NATIONAL UNIVERSITY OF SINGAPORE

PC4248 RELATIVITY

(Semester I: AY 2011–12)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR** questions and comprises **THREE** printed pages.
2. Answer any **THREE** questions.
3. Answers to the questions are to be written in the answer books.
4. This is a **closed book** examination.
5. Students are allowed to bring in one A4-sized double-sided help sheet.

1. Let S be an inertial frame with coordinates (t, x, y, z). Consider performing a boost in the direction $\frac{1}{\sqrt{2}}(0, 1, 1, 0)$ with speed v, to obtain a new inertial frame S' with coordinates (t', x', y', z').

(a) Show that the coordinates of S and S' are related by

$$\begin{aligned}
t' &= \gamma\left(t - v\,\frac{x+y}{\sqrt{2}}\right), \\
x' &= \gamma\left(-\frac{vt}{\sqrt{2}} + \frac{x+y}{2}\right) + \frac{x-y}{2}, \\
y' &= \gamma\left(-\frac{vt}{\sqrt{2}} + \frac{x+y}{2}\right) - \frac{x-y}{2}, \\
z' &= z,
\end{aligned}$$

where $\gamma \equiv 1/\sqrt{1-v^2}$. [*Hint: Rotate the x-y plane by $-45°$, then perform the usual boost, and finally rotate back again.*]

(b) Now consider a rod of proper length L that is at rest in S and aligned with the x-axis. Using the results of part (a), find the apparent length of the rod as measured by an observer co-moving with S'.

2. (a) Write down the transformation law for a tensor of type (r, s) under a general coordinate transformation.

(b) The covariant derivative of a vector V^a is defined to be

$$\nabla_b V^a = \partial_b V^a + \Gamma^a_{bc} V^c,$$

where Γ^a_{bc} is the connection. By requiring that $\nabla_b V^a$ be a tensor of type (1,1), derive the transformation law for Γ^a_{bc}.

(c) Suppose V^a, W^a are vectors, and B^{ab} is an anti-symmetric type-(2,0) tensor. Show that the following are valid tensorial expressions:

$$\begin{aligned}
&\text{(i)} \qquad V^b \partial_b W^a - W^b \partial_b V^a; \\
&\text{(ii)} \qquad \Gamma^c_{ab} B^{ab}.
\end{aligned}$$

Do <u>not</u> assume that Γ^c_{ab} is symmetric in a and b.

3. Consider a two-dimensional surface given by the metric:

$$ds^2 = y^p \, dx^2 + x^q \, dy^2,$$

where p and q are constants. Set $x^1 \equiv x$ and $x^2 \equiv y$.

(a) Calculate the Christoffel symbols Γ^a_{bc} for this metric.

(b) Find all the non-zero components of the Riemann tensor. [*Recall that* $R_{abc}{}^e = \partial_b \Gamma^e_{ac} - \partial_a \Gamma^e_{bc} + \Gamma^f_{ac} \Gamma^e_{bf} - \Gamma^f_{bc} \Gamma^e_{af}$.]

(c) For what values of p and q is this surface flat?

4. The Schwarzschild metric describing a black hole is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \, .$$

(a) Derive the equations of motion for a photon moving in the equatorial plane $\theta = \pi/2$. In particular, use the equations of motion to show that

$$\dot{r}^2 = E^2 \left[1 - \frac{b^2}{r^2} \left(1 - \frac{2m}{r}\right) \right] , \qquad (1)$$

where E and b are constants.

(b) Consider a photon approaching the black hole from infinity. (In such a situation, note that b can be interpreted as the impact parameter of the photon.) Using Equation (1), show that:

(i) if $b^2 > 27m^2$, the photon is deflected but not captured by the black hole;

(ii) if $b^2 < 27m^2$, the photon is captured by the black hole.

Can you guess what happens in the critical case $b^2 = 27m^2$?

(ET)

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