NATIONAL UNIVERSITY OF SINGAPORE

PC4242: Electrodynamics

(Semester II: AY 2014-15)

Time allowed: 2 hours

INSTRUCTIONS TO STUDENTS

1. Please write your student number only. **Do not write your name.**
2. This assessment paper contains **TWO** questions and comprises **THREE** printed pages.
3. Students are required to answer all **TWO** questions.
4. Students should write the answers for each question on a new page.
5. This is a **CLOSED BOOK** examination.
6. Some useful formulas are provided on Page 2.
7. The use of **electronic equipment** of any kind **is not permitted**.

Formulas

$$\nabla\cdot\mathbf{E}=4\pi\rho$$

$$\nabla\cdot\mathbf{B}=0$$

$$\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}=\frac{4\pi}{c}\mathbf{J}$$

$$\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}=0$$

$$\frac{\partial}{\partial t}\rho+\nabla\cdot\mathbf{J}=0$$

$$\mathbf{E}=-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}-\nabla\phi$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\phi+\nabla\cdot\mathbf{A}=0$$

$$\mathbf{f}=\rho\mathbf{E}+\frac{1}{c}\mathbf{J}\times\mathbf{B}$$

$$\begin{pmatrix} ct' \\ \mathbf{x}' \end{pmatrix}=\begin{pmatrix} \gamma & \gamma\frac{\mathbf{v}^T}{c} \\ \gamma\frac{\mathbf{v}}{c} & \mathbf{1}+(\gamma-1)\frac{\mathbf{v}\mathbf{v}^T}{v^2} \end{pmatrix}\begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}$$

$$\mathcal{L}=\frac{1}{c}A_\mu J^\mu-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$$

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}=\frac{1}{4\pi c^3}\left|\mathbf{n}\times\int\mathrm{d}^3x'\frac{\partial}{\partial t}\mathbf{J}(\mathbf{x}',t_r)\right|^2$$

$$P=\frac{2}{3}\frac{e^2}{c^3}|\dot{\mathbf{v}}|^2$$

$$\mathbf{d}(t)=\int\mathrm{d}^3x\,\mathbf{x}\,\rho(\mathbf{x},t)$$

$$\boldsymbol{\mu}(t)=\frac{1}{2c}\int\mathrm{d}^3x\,\mathbf{x}\times\mathbf{J}(\mathbf{x},t)$$

$$\mathbf{Q}(t)=\int\mathrm{d}^3x\,(3\mathbf{x}\mathbf{x}-\mathbf{1}x^2)\,\rho(\mathbf{x},t)$$

$$f(t)=\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}f(\omega)$$

$$f(\omega)=\int\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}f(t)$$

$$f(\mathbf{x})=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}f(\mathbf{k})$$

$$f(\mathbf{k})=\int\mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}f(\mathbf{x})$$

$$\begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix}(\mathbf{x},\omega)=\frac{\mathrm{e}^{ikr}}{r}\begin{pmatrix} \rho \\ \frac{1}{c}\mathbf{J} \end{pmatrix}(\mathbf{k},\omega)$$

$$\frac{\mathrm{d}\mathcal{E}(\omega)}{\mathrm{d}\Omega}=\frac{\omega^2}{4\pi^2c^3}|\mathbf{n}\times\mathbf{J}(\mathbf{k},\omega)|^2$$

$$\frac{\mathrm{d}\sigma_{\mathrm{Th}}}{\mathrm{d}\Omega}=\frac{e^4}{(mc^2)^2}\frac{|\mathbf{n}\times\mathbf{E}|^2}{|\mathbf{E}|^2}$$

$$\sigma_{\mathrm{Ray}}=\sigma_{\mathrm{Th}}\frac{\omega^4}{(\omega^2-\omega_0^2)^2+(\gamma\omega)^2}$$

$$\begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix}(\mathbf{x},t)=\int\mathrm{d}^3x'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\begin{pmatrix} \rho \\ \frac{1}{c}\mathbf{J} \end{pmatrix}(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c)$$

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega}=\frac{1}{4\pi c^3}\left|\mathbf{n}\times\left[\ddot{\mathbf{d}}(t_e)-\mathbf{n}\times\ddot{\boldsymbol{\mu}}(t_e)+\frac{1}{6c}\dddot{\mathbf{Q}}(t_e)\cdot\mathbf{n}\right]\right|^2$$

$$\frac{\mathrm{d}P(\omega,T)}{\mathrm{d}\Omega}=\frac{1}{4\pi^2c}\int_{-\infty}^{\infty}\mathrm{d}\tau\,\mathrm{e}^{-\mathrm{i}\omega\tau}\left[\mathbf{k}\times\mathbf{J}^*\left(\mathbf{k},T+\tfrac{1}{2}\tau\right)\right]\cdot\left[\mathbf{k}\times\mathbf{J}\left(\mathbf{k},T-\tfrac{1}{2}\tau\right)\right]$$

$$E_x(\mathbf{x})=-\frac{E_0}{2\pi}\frac{\mathrm{e}^{ikr}}{r}\mathrm{i}k\cos\theta\int_{\text{apertures}}\mathrm{d}^2x'_\perp\,\mathrm{e}^{-ik\mathbf{n}\cdot\mathbf{x}'_\perp}$$

$$\mathbf{E}(\mathbf{x})=\mathbf{E}_{\mathrm{inc}}(\mathbf{x})+\mathrm{i}k\left(\mathbf{1}+\frac{\nabla\nabla}{k^2}\right)\cdot\int\mathrm{d}^2x'_\perp\,\frac{\mathrm{e}^{ik|\mathbf{x}-\mathbf{x}'_\perp|}}{|\mathbf{x}-\mathbf{x}'_\perp|}\frac{1}{c}\mathbf{K}(\mathbf{x}'_\perp)$$

1: The tensor $\epsilon^{\mu\nu\rho\lambda}$ is a four-dimensional extension of the Levi-Civita symbol. It is totally antisymmetric in its indices, and $\epsilon^{0123} = +1$. Define the dual field of the electromagnetic tensor $\tilde{F}$ as

$$\tilde{F}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}F_{\rho\lambda},$$

where $F_{\rho\lambda} = \partial_\rho A_\lambda - \partial_\lambda A_\rho$ is the normal electromagnetic tensor.

(a) Show that $\tilde{F}$ satisfies

$$\partial_\nu \tilde{F}^{\mu\nu} = 0.$$

[10 marks]

(b) Find all the components of $\tilde{F}^{\mu\nu}$ in terms of the **E** and **B**.

[10 marks]

2: An antenna model has the current density

$$\mathbf{J}(\mathbf{x},t) = \mathbf{e}_z I\cos(\omega t)\delta(x)\delta(y)\cos\left(\frac{\pi z}{L}\right)\eta\left(L^2 - 4z^2\right),$$

where $\eta(x)$ is the Heaviside step function.

(a) Find the angular distribution of the radiated power, averaged over one period of the oscillation.

[15 marks]

(b) Simplify your result in **(a)** for a so-called "half-wave antenna," specified by $L = \frac{1}{2}\lambda$.

[5 marks]

(c) Two half-wave antennas are parallel to the z axis at a distance $a > 0$, with their centers at $x = \pm\frac{1}{2}a$, so that the electric current density is given by $\mathbf{J}(\mathbf{x},t) = \mathbf{J}_+(\mathbf{x},t) + \mathbf{J}_-(\mathbf{x},t)$ with

$$\mathbf{J}_\pm(\mathbf{x},t) = \mathbf{e}_z I\cos\left(\omega t \mp \tfrac{1}{2}\beta\right)\delta\left(x \mp \tfrac{1}{2}a\right)\delta(y)\cos\left(\frac{2\pi z}{\lambda}\right)\eta\left(\lambda^2 - 16z^2\right),$$

where β is the relative phase between the currents in the two antennas. Find the angular distribution of the radiated power, averaged over one period of the oscillation.

[15 marks]

(d) Determine a and β such that your result in **(c)** is particularly large in the direction $\mathbf{n} = \mathbf{e}_x$ and particularly small in the direction $\mathbf{n} = -\mathbf{e}_x$.

[5 marks]

[WQh]

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