MATH 4120-6120
SPRING 2021
EXAM 1

Friday, February 12, 2021

1. Do the following computations or give definitions for each of the following.
   a) (3 pts) What is a group?
   b) (3 pts) Give the definition of a normal subgroup of the group G.
   c) (3 pts) What does it mean for a group G to act on a set S?
   d) (3 pts) State the First Isomorphism Theorem.
   e) (3 pts) State the Third Isomorphism Theorem.
   f) (3 pts) State Lagrange's Theorem.
   g) (3 pts) Let G be a group and $H \leq G$ a subgroup. What is $N_G(H)$?
   h) (3 pts) Suppose that the order of a group is 3500. For what values of n does Cauchy's Theorem guarantee an element of this group of order n.
   i) (3 pts) Given two elements x, y in the group G, what is the commutator $[x, y]$?
   j) (3 pts) Compute the element $(1\ 2\ 5\ 7)(3\ 2\ 1\ 4)(2\ 7)(1\ 3\ 2)(5\ 6\ 1\ 4\ 3)(2\ 8)(1\ 3\ 4)$ in S_8.
   k) (3 pts) What is the order of the element you found in the part j)?
   l) (3 pts) Find an element of order 15 in S_8 or explain why such an element cannot exist.
   m) (3 pts) If the group G acts on the set S and $x \in S$, what is the stabilizer of x in S?

2. Let G be a group and $H \leq G$ a subgroup.
   a) (5 pts) Show that for all $g \in G$, the set gHg^{-1} is also a subgroup of G (gHg^{-1} is called a *conjugate* of H).
   b) (5 pts) Consider the set S of all conjugates of H (that is $S = \{gHg^{-1} | g \in G\}$). Show that the function $G \times S \longrightarrow S$ given by $x \circ gHg^{-1} = x(gHg^{-1})x^{-1}$ (where $x, g \in G$) is an action of G on S.
   c) (5 pts) Find the stabilizer of H under this action.

3. Suppose that G is a finite group and $\phi : G \longrightarrow H$ is a homomorphism.
   a) (5 pts) Show that $\phi(G) = \mathrm{im}(\phi) \leq H$ is a subgroup.
   b) (5 pts) Show that $|\mathrm{im}(\phi)|$ must divide $|G|$.

4. (5 pts) Show that no group can be written as the union of two of its proper subgroups (that is, if $H, K < G$ are subgroups that are *strictly* contained in G, then $H \bigcup K$ is also strictly contained in G).

5. (5 pts) Suppose that G is a group and $N \trianglelefteq G$ is a normal subgroup with $[G : N] = p^2$. Show that there must exist a subgroup H of G containing N ($N \leq H \leq G$) with $[G : H] = p$.