PH 102 : Electromagnetism
Tutorial 8
19/03/19

Department of Physics
Indian Institute of Technology Guwahati

1 Solutions to Selective Take Home Problems

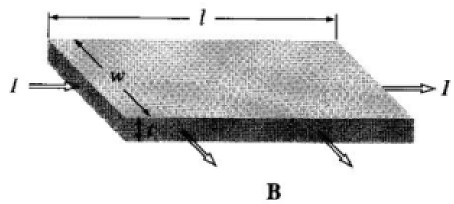

Figure 1: Figure for take home problem 1.

1. A current I flows to the right through a rectangular bar of conducting material, in the presence of a uniform magnetic field $\vec{B}$ pointing out of the page, as shown in figure 1.

   (a) If the moving charges are positive, in which direction are they deflected by the magnetic field? This deflection results in an accumulation of charge on the upper and lower surfaces of the bar, which in turn produces an electric force to counteract the magnetic one. Equilibrium occurs when the two exactly cancel each other. (A phenomenon known as the Hall effect, to be studied as a part of an experiment in PH 110).

   (b) Find the resulting potential difference (the Hall voltage) between the top and bottom of the bar, in terms of B, v (the speed of the charges), and the relevant dimensions of the bar.

   (c) How would your analysis change if the moving charges were negative? (The Hall effect is the classic way of determining the sign of the mobile charge carriers in a material, which is also one of the objectives of the Hall effect related experiment in PH 110).

   Solution: (a) Positive charges are deflected downwards as evident from the direction of $\vec{v} \times \vec{B}$ in figure 1. Therefore, the bottom edge of the bar acquires a net positive charge.

   (b) At equilibrium magnetic force equals electric force that is, $qvB = qE \implies E = vB$. The potential difference between the top and bottom ends of the bar is $V = Et = vBt$ with the bottom at higher potential.

   (c) If the moving charges are negative, by convention, they move to the left (as current is towards right). The direction of magnetic force $-q(\vec{v} \times \vec{B})$ is again downward like before. But this time, the bottom edge of the bar will acquire a net negative charge and hence at a lower potential compared to the top edge that is at higher potential.

2. (a) A rotating disk (angular velocity ω) carries a uniform density of "static electricity" σ. Find the surface current density K at a distance r from the center.
   (b) Consider a uniformly charged solid sphere of radius R and total charge Q, centered at the origin and spinning at a constant angular velocity ω about the z axis. Find the current density $\vec{J}$ at any point (r, θ, ϕ)

Solution:

(a) The surface current density, $\vec{K}$, i.e, the current per unit width - perpendicular to the flow can be written as

$$\vec{K} = \sigma \vec{v},$$

here $\vec{v}$ is the velocity of the surface charge and σ is the surface charge density. For the rotating disk $v = \omega r$, thus $K = \sigma \omega r$ with direction along $\hat{\phi}$ in spherical polar coordinates.
(b) For the sphere rotating about the z axis, the velocity at any point on the sphere $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$, with $\dot{r} = \dot{\theta} = 0$. Therefore, current density $\mathbf{J} = \rho\mathbf{v} = \rho r\,\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$, where $\rho = Q/((4/3)\pi R^3)$.

3. Find the magnetic field at a point $z > R$ on the axis of (a) the rotating disk and (b) the rotating sphere, in problem 2.
Solution:

(a) Consider the shaded region of the disk with a width of dr. This ring of charge dq $= 2\pi r\sigma dr$ will have the current $I = dq/dt = \sigma\omega r dr$, where $dt = 2\pi/\omega$. The magnetic field on the z axis for this ring is $d\vec{B} = \frac{\mu_0}{2}\sigma\omega r\ \frac{r^2}{(r^2+z^2)^{3/2}}\ dr\hat{z}$ (Discussed in the class). Therefore the total field of the disk is,

$$\vec{B} = \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^3 dr}{(r^2+z^2)^{3/2}}\hat{z} = \frac{\mu_0\sigma\omega}{4}[2(\frac{u+2z^2}{\sqrt{u+z^2}})]\Big|_0^{R^2} = \frac{\mu_0\sigma\omega}{2}[(\frac{R^2+2z^2}{\sqrt{R^2+z^2}}) - 2z]\hat{z}. \quad (\text{with } u = r^2)$$

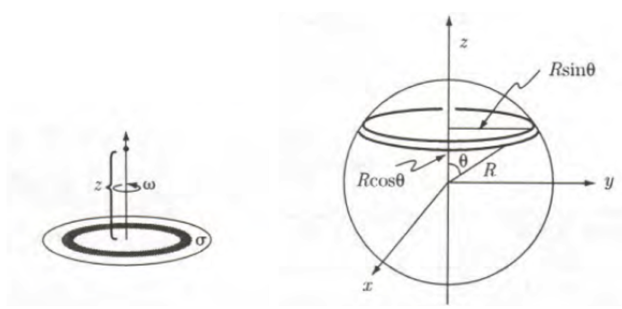

Figure 2: Figure for take home problem 3.

(b) We use the above disk solution to solve the sphere one. Slice the sphere in infinitesimal slabs (see. figure 2) and summing over all such slabs will give the solution. The thickness of such slabs $|d(Rcos\theta)| = Rsin\theta d\theta$; $\sigma \to \rho|d(Rcos\theta)| = \rho Rsin\theta d\theta$; $R \to Rsin\theta$; $z \to z - Rcos\theta$ and $R^2 + z^2 \to R^2 + z^2 - 2Rz\ cos\theta$.

From 'part a' solution, the field for such a slab would consist the term $[(\frac{R^2+2z^2}{\sqrt{R^2+z^2}}) - 2z]$ $= 2[\sqrt{R^2+z^2} - \frac{R^2/2}{\sqrt{R^2+z^2}} - z]$. Rewriting this term with the above substitutions,

$$2[\sqrt{R^2+z^2-2Rz\ cos\theta} - \frac{(R^2/2)sin^2\theta}{\sqrt{R^2+z^2-2Rz\ cos\theta}} - (z - R\ cos\theta)].$$

Therefore, total B_z is given by,

$$= \frac{\mu_0\rho\omega R}{2}2\int_0^\pi sin\theta\ d\theta\big[\sqrt{R^2+z^2-2Rz\ cos\theta} - \frac{(R^2/2)sin^2\theta}{\sqrt{R^2+z^2-2Rz\ cos\theta}} - (z - R\ cos\theta)\big]$$

$$= \mu_0\rho\omega R\int_{-1}^1 \big[\sqrt{R^2+z^2-2Rzu} - \frac{(R^2/2)sin^2\theta}{\sqrt{R^2+z^2-2Rzu}} - z + Ru)\big]du \qquad (u = cos\theta)$$

$$= \mu_0\rho\omega R\big[I_1 - \frac{R^2}{2}(I_2 - I_3) - I_4 + I_5\big].$$

$$I_1 = \int_{-1}^1 \sqrt{R^2+z^2-2Rzu}\ du = -\frac{1}{3Rz}(R^2+z^2-2Rzu)^{3/2}\Big|_{-1}^1 = \frac{2}{3z}(3z^2+R^2).$$

$$I_2 = \int_{-1}^1 \frac{1}{\sqrt{R^2+z^2-2Rzu}}\ du = -\frac{1}{Rz}(R^2+z^2-2Rzu)^{1/2}\Big|_{-1}^1 = \frac{2}{z}.$$

$$I_3 = \int_{-1}^1 \frac{u^2}{\sqrt{R^2+z^2-2Rzu}}\ du$$

$$= -\frac{1}{60R^3z^3}(8(R^2+z^2)^2 + 4(R^2+z^2)2Rzu + 3(2Rz)^2u^2)(R^2+z^2-2Rzu)^{1/2}\Big|_{-1}^1$$

$$= \frac{4}{15z^3}(R^2 + (5/2)z^2).$$

$$I_4 = z\int_{-1}^1 du = 2z; I_5 = R\int_{-1}^1 udu = 0.$$

$$B_z = \frac{2\mu_0\rho\omega R^5}{15z^3} \implies \vec{B} = \frac{2\mu_0\rho\omega R^5}{15z^3}\hat{z}.$$

4. A semicircular wire carries a steady current $\vec{I}$. Find the magnetic field at a point P on the other semicircle (see figure 3). The semicircular wire must be connected to some other wire to complete the circuit. Neglect this wire needed to complete the circuit.

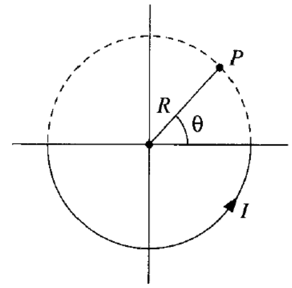

Figure 3: The path

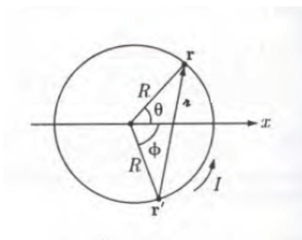

Figure 4: Solution to problem 4.

Solution:

According to the diagram, the vector connecting dl and the center is $\vec{r}' = R\ cos\phi\ \hat{x} - R\ sin\phi\ \hat{y}$, and the coordinate of point P is $\vec{r} = R\ cos\theta\ \hat{x} + R\ sin\theta\ \hat{y}$. Thus vector connecting $\vec{dl}$ and the point P is $\vec{\mathscr{r}} = \vec{r} - \vec{r}' = R\ (cos\theta - cos\phi)\ \hat{x} + R\ (sin\theta + sin\phi)\ \hat{y}$ and $\vec{dl} = \hat{z} \times \vec{r}' d\phi = R\ sin\phi\ d\phi\ \hat{x} + R\ cos\phi\ d\phi\ \hat{y} = R\ (sin\phi\ \hat{x} + cos\phi\ \hat{y})d\phi$.

$$\vec{dl} \times \vec{\mathscr{r}} = R^2 d\phi\big[1 - cos(\theta + \phi)\big]\hat{z}.$$

Therefore,

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{\vec{dl} \times \vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I}{4\pi} R^2 \hat{z}\int_0^\pi \frac{\big[1 - cos(\theta+\phi)\big]}{\big[2R^2 - 2R^2 cos(\theta+\phi)\big]^{3/2}} d\phi = \frac{\mu_0 I}{8\sqrt{2}\pi R}\hat{z}\int_0^\pi \frac{d\phi}{\sqrt{2} sin\frac{\theta+\phi}{2}}$$

$$= \frac{\mu_0 I}{16\pi R}\hat{z}\left(2\ ln\big[tan\frac{\theta+\phi}{4}\big]\right)\Bigg|_0^\pi = \frac{\mu_0 I}{8\pi R}\frac{tan\frac{\theta+\pi}{4}}{tan\frac{\theta}{4}}\hat{z}.$$

5. Two long coaxial solenoids each carry current I, but in opposite directions, as shown in figure 5. The inner solenoid (radius a) has n_1 turns per unit length, and the outer one (radius b) has n_2. Find $\vec{B}$ in each of the three regions: (i) inside the inner solenoid, (ii) between them, and (iii) outside both.

Solution:

As discussed in the class, the field inside a solenoid is $\mu_0 n I\hat{z}$ and outside it is zero, where the direction of $\hat{z}$ with respect to the current direction is shown in figure 6.

(i) Inside the inner solenoid, both the solenoid will contribute to the field. Since the

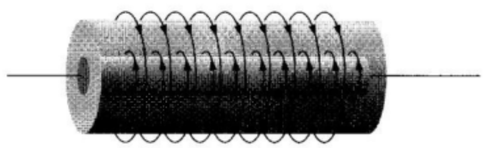

Figure 5: Figure for take home problem 5.

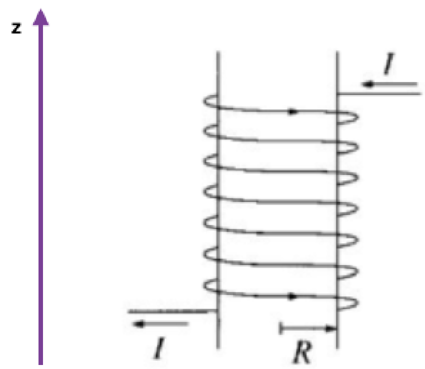

Figure 6: Figure for solution to take home problem 5.

currents in the two solenoids are in opposite directions, the field due to the inner solenoid points towards left (assumed to be the direction $\hat{z}$) whereas the one due to the outer solenoid points towards right ($-\hat{z}$). Thus the net field is $\vec{B} = \mu_0 I(n_1 - n_2)\hat{z}$.
(ii) Between the two solenoids, only the outer solenoid contributes to the field. Therefore $\vec{B} = -\mu_0 I n_2 \hat{z}$.
(iii) Outside both the solenoids, none of them contribute to the field. So $\vec{B} = 0$.

6. Just as $\vec{\nabla} \cdot \vec{B} = 0$ allows us to express $\vec{B}$ as the curl of a vector potential ($\vec{B} = \vec{\nabla} \times \vec{A}$), so $\vec{\nabla} \cdot \vec{A} = 0$ permits us to write $\vec{A}$ itself as the curl of a higher potential: $\vec{A} = \vec{\nabla} \times \vec{W}$.
(a) Find the general formula for $\vec{W}$ (as an integral over $\vec{B}$), which holds when $\vec{B} \to 0$ at ∞.
(b) Determine $\vec{W}$ for the case of a uniform magnetic field $\vec{B}$.
(c) Find $\vec{W}$ inside and outside an infinite solenoid.

Solution:
(a) As discussed in the class, for $\vec{\nabla} \cdot \vec{B} = 0, \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}, \vec{\nabla} \cdot \vec{A} = 0, \vec{\nabla} \times \vec{A} = \vec{B}$ and $\vec{J} \to 0$ at ∞, the magnetic vector potential $\vec{A}$ is given by

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'})}{\mathscr{r}} d\tau'$$

Similarly, for $\vec{\nabla} \cdot \vec{A} = 0, \vec{\nabla} \times \vec{A} = \vec{B}, \vec{\nabla} \cdot \vec{W} = 0, \vec{\nabla} \times \vec{W} = \vec{A}$ and $\vec{B} \to 0$ at ∞, we can write the higher potential $\vec{W}$ as

$$\vec{W}(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{B}(\vec{r'})}{\mathscr{r}} d\tau'$$

(b) Since $\vec{A} = \vec{\nabla}\times\vec{W}$ and $\vec{\nabla}\times\vec{A} = \vec{B}$, $\vec{W}$ will be proportional to $\vec{B}$ and two factors of $\vec{r}$ as differentiation of $\vec{W}$ twice should give $\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{W})$. One can therefore choose a minimal form of $\vec{W}$ as

$$\vec{W} = \alpha\vec{r}(\vec{r}\cdot\vec{B}) + \beta r^2\vec{B}$$

where α, β are constants which can be evaluated by using the constraints $\vec{\nabla}\cdot\vec{W} = 0, \vec{\nabla}\times\vec{W} = \vec{A}$. Divergence and curl of $\vec{W}$ can be calculated as

$$\vec{\nabla}\cdot\vec{W} = \alpha\Big[(\vec{r}\cdot\vec{B})(\vec{\nabla}\cdot\vec{r}) + \vec{r}\cdot\vec{\nabla}(\vec{r}\cdot\vec{B})\Big] + \beta\Big[r^2(\vec{\nabla}\cdot\vec{B}) + \vec{B}\cdot\vec{\nabla}(r^2)\Big]$$

Now, $\vec{\nabla}\cdot\vec{r} = 3$ and

$$\vec{\nabla}(\vec{r}\cdot\vec{B}) = \vec{r}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times r) + (\vec{r}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{r}$$

. Here all terms involving the derivative of $\vec{B}$ vanish owing to the fact that the field is uniform. Also $\vec{\nabla}\times\vec{r} = 0$. Therefore, we can write

$$\vec{\nabla}(\vec{r}\cdot\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{r} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z}) = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} = \vec{B}$$

. The last term in the expression for $\vec{\nabla}\cdot\vec{W}$ can be evaluated as

$$\vec{\nabla}(r^2) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2\vec{r}$$

. Therefore,

$$\vec{\nabla}\cdot\vec{W} = \alpha\Big[3(\vec{r}\cdot\vec{B}) + (\vec{r}\cdot\vec{B})\Big] + \beta\Big[0 + 2(\vec{r}\cdot\vec{B})\Big] = 2(\vec{r}\cdot\vec{B})(2\alpha + \beta)$$

which can be zero if $(2\alpha + \beta) = 0$. The curl of $\vec{W}$ is

$$\vec{\nabla}\times\vec{W} = \alpha\Big[(\vec{r}\cdot\vec{B})(\vec{\nabla}\times\vec{r}) - \vec{r}\times\vec{\nabla}(\vec{r}\cdot\vec{B})\Big] + \beta\Big[r^2(\vec{\nabla}\times\vec{B}) - \vec{B}\times\vec{\nabla}(r^2)\Big]$$

$$\implies \vec{A} = \vec{\nabla}\times\vec{W} = \alpha\Big[0 - (\vec{r}\times\vec{B})\Big] + \beta\Big[0 - 2(\vec{B}\times\vec{r})\Big] = -(\vec{r}\times\vec{B})(\alpha - 2\beta)$$

For uniform magnetic field $\vec{B}$, one can show that $\vec{A} = -\frac{1}{2}(\vec{r}\times\vec{B})$. It is straightforward to show that such $\vec{A}$ results in $\vec{\nabla} \cdot \vec{A} = 0, \vec{\nabla} \times \vec{A} = \vec{B}$. Using this expression for $\vec{A}$ in the expression for $\vec{\nabla} \times \vec{W}$ above, we get

$$\vec{A} = 2\vec{A}(\alpha - 2\beta) \implies (\alpha - 2\beta) = \frac{1}{2}$$

Solving $(2\alpha + \beta) = 0, (\alpha - 2\beta) = \frac{1}{2}$, we get $\alpha = \frac{1}{10}, \beta = -\frac{1}{5}$. Therefore, the higher potential is given by

$$\vec{W} = \frac{1}{10}\left[\vec{r}(\vec{r} \cdot \vec{B}) - 2r^2\vec{B}\right]$$

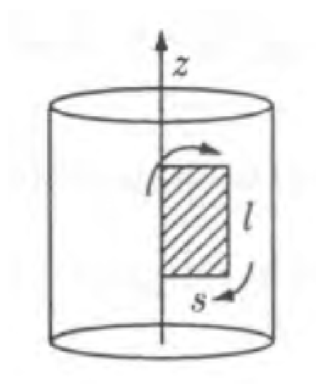

Figure 7: Figure for solution to take home problem 6 (c).

(c)

$$\vec{\nabla} \times \vec{W} = \vec{A} \implies \int (\vec{\nabla} \times \vec{W}) \cdot d\vec{a} = \int \vec{A} \cdot d\vec{a} \implies \oint \vec{W} \cdot d\vec{l} = \int \vec{A} \cdot d\vec{a}$$

Now, for an infinite solenoid having n turns per unit length, radius R and current I, we can calculate the vector potential by considering an amperian loop of radius s inside the solenoid

$$\oint \vec{A} \cdot d\vec{l} = A(2\pi s) = \int \vec{B} \cdot d\vec{a} = \mu_0 n I(\pi s^2)$$

where we have used the fact that inside an inifinite solenoid, the magnetic field is uniform and given by $\mu_0 n I$. Using this, we can write $\vec{A} = \frac{\mu_0 n I}{2} s\hat{\phi}$ inside the solenoid. For an amperian loop outside the solenoid, the magnetic flux is constant $\int \vec{B} \cdot d\vec{a} = \mu_0 n I(\pi R^2)$ as the field is zero outside. Thus the vector potential outside the solenoid is $\vec{A} = \frac{\mu_0 n I}{2} \frac{R^2}{s}\hat{\phi}$.

Now, coming back to the line integral of $\vec{W}$, we can integrate around the amperian loop shown in figure 7, taking $\vec{W}$ to point parallel to the axis and $\vec{W} = 0$ on the axis (using the freedom in choosing vector potential). This gives

$$-Wl = \int_0^s \frac{\mu_0 nI}{2} s' l ds' = \frac{\mu_0 nI}{2} \frac{s^2 l}{2}$$

$$\implies \vec{W} = -\frac{\mu_0 nIs^2}{4}\hat{z} \quad (s < R)$$

Outside the solenoid ($s > R$), we can split the surface integral of $\vec{A}$ into two parts:

$$\oint \vec{W} \cdot d\vec{l} = -Wl = \int_0^R \vec{A}_{\text{inside}} \cdot d\vec{a} + \int_R^s \vec{A}_{\text{outside}} \cdot d\vec{a}$$

$$\implies -Wl = \frac{\mu_0 nIR^2 l}{4} + \int_R^s \frac{\mu_0 nI}{2} \frac{R^2}{s'} l ds' = \frac{\mu_0 nIR^2 l}{4} + \frac{\mu_0 nIR^2 l}{2} \ln(s/R)$$

$$\implies \vec{W} = -\frac{\mu_0 nIR^2}{4}\left[1 + 2\ln(s/R)\right]\hat{z} \quad (s > R)$$