Homework 8

Anthony Jones

1. Let G be the infinite group defined as

$$G = \langle g_n, n \in \mathbb{N} \mid g_1 = e, (g_a)^2 = g_{a-1}, 1 < a \rangle$$

In other words, for some $m \in \mathbb{N}$,

$$(g_m)^2 = g_{m-1}$$

$$(g_m)^4 = (g_{m-1})^2 = g_{m-2}$$

$$(g_m)^8 = (g_{m-1})^4 = (g_{m-2})^2 = g_{m-3}$$

$$(g_m)^{16} = (g_{m-1})^8 = \cdots = g_{m-4}$$

$$\cdots$$

$$(g_m)^{2^i} = \cdots = g_1 = e, \quad \text{for some } i \in \mathbb{Z}^+.$$

Thus it is easy to see that each element of G has order of the form p^i for $p = 2$; and hence G is a p-group.

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2. Let G be a group of order pqr.

Asoume that G is simple.

Observe, by Sylow theorems, that

$n_r \equiv 1 \bmod(r)$ and $n_r \mid pq$.

First, $n_r \neq p, q$, because $n_r \equiv 1 \bmod(r)$ implies $n_r > r > q > r$. $(r \mid (n_r - 1))$.
Second, $n_r \neq 1$, because then according to ~~so~~ corollary 6.1.4, $R \trianglelefteq G$ and thus G would not be simple. Thus $n_r = pq$.

Now consider ~~$n_p \equiv 1 \bmod(r)$ and $n_p \mid pq$~~

$n_q \equiv 1 \bmod(q)$ and $n_q \mid pr$.

By the same reasoning, $n_q \neq 1$ and $n_q \neq p$ as $n_q > q > p$. Thus, n_q is at least r.
Similarly, n_p is at least q.

Therefore there are $pq(r-1)$ elements of order r in G [number of Sylow subgroups multiplied by number of non-identity elements], and at least $r(q-1)$ elements of order q and $q(p-1)$ elements of order p in G.

Thus $$|G| \geq pq(r-1) + r(q-1) + q(p-1) + 1$$
$$= pqr - pq + rq - r + qp - 1 + 1$$
$$= pqr + rq - r.$$

Since $r > q > p$, $r > q > 1$ and $rq > r$.
Thus this is a contradiction and hence G is not simple.

3. $|G| = pq$

$\Rightarrow n_p \equiv 1 \mod(p)$ & $n_p | q \longrightarrow n_p = 1$

$\exists!$ Sylow p-subgroup of order p accounting for p elements of G.

$\Rightarrow n_q = 1 \mod(q)$ & $n_q | p \longrightarrow n_q = 1$

$\exists!$ Sylow q-subgroup of order q accounting for q elements of G.

This plus identity gives $p+q+1$ elements in G.

$pq = p+q+1$

$q \not\equiv 1 \mod(p)$ $\quad q / p$

$p | pq \Rightarrow p | (p+q+1)$

$q \neq kp+1$ $\quad R(q/p) = R(1/p) = 1$

$p | (q+1)$ $\quad a = p(q+1) = pq + p$

$q \equiv 1 \mod(p)$

$q+1 = pq$

$p | (q-1)$

$R(n_r/r)$ and $R(1/r)$

$p | (p+q+1) \Rightarrow p | (q+1)$

$r(n_r - 1)$

$80 = 8 \cdot 10 = 2^5 \cdot 5 = 2^4 \cdot 5$

4. Let G be a ~~simple~~ group of order 80.

Then there are either 1 or 5 Sylow-2 subgroups as $80 = 2^4 \cdot 5$.

If there is only one Sylow-2 subgroup, then G is not simple because of corollary 6.1.4. Therefore there are 5 Sylow-2 subgroups of G.

Let $\phi : G \to S_5$ be a homomorphism defined as the collection of cosets of each of these subgroups

$$\phi(g) \rightarrow \{gH_1, gH_2, gH_3, gH_4, gH_5\}$$

where $H \leq G$ are the Sylow subgroups. Then the kernal of ϕ is easily shown as non-trivial, as the $\operatorname{im}(\phi)$ does not injectively map to G as $|G| = 80$ and $|S_5| = 120$ $[80 \nmid 120]$

Thus the kernal is a normal subgroup of G and G is not simple.

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5. Because G is simple, there are not any Sylow groups unique to another. Thus as $72 = 2^3 3^2$

The number of Sylow 3-subgroups is defined by the Sylow Theorems as

$$n_3 \equiv 1 \bmod (3) \text{ and } n_3 \mid 8.$$

Therefore $n_3 = 2$, $n_3 = 4$, or $n_3 = 8$.

However $(n_3 - 1) = 3$, so the only possibility is $n_3 = 4$. Recall that $n_3 \mid 1$ is not an option from corollary 6.1.4.

Therefore $n_3 = 4$ and there are 4 Sylow 3 subgroups of G.