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1a) Let R be a commutative ring with 1, and R be Noetherian and $I \subsetneq R$ be an ideal of R. Then I is finitely generated by some set $A \subseteq R$ such that $I = \langle A \rangle$ and

$$\langle A \rangle = \{a_1 r_1 + \cdots + a_n r_n \mid n \in \mathbb{N}, a_i \in A, r_i \in R\}$$

where $|A| < \infty$. Consider R/I. By the third isomorphism theorem, there is a one-to-one correspondence between the ideals containing I and the ideals of R/I given by

$$f : R \to R/I \; ; \; f(r) = r + I.$$

Consider the corresponding ideal of R/I that is generated by some set B. Then

$$\langle B \rangle = \{b_1 \bar{r}_1 + \cdots b_n \bar{r}_n \mid n \in \mathbb{N}, b_i \in B, \bar{r}_i \in R/I\}$$

Observe however that because $|A|$ is finite, the image of f restricted to A:

$$f\big|_{A \subseteq R} = \{a_n \mid n \in \mathbb{N}, a_i \in A\}$$

is finite as well, and correspondes to the ideal generated by B. Therefore for every ideal of R/I there is a correspondins, finitely generated ideal I and hence R/I is also Noetherian.

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1b) Let $\phi : R \to R/I$ be a homomorphism where $\phi(r) = r + I$, for some ideal $I \subsetneq R$. (Observe from the First Isomorphism Thrm that this is an epimorphism). Thus $R/I = \phi(R)$

Let $K \subset R/I$ be an ideal. Observe that $\phi^{-1}(K) \subset R$ is an ideal of R:

Let $a, b \in \phi^{-1}(K)$. Then $\phi(a), \phi(b) \in K$. Consider any element $r \in R$:

$$\begin{aligned} \phi(a + rb) &= \phi(a) + \phi(rb) \\ &= \phi(a) + \phi(r)\phi(b) \in K \end{aligned}$$

as $\phi(a), \phi(b) \in K$, $\phi(r) \in R/I$, and $K \subseteq R/I$ is an ideal. Therefore

$$\phi(a + rb) \in K \Rightarrow a + rb \in \phi^{-1}(K),$$

and hence $\phi^{-1}(K) \subset R$ is an ideal. Since R is a PIR, $\phi^{-1}(K)$ is principle and thus $\phi^{-1}(K) = \langle a \rangle$ for some $a \in R$.

But then $K \subset \langle \phi(a) \rangle$, as for any $ra \in \phi^{-1}(K)$, $\phi(ra) \in K$. However as ϕ is onto, any $\phi(r)\phi(a) \in R/I$ follows that $\phi(r)\phi(a) = \phi(ra) \in K$, and hence $\langle \phi(a) \rangle \subset K$. Thus K is principle for any ideal of R/I and R/I is a PIR.

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2a) No: consider the ring $\mathbb{R} \times \mathbb{R}$.

$(1,0)$ and $(0,1)$ are both zero divisors of $\mathbb{R} \times \mathbb{R}$:

$$(1,0) \cdot (0,1) = (0,0) \quad \text{and}$$
$$(0,1) \cdot (1,0) = (0,0).$$

However the zero divisors do not form a subgroup of $\mathbb{R} \times \mathbb{R}$ under addition:

$$(1,0) + (0,1) = (1,1) \notin I.$$

Hence it is not generally true.

2b) Let P be a prime ideal of R.

Suppose $x \in N$ and is nilpotent. Then $x^n = 0$ for some $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ the smallest number such that $x^m \in P$, which is at most n as $0 \in P$.

Then $x^m = x \cdot x^{m-1} \in P$, and as P is prime it is implied therefore that either $x \in P$ or $x^{m-1} \in P$. However, x^{m-1} cannot be a nontrivial element of P as m is the smallest number for which $x^m \in P$. Therefore $x \in P$ and hence $N \subseteq P$ is contained in every prime ideal.

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3a) We will prove by contradiction.

Suppose $x \notin R \setminus I$. Thus $x \in I$, and By definition of an ideal, $xy \in I$ for all $y \in R$. Thus $xy \notin R \setminus I$.

Therefore, by converse, if $xy \in R \setminus I$ then $x \in R \setminus I$, and symmetrically $y \in R \setminus I$ as well. Hence $R \setminus I$ is saturated.

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b) Let $I \subsetneq R$ be a proper and prime ideal. Then for all $a, b \in R$,

$$ab \in P \Rightarrow a \in P \text{ or } b \in P.$$

Suppose $x, y \in R \setminus I$. From above, we know that $R \setminus I$ is saturated, and by the converse of the prime property, as $x \notin I$ and $y \notin I$ then $xy \notin I$ and thus $xy \in R \setminus I$. Thus $R \setminus I$ is saturated and multiplicatively closed.

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3b) Now consider the opposite direction. Suppose $R \setminus I$ is both saturated and multiplicatively closed for some proper ideal I. Therefore for all $s, t \in R \setminus I$, $st \in R \setminus I$ and for all $x, y \in R$, if $xy \in R \setminus I$ then $x, y \in R \setminus I$.

~~Consider any $a, b \in I$. Observe by definition $a, b \notin R \setminus I$. Also observe that because $R \setminus I$ is saturated, because $a \notin R \setminus I$ and $b \notin R \setminus I$, $ab \notin R \setminus I$. Hence $ab \in I$~~

~~Consider any $a, b \in R$ such that $ab \in R \setminus I$.~~

~~Thus~~ Consider $a, b \in R$ such that both $a \in R \setminus I$ and $b \in R \setminus I$. Because $R \setminus I$ is multiplicatively closed, $ab \in R \setminus I$. Thus $ab \in I$ iff either $a \in I$ or $b \in I$, and therefore I is prime.

3c) Let S be multiplicatively closed and I be maximal with respect to $I \cap S = \emptyset$.

Then for all $s,t \in S$, $st \in S$.

Suppose $ab \in I$, that neither $a \notin I$ nor $b \notin I$. Since I is maximal to $I \cap S = \emptyset$, $I \subsetneq (I + Sa) \Rightarrow (I+Sa) \cap S \neq \emptyset$ as any ideal containing I must intersect S nontrivially. Therefore there exists some $x \in (I + Sa) \cap S$ and similarly $y \in (I+Sb) \cap S$

$x = i + as$ for $i \in I$, $s \in S$
$y = j + bt$ for $j \in I$, $t \in S$

where x and y are the nontrivial elements of the intersection. Because $x, y \in S$ and S is multiplicatively closed,

$$xy = ij + ibt + jas + abst \in S,$$

and because $(I + Sa)$ and $(I + Sb)$ both contain I, $xy \in I$. Thus $xy \in S \cap I$ and this is a contradiction from above.

Hence for $ab \in I$, either $a \in I$ or $b \in I$, and therefore I is prime.