1a) A group is a non-empty set that fits four conditions

1) G is closed meaning $x, y \in G \Leftrightarrow xy \in G$

2) G is associative $a(bc) \Leftrightarrow (ab)c$

3) Inverses exist : $x \in G \Leftrightarrow x^{-1} \in G$

4) Identity exists : $(ae) = e$, $\forall a \exists e$, $a, e \in G$

b) A normal subgroup N of a group G is a subgroup $N \leq G$ where $g^{-1}Ng = N$ for all $g \in G$
Or specifically $H = \{g^{-1}hg, g \in G, h \in H\} \Leftrightarrow H \trianglelefteq G$

c) If G acts on a set S then G_S is the result $G_S := \{ga \mid g \in G, a \in S\}$.

d) Let $\phi : G \to H$ be a homomorphism of group G onto H. Then $G/\ker(\phi) \cong \mathrm{im}(\phi)$, when $H \trianglelefteq G$ then $G/H \cong H$.

e) Let $N \trianglelefteq G$. Then the union N/G and H/G, where $H \leq G$, forms an isomorphism of G : $f : N/G \xrightarrow{\sim} H/G$.

Anthony J

f) Lagranges theorem states that if $H \leq G$ and $|G| = n$, where n is finite, then $|H| \mid n$; or moreso $|H| \mid |G|$.

g) $N_G(H) := \{g \mid g^{-1}Hg = H \text{ for } g \in G\}$

h) $|G| = 3500$. Cauchy's theorem gaurentees an element for values of $n = 1, 2, 5, 7$.

i) $[x,y] = x^{-1}y^{-1}xy$, $x, y \in G$.

j) $(1257)(3214)(27)(132)(56143)(23)(134)$
$(1728431)(56)$

k) $|(1728431)(56)| = 2 \cdot 7 = 14$

l) $|(123)(45678)| = 3 \cdot 5 = 15$.

m) stabilizes of x in S is
$$H := \{g \in G \mid gs = x \text{ for } s, x \in S\}.$$

2a) Observe that $gHg^{-1} \subseteq G$ as for every $h \in H$, $(ghg^{-1}) \in G$. Now observe that the properties of a group apply to gHg^{-1}:

1) gHg^{-1} is closed. Let $x, y \in gHg^{-1}$. Then $xy = g_x h_x g_x^{-1} g_y h_y g_y^{-1}$, which is a product of elements of G. Assume $xy \notin gHg^{-1}$: then g_{xy} and g_{xy}^{-1} cannot both exist in G. But since G is a group, we know that if either g_{xy} or g_{xy}^{-1} exists, the other does too. Thus gHg^{-1} is closed.

2) gHg^{-1} inherits the same associative properties as G and H.

3) As mentioned in (1), we know that if $x \in gHg^{-1}$ then $x^{-1} = (g_x h_x g_x^{-1})^{-1} \in gHg^{-1}$.

4) Consider $geg^{-1} = gg^{-1} = e$. $e \in gHg^{-1}$.

Thus gHg^{-1} is a subgroup of G.

Anthony J

3a) Let $x, y \in G$. Observe that

$\phi(x), \phi(y) \in \mathrm{im}(\phi)$ and that ~~$\phi(y^{-1})$~~

$\phi(y^{-1}) = (\phi(y))^{-1} \in \mathrm{im}(\phi)$. ~~Thus by the subgroup criterea, any $a = \phi(x) \in \mathrm{im}(\phi)$ and~~

Finally consider $\phi(xy^{-1}) = \phi(x)(\phi(y))^{-1} \in \mathrm{im}(\phi)$.
$xy^{-1} \in G$ so $\phi(xy^{-1}) \in \mathrm{im}(\phi)$; therefore by the subgroup criterea any element $\phi(x)$ and $\phi(y)$ implies $\phi(xy^{-1}) \in \mathrm{im}(\phi)$. Thus $\mathrm{im}(\phi) \leq G$.

3b) $\mathrm{im}(\phi) \leq G$, by which Lagranges Theorem shows that $|\mathrm{im}(\phi)| \mid |G|$.

Anthony J

4) Let $x \in H \setminus K$ and $y \in K \setminus H$ where $H, K < G$. Observe that if $H \cup K = G$, then $xy \in H \cup K \in G$, as G is a closed set/group. However, we know that neither $x, y \in H \cup K$, meaning any elements $d \in K \cap H$ are not elements of G.

5) Observe that $N/G = \{gN \mid g \in G\}$, and that G and N are abelian (p^2).