PH 102 : Electromagnetism | **Department of Physics**
Tutorial 9 | **Indian Institute of Technology Guwahati**
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1. A thin uniform donut, carrying charge Q and mass M, rotates about its axis as shown in the figure.
(a) Find the gyromagnetic ratio (g), i.e. the ratio of its magnetic dipole moment to its angular momentum.
(b) What is the gyromagnetic ratio a uniform spinning sphere of total charge Q and mass M?
(c) According to quantum mechanics, the angular momentum of a spinning electron is $\frac{\hbar}{2}$, where $\hbar$ is Planck's constant. What, then, is the electron's magnetic dipole moment (in units of $A.m^2$)?
Solution:

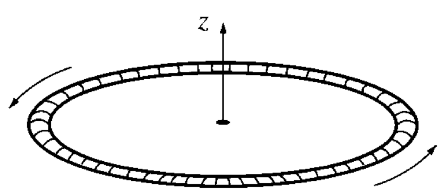

Figure 1: Figure for problem 1.

(a) Angular momentum $\vec{L} = RMv\hat{z} = M\omega R^2\hat{z}$, current is $I = \frac{Q}{2\pi/\omega} = \frac{Q\omega}{2\pi}$ and area $a = \pi R^2$. The magnetic dipole moment is $\vec{m} = I\vec{a} = \frac{Q\omega}{2\pi}\pi R^2\hat{z} = \frac{Q}{2}\omega R^2\hat{z}$.

$$g = \frac{m}{L} = \frac{Q}{2}\frac{\omega R^2}{M\omega R^2} = \frac{Q}{2M}$$

(b) g is independent of radius, thus the same ratio would have applied to all "donuts", and hence to the whole sphere (made out of the disks which are made out of the donuts) or any other revolving figure, i.e, $g = \frac{Q}{2M}$. This can also be checked explicitly by taking the result for magnetic dipole moment of spinning sphere in problem 2 and dividing it by its angular momentum (students should check it by themselves).
(c) Then for electron, $g = \frac{e}{2m_e}$ and $L = \frac{\hbar}{2}$. Thus, the magnetic dipole moment of electron $m = \frac{e}{2m_e}\frac{\hbar}{2} = \frac{e\hbar}{4m_e} = \frac{(1.60\times10^{-19})(1.05\times10^{-34})}{4(9.11\times10^{-31})}$ Am^2 $= 4.61 \times 10^{-24}$ Am^2

2. Find the magnetic dipole moment of a spherical shell, of radius R, carrying a uniform surface charge σ which is set to spin at angular velocity $\vec{\omega}$. Show that for points $r > R$,

the vector potential is same as that of a perfect dipole. Hint: The vector potential for $r > R$ is:

$$\vec{A}(\vec{r}) = \frac{\mu_0 R^4 \sigma}{3r^3}(\vec{\omega} \times \vec{r})$$

Solution:

Consider the elemental ring of radius $R\sin\theta$, thickness $Rd\theta$ so that the charge on its surface is $dq = \sigma(2\pi R\sin\theta)Rd\theta$. Time period of the spinning sphere is $dt = 2\pi/\omega$. Therefore, the current in the ring is $I = dq/dt = \sigma\omega R^2 \sin\theta d\theta$. The area of the ring is $\pi(R\sin\theta)^2$. The magnetic dipole moment of the ring is therefore, $dm = Ia = (\sigma\omega R^2 \sin\theta d\theta)\pi(R\sin\theta)^2$.

The total magnetic dipole moment of the shell can be found by integrating over θ:

$$m = \sigma\omega\pi R^4 \int_0^\pi \sin^3\theta d\theta = (4/3)\sigma\omega\pi R^4$$

$$\implies \vec{m} = \frac{4\pi}{3}\sigma\omega R^4 \hat{z}$$

The dipole contribution to the vector potential is

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{4\pi}{3}\sigma\omega R^4 \frac{\sin\theta}{r^2}\hat{\phi}$$

$$\implies \vec{A} = \frac{\mu_0\sigma\omega R^4}{3}\frac{\sin\theta}{r^2}\hat{\phi}$$

which is same as the vector potential outside the shell.

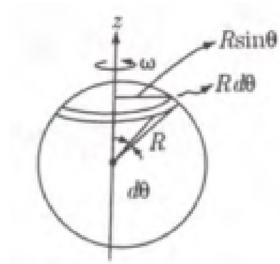

Figure 2: Solution to problem 2.

3. A uniform current density $\vec{J} = J_0\hat{z}$ fills a slab straddling the yz plane as shown in figure 3, from $x = -a$ to $x = +a$. A magnetic dipole $\vec{m} = m_0\hat{x}$ is situated at the origin.
(a) Find the force on the dipole.
(b) Do the same for a dipole pointing in the y direction: $\vec{m} = m_0\hat{y}$.
(c) In the *electrostatic case*, the expressions $\vec{F} = \vec{\nabla}(\vec{p}.\vec{E})$ and $\vec{F} = (\vec{p}.\vec{\nabla})\vec{E}$ are equivalent (prove it), but this is not the case for the magnetic analogues (explain why). As an example, calculate $(\vec{m}.\vec{\nabla})\vec{B}$ for the configurations in (a) and (b).

Solution:

(a) Since current is in the z direction, from the figure it is clear that for $x > 0(x < 0)$, the field will be in $+\hat{y}(-\hat{y})$ direction. At $x = 0$, the field is zero. Taking an Amperian loop in the x-y plane with one edge coinciding with the y-axis, we can find:

$$\oint \vec{B}\cdot d\vec{l} = Bl = \mu_0 I_{\text{enc}} = \mu_0 J_0 l x \implies \vec{B} = \mu_0 J_0 x\hat{y}$$

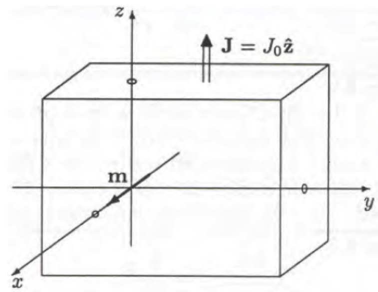

Figure 3: Figure for problem 3.

Similar problems were discussed in the class as well (for example, use Ampere's law for infinite two dimensional current sheet). Therefore, $\vec{m}\cdot\vec{B} = 0$ and force $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = 0$.

(b) $\vec{m}.\vec{B} = m_0\mu_0 J_0 x$, hence $\vec{F} = m_0\mu_0 J_0\hat{x}$.

(c) $\vec{\nabla}(\vec{p}.\vec{E}) = \vec{p}\times(\vec{\nabla}\times\vec{E}) + \vec{E}\times(\vec{\nabla}\times\vec{p}) + (\vec{p}\cdot\vec{\nabla})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{p}$.
$\vec{p}$ is independent of (x, y, z) (2nd and 4th term vanish) and $\vec{\nabla}\times\vec{E} = 0$ in electrostatics. Hence, $\vec{\nabla}(\vec{p}.\vec{E}) = (\vec{p}\cdot\vec{\nabla})\vec{E}$.
However, for magnetic field $\vec{\nabla}\times\vec{B} \neq 0$ and this argument doesn't hold. Rather, $\vec{\nabla}(\vec{m}\cdot\vec{B}) = (\vec{m}\cdot\vec{\nabla})\vec{B} + \vec{m}\times(\vec{\nabla}\times\vec{B}) = (\vec{m}\cdot\vec{\nabla})\vec{B} + \mu_0(\vec{m}\times\vec{J})$.
Then for
(a) $(\vec{m}\cdot\vec{\nabla})\vec{B}_a = m_0\partial_x(\vec{B}) = m_0\mu_0 J_0\hat{y}$.

(b) $(\vec{m}\cdot\vec{\nabla})\vec{B}_b = m_0\partial_y(\mu_0 J_0 x\hat{y}) = 0$.
Clearly, different than $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$ in 3(a) and 3(b).

4. An iron rod of length L and square cross section (side a), is given a uniform longitudinal magnetisation $\vec{M}$, and then bent around into a circle with a narrow gap (width w), as shown in figure 4. Find the magnetic field at the centre of the gap, assuming $w \ll a \ll L$.
Solution:

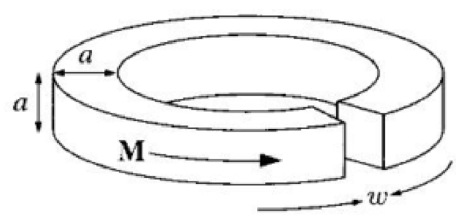

Figure 4: Figure for problem 4.

The magnetic field at the centre of the gap cam be found by using the principle of linear superposition. The given configuration can be then replaced by a complete torus/ring plus a square loop of side length a having current in opposite direction. Since there is no free current in the system, the only current that generates magnetic field is the bound current. It is given by $\vec{K}_b = \vec{M} \times \hat{n}$. Looking at the direction of $\vec{M}$, this bound current will be in anti-clockwise direction around $\vec{M}$. Magnetic field due to this bound current is $\vec{B} = \mu_0 \vec{M}$. This is in fact, the net magnetic field at the given location, originating from the full torus (without any gap).

To consider the contribution from the gap, let us take the square loop of edge a having current $I = Kw = Mw$. Magnetic field due to this loop at the centre can be found by adding the contribution from each edge. As done in the class (Lecture 15), the magnetic field at a distance s from a finite wire carrying current I is $B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$ where $\theta_{1,2}$ are the angles subtended by the wire at the point where magnetic field is being found out. At the centre of the loop, these angles correspond to $\theta_2 = -\theta_1 = \pi/4$ and $s = a/2$. Adding over the four sides, the net magnetic field is

$$B = 4 \times \frac{2\mu_0 I}{4\pi a}\frac{2}{\sqrt{2}} = \frac{2\sqrt{2}\mu_0 I}{\pi a}.$$

To consider the gap we need to consider this loop to have current in a direction opposite to the bound current $\vec{K}_b$ and hence generating a magnetic field in the opposite direction to $\vec{M}$. Therefore, the net magnetic field at the gap is

$$B = \mu_0 M - \frac{2\sqrt{2}\mu_0 I}{\pi a} = \mu_0 M\left(1 - \frac{2\sqrt{2}w}{\pi a}\right).$$

5. Consider the following similarities between electrostatics and magnetostatics (in matter):

$$\vec{\nabla}\cdot\vec{D} = 0,\ \ \vec{\nabla}\times\vec{E} = 0,\ \ \epsilon_0\vec{E} = \vec{D} - \vec{P}$$
$$\vec{\nabla}\cdot\vec{B} = 0,\ \ \vec{\nabla}\times\vec{H} = 0,\ \ \mu_0\vec{H} = \vec{B} - \mu_0\vec{M}$$

Thus, the transcription $\vec{D} \to \vec{B}, \vec{E} \to \vec{H}, \vec{P} \to \mu_0\vec{M}, \epsilon_0 \to \mu_0$ turns an electrostatic problem into an analogous magnetostatic one. Use this observation, together with the analogous electrostatic results (namely, (i) electric field inside a uniformly polarised sphere $\vec{E} = -\frac{\vec{P}}{3\epsilon_0}$, (ii) electric field inside a sphere of linear dielectric in an otherwise uniform electric field E_0 is $\vec{E} = \frac{3}{\epsilon_r+2}\vec{E}_0 = \frac{1}{1+\frac{\chi_e}{3}}\vec{E}_0$) to rederive

(a) the magnetic field inside a uniformly magnetised sphere.

(b) the magnetic field inside a sphere of linear magnetic material in an otherwise uniform magnetic field.

(c) the average magnetic field over a sphere, due to steady currents within the sphere.

Solution:

(a) Given that the electric field inside a uniformly charged sphere is $\vec{E} = -\frac{\vec{P}}{3\epsilon_0}$. With the transcription given in the problem, we can write

$$\vec{H} = -\frac{\mu_0 \vec{M}}{3\mu_0} = -\frac{\vec{M}}{3}$$

. Thus, the magnetic field inside a uniformly magnetised sphere is

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(-\frac{\vec{M}}{3} + \vec{M}) = \frac{2}{3}\mu_0 \vec{M}.$$

(b) The electric field inside a sphere of linear dielectric in an otherwise uniform electric field E_0 is given as $\vec{E} = \frac{3}{\epsilon_r + 2}\vec{E}_0 = \frac{1}{1+\frac{\chi_e}{3}}\vec{E}_0$. Using the transcription one can translate χ_e to χ_m. Therefore, we can now write

$$\vec{H} = \frac{\vec{H}_0}{1 + \frac{\chi_m}{3}}$$

Similarly $\vec{P} = \epsilon_0 \chi_e \vec{E}$ can be translated to

$$\mu_0 \vec{M} = \mu_0 \chi_m \vec{H} \implies \vec{M} = \chi_m \vec{H}.$$

Therefore, the magnetic field can be written as

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1 + \chi_m)\vec{H} = \mu_0(1 + \chi_m)\frac{\vec{H}_0}{1 + \frac{\chi_m}{3}} = \vec{B}_0 \frac{1 + \chi_m}{1 + \frac{\chi_m}{3}}.$$

where we have used $\vec{B}_0 = \mu_0 \vec{H}_0$ for the applied uniform magnetic field.

(c) The average electric field insider a uniformly charges sphere (as in part (a)) is $\vec{E} = -\frac{\vec{P}}{3\epsilon_0}$. The dipole moment of the sphere is $\vec{p} = \frac{4}{3}\pi R^3 \vec{P}$. Using this, the electric field can be written as $\vec{E} = -\frac{\vec{p}}{4\pi\epsilon_0 R^3} = -\frac{1}{4\pi\epsilon_0 R^3}\int \vec{P} d\tau$. Now, using the given transcription,

$$\vec{H} = -\frac{1}{4\pi\mu_0 R^3}\int \mu_0 \vec{M} d\tau = -\frac{\vec{m}}{4\pi R^3}.$$

Now, the magnetic field can be written as

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = -\mu_0 \frac{\vec{m}}{4\pi R^3} + \mu_0 \vec{M} = -\mu_0 \frac{\vec{m}}{4\pi R^3} + \mu_0 \frac{\vec{m}}{\frac{4}{3}\pi R^3} = \frac{\mu_0}{4\pi}\frac{2\vec{m}}{R^3}.$$

6. At the interface between one linear magnetic material and another, the magnetic field lines bend as shown in figure 5. Assuming there is no free current at the boundary, show that $\tan\theta_2/\tan\theta_1 = \mu_2/\mu_1$.

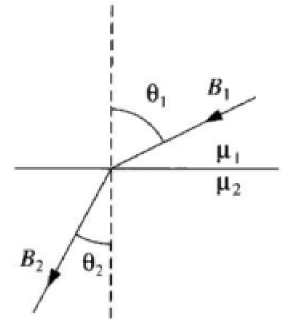

Figure 5: Figure for problem 6.

Solution:

The boundary conditions for fields:

$$B_1^{\perp} = B_2^{\perp}$$

$$H_1^{\|} = H_2^{\|}, \ \ (K_f = 0)$$

$$\implies \frac{1}{\mu_1}B_1^{\|} = \implies \frac{1}{\mu_2}B_2^{\|}$$

From figure 5, it is clear that $\tan\theta_1 = B_1^{\|}/B_1^{\perp}, \tan\theta_2 = B_2^{\|}/B_2^{\perp}$. Therefore,

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{B_2^{\|}/B_2^{\perp}}{B_1^{\|}/B_1^{\perp}} = \frac{B_2^{\|}}{B_1^{\|}} = \frac{\mu_2}{\mu_1}$$