Overall: C

MATH 4560/6560 Topology
Spring 2022
Exercises 2
Target date: Thu 03 Feb 11:59PM

Exercise 1. (a) Find an online news story from 2022 that includes the word "surface." Give the link and write a summary of the article. Does the article's use of the word "surface" match with how we use it in class? How are they different and how are they similar?

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Source: This article talks about the semantic study of discourse, and how certain sentences or groups of words relate to other ones conceptually. This breakdown of relations is used to describe how patterns and signals come about in normal discourse. The word *surface* is used in this article six times; in fact, it is in the title of the work. The best translations of this word as it is being used in the article are *"beginning", "shallow",* or *"introductory"*. It is used to say "surface structure signals", which simply denote the very first or most elementary of such signals. This contrasts with our definition of surface, which is tied to the notions of curvature and planes.

(b) Find a Wikipedia article about a mathematician whose research includes topology. (Bonus points if the mathematician is from group that has been historically excluded from mathematics, e.g., a woman or non-binary person; a Black person or member of another minority race or ethnicity; a member of the LGBTQIA+ community; or a person with a disability.) Give the link and write a summary of the person's accomplishments. Does their research involve curves or surfaces?

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John Robert Kline, as well as **one of his papers** and **an article explaining his involvement in topology**. Kline was a Black Mathematician in the 1920s and the peer and mentor of **Dudley Weldon Woodard**, the second African-American to earn a PhD in Mathematics. Their work targeted structures like the **Jordan curve**, a concept in topology that involves **plane curves**. So indeed his research does deal with curves and surfaces.

Exercise 2. Consider Definition I.B.4 from class, the informal definition of the term "surface."

(a) Use this definition to propose an informal definition of the term "curve."

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A curve is a subset $C \subseteq R^n$ such that, for every point $P \in C$, there exists an open ball $B \subseteq R^n$ where $B \cap C$ *looks like* (is homeomorphic to) an open line segment $L \in R$. In other words, as we zoom closer to P, C looks more and more line a straight line.

(b) Do the graphs of the following equations define curves according to your definition? Why or why not?

(i) $y = x^2$ with $-\infty < x < \infty$.
Yes. In fact all polynomials are curves, since all polynomials are homeomorphic to a straight line through differentiation. Thus for any $B \cap C$, where C is defined by $y = x^2$, we have $L = \{2i : a < i < b\}$ for the endpoints (a, a^2) and (b, b^2) of the intersection.

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(ii) $y = |x|$ with $-\infty < x < \infty$.
Yes. Consider the function $f : (-\infty, \infty) \to R^2$ given by $f(x) = y$ if $x \geq 0$ and

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$f(x) = -y$ if $x < 0$, projecting onto the curve $y = x$. Then it's easy to see that for each point there exists an intersection with an open ball that is an open line segment.

(iii) (node) $y^2 = x^2(x+1)$ with $-\infty < x < \infty$.
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No, because at $x = 0$ every open ball has an intersection which has five vertices, and so it can't *look like* an open line segment, which has only two vertices.

(iv) $y = x^2$ with $0 < x < 1$.
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Yes, as $0 < x < 1$ implies openness, and therefore we can always have an open ball, similar to problem (i), such that the bounds do no matter.

(v) $y = x^2$ with $0 \leq x \leq 1$.
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No. In this case, we see the weakness of our informal definition (in my opinion): since, at the endpoints $x = 0$ and $x = 1$, every open ball creates an intersection that is a half-open line segment, which is not open. This is similar to how, given the informal definition of a surface, a square with finite area is not a surface: since at every point on its perimeter the intersections with any open ball are half-open disks (and are therefore not open disks).

Exercise 3. Describe how to use a polygon to describe a 5-holed torus. Justify your response.

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Suppose we had five toruses $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4, \mathbb{T}_5$. We know from class that these can be represented as polygons, with each letter corresponding to a different edge of the polygon; thus each torus can be represented by the four-sided polygons

$$\mathbb{T}_i = a_i b_i a_i^{-1} b_i^{-1}.$$

Furthermore, in class we learned how to combine two toruses into a 2-holed torus with the polygon representation

$$\mathbb{T}^{2\#2} = abcdc^{-1}d^{-1}a^{-1}b^{-1}.$$

We can think of this transformation as an algebraic operation on the two torus representations, illustrated below:

$$\mathbb{T}^{2\#2} = \mathbb{T}\#\mathbb{T} = aba^{-1}b^{-1}\#cdc^{-1}d^{-1} = ab(cdc^{-1}d^{-1})a^{-1}b^{-1}.$$

Hence combining two toruses can be thought of as "placing one torus in the middle of another" algebraically. Using this operation, we find the representation for a 3-holed torus as

$$\mathbb{T}^{2\#3} = \mathbb{T}^{2\#2}\#\mathbb{T} = abcdc^{-1}d^{-1}a^{-1}b^{-1}\#efe^{-1}f^{-1} = abcd(efe^{-1}f^{-1})c^{-1}d^{-1}a^{-1}b^{-1},$$

a 12-sided polygon. Thus the combination of $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4, \mathbb{T}_5$, using the same operation 4 times, is given as

$$\mathbb{T}^{2\#5} = a_1b_1(a_2b_2(a_3b_3(a_4b_4(a_5b_5a_5^{-1}b_5^{-1})a_4^{-1}b_4^{-1})a_3^{-1}b_3^{-1})a_2^{-1}b_2^{-1})a_1^{-1}b_1^{-1}$$

a 20-sided polygon.