HW5 MATH 4000

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1 Problems

5.3 For the first case, $(2x-x^3) > 0$ for $x = 1$ and $(2x-x^3) < 0$ for $x = 2$; hence no matter the value of C, there exists at least one $x \in (0, 5/2)$ such that $f(x) < 0$. For the second case, $(2x - x^2) > 0$ for $x = 1$ and $(2x - x^2) < 0$ for $x = 49/20$; hence similarly there exists at least one $x \in (0, 5/2)$ such that f is negative.

5.6
- a $E[x] = \int_0^\infty \frac{1}{4}x^2 e^{-x/2}\,\mathrm{dx} = 2 * \Gamma(3) = 4$
- b $E[x] = \int_{-1}^{1}(xc - x^3 c)\,\mathrm{dx} = (\frac{c}{2}x^2 - \frac{1}{4}x^4)\mid_{-1}^{1} = 0$
- c $E[x] = \int_5^\infty \frac{5}{x}\,\mathrm{dx} = 5\ln(x)\mid_5^\infty = \infty$

5.10
- a $P(\frac{1}{12} \leq X \leq \frac{1}{4}) + P(\frac{2}{6} \leq X \leq \frac{1}{2}) + P(\frac{7}{12} \leq X \leq \frac{3}{4}) = \frac{40}{60}$
- b This doesn't change the distribution, and hence the proportion will be the same.

5.20 Because n is small and p is large, we use the normal approximation with parameters $n = 100$ and $p = 0.65$:

- a $P(50 \leq X)$
- b $P(60 \leq X \leq 70) = P(X \leq 70) - P(X \leq 60) + P(X = 60) = 0.7511$
- c $P(X < 75) = P(X \leq 75) - P(X = 75) = 0.9768$

5.24 $P(X < 1.8 \times 10^6) = \phi\left(\dfrac{1.8 \times 10^6 - 1.4 \times 10^6}{3 \times 10^5}\right) = 0.9081$

5.26
- a $P(X \geq 525) = \sum_{i=525}^{1000} \binom{1000}{i} 0.5^i (0.5)^{1000-i} = 0.0606$
- b $P(X < 525) = \sum_{i=0}^{525} \binom{1000}{i} 0.55^i (0.45)^{1000-i} = 0.0526$

5.28 We assume that X can be normally approximated. Then

$$P(X > 20) = P(X > 19.5)$$

$$= P\left(\frac{X - 200(0.12)}{\sqrt{200(0.12)(0.88)}} > \frac{19.5 - 200(0.12)}{\sqrt{200(0.12)(0.88)}}\right) = P(Z > -0.9792) = P(Z \leq 0.9792) = 0.8365$$

5.34 a $P(X > 20) = 1 - F(20) = e^{-1}$

b $P(X > 30)/(3/4) = \left(\int_{30}^{40} 1/40\right)/(3/4) = 0.33$

2 Theoretical Exercises

5.22 The hazard rate function of a gamma random variable is given as

$$h(t) = \frac{f(t)}{1-F(t)} = \frac{\lambda e^{-\lambda t}(\lambda t)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{\int_t^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}\,dx}$$

$$= \frac{1}{\int_t^\infty \lambda e^{-\lambda(x-t)}(\lambda(x/t))^{\alpha-1}\,dx}.$$

Suppose $\alpha \geq 1$. Then note when $t_1 > t_2$,

$$0 < \lambda e^{-\lambda(x-t_1)} < \lambda e^{-\lambda(x-t_2)}$$

and

$$0 < (\lambda(x/t_1))^{\alpha-1} < (\lambda(x/t_2))^{\alpha-1}.$$

Hence it follows that $h(t_1) < h(t_2)$, and thus the hazard rate is increasing. But when $\alpha \leq 1$, then

$$0 < (\lambda(x/t_2))^{\alpha-1} < (\lambda(x/t_1))^{\alpha-1};$$

and hence so is the hazard rate decreasing.

5.25 Note that the cumulative distribution function $F(y)$ of an exponential random variable is given by

$$F(y) = 1 - e^{-y}$$

when $\lambda = 1$. Consider also when

$$y = \left(\frac{x-v}{\alpha}\right)^\beta,$$

that the distribution function becomes

$$F(y) = 1 - \exp\left(-\frac{x-v}{\alpha}\right)^\beta,$$

the cdf for the Weibull distribution. Because the distribution functions are the same, we can simply work backwards starting at the Weibull variable and similarly derive the exponential solution. Hence both are equivalent.