Homework 11

AJ

1. The ideals of $\mathbb{Z}$ are all subgroups of $\mathbb{Z}$ for which I is the subgroup and $zI \subseteq I$ or $Iz \subseteq I$ for $z \in \mathbb{Z}$, all elements of $\mathbb{Z}$.

Consider first all the subgroups of $\mathbb{Z}$. Because $\mathbb{Z}$ is cyclic, the subgroups of $\mathbb{Z}$ take on the form $n\mathbb{Z}$ for $n \in \mathbb{Z}$, or $\langle n \rangle = \langle n^i \mid n \in \mathbb{Z}, i \in \mathbb{Z} \rangle$

Now consider any integers $b \in \mathbb{Z}$ and some subgroup $A \leq \mathbb{Z}$, generated by $a \in \mathbb{Z}$. Because $A = \langle a \rangle$, the elements of A take on the form

$$x = a^n \mid x \in A,\ n \in \mathbb{Z}.$$

Observe that the ideal for any subgroup $A \subseteq \mathbb{Z}$ can be expressed by terms of its generator a:

$$z \circ (a) = a^z, \quad z \in \mathbb{Z}$$

where $\circ$ describes multiplication and a^z is some element of $\mathbb{Z}$ generated by the operation of addition within the subgroup.

Thus $z\langle a \rangle \subseteq \langle a \rangle$, and the ideals of $\mathbb{Z}$ are each of its subgroups.

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2a) To see that R^{op} is a ring, first observe that $(R,+)$ and $(R^{opp},+)$ have the same addition operation and therefore $(R^{op},+)$ is an abelian group.

Let $x,y,z \in R^{op}$: $(x \circ y) \circ z = (y\cancel{x}) \circ z = zyx$ and $x \circ (y \circ z) = x \circ (zy) = zyx$ and so multiplication is associative in R^{op}.

Finally consider the same x, y, z:

$x \cdot (y+z) = (y+z)x = yx + zx$ under the operation of R, and therefore $x \cdot (y+z) = x \cdot y + x \cdot z$.

$(x+y) \cdot z = z(x+y) = zx + zy$ under the same operation of R, and therefore $(x+y) \cdot z = x \cdot z + y \cdot z$.

Hence R^{opp} is a ring.

Now consider $(R^{op})^{op} = S$.

S has the same addition operation as R and R^{op}, and its multiplication operation is defined by $x \cdot y = y \circ x = xy$.

Therefore $(R^{op})^{op}$ shares the same abelian group as $(R,+)$ and has the same multiplication operation, so $(R^{op})^{op} \cong R$.

2) Let $f: R \to S$ be an isomorphism defined by $f(x) = s$ for $x \in R$, $s \in S$.

Observe that, because f is an isomorphism, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for $x, y \in R$.

Consider $f(x \circ y)$, where $\circ$ describes the same operation as multiplication in R^{op} and S^{op}:

$$f(x \circ y) = f(yx) = f(y)f(x) = f(x) \circ f(y).$$

Thus the function $g: R^{op} \to S^{op}$ where g describes $g(x) = s$ for $x \in R^{op}$, $s \in S^{op}$ is homomorphism. ~~Furthermore, because~~

$$\cancel{(R^{op})^{op} \cong R \text{ and } (S^{op})^{op} \cong S,}$$

~~there must exist some isomorphisms f_R and f_S that~~.

To show that g is one-to-one, consider $r \in R$ given by $ab = r$ for some $a, b \in R$. Then $a \circ b \in R^{opp}$, and because $f(r)$ is one-to-one, $g(a \circ b)$ must be as well.

Similarly, given $\mathrm{im}(f) \subseteq S$, $f(r) \in S$; and hence $g(a \circ b) = g(a) \circ g(b) = x \circ y$ for $x, y \in S^{opp}$, and thus $\mathrm{im}(g) \subseteq S^{opp}$.

2c) Let I be a left ideal of R.

Then $rI \subseteq I$ for all $r \in R$, where I is a subgroup of R.

Then for every element $a \in I$, $ra \in I$ and thus $ra = b$ for $b \in I$.

Consider $a \circ r = ra \in R^{op}$:

$a \circ r \in I$ for any $a \in I$ and $r \in R^{op}$, and hence $Ir \subseteq I$ is a right ideal of R^{op}

Let I be a right ideal of R^{op}.

Then $Ir \subseteq I$ for ~~some~~ all $r \in R^{op}$ where I is a subgroup of R^{op}.

Then $a \circ r = b$ for $a, b \in I$ ~~and $r \in R^{op}$~~, every $r \in R^{op}$.

Consider $ra = a \circ r = b \in R$:

$ra \in I$ for any $a \in I$ and $r \in R$, and hence $rI \subseteq I$ is a left ideal of R.

3a) To show that IJ is an ideal we must show that $IJ \subseteq R$ ~~and~~ is a subgroup of R and that $rIJ \subseteq R$ and $IJr \subseteq R$ for all $r \in R$.

Consider $x \in IJ$ for

$$x = \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots \alpha_n\beta_n.$$

Because I and J are both ideals, they are also themselves additive subgroups of R, and therefore for every α_k, β_k : $-\alpha_k \in I$ and $-\beta_k \in J$, and thus there is a $y \in IJ$ for

$$y = (-\alpha_0)\beta_0 + (-\alpha_1)\beta_1 + \cdots (-\alpha_n)\beta_n$$
$$= -x,$$

Thus $(-x)y \in R$ and therefore IJ is a subgroup.

To show that IJ is an left ideal, consider ra for $a \in IJ$:

$$\begin{aligned} ra &= r(\alpha_0\beta_0) + r(\alpha_1\beta_1) + \cdots + r(\alpha_n\beta_n) \\ &= (r\alpha_0)\beta_0 + (r\alpha_1)\beta_1 + \cdots + (r\alpha_n)\beta_n \\ &= \phi_0\beta_0 + \phi_1\beta_1 + \cdots + \phi_n\beta_n, \ \phi_k \in I. \end{aligned}$$

Thus $ra \in IJ$.

3a) Similarly, consider ar:

$$ar = (\alpha_0\beta_0)r + (\alpha_1\beta_1)r + \dots + (\alpha_n\beta_n)r$$
$$= (\alpha_0(\beta_0 r) + \alpha_1(\beta_1 r) + \dots + (\alpha_n(\beta_n r)$$
$$= \alpha_0 w_0 + \alpha_1 w_1 + \dots + \alpha_n w_n, w_k$$

Thus IJ is both a left and right ideal and is a two-sided ideal.