Ordinary Differential Equations (ODEs)

Department of Mathematics
IIT Guwahati

RA/RKS/MGPP/KVK

Texts/References:

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2. W. E. Boyce and R. C. Diprima, Elementary Differential Equations and Boundary Value Problems, John Wiley & Son, 2001.
3. E. A. Coddington, An Introduction to Ordinary Differential Equations, Prentice Hall India, 1995.
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Topics:

- Basic Definitions
- Classification of Differential Equations
- Different Types of Solution
- Initial Value Problem
- Some Applications of ODEs

Definition: An equation containing the derivatives or differentials of functions is said to be a differential equation(DE).

Definition: A DE involving ordinary derivatives w.r.t a single independent variable is called an ordinary differential equation(ODE).

A general form of the nth order ODE:

$$F(x, y(x), y'(x), y''(x), \cdots, y^{(n)}(x)) = 0, \quad x \in I = (a, b), \quad (1)$$

where $y'(x) = \frac{dy}{dx}$, $y''(x) = \frac{d^2y}{dx^2}, \cdots, y^{(n)}(x) = \frac{d^ny}{dx^n}$.

It is assumed that Eq. (1) holds for all $x \in (a, b)$. In other words,

$$y, y', y'', \ldots, y^{(n)} \in C(a, b)$$

- The order of a DE is the order of the highest derivative that occurs in the equation.
- The degree of a DE is the power of the highest order derivative occurring in the differential equation.
- Eq. (5) is linear if F is linear in $y, y', y'', \ldots, y^{(n)}$, with coefficients depending on the independent variable x. It is called nonlinear if it is not linear.

Examples:

- $y''(x) + 3y'(x) + xy(x) = 0$
  (second-order, first-degree, linear)
- $y''(x) + 3y(x)y'(x) + xy(x) = 0$
  (second-order, first-degree, nonlinear)
- $(y''(x))^2 + 3y'(x) + xy^2(x) = 0$
  (second-order, second-degree, nonlinear)

Definition: A DE involving partial derivatives w.r.t more than one independent variable is called a partial differential equation(PDE).
A PDE for a function $u(x_1, x_2, \ldots, x_n)$ $(n \geq 2)$ is a relation of the form

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_1 x_1}, u_{x_1 x_2}, \ldots,) = 0, \qquad (2)$$

where F is a given function of the independent variables $x_1, x_2, \ldots, x_n$, the unknown function u and a finite number of its partial derivatives.
Examples:

- $xu_x + yu_y = 0$ (first-order equation)
- $u_{xx} + u_{yy} = 0$ (second-order equation)

We shall consider only ODE.

Understanding ODE via Linear Algebra

Let $I = (a, b)$. Denote $C^n(I) = \{y \mid y', y'', \dots, y^{(n)} \in C(I)\}$.
Consider first order ODE:

$$F(x, y(x), y'(x)) = 0, \quad x \in I, \tag{3}$$

which can be put in the form

$$G(y(x), y'(x)) = f(x).$$

Consider the operator $T : C^1(I) \to C(I)$ given by

$$T(y)(x) := G(y(x), y'(x)).$$

Eq. (3) is linear if T is linear for each $x \in I$. That is, for each $x \in I$,

$$T(y_1 + y_2)(x) = T(y_1)(x) + T(y_2)(x); \quad T(cy)(x) = cT(y)(x), \quad c \in \mathbb{R}.$$

Exercise: Check the linearity of the following DEs:

$$(i)\ \ y'(x) + y(x) = 0; \quad (ii)\ \ y(x)y'(x) + 5x = 0.$$

Note: The above definition can be extended to higher order ODE

Consider the linear differential equation:

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_0(x)y(x) = f(x), \quad x \in I. \tag{4}$$

In operator notation Eq. (4) is written as

$$T(y)(x) = f(x),$$

where the operator $T : C^n(I) \to C(I)$ given by

$$T(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y.$$

It is easy to verify that

$$T(y_1 + y_2) = T(y_1) + T(y_2), \quad T(cy) = cT(y).$$

Thus, T is a linear operator and $T(y) = f$ is referred as a linear DE. The operator T itself is sometimes written as

$$T = D^n + a_{n-1}D^{n-1} + \cdots + a_0,$$

where $D^k = \frac{d^k}{dx^k}$ denotes the kth derivative operator.

Different Types of Solution

Definition: The general solution (GS) of

$$F(x, y(x), y'(x), y''(x), \cdots, y^{(n)}(x)) = 0, \quad x \in I = (a, b), \quad (5)$$

involves n arbitrary constants, i.e.,

$$y(x) = \phi(x, c_1, c_2, \cdots, c_n), \quad c_i \in \mathbb{R}, \; i = 1, \ldots, n,$$

where $\phi \in C^n(I)$, and satisfies

$$F(x, \phi(x), \phi'(x), \phi''(x), \cdots, \phi^{(n)}(x)) = 0, \quad x \in I$$

Definition: A function $y = \phi \in C^n(I)$ that satisfies

$$F(x, \phi(x), \phi'(x), \phi''(x), \cdots, \phi^{(n)}(x)) = 0, \quad x \in I$$

is called an explicit solution to the equation on I.

Example: $\phi(x) = x^2 - x^{-1}$ is an explicit solution to $y''(x) - 2\frac{y}{x^2} = 0$. Note that $\phi(x)$ is an explicit solution on $(-\infty, 0)$ and also on $(0, \infty)$.

Definition: A relation $\psi(x, y) = 0$ is said to be an implicit solution to $F(x, y(x), y'(x), y''(x), \cdots, y^{(n)}(x)) = 0$ on I if it defines one or more explicit solutions on I.

Examples:

- $x + y + e^{xy} = 0$ is an implicit solution to

$$(1 + xe^{xy})y' + 1 + ye^{xy} = 0.$$

- $4x^2 - y^2 = c$, where c is an arbitrary constant, an implicit solution to $yy' - 4x = 0$.

Definition: Let the GS of the first order DE

$$F(x, y(x), y'(x)) = 0 \tag{6}$$

be given by $\phi(x, y, c) = 0$. Let $\psi(x, y) = 0$ be the equation obtain by eliminating c from $\phi(x, y, c) = 0$ and $\frac{\partial \phi}{\partial c}(x, y, c) = 0$.

If $\psi(x, y)$ satisfies $F(x, y(x), y'(x)) = 0$ but it does not belong to the family $\phi(x, y, c) = 0$, then this function $\psi(x, y)$ is called a singular solution.

Example: Consider

$$\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{y}.$$

The GS is given by $(x + c)^2 + y^2 = 1, \quad c \in \mathbb{R}$.
The singular solution is $y = \pm 1$.

Initial Value Problem (IVP)

Definition: Find a solution $y(x) \in C^n((a,b))$ that satisfies

$$F(x, y, y'(x), \cdots, y^{(n)}(x)) = 0, \quad x \in (a,b)$$

and the n initial conditions(IC)

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1},$$

where $x_0 \in (a,b)$ and $y_0, y_1, \ldots, y_{n-1}$ are given constants.

First-order IVP: $\quad F(x, y, y'(x)) = 0, \quad y(x_0) = y_0.$

Second-order IVP: $\quad F(x, y, y'(x), y''(x)) = 0,$
$y(x_0) = y_0, \; y'(x_0) = y_1.$

Example: The function $\phi(x) = \sin x - \cos x$ is a solution to IVP: $y''(x) + y(x) = 0, \;\; y(0) = -1, \; y'(0) = 1.$

Applications

A typical application of DE proceeds as follows.

$$\boxed{\text{Real World Situation}}$$
$$\downarrow$$
$$\boxed{\text{Mathematical Model}}$$
$$\downarrow$$
$$\boxed{\text{Solution of Mathematical Model}}$$
$$\downarrow$$
$$\boxed{\text{Interpretation of Solution}}$$

Example 1. In case of radioactive decay, the rate of decay is proportional to the amount of radioactive substance present. This leads to the equation

$$-\frac{dR}{dt} = kR, \quad k > 0,$$

where $R(> 0)$ is the unknown amount of radioactive substance present at time t and k is the proportionality constant. Solving for R yields

$$R = R(t) = Ce^{-kt}.$$

The value of C is determined if the initial amount amount of radioactive substance is given.

Example 2. Newton's second law can be applied to a falling object leads to the equation

$$m\frac{d^2h}{dt^2} = -mg,$$

where m is the mass of the object, h is its height above the ground, $\frac{d^2h}{dt^2}$ is its acceleration, $-mg$ is the force due to gravity.

Integrating twice w.r.t t, we obtain

$$h = h(t) = -\frac{1}{2}gt^2 + c_1 t + c_2,$$

where the integration constants c_1 and c_2 are determined if we know the initial height and initial velocity of the object.

Example 3. Some physical scenarios for which we can derive linear ODEs of the form

$$y' = a(t)y + f(t).$$

Mathematical finance:

- $y(t)$ is the amount of money in the account at time t.
- $a(t)$ is the interest rate (most often constant)
- $f(t)$ corresponds to rate of withdrawals from $(f(t) < 0)$ and/or deposits into $(f(t) > 0)$ the account.

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