Lecture 15 Duality theory I

- Symmetry

$$
\begin{array}{l} \max\ c^T x \\ \text{s.t. } Ax \le b \\ \quad\ \ x \ge 0 \end{array} \ (P)
\ \overset{⑤}{\Leftrightarrow}\
\begin{array}{l} -\min\ -c^T x \\ -Ax \ge -b \\ \quad x \ge 0 \end{array}
\ \overset{④}{\Leftrightarrow}\
\begin{array}{l} -\min\ c^T y \\ \text{St } (A^T)^T y \ge -b \\ \qquad\quad y \le 0 \end{array}
$$

① dual ↓

$$
\begin{array}{l} \min\ b^T \pi \\ \text{St } A^T \pi \ge c \\ \quad\ \ \pi \ge 0 \end{array} \ (D)
\ \overset{②}{\Leftrightarrow}\
\begin{array}{l} -\max\ -b^T \pi \\ \text{St } A^T \pi \ge c \\ \quad\ \ \pi \ge 0 \end{array}
$$

③ dual ↗

Thm The dual of the dual is the primal

Therefore, we can also read the table in last lecture from right to left

e.g.: The dual problem of

$$
\begin{array}{ll} \min & Z = 3x_1 + 2x_2 - 4x_3 \\ \text{St} & 5x_1 - 7x_2 + x_3 \ge 12 \\ & x_1 - x_2 + 2x_3 = 18 \\ & 2x_1 \qquad - x_3 \le 6 \\ & x_1, x_2, x_3 \ge 0 \end{array}
$$

is

$$
\begin{array}{ll} \max & Z = 12\pi_1 + 18\pi_2 + 6\pi_3 \\ \text{St} & 5\pi_1 + \pi_2 + 2\pi_3 \le 3 \\ & -7\pi_1 - \pi_2 \qquad \le 2 \\ & \pi_1 + 2\pi_2 - \pi_3 \le -4 \\ & \pi_1 \ge 0, \quad \pi_3 \le 0 \end{array}
$$

- Weak Duality

Consider the following primal-dual pair

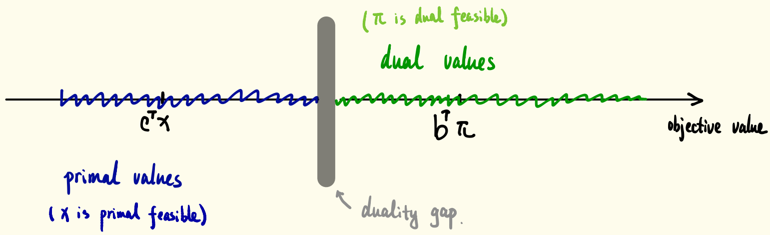
$$\begin{array}{lll} \max & z = c^T x & \\ \text{s.t.} & Ax \le b & (P) \\ & x \ge 0 & \end{array} \quad \longleftrightarrow \quad \begin{array}{lll} \min & Z = b^T \pi & \\ \text{s.t.} & A^T\pi \ge c & (D) \\ & \pi \ge 0 & \end{array}$$

Thm If $\bar{x}$ is a feasible solution to the maximization problem (P) and $\bar{\pi}$ is a feasible solution to the minimization problem (D), then $c^T\bar{x} \le b^T\bar{\pi}$

Pf $$c^T\bar{x} \underset{\substack{\uparrow \\ A^T\bar{\pi} \ge c \\ \text{and } \bar{x} \ge 0}}{\le} (A^T\bar{\pi})^T\bar{x} \underset{\substack{\uparrow \\ \text{property of} \\ \text{transpose}}}{=} \bar{\pi}^T A\bar{x} \underset{\substack{\uparrow \\ A\bar{x} \le b \\ \text{and } \bar{\pi} \ge 0}}{\le} \bar{\pi}^T b$$

Cor If the primal problem is unbounded, then the dual problem is infeasible

Pf We prove by contradiction. If the dual problem is feasible, then for any feasible solution $\bar{\pi}$ of (D), $c^T x \le \bar{\pi}^T b$ for any x feasible for (P). Therefore, $c^T x$ is bounded over the feasible region of (P). Contradiction



Remark: The converse of the corollary is not true.

eg
$$\begin{array}{ll} \max & x_1+3x_2 \\ \text{St} & x_1+2x_2=1 \\ & x_1+2x_2=2 \end{array} \quad (P) \quad \xleftrightarrow{\text{dual}} \quad \begin{array}{ll} \min & \pi_1+2\pi_2 \\ \text{St} & \pi_1+\pi_2=1 \\ & 2\pi_1+2\pi_2=3 \end{array} \quad (D)$$

Both (P) and (D) are infeasible

Corrollary [optimality property]:

If $\bar{x}$ is feasible to (P), and $\bar{\pi}$ is feasible to (D), and $c^T\bar{x} = b^T\bar{\pi}$, then $\bar{x}$ is an optimal solution to (P) and $\bar{\pi}$ is an optimal solution to (D).

Pf For any x feasible to (P), by weak duality, $c^T x \le b^T\bar{\pi} = c^T\bar{x}$ Therefore, $\bar{x}$ is an optimal solution to (P) Similarly, $b^T\pi \ge c^T\bar{x} = b^T\bar{\pi}$ for any π feasible to (D) Therefore, $\bar{\pi}$ is an optimal solution to (D)

eg
$$\begin{array}{ll} \max & -4x_1-2x_2-x_3 \\ \text{St} & -x_1-x_2+2x_3 \le -3 \\ & -4x_1-2x_2+x_3 \le -4 \\ & x_1+x_2-4x_3 \le 2 \\ & x_1,x_2,x_3 \ge 0 \end{array} \quad (P) \quad \xleftrightarrow{\text{dual}} \quad \begin{array}{ll} \min & -3y_1-4y_2+2y_3 \\ \text{S.t} & -y_1-4y_2+y_3 \ge -4 \\ & -y_1-2y_2+y_3 \ge -2 \\ & 2y_1+y_2-4y_3 \ge -1 \\ & y_1,y_2,y_3 \ge 0 \end{array} \quad (D)$$

Given a pair of solutions $\bar{x} = \begin{bmatrix} 0 \\ 4 \\ \frac{1}{2} \end{bmatrix}$ and $\bar{y} = \begin{bmatrix} \frac{9}{2} \\ 0 \\ \frac{5}{2} \end{bmatrix}$ We can justify that $\bar{x}$ is feasible to (P) and $\bar{y}$ is feasible to (D) Note that $-4\times 0 - 2\times 4 - \frac{1}{2} = -\frac{17}{2} = -3\times\frac{9}{2} - 4\times 0 + 2\times\frac{5}{2}$

We can conclude that $\bar{x}$ is optimal to (P) and $\bar{y}$ is optimal to (D)

- Strong duality

Thm If an LP has an optimal solution, so does its dual, and the respective optimal values are equal

Pf: We prove it for problems in standard form

$$p^* = \max \ c^T x \quad \text{s.t.} \ Ax = b, \ x \geq 0 \quad (P) \qquad d^* = \min \ b^T \pi \quad \text{s.t.} \ A^T \pi \geq c \quad (D)$$

By the fundamental theorem of LP, there exists a basic optimal solution x^* since (P) has an optimal solution

Recall the simplex tableau

	z	x	RHS
z	1	$c_B^T B^{-1} A - c^T$	$c_B^T B^{-1} b$
x_B	$\vec{0}$	$B^{-1} A$	$B^{-1} b$

Note that $c^T x^* = c_B^T B^{-1} b = b^T (B^{-1})^T c_B$

Consider $\bar{\pi} = (B^{-1})^T c_B$ If $\bar{\pi}$ is feasible to (D), then by the optimality property, $\bar{\pi}$ is optimal to (D) and $p^* = d^*$

In fact, $A^T \bar{\pi} = A^T (B^{-1})^T c_B = (c_B^T B^{-1} A)^T$ Since x^* is an optimal solution to (P), the reduced costs $\bar{c} = (c_B^T B^{-1} A - c^T)^T \geq 0$ Therefore, $(c_B^T B^{-1} A)^T \geq c$ That is, $\bar{\pi}$ is feasible to (D) $\square$

Cor: If both primal and dual are feasible, then they both have optimal solutions and the optimal values are equal.

Pf: Since the dual is feasible, by weak duality, the primal is not unbounded. Since the primal is feasible, it has an optimal solution. Then the result follows from strong duality. $\square$

Summary different possibilities for the primal and the dual

Primal \ Dual	Finite optimum	Unbounded	Infeasible
Finite optimum	✓	✗	✗
Unbounded	✗	✗	✓
Infeasible	✗	✓	✓

Remark: When $A = [D \;\; I]$ and $c = \begin{bmatrix} d \\ 0 \end{bmatrix}$, note that $\bar{c}^T = c_B^T B^{-1} A - c^T = [c_B^T B^{-1} D - d \;\;\; c_B^T B^{-1}]$.

identity matrix

For a problem in canonical form

$$\begin{aligned} \max \quad & d^T x \\ \text{s.t.} \quad & Dx \leq b \\ & x \geq 0 \end{aligned} \qquad \qquad \begin{aligned} \max \quad & d^T x + 0^T s \\ \text{s.t.} \quad & [D \;\; I] \begin{bmatrix} x \\ s \end{bmatrix} = b \\ & \begin{bmatrix} x \\ s \end{bmatrix} \geq 0 \end{aligned}$$

The corresponding dual solution $\pi = (c_B^T B^{-1})^T$ can be read directly from the simplex tableau.

z	x	s	RHS
1	$c_B^T B^{-1} D - d^T$	$\boxed{c_B^T B^{-1}}$	$c_B^T B^{-1} b$
$\vec{0}$	$B^{-1} D$	B^{-1}	$B^{-1} b$

For example, consider a primal-dual pair:

$$\begin{aligned} \max \quad & 3x_1 + 15x_2 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 2 \\ & x_1 + 8x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned} \quad (P) \quad \xleftrightarrow{\text{dual}} \quad \begin{aligned} \min \quad & 2\pi_1 + 3\pi_2 \\ \text{s.t.} \quad & \pi_1 + \pi_2 \geq 3 \\ & 4\pi_1 + 8\pi_2 \geq 15 \\ & \pi_1, \pi_2 \geq 0 \end{aligned} \quad (D)$$

The initial and optimal simplex tableaus of (P) are as follows:

$$
\begin{array}{|c|cccc|c|}
\hline
1 & -3 & -15 & 0 & 0 & 0 \\
\hline
0 & 1 & 4 & 1 & 0 & 2 \\
0 & 1 & 8 & 0 & 1 & 3 \\
\hline
\end{array}
\longrightarrow \cdots \longrightarrow
\begin{array}{|c|cccc|c|}
\hline
1 & 0 & 0 & \boxed{\frac{9}{4} \quad \frac{3}{4}} & & \frac{27}{4} \\
\hline
0 & 1 & 0 & 2 & -1 & 1 \\
0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\hline
\end{array}
$$

In the optimal tableau, we can read the primal optimal solution $x^* = (1, \frac{1}{4})$ as well as the dual optimal solution $\pi^* = (\frac{9}{4}, \frac{3}{4})$.