CS528
Multiprocessor Task Scheduling

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Outline

- P_m| prec, p_j = 1 | C_{max}
  - 2 Approx
- P_m| p_j | C_{max}
  - ILP Solution : Exponential
  - 2 Approx, 2-1/m approx.
  - LPT : 3/2 and 4/3 Approx
- $P_m|p_j=1|\Sigma w_jU_j$ Optimal Solution
- $P_m|p_j|\Sigma U_j$ NPC, Heuristic and Counter example
- $P_m|pmtn, p_j|\Sigma U_j$ in NPC
- $Q_m|ptmn|\Sigma C_j$ Optimal Solution

P_m | prec, $p_j = 1$ | C_{max}

Theorem 1

Pm | prec, $p_j = 1$ | C_{max} is NP-complete.

1. Ullman (1976)

   3SAT ≤ Pm | prec, $p_j = 1$ | C_{max}

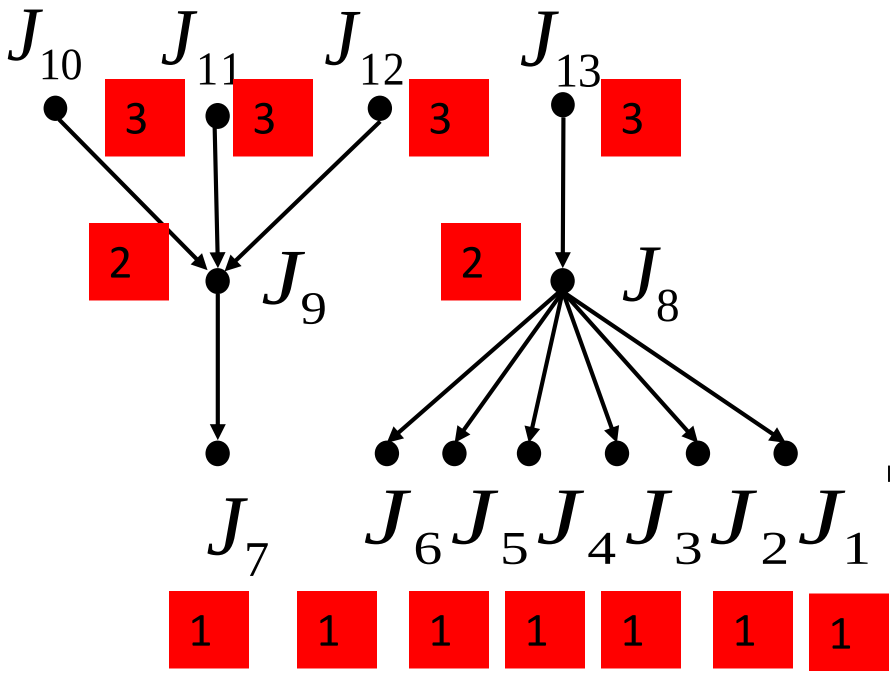
2. Lenstra and Rinooy Kan (1978)

   k-clique ≤ Pm | prec, $p_j = 1$ | C_{max}

P_m | prec, pj = 1 | C_{max} is NP-complete.

Proof: out of Syllabus

HLF/CP algorithm : Example



M2	J_{10}	J_{13}	J_8	J_6	J_3
M2	J_{11}	J_9	J_7	J_5	J_2
M1	J_{12}			J_4	J_1

$$L = (\overbrace{J_{10}, J_{11}, J_{12}, J_{13}}^{\text{Level 3}}, \overbrace{J_9, J_8}^{\text{Level 2}}, \overbrace{J_7, J_6, J_5, J_4, J_3, J_2, J_1}^{\text{Level 1}})$$

HLF/CP algorithm

- **Time complexity**

  O(|V|+|E|)   (|V| is the number of jobs and |E| is the number of edges in the precedence graph)

- **Theorem (Hu, 1961) : HLF/CP for Tree**
  - **The HLF algorithm is optimal for $P_m \mid p_j = 1$, in-tree (out-tree) $\mid C_{max}$.**
  - **The HLF algorithm is optimal for $P_m \mid p_j = 1$, in-forest (out-forest) $\mid C_{max}$.**

HLF/CP algorithm

- N.F. Chen & C.L. Liu (1975)

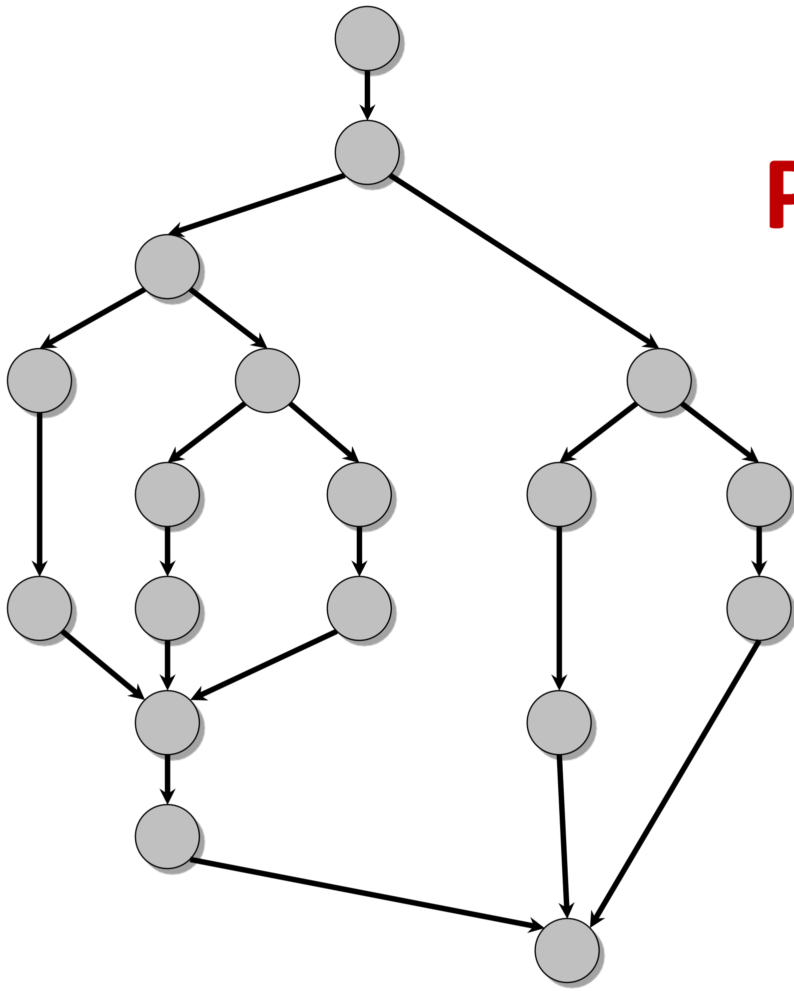
The approximation ratio of HLF algorithm for the problem with general precedence constraints:

$$\text{If } m = 2,\ \delta_{HLF} \le 4/3.$$

$$\text{If } m \ge 3,\ \delta_{HLF} \le 2 - 1/(m-1).$$

CP Algo: CLR Book Page 779-783

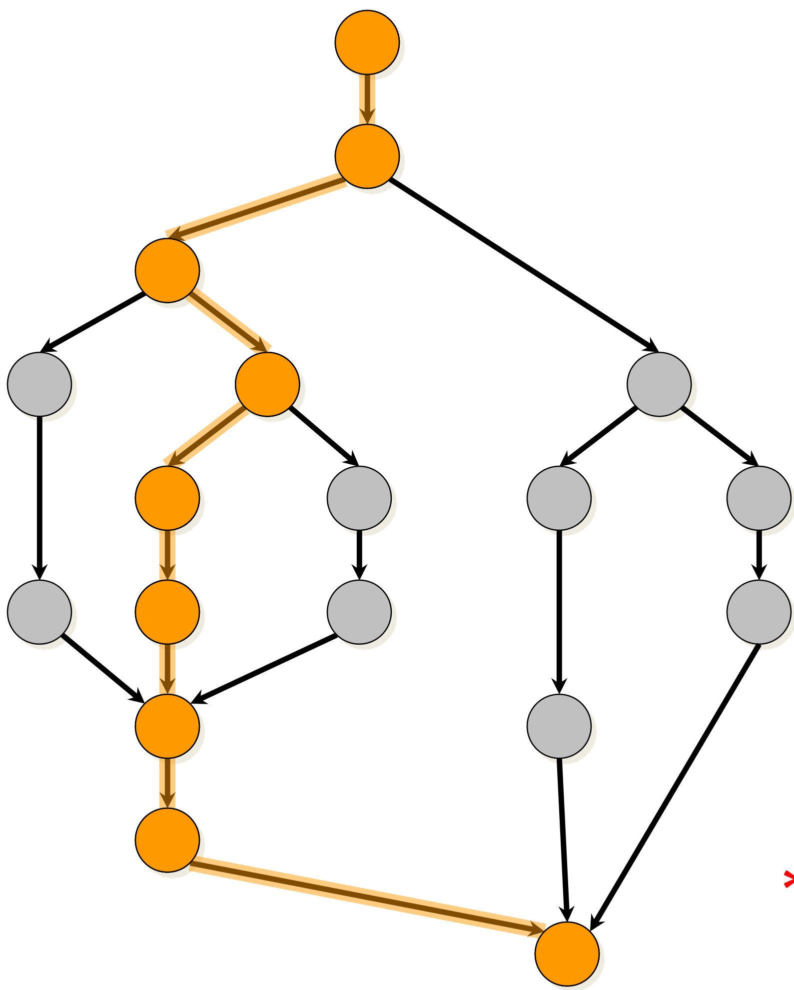
T_P = execution time on P processors



$Pm \mid p_j = 1\text{ , prec} \mid C_{max}$

Algorithmic Complexity Measures

T_P = execution time on P processors



T_1 = ***work*** ***= 18***

T_∞ = ***span**** ***= 9***

Example: P=3

LOWER BOUNDS

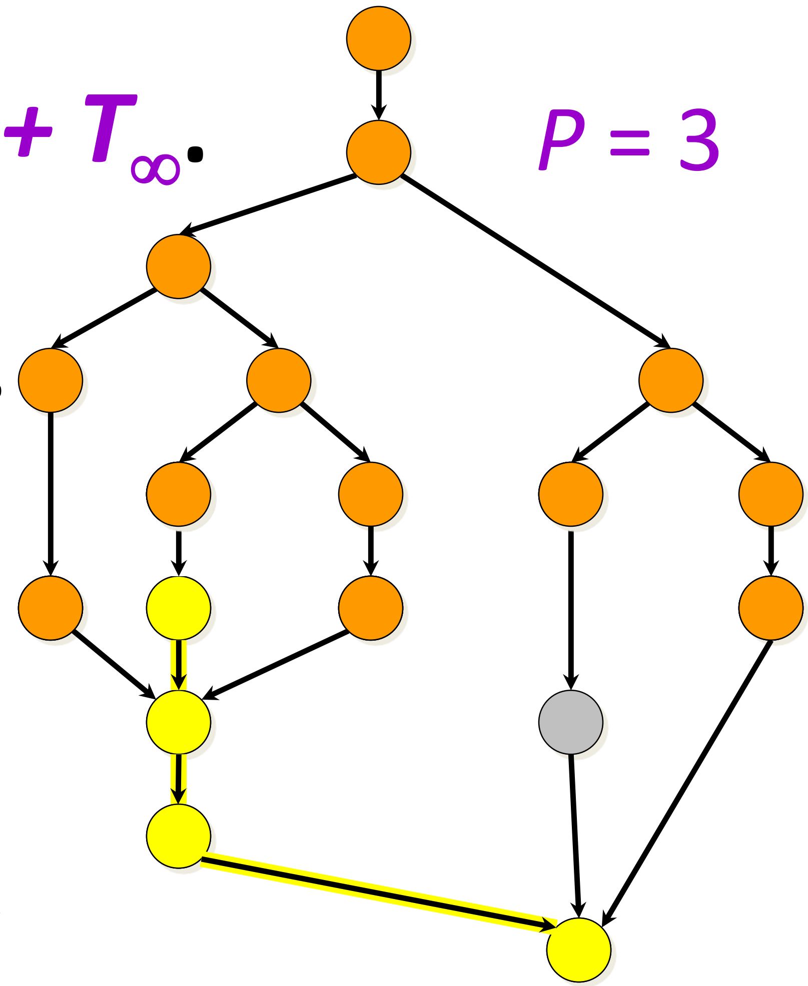
- $T_P \geq T_1/P = T_3 \geq 6$
- $T_P \geq T_\infty = 9$

* Also called ***critical-path length*** or ***computational depth***.

CP: Greedy-Scheduling Theorem

Theorem [Graham '68 & Brent '75]. Any greedy scheduler achieves

$$T_P \le T_1/P + T_\infty.$$



$T_P \le$ # complete steps $+$ # incomplete steps.

Proof. # complete steps $\le T_1/P$, since each complete step performs P work.

- # incomplete steps $\le T_\infty$, since each incomplete step reduces the span of the unexecuted dag by 1. ■

CP: Optimality of Greedy

Any greedy scheduler achieves within a factor of 2 of optimal.

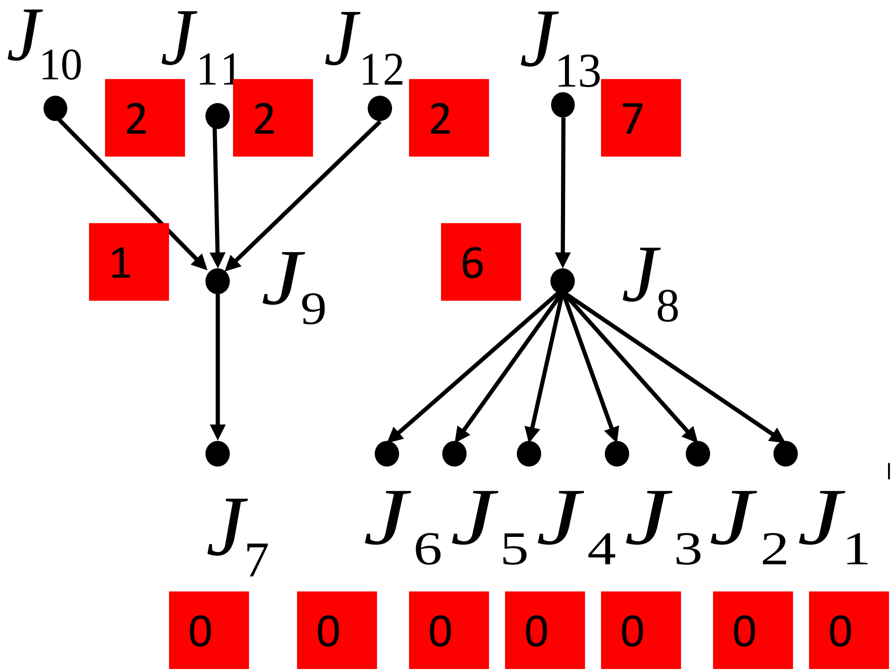
Proof. Let $T_P{}^*$ be the execution time produced by the optimal scheduler. Since $T_P{}^* \geq \max\{T_1/P, T_\infty\}$ (lower bounds), we have

$$\begin{aligned} T_P &\leq T_1/P + T_\infty \\ &\leq 2.\max\{T_1/P, T_\infty\} \\ &\leq 2T_P{}^* . \quad \blacksquare \end{aligned}$$

Most Successors First (MSF)

- Algorithm:
  - Set up a priority list L by nonincreasing order of the jobs' successors numbers.
    - (i.e. the job having more successors should have a higher priority in L than the job having fewer successors)
  - Execute the list scheduling policy based on this priority list L.

Most Successors First algorithm



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M2	J_{12}	J_8	J_6	J_4	J_1
M1	J_{11}		J_5	J_3	

$$L=(J_{13},J_8,J_{12},J_{11},J_{10},J_9,J_7,J_6,J_5,J_4,J_3,J_2,J_1)$$

7 6 2 2 2 1 0 0 0 0 0 0 0

$Pm|\ p_j|\ Cmax$
Minimum makespan scheduling

- $P_m|p_j|C_{max}$ in NPC
- Given processing times for n jobs, $p_1, p_2, \ldots, p_n$, and an integer m
- Find an assignment of the jobs to m identical machines
- So that the completion time, also called the makespan, is minimized.

0-1 Linear Programming Solution to Scheduling Problem

$$x_{ij} = \{0, 1\}$$

whether job j is scheduled in machine i

$$\min \quad T$$

$$\sum_{i=1}^{m} x_{ij} = 1$$

for each job j

Each job is scheduled in one machine.

$$\sum_{j=1}^{n} x_{ij} \cdot p_{ij} \leq T$$

for each machine i

Each machine can finish its jobs by time T

$$0 \leq x_{ij}$$

for each job j, machine i

Minimum makespan scheduling: Arbitrary List

- List Scheduling : Approximation
- Algorithm
  - 1. Order the jobs arbitrarily.
  - 2. Schedule jobs on machines in this order, scheduling the next job on the machine that has been assigned the least amount of work so far.
- Above algorithm achieves an approximation guarantee of 2

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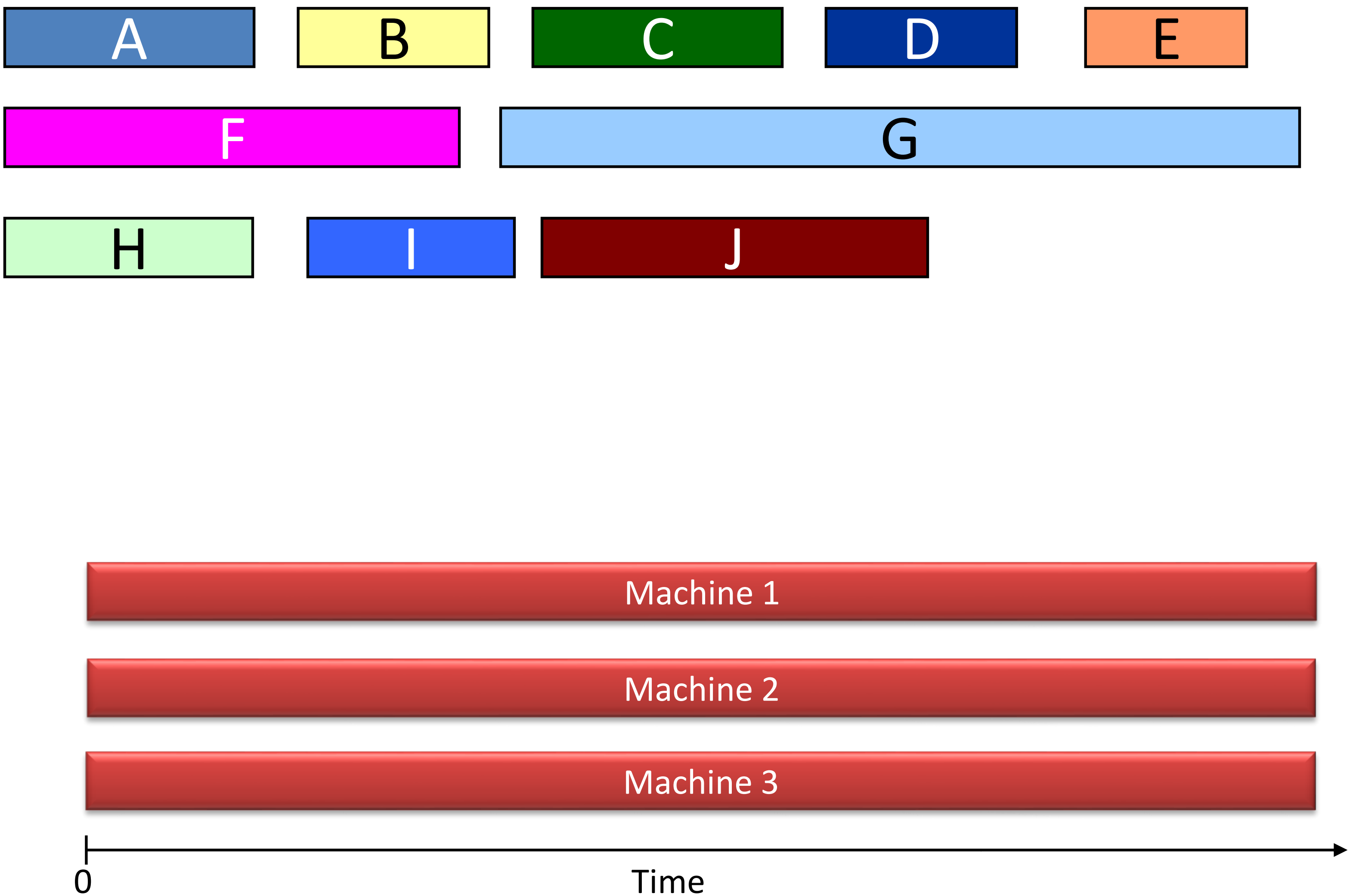
Load Balancing: List Scheduling



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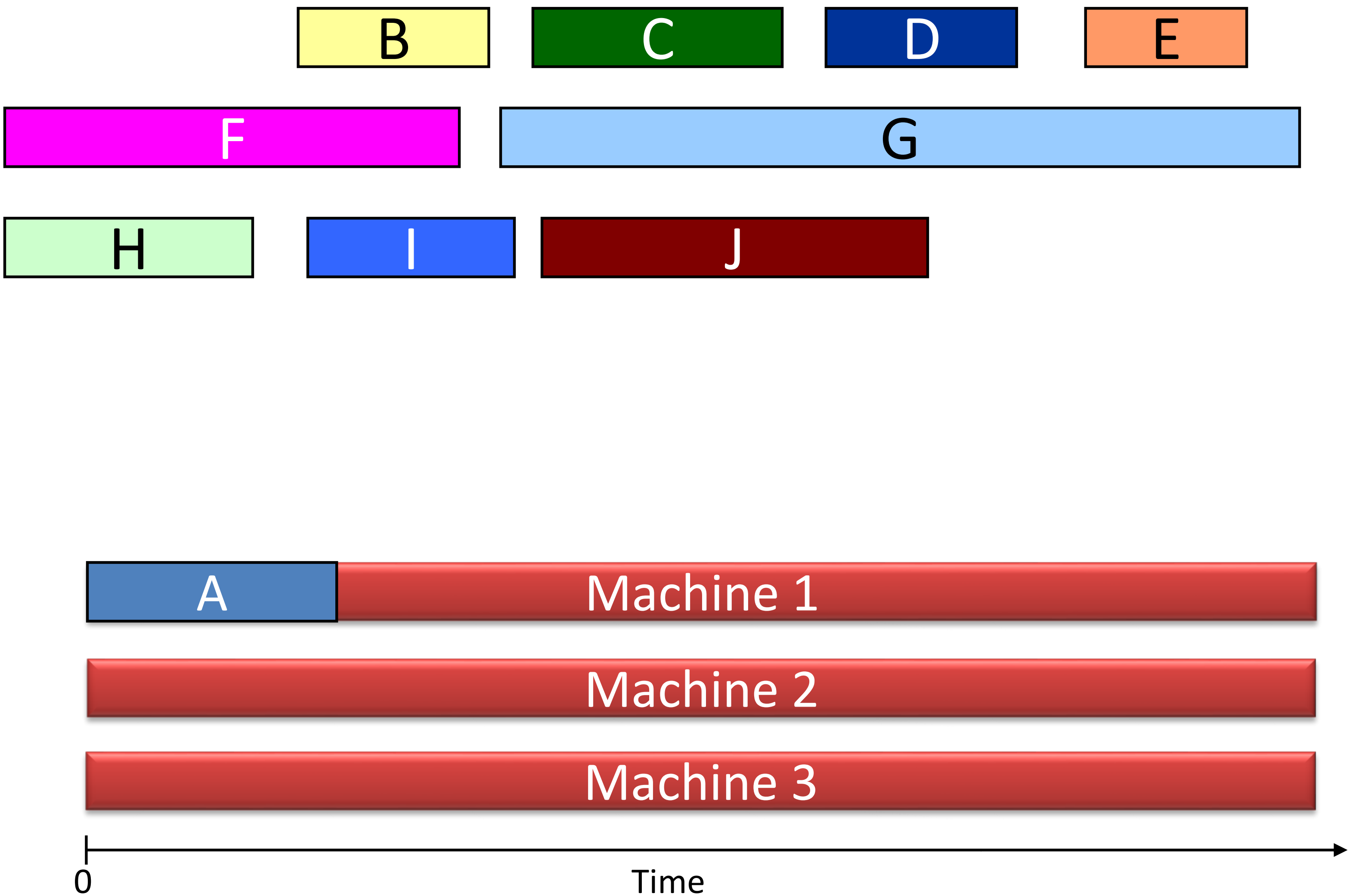

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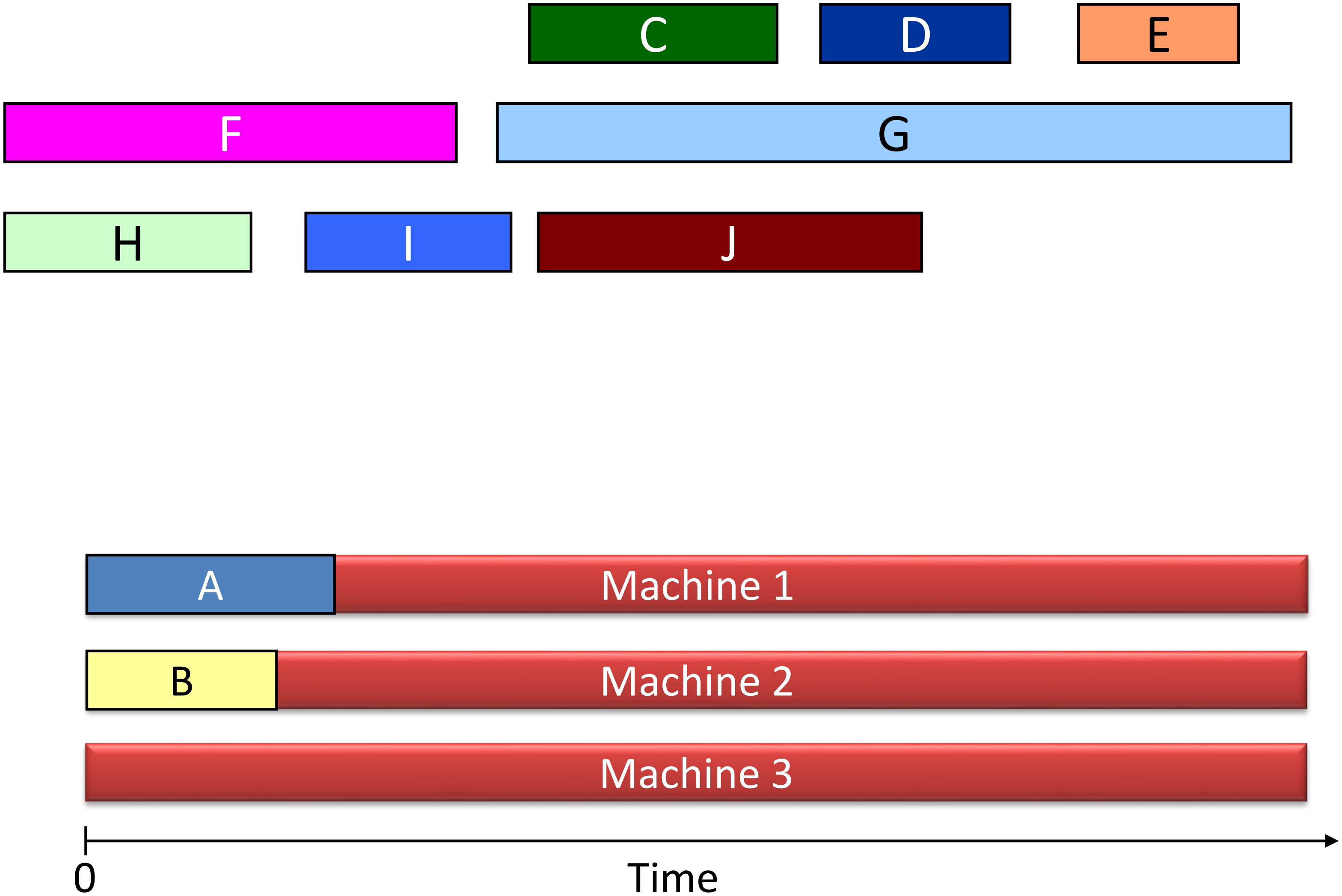

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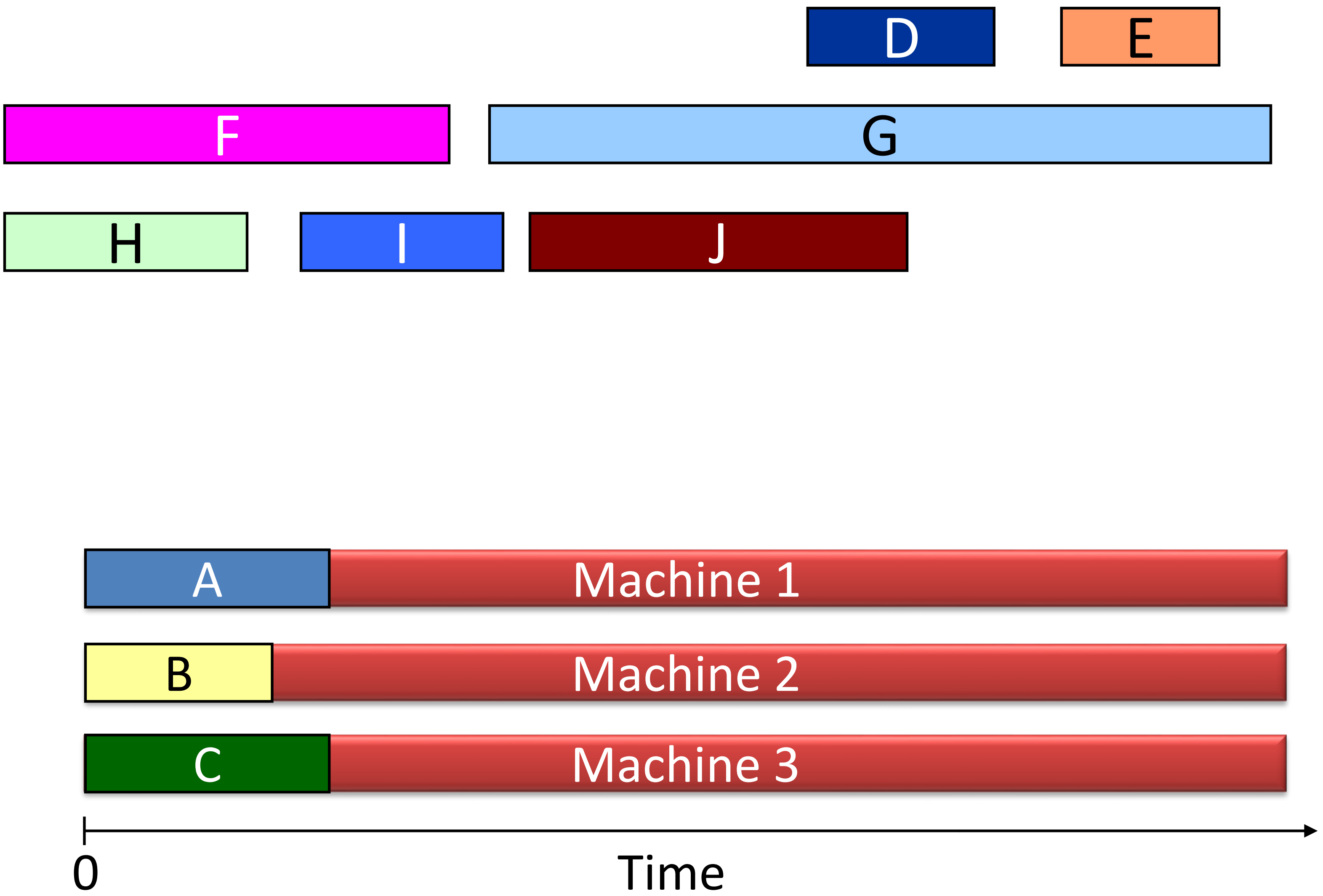

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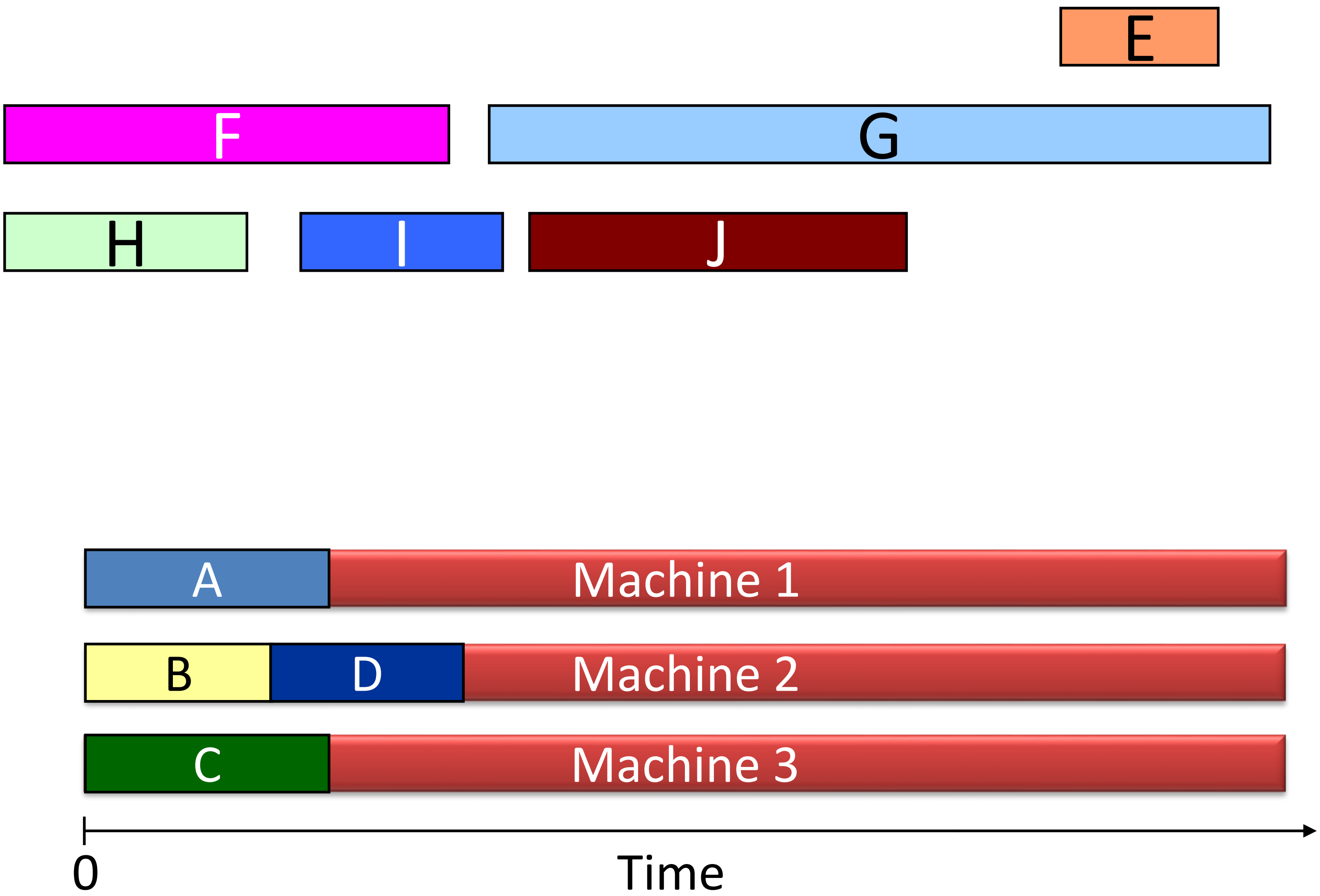

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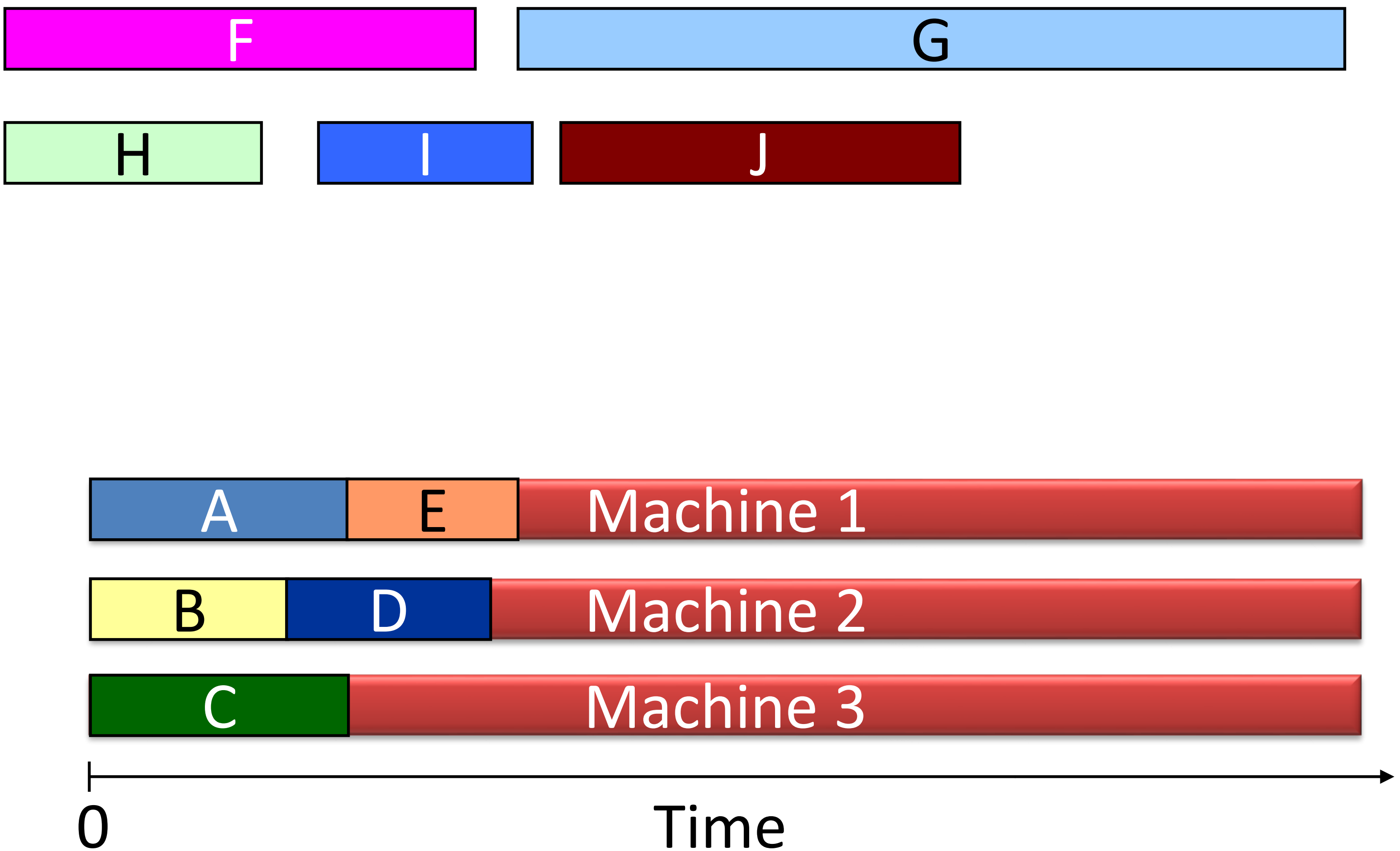

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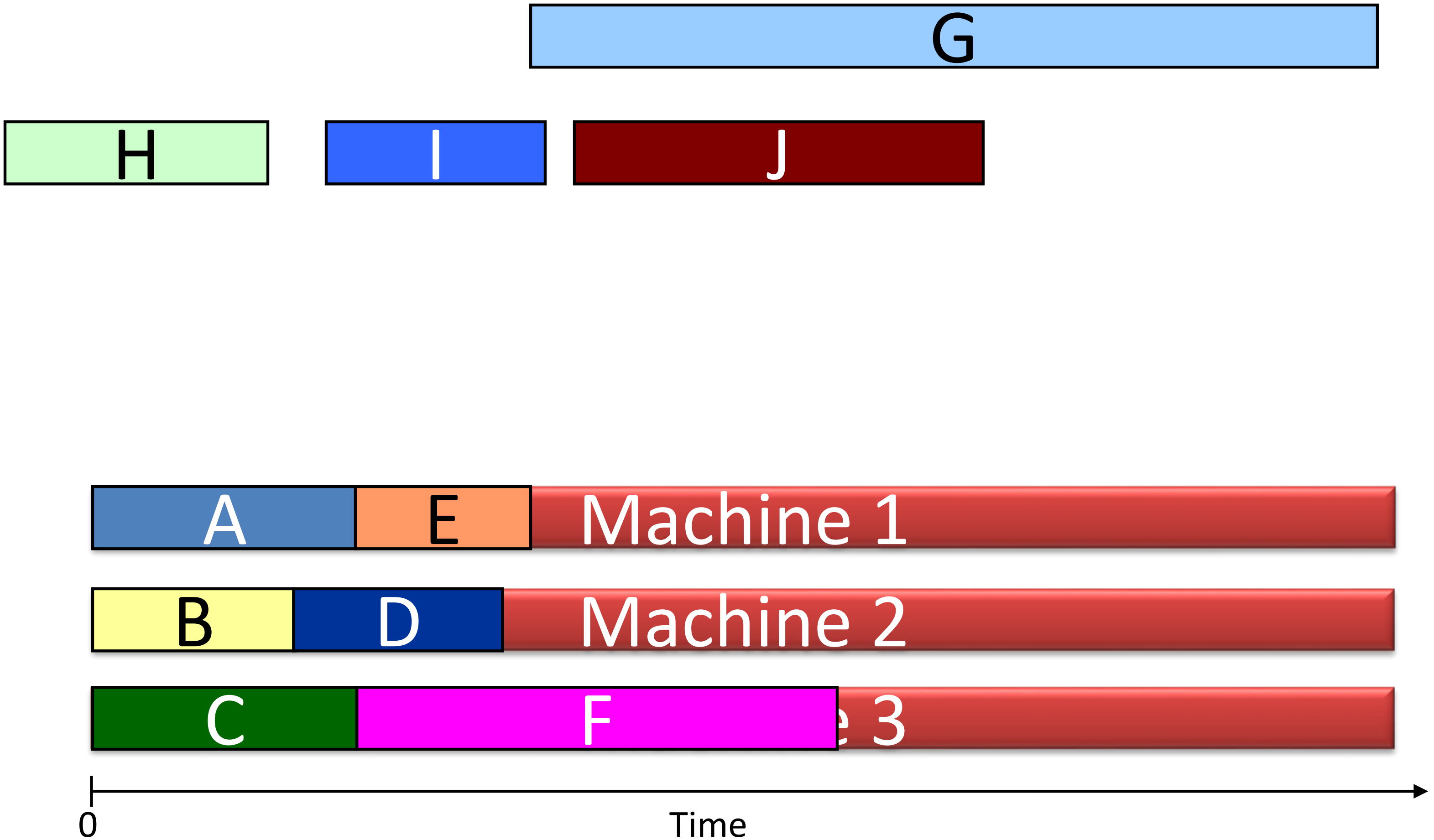

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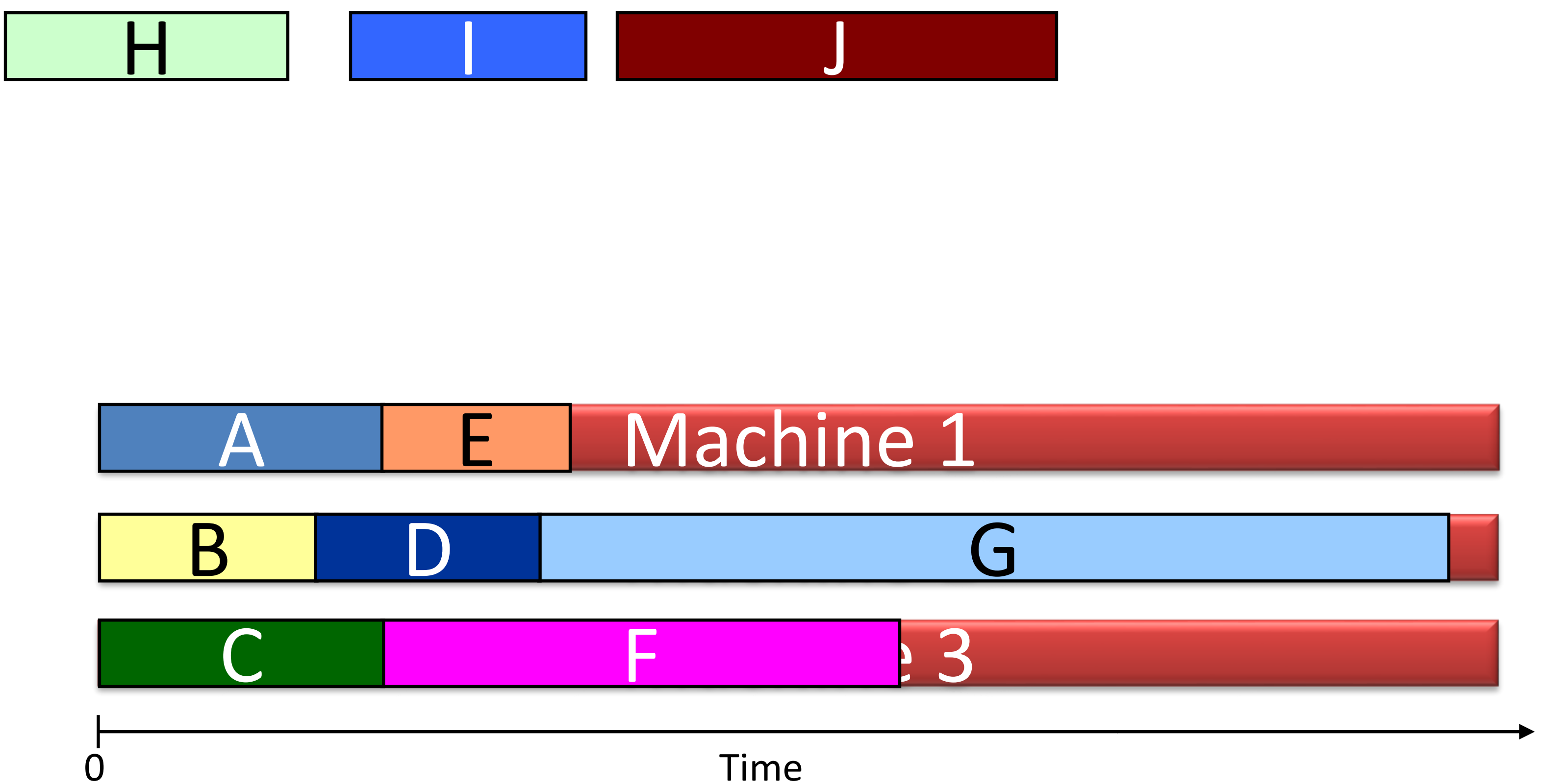

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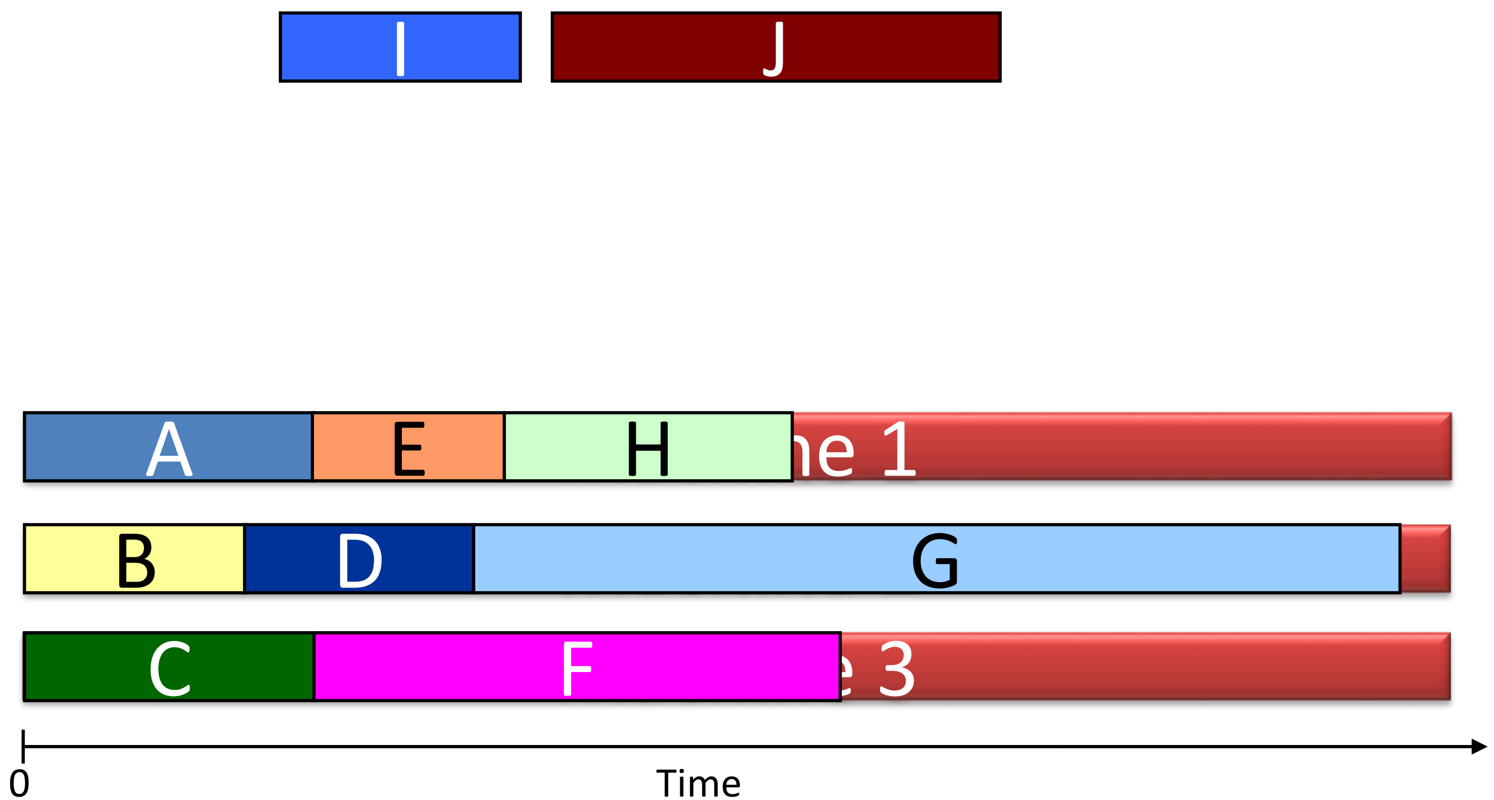

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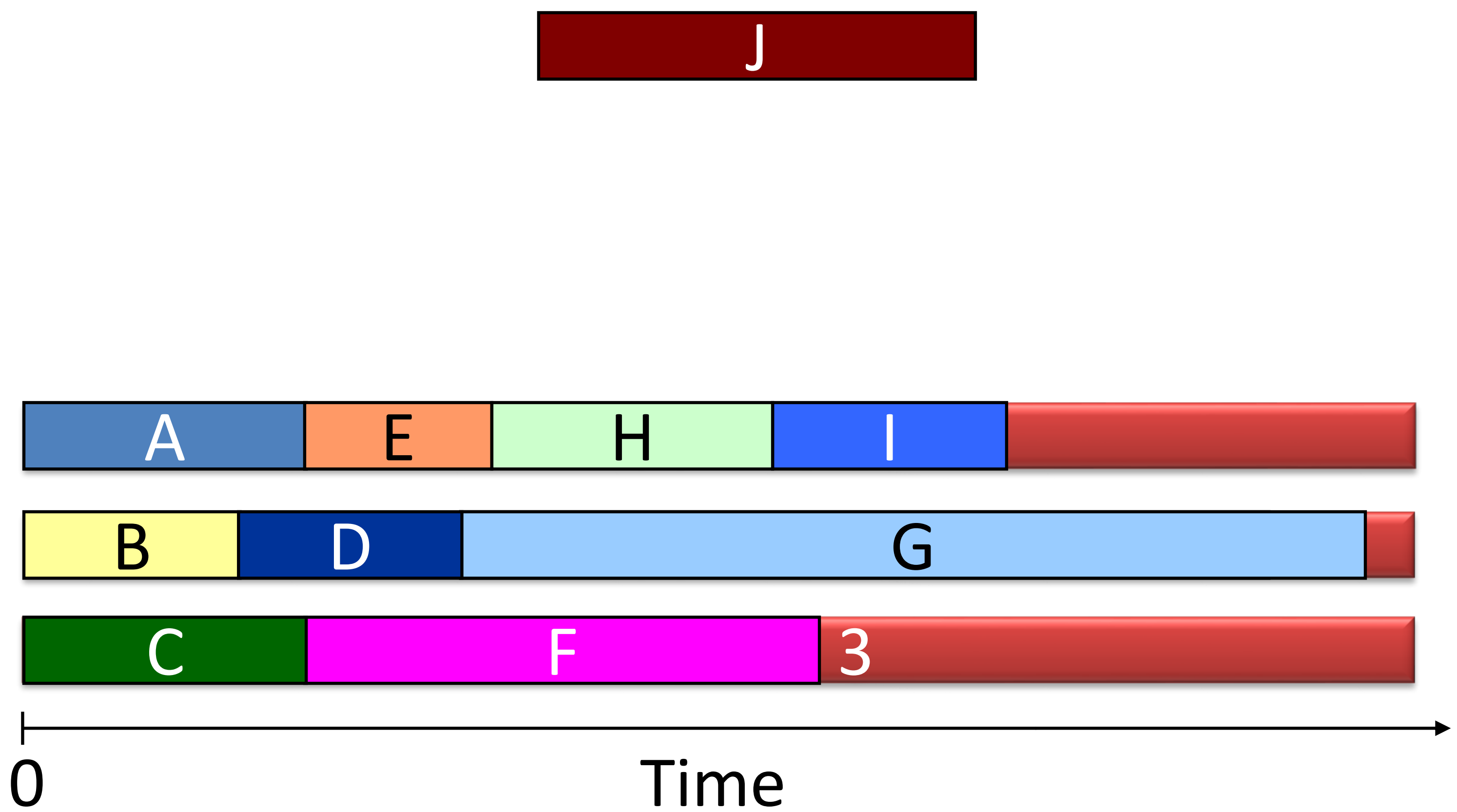

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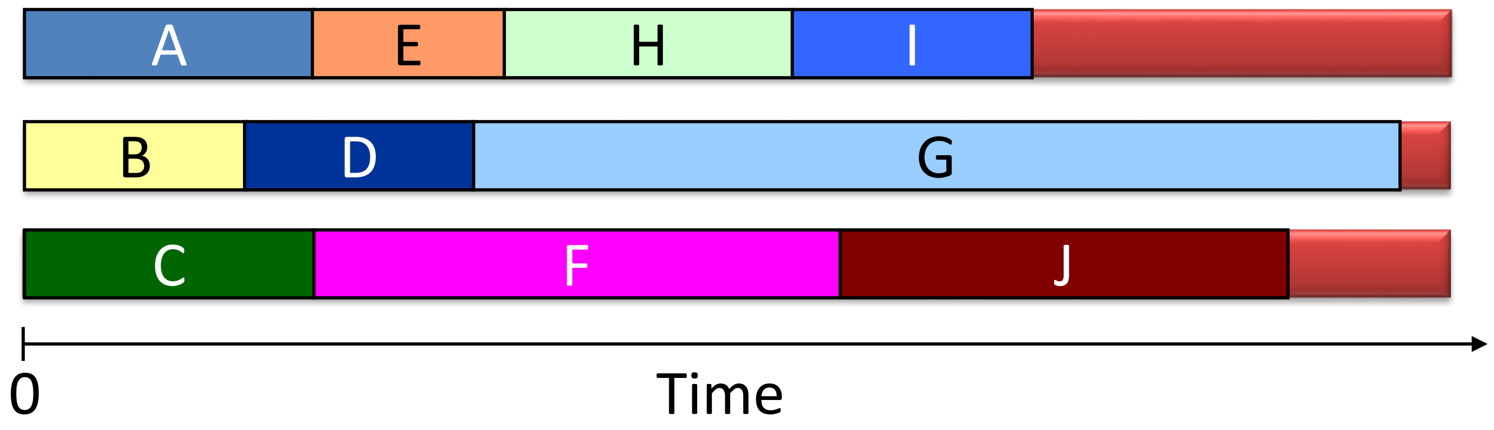

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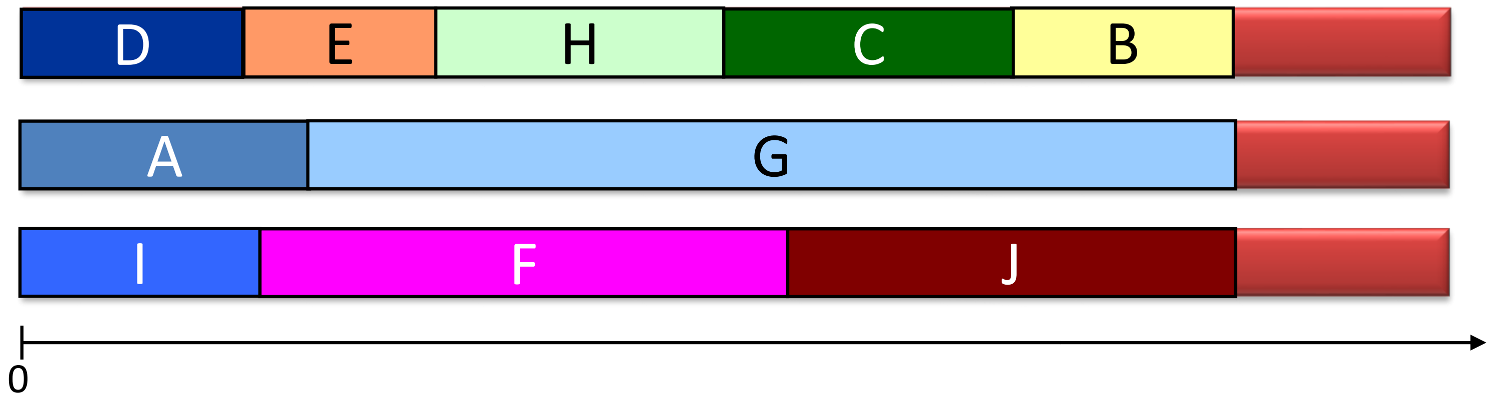
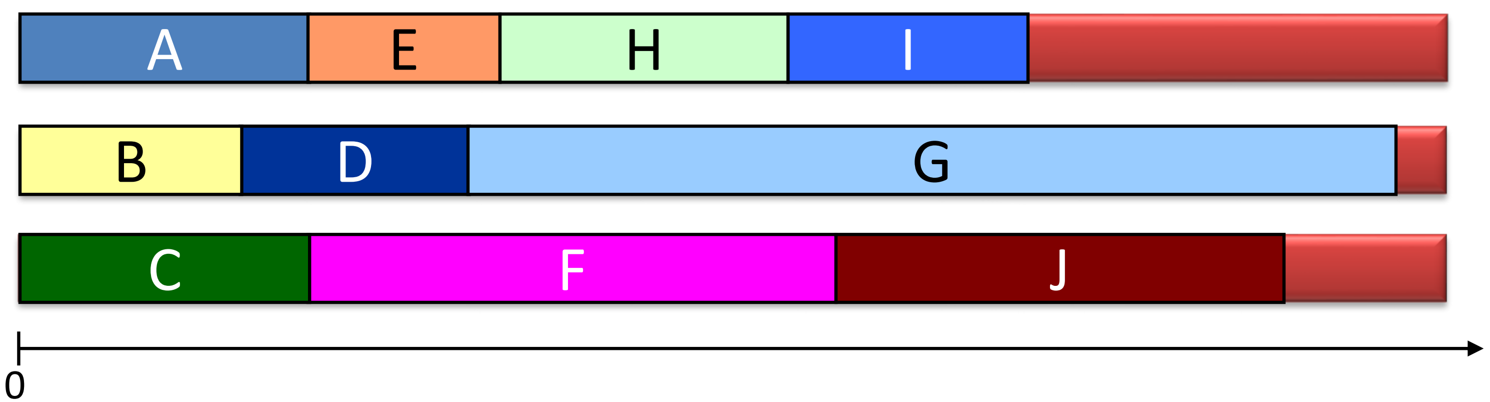


Optimal Schedule



List schedule

LS is 2 APPRX

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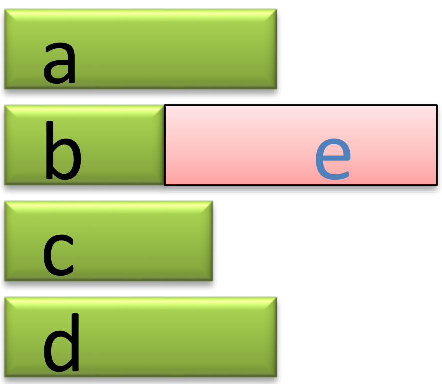
Algorithm: List scheduling

Basic idea: In a list of jobs,

schedule the next one as soon as a machine is free



machine 1

machine 2

machine 3

machine 4

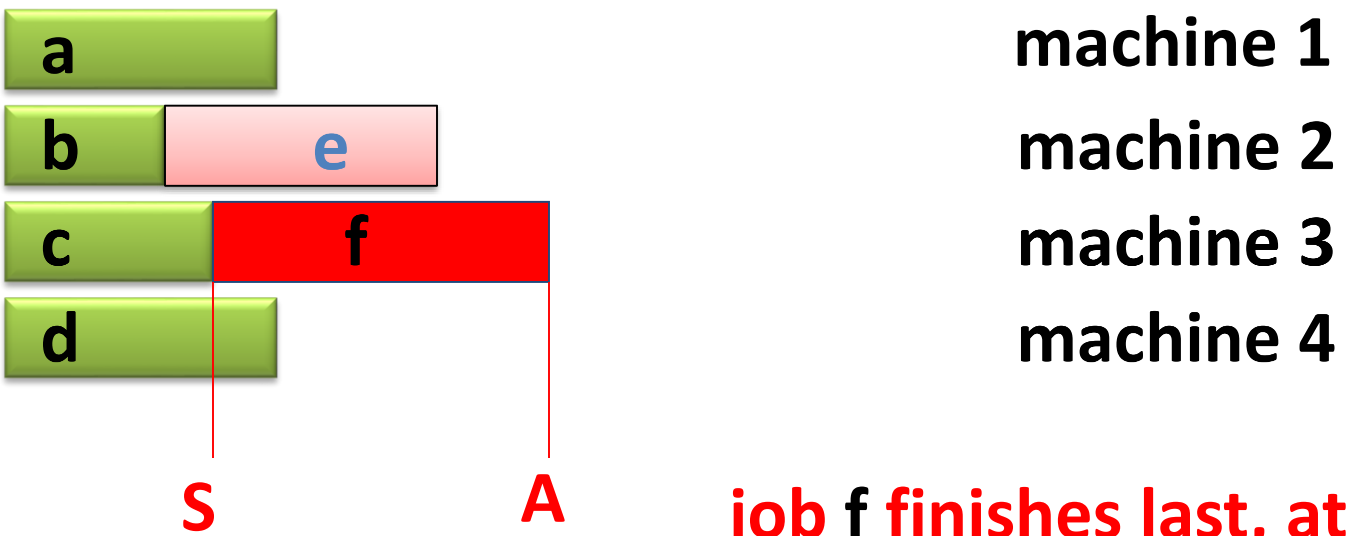
Good or bad ?

List Scheduling is "2-approximation" (Graham, 1966)

Algorithm: **List scheduling**

Basic idea: In a list of jobs,

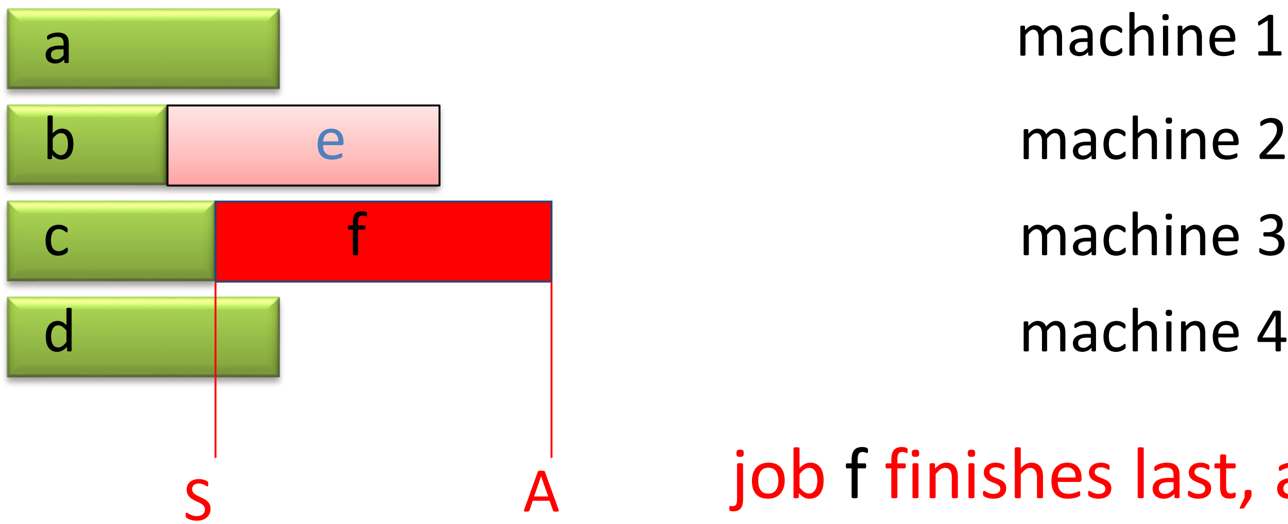
schedule the next one as soon as a machine is free



job **f** finishes last, at time A

compare to time OPT of best schedule: how ?

List Scheduling is "2-approximation"



job f finishes last, at time A

compare to time OPT of best schedule: how ?

(1) job f must be scheduled in the best schedule at some time:
f <= OPT. ➔ **A – S <= OPT.**

(2) up to time S, all machines were busy all the time, and OPT cannot beat that, and job f was not yet included: **S < OPT.**

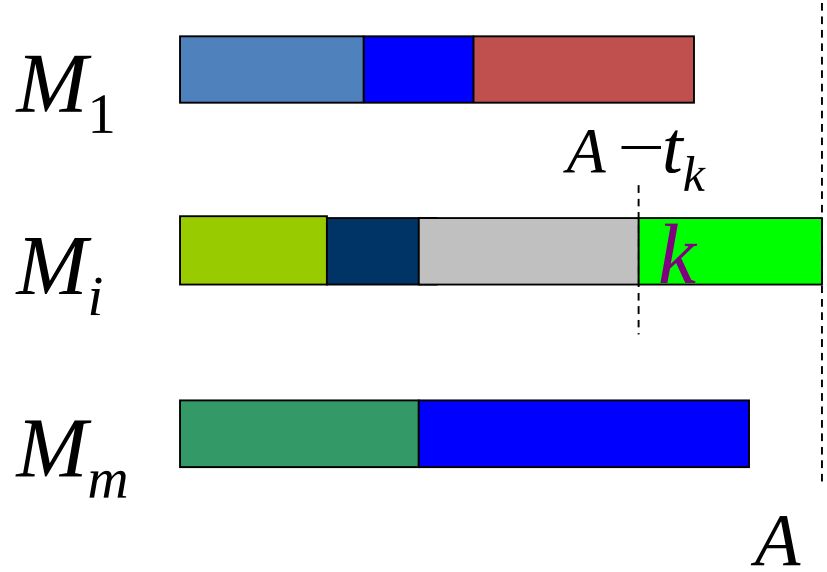
(3) both together: **A = A – S + S = (A-S) +S < 2*OPT.**

"2-approximation" (Graham, 1966)

LS is (2-1/m) APPRX

LS achieves a perf. ratio $2-1/m$.

So all machines are busy from time 0 through A-t_k
Consequently,



Let $\boldsymbol{T} = \sum \boldsymbol{t_i}$, i=1,2...,n

$$T\text{-}t_k \geq m(A-t_k) \quad \rightarrow T\text{-}t_k \geq mA - mt_k$$

$$\rightarrow \quad T\text{-}t_k + mt_k \geq mA \;\rightarrow\; T + (m\text{-}1)t_k \geq mA$$

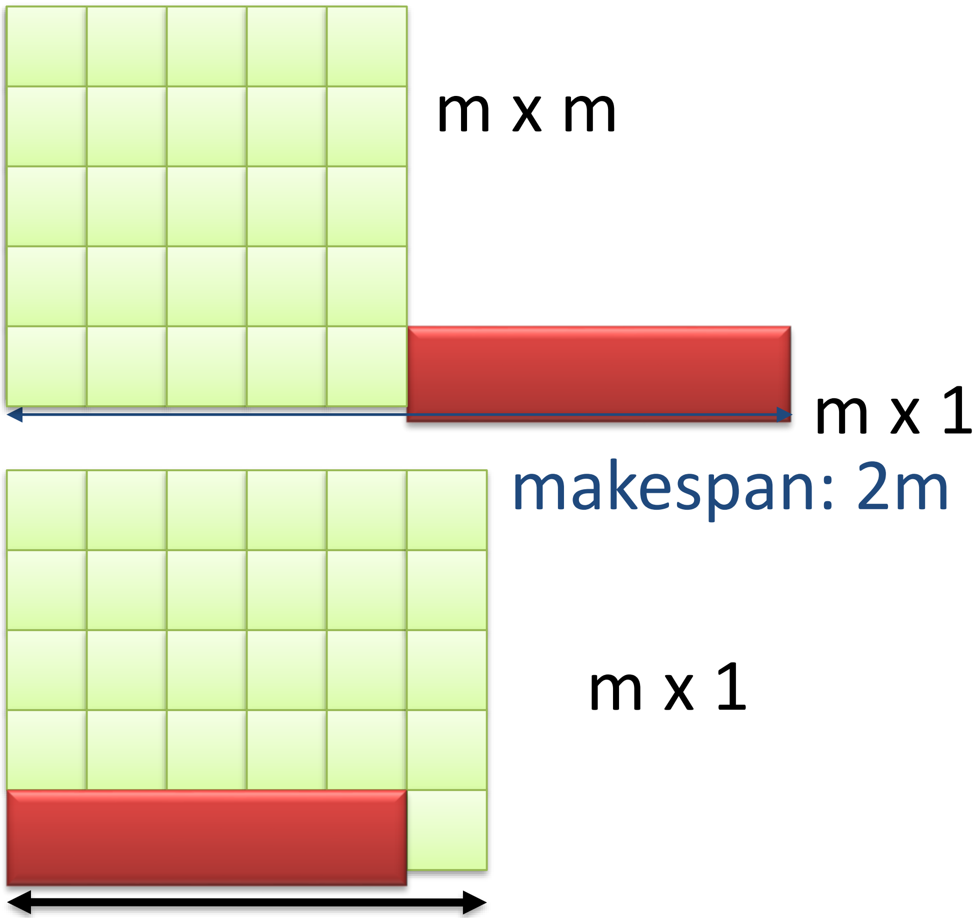
As m. T* ≥ T. So, $T^* \geq T/m$.
Also $T^* \geq t_k$ for every k.

So, $$A \leq T/m + t_k\,(m\text{-}1)/m$$

$$\leq\; T^* + (1\text{-}1/m)\,T^*$$

$$\color{red}{A \leq (2\text{-}1/m)\,T^*}$$

Example: Worst Case



makespan: m+1

LPT Rule: List with LPT

- List scheduling can do badly if long jobs at the end of the list spoil an even division of processing times.
- We now assume that the jobs are all given ahead of time, i.e. the LPT rule works only in the off-line situation. Consider the "***Largest Processing Time first***" or LPT rule that works as follows.

LPT Rule: List with LPT

LPT Algorithm

1 sort the jobs in order of decreasing processing times: $t_1 \geq t_2 \geq . . . \geq t_n$

2 execute list scheduling on the sorted list

3 **return** the schedule so obtained.

- The LPT rule achieves 3/2-Approx **Sec 11.1 of Eva Tardos Algo Book, Appx Algo Chapter**
- The LPT rule achieves a performance ratio $4/3-1/(3m)$. **Prove out of Syllabus**

LPT 3/2-Approx: Jobs are sorted

- Job Time: $t_1 \geq t_2 \geq t_3 \geq \ldots \geq t_j$
- Suppose j (=m+1) jobs (j>m), in LPT $T^* \geq 2.t_{m+1}$

- Examples: m =5 , j =6

  10, 9, 8,7, 5, 4,...

  $t_{m+1}=4$

  $T^* \geq 2*4=8$

LPT 3/2-Approx: Jobs are sorted

- Job Time: $t_1 \geq t_2 \geq t_3 \geq \ldots \geq t_j$
- Suppose j (=m+1) jobs (j>m), in LPT $T^* \geq 2.t_{m+1}$
- Suppose a machine M_i have at least two jobs and t_j be last job ($j \geq m+1$) assigned to M_i

$$t_j \leq t_{m+1} \leq T^*/2$$

- Also we have $t_j \leq T^*$ and $T_i - t_j \leq T^*$, where T_i is sum of ET of task assigned to M_i
- $T_i - t_j \leq T^* \Rightarrow T_i \leq T^* + t_j \Rightarrow T_i \leq T^* + T^*/2$

$$T_i \leq (3/2)\, T^*$$

$P|p_j=1|\Sigma w_j U_j$

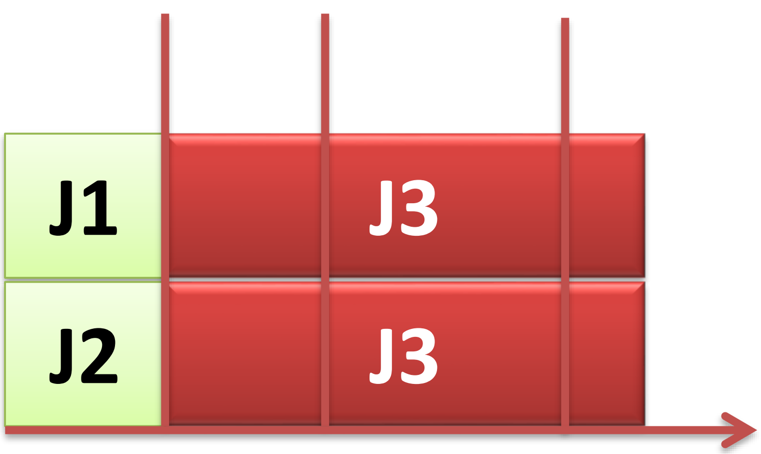
- Sorting task based on d_i and $d_1 \le d_2 \le \ldots \le d_n$
- Approach 1: Simply scheduling and rejecting the unfit task will not minimize w_i
  - **Will not work : you need to take care of weight**
- Approach 2: Sorting task based on w_i/d_i
  - Gives priority of task with higher weight but
  - Simply may reject a task based on deadline
  - **Will not work : for optimality**

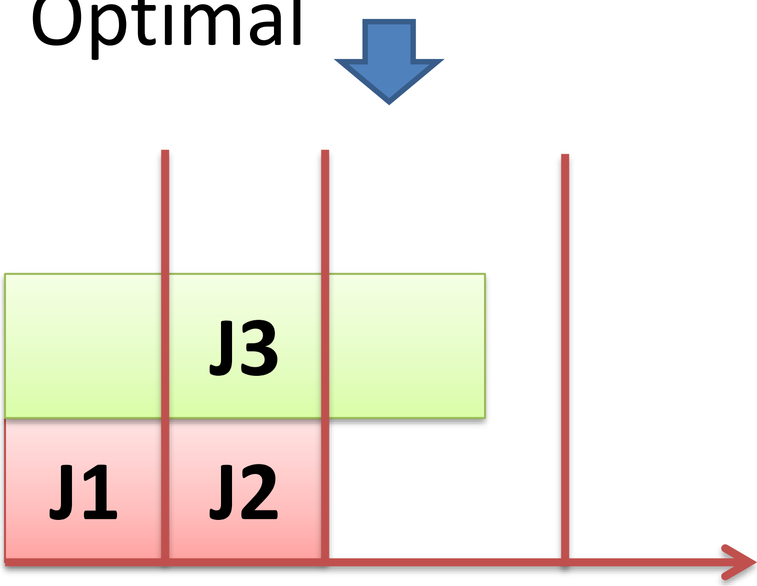
$P|p_j=1|\Sigma w_j U_j$

- Sort all the jobs with $d_1 \le d_2 \le \ldots \le d_n$
- Set $S=\Phi$
- For i=1 to n do
  - If (i_{th} task is late when scheduled in the earliest time slot on a machine)
    - Find a task i* with w_i^* = min weight of tasks in the already scheduled tasks of the set S
    - If ($w_i^* < w_i$) replace i* with i_{th} task in the schedule and in S.
  - else add i_{th} task to S and schedule the task in the earliest time slot

P||ΣU_j

- NPC: Sorting based on deadlines is excellent heuristics for most of the case, Experimentally
- But not optimal
- Counter example: J(p_j, d_j): J1(1,1), J2(1,2) and J3(3,3.5) on two processor
- EDF (J3 misses) but the Optimal





$P|ptmn|\Sigma U_j$

- In NPC

$Q|ptmn|\Sigma C_j$

- **LPT on High speed is good to optimize Σe_j the sum of task execution time but not ΣC_j**
- **Modified version of SPT** (shortest remaining time) rule. **As ΣC_j include waiting time of all the tasks**
- Order the tasks according to non-decreasing processing time.
- Schedule task 1 on available highest speed machine up to time $t_1=p_1/s_1$.
- Schedule 2nd task on M2 for t_1 time and then on M_1 from time t1 to time $t_2 \geq t_1$ until it is completed and same process continues

$Q|ptmn|\Sigma C_j$

- Example m=3, s1=3, s2=2, s3=1 and n=4, p1=10, p2=8, p3=8, p4=3
- SRT Job J_4 get scheduled on M1 with speed s1 for 1 time unit. Job 3 get scheduled on M_2 upto time 1 and then shifted to M1. Gant chat is given bellow with ΣC_i =14