Continuous-time Markov Chain

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Continuous-time Markov Chain

Consider a random process $\{X(t)\}, t \geq 0$ where state space V is either finite or countable.

$\{X(t)\}$ is called a continuous-time Markov chain if, given time instances $t_1 < t_2 < \ldots < t_n < s < s+t$ and integers $i_1, i_2, .., i_n, i, j \in V$, we have

$$P\left(X(s+t)=j \middle| X(s)=i, X(t_k)=i_k, k=1,2,..,n\right)=P\left(X(s+t)=j \middle| X(s)=i\right)$$

The probability $p_{ij}(s,t)=P\left(\left\{X(s+t)=j \middle| X(s)=i\right\}\right)$ is called the transition probability.

Example
Independent increment process

$P(X(s+t)-X(s)=j-i)$

Homogenous CTMC

If $p_{ij}(s,t)$ is independent of *s* but dependent on

the chain to *be homogeneous*. If $\{X(t),\ t \geq$

homogenous CTMC, then the transition probab

$$p_{ij}(t) = P\left(X(s+t) = j \mid X(s) = i\right)$$
$$= P\left(X(t) = j \mid X(0) = i\right)$$

The probability of a state at time *t* is given by

$$p_j(t) = P\left(X(t) = j\right)$$
$$= \sum_i p_i(0)\, p_{ij}(t)$$

Example Independent Increment Process

$$p_{ij}(s,t) = P\left(X(t+s) = j \middle| X(s) = i\right)$$

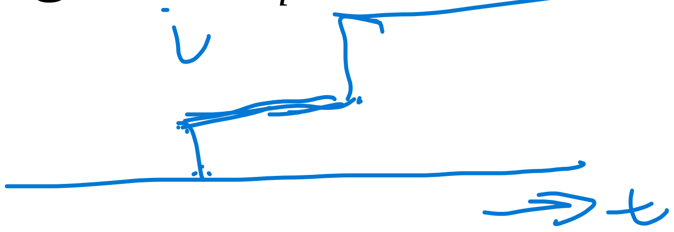
$$= \frac{P\left(X(t+s) = j, X(s) = i\right)}{P(X(s) = i)}$$

$$= \frac{\cancel{P\left(X(s) = i\right)}P\left(X(t+s) - X(s) = j - i\right)}{\cancel{P(X(s) = i)}}$$

$$= P\left(X(t+s) - X(s) = j - i\right)$$

State-holding Time

When the CTMC enters a state i, the time it spends there before it leaves the state i is called the holding time in the state i. The holding time T_i of the state i is a continuous random variable.



Theorem (a) T_i is memory-less. In other words

$$P(T_i > t+s / T_i > s) = P(T_i > t)$$

(b) $f_{T_i}(t) = \nu_i e^{-\nu_i t} u(t)$ where $\nu_i > 0$ is a constant

Proof:(a)

$$P(T_i > s+t / T_i > s) = P(X(u)=i, 0 \le u \le s+t / X(u)=i, 0 \le u \le s)$$

$$= P(X(u)=i, s < u \le s+t / X(u)=i, 0 \le u \le s)$$

Markov property

$$= P(X(u)=i, s < u \le s+t / X(s)=i)$$

homogeneous property

$$= P(X(u)=i, 0 < u \le t / X(0)=i)$$

$$= P(T_i > t)$$

Thus T_i is memory-less.

Proof of Part (b)

$$P\left(\{T_i > t+s\}\right) = P\left(\{T_i > t+s\} \cap \{T_i > s\}\right)$$
$$= P\left(\{T_i > s\}\right) P\left(\{T_i > t+s\} \middle| \{T_i > s\}\right)$$
$$= P\left(T_i > s\right) P\left(T_i > t\right)$$

Memory property

$T_i > t+s$

$s \qquad t+s$

$\{T_i > t+s\} = \{T_i > s\} \cap \{T_i > t+s\}$

In terms complementary CDF we get,

$$F_{T_i}^c(s+t) = F_{T_i}^c(s) F_{T_i}^c(t)$$

Taking logarithm, $\log_e F_{T_i}^c(s+t) = \log_e F_{T_i}^c(s) + \log_e F_{T_i}^c(t)$

The only function that satisfies the above relationship for arbitrary *t* and *s* is

$$\log_e F_{T_i}^c(t) = -\nu_i \times t$$

$$\therefore F_{T_i}^c(t) = e^{-\nu_i t} \quad t \geq 0$$

$\Rightarrow F_{T_i}(t) = 1 - e^{-\nu_i t} \quad t \geq 0$

$f_{T_i}(t) = \nu_i e^{-\nu_i t} \quad t \geq 0$

Structure of a homogeneous CTMC

The operation of a CTMC is as follows:

(1) Once CTMC enters at state *i* , it stays at the state for a time $T_I \sim \exp(\nu_i)$.

(2) Once the CTMC leaves state *i*, it enters one of the state *j* with the transition probability $P_{i,j}, j \neq i$ such that $\sum_{j \neq i} P_{ij} = 1.$

The two events of leaving the state *i* and entering the state *j* are independent because of Markovian assumption

The process of jumping to the state *j* from state *i* is like a discrete-time Markov chain and sometimes called *an embedded Markov chain*.

The structure of theis embedded MC determines the class property of a CTMC.

Example   Poisson process

Suppose the Poisson process has entered state *i* at time O. It will remain in same state until the next arrival with $T_I \sim \exp(\lambda)$. Once an arrival takes place, the state become *i*+1

Thus, for $j \neq i$, $P_{i,j} = \begin{cases} 1, & j = i+1 \\ 0 & \text{otherwise} \end{cases}$

Short-time Behaviour of the chain at a time interval $(t, t+\Delta t)$

$$p_{ii}(\Delta t) = p(T_i > \Delta t) + o(\Delta t)$$

$$= e^{-\nu_i \Delta t} + o(\Delta t)$$

$$= 1 - \nu_i \Delta t + \underset{\Delta t}{o}(\Delta t)$$

$$\lim_{\Delta t \to 0} \frac{1 - P_{ii}(\Delta t)}{\Delta t} = \nu_i$$

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$$

$$\frac{\text{Prob of leaving state } i}{\Delta t} \to \text{Prob of leave per unit time}$$

For $j \neq i$

$$p_{ij}(\Delta t) = p(T_i < \Delta t) \times P_{ij}$$
$$= (1 - e^{-v_i \Delta t}) \times P_{ij}$$
$$= (v_i \Delta t + o(\Delta t)) \times P_{ij}$$
$$= (v_i \Delta t) P_{ij} + o(\Delta t)$$
$$= q_{ij} \Delta t + o(\Delta t)$$

$\nu_i P_{ij} = q_{ij}$

where $q_{ij} = v_i P_{ij}$ is the probability rate function.

$q_{ij} = \nu_i P_{ij}$

Note that $\sum_{j \neq i} q_{ij} = v_i \sum_{j \neq i} P_{ij} = v_i$

$\sum_{j \neq i} q_{ij} = \nu_i \sum P_{ij}$

$= \nu_i$

Denoting $v_i = -q_{ii}$, we get $\sum_j q_{ij} = 0$

$\boxed{q_{ii} = -\nu_i}$

$\sum q_{ij} = 0$

Considering $p_{ij}(\Delta t) = q_{ij} \Delta t + o(\Delta t)$, we get

$$\lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t)}{\Delta t} = q_{ij}$$

Short-term behavior lemmas

Lemma 1: $$\lim_{\Delta t\to 0}\frac{1-P_{ii}(\Delta t)}{\Delta t}=v_i$$

rate of leaving state i

Lemma 2: $$\lim_{\Delta t\to 0}\frac{P_{ij}(\Delta t)}{\Delta t}=q_{ij}$$

Chapman Kolmogorov equations

- Chapman Kolmogorov Equation:

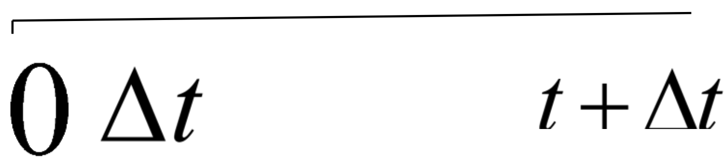
$$p_{ij}(s+t) = \sum_k p_{ik}(s)\, p_{kj}(t)$$

- The above transition probabilities are function of time-duration and not the number of steps.
- Use of this difference equation is difficult.

The dynamics is better studied in terms of two differential equations:

Kolmogorov backward equation and *Kolmogorov forward equation*

Kolmogorov Backward Equation



$$
\begin{aligned}
p_{ij}(t+\Delta t) &= P(X(t+\Delta t)=j \mid X(0)=i) \\
&= \sum_k p_{ik}(\Delta t)\, p_{kj}(t) \\
&= p_{ii}(\Delta t)\, p_{ij}(t) + \sum_{k\neq i} p_{ik}(\Delta t)\, p_{kj}(t) \\
&= (1 - v_i \Delta t + o(\Delta t))\, p_{ij}(t) + \sum_{k\neq i} q_{ik} \Delta t p_{kj}(t)
\end{aligned}
$$

$$\therefore \lim_{\Delta t \to 0} \frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = -v_i\, p_{ij}(t) + \sum_{k\neq i} q_{ik}\, p_{kj}(t)$$

$$\therefore p_{ij}\,'(t) = -v_i\, p_{ij}(t) + \sum_{k\neq i} q_{ik}\, p_{kj}(t)$$

Substituting $-v_i = q_{i,i}$, we get

$$p_{ij}\,'(t) = \sum_k q_{ik}\, p_{kj}(t)$$

Forward Kolmogorov Equation

$0 \qquad\qquad t \qquad\qquad t+\Delta t$

Consider the figure as shown above. Here,

$$p_{ij}(t+\vartriangle t) = \sum_{k} p_{ik}(t)\, p_{kj}(\vartriangle t) = p_{ij}(t)\, p_{jj}(\vartriangle t) + \sum_{k \neq j} p_{ik}(t)\, p_{kj}(\vartriangle t)$$

$$= (1 - v_j \vartriangle t + o(\vartriangle t))\, p_{ij}(t) + \sum_{k \neq j} p_{ik}(t)(q_{ik} \vartriangle t + o(\vartriangle t))$$

$$\therefore\ p_{ij}\,'(t) = -v_j\, p_{ij}(t) + \sum_{k \neq j} p_{ik}(t) q_{kj}$$

Putting $q_{jj} = -v_j$, we can rewrite the above differencial equations as:

Forward Kolmogorov Equation $p_{ij}\,'(t) = \sum_{k} p_{ik}(t) q_{kj}$

To Summarise…

- When a CTMC enters a state *i*, it spends a random duration T_i called the state holding time

Distributed as $f_{T_i}(t) = v_i e^{-v_i t}$ $\quad v_i > 0$, $t \geq 0$

- Once the CTMC leaves state *i*, it enters one of the state *j* with the transition probability $P_{i,j}$, $j \neq i$ such that $\sum_{j \neq i} P_{ij} = 1$.

- Short-time behavior

$$\lim_{\Delta t \to 0} \frac{1 - P_{ii}(\Delta t)}{\Delta t} = v_i$$

$$\lim_{\Delta t \to 0} \frac{P_{ij}(\Delta t)}{\Delta t} = q_{ij} \leftarrow \text{rate function}$$

Chapman Kolmogorov Equation.

we cannot use

To Summarise…

- To characterize the transition probabilities dynamically, Kolmogorov backward and forward differential equations are used.

- **Backward Kolmogorov Equation**

$$p_{ij}'(t) = \sum_k q_{ik} p_{kj}(t)$$

- **Forward Kolmogorov Equation** $p_{ij}'(t) = \sum_k p_{ik}(t) q_{kj}$

THANK YOU