Continuous-time Markov Chain: Poisson Process

SIDNEY CHAPMAN

ANDREY KOLMOGOROV

- One important example of the CTMC is the Poisson process.
- A Poisson process $\{N(t)\}$ is a counting process representing the number of occurrences of an event up to time t (over time interval$(0, \mathrm{t}]$.
- Used to model events that occur approximately at a certain rate, but at a completely random instants.
- Suppose you receive at an average 10 emails per day, but at random instants of time. The Poisson process may be good model to represent the number of emails in a time durarion.
- Some simple examples are:
  - Number of alpha particles emitted by a radio-active substance.
  - Number of packets received at switching node of a communication network.
  - Number of earthquakes occurring during a month in an earth-quake prone zone.

Review of Poisson RVs

- A discrete random variable *X* is called a *Poisson random variable* with the parameter $\mu > 0$ if $p_X(k) = \frac{e^{-\mu}\mu^k}{k!}, \quad k = 0,1,2,\ldots.$
- *X* is denoted as $X \sim Poi(\mu)$
- EX and $\text{var}(X)$ given by
  $EX = \mu$ and $\text{var}(X) = \mu$
- The MGF $M_X(s)$ is given by

$$M_X(s) = Ee^{sX} = \sum_k e^{sk} \frac{e^{-\mu}\mu^k}{k!} = e^{\mu(e^s - 1)}$$

- If $X_i \sim Poi(\mu_i)$, $i = 12,\ldots,n$, and independent, then

$$X_1 + X_2 + \ldots + X_n \sim Poi(\mu_1 + \mu_2 + \ldots + \mu_n)$$

Binomial RV

$X \sim Bi(n,p)$

$EX = np$, $\text{var}(X) = np(1-p)$

when $n \to \infty$, $np \to \lambda$,

Then $X \sim Poi(\lambda)$

$$M_{X_1+X_2+\cdots+X_n}(s) = Ee^{s(X_1+X_2+\cdots+X_n)} = \prod_{i=1}^n Ee^{sX_i} = \prod_{i=1}^n e^{\mu_i(e^s-1)} = e^{\left(\sum_{i=1}^n \mu_i\right)(e^s-1)}$$

Poisson Process as a CTMC

- $\{N(t)\}$ is a Markov chain because of the independent increment property.
- $\{N(t)\}$takes values from the infinite state space $V = \{0,1,...\}$ and the probability $p_j(t) = P(N(t) = j)$ is given by

$$p_j(t) = e^{-\lambda t} \frac{(\lambda t)^j}{j!}, j \in V$$

Thus at any instant *t*, $\{N(t)\}$ is a Poisson RV.

Poisson process as a CTMC

Suppose the Poisson process has entered state i at time $t = 0$. It remains in same state until the next arrival with $T_i \sim \exp(\lambda)$. Once an arrival takes place, the state become i+1

Thus, for $j \neq i$, $P_{i,j} = \begin{cases} 1, & j = i+1 \\ 0 & \text{otherwise} \end{cases}$. The transition rates are given by

$$q_{i,j} = \lambda P_{i,j} \qquad \text{for } j \neq i$$

$$\Rightarrow q_{i,i+1} = \lambda,\ q_{i,i} = -\lambda,\ q_{i,j} = 0,\ \ j \neq i, i+1$$

Further, $N(0) = 0$ with probability 1

$$p(N(0)=0) = 1 \qquad P_0(0) = 1$$

$$P_i(0) = 0 \text{ for } i \neq 0.$$

Forward Kolmogorov Equation for Poisson Process $p_{ij}'(t) = \sum_k p_{ik}(t) q_{kj}$

$$p_{0,j}'(t) = \sum_k p_{0,k}(t) q_{kj} = -\lambda p_{0,j}(t) + \lambda p_{0,j-1}(t)$$

Since $p_0(0) = 1$ and $p_j(0) = 0,\ j \neq 0$, we have $p_j(t) = p_{0,j}(t)$. Therefore, in terms of the state probabilities,

$$p_j'(t) = -\lambda p_j(t) + \lambda p_{j-1}(t),\ j = 1, 2, \ldots \qquad (1)$$

For $j = 0$,

$$p_0'(t) = -\lambda p_0(t) \qquad (2)$$

The solution of the differential equation (2) with $p_0(0) = 1$ and $p_j(0) = 0,\ j > 0$

Is given by $p_0(t) = e^{-\lambda t} \quad t \geq 0$

For j=1

$$p_1'(t) = -\lambda p_1(t) + \lambda p_0(t)$$

Substituting $p_0(t) = e^{-\lambda t}\ \ t \geq 0$, we get

$$p_1'(t) = -\lambda p_1(t) + \lambda e^{-\lambda t}$$

Solving the above linear differential equation with the initial condition $p_1(0) = 0$,

we get

$$p_1(t) = \lambda t e^{-\lambda t} = \frac{(\lambda t)^1 e^{-\lambda t}}{1!}\ \ t \geq 0$$

j = 1, The formula is correct –

Thus $p_1(t)$ satisfies the formula. Now assuming $p_{j-1}(t) = \frac{(\lambda t)^{j-1} e^{-\lambda t}}{j-1!}$ $t \geq 0$, we can solve the

differential equation $p_j'(t) = -\lambda p_j(t) + \lambda \frac{(\lambda t)^{j-1} e^{-\lambda t}}{j-1!}$, we get

$$p_j(t) = \frac{(\lambda t)^j e^{-\lambda t}}{j!}\ \ t \geq 0$$

$$p_j(t) = \frac{(\lambda t)^j e^{-\lambda t}}{j!}\ \ t \geq 0.$$

We can derive the same differential equations for the Poisson process using the following postulates:

Postulates of Poisson Process

(i) $N(0)=0$ with probability 1.

(ii) $N(t)$ is an *independent increment* process.

(iii) $P\left(\{N(\Delta t)=1\}\right)=\lambda\Delta t+o(\Delta t)$

where $o(\Delta t)$ implies any function such that $\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}=0.$

(iv) $P\left(\{N(\Delta t)\geq 2\}\right)=o\left(\Delta t\right)$

Remark

- The parameter λ is called the rate or intensity of the Poisson process. It can be shown that

$[t_1, t_2]$

$$P(N(t_2) - N(t_1) = j) = \frac{(\lambda(t_2 - t_1))^j e^{-\lambda(t_2 - t_1)}}{j!}$$

Thus, the probability of the increments depends on the length of the interval $t_2 - t_1$ and not on the absolute times t_2 and t_1. The Poisson process is a process with *stationary increments.*

- The independent and stationary increment properties help us to compute the joint probability mass function of $N(t)$. For example,

$$\begin{aligned}
p_{N(t),N(t_2)}(n_1, n_2) &= P\left(\{N(t_1) = n_1, N(t_2) = n_2\}\right) \\
&= P\left(\{N(t_1) = n_1\}\right) P(\{N(t_2) = n_2\} \mid \{N(t_1) = n_1\}) \\
&= P\left(\{N(t_1) = n_1\}\right) P(\{N(t_2) - N(t_1) = n_2 - n_1\} \mid \{N(t_1) = n_1\}) \\
&= P\left(\{N(t_1) = n_1\}\right) P(\{N(t_2) - N(t_1) = n_2 - n_1\}) \\
&= \frac{(\lambda t_1)^{n_1} e^{-\lambda t_1}}{n_1!} \frac{(\lambda(t_2 - t_1))^{n_2 - n_1} e^{-\lambda(t_2 - t_1)}}{(n_2 - n_1)!}
\end{aligned}$$

Mean, Variance and Autocorrelation function of {N(t)}

We have

$$p_j(t) = \frac{(\lambda t)^j e^{-\lambda t}}{j!} \quad t \geq 0$$

At any time $t > 0,\ N(t)$ is a Poisson random variable with the parameter λt.

Therefore, $EN(t) = \lambda t$ and $\operatorname{var} N(t) = \lambda t$

Thus both the mean and variance of a Poisson process varies linearly with time.

We can use the independent increment property to find the autpcovariance

We can show that

$$C_N(t_1,t_2) = \lambda \min(t_1,t_2)$$

$$C_N(t_1,t_2) = Cov(N(t_1), N(t_2))$$

$t_2 > t_1$

$= cov(N(t_1), N(t_2) - N(t_1) + N(t_1))$

$= cov(N(t_1), N(t_2) - N(t_1)) + cov(N(t_1), N(t_1))$

$= 0 + var\,N(t_1) = \lambda t_1$

If $t_1 > t_2$

$cov(t_1, t_2) = \lambda t_2$

$$\therefore R_N(t_1,t_2) = C_N(t_1,t_2) + EN(t_1)EN(t_2)$$

$$= \lambda \min(t_1,t_2) + \lambda^2 t_1 t_2$$

Example A petrol pump serves on the average 30 cars per hour. Assuming the Poisson model, find the probability that during a period of 5 minutes (i) no car comes to the station, (ii) exactly 3 cars come to the station and (iii) more than 3 cars come to the station.

Average arrival = 30 cars/hr= ½ car/min

Probability of no car in 5 minutes

(i) $P\left\{N(5)=0\right\}=e^{-\frac{1}{2}\times 5}=e^{-2.5}=0.0821$

(ii) $P\left\{N(5)=3\right\}=\dfrac{\left(\dfrac{1}{2}\times 5\right)^3}{3!}e^{-2.5}=0.2138$

(iii)

$$P(\{N(5)>3\})=1-P(\{N(5)\leq 3\})$$

$$=1-P(\{N(5)=0\})-P(\{N(5)=1\})-P(\{N(5)=2\})-P(\{N(5)=3\})$$

$$=1-0.08-\ 0.2052\text{- }0.2565\text{-}0.21$$

$$=0.2424$$

To Summarise

- To characterize the transition probabilities dynamically, Kolmogorov backward and forward differential equations are used.

- Poisson process $\{N(t)\}$ is the well-known CTMC with

$$P_{i,j} = \begin{cases} 1, & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$ and transition rates

$$q_{i,i+1} = \lambda,\ q_{i,i} = -\lambda,\ q_{i,j} = 0,\quad j \neq i, i+1$$

The state probabilities are given by

$$p_j(t) = e^{-\lambda t}\frac{(\lambda t)^j}{j!},\ j = 0,1,..$$

- The independent increment property can be used to find the joint probability mass functions, autocvariance functions.