Lecture 16 Duality theory II

- Complementary slackness (CS)

Consider the following primal-dual pair (A is $m \times n$)

$$\begin{array}{ll} \max & z = c^T x \\ \text{s.t} & Ax \le b \\ & x \ge 0 \end{array} \quad (P) \quad \longleftrightarrow \quad \begin{array}{ll} \min & Z = b^T \pi \\ \text{s.t} & A^T \pi \ge c \\ & \pi \ge 0 \end{array} \quad (D)$$

Thm (complementary slackness): Let $\bar{x}$ be feasible to (P) and $\bar{\pi}$ be feasible to (D), then $\bar{x}$ is optimal to (P) and $\bar{\pi}$ is optimal to (D) if and only if the following <u>complementary slackness conditions</u> are satisfied.

$$\bar{x}_i (A^T \bar{\pi} - c)_i = 0 \quad \forall\, i = 1, \ldots, n$$

$$\bar{\pi}_j (b - A\bar{x})_j = 0 \quad \forall\, j = 1, \ldots, m$$

e.g

$$\begin{array}{ll} \max & -4x_1 - 2x_2 - x_3 \\ \text{s.t} & -x_1 - x_2 + 2x_3 \le -3 \\ & -4x_1 - 2x_2 + x_3 \le -4 \\ & x_1 + x_2 - 4x_3 \le 2 \\ & x_1, x_2, x_3 \ge 0 \end{array} \quad (P) \quad \overset{\text{dual}}{\longleftrightarrow} \quad \begin{array}{ll} \min & -3y_1 - 4y_2 + 2y_3 \\ \text{s.t} & -y_1 - 4y_2 + y_3 \ge -4 \\ & -y_1 - 2y_2 + y_3 \ge -2 \\ & 2y_1 + y_2 - 4y_3 \ge -1 \\ & y_1, y_2, y_3 \ge 0 \end{array} \quad (D)$$

A pair of optimal solutions are $\bar{x} = \begin{bmatrix} 0 \\ 4 \\ \frac{1}{2} \end{bmatrix}$ and $\bar{y} = \begin{bmatrix} \frac{9}{2} \\ 0 \\ \frac{5}{2} \end{bmatrix}$

$\bar{x}_1 = 0$; $(A^T\bar{y} - c)_1 = -\bar{y}_1 - 4\bar{y}_2 + \bar{y}_3 - (-4) = 2 > 0$. $\quad \bar{x}_1 (A^T\bar{y} - c)_1 = 0$

$\bar{x}_2 > 0$; $(A^T\bar{y} - c)_2 = -\bar{y}_1 - 2\bar{y}_2 + \bar{y}_3 - (-2) = 0$. $\quad \bar{x}_2 (A^T\bar{y} - c)_2 = 0$

$\bar{x}_3 > 0$; $(A^T\bar{y} - c)_3 = 2\bar{y}_1 + \bar{y}_2 - 4\bar{y}_3 - (-1) = 0$ $\quad \bar{x}_3 (A^T\bar{y} - c)_3 = 0$

$\bar{y}_1 > 0$; $(b - A\bar{x})_1 = -3 - (-\bar{x}_1 - \bar{x}_2 + 2\bar{x}_3) = 0$ $\quad \bar{y}_1 (b - A\bar{x})_1 = 0$

$\bar{y}_2 = 0$; $(b - A\bar{x})_2 = -4 - (-4\bar{x}_1 - 2\bar{x}_2 + \bar{x}_3) = 9 > 0$ $\quad \bar{y}_2 (b - A\bar{x})_2 = 0$

$\bar{y}_3 > 0$; $(b - A\bar{x})_3 = 2 - (\bar{x}_1 + \bar{x}_2 - 4\bar{x}_3) = 0$. $\quad \bar{y}_3 (b - A\bar{x})_3 = 0$

Remark: 1. For a feasible solution $\bar{x}$, we call an inequality constraint an active constraint if the constraint is satisfied at equality. The complementary slackness conditions can be rephrased as:

> If a primal/dual variable is nonzero, then the corresponding dual/primal constraint is active.
> If a dual/primal constraint is inactive, then the corresponding primal/dual variable is zero.

2. For primal feasible $\bar{x}$ and dual feasible $\bar{\pi}$, since $\bar{x} \geq 0$ and $A^T\bar{\pi} - c \geq 0$, conditions

$$\bar{x}_i (A^T\bar{\pi} - c)_i = 0 \quad \forall\, i = 1, \ldots, n$$

are equivalent to a single condition

$$\bar{x}^T (A^T\bar{\pi} - c) = 0$$

Similarly, conditions

$$\bar{\pi}_j (b - A\bar{x})_j = 0 \quad \forall\, j = 1, \ldots, m$$

are equivalent to

$$\bar{\pi}^T (b - A\bar{x}) = 0$$

Furthermore, the complementary slackness conditions are equivalent to

$$\bar{x}^T (A^T\bar{\pi} - c) + \bar{\pi}^T (b - A\bar{x}) = 0$$

Proof of the theorem:

$$\text{Duality gap} = b^T\bar{\pi} - c^T\bar{x} = \left[\bar{\pi}^T(b-A\bar{x}) + \bar{\pi}^T A\bar{x}\right] - \left[(A^T\bar{\pi})^T\bar{x} - (A^T\bar{\pi}-c)^T\bar{x}\right]$$

$$= \bar{\pi}^T(b-A\bar{x}) + \bar{x}^T(A^T\bar{\pi}-c)$$

If $\bar{x}$ and $\bar{\pi}$ are optimal solutions to (P) and (D), respectively, then $b^T\bar{\pi} = c^T\bar{x}$ by strong duality Then the complementary slackness conditions are satisfied.

On the other hand, if the complementary slackness conditions are satisfied, then $b^T\bar{\pi} = c^T\bar{x}$.

By the optimality property, $\bar{x}$ and $\bar{\pi}$ are optimal to (P) and (D) $\square$

eg Application: Constructing the primal optimal solution from the dual optimal solution.

$$\max \quad z = x_1 + 2x_2 - 9x_3 + 8x_4 - 36x_5$$

$$\begin{aligned} \text{s.t} \quad & 2x_2 - x_3 + x_4 - 3x_5 \le 40 \qquad \text{(P)} \\ & x_1 - x_2 + 2x_4 - 2x_5 \le 10 \\ & x_1, \ \ , x_5 \ge 0 \end{aligned}$$

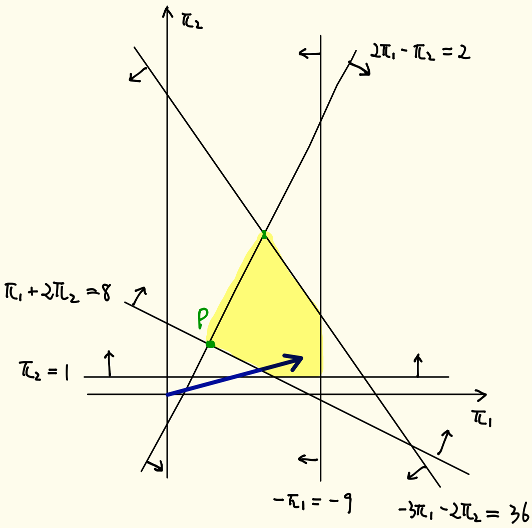
We can solve the problem without using the simplex tableau

The dual problem is

$$\min \quad 40\pi_1 + 10\pi_2$$

$$\begin{aligned} \text{s t} \quad & \pi_2 \ge 1 \\ & 2\pi_1 - \pi_2 \ge 2 \qquad \text{(D)} \\ & -\pi_1 \ge -9 \\ & \pi_1 + 2\pi_2 \ge 8 \\ & -3\pi_1 - 2\pi_2 \ge -36 \\ & \pi_1, \pi_2 \ge 0 \end{aligned}$$

The dual problem has only two variables, so we can solve it graphically



$$P \begin{cases} 2\pi_1 - \pi_2 = 2 \\ \pi_1 + 2\pi_2 = 8 \end{cases}$$

$$\pi^* = \left(\frac{12}{5}, \frac{14}{5} \right)$$

By strong duality,

the primal problem has an optimal solution We call it x^*

By complementary slackness,

Since $\pi_1^* \neq 0$ and $\pi_2^* \neq 0$,

both constraints in (P) are active at x^*

$$2x_2^* - x_3^* + x_4^* - 3x_5^* = 40$$
$$x_1^* - x_2^* + 2x_4^* - 2x_5^* = 10$$

Consider the constraints in the dual problem,

$$\begin{aligned}
& \pi_2^* = \frac{14}{5} > 1 \quad \text{inactive} \quad \overset{CS}{\Longrightarrow} \quad x_1^* = 0 \\
& 2\pi_1^* - \pi_2^* = 2 \\
& -\pi_1^* = -\frac{12}{5} > -9 \quad \text{inactive} \quad \overset{CS}{\Longrightarrow} \quad x_3^* = 0 \\
& \pi_1^* + 2\pi_2^* = 8 \\
& -3\pi_1^* - 2\pi_2^* = -\frac{64}{5} > -36 \quad \text{inactive} \quad \overset{CS}{\Longrightarrow} \quad x_5^* = 0
\end{aligned}$$

Therefore, $\begin{cases} 2x_2^* + x_4^* = 40 \\ -x_2^* + 2x_4^* = 10 \end{cases}$

Solving the linear system, we get $\begin{cases} x_2^* = 14 \\ x_4^* = 12 \end{cases}$

$x^* = (0, 14, 0, 12, 0)$ is the optimal solution of the primal problem.

e.g.: Check whether $\bar{x} = (5, 0, 0)^T$ is an optimal solution of the following linear program.

$$
\begin{aligned}
\max \quad & 6x_1 + 3x_2 + 4x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + x_3 \le 8 \\
& 5x_1 + 4x_2 + 3x_3 \le 25 \qquad (P) \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

First, we check the feasibility of $\bar{x}$.

$$5 + 2\times 0 + 0 = 5 \le 8 \quad \checkmark$$

$$5\times 5 + 4\times 0 + 3\times 0 = 25 \le 25 \quad \checkmark$$

$$5 \ge 0,\ 0 \ge 0,\ 0 \ge 0 \quad \checkmark$$

Therefore, $\bar{x}$ is feasible to (P).

Next, by the complementary slackness theorem, if $\bar{x}$ is an optimal solution of (P), then there exists a dual feasible solution $\bar{\pi}$ such that $\bar{x}$ and $\bar{\pi}$ satisfy the complementary slackness conditions. That is, there exists

$\bar{\pi}$ feasible to
$$
\begin{aligned}
\min \quad & 8\pi_1 + 25\pi_2 \\
\text{s.t.} \quad & \pi_1 + 5\pi_2 \ge 6 \\
& 2\pi_1 + 4\pi_2 \ge 3 \qquad (D) \\
& \pi_1 + 3\pi_2 \ge 4 \\
& \pi_1, \pi_2 \ge 0.
\end{aligned}
$$

and satisfies

$$
\begin{cases}
\bar{x}_1 (\bar{\pi}_1 + 5\bar{\pi}_2 - 6) = 0 \\
\bar{x}_2 (2\bar{\pi}_1 + 4\bar{\pi}_2 - 3) = 0 \\
\bar{x}_3 (\bar{\pi}_1 + 3\bar{\pi}_2 - 4) = 0 \\
\bar{\pi}_1 (8 - \bar{x}_1 - 2\bar{x}_2 - \bar{x}_3) = 0 \\
\bar{\pi}_2 (25 - 5\bar{x}_1 - 4\bar{x}_2 - 3\bar{x}_3) = 0
\end{cases}
\qquad (CS)
$$

Given $\bar{x} = (5, 0, 0)^T$, the system (CS) reduces to $\begin{cases} \bar{\pi}_1 + 5\bar{\pi}_2 = 6 \\ \bar{\pi}_1 = 0 \end{cases}$

Therefore, the only vector $\bar{\pi}$ which satisfies (CS) with $\bar{x}$ is $\bar{\pi} = (0, 6/5)^T$.

Note that $\bar{\pi}$ is infeasible to (D) as $\bar{\pi}_1 + 3\bar{\pi}_2 = \frac{18}{5} < 4$.

Consequently, there is no such $\bar{\pi}$ that is feasible to (D) and satisfies (CS) with $\bar{x}$.

Therefore, $\bar{x}$ is not optimal to (P).