Discrete-time Markov Chain 2

DTMC

A *discrete-time random process* $\{X_n, n \geq 0\}$ taking values from a countable set V is said to be a DTMC if

$$P\left(X_{n+1}=j \mid X_0=i_0, X_1=i_1, \cdots, X_n=i\right)=P\left(X_{n+1}=j \mid X_n=i\right).$$

Evolution of a DTMC

By chain rule, the joint PMF of the states up to instant *n* is given as

$$P(X_0 = i_0, X_1 = i_1, \cdots, X_n = i, X_{n+1} = j)$$
$$= P(X_0 = i_0) P(X_1 = i_1 / X_0 = i_0) \ldots$$
$$P(X_n = i / X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1})$$
$$P(X_{n+1} = j / X_0 = i_0, X_1 = i_1, \ldots, X_n = i)$$

Using the Markovian property

$$P(X_0 = i_0, X_1 = i_1, \cdots, X_n = i, X_{n+1} = j)$$
$$= P(X_0 = i_0) P(X_1 = i_1 / X_0 = i_0) P(X_2 = i_2 / X_1 = i_1) \ldots$$
$$\ldots P(X_n = i / X_{n-1} = i_{n-1}) P(X_{n+1} = j / X_n = i)$$

Transition Probability

The conditional probability $P(X_{n+1} = j | X_n = i)$ is called the *one-step transition probability* of the chain at the instant n and denoted by $p_{i,j}$. Clearly $\sum_{j \in V} p_{i,j} = 1.$

Note that $p_{i,j}$ is generally a function of *n*.

Similarly, the *m*-step transition probability $p_{i,j}^{(m)}$ is defined by

$$p_{i,j}^{(m)} = P\left(X_{n+m} = j \mid X_n = i \right)$$

Homogeneous Markov chain

If $p_{i,j}$ does not depend upon n, then this transition probability is stationary and $\left\{ X_n, n \geq 0 \right\}$ is called *a homogeneous Markov chain*.

For a homogeneous MC,

$$P\left(X_0 = i_0, X_1 = i_1, \cdots, X_n = i, X_{n+1} = j \right)$$

$$= P(X_0 = i_0) P(X_1 = i_1 / X_0 = i_0) \cdots P(X_{n+1} = j / X_n = i)$$

$$= P(X_0 = i_0) p_{i_0,i_1} p_{i_1,i_2} \cdots p_{i,j}$$

Probabilistic evolution of a homogeneous MC up to *n*+1 can completely be described by

(a) The initial probability $P(X_0 = i_0)$ and

(b) the transition probabilities at instances up to *n*

State Transition Matrix or Transition Probability Matrix

One-step transition probabilities of an MC can be represented compactly in terms of the *state transition matrix*

$$\mathbf{P} = \begin{bmatrix} p_{0,0} & p_{0,1} \cdots & p_{0,k} \cdots \\ p_{1,0} & p_{1,1} \cdots & p_{1,k} \cdots \\ & \cdots & \\ p_{k,0} & p_{k,1} \cdots & p_{k,k} \cdots \\ & \vdots & \end{bmatrix}$$

$\mathbf{P}$ is a *stochastic matrix*

m-step transition probabilities can be represented by the matrix

$$\mathbf{P}^{(m)} = \begin{bmatrix} p_{0,0}^{(m)} & p_{0,1}^{(m)} & \cdots & p_{0,k}^{(m)} & \cdots \\ p_{1,0}^{(m)} & p_{1,1}^{(m)} & \cdots & p_{1,k}^{(m)} & \cdots \\ \vdots & & & & \\ p_{k,0}^{(m)} & p_{k,1}^{(m)} & \cdots & p_{k,k}^{(m)} & \cdots \\ \vdots & & & & \end{bmatrix}$$

Example Random Walk (RW) Processes

For a *simple RW* process $\{X_n, n \ge 0\}$

$$p_{i,j} = \begin{cases} p, j{=}i{+}1 \\ 1{-}p, j{=}i{-}1 \\ 0 \ \text{ otherwise} \end{cases}$$

The state transition matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} \cdots 1-p & 0 & p & 0 \cdots \\ \vdots & 1-p & 0 & p \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

RW Process with Barriers

$\{X_n, n \geq 0\}$ with the finite state space $V = \{0,1,\dots,N\}$

State 0 State N are barriers.

The state transition matrix is given by

$$\mathbf{P} = \begin{bmatrix} a & 1-a & 0 & \dots & 0 & 0 & 0 \\ 1-p & 0 & p & \dots & 0 & 0 & 0 \\ & \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & \dots & 0 & p & 0 \\ 0 & 0 & 0 & \dots & 0 & 1-b & b \end{bmatrix}$$

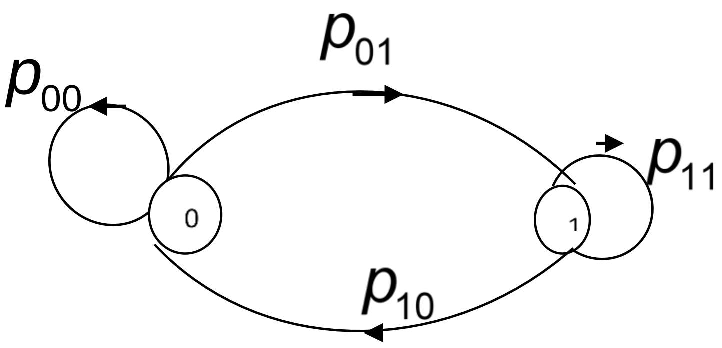
State Transition Graph

A graph where each node denotes a state and each directed denotes the one-step transition probability.

Example State transition graph for a 2-state MC

$$V = \{0,1\}$$

$$\mathbf{P} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$



- State transition graph is a tool to visualize an MC and also for studying its properties

Theorem Chapman-Kolmogorov Equation

For a homogenous MC $\{X_n, n \ge 0\}$,

$$p_{i,j}^{(m+n)} = \sum_k p_{i,k}^{(m)} p_{k,j}^{(n)}$$

Proof:

$$\begin{aligned}
P(X_0 = i, X_{m+n} = j) &= P\bigcup_k (X_0 = i, X_m = k, X_{m+n} = j) \\
&= \sum_k P(X_0 = i, X_m = k, X_{m+n} = j) \\
&= \sum_k P(X_0 = i) P(X_m = k \mid X_0 = i) P(X_{m+n} = j \mid X_0 = i, X_m = k) \\
&= \sum_k P(X_0 = i) P(X_m = k \mid X_0 = i) P(X_{m+n} = j \mid X_m = k) \\
&= \sum_k P(X_0 = i) p_{ik}^{(m)} p_{kj}^{(n)}
\end{aligned}$$

Dividing by $P(X_0 = i)$, we get

$$P(X_{n+m} = j \mid X_0 = i) = \sum_k p_{ik}^{(m)} p_{kj}^{(n)}$$

Particularly with $m = n-1$ and $n = 1$,

$$p_{i,j}^{(n)} = \sum_k p_{i,k}^{(n-1)} p_{k,j}$$

In the matrix notation

$$\mathbf{P}^{(n)} = \mathbf{P}^{(n-1)}\mathbf{P}$$

Applying CK equation for $p_{i,j}^{(n-1)}$, we can write

$$\mathbf{P}^{(n-1)} = \mathbf{P}^{(n-2)}\mathbf{P}$$

$$\therefore \mathbf{P}^{(n)} = \mathbf{P}^{(n-2)}\mathbf{P}^2$$

Continuing in a similar manner

$$\mathbf{P}^{(n)} = \underbrace{\mathbf{P}\mathbf{P}\ldots\mathbf{P}}_{n-times} = \mathbf{P}^n$$

Thus for every n,

$$\mathbf{P}^{(n)} = \mathbf{P}^n$$

which is the *CK equation in matrix form*.

State probabilities at instant n

Suppose $p_i^{(0)}, i \in V$ is the initial PMF.

PMFs of the states at $n = 0$ are represented as

$$\mathbf{p}^{(0)} = \left[p_0^{(0)}\ p_1^{(0)} \ldots\ p_k^{(0)} \ldots. \right]$$

We have

$$P\left(X_0 = i, X_n = j\right) = P\left(X_0 = i\right) P\left(X_n = j \mid X_0 = i\right)$$

$$= p_i^{(0)} p_{ij}^{(n)}$$

The state probability at instant *n* is given by the marginal PMF

$$P\left(X_n = j\right) = \sum_i P\left(X_0 = i, X_n = j\right) = \sum_i p_i^{(0)} p_{ij}^{(n)}$$

In matrix notation,

$$\mathbf{p}^{(n)} = \mathbf{p}^{(0)} \mathbf{P}^{(n)}$$

$$= \mathbf{p}^{(0)} \mathbf{P}^{n}$$

Example A 3-state MC $\{X_n\}$ with $V = \{0,1,2\}$

Given $\mathbf{P} = \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.4 & 0.6 \end{bmatrix}$ and $\mathbf{p}^{(0)} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \end{bmatrix}$

$$\therefore \mathbf{p}^{(2)} = \mathbf{p}^{(0)}\mathbf{P}^2 = \begin{bmatrix} 0.2 & 0.6 & 0.2 \end{bmatrix} \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.4 & 0.6 \end{bmatrix}^2$$

$$= \begin{bmatrix} 0.256 & 0.384 & 0.36 \end{bmatrix}$$

Contd..

If we continue in the same manner,

$$\mathbf{p}^{(20)} = \mathbf{p}^{(0)}\mathbf{P}^{(20)} = [0.2105 \quad 0.3158 \quad 0.4737]$$

$$\mathbf{p}^{(21)} = [0.2105 \quad 0.3158 \quad 0.4737]$$

and so on.

This example suggests the convergence of $\mathbf{p}^{(n)}$

To Summarise

- A DTMC is described by *one-step transition probability* of the MC at the instant n defined by $p_{i,j} = P(X_{n+1} = j \mid X_n = i)$
- *m*-step transition probability $p_{ij}^{(m)}$ is defined by

$$p_{i,j}^{(m)} = P\left(X_{n+m} = j \mid X_n = i\right)$$

- State Transition Matrix is represented by

$$\mathbf{P} = \begin{bmatrix} p_{0,0} & p_{0,1} \cdots & p_{0,k} \cdots \\ p_{1,0} & p_{1,1} \cdots & p_{1,k} \cdots \\ & \cdots & \\ p_{k,0} & p_{k,1} \cdots & p_{k,k} \cdots \\ & \vdots & \end{bmatrix}$$

Transition Probabilty Matrix

To Summarise…

For *a homogeneous MC* ,

- $p_{i,j}$ does not depend on n.
- A state transition graph pictorially represents the states and one-step state transition probabilities.

- For a homogenous MC $\{X_n, n \geq 0\}$,

$$p_{i,j}^{(m+n)} = \sum_k p_{i,k}^{(m)} p_{k,j}^{(n)}$$ (CK equation)

$$\mathbf{P}^{(n)} = \mathbf{P}^n$$ (CK equation in matrix form)

- The PMF at instant *n* is

$$P\left(X_n = j\right) = \sum_i p_i^{(0)} p_{ij}^{(n)}$$

- In matrix notation, $\mathbf{p}^{(n)} = \mathbf{p}^{(0)} \mathbf{P}^n$