The Method of Frobenius

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If either $p(x)$ or $q(x)$ in

$$y'' + p(x)y' + q(x)y = 0$$

is not analytic near x_0, then power series solutions near x_0 may or may not exist.

Example: Try to find a power series solution of

$$x^2y'' - y' - y = 0 \tag{1}$$

about the point $x_0 = 0$.

Assume that a solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

exists.

Substituting this series in (1) leads to the recursion formula

$$a_{n+1} = \frac{n^2 - n - 1}{n+1} a_n.$$

The ratio test shows that this power series converges only for $x = 0$. Thus, there is no power series solution valid in any open interval about $x_0 = 0$. This is because (1) has a singular point at $x = 0$.

The method of Frobenius is a useful method to treat such equations.

Cauchy-Euler equations revisited

Recall that a second order homogeneous Cauchy-Euler equation has the form

$$ax^2y''(x) + bxy'(x) + cy(x) = 0, \quad x > 0, \tag{2}$$

where $a(\neq 0)$, b, c are real constants. Writing (2) in the standard form as

$$y'' + p(x)y' + q(x)y = 0, \text{ where } p(x) = \frac{b}{ax}, \ \ q(x) = \frac{c}{ax^2}.$$

Note that $x = 0$ is a singular point for (2). We seek solutions of the form

$$y(x) = x^r$$

and then try to determine the values for r.

Set

$$L(y)(x) := ax^2y''(x) + bxy'(x) + cy(x) \text{ and } w(r,x) := x^r.$$

Now

$$\begin{aligned} L(w)(x) &= ax^2r(r-1)x^{r-2} + bxrx^{r-1} + cx^r \\ &= \{ar^2 + (b-a)r + c\}x^r. \end{aligned}$$

Thus,

$$w = x^r \text{ is a solution } \iff r \text{ satisfies}$$

$$ar^2 + (b-a)r + c = 0. \tag{3}$$

The equation (3) is known as the auxiliary or indicial equation for (2).

Case I: When (3) has two distinct roots r_1, r_2. Then

$$L(w)(x) = a(r - r_1)(r - r_2)x^r.$$

The two linearly independent solutions are

$$y_1(x) = w(r_1, x) = x^{r_1}, \quad y_2(x) = w(r_2, x) = x^{r_2} \text{ for } x > 0.$$

Case II: When $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$. Then

$$\begin{aligned} x^{\alpha+i\beta} = e^{(\alpha+i\beta)\ln x} &= e^{\alpha \ln x} \cos(\beta \ln x) + i\, e^{\alpha \ln x} \sin(\beta \ln x) \\ &= x^\alpha \cos(\beta \ln x) + i\, x^\alpha \sin(\beta \ln x). \end{aligned}$$

Thus, two linearly independent real-valued solutions are

$$y_1(x) = x^\alpha \cos(\beta \ln x), \quad y_2(x) = x^\alpha \sin(\beta \ln x).$$

Case III: When $r_1 = r_2 = r_0$ is a repeated roots. Then

$$L(w)(x) = a(r - r_0)^2 x^r.$$

Setting $r = r_0$ yields the solution

$$y_1(x) = w(r_0, x) = x^{r_0}, \quad x > 0.$$

To find the second linearly independent solution, we note that

$$\frac{\partial}{\partial r}\{L(w)(x)\}|_{r=r_0} = \{a(r - r_0)^2 x^r \ln x + 2a(r - r_0)x^r\}|_{r=r_0} = 0.$$

Since $\frac{\partial}{\partial r}L(w) = L\left[\frac{\partial w}{\partial r}\right]$ ($\frac{\partial}{\partial r}$ and L commute), we obtain

$$L\left[\frac{\partial w}{\partial r}\right]\Bigg|_{r=r_0} = 0.$$

A second linearly independent solution is

$$y_2(x) = \frac{\partial w}{\partial r}(r_0, x) = \frac{\partial}{\partial r}(x^r)\bigg|_{r=r_0} = x^{r_0} \ln x, \quad x > 0.$$

Example: Find a general solution to

$$4x^2 y''(x) + y(x) = 0, \quad x > 0.$$

Note that

$$L(w)(x) = (4r^2 - 4r + 1)x^r.$$

The indicial equation has repeated roots $r_0 = 1/2$. Thus, the general solution is

$$y(x) = c_1\sqrt{x} + c_2\sqrt{x}\ln x, \quad x > 0.$$

The Method of Frobenius

To motivate the procedure, recall the Cauchy-Euler equation in the standard form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \tag{4}$$

where

$$p(x) = \frac{p_0}{x}, \; q(x) = \frac{q_0}{x^2} \text{ with } p_0 = b/a \;\; q_0 = c/a.$$

The indicial equation is of the form

$$r(r-1) + p_0 r + q_0 = 0. \tag{5}$$

If $r = r_1$ is a root of (5), then $w(r_1, x) = x^{r_1}$ is a solution to (4).

Assume that $xp(x)$ and $x^2q(x)$ (instead of being constants) are analytic functions. Then

$$xp(x) = p_0 + p_1x + p_2x + \cdots = \sum_{n=0}^{\infty} p_n x^n, \tag{6}$$

$$x^2q(x) = q_0 + q_1x + q_2x^2 + \cdots = \sum_{n=0}^{\infty} q_n x^n \tag{7}$$

in some neighborhood of $x = 0$. Then, it follows that

$$\lim_{x\to 0} xp(x) = p_0 \text{ and } \lim_{x\to 0} x^2q(x) = q_0.$$

Therefore, it is reasonable to expect that the solutions to (2) will behave (for x near 0) like the solutions to the Cauchy-Euler equation

$$x^2y''(x) + p_0xy'(x) + q_0y(x) = 0.$$

When $p(x)$ and $q(x)$ satisfy (6) and (7), we say that the singular point $x = 0$ is regular. This observation leads to the following definition.

Definition: A singular point x_0 of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

is said to be a regular singular point if both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0. Otherwise x_0 is called an irregular singular point.

Example: Classify the singular points of the equation

$$(x^2 - 1)^2 y''(x) + (x + 1)y'(x) - y(x) = 0.$$

The singular points are 1 and -1. Note that $x = 1$ is an irregular singular point and $x = -1$ is a regular singular point.

Series solutions about a regular singular point

Assume that $x = 0$ is a regular singular point for

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

so that

$$p(x) = \sum_{n=0}^{\infty} p_n x^{n-1}, \quad q(x) = \sum_{n=0}^{\infty} q_n x^{n-2}.$$

In the method of Frobenius, we seek solutions of the form

$$w(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad x > 0.$$

Assume that $a_0 \neq 0$. We now determine r and a_n, $n \geq 1$.

Differentiating $w(r,x)$ with respect to x, we have

$$w'(r,x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1},$$

$$w''(r,x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Substituting w, w', w'', $p(x)$ and $q(x)$ into (4), we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(\sum_{n=0}^{\infty} p_n x^{n-1} \right) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} q_n x^{n-2} \right) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0.$$

Group like powers of x, starting with the lowest power, x^{n-2}. We find that

$$[r(r-1) + p_0 r + q_0]a_0 x^{r-2} + [(r+1)ra_1 + (r+1)p_0 a_1 + p_1 ra_0 + q_0 a_1 + q_1 a_0]x^{r-1} + \cdots = 0.$$

Considering the first term, x^{r-2}, we obtain

$$\{r(r-1) + p_0 r + q_0\}a_0 = 0.$$

Since $a_0 \neq 0$, we obtain the indicial equation.

Definition: If x_0 is a regular singular point of $y'' + p(x)y' + q(x)y = 0$, then the indicial equation for this point is

$$r(r-1) + p_0 r + q_0 = 0,$$

where

$$p_0 := \lim_{x \to x_0} (x - x_0)p(x), \quad q_0 := \lim_{x \to x_0} (x - x_0)^2 q(x).$$

Example: Find the indicial equation at the the singularity $x = -1$ of

$$(x^2 - 1)^2 y''(x) + (x + 1)y'(x) - y(x) = 0.$$

Here $x = -1$ is a regular singular point. We find that

$$p_0 = \lim_{x \to -1} (x + 1)p(x) = \lim_{x \to -1} (x - 1)^{-2} = \frac{1}{4},$$

$$q_0 = \lim_{x \to -1} (x + 1)^2 q(x) = \lim_{x \to -1} [-(x - 1)^{-2}] = -\frac{1}{4}.$$

Thus, the indicial equation is given by

$$r(r - 1) + \frac{1}{4}r - \frac{1}{4} = 0.$$

The method of Frobenius

To derive a series solution about the singular point x_0 of

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0, \quad x > x_0. \qquad (8)$$

Set $p(x) = a_1(x)/a_2(x), \; q(x) = a_0(x)/a_2(x)$.

If both $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0, then x_0 is a regular singular point and the following steps apply.

Step 1: Seek solution of the form

$$w(r, x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}.$$

Using termwise differentiation and substitute $w(r, x)$ into (8) to obtain an equation of the form

$$A_0(x - x_0)^{r+J} + A_1(x - x_0)^{r+J+1} + \cdots = 0.$$

Step 2: Set $A_0 = A_1 = A_2 = \cdots = 0$. (Notice that $A_0 = 0$ is a constant multiple of the indicial equation $r(r-1) + p_0 r + q_0 = 0$).

Step 3: Use the system of equations

$$A_0 = 0, \quad A_1 = 0, \quad \ldots, \quad A_k = 0$$

to find a recurrence relation involving a_k and a_0, a_1, ..., a_{k-1}.

Step 4: Take $r = r_1$, the larger root of the indicial equation, and use the relation obtained in Step 3 to determine a_1, a_2, ... recursively in terms of a_0 and r_1.

Step 5: A series expansion of a solution to (8) is

$$w(r_1, x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad x > x_0,$$

where a_0 is arbitrary and a_n's are defined in terms of a_0 and r_1.

Theorem: Let x_0 be a regular singular point for

$$y'' + p(x)y'(x) + q(x)y(x) = 0$$

and let r_1 and r_2 be the roots of the associated indicial equation, where $r_1 \geq r_2$ or $Re\, r_1 \geq Re\, r_2$.

Case a: If $r_1 - r_2$ is not an integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}; \quad a_0 \neq 0,$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \quad b_0 \neq 0.$$

Case b: If $r_1 = r_2$, then there exist two linearly independent solutions of the form

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}; \quad a_0 \neq 0, \\ y_2(x) &= y_1(x) \ln(x - x_0) + \sum_{n=1}^{\infty} b_n (x - x_0)^{n+r_1}. \end{aligned}$$

Case c: If $r_1 - r_2$ is a positive integer, then there exist two linearly independent solutions of the form

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}; \quad a_0 \neq 0, \\ y_2(x) &= C y_1(x) \ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \quad b_0 \neq 0, \end{aligned}$$

where C is a constant that could be zero.

In the following three examples, we shall use the Frobenius method to obtain first few terms in the series expansion about the regular singular point $x = 0$.

Example 1: Consider the DE

$$(x+2)x^2y''(x) - xy'(x) + (1+x)y(x) = 0, \quad x > 0.$$

We have $\lim_{x\to 0} xp(x) = p_0 = -\frac{1}{2}$ and $\lim_{x\to 0} x^2q(x) = q_0 = \frac{1}{2}$.
The indicial equation is
$r(r-1) + p_0 r + q_0 = 0 \Rightarrow 2r^2 - 3r + 1 = 0$ has roots
$r_1 = 1, \ \ r_2 = \frac{1}{2}$. Here $r_1 - r_2 = \frac{1}{2}$ not an integer.
The method provides two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}; \quad a_0 \neq 0, \ \text{ and } \ y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2},$$

where the coefficients a_n's and b_n's are to determined.

Proceed as before with $r_1 = 1$ and $a_0 = 1$ we obtain $y_1(x)$ as

$$y_1(x) = x - \frac{1}{3}x^2 + \frac{1}{10}x^3 - \frac{1}{30}x^4 + \cdots$$

With $r_2 = \frac{1}{2}$ and $b_0 = 1$, the second solution is given by

$$y_2(x) = x^{1/2} - \frac{3}{4}x^{3/2} + \frac{7}{32}x^{7/2} - \frac{133}{1920}x^{9/2} + \cdots$$

A general solution (GS) is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad x > 0,$$

where $y_1(x)$ and $y_2(x)$ are two series solutions obtained as above.

Example 2: Consider the DE

$$x^2y''(x) - xy'(x) + (1-x)y(x) = 0, \quad x > 0.$$

We have $\lim_{x\to 0} xp(x) = p_0 = -1$ and $\lim_{x\to 0} x^2q(x) = q_0 = 1$.
The indicial equation is $r(r-1) + p_0r + q_0 = 0 \Rightarrow (r-1)^2 = 0$ has two equal roots $r_1 = r_2 = 1$.

To obtain the first series solution, take

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

and determine the coeffcients a_n's as before and obtain the first series solution is of the form

$$y_1(x) = x + x^2 + \frac{1}{4}x^3 + \frac{1}{36}x^4 + \frac{1}{576}x^5 + \cdots = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}x^{k+1}.$$

The second linearly indpendent solution is of the form

$$y_2(x) = y_1(x)\ln x + \sum_{n=1}^{\infty} b_n x^{n+1},$$

where the coeffcients b_n are to be determined. Compute $y_2'(x) = y_1'(x)\ln x + x^{-1}y_1(x) + \sum_{n=1}^{\infty}(n+1)b_n x^n$.

$$y''(x) = y_1''(x)\ln x - x^{-2}y_1(x) + 2x^{-1}y_1'(x) + \sum_{n=1}^{\infty} n(n+1)b_n x^{n-1}.$$

Substituting $y_2(x)$, $y_2'(x)$ and $y_2''(x)$ in the differential equation and after simplication it leads to

$$\{x^2y_1''(x) - xy_1'(x) + (1-x)y_1(x)\}\ln x - 2y_1(x) + 2xy_1'(x) + \sum_{n=1}^{\infty} n(n+1)b_n x^{n+1} - \sum_{n=1}^{\infty}(n+1)b_n x^{n+1} + \sum_{n=1}^{\infty} b_n x^{n+1} - \sum_{n=1}^{\infty} b_n x^{n+2} = 0.$$

Uing the fact y_1 is a solution and a shift in the indices of summation gives

$$2xy_1'(x) - 2y_1(x) + b_1x^2 + \sum_{k=2}^{\infty}(k^2 b_k - b_{k-1})x^{k+1} = 0.$$

Substituting the series expansions for $y_1(x)$ and $y_1'(x) = \sum_{k=0}^{\infty}(k+1)\frac{x^k}{(k!)^2}$, we have

$$(2 + b_1)x^2 + \sum_{k=2}^{\infty}\left[\frac{2k}{(k!)^2} + k^2 b_k - b_{k-1}\right]x^{k+1} = 0.$$

Equating the coefficients equal to zero, we get

$$b_1 = -2, \quad b_k = \frac{1}{k^2}\left[b_{k-1} - \frac{2k}{(k!)^2}\right], \quad k \geq 2.$$

Taking $k = 2$ and 3, compute

$$b_2 = \frac{1}{2^2}(b_1 - 1) = \frac{-3}{4}, \quad b_3 = \frac{1}{9}\left[-\frac{3}{4} - \frac{6}{36}\right] = \frac{-11}{108}.$$

Thus, a second linearly independent solution is

$$y_2(x) = y_1(x)\ln x - 2x^2 - \frac{3}{4}x^3 - \frac{11}{108}x^4 + \cdots .$$

A GS is given by $y(x) = c_1 y_1(x) + c_2 y_2(x)$, $x > 0$, where $y_1(x)$ and $y_2(x)$ are two series solutions obtained as above.

Example 3: Consider the DE

$$xy''(x) + 4y'(x) - xy(x) = 0, \quad x > 0.$$

The roots of the indicial equations $r_1 = 0$ and $r_2 = -3$. Here $r_1 - r_2$ is a positive integer. With $r_1 = 0$, the first series solution is given by

$$y_1(x) = 1 + \frac{1}{10}x^2 + \frac{1}{280}x^4 + \cdots .$$

Since $r_1 - r_2 = 3$ is a positive integer, the second linearly independent solution is of the form

$$y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n-3}.$$

Subsititute the expression for $y_2(x)$, $y_2'(x)$ and $y_2''(x)$ in the differential equation leads to

$$\{xy_1''(x) + 4y_1'(x) - xy_1(x)\}C\ln x + 3Cx^{-1}y_1(x) + 2Cy_1'(x) + \sum_{n=0}^{\infty}(n-3)(n-4)b_n x^{n-4} + \sum_{n=0}^{\infty}4(n-3)b_n x^{n-4} - \sum_{n=0}^{\infty} b_n x^{n-2} = 0.$$

As $y_1(x)$ is a solution to the differential equation, combine the summations and simplify to obtain

$$3Cx^{-1}y_1(x) + 2Cy_1'(x) - 2b_1 x^{-3} + \sum_{k=2}^{\infty}[k(k-3)b_k - b_{k-2}]x^{k-4} = 0.$$

Substituting the series for $y_1(x)$ and equating the coefficients equal to zero, we have

$$b_1 = 0, \quad b_2 = -\frac{1}{2}b_0, \quad C = \frac{1}{3}b_1 = 0, \quad b_4 = \frac{1}{4}b_2 = -\frac{1}{8}b_0,$$
$$b_5 = \frac{b_3 - \frac{7}{10}C}{10} = \frac{1}{10}b_3, \quad b_6 = \frac{1}{18}b_4 = -\frac{1}{144}b_0,$$
$$b_7 = \frac{b_5 - \frac{11}{280}C}{28} = \frac{1}{280}b_3.$$

Collecting the values of for the b_n's and $C = 0$ gives

$$y_2(x) = b_0\left\{x^{-3} - \frac{1}{2}x^{-1} - \frac{1}{8}x - \frac{1}{144}x^3 + \cdots\right\}$$
$$+b_3\left\{1 + \frac{1}{10}x^2 + \frac{1}{280}x^4 + \cdots\right\},$$
$$= b_0\left\{x^{-3} - \frac{1}{2}x^{-1} - \frac{1}{8}x - \frac{1}{144}x^3 + \cdots\right\} + b_3 y_1(x),$$

where b_0 and b_1 are arbitrary constants. In order to obtain a second linearly independent solution, choose b_0 to be nonzero. Taking $b_0 = 1$ and $b_3 = 0$ gives

$$y_2(x) = x^{-3} - \frac{1}{2}x^{-1} - \frac{1}{8}x - \frac{1}{144}x^3 + \cdots .$$

Thus, a GS is $y(x) = c_1 y_1(x) + c_2 y_2(x), \quad x > 0$, with $y_1(x)$ and $y_2(x)$ obtained as above.

*** End ***