Overall: C

MATH 4560/6560 Topology
Spring 2022
Exercises 6
Target date: Thu 10 Mar 11:59PM

Instructions:
- For each item in each exercise, make sure to include
  - any sources used,
  - any collaborators worked with, and
  - a self-assessment of your work: C (correct), R (revision needed), or M (major revision needed).
- For the entire homework set, make sure to also include:
  - a reflection on the assignment and your solutions. Reflections may include
    - · discussion of how routine or challenging the assignment was,
    - · approximation of time spent on the assignment or on individual exercises,
    - · details about particular difficulties or false starts,
    - · explanations of why solutions are incomplete or incorrect, etc.
  - a self-assessment (C/R/M) for the entire assignment.

Exercise 1. Find an online news story from 2022 that includes the word "continuous." Give the link and write a summary of the article. Does the article's use of the word "continuous" match with how we use it in class? How are they different and how are they similar?
Source. The article discusses shifts in the paradigms behind bio-pharmaceutical processing techniques, most notably batch production versus continuous production. Batch processing is the standard for most biopharm companies, but this appears to be changing. Continuous processing enables better control and economic gains, requiring technical (complex engineering and manufactoring) principles. The way that the article uses continuous in this context is very similar to how we use it in class. Continuity within mathematics is generally concerned with elements of smoothness, closeness, and nondiscreteness. In topology we define continuity in terms of mapping open sets of one set to another, as open sets are one way of abstracting notions of distance and precision out of closeness in mathematics. Within the article, the word continuous is opposed to the word batched, which is more synonymous with the concepts of discreteness in mathematics. Therefore, this opposition is both upheld within class and within the article.

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Exercise 2. Let $S_1, \ldots, S_m$ be metric spaces. For $i = 1, \ldots, m$, let $f_i \colon S_i \to \mathbb{R}$ be a continuous function, and let $a_i \in \mathbb{R}$.

(a) Prove that the function $f_1 + f_2 \colon S_1 \times S_2 \to \mathbb{R}$ given by

$$(f_1 + f_2)(P_1, P_2) = f_1(P_1) + f_2(P_2)$$

is continuous.

Pf: Suppose $P_1 \in S_1$, $P_2 \in S_2$, and $\epsilon > 0$. Then because f_1 and f_2 are continuous mappings between metric spaces, it follows that

$$|f_1(P_1) - f_1(P_x)| < \frac{\epsilon}{2}$$

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and

$$|f_2(P_2) - f_2(P_y)| < \frac{\epsilon}{2}$$

whenever $d_{S_1}(P_1, P_x) < \delta$ and $d_{S_2}(P_2, P_y) < \delta$ for some $\delta > 0$ and all $P_x \in S_1, P_y \in S_2$. Recall that, by the triangle inequality,

$$|f_1(P_1) - f_1(P_x) + f_2(P_2) - f_2(P_y)| \leq |f_1(P_1) - f_1(P_x)| + |f_2(P_2) - f_2(P_y)|.$$

Thus, because $|(f_1+f_2)(P_1, P_2)-(f_1+f_2)(P_x, P_y)| = |f_1(P_1)-f_1(P_x)+f_2(P_2)-f_2(P_y)|$, it follows that whenever $d_{S_1}(P_1, P_x) < \delta$ and $d_{S_2}(P_2, P_y) < \delta$, for all $(P_x, P_y) \in S_1 \times S_2$,

$$|(f_1 + f_2)(P_1, P_2) - (f_1 + f_2)(P_x, P_y)| < \epsilon.$$

Therefore $(f_1 + f_2)$ is continuous.

(b) Prove that the function $a_1 f_1 \colon S_1 \to \mathbb{R}$ given by

$$(a_1 f_1)(P_1) = a_1 \cdot (f_1(P_1))$$

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is continuous.

Pf: Note that when $a_1 = 0$, $(a_1 f_1) = 0$ for all $P_1 \in S_1$; and thus $(a_1 f_1)$ is a constant (cf. continuous) function. Suppose instead that $a_1 \neq 0$. Then there exists $\delta > 0$ such that, for all $P_x \in S_1$,

$$d_{S_1}(P_1, P_x) < \delta \implies |f_1(P_1) - f_1(P_x)| < \frac{\epsilon}{|a_1|}.$$

After multiplying by $|a_1|$ on both sides, we derive

$$|a_1||f_1(P_1) - f_1(P_x)| = |a_1 \cdot (f_1(P_1)) - a_1 \cdot (f_1(P_x))| < \epsilon.$$

Therefore $(a_1 f_1)$ is continuous.

(c) Prove that the function $\sum_{i=1}^{m} a_i f_i \colon S_1 \times \cdots \times S_m \to \mathbb{R}$ given by

$$(\textstyle\sum_{i=1}^{m} a_i f_i)(P_1, \ldots, P_m) = \sum_{i=1}^{m} a_i \cdot (f_i(P_i))$$

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is continuous.

Pf: From (b), we know that the functions $a_i f_i \colon S_i \to \mathbb{R}$ are each continuous. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that, for all $P_{xi} \in S_i$, for $1 \leq i \leq m$,

$$d_{S_i}(P_i, P_{xi}) < \delta \implies |a_i \cdot (f_i(P_i)) - a_i \cdot (f_i(P_{xi}))| < \frac{\epsilon}{m}.$$

It hence follows that

$$\left|\sum_{i=1}^{m} a_i \cdot (f_i(P_i)) - \sum_{i=1}^{m} a_i \cdot (f_i(P_{xi}))\right|$$
$$= \sum_{i=1}^{m} |a_i \cdot (f_i(P_i)) - a_i \cdot (f_i(P_{xi}))| < \epsilon.$$

Therefore $\sum_{i=1}^{m} a_i f_i$ is continuous.

Exercise 3. Let $S, T_1, \ldots, T_n$ be metric spaces. For $i = 1, \ldots, n$, let $g_i \colon S \to T_i$ be a continuous function. Prove that the function $\underline{g} \colon S \to T_1 \times \cdots \times T_n$ given by

$$\underline{g}(P) = (g_1(P), \ldots, g_n(P))$$

is continuous.

Pf: Since the functions g_i are each continuous, we find that for any $\epsilon > 0$ and $P, Q \in S$, there exists $\delta > 0$ such that

$$d_S(P,Q) < \delta \implies |g_i(P) - g_i(Q)| < \frac{\epsilon}{\sqrt{n}}$$

for all such i. Hence, if we take the same P and Q, then by recalling the formula for euclidean distance, we derive

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$$\begin{aligned} |\underline{g}(P) - \underline{g}(Q)| &= \sqrt{\sum_{i=1}^{n}(g_i(P) - g_i(Q))^2} \\ &< \sqrt{\sum_{i=1}^{n}(\frac{\epsilon}{\sqrt{n}})^2} = \epsilon. \end{aligned}$$

Therefore $\underline{g}$ is continuous.

Exercise 4. Let $\phi\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Use Exercises 2–3 to prove that ϕ is continuous.

Pf: Let $\phi_i\colon \mathbb{R}^m \to \mathbb{R}$ be the linear transformations that maps a vector $\vec{u} \in \mathbb{R}^m$ to the i-th copy of $\mathbb{R}$ within $\mathbb{R}^n$ for $1 \leq i \leq n$. Then by property (2c) and the definition of a linear mapping, $\phi_i := \sum_{i=1}^{m} a_i f_i \colon \mathbb{R}_1 \times \cdots \times \mathbb{R}_m \to \mathbb{R}$, and hence ϕ_i is continuous. Note now that

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$$\phi(\vec{u}) = (\phi_1(\vec{u}), \ldots, \phi_n(\vec{u}))$$

and thus by property (3) we find that ϕ is continuous.

Exercise 5 (Bonus)**.** Let $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let $e_i, \ldots, e_n \in \mathbb{R}^n$ be the standard basis for $\mathbb{R}^n$. Prove that ϕ is an isometry if and only if the list $\phi(e_1), \ldots, \phi(e_n)$ is an orthonormal basis for $\mathbb{R}^n$.