HW7 MATH 4540

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1 Exercises

1. Assume, for sake of contradiction, that $f_n(x) := nx(1-x^2), n \in \mathbb{N}$ does converge uniformly to zero on $[0,1]$. Then for every $\epsilon > 0$ there is some integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| = |f_n(x)| \leq \epsilon$$

for all $x \in [0,1]$. Note that for each $n \geq N$, $1/n \in [0,1]$. Hence as $n \to \infty$:

$$|f_n(1/n)| = |n(1/n)(1-(1/n)^2)^n| = |(1-1/n^2)^n| \to 1;$$

thus for any $\epsilon < 1/2$, with sufficiently large n we find that $1 > |f_n(1/n)| = 1 - \epsilon > \epsilon$, which is a contradiction. Hence the sequence does not converge uniformly to zero.

2. Suppose $\{f_n\}$ was some uniformly convergent sequence of bounded functions on a set E. Then by the Cauchy criterion for uniform convergence, for every $\epsilon > 0$ there is some integer N such that $n, m \geq N$ implies

$$|f_n(x) - f_m(x)| \leq \epsilon.$$

Note that

$$|f_n(x)| - |f_m(x)| \leq |f_n(x) - f_m(x)| \leq 1,$$

and hence it follows that

$$|f_n(x)| \leq |f_N(x)| + 1 = M_N + 1$$

for $m = N$ and its relative boundary $M_N > 0$. Therefore f_n is bounded by $M_N + 1$ for all $n \geq N$. Consider now $M = \max(M_1, M_2, \ldots, M_N) + 1$, where each M_i is the bound for the relative function f_i. Then

$$|f_n(x)| \leq M$$

and hence the sequence is uniformly bounded.

3. Clearly as $n \to \infty$, $f_n \to 0$, and hence we will show that the sequence converges uniformly to zero for all reals. Take $\epsilon > 0$. Then

$$|f_n(x) - 0| = \left|\frac{x}{1+nx^2}\right|$$

4. Suppose the series converged uniformly on $[0, 1]$. Then by the Cauchy criterion, where $s_n(x)$ are the partial sums and for some $n, m \geq N$, $\epsilon > 0$

$$|s_n(x) - s_m(x)| < \epsilon$$

Suppose without loss of generality that $n > m$. Then

$$|s_n(x) - s_m(x)| = \left|\sum_{k=m}^{n} x(1-x)^k\right|.$$

Note that $1/N \in [0, 1]$, and hence for $x = 1/N$,

$$|s_n(1/N) - s_m(1/N)| = \left|\sum_{k=m}^{n} 1/N(1-1/N)^k\right|.$$

Observe however that

$$\left|\frac{(n-m)}{N}(1-1/N)^n\right| < \left|\sum_{k=m}^{n} 1/N(1-1/N)^k\right|$$

and hence if we choose $m = N$ and let $n \to \infty$, then

$$\left|\frac{(n-m)}{N}(1-1/N)^n\right| \to \left|\frac{(n-N)}{N}\frac{1}{e}\right| \to \infty$$

Thus with sufficiently large n, it follows that at $x = 1/N$ the series does not converge uniformly.

5. Let E be a bounded set in the reals. Then for any $x \in E$, it follows that $|x| \leq M$ for some $M > 0$; furthermore, observe that

$$\sum_{k=0}^{n} \frac{x^k}{k!} \leq M_n = \sum_{k=0}^{n} \frac{M^k}{k!}.$$

Recall that $\sum f_n$ converges uniformly if $\sum M_n$ converges (Rudin's 7.10). Being the power series representation of the exponential function, we know however that $M_n \to e^M$, and thus the sums converge uniformly. Consider now the sequence $\{f_n'\}$. Note that if

$$f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

then

$$f_n'(x) = \sum_{k=0}^{n} \frac{x^{k-1}}{(k-1)!} = \sum_{k=m-1}^{n} \frac{x^k}{k!}.$$

Hence by the Cauchy criterion for the uniform convergence of the sums, we find for all $\epsilon > 0$ and $n, m > N$,

$$\left| \sum_{k=0}^{n} \frac{x^k}{k!} - \sum_{k=0}^{m} \frac{x^k}{k!} \right| < \epsilon;$$

yet if $m' = m - 1$, then it follows that

$$\left| \sum_{k=0}^{n} \frac{x^k}{k!} - \sum_{k=0}^{m-1} \frac{x^k}{k!} \right| = \sum_{k=m'}^{n} \frac{x^{k-1}}{(k-1)!} < \epsilon,$$

and hence the differentiation of the series converges uniformly as well. Furthermore,

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

(Rudin's 7.17), and thus the derivative of the sum can be obtained by term-by-term differentiation of the series.