Anthony J

1. Find the Laplace transform of each:

(a) $f(t) = 5t^3 + 2e^{-4t} - \cos(\pi t) + \sin(t)$

$$F(s) = \mathcal{L}\{5t^3 + 2e^{-4t} - \cos(\pi t) + \sin(t)\}(s)$$

$$= \frac{30}{s^4} + \frac{2}{s+4} - \frac{s}{s^2+\pi^2} + \frac{1}{s^2+1}.$$

(b) $f(t) = 4u(t-2) + (t-5)u(t-5) + (t+1)u(t-7)$

$$\mathcal{L}\{4u(t-2)\}(s) = \frac{4e^{-2s}}{s}$$

$$\mathcal{L}\{(t-5)(u(t-5))\}(s) = e^{-5s}\mathcal{L}\{t\}(s)$$

$$= e^{-5s}\left(\frac{1}{s^2}\right)$$

$$\mathcal{L}\{(t+1)u(t-7)\}(s) = \mathcal{L}\{((t-7)+8)u(t-7)\}(s)$$

$$= e^{-7s}\mathcal{L}\{t+8\}(s)$$

$$= e^{-7s}\left(\frac{1}{s^2} + \frac{8}{s}\right)$$

$$F(s) = \frac{4e^{-2s}}{s} + e^{-5s}\left(\frac{1}{s^2}\right) + e^{-7s}\left(\frac{1}{s^2} + \frac{8}{s}\right)$$

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1. (continued)

(c) $F(t) = t^4 e^{3t} + (t * e^{3t})$

$$\mathcal{L}\{t^4 e^{3t}\}(s) = \frac{4!}{(s-3)^5}$$

$$\mathcal{L}\{t * e^{3t}\}(s) = (\mathcal{L}\{t\}(s))(\mathcal{L}\{e^{3t}\}(s))$$
$$= \left(\frac{1}{s^2}\right)\left(\frac{1}{s-3}\right)$$

$$F(s) = \frac{4!}{(s-3)^5} + \frac{1}{s^2(s-3)}$$

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2. Find the inverse Laplace transform:

(a) $F(s) = \dfrac{s}{(s-3)^2+9} = \dfrac{(s-3)}{(s-3)^2+9} + \dfrac{3}{(s-3)^2+9}$

$\mathcal{L}^{-1}\{F(s)\} = e^{3t}\cos(3t) + e^{3t}\sin(3t)$

(b) $F(s) = \dfrac{3}{s^2+4s+13} = \dfrac{3}{(s+2)^2+9}$

$\mathcal{L}^{-1}\{F(s)\} = e^{-2t}\sin(3t)$

(c) $F(s) = \dfrac{2}{s-6} + \dfrac{e^{-5s}}{s-6}$

$\mathcal{L}^{-1}\left\{\dfrac{2}{s-6}\right\} = 2e^{6t}$

$\mathcal{L}^{-1}\left\{\dfrac{e^{-5s}}{s-6}\right\} = g(t)\,u(t-5)$

$= e^{6(t-5)}u(t-5)$

$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{2}{s-6}\right\} + \mathcal{L}^{-1}\left\{\dfrac{e^{-5s}}{s-6}\right\}$

$= 2e^{6t} + e^{6(t-5)}u(t-5)$

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3. (a) $H(s) = \frac{1}{(s-1)(s+6)}$

$H(s) = \frac{A}{s-1} + \frac{B}{s+6}$;

$\begin{cases} A + B = 0 \\ 6A + (-B) + (-1) = 0 \end{cases}$ $A = 1/7$, $B = -1/7$

$H(s) = \frac{1}{7}\left(\frac{1}{s-1}\right) - \frac{1}{7}\left(\frac{1}{s+6}\right)$

$\mathcal{L}^{-1}\{H(s)\} = \frac{1}{7}e^{t} - \frac{1}{7}e^{-6t}$

(b) $H(s) = \frac{1}{(s-1)(s+6)} = \left(\frac{1}{s-1}\right)\left(\frac{1}{s+6}\right)$

~~$\mathcal{L}^{-1}\{H(s)\}$~~

$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}$ $\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\} = e^{-6t}$

$$\mathcal{L}^{-1}\{H(s)\} = \int_0^t e^{\tau} e^{-6(t-\tau)}\, d\tau$$

$$= \int_0^t e^{7\tau - 6t}\, d\tau$$

$$= \frac{1}{7}\left(e^{7\tau - 6t}\right)\Big|_0^t$$

$$= \frac{1}{7}\left(e^{t} - e^{-6t}\right)$$

Anthony J

4. Solve the IVP:

$$y'' - 4y' + 4y = t^3 e^{2t}, \quad y(0) = y'(0) = 0$$

- Take the Laplace transform of both sides:

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \frac{3!}{(s-2)^4}$$

- Evaluate laplace transforms of y' and y'':

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}\{y''\} = s\{y'\} - y''(0) = s^2 Y(s)$$

- Substitute the o.d.e:

$$s^2 Y(s) - 4sY(s) + 4Y(s) = \frac{3!}{(s-2)^4}$$

$$Y(s) = \frac{6}{(s-2)^4}\left(\frac{1}{s^2 - 4s + 4}\right) = \frac{6}{(s-2)^4 (s-2)^2}$$

$$Y(s) = \frac{6}{(s-2)^6}$$

- Find the solution:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{120} t^5 e^{2t}.$$

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5. (a) $g^*(t) = t\left(u(t) - u(t-3)\right) + \left(u(t-3) - u(t-6)\right)$

$$g^*(t) = t\,u(t) + (1-t)\,u(t-3) - u(t-6)$$

(b) $[0,3)$ t $g^*(t) = t$

$[3,6)$ t $g^*(t) = t + (1-t) = 1$

$(6,\infty)$ t $g^*(t) = t + (1-t) - 1 = 0$

Therefore $g^*(t) = g(t)$ ✓.

(c) $\mathcal{L}\{g(t)\} = \mathcal{L}\{g^*(t)\}$

$= \mathcal{L}\{t\,u(t)\} + \mathcal{L}\{((t-3)+2)\,u(t-3)\} - \mathcal{L}\{u(t-6)\}$

$= \frac{1}{s^2} - e^{-3s}\left(\frac{1}{s^2} + \frac{2}{s}\right) - \frac{e^{-6s}}{s}$.

(d) $\mathcal{L}\{f(t)\} = \frac{F_T(s)}{1-e^{-sT}}$

$$= \frac{1}{(1-e^{-sK})s^2} - \frac{e^{-3s}\left(\frac{1}{s^2} + \frac{2}{s}\right)}{(1-e^{-sK})} - \frac{e^{-6s}}{(1-e^{-sK})s}$$

For $K > 0$, where K is the periodic repeat of $g(t)$.

Anthony Jon

Bonus Problem:

Solve the IVP:

$$y' - 2y = e^{3t}, \quad y(0) = 0.$$

- Take the Laplace transform of both sides:

$$\mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \frac{1}{s-3}$$

- Evaluate laplace transforms of y':

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)$$

- Substitue into the ode:

$$sY(s) - 2Y(s) = \frac{1}{s-3}$$

$$Y(s) = \frac{1}{s-3}\left(\frac{1}{s-2}\right)$$

- complete using Partial Fractions

$$Y(s) = \frac{A}{s-3} + \frac{B}{s-2};$$

$$\left\{\begin{matrix} 0 = A + B \\ 1 = -2A - 3B \end{matrix}\right\} \quad \begin{matrix} A = 1 \\ B = -1 \end{matrix}$$

- Find the Solution:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{3t} - e^{2t}.$$