PH 102 : Electromagnetism
Tutorial 8
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1. Consider the motion of a charged particle in the simultaneous presence of magnetic field $\vec{B}$ (in x-direction) and electric field $\vec{E}$ (in z-direction). Find and sketch the trajectory of the particle if it starts at the origin with velocity (a) $\vec{v}(0) = (E/B)\hat{y}$, (b) $\vec{v}(0) = (E/2B)\hat{y}$, (c) $\vec{v}(0) = (E/B)(\hat{y} + \hat{z})$.

Solution: The general solution for trajectory of a charged particle in the presence of $\vec{B}, \vec{E}$ (Example 5.2 in Introduction to Electrodynamics, D J Griffiths, discussed in the class) is

$$y(t) = C_1 \cos\omega t + C_2 \sin\omega t + \frac{E}{B}t + C_3$$
$$z(t) = C_2 \cos\omega t - C_1 \sin\omega t + C_4$$

Since the particle starts from origin in all the cases, it implies

$$y(0) = 0 \implies C_1 + C_3 = 0 \implies C_3 = -C_1, z(0) = 0 \implies C_2 + C_4 = 0 \implies C_4 = -C_2$$

(a) The other initial condition: $\dot{y}(0) = \frac{E}{B}, \dot{z}(0) = 0$. This can be used to find the constants of integration C_1, C_2, C_3, C_4:
$\dot{y}(0) = \omega C_2 + \frac{E}{B} = \frac{E}{B} \implies C_2 = 0 \implies C_4 = 0, \dot{z}(0) = 0 \implies C_1 = 0 \implies C_3 = 0.$
Therefore,

$$y(t) = \frac{E}{B}t, z(t) = 0 \quad \text{(Equation of a straight line)}$$

(b) From the initial conditions: $\dot{y}(0) = \frac{E}{2B} = C_2\omega + \frac{E}{B} \implies C_2 = -\frac{E}{2\omega B} = -C_4$. Also, $\dot{z}(0) = 0 \implies C_1 = 0 \implies C_3 = 0$. Therefore,

$$y(t) = -\frac{E}{2\omega B}\sin\omega t + \frac{E}{B}t = \frac{E}{2\omega B}(2\omega t - \sin\omega t)$$

$$z(t) = -\frac{E}{2\omega B}\cos\omega t + \frac{E}{2\omega B} = \frac{E}{2\omega B}(1 - \cos\omega t)$$

This satisfies the equation of a circle

$$\left(y - \frac{E}{2\omega B}2\omega t\right)^2 + \left(z - \frac{E}{2\omega B}\right)^2 = \left(\frac{E}{2\omega B}\right)^2$$

$$\implies \left(y - \frac{Et}{B}\right)^2 + \left(z - \frac{E}{2\omega B}\right)^2 = \left(\frac{E}{2\omega B}\right)^2$$

with radius $\frac{E}{2\omega B}$ whose centre is $y_0 = \frac{Et}{B}, z_0 = \frac{E}{2\omega B}$. Evidently, the centre moves to the right with a constant speed.

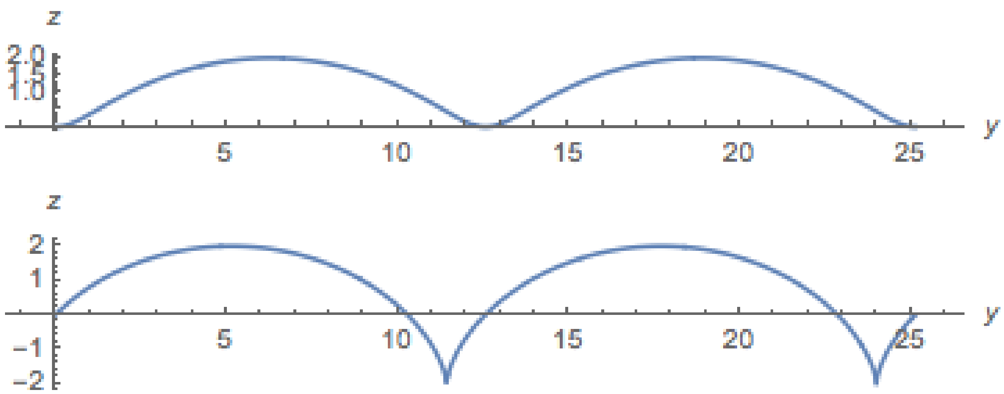

Figure 1: Upper and lower panels correspond to the solutions of Problem 1(b) and 1(c) respectively. They are plotted for $E/B = 2, \omega = 1$.

(c) From the initial conditions: $\dot{y}(0) = \dot{z}(0) = \frac{E}{B} \implies -C_1\omega = C_2\omega + \frac{E}{B} = \frac{E}{B}$. This implies $C_1 = -C_3 = -\frac{E}{\omega B}, C_2 = C_4 = 0$. Therefore,

$$y(t) = -\frac{E}{\omega B}\cos\omega t + \frac{E}{B}t + \frac{E}{\omega B}, \quad z(t) = \frac{E}{\omega B}\sin\omega t$$

This again satisfies the equation of a circle

$$\left(y - \frac{E}{\omega B}(\omega t + 1)\right)^2 + z^2 = \left(\frac{E}{\omega B}\right)^2$$

whose centre is at $y_0 = \frac{E}{\omega B}(\omega t + 1), z_0 = 0$.

2. Consider two infinite straight line charges λ, a distance d apart, moving along at a constant speed v as shown in figure 2. How large would v have to be in order for the magnetic attraction to balance the electrical repulsion?

Solution: Consider the wires to have charge per unit length λ. Since they are moving

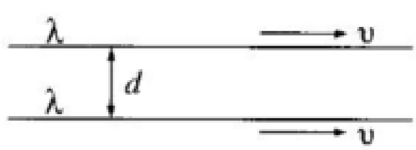

Figure 2: Figure for problem 2.

with speed v, they correspond to current $I = \lambda v$ each. Now consider the Magnetic field due to the wire 1 (carrying current I_1) at the location of the wire 2 (carrying current I_2) is (As discussed in lecture 1)

$$B = \frac{\mu_0 I_1}{2\pi d}$$

The Lorentz force on the wire 2 due to this magnetic field is

$$F = I_2 \frac{\mu_0 I_1}{2\pi d} \int dl$$

Thus, the magnetic force per unit length is

$$f_m = \frac{\mu_0 I_1 I_2}{2\pi d}$$

Now, the electric field due to a line charge at a distance s is given by

$$E = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s}$$

Electric repulsion force per unit length on the other wire of same charge per unit length is

$$f_e = \frac{1}{2\pi\epsilon_0} \frac{\lambda^2}{s}$$

Thus the electric repulsion balances the magnetic attraction when

$$\frac{\mu_0 I_1 I_2}{2\pi d} = \frac{\mu_0 \lambda^2 v^2}{2\pi d} = \frac{1}{2\pi\epsilon_0} \frac{\lambda^2}{d} \implies v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

which is nothing but the speed of light in vacuum!

3. Calculate the magnetic force of attraction between the northern and southern hemispheres of a spinning charged spherical shell, shown in figure 3. The magnetic force on a surface current $\vec{K}$ is given by

$$\vec{F} = \int (\vec{K} \times \vec{B}_{\text{avg}}) da, \quad \vec{B}_{\text{avg}} = \frac{1}{2}(\vec{B}_{\text{inside}} + \vec{B}_{\text{outside}})$$

Solution:

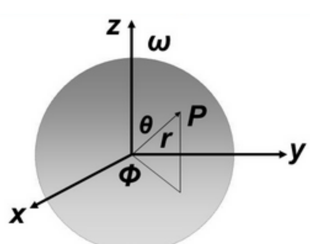

Figure 3: Figure for problem 3.

The magnetic force on a surface current $\vec{K}$ is given by

$$\vec{F} = \int (\vec{K} \times \vec{B}_{\text{avg}}) da = \int (\vec{v} \times \vec{B}) \sigma da$$

(See Section 2.5.3 (Introduction to Electrodynamics, D J Griffiths) for electrostatic force on surface charge and the reason behind taking average of the field.)

Here $\vec{K} = \sigma\vec{v} = \sigma(\vec{\omega} \times R\hat{r}) = \sigma(-\omega\sin\theta\hat{\theta} \times R\hat{r}) = \sigma\omega R\sin\theta\hat{\phi}$. The average magnetic field is $\vec{B}_{\text{avg}} = \frac{1}{2}(\vec{B}_{\text{inside}} + \vec{B}_{\text{outside}})$. The area element on the surface of the spherical shell is $da = R^2\sin\theta d\theta d\phi$.

Now, to calculate the magnetic field inside and outside the shell, recall the derivation of magnetic vector potential for such a spinning charged spherical shell done in the class:

$$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_0 R\sigma}{3}(\vec{\omega} \times \vec{r}) = \frac{\mu_0 R\omega\sigma}{3} r\sin\theta\hat{\phi} & \text{for } r \leq R \\ \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega} \times \vec{r}) = \frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi} & \text{for } r \geq R. \end{cases}$$

Using this, we can find the magnetic field as

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r^2\sin\theta}\left[\hat{r}\left(\frac{\partial}{\partial\theta} r\sin\theta A_\phi\right) - r\hat{\theta}\left(\frac{\partial}{\partial r} r\sin\theta A_\phi\right)\right]$$

$$\implies \vec{B}_{\text{inside}} = \vec{\nabla} \times \vec{A}_{\text{inside}} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\hat{r} - \sin\theta\hat{\theta})$$

$$\implies \vec{B}_{\text{outside}} = \vec{\nabla} \times \vec{A}_{\text{outside}} = \frac{\mu_0 R^4\omega\sigma}{3}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r^2}\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\hat{\theta}\right]$$

$$\implies \vec{B}_{\text{outside}} = \frac{\mu_0 R^4\omega\sigma}{3r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) = \frac{\mu_0 R\omega\sigma}{3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

after using $r = R$ in the last expression for the field just outside the shell. Thus the average field is

$$\vec{B}_{\text{avg}} = \frac{1}{2}(\vec{B}_{\text{inside}} + \vec{B}_{\text{outside}}) = \frac{\mu_0 R\omega\sigma}{6}(4\cos\theta\hat{r} - \sin\theta\hat{\theta})$$

. Therefore,

$$\vec{K} \times \vec{B}_{\text{avg}} = \sigma\omega R\sin\theta\hat{\phi} \times \frac{\mu_0 R\omega\sigma}{6}(4\cos\theta\hat{r} - \sin\theta\hat{\theta})$$

$$\implies \vec{K} \times \vec{B}_{\text{avg}} = \frac{\mu_0}{6}(\sigma\omega R)^2(4\cos\theta\hat{\theta} + \sin\theta\hat{r})\sin\theta$$

. Writing $\hat{r}, \hat{\theta}$ in terms of $\hat{x}, \hat{y}, \hat{z}$ that is $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}, \hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$ and considering only the z-component (which is also the direction of force exerted by the hemispheres on each other), we get

$$(\vec{K} \times \vec{B}_{\text{avg}})_z = \frac{\mu_0}{6}(\sigma\omega R)^2(4\cos\theta(-\sin\theta) + \sin\theta\cos\theta)\sin\theta = -\frac{\mu_0}{2}(\sigma\omega R)^2\sin^2\theta\cos\theta$$

. The net force is therefore,

$$F_z = \int(\vec{K} \times \vec{B}_{\text{avg}})_z da = -\frac{\mu_0}{2}(\sigma\omega R)^2 R^2 \int \sin^3\theta\cos\theta d\theta d\phi = -\frac{\mu_0}{2}(\sigma\omega R^2)^2(2\pi)\left(\frac{\sin^4\theta}{4}\right)\Big|_0^{\pi/2}$$

$$\implies F = -\frac{\mu_0 \pi}{4}(\sigma\omega R^2)^2 \implies \vec{F} = -\frac{\mu_0 \pi}{4}(\sigma\omega R^2)^2 \hat{z}$$

.

4. Find the magnetic field due to a current I in a coaxial cable whose inner conductor has radius a and the outer conductor has the radii $b, c (b < c)$. Also, express the magnetic field as a vector in terms of the current density.

Solution:
The current is uniformly distributed in both the conductors, see figure 4. The current

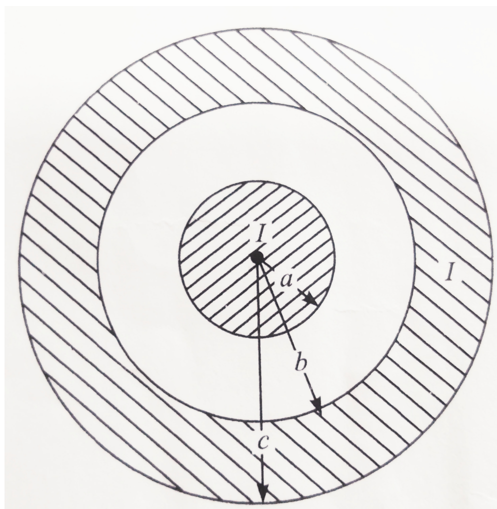

Figure 4

densities in the inner and outer conductors are $J_{\text{in}} = \frac{I}{\pi a^2}$ and $J_{\text{out}} = \frac{I}{\pi(c^2-b^2)}$, respectively with the densities are in the $\hat{z}$, $-\hat{z}$ directions respectively. Due to the symmetry of the problem the magnetic field ($\vec{B}$) is in the $\hat{\phi}$ direction.

Inside the inner conductor ($r < a$) by Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} = \mu_0 J_{\text{in}} \pi r^2$$

$$2\pi r B_\phi = \mu_0 J_{\text{in}} \pi r^2 \implies B_\phi = \mu_0 \frac{J_{\text{in}}}{2} r \implies \vec{B} = \mu_0 \frac{\vec{J}_{\text{in}} \times \vec{r}}{2}$$

.

In the region between the inner and outer conductors, $a < r < b$,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} = \mu_0 J_{\text{in}} \pi a^2$$

$$B_\phi = \mu_0 \frac{J_{\rm in}}{2} \frac{a^2}{r} \implies \vec{B} = \mu_0 \frac{\vec{J_{\rm in}} \times \vec{r}}{2} \left(\frac{a}{r}\right)^2$$

In the outer conductor, $b \leq r \leq c$,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I - I_{\rm out}\pi \left(r^2 - b^2\right)\right) = \mu_0 \left(I - \frac{I\pi \left(r^2 - b^2\right)}{\pi \left(c^2 - b^2\right)}\right) = \frac{\mu_0 I}{\pi \left(c^2 - b^2\right)} \pi \left(c^2 - r^2\right)$$

$$B_\phi = \frac{\mu_0 J_{\rm out}}{2} \frac{c^2 - r^2}{r} \implies \vec{B} = \mu_0 \frac{\vec{J_{\rm out}} \times \vec{r}}{2} \left(\frac{c^2 - r^2}{r^2}\right)$$

Note that outside the conductor $(r > c)$ $\vec{B} = 0$. The figure 5 shows the magnetic field variation with radius.

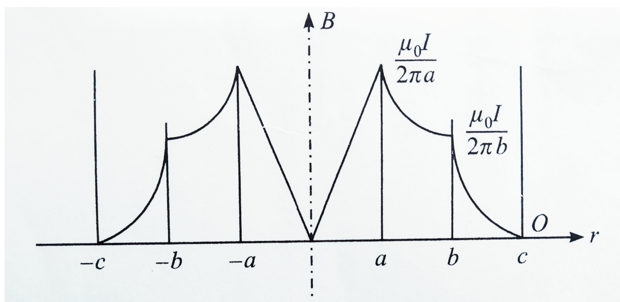

Figure 5

For Students: Solve the problem by taking the currents in outer and inner conductors to be in the same direction.

5. A long hollow coaxial wire has inner radius a and outer radius b. Uniform current I flows along its inner surface and return through the outer surface as shown in figure 6. Find vector potential at a distance s from its axis.

Solution:

Using Stoke's theorem, the flux coming out of area $d\vec{S}$ is,

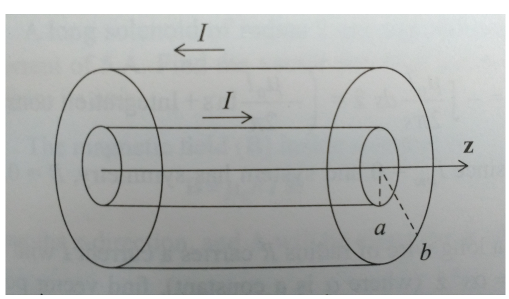

Figure 6: Figure for tutorial problem 5.

$$\Phi = \int \vec{B} \cdot d\vec{S} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \oint \vec{A} \cdot d\vec{l},$$

and using the Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} \implies \vec{B} = \frac{\mu_0 I_{\text{enc}}}{2\pi s} \hat{\phi}$$

The current is along $\hat{z}$, hence the vector potential would be also in the $\hat{z}$ direction and $A_s = A_\phi = 0$.

$$B = (\vec{\nabla} \times \vec{A})_\phi = \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial s} \right) \implies A = A_z = -\int B \; ds.$$

For $s < a$, $I_{\text{enc}} = 0$ hence $B = 0$ and $A =$ constant.

For $a < s < b$, $I_{\text{enc}} = I$ thus $\vec{A} = -\int B ds \; \hat{z} = -\int \frac{\mu_0 I}{2\pi s} ds \; \hat{z} = \left(-\frac{\mu_0 I}{2\pi} \ln s + \text{constant} \right) \hat{z}$

For $s > b$, $I_{\text{enc}} = 0$ so again $B = 0$ and $A =$ constant.

6. Show that for uniform magnetic field $\vec{B}$, the magnetic vector potential can be written as $\vec{A} = -\frac{1}{2}(\vec{r} \times \vec{B})$. Is this result unique, or are there other functions with the same properties?

Solution:

$$\vec{\nabla} \times \vec{A} = -\frac{1}{2} \vec{\nabla} \times (\vec{r} \times \vec{B}) = -\frac{1}{2} [(\vec{B} \cdot \vec{\nabla})\vec{r} - (\vec{r} \cdot \vec{\nabla})\vec{B} + \vec{r}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{r})]$$

Here $(\vec{r} \cdot \vec{\nabla})\vec{B} = 0$, as $\vec{B}$ is uniform and $\vec{r}(\vec{\nabla} \cdot \vec{B}) = 0, \vec{B}(\vec{\nabla} \cdot \vec{r}) = 3\vec{B}$. The remaining term can be written as

$$(\vec{B} \cdot \vec{\nabla})\vec{r} = \left(B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \right) (x\hat{x} + y\hat{y} + z\hat{z}) = \vec{B}$$

Therefore, $\vec{\nabla} \times \vec{A} = -\frac{1}{2}(\vec{B} - 3\vec{B}) = \vec{B}$. Also,

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{2} \vec{\nabla} \cdot (\vec{r} \times \vec{B}) = -\frac{1}{2} [\vec{B} \cdot (\vec{\nabla} \times \vec{r}) - \vec{r} \cdot (\vec{\nabla} \times \vec{B})] = 0$$

using the fact that $\vec{B}$ is uniform and $\vec{\nabla} \times \vec{r} = 0$.

Alternate Solution:

(a) Let us take the direction of uniform magnetic field (B_0) to be along the z-axis. Using $\vec{B} = \vec{\nabla} \times \vec{A}$, we can write

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0, \quad B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0, B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0.$$

Although the vector potential is not unique, there exists an interesting solution of the above equations:

$$A_x = -\frac{1}{2}yB_0, \ \ A_y = \frac{1}{2}xB_0, \ \ A_z = 0.$$

Here $\vec{B}$ is along the z-axis and $\vec{A}$ is zero along z-axis. Also, the x-component of $\vec{A}$ is proportional to $-y$ and the y-component is proportional to $+x$, indicating that $\vec{A}$ must be at right angles to the vector from the z-axis. The magnitude of $\vec{A}$ is proportional to $\sqrt{x^2 + y^2}$, the distance from the z-axis, denoted by r'. For any uniform field $\vec{B}$ the vector potential, therefore, can be written as

$$\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r}) = -\frac{1}{2}(\vec{r} \times \vec{B}).$$

(b) Another way to find $\vec{A}$ is to take its line integral around a closed circular loop and then use the Stoke's theorem:

$$\oint \vec{A} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot \hat{n} da.$$

For uniform field chosen to be perpendicular to the plane of the loop, the right hand side of the above equation is $\pi r^2 B$. Choosing the origin to be on the axis of symmetry, so that we can take $\vec{A}$ as circumferential and a function of r only, then the line integral is

$$\oint \vec{A} \cdot d\vec{l} = A(2\pi r) = \pi r^2 B \implies A = \frac{Br}{2}.$$

Since $\vec{B}$ is perpendicular to the area of the loop and $\vec{A}$ is circumferential, it is straightforward to write $\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r}) = -\frac{1}{2}(\vec{r} \times \vec{B})$.

This result is not unique, we can always change $\vec{A} \to \vec{A} + \vec{C}$ where $\vec{C}$ is a constant vector. It can also be changed to $\vec{A} \to \vec{A} + \vec{\nabla}\lambda$ where λ is a scalar function which obeys Laplace's equation $\nabla^2\lambda = 0$. Note that we are considering Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$ which the given expression for vector potential satisfies.