Stability of Linear Systems in $\mathbb{R}^2$

Department of Mathematics
IIT Guwahati

RA/RKS/MGPP/KVK

Consider the system of the form

$$x_1'(t) = f(x_1(t), x_2(t)), \quad x_2'(t) = g(x_1(t), x_2(t)), \qquad (1)$$

where $f, g \in \mathbb{R}$ and do not depend explicitly on t. Such systems are called autonomous.

A solution of the above system consists of pair of functions $\{x_1(t),\, x_2(t)\}$ that satisfy (1) for all $t \in I$. The set of points $\{(x_1(t), x_2(t)) : \; t \in I\}$ in the x_1x_2-plane is called a trajectory, and the x_1x_2-plane is referred to as the phase plane.

We shall study the behaviour of trajectories near a critical point.

Definition: A point (a, b), where

$$f(a, b) = 0 \text{ and } g(a, b) = 0,$$

is called a critical point of the system (1). If (a, b) is a critical point, then the constant functions

$$x_1(t) = a, \; x_2(t) = b$$

form a solution to this system. This is called an equilibrium solution.

Example: Consider the system

$$x_1'(t) = -x_2(x_2 - 2), \quad x_2'(t) = (x_1 - 2)(x_2 - 2).$$

The critical points are 2, 0 and the horizontal line $x_2 = 2$. The corresponding equilibrium solutions are $x_1(t) = 2, \; x_2(t) = 0$, and the family $x_1(t) = c$, $x_2(t) = 2$, where c is an arbitrary constants.

We shall study stability of critical points for the linear system only.

Definition: A solution $\mathbf{\Psi}(t)$ to the linear system

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$$

is stable for $t \geq t_0$ if for any $\epsilon > 0$ there exists $\delta(t_0, \epsilon) > 0$ such that whenever

$$\|\mathbf{x}(t_0) - \mathbf{\Psi}(t_0)\| < \delta,$$

where $\mathbf{x}(t)$ is any solution to $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$, then

$$\|\mathbf{x}(t) - \mathbf{\Psi}(t)\| < \epsilon, \quad \text{forall} \quad t \geq t_0.$$

In addition, if for any such $\mathbf{x}(t)$ we have $\mathbf{x}(t) - \mathbf{\Psi}(t) \to \mathbf{0}$ as $t \to \infty$, then $\mathbf{\Psi}(t)$ is asymptotically stable. If $\mathbf{\Psi}$ is not stable, then it is unstable.

Observe that if $\mathbf{x}(t)$ and $\mathbf{\Psi}(t)$ are solutions to

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t),$$

then $\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{\Psi}(t)$ is a solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$.

If $\mathbf{\Psi}(t)$ is stable and $\|\mathbf{x}(t) - \mathbf{\Psi}(t)\| < \epsilon \;\Rightarrow\; \|\mathbf{z}(t)\| < \epsilon$ and the zero solution is stable.

For the linear system, it is sufficient to study the stability of the zero solution to the homogeneous system.

Theorem: A solution $\mathbf{\Psi}(t)$ to the linear system

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$$

is stable (asymptotically stable) iff the zero solution $\mathbf{z}(t) \equiv \mathbf{0}$ is a stable (asymptotically stable) solution to

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

Theorem: Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the homogeneous system

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad t \geq t_0. \tag{1}$$

If there exists a constant $K > 0$ such that $\|\mathbf{\Phi}(t)\| \leq K$ for $t \geq t_0$, then the zero solution $\mathbf{x}(t) = 0$ is stable. Moreover, if $\lim_{t\to\infty} \|\mathbf{\Phi}(t)\| = 0$, then the zero solution $\mathbf{x}(t)$ is asymptotically stable.

Proof. Without loss of generality, take $\mathbf{\Phi}(t_0) = \mathbf{I}$. For any solution $\mathbf{x}(t)$ of (1), we have

$$\|\mathbf{x}(t)\| = \|\mathbf{\Phi}(t)\mathbf{x}(t_0)\| \leq \|\mathbf{\Phi}(t)\| \, \|\mathbf{x}(t_0)\| \leq K\|\mathbf{x}(t_0)\|.$$

Given $\epsilon > 0$, choose $\delta = \epsilon/K$ such that

$$\|\mathbf{x}(t_0)\| < \delta \Longrightarrow \|\mathbf{x}(t)\| < \epsilon \text{ for } t \geq t_0.$$

If $\|\mathbf{\Phi}(t)\| \to 0$ as $t \to \infty \Longrightarrow \mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$.

Theorem: If all eigenvalues of A have negative real parts, then for any t_0 the zero solution $\mathbf{x}(t) \equiv 0$ to

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad t \geq t_0,$$

is asymptotically stable. However, if at least one eigenvalue of A has positive real part, then the zero solution is unstable.

Proof. The entries of a fundamental matrix consist of functions of the form

$$e^{at}\{p(t)\cos bt + q(t)\sin bt\},$$

where $\lambda = a + ib$ is an eigenvalue of A and $p(t)$, $q(t)$ are polynomials in t (Note: If λ is real, then $b = 0$).
If $Re(\lambda_k) < 0$ for all k, then each entries of the fundamental matrix is bounded and goes to zero as $t \to \infty$, since $\lim_{t\to\infty} e^{at}p(t) = 0$.
If $Re(\lambda_k) > 0$ for some k with $\lambda_k = a + ib$, then $|e^{at}p(t)| \longrightarrow +\infty$ as $t \to \infty$.

Consider the linear system

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad x \in \mathbb{R}^2, \; A \text{ is a } 2 \times 2 \text{ matrix.} \tag{2}$$

We first describe the phase portraits (set of all solution curves in the phase space $\mathbb{R}^2$) of the linear system

$$\mathbf{y}'(t) = B\mathbf{y}, \tag{3}$$

where $B = P^{-1}AP$. The phase portrait for the linear system (2) then obtained from the phase portrait for (3) under the linear transformation $\mathbf{x} = P\mathbf{y}$.

Recall

- if $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ then the solution of IVP $\mathbf{x}'(t) = B\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x}(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \mathbf{x}_0$.

- If $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ then the solution of IVP $\mathbf{x}'(t) = B\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x}(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$.
- If $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ then the solution of IVP $\mathbf{x}'(t) = B\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{x}_0.$$

We now discuss the various phase portraits that result from these solutions.

Case I. $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda < 0 < \mu$.

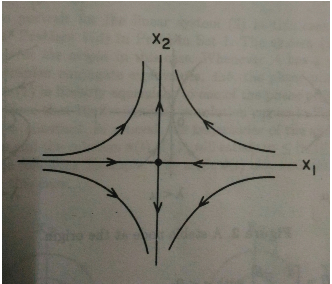

Figure 1. A saddle at the origin.

The system is said to have a saddle at the origin in this case. If $\mu < 0 < \lambda$, the arrows are reversed. Whenever A has two real eigenvalues of opposite sign, the phase portrait for the linear system is linearly equivalent to the phase portrait shown in Fig. 1.

Case II. $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda \leq \mu < 0$ or $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ with $\lambda < 0$.

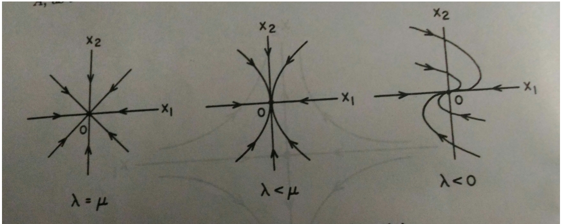

Figure 2. A stable node at the origin.

In each of these cases, the origin is refereed to as a stable node. It is called a proper node in the first case with $\lambda = \mu$ and an improper node in the other two cases. If $\lambda \geq \mu > 0$ or if $\lambda > 0$ in Case II, the arrows are reversed and the origin is referred to as an unstable node. The stability of the node is determined by the sign of the eigenvalues: stable if $\lambda \leq \mu < 0$ and unstable if $\lambda \geq \mu > 0$.

Case III. $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a < 0$.

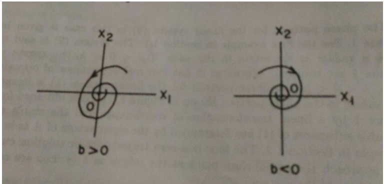

Figure 3. A stable focus at the origin.

The origin is refereed to as a stable focus in these cases.
If $a > 0$, the trajectories spiral away from the origin with increasing t. The origin is called an unstable focus. Whenever A has a pair of complex conjugate eigenvalues with nonzero real part, $a \pm ib$, with $a < 0$, the phase portraits for the system (3) is linearly equivalent to one of the phase portraits shown above.

Case IV. $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$.

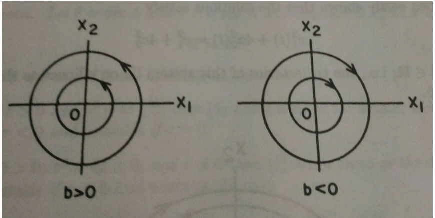

Figure 4. A center at the origin.

The system (3) is said to have a center at the origin in this case. Whenever A has a pair of purely imaginary complex conjugate eigenvalues, $\pm ib$, the phase portrait of the linear system (2) is linearly equivalent to one of the phase portraits shown above. Note that trajectories or solution curves lie on circles $\|\mathbf{x}(t)\| = \text{constant}$. In general, the trajectories of the system (2) will lie on ellipse.

Example: Consider the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, $A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$.
Note that A has eigenvalues $\lambda = \pm 2i$. The matrix P and P^{-1} are

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus

$$B = P^{-1}AP = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

The solution is then given by

$$\begin{aligned} x_1(t) &= c_1 \cos(2t) - 2c_2 \sin(2t) \\ x_2(t) &= \frac{1}{2} c_1 \sin(2t) + c_2 \cos(2t). \end{aligned}$$

It is easy to show that

$$x_1^2(t) + 4x_2^2(t) = c_1^2 + 4c_2^2, \quad \forall t \in \mathbb{R}.$$

The trajectories of this system lie on ellipses.

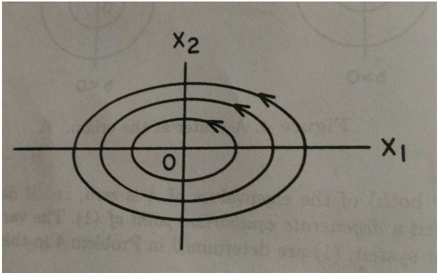

Figure 5. A center at the origin.

Eigenvalues	**Type of critical point**	**Stability**
distinct, positive	improper node	unstable
distinct, negative	improper node	asymptotically stable
opposite signs	saddle point	unstable
equal, positive	proper or improper node	unstable
equal, negative	proper or improper node	asymptotically stable
complex-valued:		
positive real part	spiral point	unstable
negative real part	spiral point	asymptotically stable
pure imaginary	center	stable

Reference:

1. Lawrence Perko, Differential Equations and Dynamical Systems, Springer-Verlag, 2001.

*** End ***