I. Introduction to $\mathbb{R}^n$: Its algebra, geometry

1. Consider the weather at any place on the earth at one instance. Weather at a place (approximately) constitutes of three components: the temperature, pressure and the humidity at that place. Also, when we say "a place", it can be precisely described by a latitude and a longitude. In this form, the weather is a "function" having three component functions namely, the temperature, pressure and the humidity, each of which in turn is a real valued function of two independent variables, the latitude and the longitude.

$$weather(latitude, longitude) = (temperature(lat., long.), pressure(lat., long.), humidity(lat., long.))$$

2. The input space to the weather function was two independent real numbers, the latitude and the longitude, although restricted to a range.

3. More generally, the input space can be (a restricted range or the entire set of) n independent real variables. In order to model this space, we define the following.

4. $\mathbb{R}^n$ is the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{R}$. A point in $\mathbb{R}^n$ is denoted by bold letters $\mathbf{x}, \mathbf{y}$ etc..

5. The real numbers $x_1, x_2, \ldots x_n$ are called the components or coordinates of the point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. In particular x_i is the i^{th} component or coordinate of this point.
   (a) $\mathbb{R}^2$ is a model to give (names) co-ordinates to the plane using rectangular co-ordinates x and y. Picture of the xy- plane and the geometric significance of the numbers x and y for a given point.
   (b) $\mathbb{R}^3$ is a model to give co-ordinates to the 3-D space using the rectangular co-ordinates x, y and z. Picture of the xyz- space and the geometric significance of the numbers x, y and z for a given point.
   (c) Analogously, $\mathbb{R}^n$ is a model to give co-ordinates to the n-dimensional space.

6. A vector is a quantity having a magnitude and direction.

7. $\mathbb{R}^n$ is also used to model the set of all vectors at a given point in the point space $\mathbb{R}^n$.

8. For each $\mathbf{x} \in \mathbb{R}^n$ there is a set $\mathbb{R}^n_{\mathbf{x}}$ of all possible vectors based at $\mathbf{x}$. We call the elements of $\mathbb{R}^n_{\mathbf{x}}$ as vectors based at $\mathbf{x}$.

9. Axiom: For $\mathbf{x}, \mathbf{y}$ in the point space $\mathbb{R}^n$, we will make no distinction between $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\mathbf{x}}$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\mathbf{y}}$. Henceforth, we will equate this vector $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\mathbf{x}}$ with the vector $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\mathbf{y}}$ and denote either of the spaces by just $\mathbb{R}^n$.

10. We will treat the points in the point space $\mathbb{R}^n$ also as "position vectors". Denote this by an arrow starting from *origin* (the point $(0, 0, \ldots, 0) \in \mathbb{R}^n$), and ending at the given point.

11. Operations of addition and scaling in $\mathbb{R}^n$.
    (a) Addition: $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
    (b) Scaling: $\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$

12. Geometric significance of addition and scaling of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

13. The dot product of vectors in $\mathbb{R}^n$: $(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = x_1y_1 + x_2y_2 + \ldots + x_ny_n$

14. Geometric significance of dot product of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

15. The length of a vector or *Euclidean norm* of a vector $\mathbf{x}$ in $\mathbb{R}^n$: $\sqrt{\mathbf{x} \cdot \mathbf{x}}$. This is denoted by $\|\mathbf{x}\|$.

16. The Cauchy Schwarz inequality: For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$. Proof: For every real t, $\|\mathbf{x} - t\mathbf{y}\|^2 \geq 0$. So $(\mathbf{x} - t\mathbf{y}) \cdot (\mathbf{x} - t\mathbf{y}) \geq 0$ *i.e.* $t^2(\mathbf{y} \cdot \mathbf{y}) - 2t\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} \geq 0$. This implies that the discriminant of the expression of the L.H.S. of the inequality is ≤ 0. This gives $4(\mathbf{x} \cdot \mathbf{y})^2 \leq 4\|x\|^2\|y\|^2$, which implies $|\mathbf{x} \cdot \mathbf{y}| \leq \|x\|\|y\|$.

17. The (Euclidean) distance d between two points $\mathbf{x}$ and $\mathbf{y}$ in the point space $\mathbb{R}^n$ is now the length of the vector $\mathbf{x}-\mathbf{y}$ *i.e.* $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \sqrt{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})}$. This gives the familiar formula for the distance between two points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ as $\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \ldots + (x_n-y_n)^2}$
18. The triangle inequality: For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
19. In $\mathbb{R}^2$, a circle with center $\mathbf{x}$ and radius r is the set $\{\mathbf{y} \in \mathbb{R}^2 : \|\mathbf{y}-\mathbf{x}\| = r\}$.
20. In $\mathbb{R}^3$, a sphere with center $\mathbf{x}$ and radius r is the set $\{\mathbf{y} \in \mathbb{R}^3 : \|\mathbf{y}-\mathbf{x}\| = r\}$.
21. The length of a straight line segment from one point $\mathbf{x}$ to another point $\mathbf{y}$ can be computed by using the length of the change vector $\mathbf{x}-\mathbf{y}$. What about the length of a curve? We will see how to do this later.
22. Angle θ between two vectors $\mathbf{x}$ and $\mathbf{y}$ at a given point is given by $\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$
23. With distance and angle at hand, areas of triangles, polygons in $\mathbb{R}^2$ and $\mathbb{R}^3$, and volumes of polyhedrons in $\mathbb{R}^3$, can be computed. What about the area of a sphere? or a more irregular surface in $\mathbb{R}^3$? We will see how to do this later.

II. Subsets of $\mathbb{R}^n$

1. Definition: A sequence in $\mathbb{R}^n$ is a function $f : \mathbb{N} \to \mathbb{R}^n$.
2. For each $k \in \mathbb{N}$, $f(k) \in \mathbb{R}^n$ is written as $\mathbf{x}(k)$ and the sequence itself is denoted by $(\mathbf{x}(k))$. Note that $\mathbf{x}(k) = (x_1(k), x_2(k), \ldots, x_n(k))$. So $x_i(k)$ is a sequence of real numbers, called the i^{th} component sequence of the sequence $(\mathbf{x}(k))$.
3. An example of a sequence in $\mathbb{R}^2$ is $(\frac{1}{k}, 3 + \frac{k}{k^2+1})$. Here $(\frac{1}{k})$ is the first component sequence and $(3 + \frac{k}{k^2+1})$ is the second component sequence. We note that both of these sequences are convergent sequences in $\mathbb{R}$.
4. Definition: A sequence $(\mathbf{x}(k))$ in $\mathbb{R}^n$ is said to converge to $\mathbf{x}$ in $\mathbb{R}^n$, if for every given $\epsilon > 0$ there is a corresponding natural number N such that $\|\mathbf{x}(k) - \mathbf{x}\| < \epsilon$ for all $k \geq N$.
5. Proposition: A sequence $(\mathbf{x}(k))$ in $\mathbb{R}^n$ converges to $\mathbf{x}$ in $\mathbb{R}^n$ if and only if the i^{th} component sequence $(x_i(k))$ of the sequence $(\mathbf{x}(k))$ converges to the i^{th} component x_i of $\mathbf{x}$ for every i such that $1 \leq i \leq n$.

   Proof: Given $|x_i(k) - x_i|^2 \leq \Sigma_{j=1}^n |x_j(k) - x_j|^2$ gives the proof in one direction. In the other direction, given $\epsilon > 0$, there is an N_i for the i^{th} component sequence, such that the WC holds. Use the max of N_i, so that the WC holds uniformly for all i. Now prove convergence.
6. Let ϵ be a positive real number. An ϵ-ball centered at $\mathbf{x} \in \mathbb{R}^n$ is the set $\{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y}-\mathbf{x}\| < \epsilon\}$. This will be denoted by $B_\epsilon(\mathbf{x})$. This is also referred to as the ϵ-neighborhood of $\mathbf{x}$.
7. A subet S of $\mathbb{R}^n$ is said to be bounded if there is a real number M such that $\|\mathbf{x}\| \leq M$ for every $\mathbf{x} \in S$. If there is no such number M for a set S, then we say that the set is unbounded. In other words, a set is unbounded if for every real number r, there is a $\mathbf{x} \in S$ such that $\|\mathbf{x}\| > r$.
8. Examples of bounded sets: unit disk.
9. Examples of unbounded sets: $\mathbb{R}^n$ itself is unbounded, horizontal strips, vertical strips, (range of) elements of a sequence, at-least one of whose components diverges to ∞.
10. Closed sets: If every convergent sequence in the set, converges to a point in the set, then we say the set is closed. Eg: Closed unit disk, rectangular strips, infinite and finite lines etc. in $\mathbb{R}^2$, closed boxes etc. in $\mathbb{R}^3$.
11. Compact sets: A subset of $\mathbb{R}^n$ is compact if it is both closed and bounded.
12. Bolzano-Weierstrass theorem: Suppose that A is a closed and bounded set in $\mathbb{R}^n$. Every sequence in A has a subsequence that converges to a point in A. Idea of proof.