NATIONAL UNIVERSITY OF SINGAPORE

PC4241 — Statistical Mechanics
(Semester I: AY 2017/18)

Exam, 1 December 2017

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Write your matric number on the answer book. **Do not write your name.**
2. This examination paper contains **FOUR** questions and comprises **THREE** printed pages.
3. Answer **ALL FOUR** questions for a total of 100 marks.
4. Start a new page for each question and show all your work in the answer book.
5. For each question, **clearly** indicate what constitutes your final answer.
6. The format of this exam is **Closed Book (with authorized materials)**: Lecture notes for PC4241 and personal notes directly related to the module may be consulted during the test, **but no other printed or written material**.
7. The use of **electronic equipment** of any kind **is not permitted**.

1. One-component thermodynamical system (20 marks)

For a system that can be characterized by entropy S, volume V, and mole number n, show that

$$v\left(\frac{\partial P}{\partial v}\right)_T = \left(\frac{\partial \mu}{\partial v}\right)_T ,$$

where $v = V/n$ is the molar volume.

2. Ideal classical gas (20=12+8 marks)

An atom in a gas has the velocity vector $\boldsymbol{v} = \boldsymbol{p}/m$ and moves at the speed $|\boldsymbol{v}|$.

(a) Find the average speed $\langle\,|\boldsymbol{v}|\,\rangle$ and also the average reciprocal speed $\langle\,|\boldsymbol{v}|^{-1}\rangle$ for the atoms of an ideal classical gas at temperature T. Confirm that $\langle\,|\boldsymbol{v}|\,\rangle\langle\,|\boldsymbol{v}|^{-1}\rangle \geq 1$.

(b) Demonstrate that, quite generally, the inequality $\langle X\rangle\langle X^{-1}\rangle \geq 1$ holds for any positive quantity X.

Hint: Consider $\left\langle\left(\lambda X^{\frac{1}{2}} - X^{-\frac{1}{2}}\right)^2\right\rangle$ and adjust the value of the parameter λ.

3. An Ising-type model (40=15+5+15+5 marks)

Consider a one-dimensional chain (or ring) of particles with N next-neighbor links and no on-site energy. The energy of the kth microstate is

$$E_k = -J\sum_j s_j s_{j+1} \quad \text{with } s_j = 0 \text{ or } +1 \text{ or } -1\,.$$

Note that we have the additional option of $s_j = 0$ here, while there is only $s_j = \pm 1$ in the standard Ising model.

(a) Show that the canonical partition function is

$$Q(K,N) = \left(\cosh(K) + \tfrac{1}{2} + \sqrt{[\cosh(K) - \tfrac{1}{2}]^2 + 2}\right)^N$$

where $K = \beta J$.

(b) What is the free energy per site?

(c) Determine the heat capacity per site at low temperatures ($K \gg 1$) and high temperatures ($K \ll 1$). In both cases, state the leading term.

(d) Confirm that this system obeys the Third Law.

Hint: A 3×3 matrix of the form $\begin{pmatrix} a & 1 & b \\ 1 & 1 & 1 \\ b & 1 & a \end{pmatrix}$ has $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ as an eigencolumn.

4. Berthelot gas (20 marks)

The equation of state of the Berthelot gas is

$$P(T, v) = \frac{RT}{v - b} - \frac{a}{v^2 RT},$$

where v is the molar volume and a and b are positive material constants. Determine all the virial coefficients $a_1(\beta)$, $a_2(\beta)$, $a_3(\beta)$, ...

End of Paper

(BG Englert)