HW6 MATH 4000

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1 Exercises

6.9 a) Note that $f(x,y)$ is nonnegative for all $x \in (0,1), y \in (0,2)$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y\mathrm{d}x = \int_0^1\int_0^2 \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\mathrm{d}y\mathrm{d}x$$

$$= \int_0^1 \frac{6}{7}\left(2x^2+x\right)\mathrm{d}x = \frac{6}{7}\left(\frac{2}{3}+\frac{1}{2}\right) = 1$$

And thus f is indeed a valid joint density function.

b)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y = \frac{6}{7}\left(2x^2+x\right)$$

c)

$$P\{X > Y\} = \int_0^1\int_0^x \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\mathrm{d}y\mathrm{d}x$$

$$= \int_0^1 \frac{6}{7}\left(x^3+\frac{x^3}{4}\right)\mathrm{d}x = \frac{6}{7}*\frac{5}{4}\left(\frac{1}{4}\right) = \frac{15}{56}$$

d) Using the definition of conditional probability:

$$P(Y > \frac{1}{2}|X < \frac{1}{2}) = \frac{P(Y > \frac{1}{2}, X < \frac{1}{2})}{P(X < \frac{1}{2})}$$

Thus we find

$$P(Y > \frac{1}{2}, X < \frac{1}{2}) = \int_0^{1/2}\int_{1/2}^2 \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\mathrm{d}y\mathrm{d}x = \frac{69}{448}$$

and

$$P(X < \frac{1}{2}) = \int_0^{1/2}\int_0^2 \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\mathrm{d}y\mathrm{d}x = \frac{5}{28};$$

hence the probability is $P(Y > \frac{1}{2}|X < \frac{1}{2}) = \frac{69*28}{5*448} \approx 0.8625$.

e)

$$E(X) = \int_0^1 \int_0^2 x * \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \mathrm{d}y\mathrm{d}x = \int_0^1 x * \frac{6}{7}\left(2x^2 + x\right) \mathrm{d}x = \frac{5}{7}$$

(Note that we integrate first in terms of y, to first derive f_X).

f)

$$E(Y) = \int_0^2 \int_0^1 y * \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \mathrm{d}x\mathrm{d}y = \int_0^2 y * \frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right) \mathrm{d}y = \frac{8}{7}$$

(Note that we integrate first in terms of x, to first derive f_Y).

6.14 Let X be the uniform random variable for the location on $(0, L)$ where the accident occurs, and let Y be the uniform random variable for the ambulance's location, also on $(0, L)$. Then because X and Y are independent, we find that

$$f(x, y) = f_X(x) * f_Y(y) = \frac{1}{L^2}.$$

Define $D = |X - Y|$ to be the distance between the two locations. Then we wish to find

$$F(d) = P\{D \leq d\} = P\{|X - Y| \leq d\}$$

for all $d \in (0, L)$. Consider that by definition

$$P\{|X - Y| \leq d\} = \int\int_{|X-Y|\leq d} f(x, y)\,\mathrm{d}y\mathrm{d}x = \frac{1}{L^2}\int\int_{|X-Y|\leq d} 1\,\mathrm{d}y\mathrm{d}x.$$

Also consider however that

$$\frac{1}{L^2}\int\int_{|X-Y|\leq d} 1\,\mathrm{d}y\mathrm{d}x = \frac{1}{L^2}\left(1 - \int\int_{|X-Y|>d} 1\,\mathrm{d}y\mathrm{d}x\right).$$

Thus we want to know for what values $|x - y| > d$. Suppose d was given. If we take $x, y \in (0, L)$ as each a coordinate inside the complex plane, then we consider now the area for which $|x - y| > d$. Clearly when $x = 0$, we must have $y > d$; and similarly when $y = L$, we must have $x < L - d$. Thus there is a region of (x, y) bounded by $x = 0$, $y = L$, and the diagonal between $(0, d)$ and $(L - d, L)$. Similarly, as the variables are uniform, we find a symmetrical region bounded by $y = 0$, $x = L$, and the diagonal between $(d, 0)$ and $(L, L - d)$. Hence

$$1 - \int\int_{|X-Y|>d} 1\,\mathrm{d}y\mathrm{d}x = 2d(L - d),$$

the square formed by the bounded regions. Thus

$$F(d) = \frac{2d(L - d)}{L^2}.$$

6.16 a) Suppose A occurred. Then note we can simply transition the beginning of the semicircle to be whichever point is nearest clockwise. At most this translation is less than πr, as we know all points fit within half the circumference of the circle. Furthermore, no points lie in the second half, by definition; hence A occurs as long as one of A_i occurs.

$$A = \bigcup_{i=1}^{n} A_i$$

b) Assuming that no points are allowed to be at the same location, then each A_i is exclusive. Suppose A_i occurs. Then if we translate the beginning of the semicircle to be the next point P_{i+1}, it follows by the same logic above that P_i is no longer contained.

c) Define P_1 to be the point furthest from all other points traveling clockwise. Then from the two statements above, it follows that $P(A) = n * P(A_1)$, since each point is equally likely to be the first. Note for all points $P_2, P_3, \ldots, P_n$, either the point belongs in the semicircle or it doesn't, with each likelihood being $1/2$. Hence the likelihood that all other points are within the semicircle is

$$P(A_1) = (1/2)^{n-1}$$

and thus $P(A) = n * (1/2)^{n-1}$.

6.39 If i is the largest of the two rolls, we know that the other roll is either equal to or less than i. Additionally, the second roll is equally split between all its possibilities. Thus $P\{Y = k|X = i\} = 1/i$ if $k \leq i$, and 0 otherwise. Notably, this means that the two are not independent, as if they were then $P\{Y = k|X = i\} = P\{Y = k\} = 1/6$. Clearly this is not the case whenever $i \neq 6$.

6.58 First consider $Y_1 = X_1 + X_2$. As Y_1 is the sum of two independent random variables,

$$f_{Y_1}(t) = \int_{-\infty}^{\infty} f_{X_1}(t - x_2) * f_{X_2}(x_2)\,\mathrm{d}x_2.$$

Note by definition $f_{X_1} = 0$ whenever $t - x_2 < 0$ and $f_{X_2} = 0$ whenever $x_2 < 0$; hence for $t \geq x_2 \geq 0$ we find

$$\int_{-\infty}^{\infty} f_{X_1}(t - x_2) * f_{X_2}(x_2)\,\mathrm{d}x_2 = \int_0^t \lambda e^{-\lambda(t-x_2)} * \lambda e^{-\lambda x_2}\,\mathrm{d}x_2 = \lambda^2 t e^{-\lambda t}.$$

Next consider $Y_2 = e^{X_1}$. Then

$$F_{Y_2}(t) = P\{Y_2 \leq t\} = P\{X_1 \leq \ln(t)\}$$

$$= \int_0^{\ln(t)} \lambda e^{-\lambda x}\,\mathrm{d}x = 1 - te^{-\lambda}.$$

Hence

$$f_{Y_2}(t) = \frac{\mathrm{d}}{\mathrm{d}t} = -e^{\lambda}.$$

As functions of random variables, note $Y_1 = g_1(x_1, x_2) = x_1 + x_2$ and $Y_2 = g_2(x_1, x_2) = \exp(x_2)$. Hence their joint density is given as

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(x_1, x_2)|J(x_1, x_2)|^{-1},$$

where J is defined as the determinate of partial derivatives of g_1 and g_2. Because X_1 and X_2 are independent, we find

$$f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1) * f_{X_2}(x_2);$$

and from the equations above find

2 Theoretical Exercises

6.22 By Bayes Theorem, we find that

$$f(w|x_1, x_2, \ldots, x_n) = \frac{f(x_1, x_2, \ldots, x_n|w) * f_W(w)}{f(x_1, x_2, \ldots, x_n)}.$$

Since whenever $W = w$, each X_i is an independent exponential variable with rate w, this yields

$$\frac{f(x_1, x_2, \ldots, x_n|w) * f_W(w)}{f(x_1, x_2, \ldots, x_n)} = \frac{we^{-wx_1} * we^{-wx_2} * \ldots * we^{-wx_n} * f_W(w)}{f(x_1, x_2, \ldots, x_n)}$$

$$= \frac{w^n e^{-w * \sum x_i} * f_W(w)}{f(x_1, x_2, \ldots, x_n)}.$$

Note that as W is a gamma random variable with parameters (t, β),

$$f_W(w) = \frac{\beta e^{-\beta w}(\beta w)^{t-1}}{\Gamma(t)}$$

and hence

$$f(w|x_1, x_2, \ldots, x_n) = \left(\frac{\beta^t}{f(x_1, x_2, \ldots, x_n)}\right)\left(\frac{e^{-(\beta + \sum x_i)w} w^{t+n-1}}{\Gamma(t)}\right).$$

Considering when X_i has a rate equal to y,

$$f(x_1, x_2, \ldots, x_n) * \Gamma(t) = \int_0^\infty ye^{-yx_i}\,\mathrm{d}x_i * \int_0^\infty e^{-y}y^{t-1}\,\mathrm{d}y$$

$$= y^n \int_0^\infty e^{-yx_i}\,\mathrm{d}x_i * \int_0^\infty e^{-y}y^{t-1}\,\mathrm{d}y = \int_0^\infty e^{-yx_i}\,\mathrm{d}x_i * \Gamma(t+n),$$

and hence it follows that W has a gamma distribution with parameters $(t + n, \beta + \sum x_i)$.

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