MA 102 (Mathematics II)

Tutorial Sheet No. 9 **Ordinary Differential Equations** **April 11, 2019**

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1. Let $P(D) = a_n D^n + \cdots + a_1 D + a_0, \;\; a_n \neq 0$, where $D = \frac{d}{dx}$.
(a) If $P(D)y = ce^{ax}$, where c is a constant then a particular solution is given by

$$y_p = \frac{1}{P(D)}(ce^{ax}) = \frac{ce^{ax}}{P(a)}, \;\; P(a) \neq 0.$$

(b) If $P(D)y = h(x)e^{ax}$, where $h(x)$ is any function in x, then

$$y_p = \frac{1}{P(D)}(h(x)e^{ax}) = e^{ax}\frac{1}{P(D+a)}h(x).$$

(c) In particular, if $P(D) = (D-a)^r P_1(D)$, where $P_1(a) \neq 0$ then $y_p = \frac{1}{P(D)}(ce^{ax}) = \frac{cx^r e^{ax}}{r!P_1(a)}$.

2. Use operator method to find a particular solution of the following ODEs.
(a) $y''' + y'' + y' + y = x^5 - 2x^2 + x$.
(b) $y''' - 5y'' + 8y' - 4y = 3e^{2x}$.
(c) $y'' - 3y' + 2y = 3\sin 2x$.

3. Find a particular solution to the following differential equations:
(a) $y'' + 4y = \tan 2x$.
(b) $y'' + y = \tan x + 3x - 1$.
(c) $y'' - 2y' + y = e^x \sin^{-1} x$.

4. Find a general solution to the differential equation given that the functions $y_1(x)$ and $y_2(x)$ are linearly independent solutions to the corresponding homogeneous equation for $x > 0$.
(a) $(\sin^2 x)y'' - 2\sin x \cos x y' + (\cos^2 x + 1)y = \sin^3 x; \quad y_1(x) = \sin x, \;\; y_2(x) = x\sin x$.
(b) $(x^2 + 2x)y'' - 2(x+1)y' + 2y = (x+2)^2; \quad y_1(x) = x + 1, \;\; y_2(x) = x^2$.

5. Use the method of variation of parameters to show that

$$y(x) = c_1 \cos x + c_2 \sin x + \int_0^x f(s)\sin(x - s)\, ds$$

is a general solution to the differential equation $y'' + y = f(x)$, where $f(x) \in C(\mathbb{R})$.

6. A differential equation and a non-trivial solution y_1 are given. Find the general solution.
(a) $x^2 y'' + xy' - y = 0, \;\; x \neq 0; \;\; y_1(x) = x$.
(b) $x^2 y'' - 2xy' - 4y = 0, \; x > 0; \;\; y_1(x) = x^{-1}$.

7. Find a general solution to the given equation for $x > 0$.
(a) $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$.
(b) $x^2 y'' - 5xy' + 8y = 2x^3$.

8. Given that $y = x$ is a solution of $x^2 y'' + xy' - y = 0, \;\; x \neq 0$, find the general solution of $x^2 y'' + xy' - y = x, \; x \neq 0$.

9. Rewrite the given scalar equation as a first-order system in normal form. Express the system in the matrix form $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$.

   (a) $y''(t) - 3y'(t) - 11y(t) = \sin t$; (b) $y^{(4)}(t) + y(t) = t^2$.

10. Determine the interval (a, b) where we are assured that there is a unique solution to the following initial value problems:

   (a) $\mathbf{x}'(t) = \begin{bmatrix} \cos t & \sqrt{t} \\ t^3 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \tan t \\ e^t \end{bmatrix}, \quad \mathbf{x}(2) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

   (b) $\mathbf{x}'(t) = \begin{bmatrix} t^2 & 1+3t \\ 1 & \sin t \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

11. The vector functions $\mathbf{x}_1 = [e^{-t}, 2e^{-t}, e^{-t}]^T$, $\mathbf{x}_2 = [e^t, 0, e^t]^T$, $\mathbf{x}_3 = [e^{3t}, -e^{3t}, 2e^{3t}]^T$ are solutions to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Determine whether they form a fundamental solution set. If they do, find a fundamental matrix for the system and give a general solution.

12. Let $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ be two fundamental matrices for the same system $\mathbf{x}'(t) = A\mathbf{x}$. Then, there exists a constant matrix $\mathbf{C}$ such that $\mathbf{X}(t) = \mathbf{Y}(t)\mathbf{C}$.