Problem One

(a) **Solution:**

$$\hat{S}\cdot\mathbf{n} = \sin\theta\cos\phi\,\hat{\sigma_x} + \sin\theta\sin\phi\,\hat{\sigma_y} + \cos\theta\,\hat{\sigma_z} \mathrel{\widehat{=}} \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$
$$= 2\cos(\theta/2)\begin{pmatrix} \cos(\theta/2) & e^{-i\phi}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & -\cos(\theta/2) \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence the possible measurement outcomes and corresponding probabilities are

$$P(E=-\hbar\beta) = |\langle E_1|\psi(0)\rangle|^2 = 1/6$$
$$P(E=\hbar\beta) = |\langle E_2|\psi(0)\rangle|^2 = 1/2$$
$$P(E=2\hbar\beta) = |\langle E_3|\psi(0)\rangle|^2 = 1/3$$

(b) **Solution:**
Let $\{|\phi_n\rangle\}$ denote the set of eigenstates of the observable $\hat{O}$. Then,

$$\langle\psi|\hat{O}|\psi\rangle = \langle\psi|\hat{O}\left(\sum_n |\phi_n\rangle\langle\phi_n|\right)|\psi\rangle = \sum_n \langle\psi|\hat{O}|\phi_n\rangle\langle\phi_n|\psi\rangle = \sum_n \langle\phi_n|\psi\rangle\langle\psi|\hat{O}|\phi_n\rangle$$
$$= \sum_n \langle\phi_n|(\hat{\rho}_\psi\hat{O})|\phi_n\rangle = \mathrm{Tr}(\hat{\rho}_\psi\hat{O}).$$

(c) Compute the probability of finding the electron at atom 2 at a later time t.

Solution:

$$|\psi(t)\rangle = \exp\left[-i\hat{H}t/\hbar\right]|\psi(0)\rangle = \sum_i \exp\left[-iE_it/\hbar\right]|E_i\rangle\langle E_i|\psi(0)\rangle$$
$$\Rightarrow |\langle x_2|\psi(t)\rangle|^2 = \left|\sum_i \exp\left[-iE_it/\hbar\right]\langle x_2|E_i\rangle\langle E_i|x_1\rangle\right|^2 = \left|-\frac{1}{3}\exp[i\beta t] + \frac{1}{3}\exp[-2i\beta t]\right|^2$$
$$= \frac{2}{9} - \frac{2}{9}\cos(3\beta t) = \frac{4}{9}\sin^2\left(\frac{3}{2}\beta t\right)$$

(d) Find the expectation value and uncertainty of the energy at time t.

Solution:
By conservation of energy, it is quite evident that the expectation value and uncertainty in the energy are independent of time and hence the answer is the same as in part (i).

(e) Are there other times $t > 0$ at which a position measurement will yield $x = -1$ (atom 1) with absolute certainty?

Solution:

$$P(x=-1) = |\langle x_1|\psi(t)\rangle|^2 = \left|\sum_i \exp\left[-iE_i t/\hbar\right] \langle x_1|E_i\rangle \langle E_i|x_1\rangle\right|^2$$

In order for the expression to be unity, we require that all the components share the same phase, that is

$$\exp\left[i\beta t\right] = \exp\left[-i\beta t\right] = \exp\left[-2i\beta t\right]$$

This clearly occurs when $t = 2n\pi/\beta$, $n \in \mathbb{Z}$.

(f) If the system is projected into the most likely energy state at $t = 0$, what is the expectation value of the position?

Solution:
After the measurement, the electron is in the state $|E_2\rangle = \left(-|x_1\rangle + |x_3\rangle\right)/\sqrt{2}$. Since there is equal probability of finding the electron at $x = -1$ (atom 1) and $x = 1$ (atom 3), the expectation value of the position is obviously 0.

Problem Eight

(a) Consider a particle in the ground state in an infinite square square well potential in the region $0 < x < a$. Write down the corresponding normalized wavefunction and energy.

Solution:

$$\psi_1(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) & 0 < x < a \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad E_1 = \frac{\pi^2 \hbar^2}{2ma^2}$$

(b) The left wall is diabatically moved to the position $x = -a$ so that the well becomes twice as wide.

(i) Write down the eigenfunctions and energies of the particle in the new potential.

Solution:

$$\phi_n(x) = \begin{cases} \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi x}{2a} + \frac{n\pi}{2}\right) & -a < x < a \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mathcal{E}_n = \frac{n^2 \pi^2 \hbar^2}{8ma^2}$$

(ii) Find the probability of finding the particle in the ground state of the new potential

Solution:

$$\begin{aligned} \int_{-a}^{a} \phi_1^*(x)\psi_1(x)\,\mathrm{d}x = \frac{\sqrt{2}}{a}\int_0^a \cos\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi x}{a}\right)\mathrm{d}x &= \frac{1}{a\sqrt{2}}\int_0^a \sin\left(\frac{\pi x}{2a}\right) + \sin\left(\frac{3\pi x}{2a}\right)\mathrm{d}x \\ &= \frac{1}{a\sqrt{2}}\left(\frac{2a}{\pi} + \frac{2a}{3\pi}\right) = \frac{4\sqrt{2}}{3\pi} \end{aligned}$$

Hence the probability is given by

$$\text{Prob} = \frac{32}{9\pi^2} \approx 0.360$$

(iii) Find the probability of finding the particle with an energy that is not higher than the original energy it had before the wall was moved.

Solution:
Only the states $\phi_1(x)$ and $\phi_2^(x)$ correspond to energies lower than or equal to the original energy. Hence,

$$\text{Prob} = \left| \int_{-a}^{a} \phi_1^*(x)\psi_1(x)\,\mathrm{d}x \right|^2 + \left| \int_{-a}^{a} \phi_2^*(x)\psi_1(x)\,\mathrm{d}x \right|^2 .$$

The first term had already been evaluated in Part (ii). Evaluating the second term explicitly yields

$$\begin{aligned} \int_{-a}^{a} \phi_2^*(x)\psi_1(x)\,\mathrm{d}x &= \frac{\sqrt{2}}{a}\int_0^a -\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\mathrm{d}x = -\frac{\sqrt{2}}{a}\int_0^a \sin^2\left(\frac{\pi x}{a}\right)\mathrm{d}x \\ &= \frac{\sqrt{2}}{a}\left(\frac{a}{2}\right) = \frac{1}{\sqrt{2}} \end{aligned}$$

Therefore,

$$\text{Prob} = \frac{32}{9\pi^2} + \frac{1}{2} \approx 0.860.$$

Problem Nine

(a) Spin-1/2 particles initially prepared in the state $|y,-\rangle$ are passed through a sequence of Stern-Gerlach selectors picking out the $|x,+\rangle$, $|n,+\rangle$ and $|z,+\rangle$ states respectively, where the second selector is oriented along the direction $\mathbf{n} = \sin\theta\,\mathbf{u}_x + \cos\theta\,\mathbf{u}_y$. What is the fraction of particles leaving the last selector for (i) $\theta = 0$, (ii) $\theta = \pi$ and (iii) $\theta = \pi/2$?

Solution:

The fraction of particles leaving the last selector is given by

$$|\langle y,-|x,+\rangle\,\langle x,+|n,+\rangle\,\langle n,+|z,+\rangle|^2 = \frac{1}{4}\,|\langle x,+|n,+\rangle|^2,$$

since $|\langle y,-|x,+\rangle|^2 = |\langle n,+|z,+\rangle|^2 = 1/2$, as $\mathbf{n}$ is orthogonal to the z-axis.

(i) $\theta = 0 :\ |\langle x,+|n,+\rangle|^2 = |\langle x,+|y,+\rangle|^2 = 1/2 \Rightarrow \text{Fraction} = 1/8$

(ii) $\theta = \pi :\ |\langle x,+|n,+\rangle|^2 = |\langle x,+|y,-\rangle|^2 = 1/2 \Rightarrow \text{Fraction} = 1/8$

(iii) $\theta = \pi/2 :\ |\langle x,+|n,+\rangle|^2 = |\langle x,+|x,+\rangle|^2 = 1 \Rightarrow \text{Fraction} = 1/4$

(b) The atoms leaving the last Stern-Gerlach selector travel at a constant velocity v_0 and enter a box of length L where they are subjected to a magnetic field described by the Hamiltonian $\hat{H} = (\hbar\omega_0/2)\,\hat{\sigma}_y$. What length L is required for all atoms leaving the box to be in the state $|x,+\rangle$?

Solution:

$$\begin{aligned}|\psi(t)\rangle &= \exp[i(\omega_0/2)t]\,|y,-\rangle\,\langle y,-|z,+\rangle + \exp[-i(\omega_0/2)t]\,|y,+\rangle\,\langle y,+|z,+\rangle \\ &= \frac{1}{\sqrt{2}}\,|y,+\rangle + (\exp[i\omega_0 t]\,|y,-\rangle)\exp[-i(\omega_0/2)t]\end{aligned}$$

Expressing $|x,+\rangle$ in terms of $|y,\pm\rangle$, we have

$$|x,+\rangle = \frac{1-i}{2}\,|y,+\rangle + \frac{1+i}{2}\,|y,-\rangle = \frac{1}{\sqrt{2}}\,(|y,+\rangle + i\,|y,-\rangle)\exp[-i\pi/4]$$

Evidently, we require that $\exp[i\omega_0 t] = i \ \Rightarrow\ \omega_0 t = (2n+1/2)\pi$ and the required length is

$$L = v_0 t = (2n+1/2)\pi\frac{v_0}{\omega_0}$$

(c) Inside the box, are any of the spin projections a constant of the motion?

Solution:

The spin projection along the y-axis is clearly a constant of the motion since $[\hat{H},\hat{S}_y] = 0$, while the other two are not as $[\hat{H},\hat{S}_x] \neq 0$ and $[\hat{H},\hat{S}_z] \neq 0$. Physically, the Hamiltonian causes the spin to precess about the y-axis.

