Overall: C

MATH 4560/6560 Topology
Spring 2022
Exercises 1
Target date: Thu 27 Jan 11:59PM

Exercise 1. (a) Create and submit something original (meme, drawing, poem, haiku, sculpture, dance, etc.) that conveys an important aspect of the notion of a curve.

Curves

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Where do you draw the line on something so simple?
From hillsides to clouds and backwater ripples:
The images of functions on each a real interval.
Is that really all, or is there more left to say?
Is it smooth? Connected? Is it Neighborhood-safe?
Does our function map to some topological space?
And what about sequences, the uniqueness of limits?
If we approached one way, would another way fit?
Would Hausdorff commend our definition of this?
Where do you draw the line on something so simple?
The art of defining what appears to be trivial.

(b) Create and submit something original (meme, drawing, poem, haiku, sculpture, dance, etc.) that conveys an important aspect of the notion of a surface.

Surfaces

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Charted by R2
To map to each neighborhood,
The surface is made

Exercise 2. Let C denote the graph in $\mathbb{R}^2$ of the equation $y = x^3 + 1$, which is a curve. Let $P = (a, b)$ and $Q = (\alpha, \beta)$ be two arbitrary points in C.

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(a) What is the distance $d(P, Q)$ between P and Q, considered as points in $\mathbb{R}^2$?
$d(P, Q) = \sqrt{(a-\alpha)^2 + (b-\beta)^2}$, which is euclidean distance in two dimensions.

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(b) How you would define the distance $\rho(P, Q)$ between P and Q, considered as points in C, that is, taking into account the path from P to Q in C? This should be different from $d(P, Q)$.
$\rho(P, Q) = |\int_a^\alpha \sqrt{1 + 9x^4}\, \mathrm{d}x|$, which is the formula for the arc length of C from a to α.

(c) Which of the following properties are satisfied?

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(1) For all $P, Q \in C$, if $P = Q$, then $\rho(P, Q) = 0$.
Yes, as if $P = Q$ then $a = \alpha$, and hence the arc length of the curve is 0.

(2) For all $P, Q \in C$, if $\rho(P, Q)$, then $P = Q$.

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Yes, because the arc length ρ is equivalent to the absolute area under $\sqrt{1 + 9x^4}$ (recall from Calculus that this is one basis for integration), which is itself definitively positive, and hence the only way that $p(P, Q) = 0$ is if $a = \alpha$, which implies $b = \beta$ (since b and β are outputs of a rule of assignment). Thus $P = Q$.

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(3) For all $P, Q \in C$, we have $\rho(P,Q) = \rho(Q,P)$.
Yes, as the Fundamental Theory of Calculus tells us that $\int_x^y f(x)\,\mathrm{d}x = -\int_y^x f(x)\,\mathrm{d}x$ for any function $f(x)$. Given that $p(P,Q)$ is defined as the absolute integral of a function $f(x) = \sqrt{1+9x^4}$, going from P to Q is equivalent to going from Q to P, since $|\int_x^y f(x)\,\mathrm{d}x| = |-\int_y^x f(x)\,\mathrm{d}x| = |\int_y^x f(x)\,\mathrm{d}x|$.

(4) For all $P, Q, R \in C$, we have $\rho(P,R) \leq \rho(P,Q) + \rho(Q,R)$.
Yes. Suppose $\rho(P,R) = |\int_x^z \sqrt{1+9x^4}\,\mathrm{d}x|$, $\rho(P,Q) = |\int_x^y \sqrt{1+9x^4}\,\mathrm{d}x|$, and $\rho(Q,R) = |\int_y^z \sqrt{1+9x^4}\,\mathrm{d}x|$. Observe that if $z = x$, it follows that $P = R$, and thus

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$$\rho(P,R) = 0 \leq \rho(P,Q) + \rho(Q,R),$$

as $\rho(A,B) \geq 0$ for all $A, B \in C$. Suppose instead that $z > x$. Then

$$\rho(P,R) = |\int_x^z \sqrt{1+9x^4}\,\mathrm{d}x| = \int_x^z \sqrt{1+9x^4}\,\mathrm{d}x$$

since $\sqrt{1+9x^4} > 0$; and thus, by the Fundamental Theory of Calculus, we have

$$\begin{aligned}\rho(P,R) &= \int_x^y \sqrt{1+9x^4}\,\mathrm{d}x + \int_y^z \sqrt{1+9x^4}\,\mathrm{d}x \\ &\leq |\int_x^y \sqrt{1+9x^4}\,\mathrm{d}x| + |\int_y^z \sqrt{1+9x^4}\,\mathrm{d}x| \\ &= \rho(P,Q) + \rho(Q,R).\end{aligned}$$

Finally, suppose that $x > z$. Then, observe that

$$\rho(P,R) = \rho(R,P) = |\int_z^x \sqrt{1+9x^4}\,\mathrm{d}x| = \int_z^x \sqrt{1+9x^4}\,\mathrm{d}x,$$

and hence by using a similar argument we have

$$\begin{aligned}\rho(P,R) &\leq |\int_y^x \sqrt{1+9x^4}\,\mathrm{d}x| + |\int_z^y \sqrt{1+9x^4}\,\mathrm{d}x| \\ &= \rho(Q,P) + \rho(R,Q) = \rho(P,Q) + \rho(Q,R).\end{aligned}$$

(5) (Bonus) For all $P, Q \in C$ and all real numbers $\epsilon > 0$, there exists a real number $\delta > 0$ such that $d(P,Q) < \delta$ implies $\rho(P,Q) < \epsilon$.
Untested.

(6) (Bonus) For all $P, Q \in C$ and all real numbers $\epsilon > 0$, there exists a real number $\delta > 0$ such that $\rho(P,Q) < \delta$ implies $d(P,Q) < \epsilon$.
Untested.

Justify your answers.