Overall: C

MATH 4560/6560 Topology
Spring 2022
Exercises 7
Target date: Thu 17 Mar 11:59PM

Instructions:

- For each item in each exercise, make sure to include
  - any sources used,
  - any collaborators worked with, and
  - a self-assessment of your work: C (correct), R (revision needed), or M (major revision needed).
- For the entire homework set, make sure to also include:
  - a reflection on the assignment and your solutions. Reflections may include
    - · discussion of how routine or challenging the assignment was,
    - · approximation of time spent on the assignment or on individual exercises,
    - · details about particular difficulties or false starts,
    - · explanations of why solutions are incomplete or incorrect, etc.
  - a self-assessment (C/R/M) for the entire assignment.

**Exercise 1.** (a) Create and submit something original (meme, drawing, poem, haiku, sculpture, dance, etc.) that conveys an important aspect of the notion of a topological space.

(b) Look up the definition and etymology of the word "topology" in a standard English dictionary (not a mathematical text). How do these compare to our definition and use of the word "topology?"

C

Source. The definition of topology as used in prior times was in reference to plants, as in the "study of the locations where plants are found," coming from Greek roots topos (place) and -logy (discourse/science). Obviously this is very different to the mathematical context of the word topology. I believe that mathematicians took the root topos to mean something else, perhaps as "placement" (the orientation and relatedness of objects) or in the sense of "location", as in comparing localities (such as through manifolds).

**Exercise 2.** Let $S_1, S_2$ be topological spaces. Prove that $S_1 \times S_2$ is homeomorhphic to $S_2 \times S_1$.

Pf: Let $f\colon : S_1 \times S_2 \to S_2 \times S_1$ be given by

$$f(s_1, s_2) = (s_2, s_1).$$

Recall that a product $S_1 \times S_2$ has basis $\mathcal{B} = \{U_1 \times U_2 \mid U_1 \subseteq S_1 \text{ and } U_2 \subseteq S_2 \text{ are open sets}\}$. Similarly, the product $S_2 \times S_1$ has basis $\mathcal{B}_{2\times 1} = \{U_2 \times U_1 \subseteq S_2 \times S_1\}$. Suppose $B \in \mathcal{B}_{2\times 1}$ was an open element of any given basis for $S_2 \times S_1$. Then, if $B = (b_2, b_1)$,

$$f^{-1}(B) = (b_1, b_2).$$

C

Since $B = (b_2, b_1) \in U_2 \times U_1$, it therefore follows that $f^{-1}(B) = (b_1, b_2) \in U_1 \times U_2$; hence the inverse element is contained in the basis for $S_1 \times S_2$, and is thus open. Hence $f$ is continuous. Note that the argument to show that $f^{-1}$ is continuous is identical to the one above, and thus $f$ is a homeomorphism between $S_1 \times S_2$ and $S_2 \times S_1$.

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**Exercise 3.** Let $S_1, S_2$ be topological spaces. Prove that $S_1 \times S_2$ is Hausdorff if and only if $S_1$ and $S_2$ are both Hausdorff.

Pf: Suppose that $S_1$ and $S_2$ are both Hausdorff. Consider $(s_1, s_2) \in S_1 \times S_2$ and $(t_1, t_2) \in S_1 \times S_2$ such that $(s_1, s_2) \neq (t_1, t_2)$. Then either $s_1 \neq t_1$ or $s_2 \neq t_2$, or both.

Assume $s_1 \neq t_1$. Then because $S_1$ is Hausdorff, there exists neighborhoods $U_1$ of $s_1$ and $V_1$ of $t_1$ such that $U_1 \cap V_1 = \emptyset$. Note that $S_2$ is by definition open, and hence $U_1 \times S_2 \subseteq S_1 \times S_2$ and $V_1 \times S_2 \subseteq S_1 \times S_2$ are both open sets containing $(s_1, s_2)$ and $(t_1, t_2)$, respectively. Furthermore, because $U_1 \cap V_1 = \emptyset$, it follows that the two neighborhoods are disjoint, and hence $S_1 \times S_2$ is Hausdorff. Assume instead $s_2 \neq t_2$. Then because $S_2$ is Hausdorff, there exists neighborhoods $U_2$ of $s_2$ and $V_2$ of $t_2$ such that $U_2 \cap V_2$. By the same argument above, we can find neighborhoods $S_1 \times U_2$ of $(s_1, s_2)$ and $S_1 \times V_2$ of $(t_1, t_2)$ which are disjoint, and hence $S_1 \times S_2$ is Hausdorff.

Now we prove the reverse direction. Suppose that $S_1 \times S_2$ is Hausdorff. Note that both $\{x\} \times S_2$ and $S_1 \times \{y\}$, where $x \in S_1$ and $y \in S_2$ are any arbitrary elements, are homeomorphic to $S_1 \times S_2$ and are thus themselves Hausdorff (the homeomorphisms are given by $f(s_1, s_2) = (x, s_2)$ and $g(s_1, s_2) = (s_1, y)$, respectively). Additionally, note that the basic elements of any product topology $T \times \{*\}$, where $\{*\}$ is a space containing only one element, are the products $U \times \{*\}$, where $U \subseteq T$ are open sets, since the topology of $\{*\} := \{\emptyset, \{*\}\}$. Suppose $s_1$ and $t_1$ were two unique elements contained $S_1$. Since $(s_1, y) \neq (t_1, y) \in S_1 \times \{y\}$ and $S_1 \times \{y\}$ is Hausdorff, there exists basic neighborhoods $U_1 \times \{y\}$ of $(s_1, y)$ and $U_2 \times \{y\}$ of $(t_1, y)$ such that $U_1 \times \{y\} \cap U_2 \times \{y\} = \emptyset$. Since both neighborhoods share the same product of $\{y\}$, this implies $U_1 \cap U_2 = \emptyset$, and thus there exists disjoint open neighborhoods which contain $s_1$ and $t_1$, respectively. Therefore $S_1$ is Hausdorff. Suppose $s_2$ and $t_2$ were two unique elements of $S_2$. Then likewise, because $\{x\} \times S_2$ is Hausdorff, there exists disjoint basic neighborhoods containing $(x, s_2)$ and $(x, t_2)$, and hence $S_2$ is also Hausdorff.