MATH 4560/6560 Topology
Spring 2022
Exercises 5
Target date: Thu 24 Feb 11:59PM

Overall: C

Exercise 1. (a) Find an online news story from 2022 that includes the word "metric." Give the link and write a summary of the article. Does the article's use of the word "metric" match with how we use it in class? How are they different and how are they similar?

Source: This article is essentially about the monitoring of COVID in Coconino County, a county in Arizona. The community transmission of the county is currently high, meaning there are more than 100 cases per 100 thousand people in the past week. However, as the article points out, this number is decreasing. The use of metric in this article reflects an idea of measuring something, which is similar to how we use it in mathematics. In this case, metrics of COVID might include hospitalizations, deaths, symptoms, and mask-wearing. Each of these factors measure the prevalence of the COVID viruses, and its societal impact within communities.

(b) Find a Wikipedia article about a mathematician whose research includes metric spaces. (Bonus points if the mathematician is from group that has been historically excluded from mathematics, e.g., a woman or non-binary person; a Black person or member of another minority race or ethnicity; a member of the LGBTQIA+ community; or a person with a disability.) Give the link and write a summary of the person's accomplishments.

(For some reason Latex will not allow me to paste his Wiki page... the person is Maurice René Fréchet, who introduced the concept of metric spaces to mathematics.) Maurice was a highly influential person in the fields of mathematics, due to his contributions for abstract spaces and analysis. He introduced both the ideas of a metric space and the concept of compactness, which are tools that we still use today in topology (and other fields). He also worked generally within Statistics, helping to lead many mathematicians further into the study of probability (notably those associated with stochastic processes, see the Kosambi–Karhunen–Loève theorem).

Exercise 2. Fix a prime number p and consider the p-adic metric on $\mathbb{Z}$ from class:

$$d(a,b) = \begin{cases} 2^{-\max\{n \,:\, p^n|(b-a)\}} & \text{if } a \neq b \\ 0 & \text{if } a = b. \end{cases}$$

Describe the open ball $B_{0.1}(0) \subseteq \mathbb{Z}$. Justify your answer.

$B_{0.1}(0) \subseteq \mathbb{Z}$ is the open ball of integers within distance of 0.1 from zero under the p-adic metric, which is equivalent to

$$\begin{aligned} B_{0.1}(0) &= \{y \in \mathbb{Z} : d(0,y) < 0.1\} \\ &= \{y \in \mathbb{Z} : \max\{n : p^n|y\} > 3\} \\ &= \{y \in \mathbb{Z} : p^4|y\}. \end{aligned}$$

This is because whenever $\max\{n : p^n|y\} \geq 4$, the distance $d(0,y) \leq 2^{-4} = 0.0625 < 0.1$, but if $\max\{n : p^n|y\} \leq 3$ then $d(0,y) \geq 2^{-3} = 0.125 > 0.1$.

Exercise 3. Let $S_1, \dots, S_n$ be metric spaces with metrics $d_1, \dots, d_n$, respectively. Consider the cartesian product $S = S_1 \times \cdots \times S_n$. Let d be the product metric from class

$$d(P, Q) = \textstyle\sum_{i=1}^n d_i(p_i, q_i)$$

and consider the function $\widetilde{d}\colon S \times S \to \mathbb{R}_{\geq 0}$ given by

$$\widetilde{d}(P, Q) = \sqrt{\textstyle\sum_{i=1}^n d_i(p_i, q_i)^2}.$$

(a) Prove that $\widetilde{d}$ is a metric on S.
To show that $\widetilde{d}$ is a metric on S, we need to prove that $\widetilde{d}$ is (1) zero for $P = Q$ and positive for $P \neq Q$, (2) symmetric, and (3) obeys the triangle inequality:

(1) Suppose that $P = Q$. Then $p_i = q_i$ for every coordinate in P and Q; and, since d_i is a metric, $d_i(p_i, q_i) = 0$. Hence $\sqrt{\sum_{i=1}^n d_i(p_i, q_i)^2} = 0$. Suppose instead that $P \neq Q$. Then it follows that $p_k \neq q_k$ for at least one coordinate in P and Q; and, similarly, $d_k(p_k, q_k) > 0$ because d_k is a metric. Hence $\sqrt{\sum_{i=1}^n d_i(p_i, q_i)^2} \geq \sqrt{d_k(p_k, q_k)^2} > 0$.

(2) Observe that for every coordinate p_i, q_i for P, Q, we have $d_i(p_i, q_i) = d_i(q_i, p_i)$. Thus

$$\widetilde{d}(P, Q) = \sqrt{\textstyle\sum_{i=1}^n d_i(p_i, q_i)^2} = \sqrt{\textstyle\sum_{i=1}^n d_i(q_i, p_i)^2} = \widetilde{d}(Q, P).$$

(3) Since for each metric we have $d_i(p_i, q_i) \leq d_i(p_i, r_i) + d_i(r_i, q_i)$, it follows that

$$\sqrt{\sum_{i=1}^n d_i(p_i, q_i)^2} \leq \sqrt{\sum_{i=1}^n (d_i(p_i, r_i) + d_i(r_i, q_i))^2}$$

$$\leq \sqrt{\sum_{i=1}^n (d_i(p_i, r_i)^2 + 2d_i(p_i, r_i)d_i(r_i, q_i) + d_i(r_i, q_i)^2)}$$

$$\leq \sqrt{\sum_{i=1}^n (d_i(p_i, r_i)^2 + d_i(r_i, q_i)^2)}$$

$$\leq \sqrt{\sum_{i=1}^n (d_i(p_i, r_i)^2} + \sqrt{\sum_{i=1}^n d_i(r_i, q_i)^2)},$$

and hence

$$\widetilde{d}(P, Q) \leq \widetilde{d}(P, R) + \widetilde{d}(R, Q).$$

(b) Let $U \subseteq S$ be an arbitrary subset. Prove that U is open with respect to $\widetilde{d}$ if and only if it is open with respect to d. *Hint:* For all $x_1, \dots, x_n \in \mathbb{R}_{\geq 0}$ we have:

(revised)

$$\sqrt{\textstyle\sum_{i=1}^n x_i^2} \leq \textstyle\sum_{i=1}^n x_i \leq \sqrt{n}\sqrt{\textstyle\sum_{i=1}^n x_i^2}$$

Pf: Assume U is open with respect to d. Then by assumption, for any points $x \in U$ and $y \in S$,

$$d(x, y) < \epsilon \implies y \in U.$$

Note however, from the inequality given above, that it thus follows that

$$\sqrt{n} * \widetilde{d}(x, y) < \epsilon \implies d(x, y) < \epsilon$$

as $d(x,y) \leq \sqrt{n} * \widetilde{d}(x,y)$. Hence for any points $x \in U$ and $y \in S$,

$$d(x,y) < \frac{\epsilon}{\sqrt{n}} \implies d(x,y) < \epsilon \implies y \in U,$$

and thus U is open with respect to $\widetilde{d}(x,y)$. Assume instead that U were open with respect to $\widetilde{d}$. Similarly, by assumption, for any points $x \in U$ and $y \in S$,

$$\widetilde{d}(x,y) < \epsilon \implies y \in U.$$

Hence because $\widetilde{d}(x,y) \leq d(x,y)$, it follows that

$$d(x,y) < \epsilon \implies \widetilde{d}(x,y) < \epsilon \implies y \in U,$$

and so is U open with respect to d.