Overall: C

MATH 4560/6560 Topology
Spring 2022
Exercises 8
Target date: Thu 31 Mar 11:59PM

Instructions:

- For each item in each exercise, make sure to include
  - any sources used,
  - any collaborators worked with, and
  - a self-assessment of your work: C (correct), R (revision needed), or M (major revision needed).
- For the entire homework set, make sure to also include:
  - a reflection on the assignment and your solutions. Reflections may include
    - · discussion of how routine or challenging the assignment was,
    - · approximation of time spent on the assignment or on individual exercises,
    - · details about particular difficulties or false starts,
    - · explanations of why solutions are incomplete or incorrect, etc.
  - a self-assessment (C/R/M) for the entire assignment.

Exercise 1. (a) Find an online news story from 2022 that includes the word "connected." Give the link and write a summary of the article. Does the article's use of the word "connected" match with how we use it in class? How are they different and how are they similar?

Source. This article is about the Dodge Hornet, a car who's unveiling has been well anticipated by consumers. The true reveal of the car has been held back by problems concerning the supply chain, and the car has been referred to as "an ideal combination of American fun and European function." The use of the word connected in this article refers to connecting ideas, and thus is similar to an associative operation. This is different to how we use the word in our class, as in class connected means that there are no gaps in the open subsets of a set. The concept of association is not really assigned to our use in class.

(b) Find an online news story from 2022 that includes the word "compact." Give the link and write a summary of the article. Does the article's use of the word "compact" match with how we use it in class? How are they different and how are they similar?

Source. I chose to do the same article, which also uses the word compact. The article actually uses the word multiple times in different ways, so I will address each. The first use of compact is the most similar to our use in class, as it refers to a "compact space". This invokes the idea of cramming things closely together, which is sort of what a compact set does. In class we learned that compact sets limit the representation of an open cover. I like to this of this as removing the holes of limiting sequences, whereas connect sets remove holes in entire subsets. The other use of compact in the article is in the phrase "compact cars," which is a size of car that roughly translates to "small, family-sized". This also sort of fits our definition, as compact sets reduce infinite sequences down to finite ones; in this way we can think of it as shrinking a collection of open sets.

Exercise 2. Let $f\colon S \to T$ be a surjective (i.e., onto) continuous function between topological spaces.

(a) Prove that if S is connected, then T is connected.

Pf: Suppose S is connected. Let $U, V \subseteq S$ be non-empty open subsets such that $U \cup V = S$. Then there exists at least one element $s \in S$ such that $s \in U$ and $s \in V$. Now suppose $X, Y \subseteq T$ were non-empty open subsets such that $X \cup Y = T$. Consider, by definition, that the pre-image of any open set under a continuous function is open. Hence $f^{-1}(X)$ and $f^{-1}(Y)$ are open subsets contained in S. Because f is surjective, it follows that $X \cup Y = T \implies f^{-1}(X) \cup f^{-1}(Y) = S$; and because $X, Y \neq \emptyset$, it also follows that both pre-images are nonempty. Thus there exists at least one element $s \in S$ such that $s \in f^{-1}(X)$ and $s \in f^{-1}(Y)$. Therefore, since $f(s) \in T$ by definition, this implies that the image of s is contained in both X and Y, and hence T is connected.

(b) Prove that if S is compact, then T is compact.

Pf: Suppose S is compact. Let $\bigcup_{i \in A} U_i$ be some open cover of T. Then for every $t \in T$, there exists some $a \in A$ such that $t \in U_a$. Consider again that the pre-image of any open set under a continuous function is open. Hence $f^{-1}(U_i)$ are open subsets contained in S for all $i \in A$. Because f is surjective, it follows that $\bigcup_{i \in A} U_i = T \implies \bigcup_{i \in A} f^{-1}(U_i) = S$; and because S is compact, there exists a finite sub-cover $A' \subseteq A$, $|A'| \in \mathbb{N}$ such that $\bigcup_{i \in A'} f^{-1}(U_i) = S$. Finally, note that $f(f^{-1}(X)) = X$ for all $X \subseteq T$; hence $\bigcup_{i \in A'} U_i = T$, as the image of any cover of S is a cover of T. Therefore T is compact.

Exercise 3. Prove that the closed interval $[0, 1] \subseteq \mathbb{R}$ is compact in the subspace topology.

Pf: Let $C = \{I_i\}_{i \in A}$ be an open cover of $[0, 1]$. Define $C' \subseteq C$ to be the sub-collection that's given by removing each interval strictly contained in another interval; that is,

$$C' = \{I_i\}_{i \in A' \subseteq A} = \{I_i \mid I_i \not\subset I_j\}_{i \neq j \in A}.$$

Define $C'' \subseteq C'$ to be any sub-collection that's given by removing all but one joint interval for each interval belonging C':

$$C'' = \{I_i\}_{i \in A'' \subseteq A'} = \{I_i \mid (\exists! j \in A')[I_i \cap I_j \neq \emptyset]\}_{i, j \in A'}.$$

Recall that because of the subspace topology, the set $[0, x) \subset [0, 1]$ is open for any $x \in (0, 1)$. Because there can at most be one subset of this form in the collection, we will call this $I_1 \in C''$. Consider that each $I_i \in C$ is by definition bounded, and hence has a minimum length. Consider this length to be δ. Then, given that each $I_i < I_j < I_k \in C''$ has at most $|I_i - I_k| > \delta$, then for $n = 1/\delta$ intervals it follows that $|I_1 - I_n| > 1$. Thus C'' is finite, and so $[0, 1] \subseteq \mathbb{R}$ is compact within the subspace topology.