NATIONAL UNIVERSITY OF SINGAPORE

MA1102R — CALCULUS

(Semester 2 : AY2016/2017)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Write your matriculation/student number only. Do not write your name.
2. This examination paper contains a total of **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
3. This examination carries a total of **100** marks.
4. Answer **ALL** questions.
5. This is a **CLOSED BOOK** examination.
6. You are allowed to bring one A4-size, double-sided help sheet.
7. You may use non-graphing and non-programmable calculators. However, you should lay out systematically the various steps in the calculations.

Question 1 **[20 marks]**

The continuous function f is defined on $(-\infty, 5)$ by

$$f(x) = \begin{cases} e^{x-3}(2-x) & \text{if } x \leq 3, \\ (x-4)(x-2)^2 & \text{if } 3 < x < 5, \end{cases}$$

(i) Find the x-coordinate of each critical point of f.

(ii) Find (if any) the absolute maximum value and absolute minimum value of f.

(iii) Find the interval(s) on which f is concave upward.

(iv) Find the exact value of $\displaystyle\int_{-\infty}^{3} |f(x)|\, dx$.

Question 2 **[20 marks]**

(a) A moving particle along the x-axis is at a distance x from the origin O at time t, and $\dfrac{dx}{dt}$ is given by

$$t^2\left(10\,\frac{dx}{dt} + x^2\right) = x^2 \qquad (t > 0).$$

It is given that $x = 4$ when $t = 2$.

(i) Solve the given differential equation and show that $x = \dfrac{kt}{1+t^2}$ where k is a constant to be found.

(ii) Find the maximum distance between the particle and the origin O. Justify your answer.

(b) Show that the substitution $z = y^{-2}$ reduces the differential equation

$$x^2\,\frac{dy}{dx} - xy = 6y^3 \ln x \qquad (x > 0)$$

to

$$\frac{dz}{dx} + \frac{2z}{x} = -\frac{12\ln x}{x^2}.$$

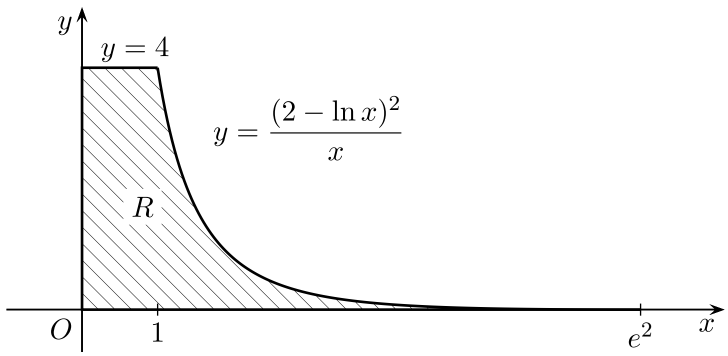
Given that $y = 1$ when $x = 1$, express y in terms of x.

Question 3 **[20 marks]**

For $n \geq 0$, let $I_n = \displaystyle\int_1^{e^2} (2 - \ln x)^n \, dx$.

(i) Show that for $n \geq 1$, $I_n = nI_{n-1} - 2^n$.

(ii) Find the exact values of I_0, I_1 and I_2.



As shown in the above diagram, R is the region bounded by the axes, the curve $y = \dfrac{(2 - \ln x)^2}{x}$ $(1 \leq x \leq e^2)$ and the horizontal line $y = 4$.

(iii) Find the exact area of the region R.

(iv) Find the exact volume of the solid formed by rotating R completely about the y-axis.

Question 4 **[20 marks]**

(a) Let a be a positive real number. Use the precise definition of limit to prove that

$$\lim_{x \to a} \sin \sqrt{x} = \sin \sqrt{a}.$$

(b) Let f be continuous on $[0, 1102]$ and differentiable on $(0, 1102)$. It is given that $f(x) > 0$ for all $x \in [0, 1102]$ and $f(0) = f(1102)$. By considering the function $g(x) = \sqrt{f(x)}\, f(1102 - x)$, show that there exists $c \in (0, 1102)$ such that

$$\frac{f'(c)}{f(c)} = \frac{2f'(1102 - c)}{f(1102 - c)}.$$

(c) Let f be continuous on $[a, b]$ and differentiable on (a, b), where $0 < a < b$, and let $\lambda \in (0, 1)$. It is given that $f(a) = a$, $f(b) = b$ and $f'(x) \neq 0$ for all $x \in (a, b)$.

(i) Show that there exists $c \in (a, b)$ such that $f(c) = \lambda a + (1 - \lambda)b$.

(ii) Show that there exist α and β in (a, b), with $\alpha < \beta$, such that $\dfrac{1 - \lambda}{f'(\alpha)} + \dfrac{\lambda}{f'(\beta)} = 1$.

Question 5 **[20 marks]**

(a) A curve C has equation $y = f(x)$, $x > 3$, where

$$f'(x) = \frac{6\sqrt{6+x}}{x-3}.$$

Show that

$$1 + (f'(x))^2 = \left(\frac{x+15}{x-3}\right)^2$$

and hence, find the exact arc length of the curve C for $4 \le x \le 5$.

(b) Let $[x]$ denote the greatest integer not exceeding x. Find the exact value of

$$\int_2^{2017} \frac{1}{[x]^2 - [x]}\, dx.$$

(c) Find the exact value of

$$\lim_{x \to 0} \left(1 + \int_{2x}^{4x} \sin(t^2)\, dt\right)^{\csc(4x^3)}.$$

(d) Let f be continuous on $[0, \infty)$ and let $t > 0$. By using the fact that

$$\int_0^t \left(f(x) - \frac{1}{t}\int_0^t f(x)\, dx\right)^2 dx \ge 0,$$

show that

$$\int_0^t (f(x))^2\, dx \ge \frac{1}{t}\left(\int_0^t f(x)\, dx\right)^2.$$

Deduce that

$$\frac{t}{\sqrt{1+t}} \ge \ln(1+t)$$

for all $t \ge 0$.

END OF PAPER