Homework 2

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1 Problems from Chapter 2

(2) The sample space is given by all finite tuples of length $n \geq 1$ whose first $n-1$ elements are the dice results 1 to 5, and whose last element is the dice result 6; as well as all infinite tuples, whose elements are only the results 1 to 5:

$$S = \{(a_1, a_2, a_3, \dots) \mid a_i \in \{1,2,3,4,5\} \text{ and } a_n = 6 \text{ for each } i < n \text{ and } n \geq 1\}$$

(Note that I'm intending to include the outcomes where $n = \infty$, the infinite tuples mentioned above). Then E_n contains all of the outcomes in S whose tuples are of a particular length n:

$$E_n = \{(a_1, a_2, a_3, \dots, a_n) \mid a_i \in \{1,2,3,4,5\} \text{ for all } i < n \text{ and } a_n = 6\},$$

and

$$(\cup_{n=1}^{\infty} E_n)^c = S - \cup_{n=1}^{\infty} E_n$$

is a subset containing all of the outcomes which are infinite.

(6) a) $S = \{(a, b) \mid a \in \{0, 1\} \text{ and } b \in \{g, f, s\}\}$
b) $A = \{(0, s), (1, s)\}$
c) $B = \{(0, g), (0, f), (0, s)\}$
d) $A \cup B^c = \{(0, s), (1, g), (1, f), (1, s)\}$

(9) $0.24 + (0.61 - 0.11) = 0.74$

(12) Let E_S, E_F, E_G denote the events that a student are in Spanish, French, or German class. Similarly, let E_{AB} represent that a student is in the classes $A, B \in \{S, F, G\}$, where $B \neq A$, and E_{SFG} denote the event that they are in all three. Finally, for any event E_X given above, let $E_{\hat{X}}$ denote that a student is only taking those classes belonging to X. Then $P(E_S) = 0.28, P(E_F) = 0.26, P(E_G) = 0.16, P(E_{SF}) = 0.12, P(E_{SG}) = 0.04, P(E_{FG}) = 0.06,$ and $P(E_{SFG}) = 0.02$.

For a chosen student in only one of the classes, we have to be careful not to double count or remove too many students:

$$
\begin{aligned}
P(E_{\hat{S}}) &= P(E_S) - P(E_{SF}) - P(E_{SG}) + P(E_{SFG}) = 0.14 \\
P(E_{\hat{F}}) &= P(E_F) - P(E_{SF}) - P(E_{FG}) + P(E_{SFG}) = 0.10 \\
P(E_{\hat{G}}) &= P(E_G) - P(E_{SG}) - P(E_{FG}) + P(E_{SFG}) = 0.08
\end{aligned}
$$

Similarly, we derive the probabilities for being in only two of the classes:

$$
\begin{aligned}
P(E_{\widehat{SF}}) &= P(E_{SF}) - P(E_{SFG}) = 0.10 \\
P(E_{\widehat{SG}}) &= P(E_{SG}) - P(E_{SFG}) = 0.02 \\
P(E_{\widehat{FG}}) &= P(E_{FG}) - P(E_{SFG}) = 0.04
\end{aligned}
$$

a) The probability that they are not in any of the classes is the complement of them being in at least one: $1.00 - P(E_S \cup E_{\hat{F}} \cup E_{\hat{G}} \cup E_{\widehat{FG}}) = 1.00 - (0.28 + 0.10 + 0.08 + 0.04) = 1.00 - 0.50 = 0.50$

b) $P(E_{\hat{S}} \cup E_{\hat{F}} \cup E_{\hat{G}}) = 0.14 + 0.10 + 0.08 = 0.32$

c) We already found that, for one student, the probability of being in at least one is $P(E_S \cup E_{\hat{F}} \cup E_{\hat{G}} \cup E_{\widehat{FG}}) = 0.50$. This means that it is equally likely that students are either taking a language class or not. Note that we can treat this basically as a coin flip. Thus for two students, either both can have a class, the first can have a class, the second can have a class, or none can have a class; hence $\frac{3}{4}$ of the outcomes are favorable. So $P = 0.75$.

(16) There are $6 * 6 * 6 * 6 * 6 = 7776$ total outcomes for the five rolls. For each exercise below, we find the probable outcomes for the event and then divide by the total. The best way to read the work is that the rolls are grouped by same value and arrangements; for example, for exercise (e), the two rolls of the pair are grouped with the number of ways to arrange them: $(6 * 1 * \binom{5}{2})$. This is read as "6 outcomes that decide the first roll, 1 outcome to decide the second roll, and $\binom{5}{2}$ ways to arrange the two". Then the remaining three rolls are grouped together, since they are a triplet, and this makes a full house.

$P(\text{no alike}) = [6 * 5 * 4 * 3 * 2]/7776 = 0.0926$

b) $P(\text{one pair}) = [(6 * 1 * \binom{5}{2})) * 5 * 4 * 3]/7776 = 0.4630$

c) $P(\text{two pair}) = [(6 * 1 * \binom{5}{2})) * (5 * 1 * \binom{3}{2}) * 4]/(2 * 7776) = 0.2315$
(Note, we divide by an additional 2 *as the arrangements between the two pairs have extra symmetry: consider, for example, the rolls* $(1, 1, 2, 2, 4)$ *and* $(1, 1, 2, 2, 4)$*, where in the first outcome the "first pair" is the pair of ones, and in the second, is the pair of twos. Because the two sets of pairs are symmetrical, we divide by two to only consider one of their arrangements.)*

d) $P(\text{three alike}) = [(6 * 1 * 1 * \binom{5}{3}) * 5 * 4]/7776 = 0.1543$

e) $P(\text{full house}) = [(6 * 1 * \binom{5}{2}) * (5 * 1 * 1)]/7776 = 0.0386$

f) $P(\text{four alike}) = [(6 * \binom{5}{1}) * 5 * 1 * 1 * 1]/7776 = 0.0193$

g) $P(\text{five alike}) = [6 * 1 * 1 * 1 * 1]/7776 = 0.0008$

(22) The obvious outcome where the ordering is preserved is the outcome $(H, H, H, \dots)$ of only flipping heads for each card. However, the order of the cards will also be preserved as long as, once a tail does get flipped, all future flips are also tails: for example, (H, H, T, T, T, T), for $n = 6$, will preserve the ordering of the cards. Suppose n was known. Then one outcome, the trivial outcome of only rolling heads, will preserve the ordering by not shuffling at all; and n additional outcomes, where from positions $1 \le i \le n$, only tails is flipped, will preserve it by reshuffling everything back into their original order. Thus there are 2^n total outcomes and $n+1$ outcomes which are probable, so

$$P = \frac{n+1}{2^n}$$

(25) There are 36 total outcomes from rolling two dice; 4 outcomes sum to 5, 6 outcomes sum to 7, and the other 26 do not sum to either. Thus

$$P(E_n) = (\frac{26}{36})^{n-1} * \frac{4}{36}$$

and hence

$$\sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} (\frac{26}{36})^{n-1} * \frac{4}{36} = \sum_{k=0}^{\infty} \frac{4}{36} * (\frac{26}{36})^k = \frac{4}{36} * \frac{(\frac{26}{36})^0}{1 - (\frac{26}{36})} = \frac{1}{9} * \frac{18}{5} = \frac{2}{5}.$$

Therefore the probability that a 5 occurs first is $P = 0.40$.

(41) The probability that a 6 comes up at least once is the complement of a 6 having never came up:

$$P = 1 - \frac{5}{6} * \frac{5}{6} * \frac{5}{6} * \frac{5}{6} = 0.518.$$

2 Theoretical Exercises from Chapter 2

(11) Consider Proposition 4.3 from the book:

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Since we know by definition that $0 \le P(E \cup F) \le 1$, we find that

$$P(E) + P(F) - P(EF) \le 1$$

and thus

$$-P(EF) \leq -P(E) - P(F) + 1.$$

Multiplying by negative one gives Bonferroni's inequality:

$$P(EF) \geq P(E) + P(F) - 1.$$

Applying this for $P(E) = 0.9$ and $P(F) = 0.8$ shows

$$P(EF) \geq 0.9 + 0.8 - 1 = 0.7.$$

(18) Suppose we tossed a coin 0 times; then there is 1 outcome which is favorable, which is trivial. Thus $f_0 = 1$. For tossing a coin 1 and 2 times, we have the sample spaces $S_1 = \{H, T\}$ and $S_2 = \{HH, HT, TT, TH\}$; and so it is easy to see that $f_1 = 2$ and $f_2 = 3$. We now construct S_3:

$$S_3 = \{AT | A \in S_2\} \cup \{BTH | B \in S_1\}.$$

To illustrate:

$$S_3 = \{HHT, HTT, TTT, THT\} \cup \{HTH, TTH\}.$$

In general, we want to prove

$$S_n = \{AT | A \in S_{n-1}\} \cup \{BTH | B \in S_{n-2}\},$$

for all $n \geq 2$. In addition, we must show that $AT \neq BTH$ for all $A \in S_{n-1}$ and $B \in S_{n-1}$, since this will show that $f_n = f_{n-1} + f_{n-2}$ as $f_n = |S_n|$ for all such n.

Suppose $C \in S_n$ is some outcome after tossing a coin $n \geq 2$ times, such that no successive heads are contained in C. Then, the very last two tosses within C is some $X \in \{TT, HT, TH\}$; notably, $C \neq HH$, as this would be an appearance of two successive heads. Additionally, the very last toss in C must be either heads or tails. Similarly, if $A \in S_{n-1}$ and $B \in S_{n-2}$, then the very last tosses of those outcomes are either heads or tails as well. Consider now the last two tosses of a new outcome AT: since A ends in either heads or tails, these two tosses must either be TT or HT. Also consider the last two tosses of a new outcome BTH, which can only be TH. Therefore, we have $AT \neq BTH$ for all $A \in S_{n-1}$ and $B \in S_{n-1}$, as the final two tosses in the two outcomes are always different.

Finally, suppose $C \neq AT$ and $C \neq BTH$ for any $A \in S_{n-1}$ and $B \in S_{n-2}$. This implies that either there is an additional outcome D after tossing a coin $n - 2$ times such that $D \notin S_{n-2}$ and $DX = C$ for some $X \in \{TT, HT, TH\}$; or, that there is an outcome E after tossing a coin $n - 1$ times such that $E \notin S_{n-1}$ and $DY = C$ for some $Y \in \{H, T\}$. This is a contradiction, however, as if $D \notin S_{n-2}$, then D must contain two successive heads, and hence so must DX; and similarly DY; and thus $C \in \{AT | A \in S_{n-1}\} \cup \{BTH | B \in S_{n-2}\}$.