NATIONAL UNIVERSITY OF SINGAPORE

PC2130 QUANTUM MECHANICS I

(Semester I: AY 2009 – 10, 25 November)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR** questions and comprises **FIVE** printed pages.
2. Answer **ALL** questions.
3. Answers to the questions are to be written in the answer books.
4. This is a CLOSED BOOK examination.

Question 1:

Consider a system whose Hamiltonian is given by the matrix

$$H = \varepsilon \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2i \\ 0 & -2i & 1 \end{pmatrix}$$

where ε is real and has the dimensions of energy. In addition, suppose it has an observable represented by the matrix

$$A = \alpha \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

where α is a real quantity.

(a) At time $t = 0$, a measurement of A is carried out. If we obtain a value of 3α, find the state of the system at a later time $t > 0$. [9]

(b) At time $t > 0$, if we measure H, what values will we obtain and with what probabilities? [2]

(c) Calculate the expectation value of the Hamiltonian $\langle H \rangle$. Does it depend on time t? Explain. [2]

(d) Calculate the expectation value $\langle A \rangle$. Does it depend on time t? Explain. [2]

Question 2:

Consider a free particle of mass m that moves in one dimension. Its initial wave function is

$$\Psi(x,0) = A\exp\left(-\frac{1}{L}|x|\right)\exp(ik_0 x)$$

where A, L and k_0 are positive real constants.

(a) Normalize $\Psi(x,0)$. [2]

(b) Calculate the expectation value of the position and momentum at time $t = 0$. [2]

(c) Compute the wave function $\Psi(x,t)$ at time $t > 0$. Leave your answer in the form of an integral. [4]

(d) Hence, or otherwise, write down the momentum wave function $\Phi(k,t)$ at time $t > 0$. [1]

(e) Calculate the expectation value of the momentum at time $t > 0$ and deduce the expectation value of the position at time $t > 0$. Explain your deduction. [3]

(f) Find the uncertainty in momentum at time $t > 0$ and deduce the minimum uncertainty in position at time $t > 0$. Explain your deduction. [3]

Hint: You may find the following formulas useful

$$\int_{-\infty}^{\infty} \frac{x}{[1 + a^2(x - x_0)^2]^2} dx = \frac{\pi}{2}\frac{x_0}{a}$$

$$\int_{-\infty}^{\infty} \frac{x^2}{[1 + a^2(x - x_0)^2]^2} dx = \frac{\pi}{2}\left(\frac{1}{a^3} + \frac{x_0^2}{a}\right)$$

Question 3:

Consider a simple harmonic oscillator of mass m with Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

Here, ω is the angular frequency of oscillation, $\hat{p}$ is the momentum operator and $\hat{x}$ the position operator.

(a) Calculate $\frac{d}{dt}\langle x\rangle$ and $\frac{d}{dt}\langle p\rangle$. [3]

(b) Given that at time $t = 0$, we have $\langle x\rangle = A$ and $\langle p\rangle = 0$, find $\langle x\rangle$ at time $t > 0$. Here, A is a constant. [2]

(c) Let

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}$$

Show that

(i) the Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

(ii) if $|n\rangle$ is an eigenket of $\hat{H}$ with eigenvalue $(n+1/2)\hbar\omega$, then $\hat{a}|n\rangle$ and $\hat{a}^\dagger|n\rangle$ are also eigenkets of $\hat{H}$ with eigenvalues $[(n-1)+1/2]\hbar\omega$ and $[(n+1)+1/2]\hbar\omega$ respectively.

[4]

(d) Given $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, show that

$$\langle n|\hat{x}|n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n'}\delta_{n,n'-1} + \sqrt{n'+1}\delta_{n,n'+1}\right),$$

$$\langle n|\hat{x}^2|n'\rangle = \frac{\hbar}{2m\omega}\left[\sqrt{n'(n'-1)}\delta_{n,n'-2} + (2n'+1)\delta_{nn'} + \sqrt{(n'+1)(n'+2)}\delta_{n,n'+2}\right].$$

[3]

(e) Hence, or otherwise, calculate $\langle 0|\exp(i\hat{x})|0\rangle$. [3]

Question 4:

The state of a spin-½ particle that is spin up along the axis whose direction is specified by the unit vector

$$\mathbf{n} = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$$

where θ and ϕ are the polar and azimuthal angles, is given by

$$|S_n;+\rangle = \cos\frac{\theta}{2}|S_z;+\rangle + \exp(i\phi)\sin\frac{\theta}{2}|S_z;-\rangle$$

(a) Write down $|S_n;-\rangle$ the state of a spin-½ particle that is spin down along **n**, and verify that $\langle S_n;-|S_n;+\rangle = 0$. [2]

(b) From your result in part (a), construct the matrix $\mathbf{S}_n$ representing the component of spin angular momentum along **n**. Express your answer in terms of $|S_z;\pm\rangle$. [3]

(c) Suppose that a measurement of S_z is carried out on a particle in the state $|S_n;+\rangle$. What is the probability that the measurement yields (i) $+\hbar/2$, and (ii) $-\hbar/2$? [2]

(d) Repeat the calculations of part (c) for measurements of S_x and S_y. [6]

(e) A beam of spin-½ particles is sent through a series of three Stern-Gerlach (S-G) apparatuses, as illustrated in Figure 1.

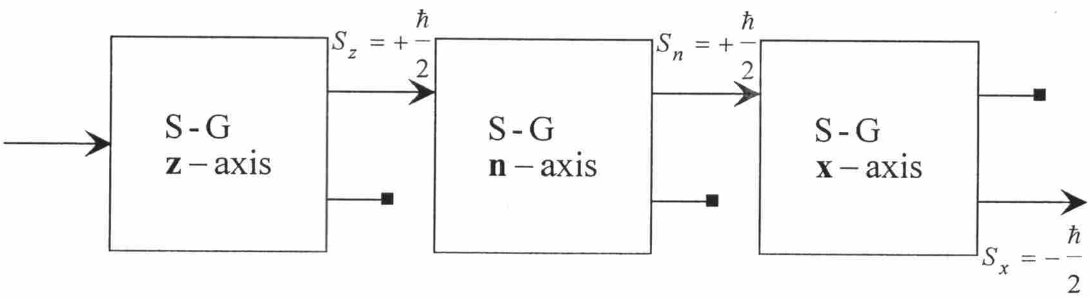

Figure 1

The first S-G **z**-axis apparatus transmits particles with $S_z = +\hbar/2$ and filters out particles with $S_z = -\hbar/2$. The second device, an S-G **n**-axis apparatus, transmits particles with $S_n = +\hbar/2$ and filters out particles with $S_n = -\hbar/2$ where the axis **n** is as given above. A last S-G **x**-axis apparatus transmits particles with $S_x = -\hbar/2$ and filters out particles with $S_x = +\hbar/2$. Find the fraction of the particles transmitted through the first S-G **z**-axis apparatus that will survive the third measurement. [2]

(YY)

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