Continuous-time Markov Chain: Poisson Process 2

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The state probabilities are given by

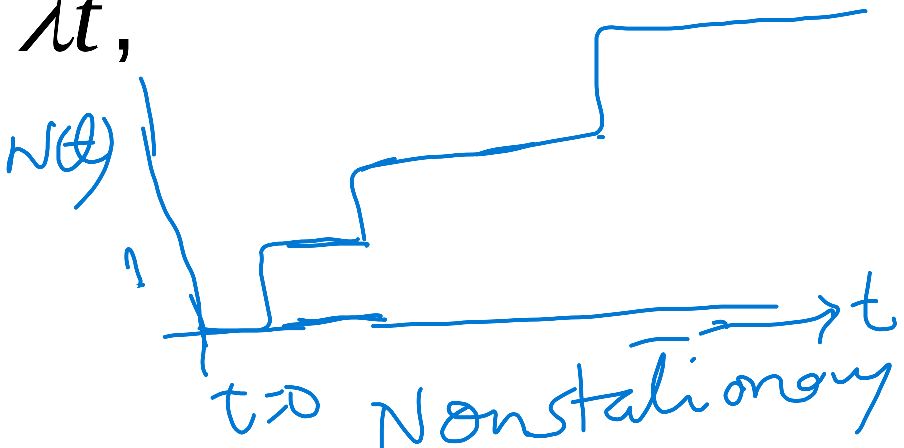
$$p_j(t) = P(N(t) = j) = e^{-\lambda t} \frac{(\lambda t)^j}{j!}, \; j = 0,1,..$$

The statistics of $\{N(t)\}$ is given by

$$EN(t) = \lambda t, \qquad \operatorname{var}(N(t)) = \lambda t,$$

$$C_N(t_1,t_2) = \lambda \min(t_1,t_2)$$

$$R_N(t_1,t_2) = \lambda \min(t_1,t_2) + \lambda^2 t_1 t_2$$



Nonstationary process

MS Continuity and Differentiability of a Poisson Process

For a Poisson process $\{N(t)\}$,

$$R_N(t_1,t_2) = \lambda \min(t_1,t_2) + \lambda^2 t_1 t_2$$

$$\therefore \quad R_N(t,t) = \lambda t \cancel{)} + \lambda^2 t^2$$

Now

$$\lim_{t_1 \to t, t_2 \to t} R_N(t_1,t_2) = \lambda \lim_{t_1 \to t, t_2 \to t} \min(t_1,t_2) + \lambda^2 \lim_{t_1 \to t, t_2 \to t} t_1 t_2$$

$$= \lambda t + \lambda^2 t^2$$

Thus the autocorrelation function of a Poisson process is continuous at each (t,t) implying that a Poisson~~er~~ process is m.s. continuous everywhere.

Differentiability

For a Poisson process $\{N(t)\}$,

$$R_N(t_1,t_2) = \lambda \min(t_1,t_2) + \lambda^2 t_1 t_2$$

$$\therefore \quad R_N(0,t_2) = \lambda \min(0,t_2)$$

$$= \begin{cases} \lambda t_2 \ \ if\ t_2 < 0 \\ \ 0 \ \ otherwise \end{cases}$$

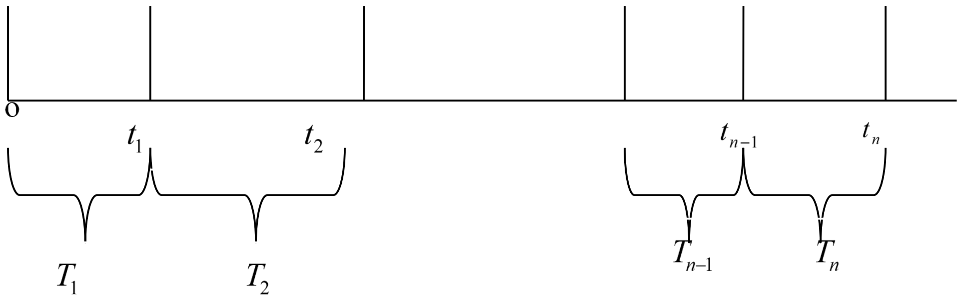
$$\therefore \frac{\partial R_N(0,t_2)}{\partial t_2} = \begin{cases} \lambda \quad \text{if } \ t_2 < 0 \\ 0 \quad \text{if } \ t_2 > 0 \\ \text{does not exist if } \ t_2 = 0 \end{cases}$$

$$\therefore \frac{\partial^2 R_N(t_1,t_2)}{\partial t_1 \partial t_2} \text{ does not exist at } \quad (t_1 = 0, t_2 = 0)$$

Thus, $\{N(t)\}$,is not m.s. differentiable.

Inter-arrival time and Waiting time for the Poisson Process

$P(\text{No event occur during } (0,t])$



$P(T_1 > t) = e^{-\lambda t}$

$f_T(t) = \lambda . e^{-\lambda t} \quad t > 0$

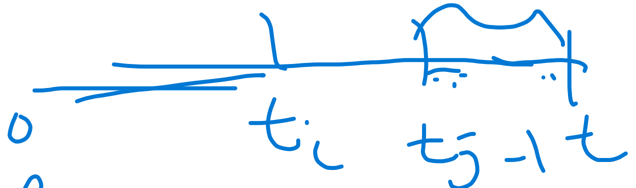
Let T_n = time elapsed between the (n-1)th event and the nth event. The random process $\{T_n, n = 1, 2, ..\}$ represent the *inter-arrival time* of the From the CTMC theory, it is clear that $T_n \sim \exp(\lambda), n = 1, 2, ..$ (identically distributed)

To prove independence of $T_n s$, consider the conditional probability $P(T_j > t \mid T_i = t_i)$

$$j > i$$

$$P(T_j > t \mid T_i = t_i) = P(T_j > t \mid N(t_i) = i)$$

$$= P(N(t_{j-1}, t] = 0 \mid N(t_i) = i)$$



Independent increment property.

$$= P(N(t_{j-1}, t] = 0)$$

$$= P(T_j > t)$$

Thus $T_n s$ are iid RVs with

$$f_{T_n}(t) = \lambda e^{-\lambda t} \qquad n > 0$$

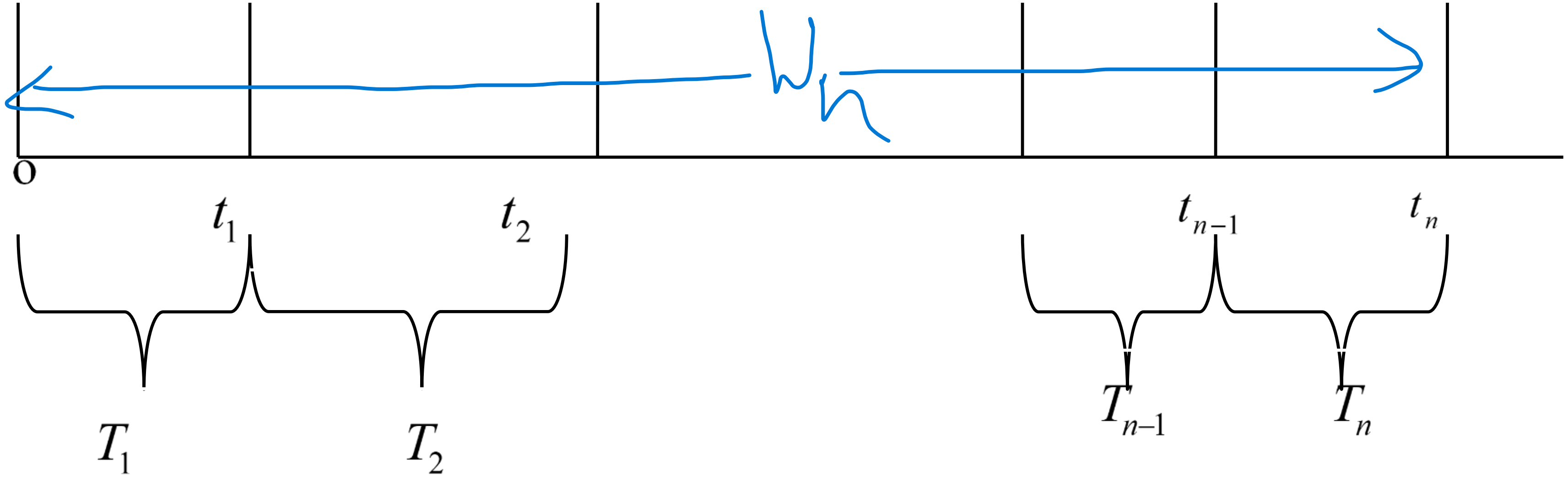
$$E T_n = \frac{1}{\lambda}$$

$$\mathrm{Var}(T_n) = \frac{1}{\lambda^2}$$

Memoryless property.

$$P(T > s+t \mid T > s) = P(T > t)$$

Waiting time



Now let us analyse the *waiting time* W_n. This is the time that elapses before the nth event occurs.

$\therefore W_n = \sum_{i=1}^{n} T_i$ (sum of n independent exponential random variables)

Gamma RV.

Proof

We note that

$$F_{W_n}(t) = P(\{W_n \le t\})$$

$$= P(\{N(t) \ge n\})$$

$$= \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$\therefore\ f_{W_n}(t) = \frac{d}{dt}\left(F_{W_n}(t)\right)$$

$$= \sum_{k=n}^{\infty}\left(\frac{\lambda(\lambda t)^{k-1} e^{-\lambda t}}{k-1!} - \frac{\lambda(\lambda t)^k e^{-\lambda t}}{k!}\right) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{n-1!} - \cancel{\frac{\lambda(\lambda t)^n e^{-\lambda t}}{n!}} + \cancel{\frac{\lambda e^{-\lambda t}(\lambda t)^n}{n!}} - \cdots$$

$$= \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{n-1!} \rightarrow \text{shape parameter}$$

`Thus W_n is a gamma random variable.

$$W_n = \sum_{i=1}^{n} T_i$$

$$E W_n = \sum_{i=1}^{n} E T_i = \sum_{i=1}^{n} \frac{1}{\lambda} = \frac{n}{\lambda}$$

$$\mathrm{Var}(W_n) = \sum_{i=1}^{n} \mathrm{Var}(T_i) = \frac{n}{\lambda^2}$$

$$\int_0^\infty f_{W_n}(t)\,dt = \frac{\int_0^\infty \lambda e^{-\lambda t}(\lambda t)^{n-1}\,dt}{\Gamma n} = \frac{\Gamma n}{\Gamma n} = 1$$

$\lambda t = u$

$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$

Gamma with integer shape parameter is known as an Erlang distribution

Example 2

The number of customers arriving at a service station is a Poisson process with a rate of 2 customers per minute.

(a) What is the mean arrival time of the customers?

(b) What is the probability that the first customer arrives after 30 second.

(c) Given that there is no arrival before 1 min, what is the probability that first arrival will be after 3 min.

(d) Given that the third customer has arrived at t=2 min, what is the probability that fourth customer will arrive after t=4 min?

(e) What is the average waiting time before the 10^{th} customer arrives?

$\lambda = 2/\text{min}$

$E\,T_i = \frac{1}{\lambda} = \frac{1}{2}\ \text{min} = 30\ \text{sec}$

$P(T_1 > 30) = e^{-\lambda t} = e^{-2 \times \frac{1}{2}} = e^{-1}$

$$P\left(T_1 > 3 \mid T_1 > 1\right) = P\left(T_1 > 2\right)$$

[Memoryless property]

$$= e^{-\lambda 2} = e^{-4}$$

$$P\left(T_4 > 4 \mid T_1 = 2\right) = P\left(T_4 > 4\right)$$

(Independence properties of T_i's)

$$EW_{10} = 10 \times \frac{1}{2} = 5 \text{ min}$$