MA 102 (Mathematics II)

Tutorial Sheet No. 8 **Ordinary Differential Equations** **April 04, 2019**

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1. Determine the largest interval (a, b) in which the given IVP is certain to have a unique solution:
   (a) $e^x y'' - \frac{y'}{x-3} + 3y = \ln x,\ y(1) = 3,\ y'(1) = 2.$
   (b) $(1-x)y'' - 3xy' + 3y = \sin x,\ y(0) = 1,\ y'(0) = 1.$
   (c) $x^2 y'' + 4y = \cos x,\ y(1) = 0,\ y'(1) = -1.$

2. Let y_1 and y_2 be two solutions of $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$ defined in the interval (a, b). Show that if their Wronskian $W(y_1, y_2) = 0$ at least one point in (a, b) then $W(y_1, y_2) = 0$ for all $x \in (a, b)$.

3. If y_1 and y_2 are linearly independent solutions of $xy'' + 2y' + xe^x y = 0$ and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.

4. (a) Verify that the functions $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ are linearly independent solutions of the differential equation $x^2y'' - 4xy' + 6y = 0$ on $(-\infty, \infty)$; (b) Show that y_1 and y_2 are linearly dependent on $(-\infty, 0)$, but are linearly independent on $(-\infty, \infty)$; (c) Although y_1 and y_2 are linearly independent, show that $W(y_1, y_2) = 0$ for all $x \in (-\infty, \infty)$. Does this violate the fact that $W(y_1, y_2) = 0$ for every $x \in (-\infty, \infty)$ implies y_1 and y_2 are linearly dependent?

5. Let $p(x), q(x) \in C(I)$. Assume that the functions $y_1, y_2 \in C^2(I)$ are solutions of the differential equations $y'' + p(x)y' + q(x)y = 0$ on an open interval I. Prove that (a) if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval; (b) if y_1 and y_2 have a common point of inflection x_0 in I, then they cannot be a fundamental set of solutions on that interval

6. Let $p(x)$ and $q(x)$ are continuous on (a, b), and let $x_0 \in (a, b)$. Let y_1, y_2 be solutions to $y'' + p(x)y' + q(x)y = 0$ on (a, b). Then y_1 and y_2 are linearly dependent on (a, b) iff the vectors $[y_1(x_0), y_1'(x_0)]^T$ and $[y_2(x_0), y_2'(x_0)]^T$ are linearly dependent.

7. Let $S = \{f : \mathbb{R} \to \mathbb{R}\,|\,L(f) = 0\}$, where $L(f) := f''' + f'' - 2$. Find the solution set S. Let $S_0 \subset S$ be the subspace of solutions g such that $\lim_{x\to\infty} g(x) = 0$. Find $g \in S_0$ such that $g(0) = 0$ and $g'(0) = 2$.

8. Find the general solution of the following differential equations.

   (a) $\dfrac{d^4y}{dx^4} + y(x) = 0.$

   (b) $\dfrac{d^5y}{dx^5} - 2\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} = 0.$

   (c) $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y(x) = 0.$

   (d) $\dfrac{d^5y}{dx^5} + 5\dfrac{d^4y}{dx^4} + 10\dfrac{d^3y}{dx^3} + 10\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + y(x) = 0.$

9. Solve the following initial-value problems:

   (a) $y'' - 2y' + y = 2xe^{2x} + 6e^x;\ y(0) = 1,\ y'(0) = 0.$

   (b) $y''(x) + y(x) = 3x^2 - 4\sin x,\ y(0) = 0,\ y'(0) = 1.$

10. If $y = \phi_1(x)$ is a particular solution of $y'' + (\sin x)y' + 2y = e^x$ and $y = \phi_2(x)$ is a particular solution of $y'' + (\sin x)y' + 2y = \cos(2x)$, then find a particular solution of $y'' + (\sin x)y' + 2y = e^x + 2\sin^2 x$.

11. Use the method of undermined coefficients to find a particular solution to the following differential equations:

    (a) $y'' - 3y' + 2y = 2x^2 + 3e^{2x}$.

    (b) $y''(x) - 3y'(x) + 2y(x) = xe^{2x} + \sin x$.

12. Use the annihilator method to determine the form of a particular solution for the equations:

    (a) $y''(x) - 5y'(x) + 6y(x) = \cos(2x) + 1$.

    (b) $y''(x) - 5y'(x) + 6y(x) = e^{3x} - x^2$.

13. In the study of a vibrating spring with damping, we are led to an IVP of the form $mx''(t) + bx'(t) + kx(t) = 0, \;\; x(0) = x_0, \;\; x'(0) = v_0$, where m is the mass of the spring system, b is the damping constant, k is the spring constant, x_0 is the initial displacement, v_0 is the initial veocity, and $x(t)$ is the displacement from equilibrium of the spring system at time t (see Figure 1). Determine the displacement after 10 sec i.e., $x(10)$ when $m = 36kg$, $b = 12$ kg/sec, $k = 37$ kg/sec^2, $x_0 = 70$ cm, and $v_0 = 10$ cm/sec.

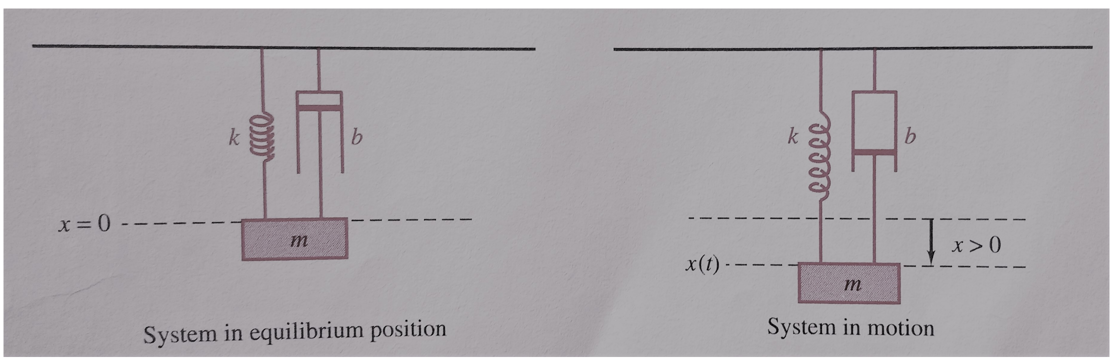

Figure 1