1a) If $x \leq 0$ and $x \geq -1$, then $x(x+1) \leq 0$. Anthony J

b) $\forall r \in \mathbb{R}, \exists s \in \mathbb{R}$ s.t. $rs = 1$.

(Negation) $\exists s \in \mathbb{R}, \forall r \in \mathbb{R}$ $rs \neq 1$.

c) We will show logical equivalence using truth tables

x	y	z	$x \wedge y$	$(x \wedge y) \to z$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

x	y	z	$\neg y$	$\neg z$	$x \wedge \neg z$	$(x \wedge \neg z) \to \neg y$
T	T	T	F	F	F	T
T	T	F	F	T	T	F
T	F	T	T	F	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	T	F	F	T	F	T
F	F	T	T	F	F	T
F	F	F	T	T	F	T

As you can see, the outcomes of both statements are equal for any input field of x, y, and z. Therefore $(x \wedge y) \to z \equiv (x \wedge \neg z) \to \neg y$.

1e)

Anthony
Jones

i. FALSE. Let $a = 9$, $b = 3$, $c = 18$.

$9 \mid 18$ and $3 \mid 18$, but $27 \nmid 18$.

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ii. TRUE. The relation R as defined is equivalent to performing the mod 10 operation on the first element of the pair to get the second element. Therefore the pair $(3, 13)$ exists with the relation, as $13 \bmod 10 = 3$. Therefore $[3]$, which gives all possible ~~outputs~~ elements such that $3 R x$, contains 13. Therefore $13 \in [3]$

2 a) TRUE. Anthony J

Let $a \in \mathbb{Z}$ s.t. a is odd. Therefore, there is an even integer c s.t. (some integer d)

$$a = c + 1 \quad \text{and}$$
$$c = 2d.$$

Multiplying a by itself gives

$$a^2 = (c+1)^2 = (2d+1)^2 = 4d^2 + 4d + 1.$$

By laws of integer arithmetic, there exists some integer g s.t.

$$g = 4d^2 + 4d = 2(2d^2 + 2d),$$

which means that $2|g$ and therefore g is even. This means that $a^2 = 2g + 1$, or therefore a^2 must be odd.

Finally, as b is even, there exists some integer q s.t.

$$b = 2q.$$

Observe that $a^2 - b$ gives

$$a^2 - b = (2g+1) - (2q)$$
$$= 2(g-q) + 1$$

which means that $a^2 - b$ must be odd.

Anthony Jones

2b) Let A, B, and C be sets.

$\Rightarrow$ Assume $A \subseteq B \cap C$.

Therefore, for any element $a \in A$, $a \in B \cap C$. Therefore, as $a \in B \cap C$, and

$$B \cap C : \{x : x \in B \wedge x \in C\},$$

$a \in B$ and $a \in C$.

As $a \in A$ and $a \in B$, and a is any element of A, A must be a subset of B, or: $A \subseteq B$. As $a \in A$ and $a \in C$, and a is any element of A, A must be a subset of C, or: $A \subseteq C$. Therefore $A \subseteq B$ and $A \subseteq C$.

$\Leftarrow$ Assume $A \subseteq B$ and $A \subseteq C$.

Therefore, for any element $a \in A$, $a \in B$ and $a \in C$. Consider the set $B \wedge C$:

$$B \wedge C : \{x : (x \in B) \wedge (x \in C)\}.$$

Seeing as $a \in B$ and $a \in C$, $a \in B \cap C$.

As $a \in A$ and $a \in B \cap C$, and a is any element of A, A must be a subset of $B \wedge C$, or: $A \subseteq B \cap C$.

Therefore $A \subseteq B \cap C \longleftrightarrow A \subseteq B$ and $A \subseteq C$.

Anthony Jones

2c) Let A, B, and C be sets.

Consider $A-C$:

$A-C: \{x : x \in A \land x \notin C\}$,

Therefore if an element $n \in A-C$, then $n \in A$ and $n \notin C$.

Consider $C-B$:

$C-B: \{y : y \in C \land y \notin B\}$,

Therefore if an element $n \in C-B$, then $n \in C$ and $n \notin B$.

Finally, consider $U \cap V$:

$U \cap V: \{z : z \in U \land z \in V\}$,

Therefore if an element $n \in U \cap V$, then $n \in U$ and $n \in V$.

Now take the set $(A-C) \cap (C-B)$:

An element n in this set would also show $n \in A-C$ and $n \in C-B$, which would show that $n \in A$, $n \notin C$, $n \in C$, and $n \notin B$. Because there are no elements which are both in and not in a set C, the set $(A-C) \cap (C-B)$ must be the empty set, or; $(A-C) \cap (C-B) = \emptyset$.

Anthony Jones

2d) Let us say, for sake of contradiction, that $\exists n \in \mathbb{N}$ s.t. $5 \nmid (6^n - 1)$.

Then, there exists some smallest number $b \in \mathbb{N}$ s.t.

$$5 \nmid (6^b - 1).$$

Observe that $b > 1$ because $5 \mid (6 - 1)$.

Consider $b - 1 < b$:

Because $b - 1 < b$ and $b \neq 1$, $5 \mid (6^{b-1} - 1)$.

Therefore there exists some number a s.t.

$$5a = 6^{b-1} - 1$$

$$5a + 1 = 6^{b-1}$$

$$6(5a+1) = 6^b$$

$$30a + 6 = 6^b$$

$$30a + 5 = 6^b - 1$$

$$5(6a+1) = 6^b - 1.$$

Because $6a + 1$ is an integer by rules of integer arithmetic, we encounter a contradiction as $5 \mid (6^b - 1)$.

Therefore b is not the smallest number that ha $5 \nmid (6^n - 1)$.

Therefore, by means of contradiction, $\forall n \in \mathbb{N}$, $6^n - 1$ is divisible by 5.

Anthony
Jones

2f) Base Case: Let $n=0$. Then

$$\sum_{i=0}^{0} \frac{i+1}{(i+2)!} = \frac{1}{2} = 1 - \frac{1}{2}.$$

Therefore the equation holds for $n=0$.
Hypothesis: assume the equation works for $[1 \ldots n]$.
Induction step: Consider $n+1$:

$$\sum_{i=0}^{n+1} \frac{i+1}{(i+2)!} = \sum_{i=0}^{n} \frac{i+1}{(i+2)!} + \frac{n+2}{(n+3)!}$$

$$= \left(1 - \frac{1}{(n+2)!}\right) + \frac{n+2}{(n+3)!}$$

$$= 1 - \frac{n+3}{(n+2)!\,(n+3)} + \frac{n+2}{(n+3)!}$$

$$= 1 - \frac{n+3}{(n+3)!} + \frac{n+2}{(n+3)!}$$

$$= 1 - \frac{1}{(n+3)!}.$$

Therefore the equation holds for $n+1$.

Therefore, by means of induction, the equation is true for $n=0$ to ∞, or; for all $n \in \mathbb{N}_0$,

$$\sum_{i=0}^{n} \frac{i+1}{(i+2)!} = 1 - \frac{1}{(n+2)!}.$$

Anthony J

3 a) Let $x, y \in \mathbb{R}$ and $\equiv$ be a relation on $\mathbb{R}$ s.t. $x \equiv y$ whenever $(x-y) \in \mathbb{Z}$.

Assume that $x \equiv y$.

Therefore there is some integer c s.t.

$$x - y = c.$$

Observe what occurs when we multiply both sides by -1:

$$-1(x-y) = -1(c)$$

$$-x + y = -c$$

$$y - x = -c.$$

By the rules of integer multiplication, there is some integer b s.t.

$$b = -c.$$

Therefore

$$y - x = b.$$

This means that $y \equiv x$.

Therefore $x \equiv y$ implies $y \equiv x$, which makes the relation an equivalence relation.

Anthony
Jones

3b) Let $a, b \in A$ and R be an equivelece relasion on set A.

Therefore $[a] = \{x : x \in A, aRx\}$
and $[b] = \{y : y \in A, bRy\}$.

$\Rightarrow$ Let $[a] = [b]$.

Then $[a]$ and $[b]$ both contain all possible relations of either a or b to other elements $x \in A$. Suppose, for sake of contradiction, that a does not R to b. Then $a \notin [b]$ and $b \notin [a]$, implying also that $a \notin [a]$ and $b \notin [b]$. Let $c \notin A$ s.t. $c \in [b]$ and $c \in [a]$. If $c \in [a]$, then aRc and cRa because of the equivalence relation. Since $c \in [b]$, bRc and cRb.

$\Leftarrow$ Let aRb.

Then bRa because of equivelency.
Therefore both a and b are possibibile relations of each other, which means that both are elements of each other's classes.
Therefore $b \in [a]$ and $a \in [b]$.
Consider any element $c \in [a]$:
Because aRc and aRb, we know that b must R c by transition.

3d) Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined as

Anthony Jones

$f(x) = 2x - 3$.

For f to be onto, every element of $\mathbb{Z}$ must be an output of f; or, rather, the codomain (image) of f must be all integers.

Take $y = 2$. Observe that for f to be onto, there must exist some integer c s.t. $f(c) = 2$.

Therefore

$$2 = 2c - 3$$

$$5 = 2c$$

$$c = 5/2.$$

However, given that the input of f must be in the domain of $\mathbb{Z}$, and $5/2 \notin \mathbb{Z}$, we have encountered a contradiction. Therefore f must not be onto.

3d) Let $g: \mathbb{Q} \to \mathbb{Z}$ be defined as

[cont]. $g(x) = 2x - 3$.

Anthony Jones

For g to be onto, every element of $\mathbb{Z}$ must be an output of g; or, rather, the codomain (image) of g must be all integers.

Consider, for sake of contradiction, that there is some integer d s.t.

$$\forall n \in \mathbb{Q},\ g(n) \neq d.$$

This would imply that

$$d \neq 2n - 3$$

$$d + 3 \neq 2n$$

$$\frac{d+3}{2} \neq n,$$

or more precisely, that $\left(\frac{d+3}{2}\right)$ is not in the domain of $\mathbb{Q}$. For any integer d, however, there is a rational number s.t.

$$n = \frac{d+3}{2};$$

Therefore we have reached a contradiction, meaning $\forall d \in \mathbb{Z}\ \exists n \in \mathbb{Q}$ s.t. $g(n) = d$.

Therefore g must be onto.