Problem 1: One-dimensional potential and phase space

A particle of mass m moves in a conservative force field with the potential

$$U(x) = a|x|\,, \quad \text{with} \quad a > 0\,.$$

(a) Find the Hamilton equations of motion.

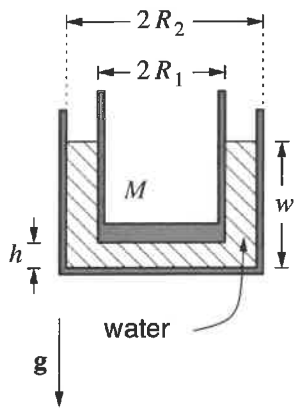
(b) For a given total energy $E > 0$, calculate the maximal/minimal momentum, and extremal positions x.

(c) Sketch the trajectory of the mass over time with its extremal points in a phase space diagram.

(d) What is the oscillation period for a given total Energy E?

Problem 2: Glass in glass

Two cylindrical glasses stacked into each other. The inner glass has a mass M and a radius R_1, whereas the outer glass has a radius R_2 with $R_2 > R_1$. The space between the glasses is filled with water of density ρ such that the inner glass floats at height h.



(a) What is the water level w between the glasses to maintain a bottom separation $h = h_0$? Glasses and water are subject to gravitational acceleration $\mathbf{g}$.

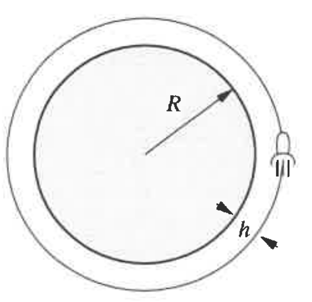
(b) With the amount of water fixed, you can now dip the inner glass down. Calculate the potential energy $U(h)$ for the system glass+water.

(c) Find the Lagrange function for the problem of the inner glass moving vertically, assuming: (i) water is incompressible and moves without friction, (ii) water between the bottoms moves only in radial direction, and its velocity does not depend on the vertical position, (iii) water in the region between R_1 and R_2 moves only vertically with a single velocity.

(d) What is the oscillation period of the inner glass moving up and down for small deviations from $h = h_0$?

Problem 3: Change of orbit

A satellite of mass $M = 100\,\text{kg}$ orbits the earth ($R = 6370\,\text{km}$, gravitational acceleration on surface $g = 10\,\text{ms}^{-2}$) in a constant height $h = 1500\,\text{km}$ above the surface.



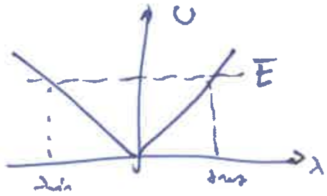
(a) What is the orbital period τ of the satellite?

(b) The satellite now fires a thruster for a time $t \ll \tau$, where it accelerates by emitting gas of mass $m = 5\,\text{kg}$ in tangential direction with a velocity of $v_g = 2500\,\text{m}^{-1}$ with respect to the satellite. What are the new minimal and maximal heights above the earth surface?

— End of paper — C.K.

PROBLEM 1:

$U(x) = a|x|$, $a>0$



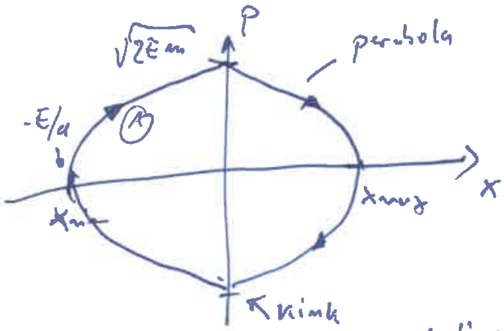
a) $H = T + U = \frac{1}{2} m \dot{x}^2 + a|x| = \frac{p^2}{2m} + a|x|$

$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$ $\qquad \dot{p} = -\frac{\partial H}{\partial x} = \begin{cases} -a, & x>0 \\ +a, & x<0 \end{cases}$

b) total energy E : ~~p_max~~ $U=0 \rightsquigarrow p_{max}/p_{min} = \pm\sqrt{2E\cdot m}$

extremal positions $\rightsquigarrow U = E$ $\qquad x_{min/max} = \pm E/a$

c)



$p = \pm at$

$\dot{x} = \pm\frac{a}{m} t \rightsquigarrow x = \pm\frac{1}{2}\frac{a}{m}t^2 = \pm\frac{p^2}{am}$

• closed trajectory in phase space

d) oscillation period is given by ~~twice~~ 4 times the time from $t=0$ to $x = x_{max}$

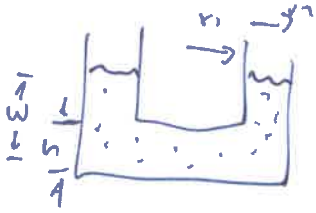
or easier: 4 times from $p=0$ to $p = p_{max}$

$p = a\cdot t$ on part (A)

$p_{max} = \sqrt{2Em} = a\cdot\frac{T}{4} \rightsquigarrow T = \frac{4}{a}\sqrt{2Em}$

2) Glass filled with water ; volume V

a) equilibrium position: displaced water mass = mass of inner glass



$A_1 = \pi R_1^2$

$A_2 = \pi R_2^2$

$\rho \cdot A_1 (w - h) = M$

$\rightarrow w = \frac{M}{\rho A_1} + h$

b) volume of water is constant $= V$

$V = A_1 h_0 + (A_2 - A_1) w_0 = A_1 h_0 + (A_2 - A_1)\left(\frac{M}{\rho A_1} + h_0\right)$

potential energy in Glass

1.) lower dish of water: $U_1 = \rho \cdot g \cdot A_1 \cdot h \cdot \frac{h}{2}$

outer ring of water: $U_2 = \rho \cdot g \cdot (A_2 - A_1) \cdot \frac{w^2}{2}$

inner glass: $U_3 = M \cdot g \cdot h$

$U = U_1 + U_2 + U_3 = \rho g A_1 \frac{h^2}{2} + \rho g (A_2 - A_1) \frac{w^2}{2} + M \cdot g \cdot h$

but: w and h are connected via constant volume V!

$\rightarrow w = \frac{V - A_1 h}{A_2 - A_1}$

$\rightarrow U = \rho g A_1 \frac{h^2}{2} + \rho g (A_2 - A_1) \left(\frac{V - A_1 h}{A_2 - A_1}\right)^2 \cdot \frac{1}{2} + M \cdot g \cdot h$

c) Lagrange function: need kinetic energy.

~~Options: 1.) Moving inner glass $T_1 = \frac{1}{2} M \dot{h}^2$~~

Dynamic variable: h $\rightarrow L = L(h, \dot{h})$

have: $U(h)$ from above

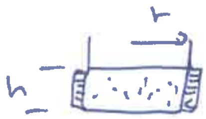
need: $T = T(h, \dot{h})$

T has three components: 1.) lower dish, radial motion of water $\rightarrow T_1$

2.) ~~vertical~~ ring, vertical motion only $\rightarrow T_2$

3.) moving glass $\rightarrow T_3$

kinetic energy of bottom disk



compression of $h \to h - \delta h$ displaces volume

$$\delta V = \pi r^2 \delta h$$

this ends up in ring volume $\delta V = 2\pi r \, \delta r \cdot h$

$$\rightsquigarrow \pi r^2 \delta h = 2\pi r h \, \delta r \quad \text{or} \quad r \delta h = 2h \delta r$$

now: express radial velocity at radius r, $v_r = \frac{\delta r}{dt}$ by $\frac{\delta h}{dt} = \dot{h}$

$$\frac{\delta r}{dt} = \frac{r}{2h} \frac{\delta h}{dt}$$

therefore, radial velocity at r: $v_r = \frac{r}{2h} \dot{h}$ ||

note: v_r depends on r!

total kinetic energy in bottom disk:

$$T_1 = \int \frac{1}{2} dm \, v_r^2 = \frac{1}{2} \rho \int_0^{R_1} (2\pi r \, dr \cdot h) \left(\frac{r}{2h}\right)^2 \dot{h}^2$$

$$= \frac{\rho}{2} \cdot 2\pi h \cdot \frac{1}{4h^2} \dot{h}^2 \cdot \int_0^{R_1} r^3 dr = \frac{\pi \rho}{4h} \dot{h}^2 \cdot \frac{R_1^4}{4} = \frac{\rho \pi R_1^4}{16 h} \dot{h}^2$$ ||

total kinetic energy of outer water ring:

constant velocity $\dot{\omega}$ across whole ring $\rightsquigarrow$

$$T_2 = \frac{1}{2} \rho (A_2 - A_1) \omega \cdot \dot{\omega}^2 \qquad \text{need } T_2 = T_2(h, \dot{h})$$

use $V = \text{const}$ $\rightsquigarrow \frac{d}{dt}\left(A_1 h + (A_2 - A_1)\omega\right) = 0$

$$\rightsquigarrow A_1 \dot{h} + (A_2 - A_1) \cdot \dot{\omega} = 0 \quad \rightsquigarrow \dot{\omega} = -\dot{h}\left(\frac{A_1}{A_2 - A_1}\right)$$ ||

$$\rightsquigarrow T_2 = \frac{1}{2} \rho (A_2 - A_1) \cdot \omega \cdot \left(\frac{A_1}{A_2 - A_1}\right)^2 \dot{h}^2$$

$$= \frac{1}{2} \rho \frac{A_1^2}{A_2 - A_1} \left(\frac{V - A_1 h}{A_2 - A_1}\right) \cdot \dot{h}^2 = \frac{1}{2} \rho \frac{A_1^2}{(A_2 - A_1)^2} (V - A_1 h) \cdot \dot{h}^2$$ ||

$$T_3 = \frac{1}{2} M \dot{h}^2$$ ||

$$\rightsquigarrow T = T_1 + T_2 + T_3 = \frac{\rho \pi R_1^4}{16 h} \dot{h}^2 + \frac{1}{2} \rho \frac{A_1^2}{(A_2 - A_1)^2} (V - A_1 h) \dot{h}^2 + \frac{1}{2} M \dot{h}^2$$

$$= \left[\frac{\rho \pi R_1^4}{16 h} + \frac{1}{2} \rho \frac{A_1^2}{(A_2 - A_1)^2} (V - A_1 h) + \frac{1}{2} M\right] \dot{h}^2$$ ||

Lagrange function $L = L(h, \dot{h})$:

$$L = T - U = \frac{1}{2}\left[\frac{\varrho \pi R_1^4}{8h} + \varrho \frac{A_1^2}{(A_2-A_1)^2}(V - A_1 h) + M\right]\dot{h}^2 -$$

$$\left\{\varrho g A_1 \frac{h^2}{2} + \frac{\varrho g}{2} \frac{(V-A_1h)^2}{A_2-A_1} + Mgh\right\} \quad \underline{\underline{\quad}}$$

d) oscillation periode for small deviations from $h = h_0$?

looking for Lagrange function of type

$$L = X \cdot \dot{h}^2 - Y \cdot (h-h_0)^2$$

then: $0 = \frac{\partial L}{\partial h} - \frac{d}{dt}\frac{\partial L}{\partial \dot{h}} = -2Y(h-h_0) - \frac{d}{dt} 2X\dot{h} = 0$

or $-Y \cdot \tilde{h} - X\ddot{\tilde{h}} = 0$ with $\tilde{h} = h - h_0$

or $\ddot{\tilde{h}} + \frac{Y}{X}\tilde{h} = 0$ the oscillation freq. $\omega = \sqrt{\frac{Y}{X}}$

or oscillation periode $T_{osc} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{X}{Y}}$ $\underline{\underline{\quad}}$

now: find X, Y for $h \approx h_0$

kinetic energy

Do taylor expansion of square bracket in T around $h = h_0$:

$$T = \frac{1}{2}\left[\frac{\varrho \pi R_1^4}{8h} + \varrho \frac{A_1^2}{(A_2-A_1)^2}(V - A_1 h) + M\right]\dot{h}^2$$

$$\left.\frac{\partial T}{\partial h}\right|_{h=h_0} = \frac{1}{2}\left[\frac{\varrho \pi R_1^4}{8h_0} + \varrho \frac{A_1^2}{(A_2-A_1)^2}(V - A_1 h_0) + M\right]\dot{h}^2$$

$$= \frac{1}{2}\left[\frac{\varrho A_1^2}{8\pi h_0} + \frac{\varrho A_1^2}{(A_2-A_1)^2}(A_2-A_1)\left(\frac{M}{\varrho A_1} + h_0\right) + M\right]\dot{h}^2$$

$$= \frac{1}{2}\left[\frac{\varrho A_1^2}{8\pi h_0} + \frac{\varrho A_1^2}{(A_2-A_1)}\left(\frac{M}{\varrho A_1} + h_0\right) + M\right]\dot{h}^2$$

$$= \underbrace{\frac{1}{2}\left[\frac{\varrho A_1^2}{8\pi h_0} + \frac{MA_2 - \varrho A_1^2 h_0}{A_2 - A_1}\right]}_{=:X}\dot{h}^2 =: X\dot{h}^2$$

consider Potential eng $U(h)$

$$U(h) = \rho g A_1 \frac{h^2}{2} + \frac{\rho g}{2}\,\frac{(V-A_1h)^2}{A_2-A_1} + Mgh$$

look for Taylor expansion of $U(h)$ around $h = h_0$

$$\frac{\partial U}{\partial h} = \rho g A_1 \cdot h + \frac{\rho g}{2}\,\frac{1}{A_2-A_1}\cdot 2(V-A_1h)(-A_1) + Mg$$

$$= \rho g A_1 h + \rho g\,\frac{-A_1}{A_2-A_1}(V-A_1h) + Mg$$

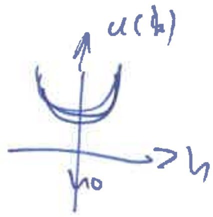
$$\frac{\partial U}{\partial h^2} = \rho g A_1 + \rho g\,\frac{-A_1}{A_2-A_1}(-A_1) = \rho g A_1 + \rho g\,\frac{A_1^2}{A_2-A_1} = \rho g A_1\left(1+\frac{A_1}{A_2-A_1}\right)$$

$$= \rho g\,\frac{A_1A_2}{A_2-A_1}$$

consistency check: $$\left.\frac{\partial U}{\partial h}\right|_{h=h_0} = \rho g A_1 h_0 + \rho g\,\frac{-A_1}{A_2-A_1}(A_2-A_1)\left(\frac{M}{\rho A_1}+h_0\right) + Mg$$

$$= \rho g A_1 h_0 - A_1 \rho g\left(\frac{M}{\rho A_1}+h_0\right) + Mg$$

$$= 0$$ ☺ As expected



Taylor-expansion of U around $h = h_0$:

$$U = U_0 + \underbrace{\frac{1}{2}\,\rho g\,\frac{A_1A_2}{A_2-A_1}}_{Y}\cdot(h-h_0)^2 \Rightarrow U_0 + Y\cdot(h-h_0)^2 \text{ with } Y = \frac{1}{2}\,\rho g\,\frac{A_1A_2}{A_2-A_1}$$

Therefore, oscillation periode T_{osc} for $h \approx h_0$:

$$T_{osc} = 2\pi\cdot\sqrt{\cancel{\tfrac{1}{2}\rho g\tfrac{A_1A_2}{A_2-A_1}}\;\frac{2(A_2-A_1)}{\rho g A_1A_2}\cdot\frac{1}{2}\left[\frac{\rho A_1^2}{8\pi h_0} + \frac{MA_2-\rho A_1^2h_0}{A_2-A_1}\right]}$$

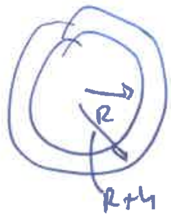
$$= 2\pi\sqrt{\frac{1}{g}}\sqrt{\frac{A_2-A_1}{\rho A_1A_2}\left[\frac{\rho A_1^2}{8\pi h_0} + \frac{MA_2-\rho A_1^2h_0}{A_2-A_1}\right]}$$

$$= 2\pi\sqrt{\frac{1}{g}}\sqrt{\frac{A_1}{A_2}\,\frac{(A_2-A_1)}{8\pi g h_0} + \frac{M}{\rho A_1} - \frac{A_1h_0}{A_2}}$$

eight

3) Change of orbit

$R = 6370\ km$ $\quad h = 1500\ km$ $\quad M = 100\ kg$ $\quad g = 10\ m\,s^{-1}$ on surface



a) orbital period of satellite

constant height h ⇝ circular orbit

expression for τ : $\tau = a^{3/2}\,\pi\sqrt{\frac{4\mu}{k}}$

a: semiaxis, here $a = R+h$ $\quad \mu$: reduced mass, here $\mu = M$

k: from $U = -\frac{k}{r} = -\frac{Me \cdot M}{r}G$ ⇝ $k = +Me\,M\,G$

on earth: $+\frac{\partial U}{\partial r} = \frac{-Me\,G\,M}{r^2} \doteq -M\cdot g$ ⇝ $g = \frac{Me\,G}{Re^2}$ ⇝ $k = +g\cdot Re^2\cdot M$

$$⇝ \tau = (h+Re)^{3/2}\cdot\pi\cdot\sqrt{\frac{4\cdot M}{g\,R_E^2\,M}} = (h+Re)^{3/2}\cdot\frac{2\pi}{R_E}\sqrt{g}^{\,-1}$$

$$= (7.87\times10^6\,m)^{3/2}\cdot 2\pi\cdot\frac{1}{6.37\cdot10^6\,m}\cdot\sqrt{\frac{1}{10\,m\,s^{-2}}}$$

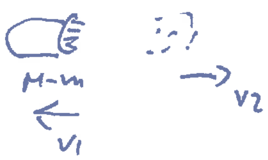
$$= 6{,}886.5\ sec = 1h\ 54'\ 46''$$

b) [illegible]: $t \ll \tau$; ejects mass m with $v_g = 2500\,m\,s^{-1}$

→ [illegible] momentum

consider momentum conservation in system rocket + gas

$$(M-m)\cdot v_1 + m\cdot v_2 = 0$$



$$v_2 - v_1 = v_g$$

$$→ (M-m)\,v_1 + m\,(v_g - v_1) = 0$$

$$M v_1 + m v_g = 0 \quad → v_1 = -v_g\cdot\frac{m}{M}$$

→ change in ~~[illegible]~~ velocity of spaceship $\quad v_0 \longrightarrow v_0 + v_1$

original velocity

and change in mass $M \longrightarrow M - m$

but: change happens at fixed R+h distance from center of earth

Also: velocity is still tangential to earth surface

$\rightsquigarrow$ spaceship must be at apocenter or pericenter after acceleration.

since the velocity increased and thereby the ℓ increased, the acceleration point must be the pericenter of the new trajectory

$$\rightsquigarrow \; r_{min} = R + h \qquad \rightsquigarrow \; h_{min} = 1500\,km$$

to find new r_{max}, make use of $V_{eff}(r_{min}) = V_{eff}(r_{max})$

$$V_{eff} = -\frac{k'}{r} + \frac{\ell'^2}{2\mu r^2} = -\frac{k'}{r} + \frac{A}{r^2} \qquad \text{with } A = \frac{\ell'^2}{2\mu} > 0, \; k' = \mu g R^2$$

$$\mu = M - m$$

$$\left(-\frac{k'}{r_{min}} + \frac{A}{r_{min}^2}\right) = -\frac{k'}{r_2} + \frac{A}{r_2^2} \qquad \text{multiply by } r_2^2$$

$$\left(-\frac{k'}{r_{min}} + \frac{A}{r_{min}^2}\right) r_2^2 + k' r_2 - A = 0$$

$$\rightsquigarrow r_2 = \frac{1}{2(\quad)^2}\left(-k' \pm \sqrt{k'^2 + 4A\left(-\frac{k'}{r_{min}} + \frac{A}{r_{min}^2}\right)}\right) = \frac{1}{2(\quad)^2}\left(-k' \pm \sqrt{\left(k' - \frac{2A}{r_{min}}\right)^2}\right)$$

$$= \frac{1}{2\left(\frac{-k'}{r_{min}} + \frac{A}{r_{min}^2}\right)}\left(-k' \pm \left(k' - \frac{2A}{r_{min}}\right)\right)$$

$$r_2^- = \frac{1}{2\left(-\frac{k'}{r_{min}} + \frac{A}{r_{min}^2}\right)}\left(-2k' + \frac{2A}{r_{min}}\right) = r_{min} \qquad \text{(not surprising...)}$$

$$r_2^+ = \frac{1}{2\left(-\frac{k'}{r_{min}} + \frac{A}{r_{min}^2}\right)}\left(-\frac{2A}{r_{min}}\right) = \frac{r_{min}}{\frac{k'}{A} r_{min} - 1} =: r_{max}$$

Now:

$$\frac{k'}{A} r_{min} = \frac{k' \cdot 2\mu}{\ell'^2} r_{min} = \frac{2k'\mu \cdot r_{min}}{\mu^2 v'^2 r_{min}^2} \qquad \text{with } v' = v_0 + |v_1|$$

from thrust: $|v_1|$; v_0: initial velocity

$$= \frac{2(\mu g R^2) \cdot \mu\, r_{min}}{\mu^2 v'^2 r_{min}^2} = \frac{2 g R^2}{v'^2 r_{min}}$$

need v_0 for circular orbit of mass M:

$$\text{use } (R+h) = \alpha = \frac{\ell^2}{Mk} = \frac{(M v_0 (R+h))^2}{M \cdot M \cdot g \cdot R^2} = \frac{v_0^2 (R+h)^2}{g R^2} \rightsquigarrow v_0 = \sqrt{\frac{g R^2}{(R+h)}}$$

Now combine values finally for numerics:

$$|v_1| = v_g \cdot \frac{m}{M} = 2500\, m s^{-1} \cdot \frac{5\,kg}{100\,kg} = 125\, m s^{-1}$$

$$v_0 = \sqrt{\frac{gR^2}{(R+h)}} = \sqrt{\frac{10\, m s^{-2}\, (6370 \cdot 10^3 m)^2}{7870 \cdot 10^3 m}} = 7180.45\, m s^{-1}$$

$$\rightsquigarrow v' = v_0 + v_1 = 7305.45\, m s^{-1}$$

$$\rightarrow \frac{k'}{A} \cdot r_{min} = \frac{2gR^2}{v'^2 (R+h)} = \frac{20\, m s^{-2} (6370 \cdot 10^3 m)^2}{(7305.45\, m/s)^2 \cdot (7870 \cdot 10^3 m)} = 1.932$$

$$\rightsquigarrow r_{max} = r_{min} \cdot \frac{1}{0.932} = \cancel{1.073}\, 1.0727\, r_{min} = 8442.1\, km = \underline{\underline{R + 2072\, km}}$$

Could try to approximate for $\frac{k'}{A} \cdot r_{min}$ since $v_1 \ll v_0$:

$$\frac{k'}{A} \cdot r_{min} = \frac{2gR^2}{(v_0+v_1)^2 (R+h)} = \frac{2 v_0^2 (R+h)}{(v_0^2 + 2v_0 v_1 + v_1^2)(R+h)}$$

$$= \frac{2 v_0^2}{v_0^2 + 2 v_0 v_1 + v_1^2} = \frac{2}{1 + 2\left(\frac{v_1}{v_0}\right) + \left(\frac{v_1^2}{v_0^2}\right)} = 2 \frac{v_0^2}{(v_0+v_1)^2}$$

$$\rightsquigarrow r_{max} = r_{min} \frac{1}{\frac{k'}{A} r_{min} - 1} = r_{min} \frac{1}{2 \frac{v_0^2}{(v_0+v_1)^2} - 1}$$

$$= r_{min} \frac{(v_0 + v_1)^2}{2v_0^2 - v_0^2 - 2v_0 v_1 - v_1^2} = r_{min} \frac{v_0^2 + 2v_0 v_1 + v_1^2}{v_0^2 - 2v_0 v_1 - v_1^2}$$

$$= r_{min} \frac{1 + 2\, v_1/v_0 + (v_1/v_0)^2}{1 - 2 v_1/v_0 - (v_1/v_0)^2} \approx r_{min} \left(1 + 4 \frac{v_1}{v_0}\right)$$

Oh, that does <u>not</u> fit into 90 minutes... <u>Sorry</u>!