# HW6 MATH 4540

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## 1 Exercises

1a. Suppose $E$ was some set with measure zero. Then by definition, for $\epsilon > 0$,

$$E \subset \bigcup_{i=1}^{\infty} I_i$$

where $I_i = (a_i, b_i)$ are a sequence of open intervals, and

$$\sum_{i=1}^{\infty} \psi(I_i) < \epsilon$$

where $\psi(I_i) = b_i - a_i$. Suppose $F \subset E$ was some subset. Then

$$F \subset E \subset \bigcup_{i=1}^{\infty} I_i$$

and hence $F$ is also measure zero, as the very same set of intervals can used to similarly construct $F$ such that the second property holds for $\epsilon > 0$.

1b. Suppose $E_1$ and $E_2$ were two sets that both have measure zero. Let $\{I_i\}_{i=1}^{\infty}$ and $\{J_i\}_{i=1}^{\infty}$ be the sequences of intervals used to construct both set, and let $\epsilon > 0$. Define $\{K_i\}_{i=1}^{\infty}$ to be the sequence of intervals given by

$$K_i := \begin{cases} I_n, & \text{if } i \text{ is odd} \\ J_n, & \text{if } i \text{ is even} \end{cases}$$

where $n = \lfloor \frac{i+1}{2} \rfloor$. Or in other words,

$$\{K_i\}_{i=1}^{\infty} = \{I_1, J_1, I_2, J_2, I_3, J_3, \dots\}.$$

Then observe that

$$E_1 \cup E_2 \subset \bigcup_{i=1}^{\infty} (I_i \cup J_i) = \bigcup_{i=1}^{\infty} K_i$$

and likewise

$$\sum_{i=1}^{\infty} \psi(K_i) = \sum_{i=1}^{\infty} \psi(I_i) + \sum_{i=1}^{\infty} \psi(J_i) < 2\epsilon.$$

Because $\epsilon$ can be made arbitrarily small, it thus suffices that

$$\sum_{i=1}^{\infty} \psi(K_i) = \sum_{i=1}^{\infty} \psi(I_i) + \sum_{i=1}^{\infty} \psi(J_i) < \epsilon'$$

for all $\epsilon' = 2\epsilon > 0$. Hence the union $E_1 \cup E_2$ has measure zero.

1c. Continuing from above, suppose that $E_n$ has measure zero for each $n \in \mathbb{N}$, and let $\epsilon > 0$. Then, for each $n$, we can find a sequence of open intervals $\{(I_i)_n\}_{i=1}^{\infty}$ where

$$E_n \subset \bigcup_{i=1}^{\infty} (I_i)_n$$

and

$$\sum_{i=1}^{\infty} \psi((I_i)_n) < \frac{\epsilon}{2^n}.$$

Therefore it follows that, for the sequence $\{I_{in}\}_{i,n=1}^{\infty}$,

$$\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} I_{in} = \bigcup_{i,n=1}^{\infty} I_{in}$$

and

$$\sum_{i,n=1}^{\infty} \psi(I_{in}) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \psi(I_{in}) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Hence $\bigcup_{n=1}^{\infty} E_n$ has measure zero.

2. Let $\epsilon > 0$. Note that because $\mathbb{Q}$ is countable, we can enumerate through all the rationals with an arbitrary sequence $(q_n) := \{q_i | q_i \in \mathbb{Q}\}_{i=1}^{n} = q_1, q_2, q_3, \dots, q_n$ as $n \to \infty$. Define $\{I_n\}_{n=1}^{\infty}$ to be the sequence of intervals given by

$$I_i := (q_i - \frac{\epsilon}{2^{i+2}}, q_i + \frac{\epsilon}{2^{i+2}}).$$

Note that for all $q_i \in (q_n)$, it follows that $q_i \in I_i$, and hence

$$\mathbb{Q} \subset \bigcup_{i=1}^{\infty} I_i.$$

Furthermore, observe that $|I_i| = \frac{2\epsilon}{2^{i+2}} = \frac{\epsilon}{2^{i+1}}$ for all such intervals, and thus

$$\sum_{i=1}^{\infty} |I_i| = \sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} = \frac{\epsilon}{2} < \epsilon.$$

Therefore $\mathbb{Q}$ has measure zero.

3. Note that $g(x)$ is continuous for all $x \notin \mathbb{Z}$, and hence the set of discontinuities of $g$ is countable. Because the discontinuities of $g(nx)$ is a subset of the discontinuities of $g(x)$, the discontinuities of $g$ has measure zero; thus $f \in \mathscr{R}$. Observe that integrating an infinite sum is equivalent to adding the relative integrals of each of its terms:

$$\int_0^1 f \, \mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty} \frac{g(nx)}{n^2} \, \mathrm{d}x = \sum_{n=1}^{\infty} \int_0^1 \frac{g(nx)}{n^2} \, \mathrm{d}x.$$

Suppose $n$ was given. Then

$$\int_0^1 \frac{g(nx)}{n^2} \, \mathrm{d}x = \frac{1}{n^2} \int_0^1 g(nx) \, \mathrm{d}x$$

as we are integrating without respect to $n$. Note that normally

$$\int_0^t g(x) \, \mathrm{d}x = \sum_{k=0}^{m-1} \int_k^{k+1} (x-k) \, \mathrm{d}x,$$

where $m \in \mathbb{N}$ is the smallest natural number less than some $t \in \mathbb{R}$. This can be thought of as adding up the integrals between each of the discontinuities of $g$, which occur for every natural number. Therefore

$$\int_0^1 g(nx) \, \mathrm{d}x = \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} (nx-k) \, \mathrm{d}x$$

$$= \sum_{k=0}^{n-1} \left[ \frac{(k+1)^2}{2n} - \frac{k(k+1)}{n} - \frac{k^2}{2n} + \frac{k^2}{n} \right]$$

$$= \sum_{k=0}^{n-1} \left[ \frac{k^2+2k+1}{2n} - \frac{2k^2+2k}{2n} - \frac{k^2}{2n} + \frac{2k^2}{2n} \right]$$

$$= \sum_{k=0}^{n-1} \frac{1}{2n} = \frac{1}{2}.$$

Recalling that the solution to the Basel problem is $\pi/6$, we derive:

$$\int_0^1 f \, \mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty} \frac{g(nx)}{n^2} \, \mathrm{d}x = \sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{\pi^2}{12}.$$

4. Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely. Then, by the Cauchy criterion for convergence, for every $\epsilon > 0$ there is an integer $N$ such that

$$\sum_{k=n}^{m} |a_k| \leq \epsilon$$

if $m \geq n \geq N$. Suppose $x \in [-1,1]$. Then for all $k \in N$, $x^k \leq 1$, and thus

$$a_k x^k \leq |a_k|.$$

Therefore

$$\left| \sum_{k=n}^{m} a_k x^k \right| \leq \sum_{k=n}^{m} |a_k| \leq \epsilon,$$

and hence the Cauchy criterion for uniform convergence on $[-1,1]$ holds.

5a. Suppose $f$ and $g$ were two bounded and continuous functions on $[a,b]$. Observe that if $f = g$, $f(x) = g(x)$ for all $x \in [a,b]$, and hence

$$d_1(f,g) = \int_a^b |f-g| \, \mathrm{d}x = \int_a^b 0 \, \mathrm{d}x = 0.$$

Suppose instead that $f(x) \neq g(x)$. Then there must exist at least one $t \in [a,b]$ such that $f(t) \neq g(t)$, and hence $|f(t) - g(t)| > 0$. Because both $f$ and $g$ are continuous, it follows that so too is $|f-g|$, and thus it's also integrable. Furthermore, because $|f(x) - g(x)| \geq 0$, and as any sub-interval containing $t$ has a supremum $M_k \geq |f(t) - g(t)| > 0$, the upper integral of $|f-g|$ must be positive:

$$\int_a^b |f-g| \, \mathrm{d}x = \inf U(f,P) \geq |f(t) - g(t)| * \triangle x_k > 0,$$

Meaning

$$d_1(f,g) > 0.$$

5b. Consider the sequence of functions $\{f_n\}$ defined by

$$f_n(x) := \begin{cases} \frac{n(x-a)}{2(b-a)}, & \text{if } x \in [a, a+\frac{b-a}{n}] \\ \frac{1}{2}, & \text{if } x \in (a+\frac{b-a}{n}, b-\frac{b-a}{n}) \\ \frac{n(x-(b-\frac{b-a}{n})}{2(b-a)}, & \text{if } x \in [b-\frac{b-a}{n}, b] \end{cases}$$

Observe that each function linearly maps the intervals $[a, a+\frac{b-a}{n}]$ and $[b-\frac{b-a}{n}, b]$ onto $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively, and that each has the value of $\frac{1}{2}$ for all other $x \in [a,b]$. Futhermore, because $0 \leq f_n(x) \leq 1$ for all $x \in [a,b]$, each $f_n$ is bounded; and because $f_n$ is a combination of line segments all containing common points, each $f_n$ is continuous. Now suppose $n \to \infty$. Then it follows that $f_n(a) = 0$ and $f_n(b) = 1$, as both are contained in their respective closed interval; yet because $(a+\frac{b-a}{n}, b-\frac{b-a}{n}) \to (a,b)$, then $f_n(r) = \frac{1}{2}$ for all $r \in (a,b)$. Hence

$$\lim_{n\to\infty} f_n(x) := \begin{cases} 0, & \text{if } x = a \\ \frac{1}{2}, & \text{if } x \in (a,b) \\ 1, & \text{if } x = b \end{cases}$$

Thus the limit of $\{f_n\}$ is discontinuous, and thereby not contained in the set of bounded and continuous functions on $[a, b]$.