# HW3 MATH 4540

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## 1 Exercises

1. Given a metric space $X$ and real valued functions $f_k : X \to \mathbb{R}$ for $k = 1, ..., n$, let $\hat{f} : X \to \mathbb{R}^n$ be given by $\hat{f}(x) := (f_1(x), ..., f_n(x))$. Show that $\hat{f}$ is continuous if and only if the component functions $f_k$ for $k = 1, ..., n$ are all continuous.

   Suppose the component functions $f_k$ for $k = 1, ..., n$ are all continuous. Then, for every $\epsilon > 0$, there exists some $\delta_k > 0$ such that $|f_k(x) - f_k(p)| < \epsilon$ whenever $d_X(x, p) < \delta_k$ (for each $x, p \in X$ and $k = 1, ..., n$). Observe that

$$d_{\mathbb{R}^n}(\hat{f}(x), \hat{f}(p)) = |\hat{f}(x) - \hat{f}(p)| = \sqrt{\sum_{k=1}^{n} |f_k(x) - f_k(p)|^2}.$$

   Suppose $\hat{\epsilon} > 0$ was another arbitrary real number. Then, for each $k$, there exists $\hat{\delta}_k > 0$ such that $|f_k(x) - f_k(p)| < \frac{\hat{\epsilon}}{\sqrt{n}}$ whenever $d_X(x, p) < \hat{\delta}_k$, as $\frac{\hat{\epsilon}}{\sqrt{n}} > 0$; and hence

$$|\hat{f}(x) - \hat{f}(p)| = \sqrt{\sum_{k=1}^{n} |f_k(x) - f_k(p)|^2} < \sqrt{\sum_{k=1}^{n} (\frac{\hat{\epsilon}}{\sqrt{n}})^2} = \sqrt{n(\frac{\hat{\epsilon}^2}{n})} = \hat{\epsilon}$$

   whenever $d_X(x, p) < \min(\delta_1, \ldots, \delta_n)$. Thus $\hat{f}$ is continuous whenever the component functions $f_k$ for $k = 1, ..., n$ are all continuous.

   We now prove the reverse assertion. Suppose $\hat{f}$ is continuous. Then

$$|f_\alpha(x) - f_\alpha(p)| \leq \sqrt{\sum_{k=1}^{n} |f_k(x) - f_k(p)|^2} = |\hat{f}(x) - \hat{f}(p)|$$

   for all $1 \leq \alpha \leq n$. Additionally, for every $\epsilon > 0$, there exists some $\hat{\delta} > 0$ such that $|\hat{f}(x) - \hat{f}(p)| < \epsilon$ whenever $d_X(x, p) < \hat{\delta}$, and hence $|f_\alpha(x) - f_\alpha(p)| \leq |\hat{f}(x) - \hat{f}(p)| < \epsilon$ as well. Thus $f_\alpha$ is also continuous.

2. Let $f : [0, 1] \to R$ be given by

$$f(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1/\ln(\frac{x}{2}) & \text{otherwise} \end{cases}$$

Note that the functions $g(x) = \ln(x)$ and $h(x) = \frac{x}{2}$ are continuous for all $x > 0$, and hence their composition $(g \circ h)(x)$ is as well. Then it follows that the reciprocal $1/(g \circ h)(x) = 1/\ln(\frac{x}{2})$ is continuous everywhere that it is defined, which is for values $x > 0$ and $x \neq 2$. To see that $f(x)$ is also continuous at zero, consider that $x = 0$ is a limit point of $(0, 1]$, and that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1/\ln(\frac{x}{2}) = 0 = f(0).$$

Hence $f(x)$ is (uniformly) continuous on $[0, 1]$. Now consider the derivative of $f$ given by $f' : (0, 1) \to R$, where

$$f'(x) = \frac{-1}{x \ln^2(\frac{x}{2})}.$$

Observe that because

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{-1}{x \ln^2(\frac{x}{2})} = -\infty,$$

then, for any real number $M \in R$, there exists some value $y \in (0, 1)$ close to zero where $|f'(y)| > M$. Therefore $f(x)$ is a uniformly continuous function on $[0, 1]$ that has an unbounded derivative on $(0, 1)$.

3. Let $f : [a, b] \to [a, b]$ be a function that is differentiable on $[a, b]$ such that $|f'(x)| \leq c$ for some given constant $0 < c < 1$ and $x \in [a, b]$. We first show that $f$ has at least one fixed point. Suppose $f(a) \neq a$ and $f(b) \neq b$. Then it follows that $a < f(a)$ and $f(b) < b$, as $f([a, b]) \subset [a, b]$. Consider now all $f(a + \delta)$ for $\delta \in (0, b - a)$, and suppose $f(a + \delta) \neq a + \delta$ as well. Then, similarly, we either have that

$$f(a + \delta) < a + \delta$$

or

$$a + \delta < f(a + \delta).$$

4. Consider the following:

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x + b) = b - 2$$

and

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (-x^2 \lfloor x \rfloor) = \lim_{x \to -2^-} (-x^2) \lim_{x \to -2^-} (\lfloor x \rfloor) = (-4)(-3) = 12.$$

Thus for $b = 14$, $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^-} f(x) \neq f(-2)$, and hence $f$ has a removable singularity at $x = 2$.

5. Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and suppose $f'(a) > 0$. Assume, for the sake of contradiction, that there are no $\delta > 0$ such that $f(x) > f(a)$ whenever $a < x < a + \delta$. Then, for every $\delta > 0$, there is at least one point $c \in (a, a + \delta)$ where $f(c) \leq f(a)$. Let $(c_n)$ be some sequence contained in $[a, b]$ such that $c_n \in (a, a + \frac{|a+b|}{n})$ and $f(c_n) \leq f(a)$ for $n \in N$. Then

$$f(c_n) - f(a) \leq 0$$

and

$$c_n - a > 0,$$

and so

$$\frac{f(c_n) - f(a)}{c_n - a} \leq 0.$$

Letting $n \to \infty$ (and thus $(c_n) \to a$), we get by definition that $f'(a) \leq 0$; which is a contradiction, as $f'(a) > 0$. Therefore our assumption must be false, and there does exist some $\delta > 0$ such that $f(x) > f(a)$ whenever $a < x < a + \delta$.