HW9 MATH 4540

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1 Exercises

1. Note that f is arbitrarily often differentiable for $x \neq 0$ as a composition of such functions. We first show that $f'(0) = 0$. By definition

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\exp(-1/x^2)}{x}.$$

Suppose $t = x^{-1}$. Then it follows that

$$f'(0) = \lim_{x \to 0} \frac{\exp(-1/x^2)}{x} = \lim_{t \to \pm\infty} t * \exp(-t^2).$$

Note however that

$$\lim_{t \to -\infty} t * \exp(-t^2) = \lim_{t \to \infty} -t * \exp(-t^2) = -\lim_{t \to \infty} t * \exp(-t^2),$$

as $t^2 > 0$ for all $t < 0$. Thus by Theorem 1f in our notes for the exponential function,

$$f'(0) = \lim_{t \to \infty} t * \exp(-t^2) = \pm 0 = 0.$$

Now suppose $n \in \mathbb{N}$ was given, and $f^{(k)}(0) = 0$ for all $k < n$. Then by definition

$$f^{(n)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x \to 0} \frac{f^{(k)}(x)}{x}.$$

For the sake of contradiction, assume $f^{(n)}(0) = A \neq 0$. Observe that both

$$\lim_{x \to 0} g(x) = f^{(n-1)}(x) = 0$$

and

$$\lim_{x \to 0} h(x) = x^2/2 = 0,$$

and that

$$f^{(n)}(0) = \lim_{x \to 0} \frac{g'(x)}{h'(x)}.$$

Hence by L'Hospital's Rule this implies

$$\lim_{x\to 0} 2 * \frac{f^{(n-1)}(x)}{x^2} = A \neq 0.$$

Note by the same reasoning, however, that this also implies

$$\lim_{x\to 0} 3! * \frac{f^{(n-2)}(x)}{x^3} = A \neq 0,$$

and so on, for all $n-k < n$; and thus ultimately that

$$\lim_{x\to 0} n! * \frac{f(x)}{x^n} = \lim_{x\to 0} n! * \frac{\exp(-1/x^2)}{x^n} = A \neq 0.$$

Recall from the solution to the first derivative that

$$\lim_{x\to 0} n! * \frac{\exp(-1/x^2)}{x^n} = \lim_{t\to\infty} n! * t^n * \exp(-t^2) = 0,$$

and hence we reach a contradiction. Thus $f^{(n)}(0) = 0$ for all such n. This example illustrates that Taylor expansions do not necessarily converge to their functions, as in this case all the derivatives zero. Hence the series sum is itself zero, which is only equivalent for $x = 0$.

2. a) Using the binomial expansion,

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} * 1^{n-k} * \left(\frac{x}{n}\right)^k \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} * \frac{x^k}{n^k} \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} * \frac{x^k}{n^k}\end{aligned}$$

Observe that as $n \to \infty$, $(n-k) \to n$ for any constant k, and thus

$$\frac{n!}{(n-k)!} = (n-1) * (n-2) * \ldots * (n-k+1) \to n^k$$

Hence

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} * \frac{x^k}{n^k} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

b) Suppose $g(x) = \log(1+x)$ and $h(x) = x$. Then clearly $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 0$, and thus by L'Hospital's Rule

$$\lim_{x\to\infty} \frac{g'(x)}{h'(x)} = \lim_{x\to\infty} \frac{g(x)}{h(x)}.$$

Hence

$$\lim_{x\to\infty} \frac{g(x)}{h(x)} = \lim_{x\to\infty} \frac{1}{1+x} = 1.$$

3. a) First we differentiate at zero:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x) - 1}{x},$$

which we derive given $f(0) = f(0+0) = f(0)f(0)f(0) = 1$. Now we consider by definition

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}.$$

Changing the variable to $h = t - x$, we derive

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)f(h) + f(x)}{h} = f(x) * \lim_{h\to 0} \frac{f(h) + 1}{h},$$

and thus $f'(x) = f(x) * f'(0)$. As a composition of differentiable functions, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log(f(x)) = \frac{f'(x)}{f(x)}$$

(Chain Rule; Rudin's 5.5). However because $f'(x) = f(x) * f'(0)$, this implies

$$\frac{\mathrm{d}}{\mathrm{d}x} \log(f(x)) = \frac{f(x) * f'(0)}{f(x)} = f'(0) = c,$$

where c is some constant value. After integrating, it is easy to see that

$$\log(f(x)) = cx,$$

and hence, by exponentiation, that

$$f(x) = \exp(\log(f(x))) = \exp(cx).$$

b)

4. Suppose $S = \sum_{(i,j)} a_{ij}$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$ (note that S may be infinite). Then for all $N \in \mathbb{N}$, it follows that

$$\sum_i \sum_{j=1}^{N} a_{ij} \leq S$$

and

$$\sum_{i=1}^{N} \sum_j a_{ij} \leq S$$

as $\sum_{(i,j)} a_{ij}$ is monotonically increasing. Notice however that similarly $S = \sum_{(j,i)} a_{ij}$, and hence

$$\sum_{i} \sum_{j=1}^{N} a_{ji} \leq S$$

and

$$\sum_{i=1}^{N} \sum_{j} a_{ji} \leq S.$$

Thus with sufficiently large N, we find that $|\sum_{i=1}^{N} \sum_{j} a_{ji} - S| < \epsilon$

5.