Newton's laws

$$\underbrace{\begin{bmatrix}0&1\\0&0\end{bmatrix}}_{A} \quad \underbrace{\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}}_{B}$$

$$\dot{X}(t) = A X(t) + B u(t) = f(x(t), u(t))$$

minimize over u $\displaystyle\int_0^{T_f} \underbrace{\left(X(t)^T Q X(t) + u(t)^T R u(t)\right)}_{c(x(t),u(t))} dt + \underbrace{X(T_f)^T S_{T_f}^{T_f} X(T_f)}_{V^{T_f}(T_f, X(T_f))}$

transpose, big T_f

stagewise cost

Final cost

in order to calculate the optimal control strategy, we'll calculate the "Value function":

$$V^{T_f}(t,x) = \min \int_t^{T_f} c(\underline{x}(\tau), u(\tau))\, d\tau + V^{T_f}(T_f, \underline{X}(T_f))$$

s.t. $\underline{X}(t) = x$, $\dot{\underline{x}}(\tau) = A\underline{X}(\tau) + Bu(\tau)$

Encodes best achievable cost if $X(t) = x$

Intuitive examples

- In navigation, value might encode shortest path length to goal from current location
- In finance, the largest expected eventual wealth given current assets.

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if $u^*(t)$ is an optimal input trajectory Then

$$V^{T_f}(t,x) = \int_t^{T_f} c(\underline{x}^*(\tau), u^*(\tau))\, d\tau + V^{T_f}(T_f, \underline{x}^*(T_f))$$

where $x^*(t) = x$, $\dot{\underline{x}}^*(\tau) = A\,\underline{x}^*(\tau) + Bu^*(\tau)$

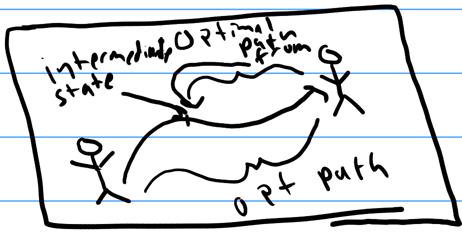
($V^{T_f}(T_f, x)$ is the value function a T_f)

if $h > 0$ $t + h \le T_f$

then $V^{T_f}(t+h, \underline{x}^*(t+h)) =$

if $0<h$, $t+h \le T_f$ then:

$$V^{T_f}(t+h, \underline{x}^*(t+h)) = \int_{t+h}^{T_f} c(\underline{x}^*(\tau), u^*(\tau))\, d\tau + V^{T_f}(T_f, \underline{x}^*(T_f))$$

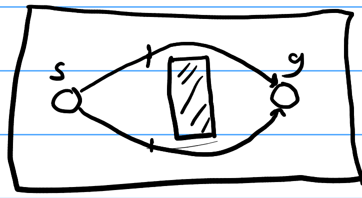


if $u^*(\tau)$ were not optimal for $\tau \ge t+h$ when starting at $\underline{x}^*(t+h)$ at time $t+h$

then a better control, $\hat{u}(\tau)$ exists and using an input of form

- $u^*(\tau)$ for $\tau \in [t, t+h)$
- $\hat{u}(\tau)$ for $\tau \in [t+h, T_f)$

would give a better value



$$\underbrace{V^{T_f}(t,x) = \int_t^{t+h} c(\underline{x}^*(\tau), u^*(\tau))\, d\tau + V^{T_f}(t+h, \underline{x}^*(t+h))}_{\text{Bellman Equation}}$$

For continuous time we can do some approximations $\quad o(h) = \text{some } g(h) \text{ s.t. } \frac{g(h)}{h} \to 0$

$$\int_t^{t+h} c(\underline{x}^*(\tau), u^*(\tau))\, d\tau = h\, c(x, u^*(t)) + o(h)$$

via Euler

$$\underline{x}^*(t+h) = x + h f(x, u^*(t)) + o(h) \quad \left(\text{since } \dot{\underline{x}}^*(\tau) = f(\underline{x}^*(\tau), u^*(\tau))\right)$$

Via Taylor

$$V^{T_f}(t+h, \underline{x}^*(t+h)) = V^{T_f}(t,x) + h \frac{\partial V^{T_f}}{\partial t}(t,x) + h \frac{\partial V^{T_f}}{\partial x}(t,x) f(x, u^*(t)) + o(h)$$

$$V^{T_f}(t,x) = V^{T_f}(t,x) + h\left(c(x, u^*(t)) + \frac{\partial V^{T_f}}{\partial t}(t,x) + \frac{\partial V^{T_f}}{\partial x}(t,x) f(x, u^*(t))\right) + o(h)$$

$$\Rightarrow \quad 0 = c(x, u^*(t)) + \frac{\partial V^{T_f}}{\partial t}(t,x) + \frac{\partial V^{T_f}}{\partial x}(t,x) f(x, u^*(t))$$

Slightly different form of Bellman

$$V^{T_f}(t,x) = \min_u \int_t^{t+h} c(\underline{x}(\tau), u(\tau))\, d\tau + V^{T_f}(t+h, \underline{x}(t+h))$$

$$\Rightarrow \quad \boxed{-\frac{\partial V^{T_f}(t,x)}{\partial t} = \min_u \left(c(x,u) + \frac{\partial V^{T_f}}{\partial x}(t,x) f(x,u) \right)}$$

Hamilton-Jacobi-Bellman Equation

return to LQR

$f(x,u) = Ax + Bu$ $\qquad R \succ 0,\ Q \succeq 0$

$c(x,u) = x^T Q x + u^T R u$ $\qquad S_{T_f}^{T_f} \succeq 0$

Guess: $V^{T_f}(t,x) = x^T S_t^{T_f} x$ for some positive semidefinite $S_t^{T_f}$

Aside: A matrix $M \succeq 0$ is positive semidefinite if
- $M = M^T$
- $x^T M x \geq 0 \quad \forall x$

it is positive definite if $(M \succ 0)$
- $M = M^T$
- $x^T M x > 0 \quad \forall x \neq 0$

positive definite $\Rightarrow$ all eigenvalues are positive
$\Rightarrow$ M is invertible

Plug into HJB

$$-x^T \frac{\partial S_t^T}{\partial t} x = \min_u \left(x^T Q x + u^T R u + 2 x^T S_t^{T_f}(Ax + Bu) \right)$$

To get minimum, we take derivative WRT u and set to 0:

symmetric

$$2 u^T R + 2 x^T S_t^{T_f} B = 0 \Rightarrow u = -R^{-1} B^T S_t^{T_f} x$$

Plug back in:

$$-x^T \frac{\partial S_t^T}{\partial t} x = x^T Q x + x^T S_t^{T_f} B R^{-1} B^T S_t^{T_f} x$$
$$+ 2 \left(x^T S_t^{T_f} (Ax - B R^{-1} B^T S_t^{T_f} x) \right)$$
$$= x^T \left(Q + A^T S_t^{T_f} + S_t^{T_f} A - S_t^{T_f} B R^{-1} B^T S_t^{T_f} \right) x$$

continuous-time Riccati equation:

$$-\frac{\partial S_t^{T_f}}{\partial t} = Q + A^T S_t^{T_f} + S_t^{T_f} A - S_t^{T_f} B R^{-1} B^T S_t^{T_f}$$