Anthony Jones

1. $\frac{dy}{dx} = \frac{y^2+1}{t-1}$, $t > 1$.

Separate the variables:

$$\frac{dy}{y^2+1} = \frac{dt}{t-1}, \quad t > 1$$

Integrate both sides:

$$\int \frac{dy}{y^2+1} = \int \frac{dt}{t-1}$$

$$\arctan(y) + a = \ln|t-1| + b \quad ; \quad b - a = c.$$

$$y = \tan(\ln|t-1| + c).$$

The solution to the ode is given by $y = \tan(\ln|t-1| + c)$ for any c.

Anthony Jones

2. $t\frac{dy}{dt} - y = t^2 \sin(t)$ for $t > 0$.

Write in ~~normal~~ linear form:

$$\frac{dy}{dt} - y\left(\frac{1}{t}\right) = t\sin(t).$$

Find integrating factor:

$$m(t) = e^{\int(-\frac{1}{t})dt} = e^{-\ln|t|} = \frac{1}{t}$$

Check your solution:

$$\frac{d}{dt}(m \cdot y) = \frac{dy}{dt} \cdot \frac{1}{t} - y\frac{1}{t^2} = m\left[\frac{dy}{dt} - y\left(\frac{1}{t}\right)\right]$$

Substitute:

$$\frac{d}{dt}(m \cdot y) = \sin(t)$$

Integrate:

$$\int \frac{d}{dt}(m \cdot y)\,dt = \int \sin(t)\,dt$$

$$m \cdot y = -\cos(t) + K$$

$$y = Kt - t\cos(t)$$

The solution to the ode is given by $y = Kt - t\cos(t)$ for any K.

Anthony Faner

3. $(xy^2 + x^2)dx + (yx^2 - y^2)dy = 0$

(a) Is the equation exact:

$M_x = xy^2 + x^2$ $\qquad$ $M_{xy} = 2xy$

$N_y = x^2y - y^2$ $\qquad$ $N_{yx} = 2xy$

The equation is exact because $M_{xy} = N_{yx}$.

This means:

$df = (xy^2 + x^2)dx + (yx^2 - y^2)dy$

$f = \int (xy^2 + x^2)dx + g(y)$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2y^2 + g(y)$

Differentiate and solve for $g'(y)$:

$\frac{d}{dy} f = \frac{d}{dt}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2y^2 + g(y)\right)$

$= x^2y + g'(y) = x^2y - y^2$

Integrate $g'(y)$:

$g'(y) = -y^2$

$g(y) = \int -y^2\, dy = -\frac{y^3}{3} + K$

$f = \frac{1}{3}x^3 + \frac{1}{2}x^2y^2 - \frac{y^3}{3} + K$

(b) $f(1,1) = \frac{1}{3} + \frac{1}{2} - \frac{1}{3} + K = 0$

$K = -\frac{1}{2}$

3. a) The implicit solution for the
cont. ode is given by $f = \frac{1}{3}x^3 + \frac{1}{2}x^2y^2 - \frac{y^3}{3} + K$
for any K.

b) The implicit solution for the
ode where $y(1) = 1$ is given by
$f = \frac{1}{3}x^3 + \frac{1}{2}x^2y^2 - \frac{y^3}{3} - \frac{1}{2}$.

Anthony Jerez

4. a) $y'' - 10y' + 25y = 0$

Solve the polynomial representation:

$$r^2 - 10r + 25 = 0$$
$$(r-5)(r-5) = 0$$

$$r = 5 \text{ (repeated).}$$

Return to differential solution:

The solution of the homogeneous ode is given by

$$y = K_1 e^{5t} + K_2 t e^{5t}.$$

b) $y'' + 6y' + 13y = 0$

Solve the polynomial representation:

$$r^2 + 6r + 13 = 0$$
$$(r+3)(r+3) + 4 = 0$$
$$(r+3)^2 + 4 = 0$$

$$r = -3 \pm 2i$$

Return to differential solution:

The solution of the homogeneous ode is given by

$$y = K_1 e^{-3t} \sin(2t) + K_2 e^{-3t} \cos(2t)$$

5 a) $y'' - y = 12e^{-2t}$

Anthony Jans

Solve the polynomial representation:

$$r^2 - 1 = 0$$
$$(r-1)(r+1) = 0$$
$$r = \pm 1.$$

Return to homogeneous solution:

$$y = K_1 e^t + K_2 e^{-t}.$$

Replace your variables with functions of t:

$$y = f(t)e^t + g(t)e^{-t}.$$

Find $y'(t)$ and $y''(t)$:

$$y' = \cancel{f'(t)e^t} + f(t)e^t + \cancel{g'(t)e^{-t}} - g(t)e^{-t}$$
$$= f(t)e^t - g(t)e^{-t}$$

$$y'' = f'(t)e^t + f(t)e^t - g'(t)e^{-t} + g(t)e^{-t}$$

$$y'' - y = f'(t)e^t + \cancel{f(t)e^t} - g'(t)e^{-t} + \cancel{g(t)e^{-t}}$$
$$\qquad - \cancel{f(t)e^t} \qquad - \cancel{g(t)e^{-t}}$$

$$= f'(t)e^t - g'(t)e^{-t} = 12e^{-2t}$$

$f'(t) = K_1 e^{-2t}$; $\cancel{y = K_1 e^{-2t} + (12-K_1)}$

$g'(t) = 12 - K_1$;

Anthony Jones

6) a) $y' - y = 2\cos(5t)$, $y(0) = 0$

$\mathcal{L}(y') - \mathcal{L}(y) = 2\mathcal{L}(\cos(5t))$

$sY(s) - Y(s) = \frac{2s}{s^2+25}$

$Y(s) = \frac{2s}{(s-1)(s^2+25)}$

Expand the Fraction into a polynomial:

$\frac{2s}{(s-1)(s^2+25)} = \frac{A}{s-1} + \frac{B}{s^2+25}$

$2s = A(s^2+25) + B(s-1)$

$2 = 26A$, for $s = 1$

$A = \frac{1}{13}$

$10i = B(5i - 1)$

$B = -50 - 10i$, For $s = 5i$

$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{1/13}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{(-50-10i)}{s^2+25}\right)$

$y = \frac{1}{13}e^t + (-50-10i)\cos(5t)$

The solution to the ode for $y(0) = 0$ is given by $y = \frac{1}{13}e^t + (-50-10i)\cos(5t)$.

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7) $\vec{x}\,' = \begin{bmatrix} 2 & -3 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \vec{x}$, where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$.

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8) $\vec{x}' = A\vec{x}$, where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$.

$$\vec{x}(t) = K_1 e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + K_2 e^{2t}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + K_3 e^{3t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) The eigenvalues for the matrix equation are $\lambda = 1, 2, 3$.

These corresponding eigenvectors are:

$\lambda = 1 \; : \; \vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 2 \; : \; \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$\lambda = 3 \; : \; \vec{q} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

(b) $\vec{x}(a) = \begin{bmatrix} 0 \\ ae^{t} \\ 0 \end{bmatrix}$

$\vec{x}(b) = \begin{bmatrix} be^{2t} \\ 2be^{2t} \\ 2be^{2t} \end{bmatrix}$

$\vec{x}(c) = \begin{bmatrix} -ce^{3t} \\ ce^{3t} \\ ce^{3t} \end{bmatrix}$, $\quad \vec{x} = \vec{x}(a) + \vec{x}(b) + \vec{x}(c)$

9) a) $\det(A - \lambda I) = (-5-\lambda)(-1-\lambda) + 8 = 0$

$$5 + 6\lambda + \lambda^2 + 8 = 0$$

$$(\lambda+3)^2 + 4 = 0$$

$$\lambda = -3 \pm 2i$$

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b) $\lambda = -3 + 2i$

$$\begin{bmatrix} -5 & -2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = (-3+2i)\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left.\begin{aligned} -5a - 2b &= (-3+2i)a \\ 4a - b &= (-3+2i)b \end{aligned}\right\} -ai + a = bi$$

$a = 1 + i$

$b = 0 - 2i$

Eisenvector for $(-3+2i)$ is

$$\vec{u} = \begin{bmatrix} 1+i \\ -2i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} i \\ -2i \end{bmatrix}$$

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c) $$\vec{x} = k_1 e^{-3t} \sin(2t)\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} t\right] + k_2 e^{-3t} \cos(2t)\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} t\right].$$

The real solutions for the matrix equation is given by $\vec{x}$ for any K_1 and K_2.

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10) a) $\det(A-\lambda I) = -\lambda\left[-4\lambda+\lambda^2\right] - \left[4-\lambda\right] = 0$

$= -\lambda^3 + 4\lambda^2 + \lambda - 4 = 0$

$= \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$= (\lambda-4)(\lambda^2-1) = 0$

$= (\lambda-4)(\lambda+1)(\lambda-1) = 0$

The eigenvalues are $\lambda = 4, 1, -1$.

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b) $\lambda = 4$

$$A - 4I = \begin{bmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 0 & -8 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2/8 \\ 0 & 1 & 1/8 \\ 0 & 0 & 0 \end{bmatrix}$$

$a + 2/8c = 0$

$b + 2/8c = 0$

$a = -2/8c$

$b = -2/8c$

$c = c = 4$

$\vec{u} = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$ for $\lambda = 4$.

11) a) The third solution corresponding to $\lambda_1 = \lambda_2 = 1$ is given by the format

$$\vec{x}_2 = e^t\left(t\begin{bmatrix}0\\1\\-1\end{bmatrix} + \vec{w}\right)$$

where $\vec{w}$ is linearly independent from $\begin{bmatrix}0\\1\\-1\end{bmatrix}$.

$$\vec{w} := (A-I)\vec{w} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$\begin{bmatrix}0 & 3 & 3\\1 & 0 & 0\\-1 & 1 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$\begin{array}{r|l} 3b+3c=0 & a=1 \\ a=1 & b=-c \\ -a+b+c=-1 & c=c \end{array}$$

The third solution is

$$\vec{x}_2 = e^t\left(t\begin{bmatrix}0\\1\\-1\end{bmatrix} + \begin{bmatrix}1\\-1\\1\end{bmatrix}\right).$$

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b) The general solution to the system is

$$\vec{x} = k_1 e^{2t}\begin{bmatrix}3\\3\\-2\end{bmatrix} + k_2 e^t\begin{bmatrix}0\\1\\-1\end{bmatrix} + k_3 e^t\left[\begin{bmatrix}0\\1\\-1\end{bmatrix} + \begin{bmatrix}1\\-1\\1\end{bmatrix}\right].$$