Homework 9

AJ

1. All abelian groups of orders 2250 are composed of direct products of their Sylow subgroups (as per Corollary 7.2.4), which are grouped into orders 2, $3^2 = 9$, and $5^3 = 125$:

Order 2	Order 9	Order 125
$\mathbb{Z}_2$	$\mathbb{Z}_9$	$\mathbb{Z}_{125}$
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_5 \times \mathbb{Z}_{25}$
		$\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

Therefore the different abelian groups of order 2250 are as follows:

1) $\mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_{125}$
2) $\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$
3) $\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_9$
4) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{125}$
5) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{25}$
6) $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

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2. $G = \mathbb{Z}_{60} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{100} \oplus \mathbb{Z}_{56} \oplus \mathbb{Z}_{48}$

The prime factorization of each are as follows:

$$60 = 2 \cdot 2 \cdot 3 \cdot 5 = \cancel{2^2} \cdot \cancel{3} \cdot \cancel{5}$$
$$24 = 2 \cdot 2 \cdot 2 \cdot 3 = \cancel{2^3} \cdot \cancel{3}$$
$$36 = 2 \cdot 2 \cdot 3 \cdot 3 = \cancel{2^2} \cdot \cancel{3^2}$$
$$100 = 2 \cdot 2 \cdot 5 \cdot 5 = \cancel{2^2} \cdot \cancel{5^2}$$
$$56 = 2 \cdot 2 \cdot 2 \cdot 7 = \cancel{2^3} \cdot \cancel{7}$$
$$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = \cancel{2^4} \cdot \cancel{3}$$

a) The invariant factor decomposition is therefore

$$G \cong \mathbb{Z}_{(2^4 \cdot 3^2 \cdot 5^2 \cdot 7)} \oplus \mathbb{Z}_{(2^3 \cdot 3 \cdot 5)} \oplus \mathbb{Z}_{(2^3 \cdot 3)} \oplus \mathbb{Z}_{(2^2 \cdot 3)} \oplus \mathbb{Z}_{(2^2)} \oplus \mathbb{Z}_{(2^2)}$$

b) The elementery divisor decomposition is therefore

$$G \cong (\mathbb{Z}_{2^4} \oplus \mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^3} \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^2})$$
$$\oplus (\mathbb{Z}_{3^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3) \oplus \cancel{\mathbb{Z}_{5^2}} \oplus \mathbb{Z}_7$$
$$(\mathbb{Z}_{5^2} \oplus \mathbb{Z}_5)$$

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3a) First observe that any n derived subgroup

$$G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$$

must contain at least the identity element, which is to say a solvable group that terminates to the trivial group $\{e\}$ will continue to derive $\{e\}$:

$$G^{(n)} = \langle x^{-1}y^{-1}xy \mid x,y \in G^{(n-1)} \rangle,$$

$$[e,e] = e \implies e \in G^{(n)}.$$

Secondly, observe that for any $N^{(i)} \leq G^{(i)}$ where $i \in \mathbb{N}$, $N^{(i+1)} \leq G^{(i+1)}$:

Let $a, \in N^{(i+1)}$. Then

$$a = x^{-1}y^{-1}xy, \quad x,y \in N^{(i)}.$$

However since $N^{(i)} \leq G^{(i)}$, $x,y \in G^{(i)}$ and

$$a \in G^{(i+1)} := \langle x^{-1}y^{-1}xy \mid x,y \in G^{(i)} \rangle.$$

Therefore, given that G is solvable,

$$N^{(m)} \leq G^{(m)} = \{e\},$$

And from the first observation N is solvable, As $N \leq G$ implies $N' \leq G'$ which implies $N'' \leq G''$, and so on until $N^{(i)} = \{e\}$

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3b) Let $\pi : G \to G/N$ be defined as

$$\pi(g) = gN \quad \text{for } g \in G.$$

Observe that π is a homomorphism:

Let $g, h \in G$.

$$\pi(gh) = ghN.$$

Because $N \trianglelefteq G$ is a normal subgroup, the left cosets are equal to the right:

$$ghN = g(hN) = gNh$$
$$= g(N \cdot N)h = (gN)(Nh)$$
$$= (gN)(hN) = \pi(g)\pi(h).$$

Next observe that $\pi(G^{(n)}) = (G/N)^{(n)}$ for some $n \in \mathbb{N}$:

$$\pi(G^{(n)}) = \operatorname{im}(\pi|_{G^{(n)}}) = \langle x^{-1}y^{-1}xyN \mid x, y \in G^{(n)} \rangle$$
$$= \left\langle a^{-1}b^{-1}ab \;\middle|\; \begin{array}{l} a = xN \in G/N \\ b = yN \in G/N \end{array} \right\rangle$$
$$= (G/N)^{(n)}$$

Finally consider $\pi(G^{(m)}) = \operatorname{im}(\pi|_{\{e\}}) = N$, whereby $(G/N)^{(n)} = N$ and therefore G/N is solvable.

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3d) $675 = 3 \cdot 3 \cdot 3 \cdot 5 \cdot 5$.

By the Sylow theorems, the number of sylow 5-subgroups of G, n_5, is given by

$$n_5 \equiv 1 \bmod 5 \quad \text{and} \quad n_5 \mid 27 \in \{1,3,9,27\}$$

Given $5 \nmid 2$, $5 \nmid 8$, and $5 \nmid 26$, the sylow subgroup N of G is unique and therefore normal.

Assume ~~Observe~~ that ~~$|N| = 25$~~ since N is a p-group it is soluable (graduate proof). Observe, also, that G/N has order

$$|G|/|N| = 27$$

which means G/N is also a p-group, and therefore soluable. Therefore, by the previous problem, G is soluable.

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4) Let $\pi : HK \to H \rtimes_\phi K$ be defined by

$$\pi(hk) = (h, k).$$

Similar to theorem 7.1.7, this function is well defined as if $hk = h_1 k_1$ then $h_1^{-1} h = k_1 k^{-1}$ which is contained in the intersection of H and K, and therefore $h = h_1$ and $k = k_1$.

To have π be a homomorphism, consider $\Phi(k) = f$, where $f : H \to H$ defined by

$$f(h) = khk^{-1}.$$

Observe that such a definition is valid on its own as $H \trianglelefteq G$, and therefore $ghg^{-1} \in H$ for all $g \in G$.

Thus $\pi(h_1 k_1)\pi(h_2 k_2) = (h_1, k_1)(h_2, k_2)$

$$= (h_1 \Phi(k_1)(h_2), k_1 k_2)$$

$$= (h_1 k_1 h_2 k_1^{-1}, k_1 k_2)$$

Because $H \trianglelefteq G$, and $G = HK$, we know

$h_1 k_1 h_2 k_2 \in G$ is also an element of HK, and can be written $h_1 k_1 h_2 k_2 = h_a k_a$.

Similar to above, $(h_1 k_1) h_2 (h_1 k_1)^{-1} \in H$, so $h_1 k_1 h_2 k_1^{-1} h_1^{-1} = h_b$ and $h_1 k_1 h_2 k_1^{-1} = h_b h_1$.