Anthony Jones

1a) Find explicit solution: $xy' - y^2 \ln(x) = 0$

Normal Form $\longrightarrow$ $y' = \frac{y^2 \ln(x)}{x} = f(x)\,g(y)$

Solve using seperable equation:

$$\frac{dy}{dx} = y^2 \cdot \frac{\ln(x)}{x}$$

$$\int \frac{dy}{y^2} = \int \frac{\ln(x)}{x}\,dx \qquad \begin{aligned} u &= \ln(x) \\ du &= \tfrac{1}{x}\,dx \end{aligned} \qquad \boxed{\begin{array}{c}\text{LOST SOL} \\ y \neq 0\end{array}}$$

$$-y^{-1} + C_0 = \int u\,du = \tfrac{1}{2}u^2 + C_1 = \tfrac{1}{2}(\ln(x))^2 + C_1$$

$$-y^{-1} = \tfrac{1}{2}(\ln(x))^2 + C_2 \qquad C_2 = C_1 - C_0$$

$$y = -\frac{1}{\frac{1}{2}(\ln(x))^2 + C_2} \neq 0 \qquad \boxed{\text{SOL NOT RECOVERED}}$$

Using seperable equations method, the general explicit solution to the ode is $y = -\frac{1}{\frac{1}{2}(\ln(x))^2 + C_2}$, for $C_2 \neq \frac{1}{2}(\ln(x))^2$.

1b) $y(e) = 1$; $x = e$, $y = 1$

$$1 = -\frac{1}{\frac{1}{2}(1)^2 + C_2} = -\frac{1}{C_2 + \frac{1}{2}}$$

$$C_2 + \tfrac{1}{2} = -1$$

$$C_2 = -\tfrac{3}{2} \quad \Rightarrow \quad \boxed{-\tfrac{3}{2} \neq \tfrac{1}{2}(1)^2}$$ Check initial value input

The specific solution satisfying $y(e) = 1$ is

$$y = -\frac{1}{\frac{1}{2}(\ln(x))^2 - \frac{3}{2}}.$$

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2) Find explicit solution: $t\frac{dy}{dt} - y = t^2\sin(t)$

Standard Form $\rightarrow$ $\frac{dy}{dt} - \frac{1}{t}y = t\sin(t)$

(Integrating Factor) $\frac{dy}{dt} + p(t)y = q(t)$

$$\mu(t) = e^{\int -\frac{1}{t}dt} = e^{-\ln|t|} = \frac{1}{t}$$

Multiply integrating Factor:

$$\underbrace{\frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}y}_{} = \sin(t)$$

Check:

$$\frac{d}{dt}\left(\frac{y}{t}\right) = \underbrace{\frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}y}_{} \quad \boxed{\checkmark}$$

Substitue:

$$\int \frac{d}{dt}\left(\frac{y}{t}\right)dt = \int \sin(t)\,dt$$

$$\frac{y}{t} = -\cos(t) + C$$

$$y = Ct - t\cos(t)$$

Using integrating Factor method, the general explicit solution to the ode is $y = Ct - t\cos(t)$ for all values C.

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3a) $\left(6xy + \frac{x}{x^2-1}\right)dx + \left(3x^2 - \frac{1}{y^2+1}\right)dy = 0$

$M = 6xy + \frac{x}{x^2-1}$ $\qquad M_y = 6x$

$N = 3x^2 - \frac{1}{y^2+1}$ $\qquad N_x = 6x$

The ode is exact because $M_y = N_x$.

3b) Because the ode is exact, there is some $z = f(x,y)$ s.t.

$$f' = f_x \frac{\delta x}{\delta z} + f_y \frac{\delta y}{\delta z} = M\,dx + N\,dy$$

$$f' = \left(6xy + \frac{x}{x^2-1}\right)dx + \left(3x^2 - \frac{1}{y^2+1}\right)dy$$

$$f = \int \left(6\cancel{x}xy + \frac{x}{x^2-1}\right)dx + g(y)$$

$$f = 3x^2y + \frac{1}{2}\ln|x^2-1| + g(y)$$

We know that $f_y \frac{\delta}{\delta y} = N\,dy$ so differentiate:

$$\frac{d}{dy} f = 3x^2 + g'(y) = 3x^2 - \frac{1}{y^2+1}$$

$$g'(y) = -\frac{1}{y^2+1}$$

$$g(y) = -\int \frac{1}{y^2+1}\,dy = -\arctan(y) + K$$

The implicit solution to the ode is

$$3x^2y + \frac{1}{2}\ln|x^2-1| - \arctan(y) = K$$ for any value K.

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4a) $\frac{dy}{dx} = \frac{y}{x-y} = \frac{y/x}{1-y/x} = g(y/x)$

The ode is homogeneous because it can be expressed soley in terms of y/x.

4b) $\frac{dy}{dx} = \frac{y/x}{1-y/x}$; $v = y/x$; $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute: $v + x\frac{dv}{dx} = \frac{v}{1-v}$

$x\frac{dv}{dx} = \frac{v}{1-v} - v = \frac{v - v + v^2}{1-v} = \frac{v^2}{1-v}$

Seperate and solve:

$\int \frac{1-v}{v^2} dv = \int \frac{1}{x} dx$

$\int \left(\frac{1}{v^2} - \frac{1}{v}\right) dv = \ln|x| + c$

$-v^{-1} - \ln|v| = \ln|x| + c$

$-\frac{x}{y} - \ln\left|\frac{y}{x}\right| = \ln|x| + c$

The general solution to the ode is

$-\frac{x}{y} - \ln\left|\frac{y}{x}\right| = \ln|x| + c$, for any value c.

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5a) Solve: $xy' + 4y = 4xy^{\frac{1}{2}}$

Standard $\rightarrow$ $y' + \frac{4}{x}y = 4y^{\frac{1}{2}}$ Bernoullit, $n = \frac{1}{2}$

$y^{-\frac{1}{2}}y' + \frac{4}{x}y^{\frac{1}{2}} = 4$; $v = y^{1-n} = y^{\frac{1}{2}}$

$\frac{1}{2}y^{-\frac{1}{2}}y' + \frac{2}{x}y^{\frac{1}{2}} = 2$ $\frac{dv}{dx} = \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}$

Substitute v and v':

$\frac{dv}{dx} + \frac{2}{x}v = 2$ $\longrightarrow$ $\boxed{\text{Linear!}}$

> Find Integrating factor:

> $\mu(x) = e^{\int \frac{2}{x}dx} = x^2$

$x^2\frac{dv}{dx} + 2xv = 2x^2$

> Check the solution:

$\frac{d}{dx}(x^2 v) = x^2\frac{dv}{dx} + 2xv$

> Substitue:

$\int \frac{d}{dx}(x^2 v)\,dx = \int 2x^2\,dx$

$x^2 v = \frac{2}{3}x^3 + c$

Substitute for V:

$y^{\frac{1}{2}} = \frac{2}{3}x + \frac{c}{x^2}$ $\Rightarrow$ $y = \left(\frac{2}{3}x + \frac{c}{x^2}\right)^2$

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5a) The explicit solution to the ode is
$y = \left(\frac{2}{3}x + \frac{c}{x^2}\right)^2$ for any value c.

5b) $y(1) = 1$; $x = 1$, $y = 1$

$$1 = \left(\frac{2}{3} + c\right)^2$$

$$c = \frac{1}{3}, -\frac{5}{3}$$

The solutions to the ode satisfying
$y(1) = 1$ are $y = \left(\frac{2}{3}x + \frac{1}{3x^2}\right)^2$ and
$y = \left(\frac{2}{3}x - \frac{5}{3x^2}\right)^2$.