MA 102 (Mathematics II)

Tutorial Sheet No. 11 **Ordinary Differential Equations** **May 02, 2019**

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1. Classify the singular points of the following differential equations:
   (a) $(x-1)^2y'' + \frac{1}{x^2}y' + 5y = 0$; (b) $(x^2-3x)y'' - (x+2)y' + y = 0$.
   (c) $(x^4 - 2x^3 + x^2)y'' + 2(x-1)y' + x^2y = 0$; (d) $(x-1)^3x^2y'' + 3x(x-1)y' - 5y = 0$.

2. Determine the convergence set of the given power series:
   (a) $\sum_{n=0}^{\infty} \frac{2^{-n}}{n+1}(x-1)^n$; (b) $\sum_{n=0}^{\infty} \frac{n^2}{2^n}(x+2)^n$; (c) $\sum_{n=1}^{\infty} \frac{3}{n^3}(x-2)^n$.

3. Compute the indicial equation and their roots of the given differential equations:
   (a) $(x^2-x-2)y'' + (x^2-4)y' - 6xy = 0$ at $x = 2$; (b) $x^2y'' + xy' + x^2y = 0$.

4. Find a series solution about the regular singular point $x = 0$ of the following equations:
   (a) $xy'' + 4y' - xy = 0$, $x > 0$; (b) $(x+2)x^2y'' - xy' + (1+x)y = 0$, $x > 0$.

5. Prove the following properties of the Legendre polynomials.
   (a) $\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases}$
   (b) If $f(x)$ is a polynomial of degree n, we have $f(x) = \sum_{k=0}^{n} c_kP_k(x)$, where $c_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)dx$.
   (c) Use orthogonality relation to show that $\int_{-1}^{1} g(x)P_n(x)dx = 0$ for every polynomial $g(x)$ with $\deg(g(x)) < n$.

6. Show that the value of the integral $\int_{-1}^{1} P_n(x)P'_{n+1}(x)\,dx$ is independent of n.

7. Find a solution of $y''(x) + \left(1 + \dfrac{1-4k^2}{4x^2}\right)y(x) = 0$, $k > 0$ a real constant, using the Bessel function of the first kind.

8. Using the series definition for J_α, prove the following identities:
   (a) $\frac{d}{dx}(x^\alpha J_\alpha(x)) = x^\alpha J_{\alpha-1}(x)$; (b) $\frac{d}{dx}(x^{-\alpha}J_\alpha(x)) = -x^{-\alpha}J_{\alpha+1}(x)$.

9. From the relation in Problem 8, deduce the recurrence relations.
   (a) $\frac{\alpha}{x}J_\alpha(x) + J'_\alpha(x) = J_{\alpha-1}(x)$; (b) $\frac{\alpha}{x}J_\alpha(x) - J'_\alpha(x) = J_{\alpha+1}(x)$.
   (c) $J_{\alpha-1}(x) + J_{\alpha+1}(x) = \frac{2\alpha}{x}J_\alpha(x)$; (d) $J_{\alpha-1}(x) - J_{\alpha+1}(x) = 2J'_\alpha(x)$.

10. Show that
   $$\int ax^\alpha J_{\alpha-1}(ax)\,dx = x^\alpha J_\alpha(ax) + C,$$
   where $a > 0$ and C is an arbitrary constant.

11. Using the series definition of $J_\alpha(x)$, show that
   (a) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$; (b) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$.

12. Show that between two consecutive positive roots of $J_0(x)$, there is a root of $J_1(x)$.