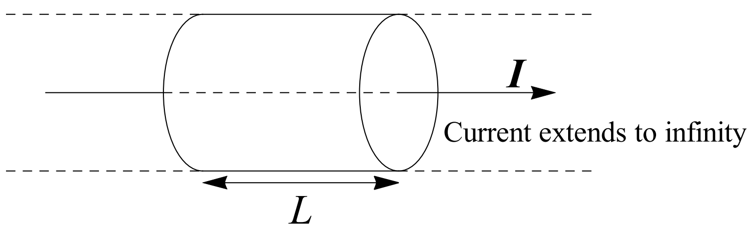
PC2131 (AY2012/2013 sem 2)
Suggested solutions

Question 1

1a. We shall use cylindrical coordinates, with the cylinder's axis centred on the z-axis and $\boldsymbol{I}$ pointing in the positive z-direction.

For ease of analysis, it is assumed that the cylinder under consideration is only a section of an infinite current-carrying cylinder with steady current (see diagram). In that case, the situation is magnetostatic since there are no changes in charge density anywhere, and all currents are steady.



For the electric field on the surface of the cylinder, letting the conductivity of the material be σ, we have $\boldsymbol{J} = \sigma \boldsymbol{E}$, i.e. $\boldsymbol{E} = \frac{I}{\sigma \pi R^2} \hat{\boldsymbol{e}}_{\boldsymbol{z}}$.

(This next part is simply rederiving the magnetic field of an infinite current-carrying cylinder, so perhaps it might not be critical to present all steps in so much detail. However, the groundwork of this argument here is used again in questions 3 and 4.)

Since the situation is magnetostatic, we can apply the Biot-Savart law, from which we note that $\boldsymbol{B}$ is linearly related to $\boldsymbol{J}$ and thus $\boldsymbol{I}$. Hence by symmetry, $\boldsymbol{B}$ can have no radial component, because if it did, reversing the direction of $\boldsymbol{I}$ would reverse the direction of this radial component (since $\boldsymbol{B}$ is linearly related to $\boldsymbol{I}$), but reversing $\boldsymbol{I}$ is equivalent to flipping the wire upside-down and thus cannot affect the radial component of $\boldsymbol{B}$.

The Biot-Savart law also shows that $\boldsymbol{B}$ has no component in the z-direction, because $\boldsymbol{I}$ is in the z-direction everywhere and $\boldsymbol{B}$ is related to $\boldsymbol{I}$ by a cross product, thus it must be perpendicular to $\boldsymbol{I}$ and hence the z-direction.

Therefore $\boldsymbol{B}$ is purely "circumferential", i.e. $\boldsymbol{B} = B(\boldsymbol{r}) \hat{\boldsymbol{e}}_{\boldsymbol{\phi}}$.

By rotational symmetry about the z-axis, $B(\boldsymbol{r})$ cannot depend on ϕ, and by translational symmetry along the z-axis, $B(\boldsymbol{r})$ cannot depend on z. Therefore $\boldsymbol{B} = B(s) \hat{\boldsymbol{e}}_{\boldsymbol{\phi}}$.

Applying Ampere's law with $\frac{d\boldsymbol{E}}{dt} = 0$ to a circular Amperian loop of radius $s = R$ coaxial with the z-axis, we have (since $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = B(R)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} \cdot \mathrm{d}l\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} = B(R)\,\mathrm{d}l$ everywhere along the loop)

$$\int (\nabla \times \boldsymbol{B}) \cdot \mathrm{d}^2\boldsymbol{r} = \int (\mu_0 \boldsymbol{J} + 0) \cdot \mathrm{d}^2\boldsymbol{r} \implies B(R)(2\pi R) = \mu_0 I$$

$$B(R) = \frac{\mu_0 I}{2\pi R}$$

Thus the magnetic field on the surface of the cylinder is $\boldsymbol{B} = \frac{\mu_0 I}{2\pi R}\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}$.

1b. Since $\hat{\boldsymbol{e}}_{\boldsymbol{z}} \times \hat{\boldsymbol{e}}_{\boldsymbol{\phi}} = -\hat{\boldsymbol{e}}_{\boldsymbol{s}}$, we have the Poynting vector

$$\begin{aligned}\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B}) &= \frac{1}{\mu_0}\left(\frac{I}{\sigma\pi R^2}\hat{\boldsymbol{e}}_{\boldsymbol{z}} \times \frac{\mu_0 I}{2\pi R}\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\right) \\ &= -\frac{I^2}{2\sigma\pi^2 R^3}\hat{\boldsymbol{e}}_{\boldsymbol{s}}, \text{ showing that the vector Poynts into the cylinder}\end{aligned}$$

Integrating over the cylindrical surface to find the power flowing into the cylinder, we have (in terms of magnitude)

$$\begin{aligned}P = \left|\int \boldsymbol{S} \cdot \mathrm{d}^2\boldsymbol{r}\right| &= \left|\int \left(-\frac{I^2}{2\sigma\pi^2 R^3}\hat{\boldsymbol{e}}_{\boldsymbol{s}}\right) \cdot \left(\mathrm{d}^2 r\hat{\boldsymbol{e}}_{\boldsymbol{s}}\right)\right| \\ &= \left|-\left(\frac{I^2}{2\sigma\pi^2 R^3}\right)(2\pi R L)\right| \\ &= \frac{I^2 L}{\sigma\pi R^2} \\ &= I\left(\frac{I}{\sigma\pi R^2}\right)L \\ &= I\,|\boldsymbol{E}|\,L \\ &= IV\end{aligned}$$

Question 2

2a. We shall use spherical coordinates, with the origin at the centre of the ball.

By spherical symmetry, the fields must be radial and their magnitudes cannot depend on either θ or ϕ, i.e. we have $\boldsymbol{E} = E(r)\hat{\boldsymbol{e}}_{\boldsymbol{r}}$ and $\boldsymbol{D} = D(r)\hat{\boldsymbol{e}}_{\boldsymbol{r}}$.

Applying Gauss' law in materials to a spherical Gaussian surface of radius r centred on the origin, we have (since $\boldsymbol{D} \cdot \mathrm{d}^2\boldsymbol{r} = D(r)\hat{\boldsymbol{e}}_{\boldsymbol{r}} \cdot \mathrm{d}^2 r\hat{\boldsymbol{e}}_{\boldsymbol{r}} = D(r)\,\mathrm{d}^2 r$ everywhere on the surface)

$$\int (\nabla \cdot \boldsymbol{D})\,\mathrm{d}^3 r = \int \rho_f\,\mathrm{d}^3 r \implies D(r)\left(4\pi r^2\right) = \begin{cases} \rho\left(\frac{4}{3}\pi r^3\right) & \text{for } r < R \\ \rho\left(\frac{4}{3}\pi R^3\right) & \text{for } r > R \end{cases}$$

$$D(r) = \begin{cases} \left(\frac{\rho}{3}\right) r & \text{for } r < R \\ \left(\frac{\rho}{3}\right) \frac{R^3}{r^2} & \text{for } r > R \end{cases}$$

Since the dielectric is linear, we have $\boldsymbol{D} = \epsilon_0\epsilon_r\boldsymbol{E}$ inside the sphere ($r < R$) and $\boldsymbol{D} = \epsilon_0\boldsymbol{E}$ outside ($r > R$), therefore

$$E(r) = \begin{cases} \left(\frac{\rho}{3\epsilon_0\epsilon_r}\right) r & \text{for } r < R \\ \left(\frac{\rho}{3\epsilon_0}\right) \frac{R^3}{r^2} & \text{for } r > R \end{cases}$$

Hence for the potential difference ΔV between the centre and surface of the sphere, we have (integrating via a radial path from $r = R$ to $r = 0$)

$$\begin{aligned} \Delta V = \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} &= \int_R^0 \left(E(r)\hat{\boldsymbol{e}}_{\boldsymbol{r}}\right) \cdot \left(-\mathrm{d}r\hat{\boldsymbol{e}}_{\boldsymbol{r}}\right) \\ &= \int_0^R \left(\frac{\rho}{3\epsilon_0\epsilon_r} r\right)\,\mathrm{d}r \\ &= \frac{\rho R^2}{6\epsilon_0\epsilon_r} \end{aligned}$$

(One can verify that for $\epsilon_r = 1$ (i.e. effectively no dielectric), the above expressions reduce to those for a uniformly charged ball without the dielectric.)

2b. The energy stored in the electric field is

$$
\begin{aligned}
U = \int_{\text{all space}} \left(\frac{1}{2}\epsilon_0 E^2\right) \mathrm{d}^3 r &= \frac{1}{2}\epsilon_0 \left(\int_0^R \left(\frac{\rho}{3\epsilon_0\epsilon_r} r\right)^2 \left(4\pi r^2\right) \mathrm{d}r + \int_R^\infty \left(\frac{\rho}{3\epsilon_0}\frac{R^3}{r^2}\right)^2 \left(4\pi r^2\right) \mathrm{d}r\right) \\
&= \frac{2\pi}{9}\frac{\rho^2}{\epsilon_0}\left(\int_0^R \frac{r^4}{\epsilon_r^2}\,\mathrm{d}r + \int_R^\infty \frac{R^6}{r^2}\,\mathrm{d}r\right) \\
&= \frac{2\pi}{9}\frac{\rho^2}{\epsilon_0}\left(\frac{1}{\epsilon_r^2}\left[\frac{r^5}{5}\right]_0^R + R^6\left[-\frac{1}{r}\right]_R^\infty\right) \\
&= \frac{2\pi}{9}\frac{\rho^2}{\epsilon_0} R^5 \left(\frac{1}{5\epsilon_r^2} + 1\right)
\end{aligned}
$$

(Again, one can verify that for $\epsilon_r = 1$, this expression reduces to the energy of a uniformly charged ball without the dielectric.)

Remark: We note that since the question asks for the energy stored in the *field*, we use $\frac{1}{2}\epsilon_0 E^2$ rather than $\frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}$ for the energy density. The latter instead gives the energy it would take to assemble the uniformly charged ball if the dielectric was present throughout the assembly process (while the former, the energy stored in the field, gives the energy it would take to create a charge distribution that matches the final charge distribution produced by the ball+dielectric system).

Question 3

3a. We shall use cylindrical coordinates, with the wire's axis centred on the z-axis and $\boldsymbol{I}$ pointing in the positive z-direction.

As there are no changes in charge density anywhere and all currents are steady, it is a magnetostatic situation.

Since the material is linear, $\boldsymbol{H}$ is linearly related to $\boldsymbol{B}$ and thus $\boldsymbol{I}$. Hence by an argument similar to that in question 1a, we can show that $\boldsymbol{H}$ is purely "circumferential" and has magnitude independent of ϕ and z, i.e. we have $\boldsymbol{H} = H(s)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}$, and similarly $\boldsymbol{B} = B(s)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}$.

Applying Ampere's law in materials with $\frac{d\boldsymbol{D}}{dt} = 0$ to an Amperian loop of radius s coaxial with the z-axis, we have (since $\boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = H(s)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} \cdot \mathrm{d}l\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} = H(s)\,\mathrm{d}l$ everywhere along the loop)

$$\int (\nabla \times \boldsymbol{H}) \cdot \mathrm{d}^2\boldsymbol{r} = \int (\boldsymbol{J_f} + 0) \cdot \mathrm{d}^2\boldsymbol{r} \implies H(s)(2\pi s) = \begin{cases} I\left(\dfrac{s^2}{R^2}\right) & \text{for } s < R \\ I & \text{for } s > R \end{cases}$$

$$H(s) = \begin{cases} \left(\dfrac{I}{2\pi}\right)\dfrac{s}{R^2} & \text{for } s < R \\ \left(\dfrac{I}{2\pi}\right)\dfrac{1}{s} & \text{for } s > R \end{cases}$$

Since the material is linear, we have $\boldsymbol{B} = \mu_0\mu_r\boldsymbol{H}$ inside the wire ($s < R$) and $\boldsymbol{B} = \mu_0\boldsymbol{H}$ outside ($s > R$), therefore in summary, we have

$$\boldsymbol{H} = \begin{cases} \left(\dfrac{I}{2\pi}\dfrac{s}{R^2}\right)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} & \text{for } s < R \\ \left(\dfrac{I}{2\pi}\dfrac{1}{s}\right)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} & \text{for } s > R \end{cases} \qquad \boldsymbol{B} = \begin{cases} \left(\dfrac{\mu_0\mu_r I}{2\pi}\dfrac{s}{R^2}\right)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} & \text{for } s < R \\ \left(\dfrac{\mu_0 I}{2\pi}\dfrac{1}{s}\right)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} & \text{for } s > R \end{cases}$$

3b. Since $\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$, we can obtain $\boldsymbol{M}$ from the results of part (a):

$$\begin{aligned} \boldsymbol{M} &= \begin{cases} (\mu_r - 1)\left(\dfrac{I}{2\pi}\dfrac{s}{R^2}\right)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} & \text{for } s < R \\ 0 & \text{for } s > R \end{cases} \\ &= \begin{cases} \left(\dfrac{\chi_m I}{2\pi}\dfrac{s}{R^2}\right)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} & \text{for } s < R \\ 0 & \text{for } s > R \end{cases} \quad \text{since } \chi_m = \mu_r - 1 \end{aligned}$$

Since $\boldsymbol{J_b} = \nabla \times \boldsymbol{M}$ and $\boldsymbol{K_b} = \boldsymbol{M} \times \hat{\boldsymbol{n}}$, we hence have the bound currents

$$\begin{aligned}
\boldsymbol{J_b} &= \nabla \times \left(\frac{\chi_m I}{2\pi} \frac{s}{R^2} \,\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} \right) \text{ inside the wire} \\
&= \left[-\frac{\partial}{\partial z} \left(\frac{\chi_m I}{2\pi} \frac{s}{R^2} \right) \right] \hat{\boldsymbol{e}}_{\boldsymbol{s}} + \frac{1}{s} \left[\frac{\partial}{\partial s} \left(s \frac{\chi_m I}{2\pi} \frac{s}{R^2} \right) \right] \hat{\boldsymbol{e}}_{\boldsymbol{z}} \\
&= \left(\frac{\chi_m I}{\pi R^2} \right) \hat{\boldsymbol{e}}_{\boldsymbol{z}}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{K_b} &= \left(\frac{\chi_m I}{2\pi} \frac{R}{R^2} \,\hat{\boldsymbol{e}}_{\phi} \right) \times \hat{\boldsymbol{e}}_{\boldsymbol{s}} \text{ on the surface of the wire, since } s = R \text{ and } \hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_{\boldsymbol{s}} \text{ there} \\
&= -\left(\frac{\chi_m I}{2\pi R} \right) \hat{\boldsymbol{e}}_{\boldsymbol{z}}, \text{ since } \hat{\boldsymbol{e}}_{\boldsymbol{\phi}} \times \hat{\boldsymbol{e}}_{\boldsymbol{s}} = -\hat{\boldsymbol{e}}_{\boldsymbol{z}}
\end{aligned}$$

3c. Considering any perpendicular cross-section of the wire, the current through the cross-section is (since $\boldsymbol{J_b}$ and $\boldsymbol{K_b}$ are both perpedicular to such a cross-section)

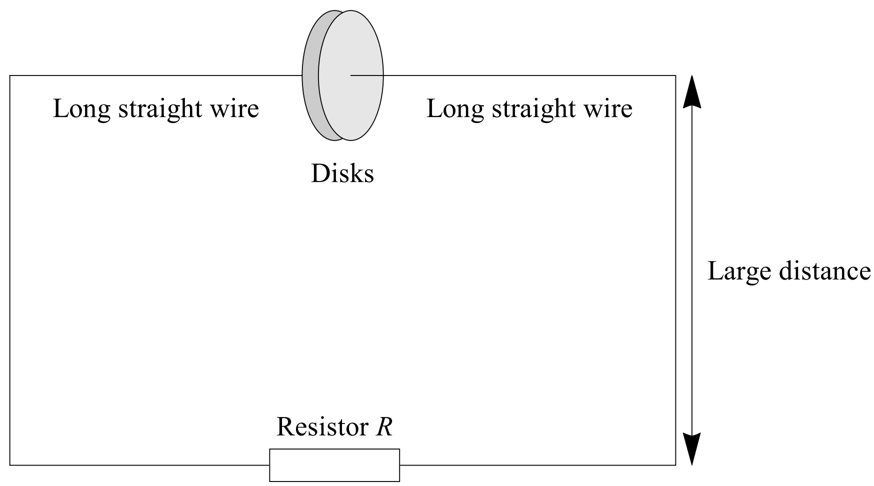
$$\begin{aligned}
\boldsymbol{I} &= \int \boldsymbol{J_b} \,\mathrm{d}^2 r + \int \boldsymbol{K_b} \,\mathrm{d}l \\
&= \int \left(\frac{\chi_m I}{\pi R^2} \right) \hat{\boldsymbol{e}}_{\boldsymbol{z}} \,\mathrm{d}^2 r + \int \left(-\frac{\chi_m I}{2\pi R} \right) \hat{\boldsymbol{e}}_{\boldsymbol{z}} \,\mathrm{d}l \\
&= \left(\left(\frac{\chi_m I}{\pi R^2} \right) (\pi R^2) + \left(-\frac{\chi_m I}{2\pi R} \right) (2\pi R) \right) \hat{\boldsymbol{e}}_{\boldsymbol{z}}, \text{ since } \hat{\boldsymbol{e}}_{\boldsymbol{z}} \text{ is independent of position} \\
&= 0
\end{aligned}$$

Therefore the total bound current along the wire is zero.

Question 4

4a. (To avoid notational ambiguity, we shall instead use L for the radius of the disks and reserve R for the resistance of the resistor.)

For ease of analysis, we shall assume that $d << L$, such that fringe effects can be ignored and the expression $C = \frac{\epsilon_0 \pi L^2}{d}$ is valid. We also assume that the wires leading to the capacitor disks are straight and connected perpendicularly to the centres of the disks, and extend outwards for distances that are large relative to the dimensions of the disks (see diagram).



The region near the capacitor then possesses approximate cylindrical symmetry, and thus we shall use cylindrical coordinates, with the z-axis coaxial to the disks. To match the sign convention defined in the question by $\boldsymbol{E}(t) = \frac{V_0}{d} e^{-\frac{t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{z}}$, the positive z-direction must be chosen to point from the positively charged disk to the negatively charged disk.

If the charging/discharging is "sufficiently slow" (which is indeed the case for typical situations, cf. Griffiths 3rd edn., page 308), the situation can be treated as effectively magnetostatic (i.e. quasistatic).

For $d << L$, the charge is approximately evenly distributed across the disks, and the electric field between the disks is approximately that produced by two infinite parallel uniformly-charged planes of opposite sign, i.e. uniform and perpendicular to the disks. Therefore we have $\boldsymbol{E}(t) \approx \frac{V_0}{d} e^{-\frac{t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{z}}$ throughout the entire region between the disks, and hence the displacement current in that region during the discharge is approximately

$$\begin{aligned}\boldsymbol{J_d}(t) = \epsilon_0 \frac{d\boldsymbol{E}}{dt} &\approx \epsilon_0 \frac{d}{dt}\left(\frac{V_0}{d} e^{-\frac{t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{z}}\right) \\ &= -\frac{\epsilon_0 V_0}{RCd} e^{-\frac{t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{z}}\end{aligned}$$

(and approximately zero elsewhere.)

By an argument similar to that in question 1a (since the region near the capacitor possesses approximate cylindrical symmetry, and is quasistatic so the Biot-Savart law can be applied), we can show that $\boldsymbol{B}$ is approximately "circumferential", and has magnitude approximately independent of ϕ and z in the region between the disks (as long as $\boldsymbol{E}(t)$, and hence $\boldsymbol{J_d}(t)$, is approximately uniform in this region). Therefore, we have $\boldsymbol{B}(\boldsymbol{r},t) \approx B(s,t)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}$.

Consider a circular Amperian loop of radius $s < L$ coaxial with the z-axis and lying between the disks. Applying Ampere's law and choosing to integrate over the planar surface enclosed by the loop, we have $\boldsymbol{J} = 0$ everywhere over the surface, and therefore (since $\boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = B(s,t)\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} \cdot \mathrm{d}l\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} = B(s,t)\,\mathrm{d}l$ everywhere along the loop)

$$\int (\nabla \times \boldsymbol{B}) \cdot \mathrm{d}^2\boldsymbol{r} = \int \left(0 + \mu_0\epsilon_0 \frac{d\boldsymbol{E}}{dt}\right) \cdot \mathrm{d}^2\boldsymbol{r} \implies B(s,t)(2\pi s) = \int \left(-\frac{\mu_0\epsilon_0 V_0}{RCd} e^{-\frac{t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{z}}\right) \cdot \mathrm{d}^2 r\hat{\boldsymbol{e}}_{\boldsymbol{z}}$$

$$B(s,t) = \frac{1}{2\pi s}\left(-\frac{\mu_0\epsilon_0 V_0}{RCd} e^{-\frac{t}{RC}}\right)\left(\pi s^2\right)$$

$$B(s,t) = -\frac{\mu_0\epsilon_0 V_0 s}{2RCd} e^{-\frac{t}{RC}}$$

(we note that the distance of the Amperian loop from either of the disks does not affect this result, as long as $\boldsymbol{E}(t)$ is approximately uniform in this region between the disks.)

Thus in summary, we have $\boldsymbol{J_d}(t) = -\frac{\epsilon_0 V_0}{RCd} e^{-\frac{t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{z}}$ and $\boldsymbol{B}(\boldsymbol{r},t) = -\frac{\mu_0\epsilon_0 V_0 s}{2RCd} e^{-\frac{t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{\phi}}$ in the region between the disks.

4b. At the rim $s = L$, we have the Poynting vector

$$\boldsymbol{S}(t) = \frac{1}{\mu_0}\left(\boldsymbol{E}(t) \times \boldsymbol{B}(L,t)\right) = \frac{1}{\mu_0}\left(\frac{V_0}{d} e^{-\frac{t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{z}} \times -\frac{\mu_0\epsilon_0 V_0 L}{2RCd} e^{-\frac{t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\right)$$

$$= \frac{\epsilon_0 V_0^2 L}{2RCd^2} e^{-\frac{2t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{s}}, \text{ since } \hat{\boldsymbol{e}}_{\boldsymbol{z}} \times \hat{\boldsymbol{e}}_{\boldsymbol{\phi}} = -\hat{\boldsymbol{e}}_{\boldsymbol{s}}$$

The vector Poynts outwards, and thus represents a flow of electromagnetic energy *out* of the region between the disks. This is as we would expect, because the capacitor is discharging and hence the energy stored in it is decreasing (flowing outwards).

4c. At time t, the power flowing out from the region between the disks is (in terms of magnitude)

$$P(t) = \left|\int \boldsymbol{S}(t) \cdot \mathrm{d}^2\boldsymbol{r}\right| = \left|\int \left(\frac{\epsilon_0 V_0^2 L}{2RCd^2} e^{-\frac{2t}{RC}} \hat{\boldsymbol{e}}_{\boldsymbol{s}}\right) \cdot \left(\mathrm{d}^2 r\hat{\boldsymbol{e}}_{\boldsymbol{s}}\right)\right|$$

$$= \left|\left(\frac{\epsilon_0 V_0^2 L}{2RCd^2} e^{-\frac{2t}{RC}}\right)(2\pi L d)\right|$$

$$= \frac{\pi\epsilon_0 V_0^2 L^2}{RCd} e^{-\frac{2t}{RC}}$$

Therefore, the energy removed from the region at time t is

$$
\begin{aligned}
\text{Energy removed} = \int_0^t P(t')\,\mathrm{d}t' &= \int_0^t \frac{\pi\epsilon_0 V_0^2 L^2}{RCd} e^{-\frac{2t'}{RC}}\,\mathrm{d}t' \\
&= \frac{\pi\epsilon_0 V_0^2 L^2}{RCd}\left[-\frac{RC}{2}e^{-\frac{2t'}{RC}}\right]_0^t \\
&= \frac{\pi\epsilon_0 V_0^2 L^2}{2d}\left(1 - e^{-\frac{2t}{RC}}\right)
\end{aligned}
$$

Over the course of the full discharge (i.e. $t \to \infty$), the energy removed from the region between the disks is hence $\dfrac{\pi\epsilon_0 V_0^2 L^2}{2d} = \dfrac{1}{2}CV_0^2$, which is the energy stored in the capacitor in the beginning, as expected.

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