HW5 MATH 4540

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1 Exercises

1. Let f be a real valued and continuous function on $[a, b]$ and α be monotonically increasing on $[a, b]$. First, suppose that $\alpha(a) = \alpha(b)$. Then because we know that α is monotonically increasing, it follows that it must take on a constant value on $[a, b]$; and hence

$$\int_a^b f \, \mathrm{d}\alpha = 0.$$

Since $\alpha(b) - \alpha(a) = 0$, in this case all values of $c \in [a, b]$ would suffice to show that $\int_a^b f \, \mathrm{d}\alpha = f(c)[\alpha(b) - \alpha(a)] = 0$. Suppose instead that $\alpha(a) \neq \alpha(b)$. Let $f(x) = M$ and $f(y) = m$ be the maximum and minimum values of f given by $x, y \in [a, b]$, respectively. Then $0 < \alpha(b) - \alpha(a)$, by definition; and furthermore, it follows that

$$m(\alpha(b) - \alpha(a)) \leq \int_a^b f \, \mathrm{d}\alpha \leq M(\alpha(b) - \alpha(a)),$$

and thus

$$f(y) \leq \frac{\int_a^b f \, \mathrm{d}\alpha}{\alpha(b) - \alpha(a)} \leq f(x).$$

Because f is continuous on $[a, b]$ and $x, y \in [a, b]$, it then follows by the Intermediate Value Theorem that there exists some $c \in [a, b]$ such that

$$f(c) = \frac{\int_a^b f \, \mathrm{d}\alpha}{\alpha(b) - \alpha(a)},$$

and hence $f(c)[\alpha(b) - \alpha(a)] = \int_a^b f \, \mathrm{d}\alpha$.

2. Start with the simple partition $P = \{-1, 0, 1\}$. Calculating the upper and lower integrals of this partition gives

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \triangle \alpha_i = M_1 + M_2$$

and

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \triangle \alpha_i = m_1 + m_2,$$

where M_1 and m_1 are the supremum and infimum of f on $[-1, 0]$, and M_2 and m_2 are likewise the extrema on $[0, 1]$. Consider now the refinements of P given by the form $P^* = \{-1, v, 0, 1\}$, where $-1 < v < 0$. Then we can recalculate the sums as

$$U(P*, f, \alpha) = \sum_{i=1}^{n} M_i \triangle \alpha_i = M_2 + M_3$$

and

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \triangle \alpha_i = m_2 + m_3,$$

where now the summed extrema span across the intervals $[v, 0]$ and $[0, 1]$, respectively. Similarly, if we were to instead consider refinements given by the form $P^* = \{-1, 0, w, 1\}$, where $0 < w < 1$, then we would find that the summed extrema span across $[-1, 0]$ and $[0, w]$. Thus by adding a new point to P, we either shorten the first or the second interval over which we calculate our extreme values, which are together then summed. Let (v_n) and (w_n) be any two arbitrary and infinite sequences such that $(v_n) \subset (-1, 0)$ is increasing and $(w_n) \subset (0, 1)$ is decreasing. Suppose now we were to refine P with all the points belonging to either sequence. Then as $n \to \infty$, it follows that $(v_n) \to 0$ and $(w_n) \to 0$; and hence

$$\inf U(P*, f, \alpha) = \max(f([(v_n), 0]) + \max(f([0, (w_n)]) \to 2f(0)$$

and

$$\sup L(P*, f, \alpha) = \min(f([(v_n), 0]) + \min(f([0, (w_n)]) \to 2f(0).$$

Thus the integral evaluates to twice the value of f at zero.

3. Consider the refinements of $P = \{0, 1\}$ given by the form

$$P^* = \{0, \frac{1}{n}, \frac{1}{n-1}, \frac{1}{n-2}, \ldots, \frac{1}{4}, \frac{1}{2}, 1\}$$

as $n \to \infty$. Observe that for $x = 1$,

$$\alpha(x) = 1/2 + 1/4 + 1/8 + \ldots + 2^{-n};$$

For all $x \in [\frac{1}{2}, 1)$, likewise,

$$\alpha(x) = 1/4 + 1/8 + \ldots + 2^{-n};$$

and for all $x \in [\frac{1}{3}, \frac{1}{2})$, likewise,

$$\alpha(x) = 1/8 + \ldots + 2^{-n}.$$

Following this pattern, it's easy to see that α only increases whenever $p \in P$ increases from a number of the form $\frac{1}{n}$ to $\frac{1}{n-1}$. Therefore, the only effect that adding a point not of that form has is affecting the ranges over which the extrema of our upper and lower integrals are summed. For example, consider the partition $P_1 = \{0, \frac{1}{2}, 1\}$ and its refinement $P_1^* = \{0, \frac{1}{2}, \frac{3}{4}, 1\}$. Then

$$U(P_1, f, \alpha) = M_1 * \triangle\alpha_1 + M_2 * \triangle\alpha_2$$

and

$$U(P_1^*, f, \alpha) = M_1 * \triangle\alpha_1 + \hat{M}_3 * \triangle\hat{\alpha}_3$$

where in this case the first terms are the same but the second terms are not (as the maximum on f from $\frac{3}{4}$ to 1 may be different than from $\frac{1}{2}$ to 1). Notice also that the additional term $\hat{M}_2 * \triangle\hat{\alpha}_2$ is omitted, as $\alpha(\frac{1}{2}) = \alpha(\frac{3}{4})$; and hence their difference is zero. Therefore from these observations we can conclude that as we add more points to P following the form above, we add additional terms, whereas when we add more points not in the form above, we change the terms such that the extreme values are evaluated along a smaller interval. Thus

$$\int_0^1 f\,\mathrm{d}\alpha = \sum_1^\infty 2^{-n} * f(n^{-1}).$$

4. (a) First we apply the smooth α theorem:

$$\int_0^{\pi/2} x\,\mathrm{d}\sin(x) = \int_0^{\pi/2} x\cos(x)\,\mathrm{d}x.$$

Observe now that for $F = x\sin(x) + \cos(x)$, $F' = \sin(x) + x\cos(x) - \sin(x) = x\cos(x)$. Therefore

$$\int_0^{\pi/2} x\cos(x)\,\mathrm{d}x = F(\pi/2) - F(0) = \pi/2 - 1.$$

(b) This is similar to problem (2), and is calculated by considering the refinements $P* = \{0, p_1, 1, p_2, 2, p_3, 3\}$, where $p_1 \to 1$, $p_2 \to 2$, and $p_3 \to 3$. As each p approaches its neighbor, the extrema between the two approaches the value of f for the neighbor that p is approaching (either $f(1)$, $f(2)$, or $f(3)$, respectively). In those cases, $\triangle\lfloor x\rfloor = 1$ because p is always less than the integer that its approaching, and in all other cases, $\triangle\lfloor x\rfloor$ is zero. Thus

$$\int_0^3 x^2\,\mathrm{d}\lfloor x\rfloor = 1^2 + 2^2 + 3^2 = 14.$$

(c) First we apply the smooth α theorem:

$$\int_1^4 (x - \lfloor x\rfloor)\,\mathrm{d}x^3 = \int_1^4 (x - \lfloor x\rfloor)3x^2\,\mathrm{d}x.$$

Then we evaluate by breaking the integral up into parts:

$$\int_1^4 (x-\lfloor x\rfloor)3x^2\,\mathrm{d}x = 3[\int_1^2 (x^3-x^2)\,\mathrm{d}x+\int_2^3 (x^3-2x^2)\,\mathrm{d}x+\int_3^4 (x^3-3x^2)\,\mathrm{d}x].$$

Notice that we made the coefficient of x^2 equal to 1, 2, or 3 depending on the range that each integral evaluates over. This is to match the respective value of the greatest integer function. Now we solve for each integral:

$$\int_1^4 (x-\lfloor x\rfloor)3x^2\,\mathrm{d}x = 3[(\frac{x^4}{4}-\frac{x^3}{3})|_1^2+(\frac{x^4}{4}-\frac{2x^3}{3})|_2^3+(\frac{x^4}{4}-x^3)|_3^4]$$

$$= 3(12-\frac{1}{4}) = 36-\frac{3}{4}.$$