MATH 4400 Homework 2

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1. $$\max\ z = -2x_1 + 3x_2 - 5x_3 - x_4$$
s.t.
$$x_1 - 3x_2 + x_3 - 2x_4 - s_1 = 12$$
$$5x_1 + x_2 + 4x_3 - x_4 - s_2 = 10$$
$$-3x_1 + 2x_2 - x_3 + x_4 = 8$$

$$x_1, x_2, x_3, x_4, s_1, s_2 \geq 0$$

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2. $$\max\ z = -x_1 - y_2 - y_3 + w_3$$
s.t.
$$x_1 - 2y_2 - y_3 + w_3 + s_1 = 3$$
$$-x_1 - y_2 + y_3 - w_3 + s_2 = 2$$
$$x_1 + y_2 = 10$$

$$x_1, y_2, y_3, w_3, s_1, s_2 \geq 0$$

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3. $$\max\ z = -c^T x$$
s.t.
$$Ax - Is = b$$
$$y - w = x$$

$$y \geq 0,\ w \geq 0,\ s \geq 0;$$

where y, w, and s are all elements of $\mathbb{R}^n$ and I is the $m \times n$ identity matrix.

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4. To prove that B is a convex set, we must show that

$$\lambda x_1 + (1-\lambda)x_2 \in B$$

for any $x_1 \in B$, $x_2 \in B$, and $\lambda \in [0,1]$.

Let $x_1 \in B$, $x_2 \in B$, and $\lambda \in [0,1]$.
Then, according to the triangle inequality,

$$\|\lambda x_1 + (1-\lambda)x_2\| \leq \|\lambda x_1\| + \|(1-\lambda)x_2\|,$$

and, according to absolute scalability,

$$\|\lambda x_1 + (1-\lambda)x_2\| \leq |\lambda| \cdot \|x_1\| + |(1-\lambda)| \cdot \|x_2\|.$$

Observe that, since x_1 and x_2 are both unit balls with respect to norm $\|\cdot\|$ in $\mathbb{R}^n$,

$$\|x_1\| \leq 1 \text{ and}$$
$$\|x_2\| \leq 1,$$

Thus

$$|\lambda| \cdot \|x_1\| + |(1-\lambda)| \cdot \|x_2\| \leq |\lambda| + |(1-\lambda)| = 1,$$

and hence $\lambda x_1 + (1-\lambda)x_2 \in B$. ◊

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5. Let x_1 and $x_2 \in S_1 \cap S_2$, and $\lambda \in [0,1]$.

Then x_1 and $x_2 \in S_1$ and x_1 and $x_2 \in S_2$. Because $x_1, x_2 \in S_1$, and S_1 is convex,

$$\lambda x_1 + (1-\lambda)x_2 \in S_1,$$

by definition. Likewise, since $x_1, x_2 \in S_2$,

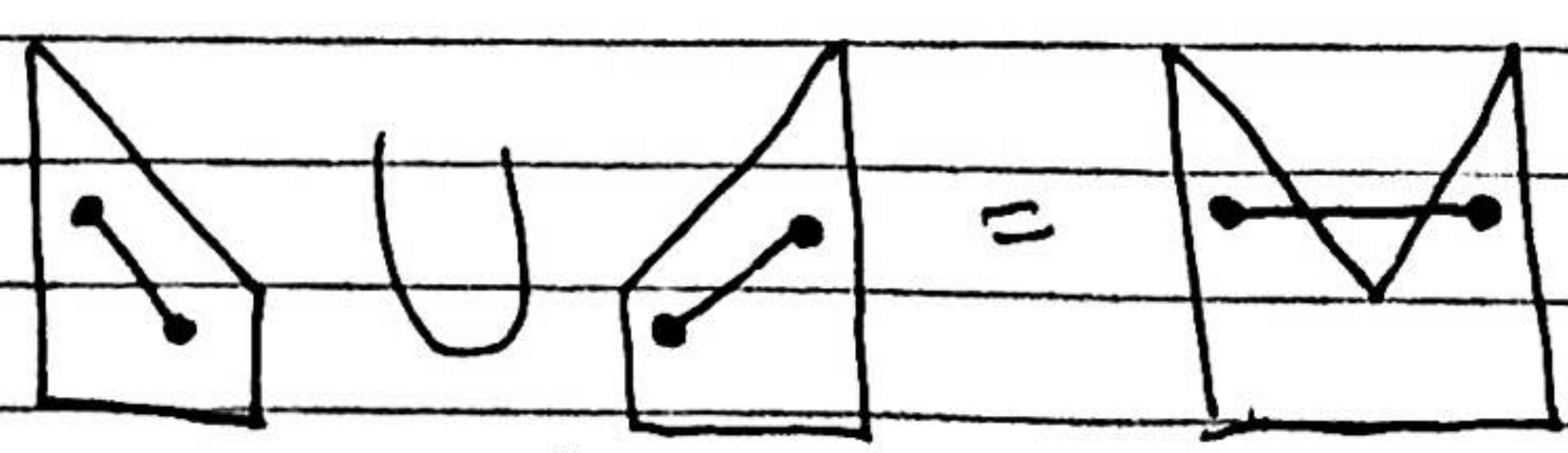
$$\lambda x_1 + (1-\lambda)x_2 \in S_2.$$

Thus $\lambda x_1 + (1-\lambda)x_2 \in S_1 \cap S_2$ for any x_1 and $x_2 \in S_1 \cap S_2$.

$S_1 \cup S_2$ is not necessarily convex, because x_1 and x_2 are no longer both contained in S_1 and S_2 by definition. Thus, if $x_1 \notin S_1$, then

$$\lambda x_1 + (1-\lambda)x_2 \in S_1$$

may not be true (and similarly for S_2, or when $x_2 \in S_1$). For example:



(convex) (convex) (nonconvex)

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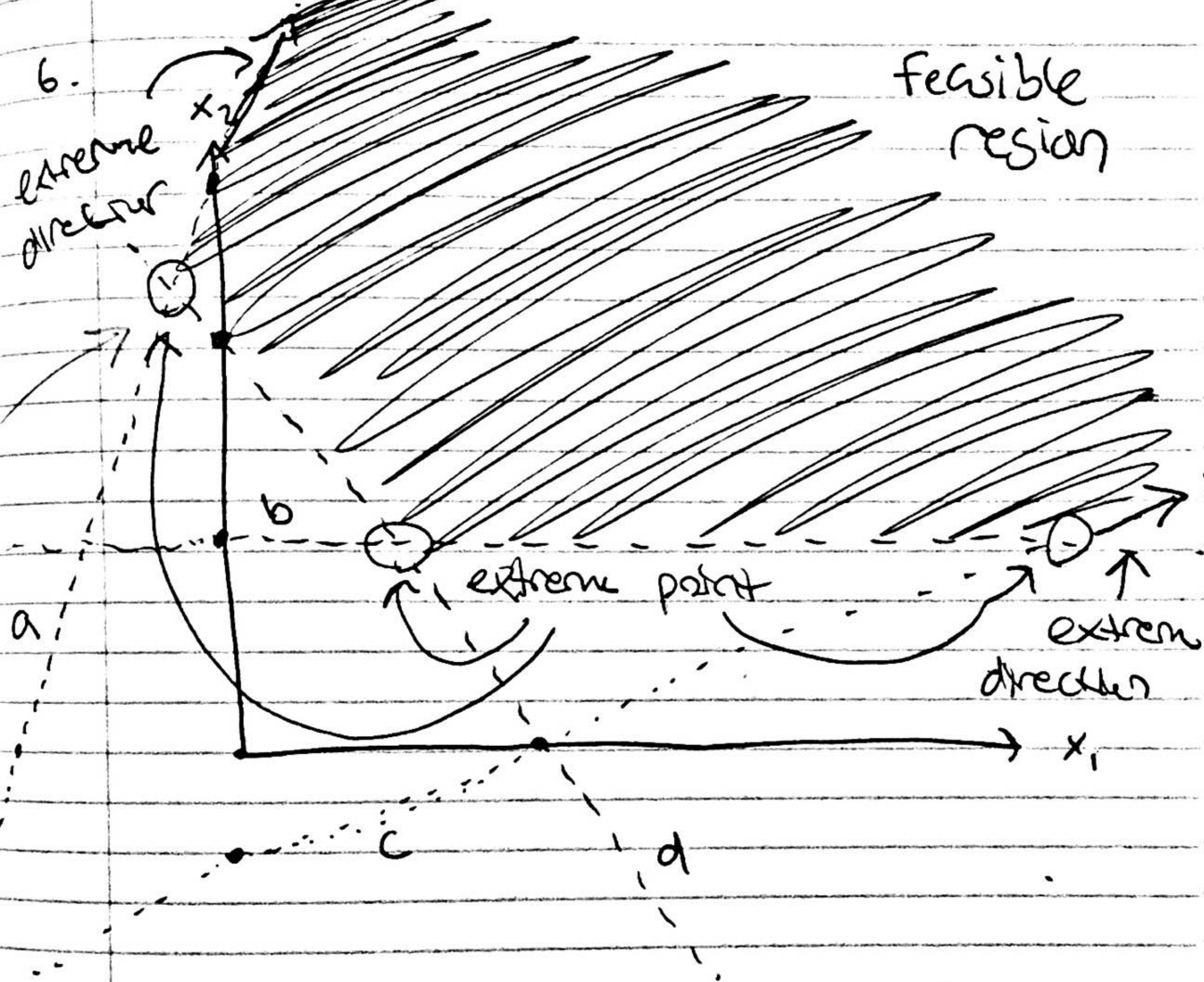
6.



There are extreme points at where line a: $-3x_1 + x_2 \leq 9$ meets line d: $2x_1 + x_2 = 6$, where b: $x_2 = 4$ meets d, and where c: $2x_1 - 4x_2 = 6$ meets b.

The extreme directions are along line a: $-3x_1 + x_2 = 9$ and c: $2x_1 - 4x_2 = 6$.