume n is odd. Now consider $2n^2-2$. We can factor $2n^2-2$ as

$$2n^2-2 = 2(n^2-1) = 2(n+1)(n-1).$$

Observe that $8 \mid 2(n+1)(n-1)$ iff $4 \mid (n+1)(n-1)$. Therefore, because n is odd, both $(n+1)$ and $(n-1)$ have to be even, meaning:

$2 \mid (n+1)$,

$2 \mid (n-1)$, and

$4 \mid (n+1)(n-1)$.

Therefore $8 \mid 2(n+1)(n-1)$, if n is odd, and by contrapositive if $8 \nmid 2(n+1)(n-1) == 8 \nmid 2n^2-2$ then n must be even.

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3b) Suppose, for sake of contradiction,

$\exists a, b \in \mathbb{Z}$ s.t. $2a + 4b = 1$.

Then dividing both sides gives

$$a + 2b = \frac{1}{2},$$

which by the rules of integer addition is impossible, meaning the supposition has to be false. $\diamond$

4a) ~~Assume~~ ~~$\exists n \in \mathbb{N}$ s.t. $8 \nmid 9^n - 1$, and~~ [illegible] ~~[illegible] smallest~~

Let b be the smallest number in the natural numbers s.t. $8 \nmid 9^b - 1$. Note $b \neq 1$ because $8 \mid (9-1)$. Also note that $b-1$ must have that $8 \mid 9^{b-1} - 1$. However:

$$9^{b-1} - 1 = 9^b \cdot 9^{-1} - 1$$

$$= \frac{9^b}{9} - 1,$$

which only ~~divides~~ can be divided by 8 if 9^b posits $8 \mid 9^b - 1$. Therefore b is not the smallest number that has $8 \nmid 9^b - 1$.

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5a) Proof: Consider the sequence $b_n = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \cdots + \frac{1}{n(n+1)}$

(a) Base case: Let $n=1$.

$$\frac{1}{1(1+1)} = \frac{1}{2} = 1 - \left(\frac{1}{1+1}\right),$$ so the statement is true for $n=1$.

(b) Inductive step.

- Hypothesis: Assume the statement is true when $n=K$. Meaning assume the sequence

$$b_K = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \cdots + \frac{1}{K(K+1)} = 1 - \left(\frac{1}{K+1}\right).$$

- We want to show that the statement is true for $K+1$, meaning

$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \cdots + \frac{1}{(K+1)(K+2)} = 1 - \left(\frac{1}{K+2}\right).$$

- Consider $K+1$:

$$b_{K+1} = b_K + \frac{1}{(K+1)(K+2)}$$

$$= 1 - \left(\frac{1}{K+1}\right) + \frac{1}{(K+1)(K+2)} \quad \text{[by hypothesis]}$$

$$= 1 - \frac{K+2}{(K+1)(K+2)} + \frac{1}{(K+1)(K+2)}$$

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$$= 1 - \frac{K+2-1}{(K+1)(K+2)} = 1 - \frac{K+1}{(K+1)(K+2)}$$

$$= 1 - \frac{1}{K+2}$$

Thus we have shown through induction that $b_n = 1 - \left(\frac{1}{n+1}\right)$, for all $n \in \mathbb{N}$.

5c) Proof: Anthony Jones

(a) Base Case:

Let $n=0$; $a_0 = 1$ and $\frac{3^0+1}{2} = 1$. So the statement is true for $n=0$.

Let $n=1$; $a_1 = 3(1)-1 = 2$ and $\frac{3^1+1}{2} = 2$. So the statement is true for $n=1$.

(b) Inductive step:

- Hypothesis: Assume the statement is true when $k \in \{0, 1, \cdots, \cancel{n}\}$. Meaning $\exists k$, $a_k = \frac{3^k+1}{2}$.
- We want to show that the statement is true for $k+1$. Meaning $a_{k+1} = \frac{3^{k+1}+1}{2}$.
- Consider $k+1$:

$$a_{k+1} = 3a_k - 1$$

$$= 3\left(\frac{3^k+1}{2}\right) - 1 \quad \text{[by hypothesis]}$$

$$= \frac{3^{k+1}+3}{2} - \frac{2}{2}$$

$$= \frac{3^{k+1}+1}{2}.$$

[didnt need strong induction...]

Thus we have shown through induction that $a_n = \frac{3^n+1}{2}$, for all $n \in \mathbb{N}_0$.