MA 102 (Mathematics II)

IIT Guwahati

Tutorial Sheet No. 4 | Linear Algebra | February 7, 2019

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1. Determine whether the following are vector spaces (under the usual operations of addition and scalar multiplication of functions) over $\mathbb{R}$.
   (a) $\{f : (a, b) \to \mathbb{R} \mid f(c) = 0\}$, where $c \in (a, b)$.
   (b) $\{f : (a, b) \to \mathbb{R} \mid f(c) \neq 0$ for any $c \in (a, b)\}$.
   (c) $\{f : (a, b) \to \mathbb{R} \mid f$ is continuous in $(a, b)\}$.
   (d) $\{f : (a, b) \to \mathbb{R} \mid f$ is continuous everywhere except at c, where $c \in (a, b)$. $\}$
   (e) $\{f : (a, b) \to \mathbb{R} \mid f$ is a one-one function $\}$.
   (f) $\{f : (a, b) \to \mathbb{R} \mid$ range of f is a finite set $\}$.
   (g) $\{f : (a, b) \to \mathbb{R} \mid f' = 0\}$.
   (h) $\{f : (a, b) \to \mathbb{R} \mid f'' - 3f' + 7f = 0\}$.

2. (a) If $\mathbb{U} := \left\{ \begin{bmatrix} a & b \\ -b & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$, then find $\mathbb{V}$ such that $\mathbb{U} \oplus \mathbb{V} = \mathcal{M}_2(R)$.
   (b) Let $\mathcal{C}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f$ is continuous $\}$ and $\mathbb{U} = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is continuous and $f(-x) = f(x)$ for all $x \in \mathbb{R}\}$. Then find $\mathbb{V}$ such that $\mathbb{U} \oplus \mathbb{V} = \mathcal{C}(\mathbb{R})$.

3. (a) Let $\mathbb{V}$ is a vector space over $\mathbb{R}$ and let $A := [a_{ij}] \in \mathcal{M}_k(\mathbb{R})$ be invertible. Show that $\mathbf{u}_1, \cdots, \mathbf{u}_k \in \mathbb{V}$ are linearly independent if and only if $\sum_{i=1}^{k} a_{i1}\mathbf{u}_i, \cdots, \sum_{i=1}^{k} a_{ik}\mathbf{u}_i$ are linearly independent.
   (b) Show that $\{\mathbf{u}, \mathbf{v}\} \subseteq \mathbb{V}$ is linearly independent iff $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

Solution:

(a) Let $U := [\mathbf{u}_1, \ldots, \mathbf{u}_k]$. For $\mathbf{x} = [x_1, \ldots, x_k]^\top \in \mathbb{R}^k$, define $U\mathbf{x} := x_1\mathbf{u}_1 + \cdots + x_k\mathbf{u}_k$.

Define $\mathbf{w}_r := \sum_{i=1}^{k} a_{ir}\mathbf{u}_i$ and $W := [\mathbf{w}_1, \ldots, \mathbf{w}_k]$. Then $W = UA$.

Suppose that U is LI. Let $\mathbf{x} \in \mathbb{R}^k$ be such that $W\mathbf{x} = \mathbf{0}$. Then $UAx = \mathbf{0} \implies A\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$ since A is invertible. This shows that W is LI.

Conversely, suppose that W is LI. Let $\mathbf{y} \in \mathbb{R}^k$ be such that $U\mathbf{y} = \mathbf{0} \implies UA(A^{-1}y) = \mathbf{0} \implies W\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$, where $\mathbf{x} = A^{-1}\mathbf{y}$. This shows that $\mathbf{y} = \mathbf{0}$. Hence U is LI.

(b) Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then $[\mathbf{u}, \mathbf{v}]A = [\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}]$.

4. Let $\mathbb{W}, \mathbb{U}$ be subspaces of $\mathbb{V}$. Show that $\mathbb{W} \cup \mathbb{U}$ is a subspace iff either $\mathbb{W} \subseteq \mathbb{U}$ or $\mathbb{U} \subseteq \mathbb{W}$. What about union of three subspaces?

Solution: ($\Rightarrow$) Suppose that $\mathbb{U} \cup \mathbb{W}$ is a subspace. We claim that either $\mathbb{U} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{U}$. Assume our claim is not true. Then $\exists$ a $u \in \mathbb{U} \setminus \mathbb{W}$ and a $w \in \mathbb{W} \setminus \mathbb{U}$. Note that $u, w \in \mathbb{U} \cup \mathbb{W}$, a subspace. So $u + w \in \mathbb{U} \cup \mathbb{W}$, a union of two sets. So either $u + w \in \mathbb{U}$ or $u + w \in \mathbb{W}$. Let $u + w \in \mathbb{U}$. As u is already in $\mathbb{U}$, we get $w = (u + w) + (-1)u \in \mathbb{U}$, a contradiction. Similarly, $u + w \in \mathbb{W}$ leads to another contradiction. Hence our claim is valid and we are done.

($\Leftarrow$) Trivial.

2nd part Answer: If and only if one of the subspaces contains the other two (take the field as $\mathbb{R}$ or $\mathbb{C}$).
If part is obvious.
Only if part. Assume $\mathbb{U} \cup \mathbb{V} \cup \mathbb{W}$ is a subspace to show that one of the subspaces contain the other two.
If not, then neither $\mathbb{U} \supseteq \mathbb{V} \cup \mathbb{W}$ nor $\mathbb{V} \supseteq \mathbb{U} \cup \mathbb{W}$ ($\mathbb{W}$ is also not a superset of $\mathbb{U} \cup \mathbb{V}$, but we will not need it).
Take $x \in \mathbb{U} \setminus (\mathbb{V} \cup \mathbb{W})$ and a $y \in \mathbb{V} \setminus (\mathbb{U} \cup \mathbb{W})$, then $x + y \in \mathbb{U} \cup \mathbb{V} \cup \mathbb{W}$. If $x + y \in \mathbb{U}$, then $y = x + y + (-1)x \in \mathbb{U}$, which is a contradiction. Similarly $x + y \in \mathbb{V}$ gives a contradiction. Hence $x + y \in \mathbb{W}$. Since $y \in \mathbb{V} \setminus (\mathbb{U} \cup \mathbb{W})$ implies $(-1)y \in \mathbb{V} \setminus (\mathbb{U} \cup \mathbb{W})$(check this), by exactly similar argument $x + (-1)y \in \mathbb{W}$. Hence $2x = (x + y) + (x + (-1)y) \in \mathbb{W}$, which contradicts that $x \in (\mathbb{U} \setminus \mathbb{V} \cup \mathbb{W})$. Hence the claim is true.

5. Let $\mathbb{V}$ be a finite dimensional vector space. Let U and W be subspaces of $\mathbb{V}$. Show that $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$.

Solution: Choose a basis B of $U \cap W$. Extend B to a basis B_1 of U. Also extend B to a basis B_2 of W. Then $B \cup (B_1 \setminus B) \cup (B_2 \setminus B)$ is a basis of $U + W$ (Check).

6. If $\mathcal{W}_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 = 2x_3 + x_2 \right\}$ and $\mathcal{W}_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid 2x_1 = 2x_3 + 3x_2 \right\}$ then determine $\mathcal{W}_1 \cap \mathcal{W}_2$ and $\mathcal{W}_1 + \mathcal{W}_2$?

7. Extend

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

to a basis of $\mathbb{R}^6$ using GJE.

Solution: Consider $A = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 3 \\ 2 & 4 & 1 & 0 & 1 & -1 \\ 3 & 6 & 2 & 1 & 2 & -1 \end{bmatrix}$. Then $\tilde{A} = \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Note that $\text{span}(S) = \text{col}(A^T) = \text{col}(\tilde{A}^T)$. Since the 2nd, 4th and 5th columns in $\tilde{A}$ are non-pivot columns, if we add rows $\mathbf{e}_2^T, \mathbf{e}_4^T, \mathbf{e}_5^T$ to $\tilde{A}$, then we get a 6×6 matrix of rank 6. Thus, $S \cup \{\mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5\}$ is a basis for $\mathbb{R}^6$.

8. Find a basis for each of the following subspaces.

   (a) $U := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 2a - c - d = 0, a + 3b = 0, a, b, c, d \in \mathbb{R} \right\}$.

   (b) $V := \{p(x) \in \mathbb{R}[x] : \deg(p(x)) \leq 4 \text{ and } p(-2) = 0\}$.

Solution:

(a) Solving $2a - c - d = 0, a + 3b = 0$ we get $[a, b, c, d]^T = [\frac{s}{2} + \frac{t}{2},\ \frac{-s}{6} + \frac{-t}{6},\ s,\ t]^T$. Thus, $\left\{ \begin{bmatrix} \frac{1}{2} & \frac{-1}{6} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{-1}{6} \\ 0 & 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 0 & 6 \end{bmatrix} \right\}$ is a basis.

(b) Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ be such that $p(-2) = 0$. Then, $a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 = 0$. Therefore,

$$\begin{aligned} p(x) &= (2a_1 - 4a_2 + 8a_3 - 16a_4) + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \\ &= a_1(2 + x) - a_2(4 - x^2) + a_3(8 + x^3) - a_4(16 - x^4). \end{aligned}$$

and we get a basis $\{2 + x,\ 4 - x^2,\ 8 + x^3,\ 16 - x^4\}$.

9. Let $\mathbb{V}$ be a vector space and S be a subset of $\mathbb{V}$. Let $L = \{\mathbb{U} |\ \ \mathbb{U} \preccurlyeq \mathbb{V},\ S \subseteq \mathbb{U}\}$. Then show that $\text{span}(S) = \bigcap\limits_{\mathbb{U} \in L} \mathbb{U}$ = the smallest subspace containing S.

Solution: As each $\mathbb{U}$ contains S, it must contain $\text{span}(S)$. Hence $\bigcap_{\mathbb{U} \in L} \mathbb{U}$ contains $\text{span}(S)$. Further, as $\text{span}(S)$ is subspace, it must appear as one $\mathbb{U}$ on the right hand side. Thus $\bigcap_{\mathbb{U} \in L} \mathbb{U}$ cannot be larger than $\text{span}(S)$.

10. Consider $\mathbb{W} = \{v \in \mathbb{R}^6 \,|\, v_1 + v_2 + v_3 = 0,\ v_2 + v_3 + v_4 = 0,\ v_5 + v_6 = 0\}$. Find a basis of $\mathbb{W}$ and extend it to a basis of $\mathbb{R}^6$.

11. Consider $S := \{1 + x, (1 + x)^2, 1 - x^2, 10\} \subseteq \mathbb{R}[x]$. Describe $\text{span}(S)$ and find its dimension.

12. Find a basis for $\text{span}(1 - 2x, 2x - x^2, 1 - x^2, 1 + x^2)$ in $\mathbb{R}_2[x]$.

13. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of a vector space $\mathbb{V}$. Show that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \ldots, \mathbf{v}_1 + \cdots + \mathbf{v}_n\}$ is also a basis of $\mathbb{V}$.

14. Determine whether the set $\mathcal{B}$ given below is a basis for $\mathcal{M}_2(\mathbb{R})$.

    (a) $\mathcal{B} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$.

    (b) $\mathcal{B} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$.

15. Find a basis for each of the following subspaces.

    a) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a - d = 0, a, b, c, d \in \mathbb{R} \right\}$.

b) $\{a + bx + cx^3 : a, b, c \in \mathbb{R}, a - 2b + c = 0\}$.

c) $\{A \in \mathcal{M}_{m\times n}(\mathbb{R}) :$ row sums of A are zero$\}$.

16. Let $U := \{A \in \mathcal{M}_3(\mathbb{R}) : A^\top = A$ and $\mathrm{Tr}(A) = 0\}$. Find two bases of U and extend these bases to bases of the real symmetric matrices of size 3×3.