# Linear Programming: Exam 3

Time to solve problems: **75 minutes**

Time to submit solution: **10 minutes**

There are five (5) questions in this exam, each of which is ten (10) points. Only four (4) questions will be graded, and the maximum total score is forty (40). Please indicate below which question should not be graded for you. For the questions you choose to answer, make sure to justify your answers with your work.

Please do not grade my Question # ________

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Name:

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(1) Write down the dual problem for each of the following linear programs.

(a)

$$\begin{aligned}
\min \quad & 3x_1 + 2x_2 - 4x_3 \\
S.t. \quad & 5x_1 - 7x_2 + \ \ x_3 \geq 12 \\
& \ \ x_1 - \ \ x_2 + 2x_3 = 18 \\
& 2x_1 \qquad\quad - \ \ x_3 \leq 6 \\
& x_1 \leq 0,\ x_2 \leq 0,\ x_3 \geq 0
\end{aligned}$$

(b)

$$\begin{aligned}
\max \quad & \mathbf{c}^T\mathbf{x} + \mathbf{d}^T\mathbf{s} \\
S.t. \quad & A\mathbf{x} + I_m\mathbf{s} = \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0},\ \mathbf{s} \geq \mathbf{0},
\end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n, \mathbf{s} \in \mathbb{R}^m$ are decision variables, $\mathbf{c} \in \mathbb{R}^n, \mathbf{d} \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n}$ are given data and $I_m$ is the $m \times m$ identity matrix.

a) $\max \quad 12\pi_1 + 18\pi_2 + 6\pi_3$

$$\begin{aligned}
\text{s.t.} \quad & 5\pi_1 + \pi_2 + 2\pi_1 \geqslant 3 \\
& -7\pi_1 - \pi_2 \geqslant 2 \\
& \pi_1 + 2\pi_2 - \pi_3 \leqslant -4 \\
& \pi_1 \geqslant 0,\ \pi_3 \leq 0.
\end{aligned}$$

b) $\min \quad b^T\pi$

$$\begin{aligned}
\text{s.t.} \quad & A^T\pi \geqslant c \\
& \pi \geqslant d
\end{aligned}$$

(2) Consider the following linear programming problem where $A$ is an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$.

$$\begin{aligned} \min \ z = \mathbf{b}^T\mathbf{w} - \mathbf{c}^T\mathbf{x} \\ S.t. \quad A\mathbf{x} \le \mathbf{b} \\ A^T\mathbf{w} \ge \mathbf{c} \\ \mathbf{x} \ge \mathbf{0} \\ \mathbf{w} \ge \mathbf{0} \end{aligned}$$

Prove that the optimal value of this problem is zero if it is feasible.

Pf: Consider two linear programs:

$$\begin{aligned} \max \ & c^T x \\ \text{s.t.} \ & Ax \le b \\ & x \ge 0 \end{aligned} \quad (P) \qquad \text{and} \qquad \begin{aligned} \min \ & b^T w \\ \text{s.t.} \ & A^T w \ge c \\ & w \ge 0 \end{aligned} \quad (D).$$

Note that (P) and (D) are a primal-dual pair.

If the problem in the question is feasible, then both (P) and (D) are feasible.

① By weak duality, for any feasible solution $(\bar{x}, \bar{w})$, $c^T\bar{x} \le b^T\bar{w}$.

Therefore, the minimum value of the problem $z^* \ge 0$.

② By strong duality, (P) and (D) have optimal solutions $x^*$ and $w^*$ and $b^T w^* = c^T x^*$.

Therefore, the minimum value of the problem $z^* \le b^T w^* - c^T x^* \le 0$.

That is, $z^* = 0$. $\square$

(3) Consider the following primal dual pair.

$$
\text{(P)} \qquad
\begin{aligned}
\max\ z = {} & 3x_1 - x_2 + 6x_3 \\
S.t. \qquad & 5x_1 + x_2 + 4x_3 \le 42 \\
& 2x_1 - x_2 + 2x_3 \le 18 \\
& x_1,\ x_2,\ x_3 \ge 0
\end{aligned}
$$

$$
\text{(D)} \qquad
\begin{aligned}
\min\ Z = {} & 42\pi_1 + 18\pi_2 \\
S.t. \qquad & 5\pi_1 + \ 2\pi_2 \ge 3 \\
& \pi_1 - \quad \pi_2 \ge -1 \\
& 4\pi_1 + \ 2\pi_2 \ge 6 \\
& \pi_1,\ \pi_2 \ge 0
\end{aligned}
$$

Given that $\boldsymbol{\pi}^* = (2/3, 5/3)$ is an optimal solution to the dual problem (D), use complementary slackness to find an optimal solution to the primal problem (P).

By complementary slackness,

$$\pi_1^* \left(42 - (5x_1^* + x_2^* + 4x_3^*)\right) = 0$$
$$\pi_2^* \left(18 - (2x_1^* - x_2^* + 2x_3^*)\right) = 0$$
$$x_1^* \left(5\pi_1^* + 2\pi_2^* - 13\right) = 0$$
$$x_2^* \left(\pi_1^* - \pi_2^* + 1\right) = 0$$
$$x_3^* \left(4\pi_1^* + 2\pi_2^* - 6\right) = 0$$

Since $\pi_1^*$ and $\pi_2^*$ are nonzero, the first two equations imply

$$\begin{cases} 5x_1^* + x_2^* + 4x_3^* = 42 \\ 2x_1^* - x_2^* + 2x_3^* = 18 \end{cases}$$

Since $5\pi_1^* + 2\pi_2^* = \frac{20}{3} > 3$, the third equation implies

$$x_1^* = 0.$$

Therefore, we can solve

$$\begin{cases} x_2^* + 4x_3^* = 42 \\ -x_2^* + 2x_3^* = 18 \end{cases}$$

and get $x_2^* = 2$ and $x_3^* = 10$.

That is, $x^* = (0, 2, 10)^T$ is an optimal solution of (P).

(4) Solve the following linear program by the dual simplex method.

$$
\begin{aligned}
\max\ z = &-3x_1 - x_2 - 2x_3 \\
S.t.\quad & 3x_1 + 2x_2 + x_3 \geq -2 \\
& x_1 - 4x_2 + 2x_3 \geq 20 \\
& 4x_1 + 6x_2 + 2x_3 \leq 30 \\
& x_1,\ x_2,\ x_3 \geq 0
\end{aligned}
$$

We first rewrite the problem:

$$
\begin{aligned}
\max\ & z = -3x_1 - x_2 - 2x_3 \\
\text{s.t.}\ & -3x_1 - 2x_2 - x_3 + x_4 = 2 \\
& -x_1 + 4x_2 - 2x_3 + x_5 = -20 \\
& 4x_1 + 6x_2 + 2x_3 + x_6 = 30 \\
& x_1, x_2, x_3, x_4, x_3, x_6 \geqslant 0.
\end{aligned}
$$

Dual simplex method.

| | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| z | 1 | 3 | 1 | 2 | 0 | 0 | 0 | 0 |
| $x_4$ | 0 | -3 | -2 | -1 | 1 | 0 | 0 | 2 |
| $x_5$ | 0 | -1 | 4 | -2 | 0 | 1 | 0 | -20 |
| $x_6$ | 0 | 4 | 6 | 2 | 0 | 0 | 1 | 30 |

$\downarrow$

| | Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| Z | 1 | 2 | 5 | 0 | 0 | 1 | 0 | -20 |
| $x_4$ | 0 | $-5/2$ | -4 | 0 | 1 | $-1/2$ | 0 | 12 |
| $x_3$ | 0 | $1/2$ | -2 | 1 | 0 | $-1/2$ | 0 | 10 |
| $x_6$ | 0 | 3 | 10 | 0 | 0 | 1 | 1 | 10 |

The optimal solution of the original problem is $x^* = (0, 0, 10)$.

The optimal value of the original problem is $z^* = -20$.

(5) Consider the following resource allocation problem and the accompanying optimal tableau, where $x_5, x_6$ and $x_7$ are the slack variables for constraints 1 through 3, respectively.

$$
\begin{array}{llr}
\max\ z = 7x_1 + 8x_2 + 3x_3 + 7x_4 & & \text{(Profit \$)} \\
S.t. & x_1 + 2x_2 + x_3 + x_4 \le 50 & \text{(Resource 1)} \\
& 2x_1 + 2x_2 + x_3 + x_4 \le 40 & \text{(Resource 2)} \\
& x_1 + 2x_2 + x_3 + 5x_4 \le 30 & \text{(Resource 3)} \\
& x_1,\ x_2,\ x_3,\ x_4 \ge 0 &
\end{array}
$$

| | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 0 | 0 | 1 | 1 | 0 | 3 | 1 | 150 |
| $x_5$ | 0 | 0 | 0 | 0 | -4 | 1 | 0 | -1 | 20 |
| $x_1$ | 0 | 1 | 0 | 0 | -4 | 0 | 1 | -1 | 10 |
| $x_2$ | 0 | 0 | 1 | 1/2 | 9/2 | 0 | -1/2 | 1 | 10 |

(a) Identify an optimal solution $\boldsymbol{\pi}^* \in \mathbb{R}^3$ of the dual problem from the tableau.

$\pi^* = (0, 3, 1)^T.$

(b) What is the range on the available units of Resource 2 ($b_2$, currently equals 40) for which the current basis remains optimal?

Let $b' = \begin{bmatrix} 50 \\ b_2 \\ 30 \end{bmatrix}$.

Note that

$$B^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

Then $\beta' = B^{-1}b' = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 50 \\ b_2 \\ 30 \end{bmatrix} = \begin{bmatrix} 20 \\ b_2 - 30 \\ -\frac{1}{2}b_2 + 30 \end{bmatrix}$

The current basis remains optimal if and only if $\beta' \ge 0$.

That is, $b_2 \in [30, 60]$.

| Maximization Problem | | Minimization Problem |
|---|---|---|
| Constraints | | Variables |
| $\leq$ | $\leftrightarrow$ | $\geq 0$ |
| $\geq$ | $\leftrightarrow$ | $\leq 0$ |
| $=$ | $\leftrightarrow$ | *unrestricted* |
| Variables | | Constraints |
| $\geq 0$ | $\leftrightarrow$ | $\geq$ |
| $\leq 0$ | $\leftrightarrow$ | $\leq$ |
| *unrestricted* | $\leftrightarrow$ | $=$ |

Table 1. Primal-dual relationships