MATH 4400/6400: Linear Programming (Fall 2021)
Course Project

Anthony Jones

Due: Friday, December 3

Description

- **Team:** A team of up to 3 members can be formed to collaborate on their course project. Each member of the team will receive the same credit.
- **Topics:** There are four topic listed in the following pages. The first 3 topics are on applications of linear programming. Each topic has 3 parts: theory (30 points), problem solving (10 points) and implementation (20 points). The last topic is on implementation of the simplex method, which consists of 3 parts as well: the basic simplex method (30 points), Bland's rule (10 points) and the two-phase method (20 points).
- **Points:** Based on the formation of a team, each team has its goal points:

(# 4400:001 students, # 4400:100 students)	goal points
(1,0) or (2,0)	40
(3,0) or (0,1) or (1,1)	60
(0,2) or (2,1)	80
(1,2) or (0,3)	100

  Each team may choose (parts of) up to 2 topics to work on. If a team choose to work on two topics, they have to complete all three parts of the first topic before moving to the next one. The total points of the parts should be no less than their goal points. If the total points exceed their goal points, the extra part will be considered as "meaningful extra work".

  Meaningful extra work will be rewarded with a bonus of up to 10% of your goal points. However, your overall score for the project will not exceed 100. "Meaningful extra work" includes but is not limited to extra implementation, extra problems solving, interesting background story, excellent presentation of the report/code, etc.

- **Submission:** A well-typed technical report is expected. In your report, please cite proper references for any of the literature you used including - but not limited to - journal papers, books, and online resources. Submit your report and code to Canvas by the due date specified above.
- **Grading:** Grades will be assigned based on completion (40%), technical correctness (50%), and presentation of the results and/or code (10%). Your report should be understandable for students who have taken this course, but is not familiar with your topic(s).

1 Linear Programming with Absolute Values

Consider a problem of the form:

$$\begin{aligned} \min\ z(\mathbf{x}) = \sum_{i=1}^{n} c_i |x_i| & \\ s.t. \qquad A\mathbf{x} = \mathbf{b} & \end{aligned} \tag{ABS}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the decision variable. The $m \times n$ matrix A, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$ are given data. We assume that $\mathbf{c} \geq \mathbf{0}$.

1. (30pts) Show that (ABS) is equivalent to a linear program. Choose one of the following two methods.

   (a) Show that (ABS) is equivalent to the following linear program (LP2a)

$$\begin{aligned} \min\ \zeta(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} c_i y_i & \\ s.t. \qquad A\mathbf{x} = \mathbf{b} & \\ \mathbf{x} \leq \mathbf{y} & \\ -\mathbf{x} \leq \mathbf{y} & \end{aligned} \tag{LP2a}$$

   Proof: We will show that the two are equivalent by proving that any feasible solution to one problem can be made into a feasible solution for the other, and that the optimal values of the two solutions are equal. Let $\bar{\mathbf{x}}$ be a feasible solution to (ABS), and let $\bar{\mathbf{y}} = |\bar{\mathbf{x}}|$ be the vector whose components are given by $\bar{y}_i = |\bar{x}_i|$. Then, because $\bar{\mathbf{x}}$ is feasible to (ABS), the first constraint $A\mathbf{x} = \mathbf{b}$ holds for the solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ in (LP2a), as $A\bar{\mathbf{x}} = \mathbf{b}$. Suppose the vector $\bar{\mathbf{x}}$ had some negative component $\bar{x}_i < 0$; then $\bar{x}_i < 0 < |\bar{x}_i| = \bar{y}_i$ and $-\bar{x}_i = |\bar{x}_i| = \bar{y}_i$, and thus both the second and third constraints are held for any negative component of $\bar{\mathbf{x}}$. Finally, consider some nonnegative component $\bar{x}_i \geq 0$; then, similar to above, $\bar{x}_i = |\bar{x}_i| = \bar{y}_i$ and $-\bar{x}_i \leq 0 \leq |\bar{x}_i| = \bar{y}_i$, and thus the second and third constraints are held for any nonnegative component of $\bar{\mathbf{x}}$. Hence $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is a feasible solution to (LP2a).

   Now consider the value of the objective function at $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$:

$$\zeta(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \sum_{i=1}^{n} c_i \bar{y}_i = \sum_{i=1}^{n} c_i |\bar{x}_i| = z(\bar{\mathbf{x}}) \tag{1}$$

   Thus we can see that the objective values of our respective solutions are equal. For the opposite direction, suppose that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a feasible solution to (LP2a). Then by definition $A\hat{\mathbf{x}} = \mathbf{b}$, and so $\hat{\mathbf{x}}$ is feasible for (ABS). Furthermore, $|\hat{\mathbf{x}}| \leq \hat{\mathbf{y}}$ as $\hat{\mathbf{x}} \leq \hat{\mathbf{y}}$ and $-\hat{\mathbf{x}} \leq \hat{\mathbf{y}}$; thus for any components $\hat{x}_i$ of $\hat{\mathbf{x}}$ and $\hat{y}_i$ of $\hat{\mathbf{y}}$, $|\hat{x}_i| \leq \hat{y}_i$.

   Now consider the value of the objective function at $\hat{\mathbf{x}}$:

$$z(\hat{\mathbf{x}}) = \sum_{i=1}^{n} c_i |\hat{x}_i| \leq \sum_{i=1}^{n} c_i \hat{y}_i = \zeta(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \tag{2}$$

Thus we can see that the objective value for some feasible solution $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ of (LP2a) is greater than or equal to the objective value for the respective solution $\hat{\mathbf{x}}$ of (ABS). Finally, we consider the optimal solutions $\bar{\mathbf{x}}^*$ for (ABS) and $(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*)$ for (LP2a), with respective values z^* and ζ^*. Because $\bar{\mathbf{x}}^*$ is feasible to (ABS), we know from above that the solution $(\bar{\mathbf{x}}^*, \bar{\mathbf{y}}^*)$ given by $\bar{\mathbf{y}}^* = |\bar{\mathbf{x}}^*|$ is feasible to (LP2a), and by (1) that

$$\zeta(\bar{\mathbf{x}}^*, \bar{\mathbf{y}}^*) = z(\bar{\mathbf{x}}^*) = z^* \tag{3}$$

Similarly, because $(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*)$ is feasible to (LP2a), we know that the solution $\hat{\mathbf{x}}^*$ is feasible to (ABS), and by (2) that

$$z(\hat{\mathbf{x}}^*) \leq \zeta(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*) = \zeta^* \tag{4}$$

Suppose the optimal values z^* and ζ^* were not equal; this implies by (4) that $\zeta^* > z^*$, as $\zeta^* \geq z(\hat{\mathbf{x}}^*) \geq z^*$. This is a contradiction, however, as we know from (3) that there exists a feasible solution to (LP2a) such that $\zeta(\bar{\mathbf{x}}^*, \bar{\mathbf{y}}^*) = z^*$, and that this value is less (and therefore more optimal) than ζ^*. Thus $z^* = \zeta^*$. ■

2. (20pts) Implement an algorithm to solve linear programs with absolute value in the form of (ABS).

   - Inputs: $\mathbf{c}, A, \mathbf{b}$.
   - Outputs: optimal solution $\mathbf{x}^*$, optimal value z^*.
   - You may call existing linear program solvers/functions in your code.

```
% Algorithm to solve linear programs with absolute value in the form of
% (ABS). Requires inputs c, A, b.

x = optimvar( x ,length(c));
y = optimvar( y ,length(c));
prob = optimproblem( Objective ,dot(c,y), ObjectiveSense , min );
prob.Constraints.c1 = A*x == b;
prob.Constraints.c2 = x ≤ y;
prob.Constraints.c3 = -x ≤ y;

sol = solve(prob);
sol.x
dot(c,sol.y)
```