Homework 13

1. The zero divisors of the ring are all cosets that multiply to give zero. Because quotient rings are operative under modular arithmetic, this is also all cosets that multiply and divide by 43200, which are each of the cosets that are offset by a number that shares a common factor:

To see this, let a be a common factor of $r \in \mathbb{Z}$ and 43200. Then $a | r$ and $a | 43200$.

Consider $[r] = (r + 43200\mathbb{Z}) \in \mathbb{Z}/43200\mathbb{Z}$

Let $b = (43200/a) \in 43200\mathbb{Z}$.

Then $[r][b] = (r(43200/a) + 43200\mathbb{Z})$, which because $a|r$ must be some offset that is a multiple of 43200 and therefore 0. For the other direction, Let $[s] \in \mathbb{Z}/43200\mathbb{Z}$ be a zero divisor

Then there exists some $[b]$ for which

$[s][b] = (sb + 43200\mathbb{Z}) = 43200\mathbb{Z}$.

Therefore sb is a multiple of 43200 and thus there exists some common factor between s and 43200 as they are not coprime.

To count the cosets which are zero divisors, consider the prime factors of 43200: 2, 3, and 5. The zero divisors are those offset by a number divisible by at least one of these numbers, and thus there are

$$43200/2 + 43200/3 + 43200/5 + 43200/30$$
$$- 43200/6 - 43200/10 - 43200/15$$
$$= 31680 \text{ zero divisors.}$$

Recall that $\mathbb{Z}$ is a commutative ring and thus $\mathbb{Z}/43200\mathbb{Z}$ is as well. Therefore, because $\mathbb{Z}/43200\mathbb{Z}$ is finite and commutative, each of its elements are either units or zero divisors. Thus there are

$$43200 - 31680 = 11520 \text{ units.}$$

Finally, consider the nilpotents $[n]$ which, similar to the zero divisors, must exponentiate into zero: thus 2, 3, and 5 must divide n as any numbers without these factors are coprime to 43200 and cannot exponentiate as such. Thus there are

$$43200 / 30 = 1440 \text{ nilpotents,}$$

the intersection of zero divisors all divisible by 2, 3, and 5.

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2a. Let $f(x), g(x) \in R[x]$, whose degrees are given as $\deg(f(x)) = n$ and $\deg(g(x)) = m$. Consider their product, $f(x)g(x)$, given by definition as

$$f(x)g(x) = \sum_{k=0}^{n+m} c_k x^k,$$

where $c_k = \sum_{i=0}^{k} r_i r'_{k-i}$.

Consider $c_{n+m} = \sum_{i=0}^{n+m} r_i r'_{n+m-i}$. Because R is an integral domain, the products of two nonzero elements are nonzero as well. Given the degrees of $f(x)$ and $g(x)$, and the definition of a degree, we know that r_n and r'_m are both nonzero. Therefore the product in the sum given by $i = n$ is also nonzero. Furthermore, they are each their functions maximal coefficients, and thus no other products given by the sum can cancel them.

Thus $c_{n+m} \neq 0$ and the degree of $f(x)g(x) = n+m$; in other words, $\text{degree}(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$.

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2a. ~~Let $f(x), g(x) \in R[x]$. Then their product $f(x)g(x)$ is given by~~

$$\cancel{\left(\sum_{k=0}^{n} r_k x^k\right)\left(\sum_{k=0}^{m} r'_k x^k\right) = \sum_{k=0}^{n+m} c_k x^k.}$$

~~Because R is an integral domain, all products of nonzero elements are nonzero. Therefore $c_{n+m} = 0$ only if $\sum_{i=0}^{m+n} r_i r'_{n+m-i} = 0$, and thus only if for $i = n$, $r_n r'_m = 0$. However because~~

2b. Let R be an integral domain. Let $f(x), g(x) \in R[x]$ be any arbitrary nonzero elements. Then they have at least one maximal term each, $f(x) \approx a_n x^n$ and $g(x) \approx b_m x^m$, such that $\deg(f(x)) = n$ and $\deg(g(x)) = m$. Therefore $f(x)g(x) \approx a_n b_m x^{n+m}$ with degree given by $n+m$; and because R is an integral domain, $a_n b_m$ is nonzero and therefore $f(x)g(x)$ is nonzero.

Similarly let $R[x]$ be an integral domain. Then $f(x) \neq 0$ and $g(x) \neq 0 \iff f(x)g(x) \neq 0$. But if $f(x) = a_1 \in R$ and $g(x) = b_1 \in R$, then by the same reason it follows that $a \neq 0$ and $b \neq 0 \iff ab \neq 0$. Therefore the constant functions embed R in $R[x]$ and R is also an integral domain

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2c) Consider the units of $R[x]$ given by $f(x)g(x) = 1$. Therefore $\deg(fg) = 0$ and thus $\deg(f(x))$ and $\deg(g(x))$ are also both zero. Thus the units of $R[x]$ are constant functions embeded in R.

Because R is an integral domain

Consider the units of R when R is an integral domain. Then $ab = 1$, where $a \in R$ and $b \in R$ are units.

Finally, suppose there were some constant functions $f(x), g(x) \notin U(R)$. Then for $f(x) = a$ and $g(x) = b$, $ab \neq 1$. However Then $f(x)g(x) \neq 1$, and thus $U(R[x]) \subset U(R)$

Thus, as $U(R[x])$ are constants equally embded by $ab = 1$, $U(R) \subset U(R[x])$ and hence $U(R[x]) = U(R)$.

2d) Let $I[x] = \{\sum_{k=0}^{n} a_k x^k \mid a_k \in I\}$, and $f(x) \in R[x]$, and $i(x) \in I[x]$.

Then $f(x)i(x) = \sum_{k=0}^{n+m} c_k x^k$,

where $\deg(f(x)) = n$ and $\deg(i(x)) = m$. However, observe that for all k,

$c_k = \sum_{i=0}^{k} r_i a_{k-i}$, and because I is an ideal, $r_i a_{k-i} \in I$. Thus $c_k \in I$ and hence $f(x)i(x) \in I[x]$ is an ideal.

2e) Let $\pi : R[x] \to (R/I)[x]$ be given by

$$\pi(f(x)) = \sum_{k=0}^{n} (r_k + I)x^k, \text{ where}$$

$$f(x) = \sum_{k=0}^{n} r_k x^k \text{ for } r_k \in R.$$

Then $\pi(f(x)+g(x))$

$$= \sum_{k=0}^{n} (r_k + r'_k + I)x^k$$

$$= \sum_{k=0}^{n} (r_k + I)x^k + \sum_{k=0}^{n} (r'_k + I)$$

$$= \pi(f(x)) + \pi(g(x)) \text{ and}$$

$$\pi(f(x)g(x)) = \sum_{k=0}^{n+m} (c_k + I)x^k$$

$$= \sum_{k=0}^{n} (r_k + I)x^k \cdot \sum_{k=0}^{m} (r'_k + I)x^k$$

$$= \pi(f(x)) \cdot \pi(g(x)),$$

and π is surjective as for any element $(R/I)[x]$, the coffect $(r_k + I)$ maps $r_k \in R$ to $(r_k + I) \in R/I$. Thus by the First Homomorphism Thrm,

$$R[x]/\ker(\pi) \cong \mathrm{im}(\pi).$$

As π is surjective, $\mathrm{Im}(\pi) = (R/I)[x]$, and $\ker(\pi) = I[x]$ as $\pi(f(x)) = 1_{(R/I)[x]}$ only when $f(x) \in I[x]$.

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2f) Let I be a prime ideal of R. Then if $ab \in I$, either $a \in I$ or $b \in I$.

Consider $I[x]$ given again by

$$\{\sum_{k=0}^{n} \alpha_k x^k \mid \alpha_k \in I\}.$$

Let $f(x)g(x) \in I[x]$ where $f(x) \in R[x]$ and $g(x) \in R[x]$. Then $c_k \in I$ for

$$c_k = \sum_{i=0}^{k} r_i r'_{k-i},$$

and because $r_i r'_{k-i} \in I$ either $r_i \in I$ or $r'_{k-i} \in I$. Therefore $f(x)$ or $g(x)$ must have coeffecient elements of I, and thus either f or g are $\in I[x]$.

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3a) Let R be an integral domain. Then any product of two nonzero elements are also nonzero. Consider $f(x), g(x) \in R[[x]]$. Then their product is given by the definition, where $c_k = \sum_{i=0}^{k} r_i r'_{k-i}$.

Observe that if $f(x)$ and $g(x)$ are nonzero, then they each contain at least one smallest term, a and b respectively, for which $a \neq 0$ and $b \neq 0$. Thus let ab be the smallest product given by the sums of products, c_k: because $a, b \in R$ and are nonzero, the product ab is also nonzero for $f(x)g(x)$'s smallest ordered term. Because no other products can sum to cancel ab, the polynomial $f(x)g(x)$ contains at least one nonzero term and thus $R[[x]]$ is an integral domain.

If you instead suppose that $R[[x]]$ is an integral domain, then $f(x)g(x) \neq 0$ $\iff$ $f(x) \neq 0$ and $g(x) \neq 0$. But if $f(x) = a$ and $g(x) = b$ are constant terms representing the elements of R, then likewise if $ab \neq 0$ then $a \neq 0$ and $b \neq 0$. Thus R is embedded within $R[[x]]$ and is also an integral domain.

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3b) The units of $R[[x]]$ are given by all function polynomials with multiplicative inverses; that is, $f(x)$ and $g(x)$ are units if $f(x)g(x) = 1$.

Suppose $f(x)$ and $g(x)$ are units. Then there must exist nonzero constant terms $u \in R$ and $v \in R$ for which
$f(x) = u + r_1 x + r_2 x^2 + \cdots$ and
$g(x) = v + s_1 x + s_2 x^2 + \cdots$ and

$u \cdot v = 1$; else $f(x)g(x)$ could not equal 1. Observe however that

~~$\sum_{k=0}^{\infty} x^k \in R[[x]]$, and therefore~~

$$f(x) = u + r_1 x + r_2 x^2 + \cdots$$
$$= u + x(r_1 + r_2 x + \cdots)$$ and

$$g(x) = v + s_1 x + s_2 x^2 + \cdots$$
$$= v + x(s_1 + s_2 x + \cdots).$$

Therefore units of $R[[x]]$ are equal to $\{u + x f(x) \mid u \in U(R),\ f(x) \in R[[x]]\}$, as $\{\sum_{k=0}^{\infty} r_{k+1} x^k \mid r_k \in R\} \subseteq R[[x]]$.