Lecture 14 LP dual problem

- The dual problem as a bound

  - Consider the following LP

$$\begin{aligned} \max \quad & 3x_1 + 15x_2 \\ \text{s.t} \quad & x_1 + 4x_2 \le 2 \\ & x_1 + 8x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

We can find an upper bound for the maximization problem.

e.g. For any feasible $x = (x_1, x_2)$, we have $x_1, x_2 \ge 0$. Therefore,

$$3x_1 + 15x_2 \le 3x_1 + 24x_2 = 3(x_1 + 8x_2) \le 9$$

9 is an upper bound of the problem

The lower/tighter the upper bound is, the more useful it is

  - To find the tightest upper bound, we consider a general nonnegative combination of the constraints.

Let $\pi_1(x_1+4x_2) + \pi_2(x_1+8x_2)$ be an upper bound of $3x_1 + 15x_2$, where $\pi_1, \pi_2 \ge 0$. Then

$$3x_1 + 15x_2 \le \pi_1(x_1+4x_2) + \pi_2(x_1+8x_2) = (\pi_1+\pi_2)x_1 + (4\pi_1+8\pi_2)x_2 \quad \text{for all } x_1, x_2 \ge 0$$

That is, $3 \le \pi_1 + \pi_2$ & $15 \le 4\pi_1 + 8\pi_2$

The upper bound corresponding to π_1, π_2 is $2\pi_1 + 3\pi_2$

To find the lowest / tightest upper bound, we should consider the following problem

$$\begin{aligned} \min \quad & 2\pi_1 + 3\pi_2 \\ \text{s.t} \quad & \pi_1 + \pi_2 \ge 3 \\ & 4\pi_1 + 8\pi_2 \ge 15 \\ & \pi_1, \pi_2 \ge 0 \end{aligned}$$

This problem is called the <u>dual problem</u>. The original problem is called the <u>primal problem</u>.

Note the relation between the primal and the dual.

$$\begin{aligned} \max\ & 3x_1 + 15x_2 \\ \text{s.t}\ & x_1 + 4x_2 \le 2 \\ & x_1 + 8x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & 2\pi_1 + 3\pi_2 \\ \text{s.t}\ & \pi_1 + \pi_2 \ge 3 \\ & 4\pi_1 + 8\pi_2 \ge 15 \\ & \pi_1, \pi_2 \ge 0 \end{aligned}$$

In general, the dual problem of

$$\begin{aligned} \max\ & z = c^T x \\ \text{s.t}\ & Ax \le b \\ & x \ge 0 \end{aligned} \qquad (P)$$

is given by

$$\begin{aligned} \min\ & z = b^T \pi \\ \text{s.t}\ & A^T \pi \ge c \\ & \pi \ge 0 \end{aligned} \qquad (D)$$

e.g. A primal-dual pair.

$$\begin{aligned} \max\ & z = 4x_1 - 3x_2 + 5x_3 \\ \text{s.t}\ & -x_1 + x_2 \le 8 \\ & x_1 + 2x_2 + x_3 \le 30 \\ & 2x_1 - x_2 - 2x_3 \le -6 \\ & x_1 + x_2 + 2x_3 \le 20 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

$$\begin{aligned} \min\ & z = 8\pi_1 + 30\pi_2 - 6\pi_3 + 20\pi_4 \\ \text{s.t}\ & -\pi_1 + \pi_2 + 2\pi_3 + \pi_4 \ge 4 \\ & \pi_1 + 2\pi_2 - \pi_3 + \pi_4 \ge -3 \\ & \pi_2 - 2\pi_3 + 2\pi_4 \ge 5 \\ & \pi_1, \pi_2, \pi_3, \pi_4 \ge 0 \end{aligned}$$

- The dual problem of an LP in general form

Consider a general form LP:

$$\begin{aligned} \max \quad & c_1x_1+c_2x_2+c_3x_3 \\ \text{s.t} \quad & a_{11}x_1+a_{12}x_2+a_{13}x_3 \le b_1 \\ & a_{21}x_1+a_{22}x_2+a_{23}x_3 \ge b_2 \\ & a_{31}x_1+a_{32}x_2+a_{33}x_3 = b_3 \\ & x_1 \ge 0, \quad x_2 \le 0 \end{aligned}$$

To find its dual problem, we first convert it to the form of

$$\begin{aligned} \max \quad & c^Tx \\ \text{s.t} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

constraints $a_{21}x_1+a_{22}x_2+a_{23}x_3 \ge b_2 \leftrightarrow -a_{21}x_1-a_{22}x_2-a_{23}x_3 \le -b_2$

$$a_{31}x_1+a_{32}x_2+a_{33}x_3 = b_3 \leftrightarrow \begin{cases} a_{31}x_1+a_{32}x_2+a_{33}x_3 \le b_3 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3 \ge b_3 \end{cases}$$

$$\downarrow$$

$$-a_{31}x_1-a_{32}x_2-a_{33}x_3 \le -b_3$$

variables: $y_2 = -x_2 \qquad x_2 \le 0 \leftrightarrow y_2 \ge 0$

$y_3 - y_4 = x_3 \qquad x_3 \in \mathbb{R} \leftrightarrow y_3 \ge 0, y_4 \ge 0$

The problem becomes

$$\begin{aligned} \max \quad & c_1x_1 - c_2y_2 + c_3y_3 - c_3y_4 \\ \text{s.t} \quad & a_{11}x_1 - a_{12}y_2 + a_{13}y_3 - a_{13}y_4 \le b_1 \\ & -a_{21}x_1 + a_{22}y_2 - a_{23}y_3 + a_{23}y_4 \le -b_2 \\ & a_{31}x_1 - a_{32}y_2 + a_{33}y_3 - a_{33}y_4 \le b_3 \\ & -a_{31}x_1 + a_{32}y_2 - a_{33}y_3 + a_{33}y_4 \le -b_3 \\ & x_1, y_2, y_3, y_4 \ge 0 \end{aligned}$$

The dual of the problem is

$$\min \quad b_1 w_1 - b_2 w_2 + b_3 w_3 - b_3 w_4$$

$$\begin{aligned}
\text{st} \quad & a_{11} w_1 - a_{21} w_2 + a_{31} w_3 - a_{31} w_4 \geq c_1 \\
& -a_{12} w_1 + a_{22} w_2 - a_{32} w_3 + a_{32} w_4 \geq -c_2 \\
& a_{13} w_1 - a_{23} w_2 + a_{33} w_3 - a_{33} w_4 \geq c_3 \\
& -a_{13} w_1 + a_{23} w_2 - a_{33} w_3 + a_{33} w_4 \geq -c_3 \\
& w_1, w_2, w_3, w_4 \geq 0
\end{aligned}$$

Rearranging, we have

$$\min \quad b_1 w_1 - b_2 w_2 + b_3 (w_3 - w_4)$$

$$\begin{aligned}
\text{st} \quad & a_{11} w_1 - a_{21} w_2 + a_{31} (w_3 - w_4) \geq c_1 \\
& a_{12} w_1 - a_{22} w_2 + a_{32} (w_3 - w_4) \leq c_2 \\
& a_{13} w_1 - a_{23} w_2 + a_{33} (w_3 - w_4) \geq c_3 \\
& a_{13} w_1 - a_{23} w_2 + a_{33} (w_3 - w_4) \leq c_3 \\
& w_1, w_2, w_3, w_4 \geq 0
\end{aligned}$$

Let $\pi_1 = w_1$, $\pi_2 = -w_2$, $\pi_3 = w_3 - w_4$ The dual problem becomes

$$\min \quad b_1 \pi_1 + b_2 \pi_2 + b_3 \pi_3$$

$$\begin{aligned}
& a_{11} \pi_1 + a_{21} \pi_2 + a_{31} \pi_3 \geq c_1 \\
& a_{12} \pi_1 + a_{22} \pi_2 + a_{32} \pi_3 \leq c_2 \\
& a_{13} \pi_1 + a_{23} \pi_2 + a_{33} \pi_3 = c_3 \\
& \pi_1 \geq 0, \ \pi_2 \leq 0
\end{aligned}$$

In general, we have the following table.

maximization	minimization
Constraints	Variables
$\leq$	$\geq$
$\geq$	$\leq$
$=$	unrestricted
Variables	Constraints
$\geq$	$\geq$
$\leq$	$\leq$
unrestricted	$=$

e.g.:

Primal

$$
\begin{aligned}
\max \quad & z = 4x_1 - 3x_2 + 5x_3 \\
\text{s.t.} \quad & -x_1 + x_2 \leq 8 \\
& x_1 + 2x_2 + x_3 \leq 30 \\
& 2x_1 - x_2 - 2x_3 \geq -6 \\
& x_1 + x_2 + 2x_3 = 20 \\
& x_1 \leq 0, \quad x_3 \geq 0
\end{aligned}
$$

$\longrightarrow$

Dual

$$
\begin{aligned}
\min \quad & z = 8\pi_1 + 30\pi_2 - 6\pi_3 + 20\pi_4 \\
\text{s.t.} \quad & -\pi_1 + \pi_2 + 2\pi_3 + \pi_4 \leq 4 \\
& \pi_1 + 2\pi_2 - \pi_3 + \pi_4 = -3 \\
& \pi_2 - 2\pi_3 + 2\pi_4 \geq 5 \\
& \pi_1, \pi_2 \geq 0, \quad \pi_3 \leq 0
\end{aligned}
$$

e.g.: Consider a LP in standard form:

Primal

$$
\begin{aligned}
\max \quad & c^T x \\
\text{s.t.} \quad & Ax = b \quad \pi \\
& x \geq 0
\end{aligned}
$$

$(A \in \mathbb{R}^{m \times n},\ c, x \in \mathbb{R}^n,\ b \in \mathbb{R}^m)$

$\longrightarrow$

Dual

$$
\begin{aligned}
\min \quad & b^T \pi \\
\text{s.t.} \quad & A^T \pi \geq c
\end{aligned}
$$

$(\pi \in \mathbb{R}^m)$