PH 102: Physics II

Lecture 23 (Spring 2019)

IIT Guwahati
Debasish Borah

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Lectures	Division
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	I, II (4-4:55 pm)
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	III, IV (9-9:55 am)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	I, II (3-3:55 pm)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	III, IV (10-10:55 am)
Tut 8	19-3-2019	Lec 15, 16			
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic Materials, Magnetization	5.4, 6.1	1	I, II (4-4:55 pm)
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic Materials, Magnetization	5.4, 6.1	1	III, IV (9-9:55 am)
Lec 18	22-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	I, II (3-3:55 pm)
Lec 18	22-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	III, IV (10-10:55 am)
Tut 9	26-3-2019	Lec 17, 18			
Lec 19	27-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	I, II (4-4:55 pm)
Lec 19	27-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	III, IV (9-9:55 am)
Lec 20	28-3-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	I, II (3-3:55 pm)
Lec 20	28-3-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	III, IV (10-10:55 am)
Tut 10	2-4-2019	Lec 19, 20			
Lec 21	3-4-2019	Maxwell's equations	7.3	1	I, II (4-4:55 pm)
Lec 21	3-4-2019	Maxwell's equations	7.3	1	III, IV (9-9:55 am)
Lec 22	4-4-2019	Discussions, problem solving	7.3	1	I, II (3-3:55 pm)
Lec 22	4-4-2019	Discussions, problem solving	7.3	1	III, IV (10-10:55 am)
Tut 11	9-4-2019	Quiz II			

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	I, II (4-4:55 pm)
Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	III, IV (9-9:55 am)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	I, II (3-3:55 pm)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	III, IV (10-10:55 am)
Tut 12	23-4-2019	Lec 23, 24			
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	I, II (4-4:55 pm)
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	III, IV (9-9:55 am)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	I, II (3-3:55 pm)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	III, IV (10-10:55 am)
Tut 13	30-4-2019	Lec 25, 26			
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	I, II (4-4:55 pm)
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	III, IV (9-9:55 am)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	I, II (3-3:55 pm)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	III, IV (10-10:55 am)

Maxwell's Equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$	Gauss's (Coulomb's) Law
$$\vec{\nabla} \cdot \vec{B} = 0$$	No name
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$	Faraday's Law
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$	Ampere's Law with Maxwell's Correction

- Faraday's law led to the definition of work done or energy stored in magnetic field $W_m = \frac{1}{2\mu_0} \int B^2 d\tau$
- Does Ampere's law have anything to add to the discussion of work/energy/energy conservation?

Continuity Equation: Conservation of Charge

Charge inside a volume: $$Q(t) = \int_{\mathcal{V}} \rho(\vec{r}, t) d\tau$$

Current flowing through a boundary should be equal to the rate of change of charge inside the volume:

$$\frac{dQ}{dt} = -\oint_{\mathcal{S}} \vec{J} \cdot d\vec{a} \implies \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau = -\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{J} d\tau$$

Since this is true for any volume, we have

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$$

which is the continuity equation for charge implying the local conservation of charge.

This can also be derived from Maxwell's equation and hence is a consequence of the laws of classical electrodynamics.

Continuity Equation: Conservation of Energy

- Just like the conservation of charge is quantitatively expressed in terms of the continuity equation, is it possible to write down the conservation of energy quantitatively?

- In analogy with the conservation of charge, can we write conservation of energy as

$$\frac{\partial u}{\partial t} = -\vec{\nabla} \cdot \vec{S}$$

where *u* is energy density (analogous to charge density ρ) and $\vec{S}$ is the energy flux: energy per unit time per unit area (analogous to $\vec{J}$: charge per unit time per unit area)?

Conservation of Energy

The energy stored in electric field or the work required to assemble a static charge distribution (against their Coulomb repulsion) is:

$$W_e = \frac{\epsilon_0}{2} \int E^2 d\tau$$

Similarly, the energy stored in magnetic field or the work required to be done to keep the current going against the back emf is

$$W_m = \frac{1}{2\mu_0} \int B^2 d\tau$$

Lecture 20

Thus, the total energy stored in electromagnetic fields is

$$U_{em} = \frac{1}{2} \int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau$$

Conservation of Energy

Consider a charge and current configuration at time *t* which produces fields $\vec{E}, \vec{B}$. The work done by the electromagnetic forces in moving the charges in time interval *dt* is

$$\vec{F} \cdot d\vec{l} = q(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v}dt = q\vec{E} \cdot \vec{v}dt$$

Magnetic force does not do work, as usual!

Using $q = \rho d\tau, \ \rho\vec{v} = \vec{J}$, the rate of work done on all the charges in the volume is

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\vec{E} \cdot \vec{J}) d\tau$$

Work done per unit time per unit volume:
Power delivered per unit volume $\equiv \vec{E} \cdot \vec{J}$

Conservation of Energy

Using Ampere's law (after Maxwell's correction)

$$\vec{E}\cdot\vec{J} = \vec{E}\cdot\left(\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right) = \frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

Using the product rule $\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$ and Faraday's law $\vec{\nabla}\times\vec{E} = -(\partial\vec{B}/\partial t)$ we get

$$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Therefore,

$$\vec{E}\cdot\vec{J} = -\epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} - \frac{1}{\mu_0}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$\implies \vec{E}\cdot\vec{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Using this in the expression for the rate of work:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\int_{\mathcal{V}}\vec{\nabla}\cdot(\vec{E}\times\vec{B})d\tau$$

$$\implies \frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}}(\vec{E}\times\vec{B})\cdot d\vec{a}$$

$\mathcal{S}$ is the surface bounding $\mathcal{V}$

Poynting's Theorem

Work-energy theorem of electrodynamics:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\vec{E}\times\vec{B})\cdot d\vec{a}$$

Here, $\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau = U_{em}$ is the total energy stored in the fields.

The second integral $\frac{1}{\mu_0}\oint_{\mathcal{S}} (\vec{E}\times\vec{B})\cdot d\vec{a}$ is the rate at which energy is carried out of the volume through the boundary surface.

Poynting's Theorem: *The work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface.*

John Henry Poynting (1852-1914)

Poynting's Theorem

The energy per unit time, per unit area, transported by the fields is called the **Poynting vector**:

$$\vec{S} \equiv \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$

and the energy per unit time crossing an infinitesimal area is $\vec{S} \cdot d\vec{a}$
In terms of Poynting vector, the Poynting theorem can be written as

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_{\mathcal{S}} \vec{S} \cdot d\vec{a}$$

Since the work done on the charged particles eventually increases their mechanical energy (kinetic, potential etc.), the left hand side can be written as

$$\frac{dW}{dt} = \frac{d}{dt}\int_{\mathcal{V}} u_{\text{mech}} d\tau$$

u_{mech} is the mechanical energy density

Poynting's Theorem

Writing the energy density in electromagnetic fields as

$$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

The Poynting's theorem can be rewritten as

$$\frac{d}{dt}\int_{\mathcal{V}} (u_{\text{mech}} + u_{em}) d\tau = -\oint_{\mathcal{S}} \vec{S} \cdot d\vec{a} = -\int_{\mathcal{V}} (\vec{\nabla} \cdot \vec{S}) d\tau$$

$$\implies \frac{\partial}{\partial t}(u_{\text{mech}} + u_{em}) = -\vec{\nabla} \cdot \vec{S}$$

Differential version of Poynting's theorem: Continuity equation for energy density (mechanical and electromagnetic)

Poynting's Theorem

The previous statement: $$\frac{dU}{dt} = -\oint \vec{S} \cdot d\vec{a}$$

allows us to interpret **S** as the energy flux density, in analogy with current density **J** in the continuity equation for electromagnetic charge:

$$\frac{dQ}{dt} = -\oint \vec{J} \cdot d\vec{a}$$

In energy flows out of the system of volume V:$\vec{S} = S\hat{n}, dU/dt < 0$ where n is the unit normal vector perpendicular to the surface enclosing the volume V.

On the other hand, if energy flows into the system: $\vec{S} = S(-\hat{n}), dU/dt > 0$

Example: Consider an inductor made from a solenoid of length *l*, radius r, n turns per unit length. Suppose at some instant the current is changing at a rate *dI/dt* >0.

The magnetic field inside is $\vec{B} = \mu_0 n I \hat{k}$

The rate of increase of the field is $\dfrac{dB}{dt} = \mu_0 n \dfrac{dI}{dt}$

According to Faraday's law, the rate of change of magnetic flux will induce an electric field.

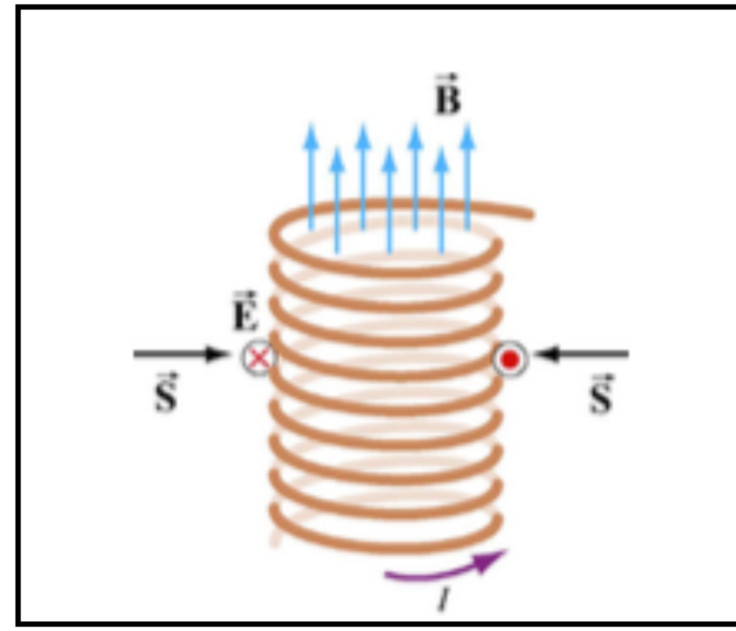
$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

$$\implies E(2\pi r) = -\mu_0 n \frac{dI}{dt}(\pi r^2) \implies \vec{E} = -\frac{\mu_0 n r}{2}\frac{dI}{dt}\hat{\phi}$$

The corresponding Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \frac{1}{\mu_0}\left(-\frac{\mu_0 n r}{2}\frac{dI}{dt}\hat{\phi}\right) \times (\mu_0 n I \hat{k}) = -\frac{\mu_0 n^2 r I}{2}\frac{dI}{dt}\hat{r}$$

The Poynting vector is in the negative radial direction, as shown in the figure.



The magnetic energy stored is

$$U_B = \left(\frac{B^2}{2\mu_0}\right)(\pi r^2 l) = \frac{1}{2}\mu_0 \pi n^2 I^2 r^2 l$$

The rate of change of stored energy

$$\frac{dU_B}{dt} = \mu_0 \pi n^2 I r^2 l \frac{dI}{dt} = I|\mathcal{E}| = P$$

where $$\mathcal{E} = -N\frac{d\Phi}{dt} = -(nl)\frac{dB}{dt}(\pi r^2) = -\mu_0 n^2 l \pi r^2 \frac{dI}{dt}$$

which is same as

$$-\oint \vec{S} \cdot d\vec{a} = \frac{\mu_0 n^2 r I}{2}\frac{dI}{dt}(2\pi r l) = \mu_0 \pi n^2 I r^2 l \frac{dI}{dt}$$

Image credit: MIT

Therefore

$$\frac{dU_B}{dt} = -\oint \vec{S} \cdot d\vec{a} > 0$$

The energy of the system increases as expected, when *dI/dt* >0.

For *dI/dt* < 0, energy of the system will decrease.

This is also reflected by the direction of the Poynting vector

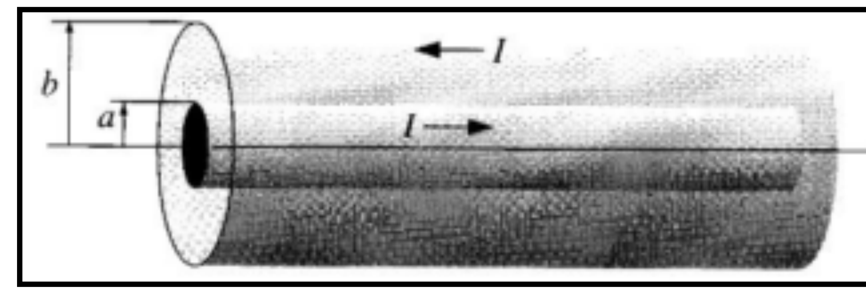
Note: While calculating the energy density stored in fields, we calculated only the magnetic energy, as the energy stored in electric field is not changing with time: $d^2I/dt^2 = 0$

Repeat the calculation taking $d^2I/dt^2 \neq 0$ and check the validity of Poynting's theorem.

Problem 8.1 (Introduction to Electrodynamics, D J Griffiths): Calculate the power transported down the cables shown in figure , assuming the outer and inner cylinder to be held at potential difference V.

Solution: Find the Poynting vector: The power per unit area transported by the fields.

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$



$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\hat{r}, \quad \vec{B} = \frac{\mu_0 I}{2\pi}\frac{1}{r}\hat{\phi}, \quad \vec{S} = \frac{\lambda I}{4\pi^2\epsilon_0}\frac{1}{r^2}\hat{z}$$

Total power transported:

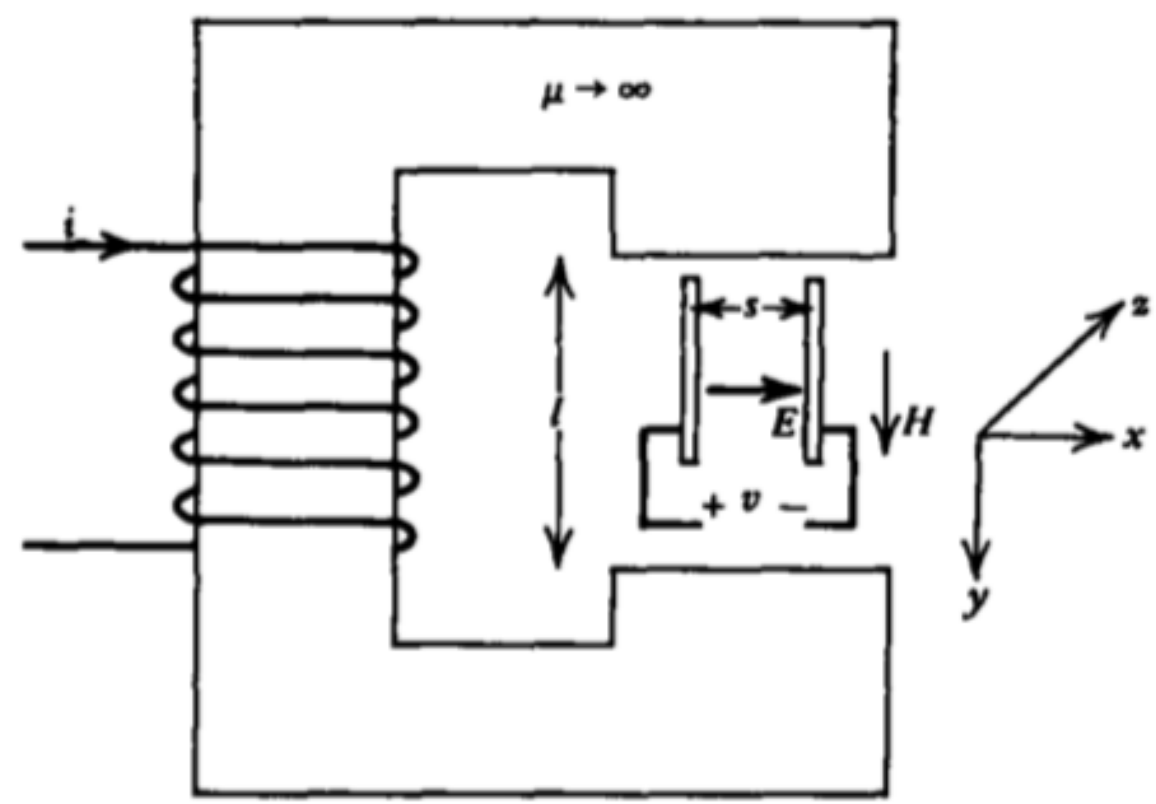
$$P = \int \vec{S} \cdot d\vec{a} = \int_a^b S(2\pi r)dr = \frac{\lambda I}{2\pi\epsilon_0}\int_a^b \frac{dr}{r} = \frac{\lambda I}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$

Charge per unit length can be absorbed in the potential difference:

$$V = \int_a^b \vec{E} \cdot d\vec{l} = \frac{\lambda}{2\pi\epsilon_0}\int_a^b \frac{dr}{r} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$

Therefore, P = VI.

Exercise: Consider a situation where a static electric field is applied perpendicularly to a static magnetic field, as in the case of a pair of electrodes placed within a magnetic circuit (see figure).



1. What are **E**, **B** and **S**?
2. What is the energy density stored in the system?
3. Verify Poynting's theorem.

Exercise: Consider an electromagnetic field present within a superconductor to be given by the relation

$$\frac{\partial J}{\partial t} = \omega^2 \epsilon \vec{E}$$

Show that the Poynting's theorem can be written in the form $\vec{\nabla} \cdot \vec{S} + \partial u/\partial t = 0$. What is u?

Exercise: A parallel plate capacitor with circular plates of radius R is being charged at a uniform rate by current I in an external circuit. Obtain an expression for the Poynting vector when it is being charged. Calculate the rate of flow of energy through an imaginary cylindrical surface at the edge of the two plates.

- Find electric field.
- Find displacement current and then find the magnetic field.
- Find Poynting vector.
- Find the direction of energy flow.

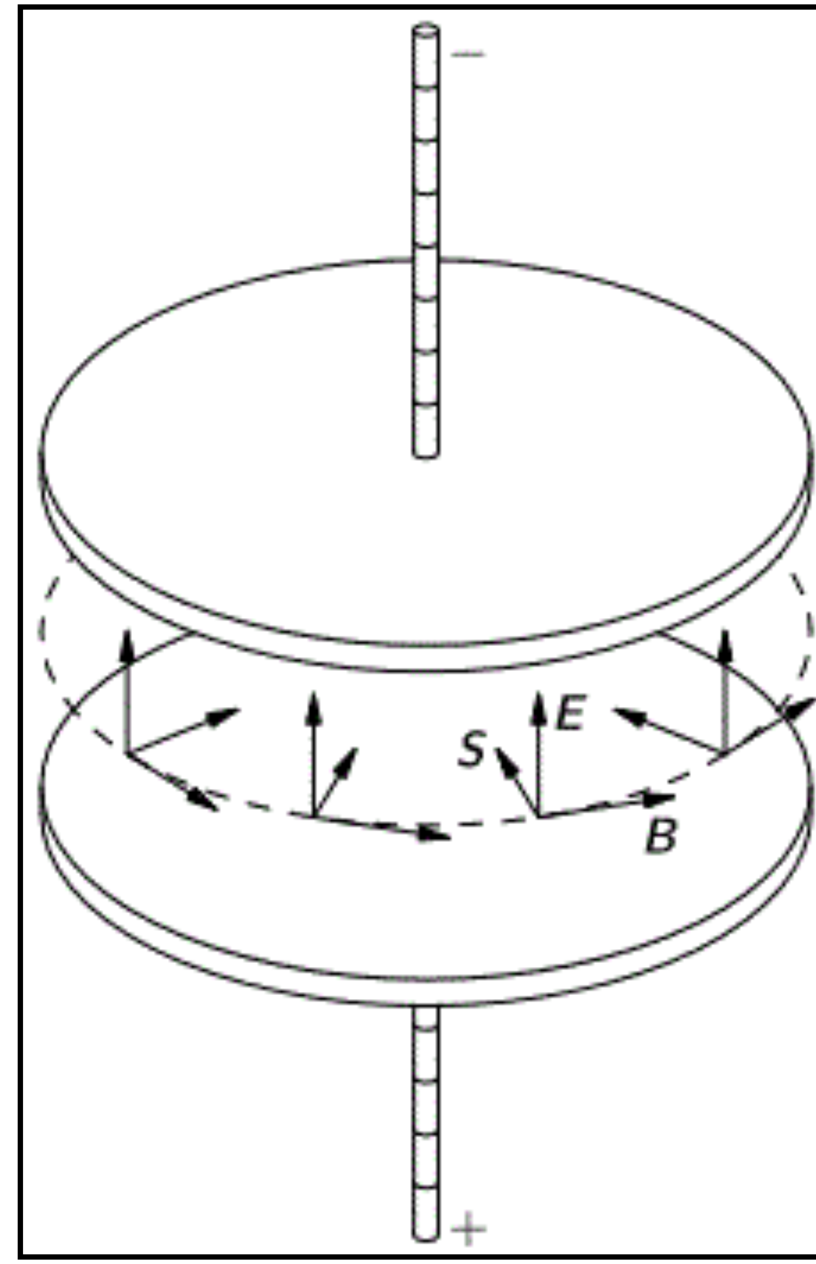


Image credit: Feynman Lectures Vol II