HW8 MATH 4540

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April 2022

1 Exercises

1. Because $\{f_n\}$ is equicontinuous, it follows that for every $\epsilon > 0$, there exists some distance $\delta > 0$ such that

$$|f_n(x) - f_n(y)| = |f(nx) - f(ny)| < \epsilon$$

whenever $x, y \in [0, 1]$ and $|x - y| < \delta$. Consider now $a = \frac{x}{n}$ and $b = \frac{y}{n}$; then clearly $a, b \in [0, 1]$ and

$$|a - b| = \left|\frac{x}{n} - \frac{y}{n}\right| = \frac{|x - y|}{n} < \delta.$$

Thus it follows similarly that

$$|f_n(a) - f_n(b)| = |f(x) - f(y)| < \epsilon.$$

Hence f is uniformly continuous. Furthermore, suppose $\epsilon = |f(x) - f(y)|$. Then we reach a contradiction, as $|f(x) - f(y)| < |f(x) - f(y)|$. Hence it must follow that $|f(x) - f(y)| \leq 0$, and thus by definition $x = y$.

2. By the Stone-Weierstrass Theorem, there exists a sequence of polynomials $\{P_n\}$ that converges uniformly to f on $[0, 1]$. Note, for each polynomial

$$P_n(x) = \sum_{k=1}^{n} a_k x^k$$

($a_k \in \mathbb{R}$ for all $1 \leq k \leq n$), that

$$\int_0^1 f(x)P_n(x)\,\mathrm{d}x = \int_0^1 f(x) * \sum_{k=1}^{n} a_k x^k\,\mathrm{d}x = \sum_{k=1}^{n} a_k * \int_0^1 f(x)x^k\,\mathrm{d}x$$

$$= \sum_{k=1}^{n} a_k * 0 = 0.$$

Furthermore, because $\{P_n\}$ converges uniformly,

$$\lim_{n\to\infty} \int_0^1 P_n(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x$$

(Rudin's 7.16), and thus

$$\lim_{n\to\infty}\int_0^1 f(x)P_n(x)\,\mathrm{d}x = \int_0^1 f^2(x)\,\mathrm{d}x.$$

Note that f^2 is continuous and bounded, and since f maps to $\mathbb{R}$, that $f^2 \geq 0$. Hence if

$$\int_0^1 f^2(x)\,\mathrm{d}x = 0,$$

it must follow that $f = 0$, as if otherwise $f(t) > 0$ for $t \in [0,1]$, then

$$\int_{t-\epsilon}^{t+\epsilon} f^2(x)\,\mathrm{d}x > 0$$

whenever $\epsilon \to 0$, and hence a contradiction is reached.

3. Because $\{f_n\}$ is a uniformly bounded set of functions, there exists a number M such that

$$|f_n(x)| < M$$

$(x \in [a,b], n = 1,2,3,\dots)$. Hence

$$|F_n(x)| = \int_a^x |f_n(t)|\,\mathrm{d}t < \int_a^x M\,\mathrm{d}t = M(x-a).$$

Thus for $\phi(x) = M(x-a)$, it follows that $F_n(x)$ is pointwise bounded, as

$$|F_n(x)| < \phi(x)$$

$(x \in [a,b], n = 1,2,3,\dots)$. Consider also, without loss of generality, that if $x < y \in [a,b]$, then

$$|F_n(x) - F_n(y)| \leq \left|\int_a^x f_n(t)\,\mathrm{d}t - \int_a^y f_n(t)\,\mathrm{d}t\right| = \left|\int_y^x f_n(t)\,\mathrm{d}t\right|.$$

Suppose $\epsilon > 0$. Then for any $x, y \in [a,b]$ such that $|x-y| < \epsilon/M$,

$$|F_n(x) - F_n(y)| \leq \left|\int_y^x M\,\mathrm{d}t\right| = M|x-y| < \epsilon;$$

and hence $\{F_n\}$ is equicontinuous. Noting that $[a,b] \subset \mathbb{R}$ is compact, it thus follows that $\{F_n\}$ must contain a uniformly convergent subsequence (Rudin's 7.25).

4. Note that

$$|2^{-n} f(3^{2n-1}t)| \leq 2^{-n}$$

and

$$|2^{-n} f(3^{2n}t)| \leq 2^{-n},$$

and that $\sum_{k=1}^{\infty} 2^{-n} \to 1$. Then it follows that both sequences converge uniformly, to $x(t)$ and $y(t)$, respectively (Rudin's 7.10). Thus each function is continuous (Rudin's 7.12), and hence so must be their product $\phi(t) = (x(t), y(t))$. To see that $\phi(t)$ is onto, we consider any arbitrary $(a, b) \in [0, 1]^2$. Then it follows that a can be represented in binary by

$$a = \sum_{k=1}^{\infty} \frac{a_k}{2^k},$$

where each $a_k \in \{0, 1\}$. Similarly, for each $b_k \in \{0, 1\}$,

$$b = \sum_{k=1}^{\infty} \frac{b_k}{2^k}.$$

Consider $k \in \mathbb{N}$. Whenever $a_k = 1$ and $b_k = 0$,

$$t = 3^{-2k} \implies f(3^{2k-1}t) = f(1/3) = 0$$

and

$$t = 3^{-2k} \implies f(3^{2k}t) = f(1) = 1;$$

and hence $x(3^{-2k}) = a_k$ and $y(3^{-2k}) = b_k$. Whenever $a_k = 0$ and $b_k = 1$,

5. Let $x_1, x_2 \in S^1$ be distinct points on the unit circle. Note then that

$$f(e^{i\theta}) = e^{i\theta}$$

is contained in the algebra provided, and hence $f(x_1) = x_1 \neq x_2 = f(x_2)$. Thus $\mathscr{A}$ separates points on S^1. Consider now

$$g(e^{i\theta}) = e^{-i\theta}.$$

Note that g is a composition of continuous functions e^x and $-ix$, and thus is continuous on S^1. Then by the Stone-Weierstrass Theorem, there exists a sequence $\{P_n\} \subseteq \mathscr{A}$ of polynomials that converged uniformly to g on S^1. Consider however that

$$\int_0^{2\pi} g(e^{i\theta})e^{i\theta}\,\mathrm{d}\theta = \lim_{n\to\infty} \int_0^{2\pi} P_n(e^{i\theta})e^{i\theta}\,\mathrm{d}\theta$$
$$= \int_0^{2\pi} e^{-i\theta}e^{i\theta}\,\mathrm{d}\theta = \int_0^{2\pi} 1\,\mathrm{d}\theta = 2\pi \neq 0$$

Hence because the integral of P_n does not converge to 0, there exists some $N \in \mathbb{N}$ such that, when $n > N$,

$$\int_0^{2\pi} P_n(e^{i\theta})e^{i\theta}\,\mathrm{d}\theta \neq 0.$$

This is a contradiction, because by assumption each $P_n \in \mathscr{A}$, and hence each integral should be zero. Thus g is not in the uniform closure of S^1.