NATIONAL UNIVERSITY OF SINGAPORE

PC1142 Introduction to Thermodynamics and Optics

(Semester I: AY 2015–16)

Time Allowed: 2 Hours

<u>INSTRUCTIONS TO CANDIDATES</u>

1. Please write your matriculation number in the answer booklet. Do not write your name.
2. This examination paper comprises 8 printed pages with 5 short questions in Part I and 3 long questions in Part II.
3. Answer <u>all</u> questions.
4. This is a <u>closed book</u> examination.
5. You may use electronic calculators.
6. A list of physical constants and formulae is given on pages 2 and 3.

Thermal physics and kinetic theory of gases:

(i) Kinetic mean free path is given by $\ell = \dfrac{1}{\sqrt{2}\pi \cdot n_v \cdot d^2}$, where n_v is the number density and d is the molecular diameter.

(ii) Maxwell–Boltzmann distribution in 3D:

$$P(v) = 4\pi \cdot v^2 \cdot \left(\frac{m}{2\pi \cdot k_B \cdot T}\right)^{3/2} \cdot \exp\left(-\frac{m \cdot v^2}{2 \cdot k_B \cdot T}\right)$$

(a) Root-mean-square speed: $v_{rms} = \sqrt{\dfrac{3k_BT}{m}}$.

(b) Average speed: $v_{av} = \sqrt{\dfrac{8k_BT}{\pi m}}$.

(c) Most-probable speed: $v_{mp} = \sqrt{\dfrac{2k_BT}{m}}$.

(iii) Ideal gas equation: $p \cdot V = n \cdot R \cdot T$.

(iv) Van der Waals equation:

$$(p + \frac{a \cdot n^2}{V^2}) \cdot (V - n \cdot b) = n \cdot R \cdot T \cdot$$

(v) One form of adiabat: $p \cdot V^\gamma = constant$.

(vi) Stefan–Boltzmann equation: $P_{rad} = \sigma \cdot A \cdot e \cdot T^4$.

(vii) Planck radiation equation:

$$I_{rad}(\lambda) = \frac{2\pi \cdot h \cdot c^2}{\lambda^5 \cdot \left(\exp\left(\dfrac{h \cdot c}{\lambda \cdot k_B \cdot T}\right) - 1\right)}.$$

(viii) Wien displacement law:

$$\lambda_{peak} = \frac{2.898x10^{-3}\, m \cdot K}{T}.$$

(ix) Convection equation: $P_{conv} = h \cdot A \cdot \Delta T$.

(x) Conduction equation: $P_{ccond} = -k \cdot A \cdot \dfrac{\Delta T}{L}$.

(xi) Linear thermal expansion: $\Delta L = \alpha \cdot L \cdot \Delta T$.

Thermodynamics:

(i) First law of thermodynamics: $\Delta U = q_{in} + W_{in}$. Gas expansion work done by the gas: $W_{out} = \int p \cdot dV$.

(ii) Carnot heat-engine efficiency $e_C = 1 - \dfrac{T_C}{T_H}$.

(iii) Entropy $dS = \dfrac{dq_{rev}}{T}$.

(iv) Enthalpy $H = E + PV$.

(v) Helmholtz free energy $F = E - TS$.

(vi) Gibbs free energy $G = H - TS$.

Geometric optics:

(i) p and q are object and image distances measured respectively on opposite sides of the lens or of the refracting surface:

(a) Object–image relation (thin lens): $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{f}$,

where $\dfrac{1}{f} = (n-1) \cdot \left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$.

(b) Refracting-surface equation:

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}.$$

(ii) Gullstrand equation:

(a) Effective power: $P_e = P_1 + P_2 - P_1 \cdot P_2 \cdot \dfrac{d}{n}$.

(b) Front-vertex refracting power:

$$P_f = P_1 + \frac{P_2}{1 - \dfrac{P_2 \cdot d}{n}}.$$

(c) Back-vertex refracting power:

$$P_b = P_2 + \frac{P_1}{1 - \dfrac{P_1 \cdot d}{n}}.$$

(iii) Spherical-mirror equation: $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$, where $\frac{1}{f}=\frac{2}{R}$.

(iv) f-number, $f/\# = \frac{f}{D}$.

(v) Numerical aperture: $NA = n \cdot \sin\theta$.

(vi) Snell's law: $n_1 \cdot \sin\theta_1 = n_2 \cdot \sin\theta_2$.

(a) Critical angle: $\sin\theta_c = n_2 / n_1$.

(b) Brewster angle: $\tan\theta_p = n_2 / n_1$.

(vii) Wave relation: $v = f \cdot \lambda$.

(viii) Abbe number: $V = \frac{n_D - 1}{n_F - n_C}$.

Wave optics:

(i) Circular aperture (Airy's disc): first diffraction minimum is at $\sin\theta = \frac{1.22\lambda}{a}$.

(ii) Slit: first diffraction minimum is at $\sin\theta = \frac{\lambda}{a}$.

(iii) N-slit intensity pattern: $I = I_o \cdot \frac{\sin^2(N \cdot \phi/2)}{\sin^2(\phi/2)}$, where $\phi = \frac{2\pi}{\lambda} \cdot d \cdot \sin\theta$.

(iv) Single-slit diffraction pattern:

$I = I_o \cdot \frac{\sin^2(\delta/2)}{(\delta/2)^2}$, where $\delta = \frac{2\pi}{\lambda} \cdot a \cdot \sin\theta$.

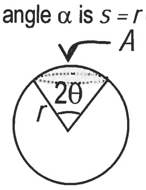
(v) Thin-film interference:

$\delta = 2 \cdot n \cdot d \cdot \cos\beta$, where β is the refracted angle.

General:

(i) Geometry:

The arc length on a circle of radius r subtended by angle α is $s = r\alpha$.



The surface area A subtended by polar angle 2θ is:

$A = 2\pi r^2 (1 - \cos\theta)$.

(ii) Logarithms and exponents :

$\log_a (b\,c) = \log_a b + \log_a c$

$\log_a b = \log_d b / \log_d a$

$a^b * a^c = a^{b+c}$

$(a^b)^c = a^{bc}$

(iii) Integrations:

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$, except for $n = -1$, where $\int x^{-1} dx = \ln x + c$.

(iv) Taylor expansions:

$\sin\theta = \theta - \frac{1}{6}\theta^3 + \dots$

$\cos\theta = 1 - \frac{1}{2}\theta^2 + \dots$

$\tan\theta = \theta + \frac{1}{3}\theta^3 + \dots$

$(1+x)^a = 1 + ax + \frac{a(a-1)}{2}x^2 + \dots$

(v) Series sum:

$\sum_{n=0}^{n=m} a_0 r^n = \frac{a_0(1-r^{m+1})}{1-r}$

(vi) Differentiations:

$d(\sin u) = \cos u\, du$

$d(\cos u) = -\sin u\, du$

$d(u^m) = m u^{m-1}\, du$

Universal constants:

Gas constant $R = 8.314$ J K^{-1} mol^{-1}

Boltzmann constant $k_B = 1.381\text{x}10^{-23}$ J K^{-1}

Stefan–Boltzmann constant $\sigma = 5.670\text{x}10^{-8}$ W m^{-2} K^{-4}

Speed of light in vacuum $c = 2.998\text{x}10^{8}$ m s^{-1}

Avogadro's number $N_A = 6.022\text{x}10^{23}$ mol^{-1}

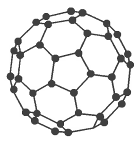
Permittivity of free space $\varepsilon_o = 8.854\text{x}10^{-12}$ Fm^{-1}

Planck constant $h = 6.626\text{x}10^{-34}$ J s

Part 1. Answer **ALL** questions. Each question carries 8 marks.

1. The earth is internally heated primarily by decay of radioactive elements (e.g. ^{238}U, ^{232}Th and ^{40}K) located deep in the crust layer which is about 30–50 km thick. The radius of earth is 6,371 km. This heat flows by conduction through the crust to the earth's surface. The average geothermal gradient just below the surface is 20 K km^{-1}. This gradient gives the average rate of temperature increase with depth. The average thermal conductivity of the crust is 3.0 W m^{-1} K^{-1}. Estimate the global geothermal power flowing to the earth's surface. [8 marks]

2. C_{60} molecules (pictured below, molecular weight, 720.6 g mol^{-1}) are confined between two non-sticking parallel walls at 298 K. The molecules exhibit two-dimensional (2D) free translational motion parallel to these walls.
   (i) Write down a statement of the energy equipartition theorem.
   (ii) Hence or otherwise, deduce the average root-mean-square speed of these molecules.



[8 marks]

3. A fuel cell is a device that converts chemical energy of a fuel directly into electrical energy through a reaction with O_2 or other oxidizers. The theoretical maximum thermodynamic efficiency of a fuel-cell reaction is defined as the ratio of the maximum electrical energy to the heat energy that can be obtained from the reaction under the specified conditions of temperature and pressure. For the H_2/ O_2 fuel-cell reaction:

   $H_2(g) + 0.5\, O_2(g) \rightarrow H_2O\,(\ell)$,

   the standard enthalpy of formation $\Delta H^o_{f,298}$, entropy S^o_{298}, and Gibbs free energy of formation $\Delta G^o_{f,298}$ of the relevant reactants and products are given for 298 K and 1.00 atm in the table on the next page.

Quantity	H_2O (l)	H_2 (g)	O_2 (g)
$\Delta H^o_{f,298}$ (kJ / mol)	−285.83	0	0
S^o_{298} (J / K mol)	69.91	130.68	205.14
$\Delta G^o_{f,298}$ (kJ / mol)	−306.66	−38.94	−61.13

[8 marks]

(i) Compute the theoretical maximum thermodynamic efficiency for the H_2/ O_2 fuel-cell reaction at 298 K and 1.00 atm.

(ii) Briefly explain why the thermodynamic efficiency of the fuel cell is not subjected to the Carnot limit.

[8 marks]

4. A Canon Powershot G1 X camera has a charge-coupled device sensor array that has an effective pixel size (diameter) of 4.3 μm. The lens aperture was set to *f*/16. The average wavelength of the image spectrum is 550 nm.

(i) Compute the size of the Airy disk on the image plane for an object point at infinity.

(ii) Hence briefly explain whether decreasing the pixel size would improve image sharpness.

[8 marks]

5. A circularly-polarised monochromatic light beam is incident in air on a thick glass slab with isotropic refractive index $n = 1.55$.

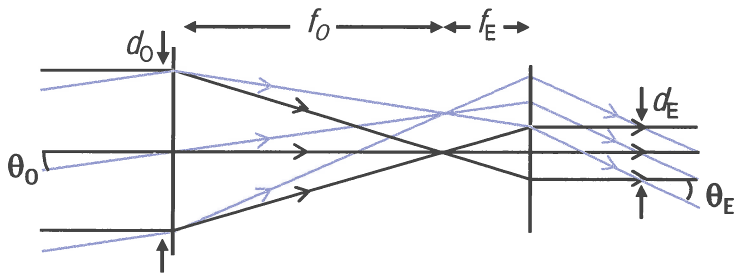
(i) Compute the required incidence angle for the reflected beam to be completely linearly *s*-polarised.

(ii) Explain the state of polarisation of the light beam that is transmitted through the glass slab.

[8 marks]

Part 2. Answer **ALL** questions. Each question carries 20 marks.

6. Binoculars are a parallel combination of two refracting telescopes containing internal image-rotating prisms to produce upright images for viewing with both eyes. Binoculars are specified by angular magnification (m) X objective lens diameter (d_O). A basic ray diagram for two paraxial object points observed through one of the telescopes omitting the prism is shown below. The two object points are separated by angle θ_O. The focal length of the objective lens group is f_O, and that of the eye lens group is f_E. The diameter of the exit pupil is d_E. The entrance pupil of the eye should be placed at this exit pupil to view through the binoculars.



A pair of binoculars is specified as 8 X 36 mm. It has f_O = 135 mm.

(i) Derive an expression for m in terms of f_O and any other necessary parameters for objects at infinity in the paraxial ray approximation. Briefly explain how the given pair of binoculars helps you view distant objects. [*Hint*: You may wish to start from: $m = \theta_E / \theta_O$] [8 marks]

(ii) The diameter of the exit pupil is given by the smallest diameter of the ray bundle exiting the eye lens group. By considering the ray bundle that is co-axial with the optical axis or otherwise, derive an expression for the ratio of d_E to d_O.

[4 marks]

(iii) The binoculars are used to view an object at infinity and then another one 3.0 m away. Compute the distance and direction that the eye lens group needs to be translated while keeping the images in focus at infinity. [4 marks]

(iv) Usually the objective lens group comprises a cemented achromat doublet, while the eye lens group comprises four or five lens elements. Briefly explain the aberration that the achromat is intended to correct. [4 marks]

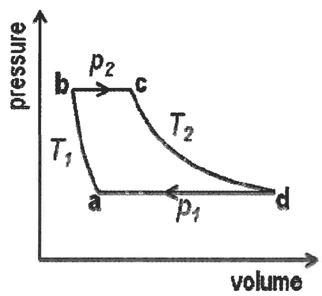
7. Gas turbines may be operated using the Ericsson cycle. This cycle comprises the following stages as shown below:

a→**b**: isothermal compression of the working gas from pressure p_1 to pressure p_2;

b→**c**: isobaric heating of the gas from temperature T_1 to temperature T_2;

c→**d**: isothermal expansion of the gas from pressure p_2 to pressure p_1; and

d→**a**: isobaric cooling of the gas from temperature T_2 to temperature T_1.



Isothermal compression of the gas is obtained using a multi-stage compressor together with intercoolers between the stages. Isobaric heating is obtained by fuel combustion. Isothermal expansion is obtained using a multi-stage turbine with reheat between the stages. Isobaric cooling is obtained at discharge.

(i) Derive an expression for the thermal efficiency of this cycle in terms of T_1, T_2 and any other necessary parameters, assuming ideal gas behavior. [8 marks]

(ii) The heat released by process **d** → **a** can be recycled to the process **b** → **c** using a heat exchanger. For perfect regeneration, all the heat released by **d** → **a** is absorbed by **b** → **c**. From your result in (i) or otherwise, derive an expression for the thermal efficiency in the case of perfect regeneration. [4 marks]

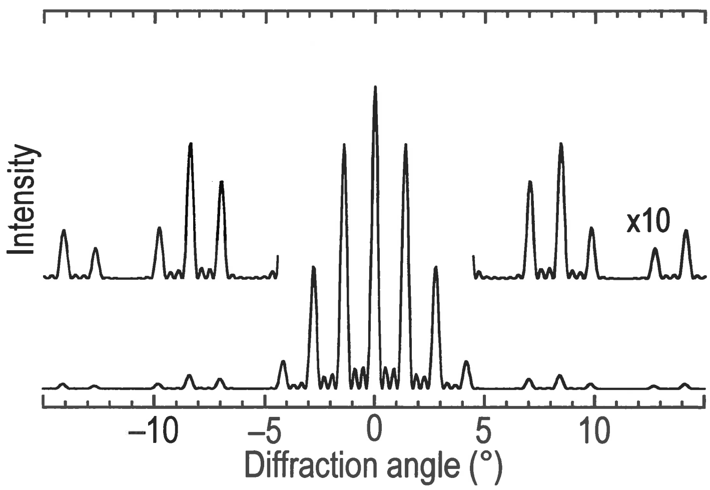
(iii) Sketch the temperature *vs* entropy state-variable diagram for this cycle. Indicate on the diagram the locations of **a, b, c** and **d**.

[4 marks]

(iv) Briefly explain what happens to the temperature of the gas if intercoolers had not been used during its compression.

[4 marks]

8. A laser beam with vacuum wavelength λ_0 of 488 nm is incident normal on an opaque screen that has *N* straight slits. Each of the slits has width *a*. The slits are separated from each other by distance Λ measured centre to centre. The intensity pattern of the laser beam in the far field is plotted as a function of the diffraction angle perpendicular to the slits as shown below. Zero corresponds to the straight through direction. The intensity of the wings has also been expanded by a factor of ten for clarity.



(i) Write down the number of slits illuminated by the laser, and briefly explain your reasoning. [8 marks]

(ii) Estimate the value of *a*. [4 marks]

(iii) Estimate the value of Λ. [4 marks]

(iv) Briefly explain how the diffraction pattern would change if the experiment were conducted in a medium with refractive index *n*. [4 marks]

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