# HW1 Math 4540

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## 1 Exercises

1. Fix a prime number $p$. Then we can write any (non zero) rational number $r \in Q$ uniquely as $\frac{a}{b}p^n$ where the fraction is in lowest terms such that $a, n \in Z$, $b \in N$, and $a$ and $b$ are coprime to $p$. The *p-valuation* $v_p(r)$ of a rational $r$ is defined by $v_p(r) := n$, if $r \neq 0$ is presented as before and $v_p(0) := \infty$. Show the following properties of $v_p(r)$ for $x, y \in Q$:

   (a) $v_p(xy) = v_p(x) + v_p(y)$

   Let $x, y \in Q$ be represented as $x = \frac{a}{b}p^n$ and $y = \frac{c}{d}p^m$ respectively, for some fixed prime $p$. Then $xy = (\frac{a}{b}p^n)(\frac{c}{d}p^m) = \frac{ac}{bd}p^{n+m}$. As $a, b, c, d$ are all coprime to $p$, we can reduce $xy$ to the form $\frac{\alpha}{\beta}p^{n+m}$, for which $\alpha$ and $\beta$ define $xy$ by the lowest terms such that $\alpha \in Z$, $\beta \in N$, and $\alpha$ and $\beta$ are coprime to $p$, as above. Thus we have $v_p(xy) = n + m = v_p(x) + v_p(y)$.

   (b) $v_p(x + y) \geq \min(v_p(x), v_p(y))$

   Let $x, y \in Q$ be represented as $x = \frac{a}{b}p^n$ and $y = \frac{c}{d}p^m$, as above. Assume, without any loss of generality, that $n \leq m$. Then $x + y = \frac{a}{b}p^n + \frac{c}{d}p^m = p^n(\frac{ad+bcp^{m-n}}{bd})$. Observe that $\frac{ad+bcp^{m-n}}{bd}$ is rational and nonzero, and can therefore be written itself as $\frac{\alpha}{\beta}p^\omega$ where the fraction is in lowest terms such that $\alpha, \omega \in Z$, $\beta \in N$, and $\alpha$ and $\beta$ are coprime to $p$. Furthermore, because by assumption $n \leq m$ and $a, b, c, d$ are all coprime to $p$, it follows that $\omega \geq m - n \geq 0$. Thus $x + y = \frac{a}{b}p^n + \frac{c}{d}p^m = p^n(\frac{\alpha}{\beta}p^\omega) = \frac{\alpha}{\beta}p^{n+\omega}$, and hence $v_p(x + y) = n + \omega \geq n$. Since we assumed in the beginning that $n \leq m$, this is equivalent to $v_p(x + y) \geq \min(v_p(x), v_p(y)) = n$.

2. Now define the *p-norm* $|r|_p$ of a rational $r \neq 0$ to be $p^{-v_p(r)}$ and to be 0 if $r = 0$. Use the properties of the valuation from the previous problem to show that $|r|_p$ has the following properties:

(a) $|xy|_p = |x|_p|y|_p$

Using the first property $v_p(xy) = v_p(x) + v_p(y)$, it follows that $|xy|_p = p^{-v_p(xy)} = p^{-v_p(x)-v_p(y)} = p^{-v_p(x)}p^{-v_p(y)} = |x|_p|y|_p$ in the case that neither $x, y$ are 0, and that $|xy|_p = |0|_p = 0$ otherwise.

(b) $|x + y|_p \leq \max(|x|_p, |y|_p)$

Using the second property $v_p(x + y) \geq \min(v_p(x), v_p(y))$, it follows that $-v_p(x + y) \leq -\min(v_p(x), v_p(y))$. Assume, without any loss of generality, that $v_p(x) = n = \min(v_p(x), v_p(y))$. Then by assumption $-v_p(x + y) \leq -n$, and therefore $|x + y|_p = p^{-v_p(x+y)} \leq p^{-n} = p^{-v_p(x)} = |x|_p$. Since we assumed that $v_p(x) = \min(v_p(x), v_p(y))$, this implies $|x|_p = \max(|x|_p, |y|_p)$ as $|y|_p = p^{-v_p(y)} \leq p^{-v_p(x)} = |x|_p$. Thus we have $|x + y|_p \leq \max(|x|_p, |y|_p)$.

(c) Finally define the *p-adic distance* to be $d_p(x, y) := |x - y|_p$. Use the properties of $|r|_p$ to show that $d_p$ is a *metric* on $Q$.

To prove $d_p(x, y) := |x - y|_p$ is a metric on $Q$, we must show that for every $x, y \in Q$:

i. $d_p(x, y) > 0$ if $x \neq y$ and $d_p(x, y) = 0$ if $x = y$,

ii. $d_p(x, y) = d_p(y, x)$, and

iii. $d_p(x, y) \leq d_p(x, q) + d_p(q, y)$, for any $q \in Q$.

Let $x, y \in Q$, and suppose $x = y$. Then $|x - y|_p = |0|_p = 0$ by definition, and hence the first condition of (i.) is met. Now suppose $x \neq y$. Then $x - y \neq 0$, and therefore $|x - y|_p = p^{-v_p(x-y)} > 0$, as $v_p(x - y) \in Z$ and $p^i > 0$ for all primes $p$ and $i \in Z$. Hence both of the conditions of (i.) are met. For (ii.), observe that $|-1|_p = p^{-v_p(-1)} = p^0 = 1$, and by the property $|xy|_p = |x|_p|y|_p$ it follows that $|x-y|_p * |-1|_p = |-(x-y)|_p$. Therefore, $d_p(x, y) = |x-y|_p = (|x-y|_p) * 1 = |x-y|_p * |-1|_p = |-(x-y)|_p = |y-x|_p = d_p(y, x)$. Finally, for (iii.), observe that by the property $|x + y|_p \leq \max(|x|_p, |y|_p)$ it follows that, for any $q \in Q$, $d_p(x, y) = |x-y|_p = |(x-q)+(q-y)|_p \leq \max(|x - q|_p, |q - y|_p) < |x - q|_p + |q - y|_p$. Therefore (iii.) is met, and hence $d_p$ is a metric on $Q$.

3. As usual denote the *p-adic ball of radius* $r$ centered at $x \in Q$ by $B_r^p(x) := \{y \in Q : d_p(x, y) < r\}$. Show that for any $x_0 \in B_r^p(x)$ we have $B_r^p(x_0) = B_r^p(x)$. In other words every point in a p-adic ball is a center point.

Let $B_r^p(x) := \{y \in Q : d_p(x, y) < r\}$ for some $x \in Q$, and suppose $x_0 \in B_r^p(x)$. Then by definition $d_p(x, x_0) = |x - x_0|_p < r$. Let $B_r^p(x_0) := \{z \in Q : d_p(x_0, z) < r\}$, and suppose $q \in B_r^p(x_0)$. Then by definition $d_p(x_0, q) = |x_0 - q|_p < r$. Observe that $|x - q|_p = |x - x_0 + x_0 - q|$.

Using the second property of the *p-norm*, it then follows that $|x-q|_p = |x - x_0 + x_0 - q| \leq \max(|x - x_0|_p, |x_0 - q|_p|)$. However, given that both $|x - x_0|_p$ and $|x_0 - q|_p$ are less than $r$, this implies that $|x-q|_p < r$, and hence $q \in B_r^p(x)$ for all $q \in B_r^p(x_0)$. Using the same argument, this also implies that $s \in B_r^p(x_0)$ for any $s \in B_r^p(x)$; thus $B_r^p(x_0) = B_r^p(x)$.

4. Show by a direct estimate that for any $n \geq 1$ and $M > 0$ the function $f(x) = x^n$ is continuous on the interval $[-M, M]$, and conclude that it is continuous for all real numbers.

   Let $n \geq 1$, $M > 0$, and $f(x) = x^n$ for all $x \in [-M, M]$. We want to show that for any arbitrarily small $\epsilon > 0$ and $x, p \in [-M, M]$, there exists some $\delta > 0$ such that $|x - p| < \delta \implies |x^n - p^n| < \epsilon$. Let $\epsilon > 0$, and choose $\delta = \frac{\epsilon}{(n+1)M^n}$. Observe that because $x, p \in [-M, M]$, $|x| \leq M$ and $|p| \leq M$, and hence $|x^n| \leq M^n$ and $|p^n| \leq M^n$. Then, taking advantage of the *triangle inequality*, $|x^n - p^n| = |\sum_{k=1}^n p^k x^{n-k}| * |x - p| \leq (\sum_{k=1}^n |p^k x^{n-k}|) * |x - p| \leq (\sum_{k=1}^n |p^k||x^{n-k}|) * |x - p| \leq (\sum_{k=1}^n M^n) * \delta < (n+1)M^n * (\frac{\epsilon}{(n+1)M^n}) = \epsilon$. Hence $|x^n - p^n| < \epsilon$ whenever $|x - p| < \delta$, and $f(x)$ is continuous for all $x \in [-M, M]$. Now suppose $r \in R$ is any real number. Then by the *archimedean property* of the reals, there exists some $M > |r| \geq 0$ such that $f(x)$ is continuous at $r$, since $r \in [-M, M]$. Thus $f(x)$ is also continuous for all $r \in R$.

5. Consider the function $f(x) = xsin(1/x)$ for $x \neq 0$ continued into $x = 0$ by $f(0) = 0$. Show that this function is continuous on $R$.

   First, we show that $f(x)$ is continuous at $x = 0$. Let $\epsilon > 0$, and suppose $|x - 0| = |x| < \delta = \epsilon$. We want to show that this implies $|f(x) - f(0)| = |xsin(1/x)| < \epsilon$. Note that if $x = 0$, $|f(0) - f(0)| = 0 < \epsilon$ by definition. Otherwise, observe that $|sin(1/x)| \leq 1$, for all $x \neq 0 \in R$. Thus $|xsin(1/x)| = |x||sin(1/x)| \leq |x| < \epsilon$, and $f(x)$ is therefore continuous at zero. Next, we show that both $g(x) = x$ and $h(x) = sin(1/x)$ are continuous for all $x \neq 0$, and hence their product $f(x) = g(x)h(x)$ is continuous by the algebraic property of continuity. Note that $g(x)$ is continuous is trivial, as $|x - p| < \epsilon$ implies $|g(x) - g(p)| = |x - p| < \epsilon$ for all $x, p \in R$. To show that $h(x)$ is continuous, we consider another property of continuity, that the composition of two continuous functions are also continuous, and write $h(x) = \psi \circ \pi(x)$ where $\psi(x) = sin(x)$ and $\pi(x) = 1/x$. We now show $\pi(x)$ is continuous. However, this is also trivial, as the inverses of continuous functions are themselves continuous, and $\pi^{-1}(x) = g(x)$. Thus $\pi(x)$ is continuous for all $x \neq 0$. Finally, we show that $\psi(x)$ is continuous. Let $\epsilon > 0$, and suppose $|x - p| < \delta = \epsilon$. Consider that $|cos(\lambda)| \leq 1$ and $|sin(\lambda)| \leq |\lambda|$ for all $\lambda \in R$. Then $|\psi(x) - \psi(p)| = |sin(x) - sin(p)| = 2|cos(\frac{p+x}{2})||sin(\frac{p-x}{2})| \leq 2|\frac{p-x}{2}| = |x - p| < \epsilon$, and hence $\psi$ is continuous for all $x \neq 0$ and $p \neq 0$. Therefore $h(x) = \psi \circ \pi(x) = sin(1/x)$, and hence $f(x) = g(x)h(x) = xsin(1/x)$ is continuous on $R$.