Linear Algebra

Department of Mathematics
Indian Institute of Technology Guwahati

January – May 2019

MA 102 (RA, RKS, MGPP, KVK)

Eigenvalues and eigenvectors

Topics:

- Determinant of Matrices
- Eigenvalues and Eigenvectors

Determinant of a matrix

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Proof: Use induction on the size of the matrix.

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Henceforth, we consider $\mathbb{F} = \mathbb{C}$.

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$$E_4 = \text{null}(A - 4I) = \left\{ \mathbf{x} : \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \mathbf{x} = \mathbf{0} \right\} = \text{span}([1, 1]^\top),$$

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Exercise

- Let A be a matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$.
    - For any positive integer n, show that λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.
    - If A is invertible, then show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$.
    - If A is invertible then show that for any integer n, λ^{-n} is an eigenvalue of A^{-n} with corresponding eigenvector $\mathbf{x}$.
- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be eigenvectors of a matrix A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, respectively. Let $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$. Show that for any positive integer k,

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \ldots + c_m\lambda_m^k\mathbf{v}_m.$$

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- Rank each page in the database, so that when a user does a search and the subset of pages in the database with the desired information has been found, the more important pages can be presented first.

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Let L_k denote the set of in-links P_k. Then the rank x_k of P_k is defined by

$$x_k = \sum_{j \in L_k} \frac{x_j}{n_j} \tag{1}$$

where n_j is the number of outgoing links (outlinks) from page P_j.
Setting $\mathbf{v} := [x_1, \ldots, x_n]^T$, equation (1) is rewritten as

PageRank Algorithm (eigenvalue problem)

Let $x_j \geq 0$ be the rank of page P_j. Then $x_j > x_k \implies P_j$ is more important than page P_k.
If page P_j contains n_j out-links, one of which links to page P_k, then page P_k's score is boosted by x_j/n_j.

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Eigenvalue problem $\mathbf{Hv} = \mathbf{v}$,

where $\mathbf{H}$ is the hyperlink matrix and $\mathbf{v}$ is the page rank vector.

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By Perron-Frobenius theorem, there is a unique vector $\mathbf{v} = [x_1, \ldots, x_n]^\top$ with $x_j > 0$ for $j = 1 : n$ such that

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The rank of the web page P_j is given by the j-th component x_j of the eigenvector $\mathbf{v}$ of the Google matrix G. Google sets $m = 0.15$.

References

For more on Google search engine, read the following paper/book.

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- K. Bryan and T. Leise, The \$25,000,000,000 Eigenvector: The Linear Algebra behind Google, SIAM Review, 48(2006), pp.569-581.

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