HW2 Math 4540

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1 Exercises

1. Suppose $f, g : [0,1] \to [0,1]$ are two continuous functions with $f(0) = g(1) = 0$ and $g(0) = f(1) = 1$. Show that the graphs $\Gamma(f)$ and $\Gamma(g)$ of the two functions intersect in $(0,1) \times [0,1]$.

   Consider the function $h(x) = f(x) - g(x)$. Note that h is continuous on the closed interval $[0,1]$, as both f and g are continuous there. Consider that $h(0) = f(0) - g(0) = -1$ and $h(1) = f(1) - g(0) = 1$; then by the Intermediate Value Theorem, since h is continuous on $(0,1)$, there exists some value $c \in (0,1)$ such that $h(c) = 0$, as $h(0) < 0 < h(1)$. Hence $f(c) = g(c) \in [0,1]$ for some $c \in (0,1)$, and thus $(c, f(c)) = (c, g(c)) \in (\Gamma(f) \cap \Gamma(g)) \cap ((0,1) \times [0,1])$.

2. Show by example that the set theoretical inverse $f^{-1}$of a continuous map $f : X \to Y$ that is 1-to-1 and onto does not need to be continuous if X is not compact. Hint: Consider a map from $[0,1)$ onto the (unit) circle.

   Suppose $f : [0,1) \to S^1$ is any continuous map that is 1-to-1 and onto from the interval $[0,1)$ to the unit circle $S^1 \subset R^2$, such as the mapping $f(t) = (cos(2\pi t), sin(2\pi t))$. We first show that S^1 is compact. This can be easily seen, since the continuous function $g : R \to S^1$ given by $g(t) = (cos(t), sin(t))$ produces the image S^1 from the closed interval $[0, 2\pi]$. Since the interval $[0, 2\pi]$ is compact, this means its image $g([0, 2\pi]) := S^1$ must also be compact, as g is a continuous function. Next we note that the half-open interval $[0,1)$ is not compact, since it does not contain the limit point $x = 1$. Consider now the inverse f^{-1} of f, defined above. Then this is some bijection from the unit circle to the half-open interval $[0,1)$. Observe, however, that f^{-1} cannot be continuous: if f^{-1} were some continuous mapping, then this would mean that $f^{-1}(S^1) := [0,1)$ would have to be compact, as S^1 is a compact set, and hence so is its image on f^{-1}. Therefore a contradiction is reached, and thus there do exist continuous maps $f : X \to Y$ that are 1-to-1 and onto that do not have inverses which are continuous whenever X is not compact.

3. Let f be a real valued function on $(a, b]$, then

   (a) if f is continuous on $(a, b]$ and $\lim_{x \to a^+} f(x)$ exists, then f is uniformly continuous on $(a, b]$.
   Let g be a real valued function that extends f for the endpoint $a \in [a, b]$, given by $g(a) = \lim_{x \to a^+} f(x)$ and $g(x) = f(x)$ whenever $x \neq a$. Because f is continuous on $(a, b]$, it follows that g is continuous there as well; now reconsider the same point $a \in [a, b]$ as before. Note that this point is a limit point, since it's contained in the closure of $(a, b]$. Then it's easy to see that $g(x)$ is also continuous when $x = a$, since $\lim_{x \to a} g(x) = g(a)$ exists. Therefore g is continuous on all of $[a, b]$, and since $[a, b]$ is compact, is thus uniformly continuous. This means that for any chosen $\epsilon > 0$, there exists some $\delta > 0$ such that $|g(x) - g(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $x, y \in [a, b]$. Notably, for $x, y \in (a, b]$ this also means $|g(x) - g(y)| = |f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$; and hence f is uniformly continuous as well.

   (b) if f is uniformly continuous on $(a, b]$, then $\lim_{x \to a^+} f(x)$ exists.
   Suppose f is uniformly continuous on $(a, b]$. Then for any chosen $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ for all $x, y \in (a, b]$. Suppose (x_n) is any (ultimately decreasing) sequence that is contained in $(a, b]$ such that $x_i \neq a$ for all $i \leq n$, and for which (x_n) approaches a from the right; then there exists some integer $N \in Z$ such that whenever $n, m > N$, $|x_n - x_m| < \delta$, for δ defined above. As f is uniformly continuous, this also implies that $f(x_n) - f(x_m) < \epsilon$ for any chosen $\epsilon > 0$; thus the sequence (x_n) converges as it tends to a from the right. Because (x_n) was arbitrary and therefore approaches a every possible way, this implies $\lim_{x \to a^+} f(x)$ exists.

4. Show that the equation $x^3 - 3x + b = 0$ has at most one root in the interval $[0, 1]$.

   Suppose there existed two roots in the interval $[0, 1]$ for the equation $x^3 - 3x + b = 0$. Then the function $f : [0, 1] \to R$ given by the equation above would have two values $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2) = 0$. Note that f is differentiable and therefore continuous on $[0, 1]$. Then by Rolle's theorem, there must exist some $c \in (x_1, x_2)$ such that $f'(c) = 0$. Observe however that $f'(x) = 3x^2 - 3$, and hence $f'(c) = 0$ is only valid for $c = \pm 1$. This is a contradiction however, as $c \in (x_1, x_2) \subset [0, 1]$, and hence $0 < c < 1$; thus there can only be at most one root for the equation in the interval $[0, 1]$.

5. Suppose f is differentiable on an interval I.

   (a) Prove that f' is bounded if and only if there exists a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in I$.

We first prove the forward direction. Let's assume there exists some constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in I$. Then

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M,$$

and thus

$$f'(y) = \lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim_{x \to y} M = M$$

whenever $x \neq y$. Since $x, y \in I$ and f is differentiable on all of I, it follows that f' is bounded by M as

$$f'(y) \leq M.$$

Now consider the other direction. Assume that f' is bounded; then for all $z \in I$, $|f'(z)| \leq M$ for some constant $M > 0$. Recall that, by the Mean Value Theorem, there exists at least one point $z \in (x, y)$ such that $|f(x) - f(y)| = |f'(z)||x - y|$ for each $x, y \in I$. Since $|f'(z)| \leq M$ for all such $z \in I$, it follows that the same constant M exists where $|f(x) - f(y)| \leq M|x - y|$ for any $x, y \in I$.

(b) Prove that $|\sin(x) - \sin(y)| \leq M|x - y|$ for all $x, y \in R$.

Recall that $f(t) = \sin(t)$ is differentiable on R, and that its derivative $f'(t) = \cos(t)$ is bounded since each value $|a| \leq 1$ where $a = cos(t)$ for $t \in R$; then, because of exercise (a), it follows that $|f(x) - f(y)| = |\sin(x) - \sin(y)| \leq M|x - y|$ for all $x, y \in R$.

(c) $|\sqrt{x} - \sqrt{y}| \leq \frac{1}{2\sqrt{a}}|x - y|$ for all $x, y \in [a, \infty)$ whenever $a > 0$.

Similiarly, recall that $f(t) = \sqrt{t}$ is differentiable on R, and that its derivative is $f'(t) = \frac{1}{2\sqrt{t}}$. Consider $z \in [a, \infty)$, where $a > 0$; then, since $z \geq a$,

$$f'(z) = \frac{1}{2\sqrt{z}} \leq \frac{1}{2\sqrt{a}} = f'(a).$$

Thus f' is bounded by $M = f'(a) = \frac{1}{2\sqrt{a}}$ whenever $z \in [a, \infty)$, and hence by exercise (a), it follows that $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| \leq \frac{1}{2\sqrt{a}}|x - y|$ for $x, y \in [a, \infty)$ whenever $a > 0$.