Lecture 17 The dual simplex method

- Simplex method revisit.

Consider

$$\begin{aligned} \max\ & c^T x \\ \text{St}\ & Ax = b \quad (P) \\ & x \geq 0 \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} \min\ & b^T \pi \\ \text{St}\ & A^T \pi \geq c \quad (D) \end{aligned}$$

The complementary slackness conditions are:

$$x_i (A^T \pi - c)_i = 0, \quad i = 1, \cdots, n.$$

For a BFS $\bar{x}$ of (P), let $\bar{\pi} := (c_B^T B^{-1})^T$. Note that

$$A^T \bar{\pi} - c = (c_B^T B^{-1} A)^T - c = \bar{c} \quad (\text{the reduced costs})$$

Since $\bar{x}_N = 0$ and $\bar{c}_B = 0$, the complementary slackness conditions are always satisfied by $\bar{x}$ and $\bar{\pi}$.

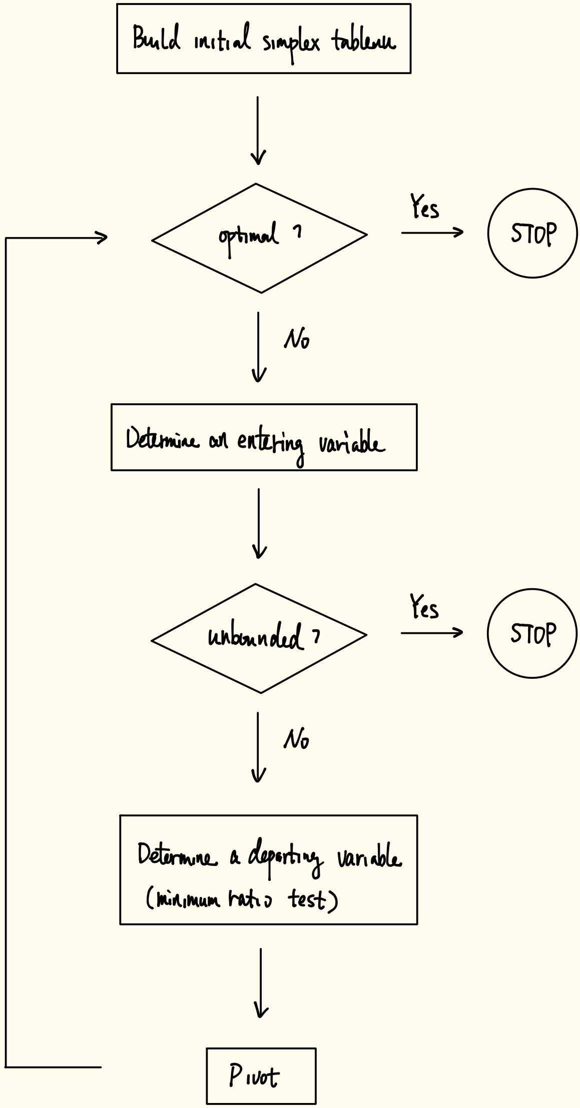
By the complementary slackness theorem, $\bar{x}$ is optimal to (P) if $\bar{\pi}$ is feasible to (D), i.e., $\bar{c} \geq 0$, which is the same as the optimality condition that we learned in the simplex method.

In summary, in the simplex method, we maintain primal feasibility and complementary slackness conditions in each iteration and make improvements to reach dual feasibility.

- Dual simplex method.

In the dual simplex method, we maintain dual feasibility and complementary slackness conditions in each iteration and make improvements to reach primal feasibility

Recall: The Simplex Method



The simplex algorithm

Step 0 We start from a basic feasible solution. $(b \geq 0)$

Step 1 Check optimality / dual feasibility

If $\bar{c} \geq 0$, then the current solution is optimal. STOP.

Step 2 Determine the entering variable x_k

Any nonbasic variable with a negative reduced cost can be the entering variable. We usually pick the one with the most negative reduced cost.

Step 3 Determine the departing variable x_r.

Positive elements in Column k are candidates for the pivot element. The departing variable is determined by a minimum ratio test:

$$\frac{b_r}{a_{rk}} = \min \left\{ \frac{b_i}{a_{ik}} \quad a_{ik} > 0 \right\}$$

If $a_{ik} \leq 0$ for all $i = 1, \cdots, n$, the problem is infeasible. STOP.

Step 4: Pivot.

Go back to Step 1.

The dual simplex algorithm.

Step 0 We start from a dual feasible basic solution. $(\bar{c} \geq 0)$

Step 1 Check primal feasibility

If $\beta = B^{-1}b \geq 0$, then the current solution is optimal. STOP.

Step 2 Determine the departing variable x_r

Any basic variable with a negative value can be the departing variable. We usually pick the most negative one.

Step 3 Determine the entering variable x_k.

Negative elements in Row r are candidates for the pivot element. The entering variable is determined by a minimum ratio test:

$$\frac{\bar{c}_k}{|a_{rk}|} = \min \left\{ \frac{\bar{c}_j}{|a_{rj}|} \quad a_{rj} < 0 \right\}$$

If $a_{rj} \geq 0$ for all $j = 1, \cdots, n$, the problem is infeasible. STOP.

Step 4: Pivot.

Go back to Step 1.

We show how the dual simplex method works with an example.

The dual simplex algorithm.

Step 0 We start from a dual feasible basic solution. ($\bar{c} \geq 0$)

Step 1 Check primal feasibility

If $\beta = B^{-1}b \geq 0$, then the current solution is optimal. STOP.

Step 2 Determine the departing variable x_r

Any basic variable with a negative value can be the departing variable. We usually pick the most negative one.

Step 3 Determine the entering variable x_k.

Negative elements in Row r are candidates for the pivot element. The entering variable is determined by a minimum ratio test:

$$\frac{\bar{c}_k}{|a_{rk}|} = \min \left\{ \frac{\bar{c}_j}{|a_{rj}|} \quad a_{rj} < 0 \right\}$$

If $a_{rj} \geq 0$ for all $j = 1, \cdots, n$, the problem is infeasible. STOP.

Step 4: Pivot.

Go back to Step 1.

eg
$$\max \ -2x_1 - 6x_2 - 10x_3$$
$$\text{s.t.} \quad -2x_1 + 4x_2 + x_3 \leq 2$$
$$4x_1 - 2x_2 - 3x_3 \leq -1$$
$$x_1, x_2, x_3 \geq 0$$

0 dual feasible

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	2	6	10	0	0	0
x_4	0	-2	4	1	1	0	2
x_5	0	4	-2	-3	0	1	$\boxed{-1}$

2 departing variable

1 primal infeasible

3 $\left.\begin{array}{l} \frac{6}{|-2|} = 3 \\ \frac{10}{|-3|} = \frac{10}{3} \end{array}\right\} \Rightarrow x_2$ is the entering variable.

4 $\downarrow$

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	14	0	1	0	3	-3
x_4	0	6	0	-5	1	2	0
x_2	0	-2	1	$\frac{3}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$

1 primal feasible

The optimal solution is $x^* = (0, \frac{1}{2}, 0)$.

Remark: 1. The objective value is non-increasing in the dual simplex method.

2. In order to be distinguished from the dual simplex method, the simplex method we learned earlier is also known as the <u>primal simplex method</u>.

3. The dual simplex method is equivalent to the primal simplex method applied to the dual problem.

Exercise: Write down the dual problem of the above example. Solve the dual problem by the primal simplex method.

- Geometry of the dual simplex method.

eg

$$\max\ -x_1 - x_2$$
$$\text{s.t.}\quad -x_1 - 2x_2 \le -2$$
$$-x_1 \le -1$$
$$x_1, x_2 \ge 0$$

dual

$$\min\ -2\pi_1 - \pi_2$$
$$\text{s.t.}\quad -\pi_1 - \pi_2 \ge -1$$
$$-2\pi_1 \ge -1$$
$$\pi_1, \pi_2 \ge 0$$

	z	x_1	x_2	x_3	x_4	RHS
z	1	1	1	0	0	0
x_3	0	-1	-2	1	0	-2
x_4	0	-1	0	0	1	-1

P

$\longrightarrow$

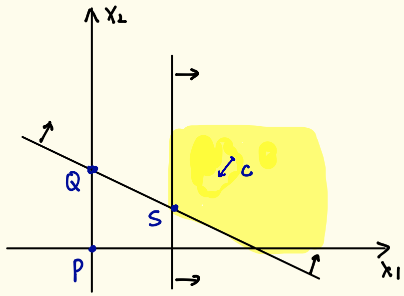
	z	x_1	x_2	x_3	x_4	RHS
z	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	-1
x_2	1	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	1
x_4	1	-1	0	0	1	-1

Q

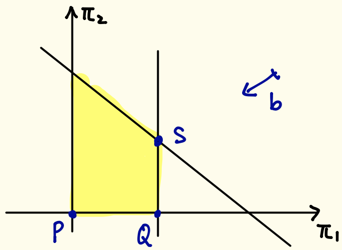
$\longrightarrow$

	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$
x_2	0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
x_1	0	1	0	0	-1	1

S



primal space



dual space