HW4 MATH 4540

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1 Exercises

1. Let P be any partition of $[0,1]$ given as $\{x_0, x_1, \ldots, x_n\}$. Let

$$M_i = \sup \chi_{\mathbb{Q}}(x)$$

and

$$m_i = \inf \chi_{\mathbb{Q}}(x),$$

where $x \in [x_{i-1}, x_i]$ for all $i \in \{1, \ldots, n\}$. Observe that, for any $x_i \in P$, there exists infinitely many such points $x = \alpha$ and $x = \beta$ where $\alpha \in \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. Therefore, $\chi_{\mathbb{Q}}(x)$ takes on both its values (0 and 1) on the interval $[x_{i-1}, x_i]$; and thus $M_i = 1$ and $m_i = 0$ for all respective sub-intervals. Consider then that

$$U(P, \chi_{\mathbb{Q}}(x)) = \sum_{i=1}^{n} M_i \triangle x_i = \sum_{i=1}^{n} \triangle x_i = x_n - x_0 = 1$$

and

$$L(P, \chi_{\mathbb{Q}}(x)) = \sum_{i=1}^{n} m_i \triangle x_i = \sum_{i=1}^{n} 0 = 0$$

for any partition P, and thus the upper and lower integrals, given by

$$\overline{\int_0^1} \chi_{\mathbb{Q}}(x)\, \mathrm{d}x = \inf U(P, \chi_{\mathbb{Q}}(x)) = 1$$

and

$$\underline{\int_0^1} \chi_{\mathbb{Q}}(x)\, \mathrm{d}x = \sup L(P, \chi_{\mathbb{Q}}(x)) = 0,$$

are not equal; hence $\chi_{\mathbb{Q}}(x)$ is not Riemann integrable on its domain.

2. Suppose $f : [a,b] \to [0, \infty)$ is continuous and that

$$\int_a^b f(x)\, \mathrm{d}x = 0.$$

Observe that the range of f is the interval $[0, \infty)$, and hence $f(x) \geq 0$ for all $x \in [a, b]$. Furthermore, since the above integral of f is defined on $[a, b]$, it is therefore Riemann integrable, and hence

$$\int_a^b f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x = 0.$$

Thus

$$\inf U(P, f(x)) = \sup L(P, f(x)) = 0$$

for all partitions P of $[a, b]$. Suppose P was an arbitrary partition of $[a, b]$, and that there existed some value $c \in [a, b]$ such that $f(c) \neq 0$; then it follows that $f(c) > 0$ and that, since f is continuous, there exists some $\delta > 0$ where

$$|f(x) - f(c)| < \frac{f(c)}{2}$$

whenever $x \in (c - \delta, c + \delta)$. Hence $f(x) > \frac{f(c)}{2}$ for such x (to see this, suppose either $f(x) = \frac{f(c)}{2}$ or $f(x) < \frac{f(c)}{2}$. Then you find that $|f(x) - f(c)| \geq \frac{f(c)}{2}$). Consider now all points $\{x_j, x_{j+1}, \ldots x_k\} \subset P$ such that $(c - \delta, c + \delta) \subset [x_{i-1}, x_i]$: by definition, it follows that

$$M_k = \sup f(x) \geq \frac{f(c)}{2} > 0$$

whenever $x \in [x_{k-1}, x_k]$, and hence

$$U(P, f(x)) = \sum_{i=1}^{n} M_i \triangle x_i \geq M_k \delta > 0,$$

as the sub-intervals span at least the length of δ. But this then implies that for any P,

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf U(P, f(x)) > 0,$$

which is a contradiction. Therefore $f(x) = 0$ for all $x \in [a, b]$.

3. (a) Since $f \in \mathscr{R}$, we know that for every $\epsilon > 0$ there exists a partition P such that $U(P, f) - L(P, f) < \epsilon$. In other words,

$$\sum_{i=1}^{n} (M_i - m_i) * \triangle x_i < \epsilon$$

where M_i and m_i are the suprema and infima of the sub-intervals associated with P. We want to show $U(P, |f|) - L(P, |f|) < \epsilon$; or, in other words,

$$\sum_{i=1}^{n} (M_i' - m_i') * \triangle x_i < \epsilon$$

for P evaluated over the absolute function $|f|$. Suppose, throughout any of the sub-intervals, that all of f is negative. Then it follows that $M_i' = -m_i$ and $m_i' = -M_i$, since

$$m_i \leq f(x) \leq M_i < 0$$

implies

$$-m_i \geq -f(x) = |f(x)| \geq -M_i > 0.$$

Similarly, when we suppose that all of f is nonnegative, it follows that $M_i' = M_i$ and $m_i' = m_i$, since

$$|f(x)| = f(x).$$

Hence whenever all of f shares the same sign on any of the sub-intervals,

$$(M_i' - m_i') = ((-m_i) - (-M_i)) = (M_i - m_i).$$

Suppose, instead, that there was a given sub-interval (x_{i-1}, x_i) such that all of f did not share the same sign. Then either $M_i' = M_i$ or $M_i' = -m_i$, since

$$M_i' = \sup |f(x)| = \max(\sup f(x), \sup -f(x)) = \max(M_i, -m_i).$$

Additionally, for at least two values $c, d \in (x_{i-1}, x_i)$, observe that

$$m_i \leq f(c) < 0 < f(d) \leq M_i.$$

Thus it follows that both $M_i < (M_i - m_i)$ and $-m_i < (-m_i + M_i)$, since we have that $m_i < 0$ and $M_i > 0$; and hence $M_i' < (M_i - m_i)$, as from above we found that $M_i' = \max(M_i, -m_i)$. Finally, since $|f(x)| \geq 0$, it follows that $m_i' \geq 0$; and thus

$$(M_i' - m_i') \leq (M_i') < (M_i - m_i).$$

Therefore, summing through each sub-interval, we find that

$$\sum_{i=1}^{n} (M_i' - m_i') * \triangle x_i \leq \sum_{i=1}^{n} (M_i - m_i) * \triangle x_i < \epsilon;$$

and hence $U(P, |f|) - L(P, |f|) < \epsilon$, meaning $|f| \in \mathscr{R}$.

(b) Consider a modification to the *Dirichlet function* given in problem 1:

$$\tilde{\chi_{\mathbb{Q}}}(x) := \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$$

Then, using the same argument presented in the solution to problem 1, we know that

$$\tilde{\chi_{\mathbb{Q}}}(x) \notin \mathscr{R};$$

however, $|\tilde{\chi_{\mathbb{Q}}}(x)| = 1$ is a constant function, which is trivially integrable (consider for example that every constant function is trivially continuous, and thus, by Theorem 6.6 in Rudin's, is also *Riemann* integrable). Therefore the converse of (a) is not true.

4. (a) Since $f \in \mathscr{R}$, we know that for every $\epsilon > 0$ there exists a partition P where $U(P, f) - L(P, f) < \epsilon$. In other words,

$$\sum_{i=1}^{n}(M_i - m_i) * \triangle x_i < \epsilon,$$

where M_i and m_i are the suprema and infima of each interval associated with P. We need to show that $U(P, f^2) - L(P, f^2) < \epsilon$; or, in other words,

$$\sum_{i=1}^{n}(M_i' - m_i') * \triangle x_i < \epsilon$$

for some P similarly evaluated over f^2. Note that because f is bounded,

$$-M \leq m_i \leq f(x) \leq M_i \leq M$$

for some $M \in \mathbb{N}$ and all x within the intervals associated with P. It thus follows that

$$\min(m_i^2, M_i^2) \leq m_i' \leq f^2(x) \leq M_i' \leq \max(m_i^2, M_i^2).$$

We now show $(M_i' - m_i') \leq 2M(M_i - m_i)$. Suppose $\min(m_i^2, M_i^2) = m_i^2$. Then

$$(M_i' - m_i') \leq (M_i^2 - m_i^2) = (M_i + m_i)(M_i - m_i) \leq 2M(M_i - m_i),$$

as both $M_i \leq M$ and $m_i \leq M$. Suppose instead that $\min(m_i^2, M_i^2) = M_i^2$. Then likewise

$$(M_i' - m_i') \leq (m_i^2 - M_i^2) = (-M_i - m_i)(M_i - m_i) \leq 2M(M_i - m_i),$$

and thus the statement is true for all intervals associated with P. Hence, for every $\epsilon > 0$, there exists a partition P such that

$$\sum_{i=1}^{n}(M_i - m_i) * \triangle x_i < \frac{\epsilon}{2M},$$

and hence

$$\sum_{i=1}^{n}(M_i' - m_i') * \triangle x_i \leq \sum_{i=1}^{n} 2M(M_i - m_i) * \triangle x_i < \epsilon.$$

Therefore $f^2 \in \mathscr{R}$.

(b) The same example from above, with the same reasoning:

$$\tilde{\chi_{\mathbb{Q}}}(x) := \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$$

(c) Yes. The difference is that f^3 ensures all negative portions of f are still integrable, whereas f^2 does not. Suppose $f^3 \in \mathscr{R}$ on $[a, b]$. Since f is bounded, it follows then that so too is f^3; and hence $m \leq f^3 \leq M$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be given as $\phi(x) = \sqrt[3]{x}$. Since ϕ is continuous on all of R, it follows that $f(x) = \phi(f^3(x))$ and, by Theorem 6.11 in Rudin's, $f \in \mathscr{R}$.

5. Suppose $f(0+) = f(0)$. We want to show that for every $\epsilon > 0$, there exists some partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) * \triangle\alpha_i < \epsilon.$$

Observe that whenever $x_i > x_{i-1} > 0$,

$$\triangle\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = 1 - 1 = 0;$$

and similarly whenever $x_{i-1} < x_i \leq 0$,

$$\triangle\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = 0 - 0 = 0.$$

Hence $\triangle\alpha_i$ is nonzero only whenever $x_{i-1} \leq 0 < x_i$, which is only once for any partition P. In this case

$$\triangle\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) = 1 - 0 = 1,$$

where x_k is the unique point in P where $\alpha(x)$ is nonzero. Thus

$$U(P, f, \alpha) - L(P, f, \alpha) = M_k - m_k.$$

Let P be a partition such that $x_{k-1} = 0$, and suppose that $U(P, f, \alpha) - L(P, f, \alpha) \geq \epsilon$ for some $\epsilon > 0$. Let (a_n) be an infinite sequence of points between 0 and x_k such that $(a_n) \to 0$. Then P^*, given as $P^* = P \cup \{a_1\}$, is a refinement of P, and now a_1 is the unique point such that $\alpha(x)$ is nonzero. Using the same process, we can continue to refine P further and further, where at each step a_i is the unique point of interest for $M_k - m_k$. Because $a_i > 0$ and $(a_n) \to 0$, it thus follows that $M_k \to f(0)$ and $m_k \to f(0)$, since $f(x) \to f(0)$ as $x \to 0$ (from the right) and thus $\sup f(x) = \inf f(x) = f(0)$.