MA 102 (Mathematics II)

Tutorial Sheet No. 10 **Ordinary Differential Equations** **April 18, 2019**

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1. Suppose that the square matrix A has a negative eigenvalue. Show that the linear system $\mathbf{x}' = A\mathbf{x}$ has at least one nontrivial solution that satisfies $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

2. If $A = P^{-1}\,\mathrm{diag}[\lambda_j]\,P$, show that $\det e^A = e^{\mathrm{trace}\,A}$. Verify this fact for any 2×2 matrix A.

3. Find a fundamental matrix of the linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ by computing e^{At}.

$$(a)\ A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix},\quad (b)\ A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix},\quad (c)\ A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad (d)\ A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}.$$

4. Solve the initial value problems $\mathbf{x}'(t) = A\mathbf{x}(t),\ \mathbf{x}(0) = \mathbf{x}_0$ for the matrix

$$(a)\ A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix},\ (b)\ A = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix},\ (c)\ A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(d)\ A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

5. Let $\mathbf{x}(t)$ be a nontrivial solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $A + A^T$ is positive definite. Prove that $\|\mathbf{x}(t)\|$ is an increasing function of t. (Here, $\|\cdot\|$ denotes the Euclidean norm.)

6. Let A be a real 3×3 matrix such that $A^T = -A$. Let $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$ be a real solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Prove that
   (a) $\|\mathbf{x}(t)\|$ is independent of t.
   (b) If $\mathbf{v} \in Ker(A)$ then $\mathbf{x}(t)\cdot\mathbf{v}$ is independent of t.

7. Solve the nonhomogeneous linear system $\mathbf{x}(t) = A\mathbf{x}(t) + \mathbf{f}(t)$ with initial condition $\mathbf{x}(0) = [1, 0]^T$, where $(a)\ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix};\ (b)\ A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} t \\ 1+2t \end{bmatrix}.$

8. Show that $\Phi(t) = \begin{bmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{bmatrix}$ is a fundamental matrix solution of the nonautonomous linear system $\mathbf{x}'(t) = A(t)\mathbf{x}$ with $A(t) = \begin{bmatrix} -2\cos^2 t & -1-\sin 2t \\ 1-\sin 2t & -2\sin^2 t \end{bmatrix}$. Find the inverse of $\Phi(t)$ and solve $\mathbf{x}'(t) = A(t)\mathbf{x} + \mathbf{f}(t),\ \mathbf{x}(0) = \mathbf{x}_0$ with $A(t)$ as given above and $\mathbf{f}(t) = [1,\ e^{-2t}]^T$.

9. Find all critical points of each of the following plane autonomous systems:
   (a) $x_1'(t) = -x_1 + x_2,\ \ x_2'(t) = x_1 - x_2$; (b) $x_1'(t) = x_1^2 + x_2^2 - 6,\ \ x_2'(t) = x_1^2 - x_2$.
   (c) $x_1'(t) = x_1^2 e^{x_2},\ \ x_2'(t) = x_2(e^{x_1} - 1)$.

10. Determine the nature of critical point $(0,0)$ of each of the linear autonomous systems $\mathbf{x}'(t) = A\mathbf{x}(t)$, and determine whether or not the critical point is stable.

$$(a)\ A = \begin{bmatrix} 5 & -3 \\ 4 & -3 \end{bmatrix},\ (b)\ A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix},\ (c)\ \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}.$$