Homework 4

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MATH 4120

1a) Let $[x,y] = 1$.
Then $x^{-1}y^{-1}xy = 1$

$$(xx^{-1})y^{-1}xy = x$$
$$y^{-1}xy = x$$
$$(yy^{-1})xy = yx$$
$$xy = yx$$

Thus, x and y commute.

Let $xy = yx$.
Observe that $xx^{-1} = yy^{-1} = 1$.
Then $x^{-1}y^{-1}yx = 1$.
But since $xy = yx$, this shows

$$x^{-1}y^{-1}xy = [x,y] = 1.$$

Thus $x, y \in G$ commute if and only if $[x,y] = 1$.

1b) Let $n \in G'$ and $g \in G$.
Then $n \in G$ as well, as $G' \leq G$.
Therefore $[n,g] \in G'$:

$$[n,g] = n^{-1}g^{-1}ng.$$

But, being a group, G' is closed under the same operation as G, meaning $n[n,g] \in G'$ where

$$n[n,g] = (nn^{-1})g^{-1}ng = g^{-1}ng.$$

Therefore, for all $n \in G'$ and $g \in G$, $g^{-1}ng \in G^{-1}$; Thus $G' \trianglelefteq G$.

1c) Let $N \trianglelefteq G$.

For the first direction, let G/N be abelian. Then, for every $a, b \in G$,

$$\begin{aligned} aNbN &= bNaN & \iff \\ abN &= baN & \iff \\ b^{-1}abN &= (b^{-1}b)aN & \iff \\ b^{-1}abN &= aN & \iff \\ a^{-1}b^{-1}abN &= (a^{-1}a)N & \iff \\ a^{-1}b^{-1}abN &= N. \end{aligned}$$

Thus $[a,b] \in N$, as N is a closed group under the same operation as G. Therefore $[a,b] \in N$ for every $a, b \in G$; Thus N contains G'.

For the other direction, let N contain G'. Then, for every $a, b \in G$, $[a,b] \in N$:

$$[a,b] = a^{-1}b^{-1}ab = n, \quad n \in N.$$

However, $nN = N$, meaning

$$\begin{aligned} [a,b]N = a^{-1}b^{-1}abN &= N & \iff \\ b^{-1}abN &= aN & \iff \\ abN &= baN. \end{aligned}$$

Therefore, for every $a, b \in G$ and $n \in N$,

$$abn = ban.$$

Thus G/N is abelian.

2) Recall that the collection of left (resp. right) cosets of H in G form a partition of G. Because the index of H in G is 2, this means that the two cosets g_1H and g_2H, $g_1, g_2 \in G$,

$$g_1H = G - g_2H \text{ and, similarly, } g_2H = G - g_1H.$$

Observe that H is a left coset of G:

Consider $e \in G$. The left coset $eH = H$.

This means that the other coset has

$$gH = G - H, \text{ for some } g \in G.$$

But observe, similarly, that the right cosets of H in G form the same partition:

Consider $e \in G$. The right coset $He = H$.

Therefore, the other right coset shows

$$Hg = G - H, \text{ for some } g \in G.$$

Thus $eH = H = He$ and $gH = G - H = Hg$, for any possible coset of H in G.

Therefore, by Proposition 4.1.9, $H \trianglelefteq G$.

$* \langle (a\ b) \rangle = \{ (), (a\ b) \}$

2 ext) Consider $[G : H] = 3$.

This is not necessarily the same, as there are now 3 cosets in each partition of G.

Consider $G = S_3$ and $H = \langle (1\ 3) \rangle$.

The left cosets of H in G are

$L = \{ e\langle(1\ 3)\rangle, (1\ 2)\langle(1\ 3)\rangle, (2\ 3)\langle(1\ 3)\rangle \}$

while the right cosets of H in G are

$R = \{ \langle(1\ 3)\rangle e, \langle(1\ 3)\rangle(1\ 2), \langle(1\ 3)\rangle(2\ 3) \}$.

However, observe that not every left coset gH is equal to the corresponding right coset Hg, $g \in G$.

Consider $g = (2\ 3)$:

$$gH = (2\ 3)\langle(1\ 3)\rangle = \{ (2,3), (1,2,3) \}$$

$$Hg = \langle(1\ 3)\rangle(2\ 3) = \{ (2,3), (1,3,2) \}$$

Thus, by proposition 4.1.9, H cannot be normal as $gH \neq Hg$.

4a) Let $H \leq Q_8$ be a proper, nontrivial subgroup. Observe that the elements $e = 1$ and $\bar{e} = -1$ must be elements of H: e is the identity of G, and for any non-identity element $h \in H$, $h^{-1} = \bar{e}h$. Thus, for H to be closed under multiplication, $\bar{e} \in H$.

Also observe that for

$$Q_8 = \{\bar{e}, i, j, k \mid \bar{e}^2 = e,\ i^2 = j^2 = k^2 = ijk = e\},$$

the relations

$$ij = k$$
$$jk = i$$
$$ki = j$$ extend to show

$$j^{-1}(ij) = j^{-1}k = -jk = \bar{e}(jk) = \bar{e}i = -i = i^{-1}$$
$$k^{-1}(jk) = k^{-1}i = -ki = \bar{e}(ki) = \bar{e}j = -j = j^{-1}$$
$$i^{-1}(ki) = i^{-1}j = -ij = \bar{e}(ij) = \bar{e}k = -k = k^{-1}.$$

Now consider any $g \in Q_8$ and $h \in H$:

If $g \in H$, $g^{-1}hg \in H$ as H is closed.
If, however, $g \notin H$, then

$$g^{-1}hg = h^{-1},$$ as shown above.

Therefore $g^{-1}hg \in H$ for every $g \in G$.
Thus, H is normal.

4c) Let $G = D_4$, the dihedral group of a square.

$G = \langle r, s \mid r^4 = s^2 = (sr)^2 = 1 \rangle$

$= \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$

Let $H = \{e, r, r^2, s, sr, sr^2\}$

and $K = \{e, s\}$

Observe that $H \trianglelefteq G$

3) Let $G/Z(G)$ be cyclic.

Then $G/Z(G) = \langle z \rangle = \{z^n \mid n \in \mathbb{Z}, z \in G/Z(G)\}$

Therefore $G/Z(G) = \{(gZ(G))^n \mid n \in \mathbb{Z}, g \in G\}$.

Let $h \in G$.

Then $hZ(G) = (gZ(G))^n = g^n Z(G)$ for some $n \in$
This, from proposition 4.1.8, implies:

$$hg^{-n} \in Z(G).$$

Therefore, for $h \in G$, there exists some $z \in Z(G)$ such that $h = g^n z$.

Let $x, y \in G$ such that

$$x = g^a z_1 \text{ and}$$
$$y = g^b z_2.$$

Then $xy = (g^a z_1)(g^b z_2)$.
Observe that since $z_1, z_2 \in G/Z(G)$:

$$\begin{aligned} xy &= g^a (z_1 g^b) z_2 \\ &= g^a (g^b z_1) z_2 \\ &= g^b g^a (z_1 z_2) \\ &= g^b g^a z_2 z_1 = yx. \end{aligned}$$

Thus G is abelian.