MA 102 (Mathematics II)
IIT Guwahati

Tutorial Sheet No. 5 | Linear Algebra | February 21, 2019

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1. True or False? Give justifications.
   (a) There exist distinct linear transformations $S, T : \mathbb{V} \to \mathbb{W}$ such that $ker(S) = ker(T)$ and $range(S) = range(T)$.
   (b) There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that none of T, T^2, T^3 is the identity transformation but $T^4 = I$ (identity transformation).
   (c) If $T : \mathbb{V} \to \mathbb{W}$ is a linear transformation then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is LI in $\mathbb{V}$ if and only if $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is LI in $\mathbb{W}$.

Solution:

(a) True. Consider $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $S([x, y]^\top) = [x, y]^\top$ and $T([x, y]^\top) = [y, x]^\top$.

(b) True. Rotate every element of $\mathbb{R}^2$ by 90 degrees, that is, $T([x, y]^\top) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [x, y]^\top$.

(c) False. If $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is LI in $\mathbb{W}$ then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is LI in $\mathbb{V}$ but the converse is not true. For instance consider the $\mathbf{0}$ transformation.

2. Determine a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ such that $range(T) = \{[x, y, z]^\top : x + 2y + z = 0\}$. If possible give two more such linear transformations with the same range.

Solution: It is enough to define T on a basis.

Consider the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ and define the LT T such that $T(\mathbf{e}_1) = [2, -1, 0]^T$, $T(\mathbf{e}_2) = [0, -1, 2]^T$ and $T(\mathbf{e}_3) = \mathbf{0}$. Then $range(T) = span\{T(\mathbf{e}_1), T(\mathbf{e}_2)\}$.

Note that $\{[2, -1, 0]^T, [0, -1, 2]^T\}$ is a basis of $S = \{[x, y, z]^T : x + 2y + z = 0\}$.
Hence $range(T) = span\{T(\mathbf{e}_1), T(\mathbf{e}_2)\} = S$ and $Ker(T) = \text{span}\{\mathbf{e}_3\}$.

If $T(\mathbf{e}_3) = \alpha(2, -1, 0)^T + \beta(0, -1, 2)^T$ then again $range(T)$ is same but $Ker(T)$ is different. For the same basis by making suitable choices one can also get $Ker(T)$ as the x-axis or the y-axis. By considering different basis of $\mathbb{R}^3$, one can get many more T's.

3. If possible, find linear transformations $S : \mathbb{R}^2 \to \mathbb{R}_2[x]$ and $T : \mathbb{R}_2[x] \to \mathbb{R}^2$ such that
   (a) $S \circ T = I$.
   (b) $T \circ S = I$.
   (c) $range(T \circ S)$ is a line.
   (d) Neither S not T is the zero transformation but $S \circ T = \mathbf{0}$.

Solution:

(a) $S \circ T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$.
Note that $range(S \circ T) \preccurlyeq range(S)$ and $rank(S) \leq 2$ (from the rank nullity theorem). Hence $rank(S \circ T) \leq 2$. An identity map from $\mathbb{R}_2[x] \to \mathbb{R}_2[x]$ will have rank 3, hence not possible.

(b) $T \circ S : \mathbb{R}^2 \to \mathbb{R}^2$.
Define $S([a,b]^\top) = a + bx$ and $T : \mathbb{R}_2[x] \to \mathbb{R}^2$ by $T(a_0 + a_1x + a_2x^2) = [a_0, a_1]^\top$. Then we have $T \circ S = I$.

(c) Define $S([a,b]^\top) = a + bx$ and $T : \mathbb{R}_2[x] \to \mathbb{R}^2$ by $T(a_0 + a_1x + a_2x^2) = [a_0, 0]^\top$. Then $range(T \circ S)$ is a line.

(d) Consider $T(a_0 + a_1x + a_2x^2) = [a_0, 0]^\top$ and $S([a,b]^\top) = bx$.

4. Let $\mathbb{V}, \mathbb{W}$ be finite dimensional vector spaces with ordered bases B and C, respectively. Let $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Show that $\text{rank}(T) = \text{rank}([T]_{C \leftarrow B})$ and $\text{nullity}(T) = \text{nullity}([T]_{C \leftarrow B})$.

Solution: For $\mathbf{v} \in \mathbb{V}$, we have $T\mathbf{v} = C[T\mathbf{v}]_C = C[T]_{C \leftarrow B}[\mathbf{v}]_B$. Since $\mathbf{v} \longmapsto [\mathbf{v}]_B$ is an isomorphism, it follows that $T\mathbf{v} = \mathbf{0} \Leftrightarrow [T]_{C \leftarrow B}[\mathbf{v}]_B = \mathbf{0}$. Hence $\text{nullity}(T) = \text{nullity}([T]_{C \leftarrow B})$.

Next, let B be given by $B = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. Then $T\mathbf{v}_j = C[T]_{C \leftarrow B}\mathbf{e}_j$ shows that $\text{rank}(T) = \text{rank}([T]_{C \leftarrow B})$.

5. True or False? Give justifications.

(a) A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T([x,y]^\top) = [x,y]^\top$ for $x \neq 0$ and $T([0,y]^\top) = [0,0]^\top$ satisfies $T(c[x,y]^\top) = cT([x,y]^\top)$ but is not a linear transformation.

(b) Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. Then for any $\mathbf{v}_1, \mathbf{v}_2$ in $\mathbb{V}$ and $\mathbf{w}_1, \mathbf{w}_2$ in $\mathbb{W}$, there exists a linear transformation $T : \mathbb{V} \to \mathbb{W}$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$.

(c) Let $\mathbb{V}$ and $\mathbb{W}$ be n-dimensional vector spaces and $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ be invertible. Then there exist ordered bases B and C of $\mathbb{V}$ and $\mathbb{W}$, respectively, such that $[T]_{C \leftarrow B} = I_n$.

Solution:

(a) True. $T(c[x,y]^\top) = cT([x,y]^\top)$ is easy to check. But for $y \neq 0$, $T([-1,y]^\top) + T([1,y]^\top) = [-1,y]^\top + [1,y]^\top = [0,2y]^\top \neq [0,0]^\top = T([-1,y]^\top + [1,y]^\top)$.

(b) False. Consider $\mathbf{v}_1 = [1,0]^T$ and $\mathbf{v}_2 = [2,0]^T$ and $\mathbf{w}_1 = [1,0]^T$ and $\mathbf{w}_2 = [0,1]^T$, then there exists no LT $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$ as $\{\mathbf{v}_1, \mathbf{v}_2\}$ is LD.

(c) True. Since T is invertible it takes every basis of $\mathbb{U}$ to a basis of $\mathbb{W}$. For any basis B of $\mathbb{U}$, consider the ordered basis $C := T(B)$ of $\mathbb{W}$. Then $[T]_{B \leftarrow C} = I_n$.

6. Determine a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ such that $Ker(T) = \{[x,y]^\top : 2x + y = 0\}$.

Solution: Take an $[x, y]^\top$ satisfying $2x + y = 0$. For example, consider $[-1, 2]^\top$ and define an LT T such that $T([-1, 2]^\top) = \mathbf{0}$ and $T([1, 0]^\top) = [1, 0]^\top$.

7. Let $\mathbb{V}$ be a vector space and $\dim(\mathbb{V}) = n$. Show that there exists an LT $T : \mathbb{V} \to \mathbb{V}$ such that $T^j \neq \mathbf{0}$ for $j = 1, 2, \ldots, n-1$ but $T^n = \mathbf{0}$.

Solution: Let $B := [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be an ordered basis of $\mathbb{V}$. Define an LT T by $T(\mathbf{v}_1) = 0$ and $T(\mathbf{v}_j) = \mathbf{v}_{j-1}$ for $j = 2, \ldots, n$. Then $T^j \mathbf{v}_{j+1} = \mathbf{v}_1$ for $j = 1, 2, \ldots, n-1$, and $T^n \mathbf{v}_j = \mathbf{0}$ for $j = 1 : n$.

8. Let $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ be defined as $T(A) := A - A^\top$ for all $A \in \mathcal{M}_2(\mathbb{R})$. Find a basis of range$(T)$ and $\ker(T)$.

Solution: Since T maps every 2×2 real matrix to a skew symmetric matrix and $T(\begin{bmatrix} x & y \\ z & w \end{bmatrix}) = \begin{bmatrix} 0 & y - z \\ z - y & 0 \end{bmatrix}$, we have $\text{range}(T) = \text{span}\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}$, and $\ker(T) = \text{span}\{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$.

9. Find the change of basis matrices $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ for the bases $B := [1, x, x^2]$ and $C := [1 + x, x + x^2, 1 + x^2]$ of $\mathbb{R}_2[x]$. Consider $p(x) := 1 + 2x - x^2$. Find $[p]_C$ using the change of basis matrix.

Solution: Note that $P_{B \leftarrow C} = [[1 + x]_B, [x + x^2]_B, [1 + x^2]_B] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

We have $P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$. Hence $[p]_C = P_{C \leftarrow B}[p]_B = [2, 0, -1]^\top$.

10. Let $\mathbb{V}$ be an n-dimensional vector space with an ordered basis $B := [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. Let $A \in \mathcal{M}_n(\mathbb{F})$ be an invertible matrix. Consider $C := BA$. Show that C is an ordered basis of $\mathbb{V}$ and that the change of basis matrix is given by $P_{B \leftarrow C} = A$.

Solution: Set $\mathbf{u}_i := BA\mathbf{e}_i$ for $i = 1 : n$. Then $P_{B \leftarrow C} = [[\mathbf{u}_1]_B, \ldots, [\mathbf{u}_n]_B] = [A\mathbf{e}_1, \ldots, A\mathbf{e}_n] = A$.

11. True or False? Give justifications.

    (a) Let $\mathbf{x}$ be a nonzero vector. Then $\mathbf{x}$ is an eigenvector of A corresponding to an eigenvalue λ if and only if $\mathbf{x}$ is an eigenvector of A^2 corresponding to the eigenvalue λ^2.

(b) Let A be a nonzero matrix such that $A^{31} = \mathbf{0}$. Then A has all eigenvalues equal to 0 and A is not diagonalizable.

(c) If A is diagonalizable then $rank(A - cI) = rank(A - cI)^2$ for all $c \in \mathbb{C}$.

Solution:

(a) False. If $A\mathbf{x} = \lambda\mathbf{x}$ then $A(A\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$ which shows that $\mathbf{x}$ is also an eigenvector of A^2 corresponding to the eigenvalue λ^2. But the converse is not true. For example, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $E_0(A^2) = \mathbb{R}^2 \neq E_0(A) = span\{[1,0]^T\}$.

(b) True. If $\lambda \neq 0$ is an eigenvalue of A then $\lambda^{31} \neq 0$ is an eigenvalue of A^{31}, which is a contradiction. Since all eigenvalues of A are equal to 0, if A is diagonalizable then A has to the $\mathbf{0}$ matrix, which is a contradiction.

(c) True. If $P^{-1}AP = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & \lambda_n \end{bmatrix}$, then $P^{-1}(A - cI)P = \begin{bmatrix} \lambda_1 - c & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & \lambda_n - c \end{bmatrix}$ and

$$P^{-1}(A - cI)^2P = \begin{bmatrix} (\lambda_1 - c)^2 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & (\lambda_n - c)^2 \end{bmatrix}.$$

Hence $rank(A - cI) = rank(P^{-1}(A - cI)P) = rank(P^{-1}(A - cI)^2P) = rank((A - cI)^2)$.

12. Let $\mathbb{V}, \mathbb{W}$ be n dimensional vector spaces with ordered bases B and C, respectively, and $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Show that T is invertible if and only if the matrix $[T]_{C \leftarrow B}$ is invertible. In such a case, show that

$$([T]_{C \leftarrow B})^{-1} = [T^{-1}]_{B \leftarrow C}.$$

Solution: Since $T\mathbf{v} = C[T\mathbf{v}]_C = C[T]_{C \leftarrow B}[\mathbf{v}]_B$ it follows that $\ker(T) = \{\mathbf{0}\} \Leftrightarrow \text{null}([T]_{C \leftarrow B}) = \{\mathbf{0}\}$. Hence T is invertible $\Leftrightarrow$ $[T]_{C \leftarrow B}$ is invertible. (Also follows from Q.4)

Next, for $\mathbf{v} \in \mathbb{V}$, we have $[\mathbf{v}]_B = [T^{-1}(T\mathbf{v})]_B = [T^{-1}]_{B \leftarrow C}[T\mathbf{v}]_C = [T^{-1}]_{B \leftarrow C}[T]_{C \leftarrow B}[\mathbf{v}]_B$. Hence $[T^{-1}]_{B \leftarrow C}[T]_{C \leftarrow B} = I_n \Rightarrow ([T]_{C \leftarrow B})^{-1} = [T^{-1}]_{B \leftarrow C}$.

13. Consider $\mathbb{U} := \mathbb{R}^3$, $\mathbb{V} := \mathcal{M}_2(\mathbb{R})$ and $\mathbb{W} := \mathbb{R}_2[x]$ with ordered bases $B := [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$, $C := \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]$ and $D := [1, x, x^2]$, respectively. Let $T : \mathbb{U} \to \mathbb{V}$ be given by $T[x, y, z]^\top = \begin{bmatrix} 0 & x \\ y & y + z \end{bmatrix}$ and $S : \mathbb{V} \to \mathbb{W}$ be given by $S\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + (b + c)x + dx^2$. Then determine $[S \circ T]_{D \leftarrow B}$, $[S]_{D \leftarrow C}$ and $[T]_{C \leftarrow B}$ and verify that $[S \circ T]_{D \leftarrow B} = [S]_{D \leftarrow C}[T]_{C \leftarrow B}$.

14. For each LT T on $\mathbb{V}$, find the eigenvalues of T and an ordered basis B of $\mathbb{V}$ such that $[T]_B$ is a diagonal matrix.

(a) $\mathbb{V} := \mathbb{R}_3[x]$ and $(Tp)(x) := xp'(x) + p''(x) - p(2)$.

(b) $\mathbb{V} := \mathcal{M}_2(\mathbb{R})$ and $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) := \begin{bmatrix} d & b \\ c & a \end{bmatrix}$.

(c) $\mathbb{V} := \mathcal{M}_2(\mathbb{R})$ and $T(A) := A^\top + 2\,\mathrm{Trace}(A) I_2$.

15. Let $T : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ be given by $(Tp)(x) := p(1) + p'(0)x + (p'(0) + p''(0))x^2$. Find eigenvalues and eigenvectors of T. Also, find an ordered basis B, if it exists, of $\mathbb{R}_2[x]$ such that $[T]_B$ is a diagonal matrix.

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