Continuous-time Markov Chain 2

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We discussed

For a CTMC,

$$P\left(X(s+t)=j \middle| X(s)=i, X(t_k)=i_k, k=1,2,..,n\right)=P\left(X(s+t)=j \middle| X(s)=i\right)$$

For a homogeneous CTMC,

$$p_{i,j}(t)=P\left(X(s+t)=j \middle| X(s)=i\right)=P\left(X(t)=j \middle| X(0)=i\right)$$

- When a CTMC enters a state *i*, it spends a random duration T_i called the state holding time

Distributed as $f_{T_i}(t)=\nu_i e^{-\nu_i t}$ $\quad \nu_i > 0$

- Once the CTMC leaves state *i*, it jumps to one of the state *j.*

The jumping process is an embedded MC with the transition probability $P_{i,j}, j \neq i$ such that $\sum_{j\neq i} P_{ij} = 1.$

We discussed

- Short-time behavior

$$\lim_{\Delta t \to 0} \frac{1 - P_{ii}(\Delta t)}{\Delta t} = v_i$$ and

prob of leaving state i during δt

$P_{ii}(\Delta t) = P(X(t+\delta t) = i \mid X(t) = i)$

$$\lim_{\Delta t \to 0} \frac{P_{ij}(\Delta t)}{\Delta t} = q_{ij}$$

= prob. rate

$j \neq i$

where $q_{ij} = v_i P_{ij}$

Chapman Kolmogorov equation for CTMC

$P(X(s+t)=j \mid X(s)=i)$

- Chapman Kolmogorov Equation:

$$p_{ij}(s+t)=\sum_k p_{ik}(s)\,p_{kj}(t)$$

- The above transition probabilities are function of time-duration and not the number of steps.
- Use of this difference equation is difficult.

The dynamics is better studied in terms of two differential equations:

Kolmogorov backward equation and *Kolmogorov forward equation*

Kolmogorov Backward Equation

$0 \quad \Delta t \qquad\qquad t+\Delta t$

$$
\begin{aligned}
p_{ij}(t+\Delta t) &= P(X(t+\Delta t)=j \mid X(0)=i) \\
&= \sum_k p_{ik}(\Delta t)\, p_{kj}(t) \\
&= p_{ii}(\Delta t)\, p_{ij}(t) + \sum_{k\neq i} p_{ik}(\Delta t)\, p_{kj}(t) \\
&= (1-v_i\Delta t + o(\Delta t))\, p_{ij}(t) + \sum_{k\neq i} q_{ik}\Delta t p_{kj}(t)
\end{aligned}
$$

$\nu_i P_{ik}$

$$\therefore \lim_{\Delta t\to 0} \frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = -v_i\, p_{ij}(t) + \sum_{k\neq i} q_{ik}\, p_{kj}(t)$$

$$\therefore p_{ij}'(t) = -v_i\, p_{ij}(t) + \sum_{k\neq i} q_{ik}\, p_{kj}(t)$$

Substituting $-v_i = q_{i,i}$, we get

$$\boxed{p_{ij}'(t) = \sum_k q_{ik}\, p_{kj}(t)}$$

Forward Kolmogorov Equation

$$0 \qquad\qquad t \qquad\qquad t+\Delta t$$

Consider the figure as shown above. Here,

$$p_{ij}(t+\Delta t)=\sum_k p_{ik}(t)\,p_{kj}(\Delta t)=p_{ij}(t)\,p_{jj}(\Delta t)+\sum_{k\neq j} p_{ik}(t)\,p_{kj}(\Delta t)$$

$$=(1-v_j\,\Delta t+o(\Delta t))\,p_{ij}(t)+\sum_{k\neq j} p_{ik}(t)(q_{ik}\,\Delta t+o(\Delta t))$$

$$\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}=0$$

$$v_i P_{ik}$$

$$\lim_{\Delta t\to 0}\frac{P_{ij}(t+\Delta t)-P_{ij}(t)}{\Delta t}$$

$$\therefore\; p_{ij}'(t)=-v_j\,p_{ij}(t)+\sum_{k\neq j} p_{ik}(t)\,q_{kj}$$

Putting $q_{jj}=-v_j$, we can rewrite the above differencial equations as:

Forward Kolmogorov Equation $p_{ij}'(t)=\sum_k p_{ik}(t)\,q_{kj}$

Backward $\quad P_{ij}'(t)=\sum_k q_{ik}P_{kj}(t)$

Matrix Form of Kolmogorov Equations

We can define the matrices

$$\mathbf{P}(t) = \begin{bmatrix} p_{00}(t) & p_{01}(t)\ldots. \\ p_{10}(t) & p_{11}(t)\ldots. \\ \vdots & \vdots \end{bmatrix}, \quad \mathbf{P}'(t) = \begin{bmatrix} p_{00}{}'(t) & p_{01}{}'(t)\ldots. \\ p_{10}{}'(t) & p_{11}{}'(t)\ldots. \\ \vdots & \vdots \end{bmatrix}$$

$$\text{and } \mathbf{Q} = \begin{bmatrix} q_{00} & q_{01}\cdots \\ q_{10} & q_{11}\ldots.. \\ \vdots & \vdots \end{bmatrix}$$

= Generator matrix
or rate matrix

$q_{ij} = \nu_i P_{ij}$

Matrix Form of Kolmogorov Equations

In matrix form, the Kolmogorov backward and forward equations can be written as

$$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$$

and $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$

Example A certain system has two states – under operation state 1 and under repair state 0. The duration of operation and repair are exponential RVs with rate parameters λ and μ respectively.

Find $\mathbf{P}(t) = \begin{bmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{bmatrix}$ and analyse it

$q_{ii} = -\lambda_i$

$q_{00} = -\mu$ $q_{01} = \lambda_0 \cdot P_{01}$
$= \mu \times 1$
$= \mu$

Solution-

The rate matrix

$\mathbf{Q} = \begin{bmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{bmatrix}$

$= \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$ $\lambda_1 P_{10} = \lambda \times 1$

The forward kolmogrov equation is given by

$\cancel{\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}}$

Solution-

The rate matrix $\mathbf{Q} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$

The forward kolmogrov equation is given by

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$$

$$\Rightarrow \begin{bmatrix} p'_{00}(t) & p'_{01}(t) \\ p'_{10}(t) & p'_{11}(t) \end{bmatrix} = \begin{bmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{bmatrix} \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

$$\therefore p'_{00}(t) = -\mu p_{00}(t) + \lambda p_{01}(t)$$
$$= -\mu p_{00}(t) + \lambda(1 - p_{00}(t)) \quad \because p_{00}(t) + p_{01}(t) = 1$$

We have to solve the linear differential equation

$p'_{00}(t) = -(\mu + \lambda) p_{00}(t) + \lambda$ with the initial condition $p_{00}(0) = 1$

$P(X(0)=0 \mid X(0)=0) = 1$

$$\therefore p_{00}(t) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

Similarly, $p_{1,1}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$

Now

$$p_{01}(t) = 1 - p_{00}(t)$$

$$= 1 - \frac{\lambda}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$= \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

Similarly,

$$p_{10}(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$\lim_{t\to\infty}\mathbf{P}(t)=\begin{bmatrix}\dfrac{\lambda}{\lambda+\mu} & \dfrac{\mu}{\lambda+\mu}\\ \dfrac{\lambda}{\lambda+\mu} & \dfrac{\mu}{\lambda+\mu}\end{bmatrix}=\begin{bmatrix}\pi_0 & \pi_1\\ \pi_0 & \pi_1\end{bmatrix}$$

We also observe that, irrespective of the values of the initial state probabilities $\begin{bmatrix} p_0(0) & p_1(0)\end{bmatrix}$, the steady-state state probabilities are given by

$$\begin{bmatrix}\pi_0 & \pi_1\end{bmatrix}$$

$$\lim_{t\to\infty}\begin{bmatrix}p_0(t) & p_1(t)\end{bmatrix}=\begin{bmatrix}p_0(0) & p_1(0)\end{bmatrix}\begin{bmatrix}\pi_0 & \pi_1\\ \pi_0 & \pi_1\end{bmatrix}$$

$$=\begin{bmatrix}\pi_0 & \pi_1\end{bmatrix}$$

The above example illustrates a remarkable property of the CTMC without proof:

If $\lim_{t \to \infty} p_{i,j}(t)$ exists, then

$\lim_{t \to \infty} p_{i,j}(t) = \pi_j$ independent of *i* where π_j is the probability of the state *j* at the steady state.

To Summarise

- To characterize the transition probabilities dynamically, Kolmogorov backward and forward differential equations are used.

- **Backward Kolmogorov Equation**

$$p_{ij}'(t) = \sum_k q_{ik} p_{kj}(t)$$

- **Forward Kolmogorov Equation** $p_{ij}'(t) = \sum_k p_{ik}(t) q_{kj}$

Summary

Matrix Form of Kolmogorov Equations

Backward equation

$$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$$

rate matrix
or Generator matrix

and forward equation

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$$

Poisson

Birth-Death process.