HW4 MATH 4000

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1 Exercises

1. Let T be a geometrically distributed random variable with parameter p. Using the probability generating function, which was derived in class, find the expected value and variance of T. Then, by definition

$$P\{T = n\} = (1-p)^{n-1}p$$

and

$$1 = \sum_{n=1}^{\infty} P\{T = n\} = \sum_{n=1}^{\infty}(1-p)^{n-1}p.$$

Recall that the expected value $E(T)$ is given as the sum of the possible values of T multiplied by each value's respective probability. Hence

$$E(T) = \sum_{n=1}^{\infty} nP\{T = n\} = \sum_{n=1}^{\infty} n(1-p)^{n-1}p.$$

Rearranging, we find

$$E(T) = \sum_{n=1}^{\infty} n(1-p)^{n-1}p$$

$$= \sum_{n=1}^{\infty}(1-p)^{n-1}p + \sum_{n=1}^{\infty}(n-1)(1-p)^{n-1}p$$

$$= 1 + \sum_{n=1}^{\infty}(n-1)(1-p)^{n-1}p$$

$$= 1 + (1-1)(1-p)^0 + \sum_{n=2}^{\infty}(n-1)(1-p)^{n-1}p$$

$$= 1 + 0 + \sum_{m=1}^{\infty} m(1-p)^m p$$

$$= 1 + (1-p)\sum_{m=1}^{\infty} m(1-p)^{m-1}p$$
$$= 1 + E(T) - pE(T)$$

Hence
$$pE(T) = 1$$
and thus the expected value is $1/p$. Recall next that
$$V(T) = E(T^2) - (E(T))^2.$$
Since $(E(T))^2 = 1/p^2$, all we need to find is $E(T^2)$:
$$E(T^2) = \sum_{n=1}^{\infty} n^2 P\{T = n\} = \sum_{n=1}^{\infty} n^2(1-p)^{n-1}p.$$
Similarly to finding the expected value of T, we rearrange:
$$E(T^2) = \sum_{n=1}^{\infty} n^2(1-p)^{n-1}p$$
$$= \sum_{n=1}^{\infty} n(n-1)(1-p)^{n-1}p + \sum_{n=1}^{\infty} n(1-p)^{n-1}p$$
$$= \sum_{n=2}^{\infty} n(n-1)(1-p)^{n-1}p + E(T)$$
$$= \sum_{m=1}^{\infty} m(m+1)(1-p)^{m}p + \frac{1}{p}$$
$$= (1-p)\sum_{m=1}^{\infty} m(m+1)(1-p)^{m-1}p + \frac{1}{p}$$
Looking closely, we find that
$$\sum_{m=1}^{\infty} m(m+1)(1-p)^{m-1}p = \sum_{m=1}^{\infty} m^2(1-p)^{m-1}p + \sum_{m=1}^{\infty} m(1-p)^{m-1}p$$
$$= E(T^2) + E(T)$$
and
$$E(T^2) = (1-p)\left(E(T^2) + \frac{1}{p}\right) + \frac{1}{p}$$
Solving for $E(T^2)$ yields
$$E(T^2) = \frac{2-p}{p^2},$$
and thus
$$V(T) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

2. Yosemite Sam (YS for short) uses his trusty six gun for target practice. The number of shots it takes YS to hit the bulls eye is $(N+2)^4$, where N has a Poisson distribution with parameter $\lambda = 2$. Find the expected number of shots for YS to hit the bulls eye.

$$P\{N = n\} = e^{-\lambda}\frac{\lambda^n}{n!}$$

Calculating the expected value of N:

$$E(N) = \sum_{n=1}^{\infty} nP\{N = n\} = \sum_{n=1}^{\infty} ne^{-\lambda}\frac{\lambda^n}{n!}$$

$$= \lambda e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}$$

$$= \lambda e^{-\lambda}\sum_{m=0}^{\infty}\frac{\lambda^m}{m!},$$

And since the infinite series above respresents e^{λ}:

$$E(N) = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

Thus

$$E\{\text{number of shots}\} = E((N+2)^4) = (\lambda+2)^4 = 256.$$

3. To pass an algebra course Roger Rabbit must complete 4 exams having no errors. The number of errors Roger makes on exams form a sequence of independent and identically distributed random variable which are Poisson distributed with parameter $\lambda = 3$. Find the probability Roger must take at least 6 exams to pass the course.

Let the number of errors that RR makes on exam i be given by X_i. Then for each exam:

$$P\{X_i = 0\} = e^{-3} \approx 0.0497$$

Observe the probability that RR must take at least 6 exams is equal to the probability that at least two of his first five have an error. Suppose N was the number of exams with an error out of the first five exams. Then

$$P\{N \geq 2\} = 1 - P\{N = 1\} - P\{N = 0\}$$

$$= 1 - \binom{5}{1}(e^{-3})^4(1-e^{-3}) - (e^{-3})^5 \approx 0.99997$$

2 Problems

4.7 (a) 1,2,3,4,5,6

(b) 1,2,3,4,5,6

(c) 2,3,4,5,6,7,8,9,10,11,12

(d) -5,-4,-3,-2,-1,0,1,2,3,4,5

4.8 Suppose the first die has been rolled. Then there is $\frac{1}{6}$ chance for the result to be any of the values. If a 6 has been rolled, then the second roll does not affect the variable. If a 5 has been rolled, then only a 6 affects it. If a 4, then only a 5 or 6; and so on. Hence, if X is the variable for the maximum, then

$$P\{X = i\} = \binom{2}{1}\left(\frac{1}{6}\right)\left(\frac{i}{6}\right) - \frac{1}{36}$$

There are 2 ways to choose the first die, there is (1/6) chance that the first die is i, and there are $(i/6)$ chance that the second die is less than or equal to i. Since there is symmetry when both dice are i, we have to subtract one outcome.

Suppose that Y was the variable for the minimum. Then

$$P\{Y = i\} = P\{X = 7 - i\} = \binom{2}{1}\left(\frac{1}{6}\right)\left(\frac{7-i}{6}\right) - \frac{1}{36}$$

Since maximum is effectively the reverse of the minimum, then all we have to do is map 1 to 6, 2 to 5, and so on. This can be achieved by $Y = 7 - X$.

Suppose that Z was the variable for the sum. Then clearly $P\{Z = 2\} = \frac{1}{36}$, the outcome of rolling two ones. Likewise, $P\{Z = 12\} = \frac{1}{36}$, the outcome of rolling two sixes. As we approach $Z = 7$, the most common sum, from either direction, we always increase the outcomes by one. This is because if $a + b = Z$, then $(a + 1) + b = Z + 1$, and so for $(a + 1) \in \{2, 3, 4, 5, 6\}$ this holds. Thus

$$P\{Z = i\} = \frac{i-1}{36}$$

if $i \leq 7$ and

$$P\{Z = i\} = \frac{13-i}{36}$$

otherwise.

4.13

$$P(X = 0) = (1 - 0.3)(1 - 0.6) = 0.28$$

$$P(X = 500) = (0.3)(1 - 0.6)(0.5) + (1 - 0.3)(0.6)(0.5) = 0.27$$

$$P(X = 1000) = (0.3)(0.6)(0.5)(0.5) + (0.3)(1-0.6)(0.5) + (1-0.3)(0.6)(0.5) = 0.315$$

$$P(X = 1500) = 2 * (0.3)(0.6)(0.5)(0.5) = 0.09$$

$$P(X = 1500) = (0.3)(0.6)(0.5)(0.5) = 0.045$$

4.17 (a)

$$P(X=1) = \left(\frac{1}{2}+0\right) - \frac{1}{4} = \frac{1}{4}$$

$$P(X=2) = \frac{11}{12} - \left(\frac{1}{2}+\frac{1}{4}\right) = \frac{1}{6}$$

$$P(X=3) = 1 - \frac{11}{12} = \frac{1}{12}$$

(b)

$$P\left(\frac{1}{2} < X < \frac{3}{2}\right) = F\left(\frac{3}{2}-\right) - F\left(\frac{1}{2}\right) = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$$

4.21 The expected value after selecting a teacher will be smaller, since it is effectively just the mean, whereas the expected value after selecting a student is more representative of the student's distribution (which is more wide spread).

$$E(X) = \frac{40}{148} * 40 + \frac{33}{148} * 33 + \frac{25}{148} * 25 + \frac{50}{148} * 50 \approx 39.283$$

$$E(Y) = \frac{1}{4} * 40 + \frac{1}{4} * 33 + \frac{1}{4} * 25 + \frac{1}{4} * 50 = 37$$

4.40

$$P(X \geq 4) = P(X=4) + P(X=5)$$

$$= \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^5 \approx 0.0452$$

4.45 Let X be the number out of N examiners who pass him. Then if $N = 3$,

$$P(X=2) = \frac{2}{3}\binom{3}{2}(0.4)^2(0.6) + \frac{1}{3}\binom{3}{2}(0.8)^2(0.2) \approx 0.448;$$

and if $N = 5$,

$$P(X=3) = \frac{2}{3}\binom{5}{3}(0.4)^3(0.6)^2 + \frac{1}{3}\binom{5}{3}(0.8)^3(0.2)^2 \approx 0.290.$$

It's better for the student to request an examination with 3 examiners.

4.60 For a Poisson random variable with parameter λ,

$$P(X=i,\lambda) = e^{-\lambda}\frac{\lambda^i}{i!}.$$

Therefore, for $X = 2$,

$$P(X=2) = \frac{0.75 * P(X=2,\lambda=3)}{0.75 * P(X=2,\lambda=3) + 0.25 * P(X=2,\lambda=5)}$$

$$P(X=2) \approx 0.888.$$

3 Theoretical Exercises

Let X be a Poisson random variable with parameter λ. Then

$$P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!}.$$

We want to first show that P increases monotonically as long as $i \leq \lambda$. Recall that P increases monotonically as long as $P(X = i) \geq P(X = i-1)$, or in other words,

$$\frac{P(X = i)}{P(X = i-1)} \geq 1$$

Therefore, if $i \leq \lambda$, then

$$\frac{P(X = i)}{P(X = i-1)} = \left(e^{-\lambda}\frac{\lambda^i}{i!}\right)\left(e^{\lambda}\frac{(i-1)!}{\lambda^{i-1}}\right) = \frac{\lambda}{i} \geq \frac{\lambda}{\lambda} = 1;$$

And hence P is monotonically increasing. Suppose instead that $i > \lambda$. Then

$$\frac{P(X = i)}{P(X = i-1)} = \left(e^{-\lambda}\frac{\lambda^i}{i!}\right)\left(e^{\lambda}\frac{(i-1)!}{\lambda^{i-1}}\right) = \frac{\lambda}{i} < \frac{\lambda}{\lambda} = 1;$$

And therefore P is monotonically decreasing.