PH 102: Physics II

Lecture 18 (Spring 2019)

IIT Guwahati
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LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Lectures	Division
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	I, II (4-4:55 pm)
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	III, IV (9-9:55 am)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	I, II (3-3:55 pm)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	III, IV (10-10:55 am)
Tut 8	19-3-2019	Lec 15, 16			
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic Materials, Magnetization	5.4, 6.1	1	I, II (4-4:55 pm)
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic Materials, Magnetization	5.4, 6.1	1	III, IV (9-9:55 am)
Lec 18	22-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	I, II (3-3:55 pm)
Lec 18	22-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	III, IV (10-10:55 am)
Tut 9	26-3-2019	Lec 17, 18			
Lec 19	27-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	I, II (4-4:55 pm)
Lec 19	27-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	III, IV (9-9:55 am)
Lec 20	28-3-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	I, II (3-3:55 pm)
Lec 20	28-3-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	III, IV (10-10:55 am)
Tut 10	2-4-2019	Lec 19, 20			
Lec 21	3-4-2019	Maxwell's equations	7.3	1	I, II (4-4:55 pm)
Lec 21	3-4-2019	Maxwell's equations	7.3	1	III, IV (9-9:55 am)
Lec 22	4-4-2019	Discussions, problem solving	7.3	1	I, II (3-3:55 pm)
Lec 22	4-4-2019	Discussions, problem solving	7.3	1	III, IV (10-10:55 am)
Tut 11	9-4-2019	Quiz II			

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	I, II (4-4:55 pm)
Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	III, IV (9-9:55 am)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	I, II (3-3:55 pm)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	III, IV (10-10:55 am)
Tut 12	23-4-2019	Lec 23, 24			
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	I, II (4-4:55 pm)
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	III, IV (9-9:55 am)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	I, II (3-3:55 pm)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	III, IV (10-10:55 am)
Tut 13	30-4-2019	Lec 25, 26			
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	I, II (4-4:55 pm)
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	III, IV (9-9:55 am)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	I, II (3-3:55 pm)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	III, IV (10-10:55 am)

Magnetisation

- Net alignment of magnetic dipoles inside a medium, in the presence of an applied magnetic field is called magnetisation.

- Although electric polarisation is almost always in the same direction as the electric field ($\vec{P} = \epsilon_0 \chi_e \vec{E}, \chi_e > 0$), magnetisation on the other hand, can happen either parallel (**paramagnets**, say Al) or opposite (**diamagnets**, say Cu) to the applied magnetic field.

- A few substance also retain their magnetisation even after the applied magnetic field is removed (**ferromagnets**). For example: iron, nickel, cobalt.

- The torque on magnetic dipoles by the external magnetic field and their subsequent alignment parallel to the field is the root of paramagnetism.

- Electrons inside atom revolve around the nucleus and hence constitute a magnetic dipole.

- Paramagnetism normally occurs in atoms and molecules with odd number of electrons.

- For even number of electrons, they usually pair up with opposite spins (Pauli's exclusion principle) and this effectively neutralises the torque on the combination.

- Magnetism is due to such tiny current loops (Ampere model) at atomic level, not due to separated magnetic monopoles (Gilbert model).

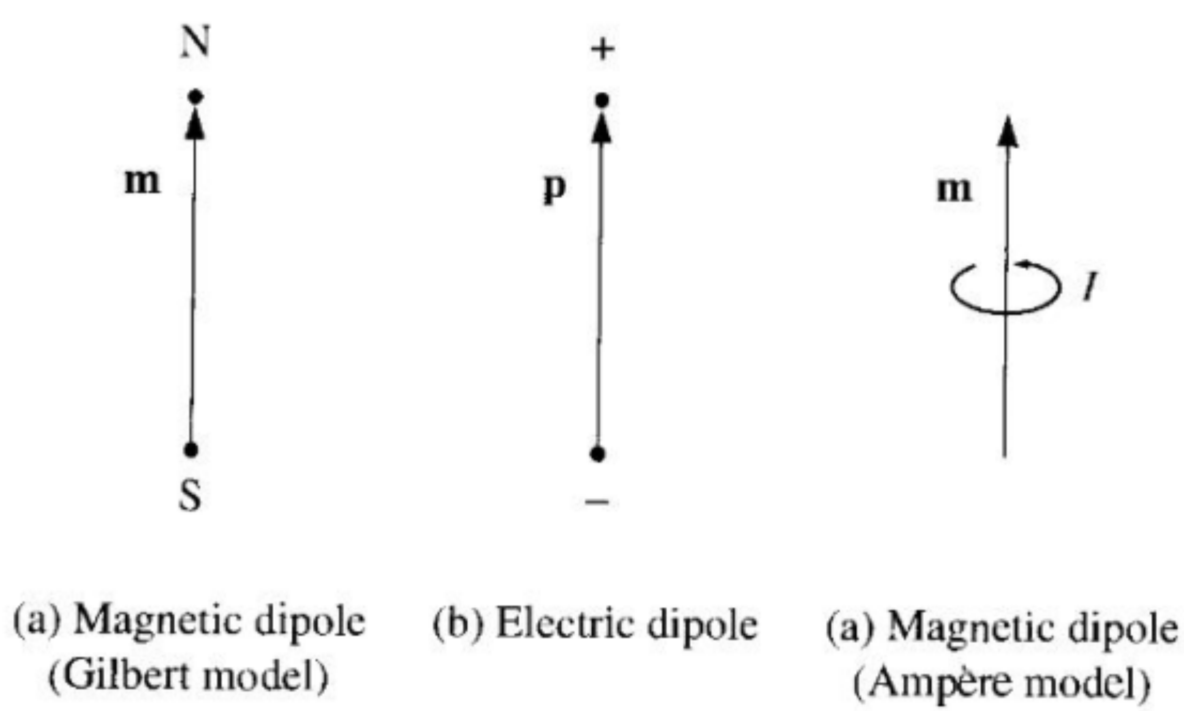


Figure 6.5, Introduction to Electrodynamics, D. J. Griffiths

Electrons moving in an orbit of radius R around the nucleus can give rise to a current $I = \frac{e}{T} = \frac{ev}{2\pi R}$

which can be approximated to be a *steady* current given the tiny T.

The orbital dipole moment is

$$\vec{m} = -\frac{1}{2} evR\hat{z}$$

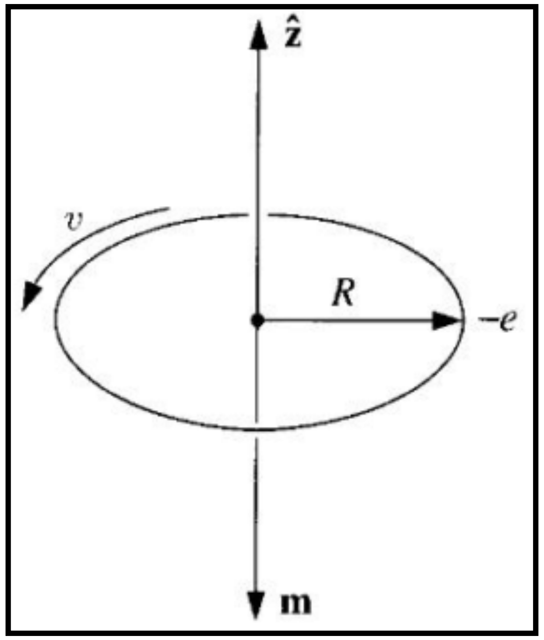


Figure 6.9, Introduction to Electrodynamics, D. J. Griffiths

External magnetic field applies a torque $(\vec{m} \times \vec{B})$ trying to tilt the dipole. The same field also affects the orbital motion of electrons by speeding it up or slowing it down. The latter effect dominates.

Tutorial 9: g=m/L=Q/2M

Electrons in atomic orbits can either speed up or slow down depending on the orientation of magnetic field. For magnetic field perpendicular to the plane of the orbit

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e\frac{\bar{v}^2}{R}$$

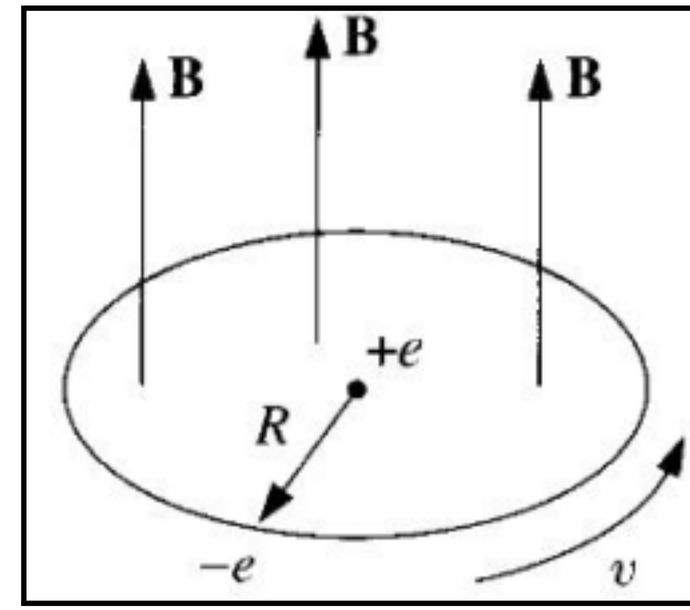


Figure 6.10, Introduction to Electrodynamics, D. J. Griffiths

The new speed v is greater (?) than the one for B=0. Therefore,

$$e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v}+v)(\bar{v}-v)$$

Assuming the change in speed $\Delta v = \bar{v} - v$ to be small, we can write

$$\Delta v = \frac{eRB}{2m_e} \qquad (\bar{v} + v = 2\bar{v} - \Delta v, (\Delta v)^2 \approx 0\,)$$

A change in orbital speed leads to a change in current and hence a change in the dipole moment

$$\Delta\vec{m} = -\frac{1}{2}e\Delta vR\hat{z} = -\frac{e^2R^2}{4m_e}\vec{B}$$

Change in dipole moment is **opposite** to the direction of magnetic field*.

Usually, the random orientations of electron orbits lead to cancelation of orbital dipole moments. In presence of magnetic field, they can acquire an extra dipole moment: **Diamagnetism**.

Diamagnetism is universal but weaker than *Paramagnetism* and is observed in atoms with even number of electrons where the latter is absent.

*Same is true even if you flip the direction of magnetic field w.r.t. the plane of the orbit: Check it!

- Easier to tilt the spin than the entire orbit: orbital contribution to paramagnetism is small!

To summarise:

- In paramagnetism, the dipoles associated with the spins of unpaired electrons experience a torque trying to align them parallel to the field.

- In diamagnetism, the orbital speed of the electrons is changed in such a way that the change in orbital dipole moment opposes the direction of the applied field.

Magnetisation (M)

Quantitatively, magnetisation is defined as the magnetic dipole moment per unit volume (M), analogous to P in electrostatics.

$$\vec{M} = \frac{1}{V}\sum_i \vec{m}_i$$

A piece of a magnetic material can be placed above a solenoid to induce magnetisation. For paramagnetic material, the induced magnetisation will be upward and hence the force is downward. ($\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$)*

The magnetisation (force) is downward (upward) for diamagnetic materials*.

Therefore, a paramagnet is attracted into the field whereas a diamagnet is repelled by the field.

The actual force is however, much smaller (4-5 order of magnitudes) compared to that on an iron sample (a ferromagnet) of similar size.

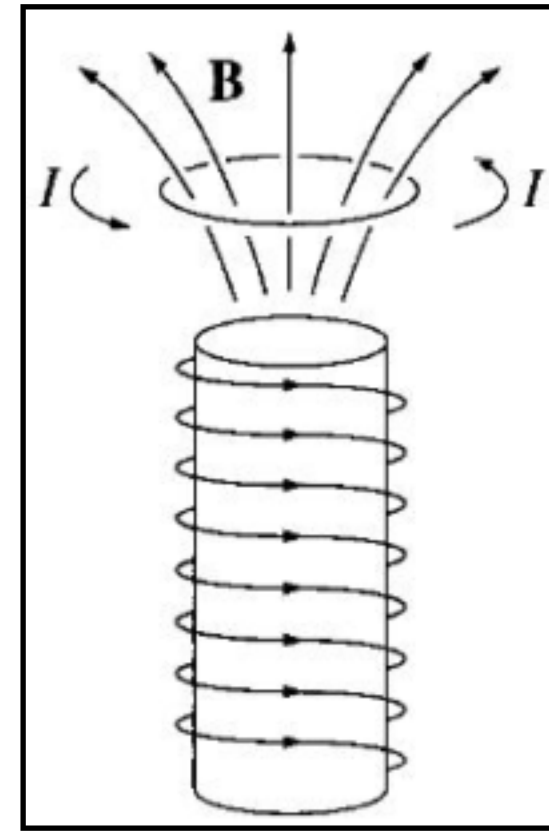


Figure 6.3, Introduction to Electrodynamics, D. J. Griffiths

*Argue this from the knowledge of force between current carrying wires as well as force on dipole in non-uniform field

The field of a magnetised object

The vector potential for a single dipole is given by

$$\vec{A}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

Since a magnetised object contains dipole moment per unit volume equal to $\vec{M}$, the net vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{M}(\vec{r'})\times\hat{\mathfrak{r}}}{\mathfrak{r}^2}d\tau'$$

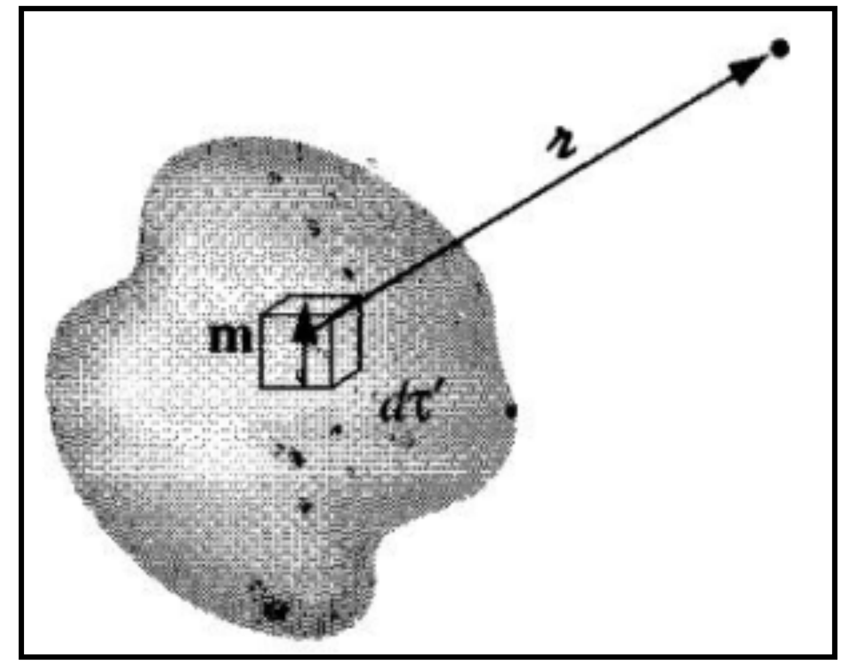


Figure 6.11, Introduction to Electrodynamics, D. J. Griffiths

The field of a magnetised object

Using the identity $\vec{\nabla}' \frac{1}{\mathfrak{r}} = \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$ the vector potential due to a magnetised object can be written as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \left[\vec{M}(\vec{r'}) \times \left(\vec{\nabla}' \frac{1}{\mathfrak{r}} \right) \right] d\tau'$$

$$\implies \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \left[\int \frac{1}{\mathfrak{r}} (\vec{\nabla}' \times \vec{M}(\vec{r'})) d\tau' - \int \vec{\nabla}' \times \left(\frac{\vec{M}(\vec{r'})}{\mathfrak{r}} \right) d\tau' \right]$$

(Using the rule $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla} f)$)

Using the identity $\int (\vec{\nabla} \times \vec{v}) d\tau = -\oint_S \vec{v} \times d\vec{a}$
we can simplify the vector potential further as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{\mathfrak{r}} (\vec{\nabla}' \times \vec{M}(\vec{r'})) d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\mathfrak{r}} (\vec{M}(\vec{r'}) \times d\vec{a'})$$

To show: $$\int (\vec{\nabla} \times \vec{v}) d\tau = -\oint_S \vec{v} \times d\vec{a}$$

Consider a constant vector $\vec{C}$ so that we can write:

$$\int \vec{\nabla} \cdot (\vec{v} \times \vec{c}) d\tau = \int (\vec{v} \times \vec{c}) \cdot d\vec{a}$$

$$\implies \int \left[\vec{c} \cdot (\vec{\nabla} \times \vec{v}) - \vec{v} \cdot (\vec{\nabla} \times \vec{c}) \right] d\tau = \int (\vec{v} \times \vec{c}) \cdot d\vec{a}$$

$$\implies \vec{c} \cdot \int (\vec{\nabla} \times \vec{v}) d\tau = -\int (\vec{c} \times \vec{v}) \cdot d\vec{a} = -\vec{c} \cdot \int \vec{v} \times d\vec{a}$$

$$\implies \int (\vec{\nabla} \times \vec{v}) d\tau = -\int \vec{v} \times d\vec{a}$$

The field of a magnetised object

The vector potential for a magnetised object is the sum of one volume and one surface term.

The volume term resembles the potential of a volume current $\vec{J}_b = \vec{\nabla} \times \vec{M}$ whereas the surface term looks like the potential of a surface current $\vec{K}_b = \vec{M} \times \hat{n}$ where $\hat{n}$ is the unit normal vector*.

The vector potential is therefore,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}_b(\vec{r'})}{\mathfrak{r}} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{K}_b(\vec{r'})}{\mathfrak{r}} da'$$

This is similar to electrostatics where the field due to a polarised object was a combination of bound volume charge $\rho_b = -\vec{\nabla} \cdot \vec{P}$ plus a bound surface charge $\sigma_b = \vec{P} \cdot \hat{n}$.

*We will see how to interpret them physically in next few slides

The field of a magnetised object

Example 6.1, Introduction to Electrodynamics, D. J. Griffiths: Find the magnetic field of a uniformly magnetised sphere.

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0, \vec{K}_b = \vec{M} \times \hat{n} = M \sin\theta\hat{\phi}$$

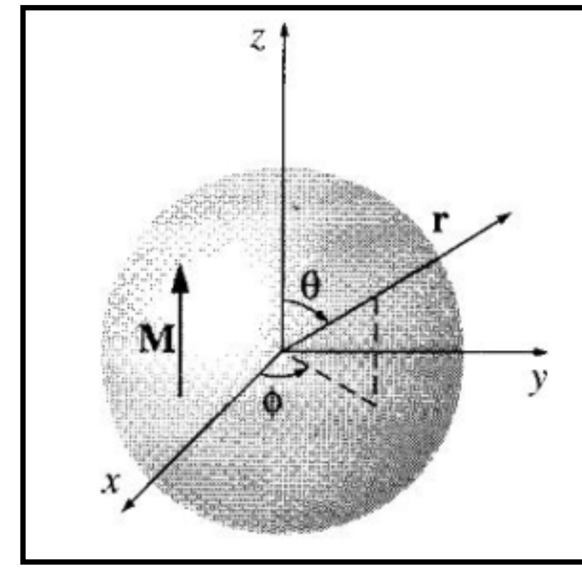


Figure 6.12, Introduction to Electrodynamics, D. J. Griffiths

The vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r'})}{\mathfrak{r}} da'$$

Comparing it to a rotating spherical shell of uniform surface charge σ that has surface current density $\vec{K} = \sigma\vec{v} = \sigma\omega R \sin\theta\hat{\phi}$ $(\vec{M} = \sigma R\vec{\omega})$

And using the result of Example 5.11 (Lecture 16, Tutorial 8):

$$\vec{A}_{\text{inside}} = \frac{\mu_0 R\omega\sigma}{3} r \sin\theta\hat{\phi}, \quad \vec{B}_{\text{inside}} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\hat{r} - \sin\theta\hat{\theta}) = \frac{2}{3}\mu_0\sigma R\omega\hat{z} = \frac{2}{3}\mu_0\sigma R\vec{\omega}$$

$$\vec{A}_{\text{outside}} = \frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi}, \quad \vec{B}_{\text{outside}} = \frac{\mu_0 R\omega\sigma}{3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) = \vec{B}_{\text{dipole}}$$

Uniformly magnetised sphere

Comparing the results, the magnetic field inside the sphere is

$$\vec{B}_{\text{inside}} = \frac{2}{3}\mu_0 \sigma R \vec{\omega} = \frac{2}{3}\mu_0 \vec{M}$$

The field outside the sphere is same as that of a pure dipole of dipole moment:

$$\vec{m} = \frac{4}{3}\pi R^3 \vec{M}$$

Lesson: Although bound current is fundamentally different from the usual free current, its effects are same as the latter. It produces a magnetic field in the same way as other current does.

Bound Currents: Physical Interpretation

Magnetic dipoles: Current loops

No cancellation of currents
at edges, leading to a net current

Internal currents cancel

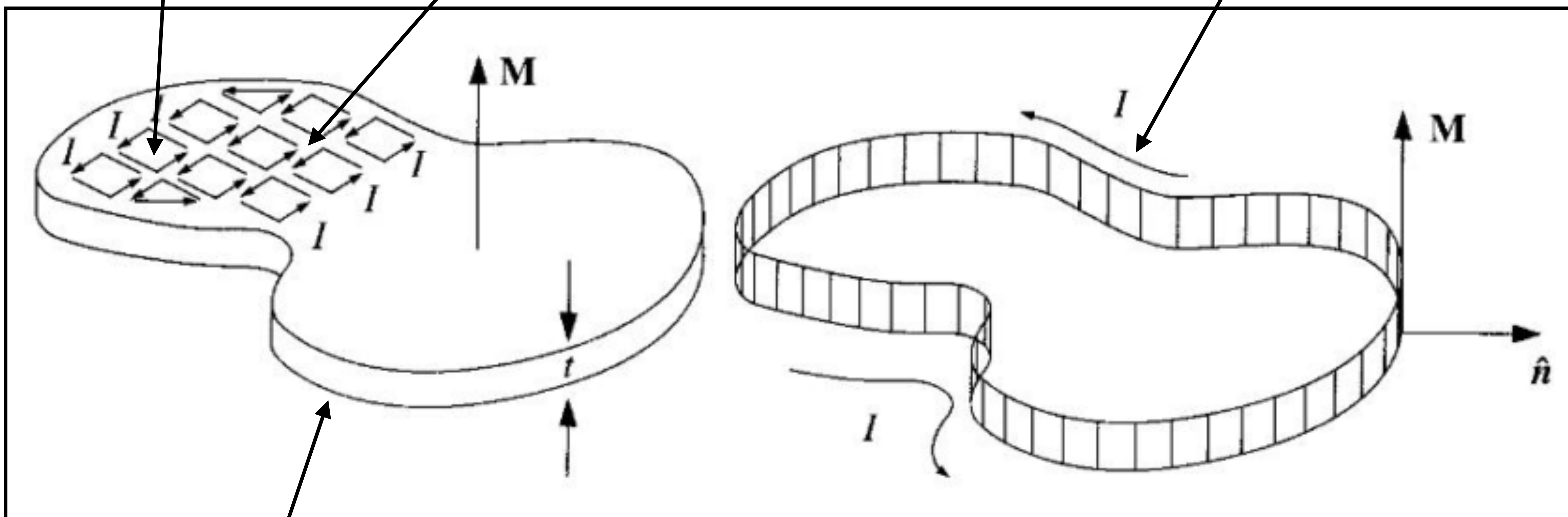


Figure 6.15, 6.16 Introduction to Electrodynamics, D. J. Griffiths

Slab of uniformly magnetised material

Bound Currents: Physical Interpretation

For each tiny current carrying loop area **a**, thickness **t**, the dipole moment in terms of magnetisation can be written as **m=Mat**.

For current **I** in each loop, we have **m=Ia**. Therefore **I=Mt** and hence the surface current is **K=I/t=M**.

In terms of the outward normal vector, the direction of surface current can be expressed as $\vec{K}_b = \vec{M} \times \hat{n}$

This also makes sure that there is no current on top or bottom surface of the slab where magnetisation is parallel to the unit normal vector.

Bound Currents: Physical Interpretation

This surface current is called bound current as the every charge that takes part in it is attached to a particular atom.

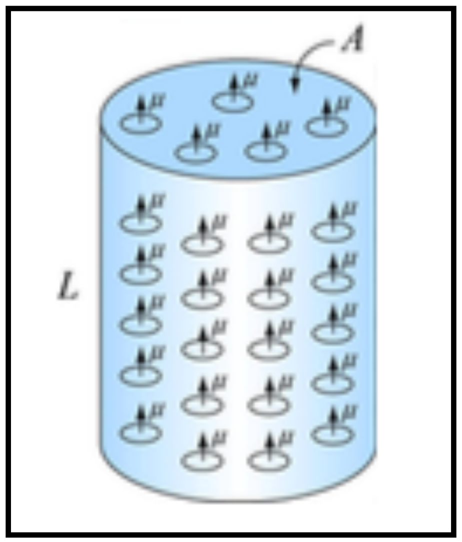
No single charge makes the whole trip over the surface, each charge moves only a tiny loop within a single atom.

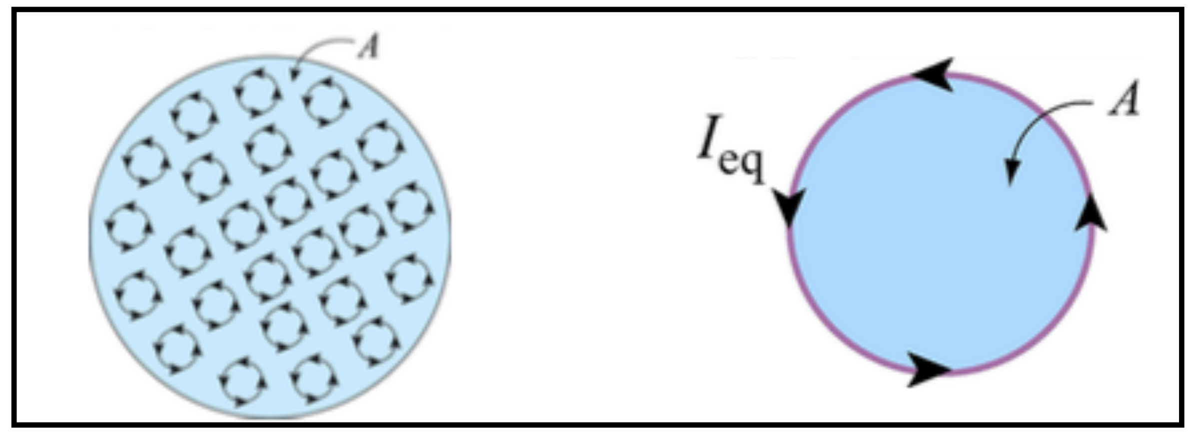
Although bound current is fundamentally different from the usual free current, its effects are same as the latter. It produces a magnetic field in the same way as other current does.

The field produced this way is the **macroscopic** field, averaged over a large enough region to contain many atoms or atomic dipoles.

Bound Currents

Thus, for uniform magnetisation, only the surface bound current contributes.





For the above figure on right (top view of the cylinder on left):

$$I_{\text{eq}} A = Nm \implies I_{\text{eq}} = \frac{Nm}{A}$$

$$K = \frac{I_{\text{eq}}}{L} = \frac{Nm}{AL} = M$$

$$B_M = \mu_0 K = \mu_0 M$$

No. of dipoles

Similar to a solenoid with surface current in circumferential direction

Effects of bound current due to uniform magnetisation is same as that of a surface current

Image credit: MIT

Bound current: non-uniform magnetisation

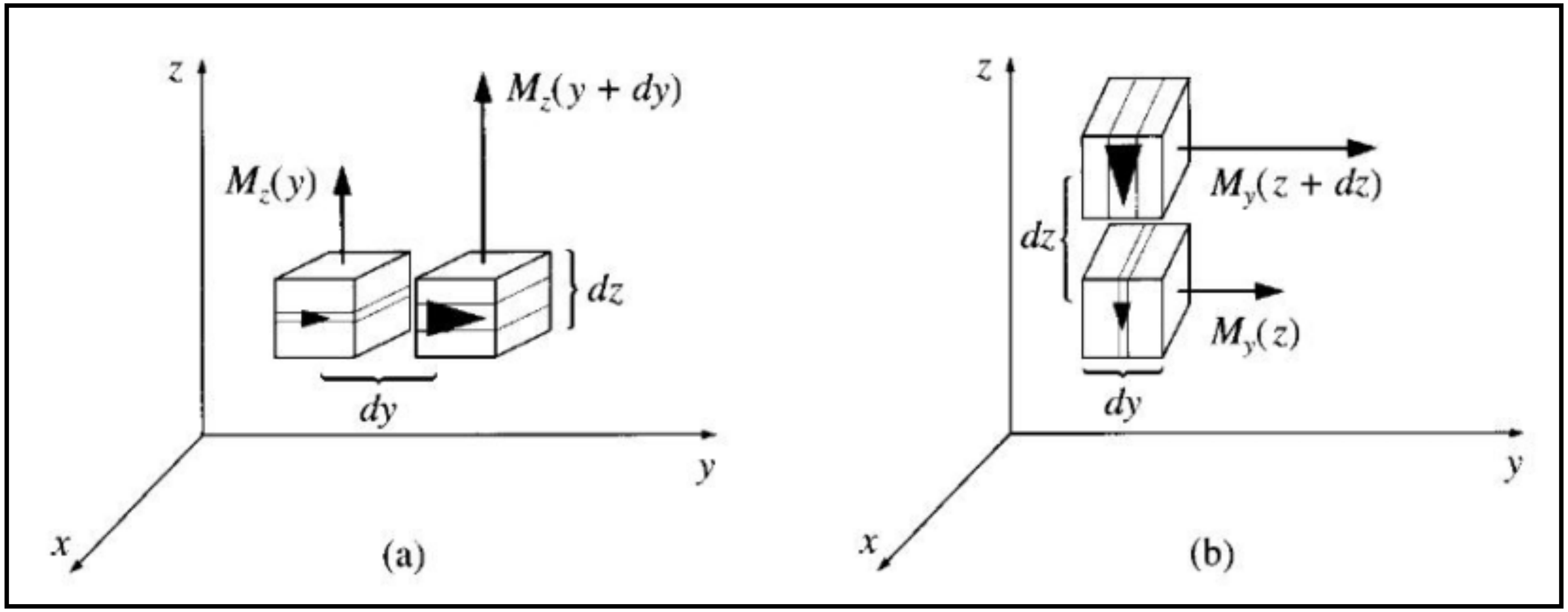


Figure 6.18, Introduction to Electrodynamics, D. J. Griffiths

For non-uniform M, the internal currents no longer cancel. On the surface where they join, there exists a net current in x-direction (fig (a))

$$I_x = \big[M_z(y+dy) - M_z(y)\big]dz = \frac{\partial M_z}{\partial y}dydz$$

Similarly, non-uniform M in y direction will contribute (fig (b))

$$-I_x = \Big[M_y(z+dz) - M_y(z)\Big]dy = \frac{\partial M_y}{\partial z}dydz$$

Bound current: non-uniform magnetisation

The volume current density corresponding to the currents derived before is

$$(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

In general, $\vec{J}_b = \vec{\nabla} \times \vec{M}$ which is same as the result obtained before.

Like any steady current, this bound current also satisfies the conservation law

$$\vec{\nabla} \cdot \vec{J}_b = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{M}) = 0$$

Ampere's law in Magnetised Materials

The total current in a medium is the summation of free and bound currents

$$\vec{J} = \vec{J}_b + \vec{J}_f$$

Ampere's law can therefore, be written as

$$\frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + (\vec{\nabla} \times \vec{M})$$

$$\implies \vec{\nabla} \times \left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \vec{\nabla} \times \vec{H} = \vec{J}_f$$

where $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ plays a role in magnetostatics analogous to $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ in electrostatics.

Ampere's law in integral form for magnetised materials can therefore be written as

$$\oint \vec{H} \cdot d\vec{l} = I_{f_{\mathrm{enc}}}$$

H: Auxiliary Field

The free current density is the source of H:

$$\vec{\nabla} \times \vec{H} = \vec{J}_f$$

The bound current density is the source of M:

$$\vec{\nabla} \times \vec{M} = \vec{J}_b$$

The total current is the source of B:

$$\vec{\nabla} \times \vec{B} = \mu_0(\vec{J}_f + \vec{J}_b)$$

- Although the Ampere's law for B and H looks similar, they are very different.

- For example, $\vec{\nabla} \cdot \vec{B} = 0$ but $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$*

- Since the divergence of $\vec{H}$ need not be zero always, a vanishing free current does not always imply a vanishing $\vec{H}$*. ($\vec{\nabla} \times \vec{H} = 0$ doesn't always mean $\vec{\nabla} \cdot \vec{H} = 0$)

- If the problem has some symmetry, one can use Ampere's law directly to find $\vec{H}$. In such cases divergence of $\vec{M}$ is zero as the free current itself determines $\vec{H}$. (Check divergence of $\vec{H}$ in Ex 6.2, slide number 30)

*This has implications for boundary conditions as well as field lines of H (slide 26, 35)

Boundary Conditions

Using $\vec{\nabla} \cdot (\vec{H} + \vec{M}) = 0 \implies \oint (\vec{H} + \vec{M}) \cdot d\vec{a} = 0$

we have, $H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$

Using Ampere's law $\oint \vec{H} \cdot d\vec{l} = I_{f_{\text{enc}}}$ we have

$$\vec{H}^{\parallel}_{\text{above}} - \vec{H}^{\parallel}_{\text{below}} = \vec{K}_f \times \hat{n}$$

Which, in the absence of materials, become

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0 \qquad \vec{B}^{\parallel}_{\text{above}} - \vec{B}^{\parallel}_{\text{below}} = \mu_0(\vec{K}_f \times \hat{n})$$

Lecture 17

Linear Magnetic Medium

For most substances, the magnetisation is proportional to the field, unless the field is too strong. However, it is not written as $\vec{M} = \frac{1}{\mu_0}\chi_m \vec{B}$

The proportionality is conventionally denoted by $\vec{M} = \chi_m \vec{H}$

The constant of proportionality χ_m is called the magnetic susceptibility. This dimensionless quantity is positive for paramagnets and negative for diamagnets.

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.6×10^{-4}	Oxygen	1.9×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.1×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.8×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.8×10^{-4}
Carbon Dioxide	-1.2×10^{-8}	Liquid Oxygen ($-200°$ C)	3.9×10^{-3}
Hydrogen	-2.2×10^{-9}	Gadolinium	4.8×10^{-1}

Table 6.1, Introduction to Electrodynamics, D. J. Griffiths

Magnetisation disappears when the field is removed: para- and dia- magnetic materials!

Linear Magnetic Medium

Materials that obey the proportionality between magnetisation and auxiliary field are called **linear media**. $\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1 + \chi_m)\vec{H}$

Thus B is proportional to H that is $\vec{B} = \mu\vec{H}$, where $\mu = \mu_0(1 + \chi_m)$ is called the permeability of the material.

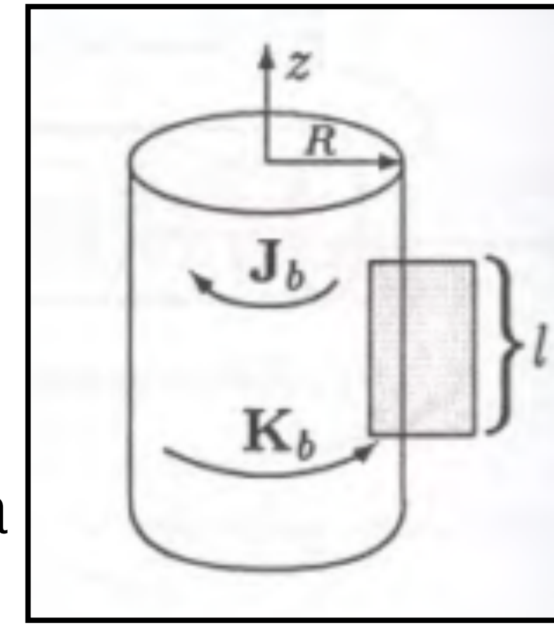
In vacuum, the susceptibility vanishes and hence the permeability is μ_0 (The permeability of free space).

The volume bound current density in a homogeneous linear material is proportional to the free current density

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \vec{\nabla} \times (\chi_m \vec{H}) = \chi_m \vec{J}_f$$

(Unless free current actually flows through the material, all bound current will be at surface)

Problem 6.12 (Introduction to Electrodynamics, D J Griffiths): An infinitely long cylinder, of radius R, carries a "frozen-in" magnetisation, parallel to the axis $\vec{M} = ks\hat{z}$, where k is a constant, s is the distance from the axis; there is no free current anywhere. Locate the bound currents and calculate the magnetic field.



Solution: The bound currents for the given magnetisation are:

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = -k\hat{\phi}, \quad \vec{K}_b = \vec{M} \times \hat{n} = kR\hat{\phi}$$

Since, current is circumferential, this can be compared with a superposition of solenoids, for which the field outside is zero.

Taking line integral of B along the Amperian loop shown in figure:

$$\begin{aligned}\oint \vec{B} \cdot d\vec{l} = Bl &= \mu_0 \left[\int J_b da + K_b l \right] \\ &= \mu_0 \left[-kl(R-s) + kRl \right] \\ &= \mu_0 kls \implies \vec{B}_{\text{inside}} = \mu_0 ks\hat{z}\end{aligned}$$

The Auxiliary Field H

Example 6.2, Introduction to Electrodynamics, D. J. Griffiths: A long copper rod of radius R carries a uniformly distributed (free) current I. Find H inside and outside the rod.

Using Ampere's law for magnetised object,

$$H(2\pi s) = I_{f_{\text{enc}}} = I\frac{\pi s^2}{\pi R^2}$$

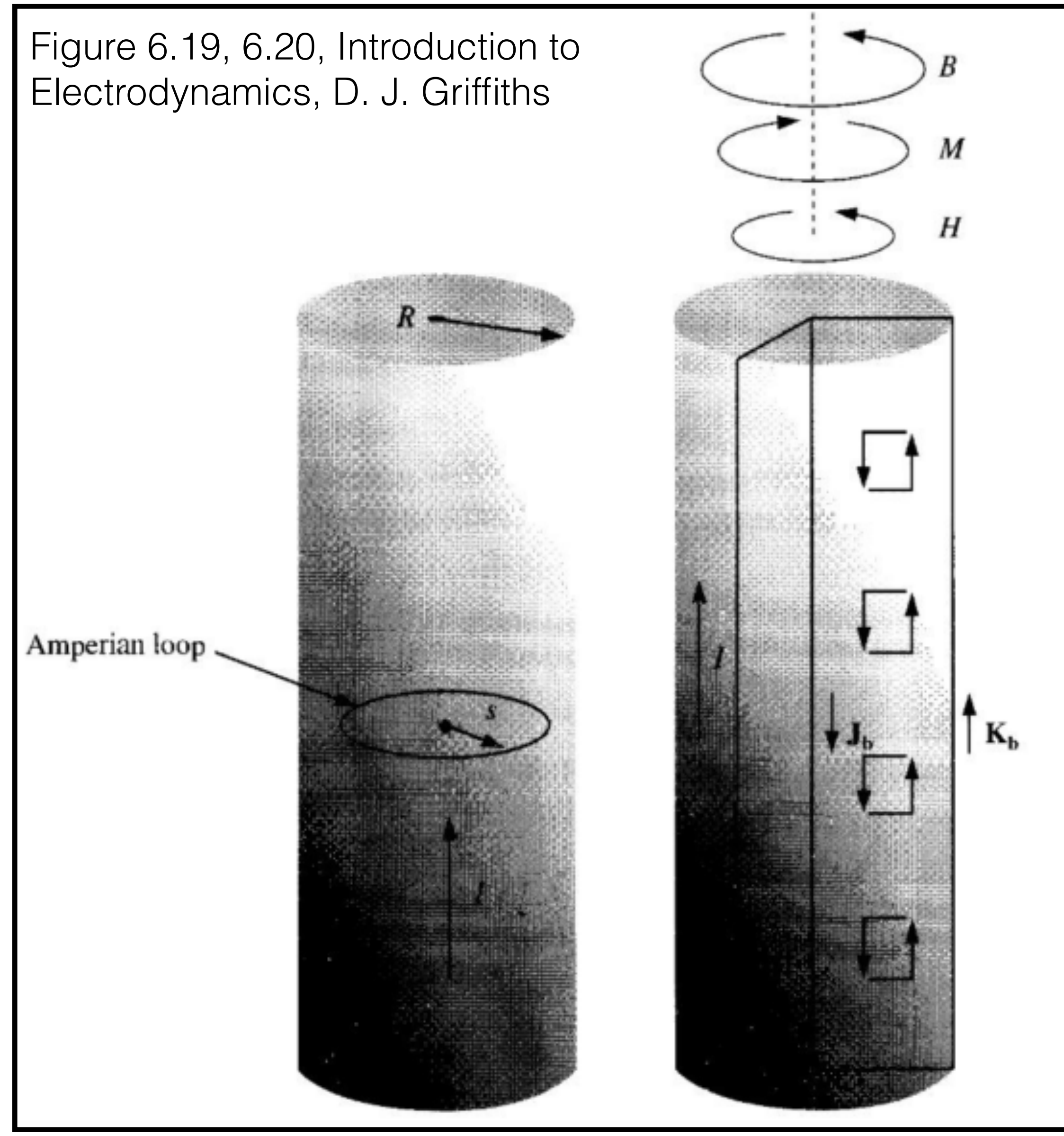
$$\vec{H} = \frac{I}{2\pi R^2} s\hat{\phi} \quad (s \leq R)$$

$$\vec{H} = \frac{I}{2\pi s}\hat{\phi} \quad (s \geq R)$$

$$\text{Check } \vec{\nabla}\cdot\vec{H}$$

As noted earlier, divergence of H should be zero in those cases where free current itself determines it.

Figure 6.19, 6.20, Introduction to Electrodynamics, D. J. Griffiths



Since M is not known, B can not be calculated. It can however be calculated outside the rod where M=0:

$$\vec{B} = \mu_0\vec{H} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

Example 6.3, Introduction to Electrodynamics, D. J. Griffiths: An infinite solenoid (n turns per unit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.

Due to the symmetry of the problem, one can use Ampere's law to find the auxiliary field as $\vec{H} = nI\hat{z}$

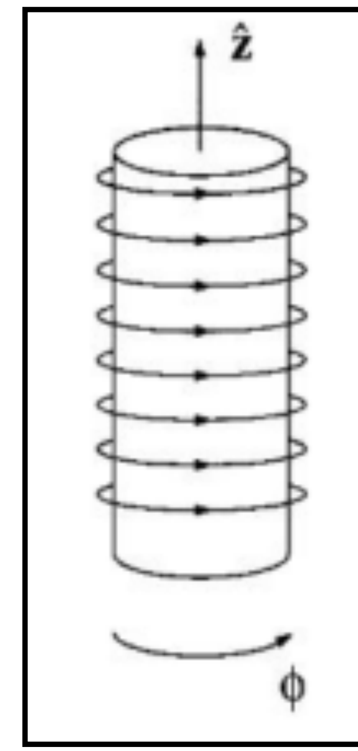


Figure 6.22, Introduction to Electrodynamics, D. J. Griffiths

Therefore $\vec{B} = \mu_0(1+\chi_m)nI\hat{z}$

For paramagnetic $(\chi_m > 0)$, field is enhanced.

For diamagnetic $(\chi_m < 0)$, field is somewhat reduced.

This reflects in surface current

$$\vec{K}_b = \vec{M} \times \hat{n} = \chi_m(\vec{H} \times \hat{n}) = \chi_m nI\hat{\phi}$$

which is in same direction as I for paramagnetic and opposite for diamagnetic.

Exercise: A line current I of infinite extent is within a cylinder of radius a that has permeability mu. The cylinder is surrounded by free space. What are the B, H, M fields everywhere? What is the bound current?

Solution: Using ampere's law and taking an amperian circular loop around the current, we can find the auxiliary field as

$$\oint \vec{H} \cdot d\vec{l} = H_\phi(2\pi r) = I \implies H_\phi = \frac{I}{2\pi r}$$

which is same both inside and outside the cylinder. The magnetic field is however different in each region due to the difference in permeability.

$$B_\phi = \begin{cases} \mu H_\phi = \frac{\mu I}{2\pi r}, & 0 < r < a \\ \mu_0 H_\phi = \frac{\mu_0 I}{2\pi r}, & r > a \end{cases}$$

The magnetisation is

$$M_\phi = \frac{B_\phi}{\mu_0} - H_\phi = \begin{cases} \left(\frac{\mu}{\mu_0} - 1\right) H_\phi = \left(\frac{\mu}{\mu_0} - 1\right) \frac{I}{2\pi r}, & 0 < r < a \\ 0, & r > a \end{cases}$$

There is no bound current in the bulk:

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = -\frac{\partial M_\phi}{\partial z}\hat{r} + \frac{1}{r}\frac{\partial}{\partial r}(rM_\phi)\hat{z} = 0, \quad 0 < r < a$$

There exists, a bound line current though at r=0:

$$I_b = \int \vec{J}_b \cdot d\vec{a} = \oint \vec{M} \cdot d\vec{l} = M_\phi(2\pi r) = \left(\frac{\mu}{\mu_0} - 1\right) I$$

There also exists a bound surface current at r=a:

$$K_b = -M_\phi(r=a) = -\frac{I_b}{2\pi a} = -\left(\frac{\mu}{\mu_0} - 1\right)\frac{I}{2\pi a}$$

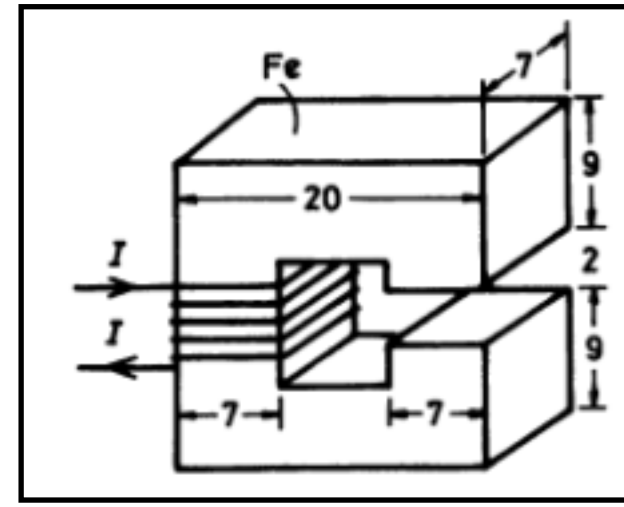
The negative sign is coming due to $\vec{K}_b = \vec{M} \times \hat{n}$ where $\vec{M}$ is along $\hat{\phi}$ and $\hat{n}$ is along $\hat{r}$.

Absence of bulk current is due to absence of bulk free current. The adjacent dipoles cancel **(?)** the currents except at r=0 and r=a.

Non-uniform magnetisation but curl of M is zero!

Magnet with a gap!

Consider N number of turns in the coil with current I, radius R and the gap length g.



Ampere's law*: $\oint \vec{H} \cdot d\vec{l} = NI$

Using $\vec{B} = \mu \vec{H}, B_1^{\perp} = B_2^{\perp}$ above

$$\frac{B}{\mu}(2\pi R - g) + \frac{B}{\mu_0} g = NI$$

$$\implies B = \frac{NI\mu}{2\pi R}\left(\frac{1}{\left[1 - \frac{g}{2\pi R}\right] + \frac{\mu}{\mu_0}\frac{g}{2\pi R}}\right)$$

Therefore $\dfrac{B_{\text{with gap}}}{B_{\text{without gap}}} \approx \left(1 + \dfrac{\mu}{\mu_0}\dfrac{g}{2\pi R}\right)^{-1}$

For ferromagnets $\mu/\mu_0 > 10^4$

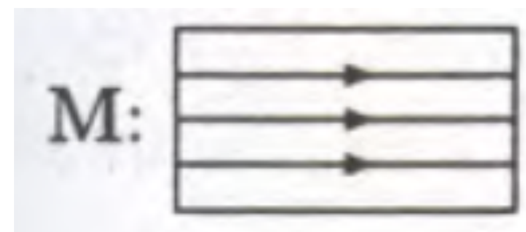
Even a tiny gap < 1% can reduce B by over 100

*Take the Amperian loop through the magnetic material in a way it is perpendicular to the surfaces near the gap

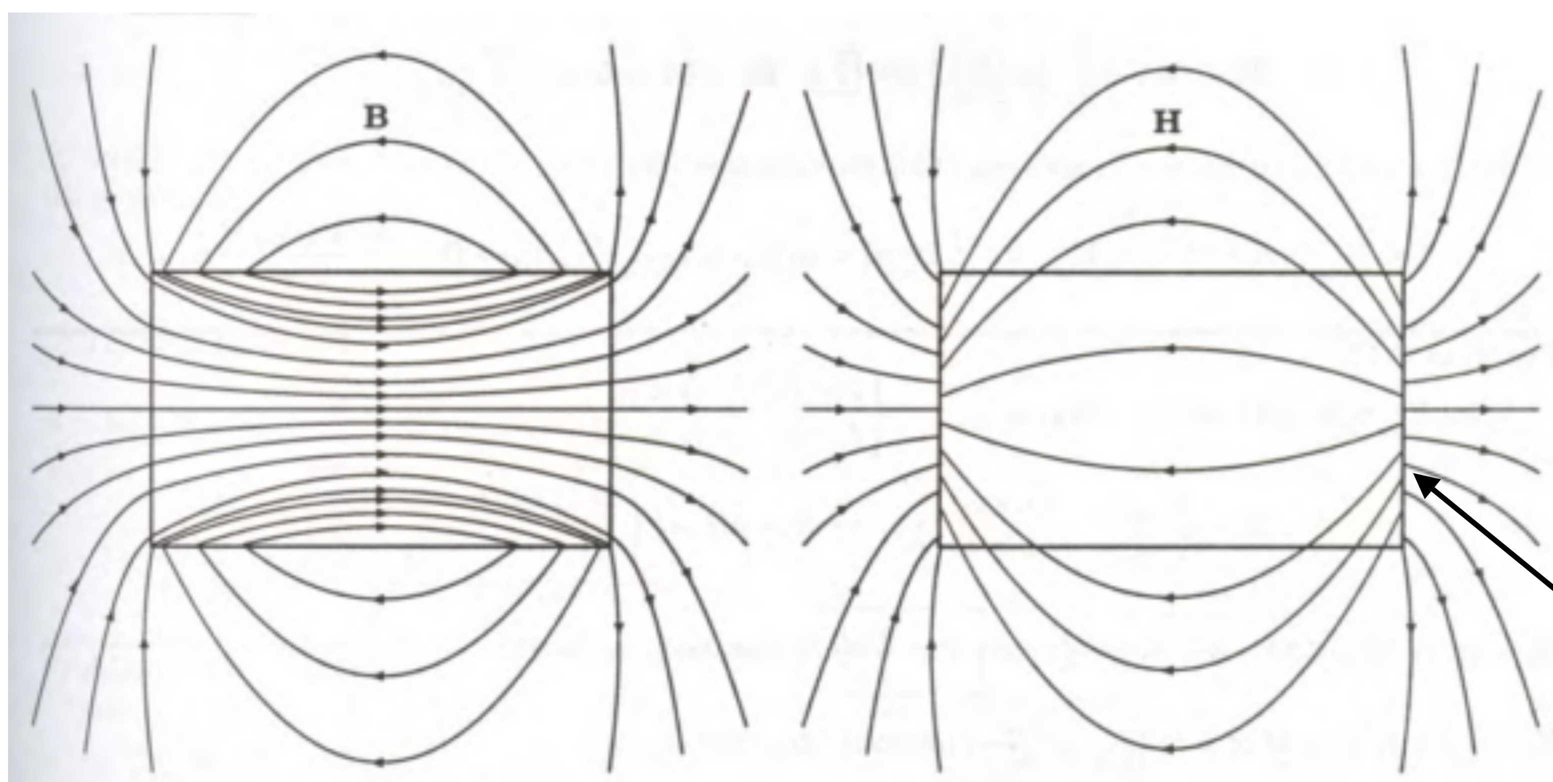
(A tiny applied field of 0.0002 T can produce 1 T field in iron!)

Field lines of **B**, **H**, **M**

A circular cylinder of radius a and length L has a frozen-in magnetisation M parallel to the axis. The field lines for L=2a looks like:



$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$



Discontinuity at the edge as M is discontinuous