PH 102 : Electromagnetism **Department of Physics**
Tutorial 10 **Indian Institute of Technology Guwahati**
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1 Solutions to Selective Take Home Problems

1. A rectangular loop of wire is situated so that one end (height h) is between the plates of a parallel plate capacitor (shown in figure 1), oriented parallel to the field $\vec{E}$. The other end is way outside, where the field is essentially zero. What is the emf in this loop? If the total resistance is R, what current flows? Explain.

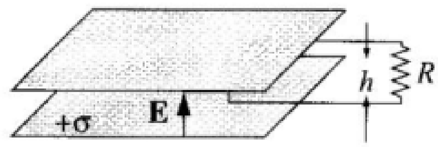

Figure 1: Figure for take home problem 1.

Solution:
For all electrostatic fields

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = 0.$$

However, the electric field in between the two plates of a parallel plate capacitor is $E = \sigma/\epsilon_0$ where σ is the surface charge density of the plates. Taking the field outside to be exactly zero, the line integral would look like $\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = (\sigma/\epsilon_0)h$. However, the field outside the plates is never exactly zero, there always exists some fringing field a the edges (figure 4.31, Introduction to Electrodynamics, D J Griffiths) that makes sure that the non-zero contribution from the left edge of the loop to the line integral gets cancelled out. This is consistent with the fact $\oint \vec{E} \cdot d\vec{l} = 0$ for electrostatics. Therefore, the emf in the loop and hence the current through the resistor R are zero.

2. A square loop is cut out of a thick sheet of aluminium. It is then placed so that the top portion is in a uniform magnetic field $\vec{B}$, and allowed to fall under gravity (shown in figure 2 where the shading indicates the field region and $\vec{B}$ points into the page). If the magnetic field is 1 T, find the terminal velocity of the loop (in m/s). Find the velocity of the loop as a function of time. How long does it take (in seconds) to reach, say, 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit? Write your final answer in numbers by using acceleration due to gravity $g = 9.8$ m/s^2, mass density of aluminium $\eta = 2.7 \times 10^3$ kg/m^3, resistivity of aluminium $\rho = 2.8 \times 10^{-8}$ Ωm.

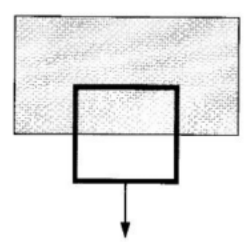

Figure 2: Figure for take home problem 2

Solution:

The magnitude of the induced emf is

$$\mathcal{E} = Bl\frac{dx}{dt} = Blv = IR.$$

The induced current is, therefore, given by $I = Blv/R$. The current flows in such a direction that opposes the motion of the loop downward (Lenz's law). The upward force on the loop is therefore $F = IlB = B^2l^2v/R$. Since this force opposes the gravitational force downward, applying Newton's law for the loop we get

$$mg - \frac{B^2l^2}{R}v = m\frac{dv}{dt} \implies \frac{dv}{dt} = g - \alpha v, \quad \alpha = \frac{B^2l^2}{mR}.$$

The terminal velocity (v_t) can be found by equating $dv/dt = 0$:

$$g - \alpha v_t = 0 \implies v_t = \frac{g}{\alpha} = \frac{mgR}{B^2l^2}.$$

The velocity as a function of time can be found by solving the above differential equation:

$$\frac{dv}{dt} = g - \alpha v \implies \frac{dv}{g - \alpha v} = dt \implies -\frac{1}{\alpha}\ln(g - \alpha v) = t + C, \quad C \equiv \text{Constant}$$

$$\implies g - \alpha v = Ae^{-\alpha t}$$

where the constant C is absorbed in A. Since the loop started from rest that is at $t = 0$, $v = 0$, we have $A = g$. Thus the velocity as a function of time is

$$v = \frac{g}{\alpha}(1 - e^{-\alpha t}) = v_t(1 - e^{-\alpha t}).$$

For 90% of terminal velocity

$$v/v_t = 0.9 = 1 - e^{-\alpha t} \implies e^{-\alpha t} = 1 - 0.9 = 0.1$$

$$\implies \ln(0.1) = -\alpha t \implies \alpha t = \ln 10 \implies t = \frac{1}{\alpha}\ln 10$$

$$\implies t_{90\%} = \frac{v_t}{g}\ln 10.$$

If A is the cross sectional area of the aluminium sheet, l is the length of each side of the square loop, the mass of the loop is $m = 4\eta Al$. The resistance of the loop is $R = 4l\rho/A$. Using these in the expression for terminal velocity

$$v_t = \frac{g}{B^2l^2}4\eta Al\frac{4l\rho}{A} = \frac{16g\eta\rho}{B^2}.$$

Using the given numerical values, we get $v_t = 1.2$ cm/s. Also $t_{90\%} = 2.8$ ms.

If the loop is cut to break the circuit, there will not be any induced current and hence no upward force opposing the fall. The loop will then fall freely with acceleration g.

3. A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I (as shown in figure 3). Now someone cuts the wire, so that I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows? If you don't like the scissors model, turn the current down gradually:

$$I(t) = \begin{cases} (1-\alpha t)I, & \text{for } 0 \le t \le 1/\alpha, \\ 0, & \text{for } t > 1/\alpha. \end{cases}$$

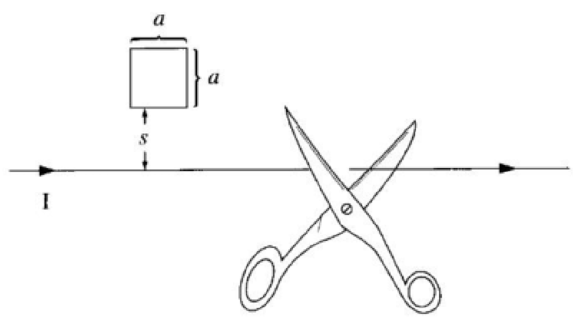

Figure 3: Figure for take home problem 3.

Solution:
The magnetic field due to an infinite current carrying wire at a distance s is given by

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}.$$

The magnetic flux through the square loop is

$$\Phi = \int \vec{B}\cdot d\vec{a} = \int_a^{2a} \frac{\mu_0 I}{2\pi s} a ds = \frac{\mu_0 I a}{2\pi}\int_a^{2a}\frac{ds}{s} = \frac{\mu_0 I a \ln 2}{2\pi}$$

The emf induced in the loop is

$$\mathcal{E} = I_{\text{loop}}R = \frac{dQ}{dt}R = -\frac{d\Phi}{dt} = -\frac{\mu_0 a \ln 2}{2\pi}\frac{dI}{dt}$$

$$\implies dQ = -\frac{\mu_0 a \ln 2}{2\pi R}dI \implies Q = -\frac{\mu_0 a \ln 2}{2\pi R}(0 - I) = \frac{I\mu_0 a \ln 2}{2\pi R}$$

On the other hand, using

$$I(t) = \begin{cases} (1-\alpha t)I, & \text{for } 0 \le t \le 1/\alpha, \\ 0, & \text{for } t > 1/\alpha. \end{cases}$$

we get

$$\frac{dQ}{dt}R = -\frac{\mu_0 a \ln 2}{2\pi}(-\alpha I)$$

$$\implies Q = \frac{\mu_0 a \ln 2}{2\pi R}\int_0^{1/\alpha} \alpha I dt = \frac{I\mu_0 a \ln 2}{2\pi R}.$$

The direction of induced current can be found by applying Lenz's law. The magnetic field at the square loop is out of the plane and is decreasing as current slowly drops to zero. Hence the induced current should be in counter clockwise direction to give rise to a magnetic field in a direction out of the plane.

4. Find the energy stored in a section of length l of a long solenoid (radius R, current I, n turns per unit length) using the following formulas discussed in the class:
(a) $W = \frac{1}{2}LI^2$ where L is the inductance.
(b) $W = \frac{1}{2}\oint(\vec{A}\cdot\vec{I})dl$ where $\vec{A}$ is the magnetic vector potential.
(c) $W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau$.
(d) $W = \frac{1}{2\mu_0}\left[\int_V B^2 d\tau - \oint_S(\vec{A}\times\vec{B})\cdot d\vec{a}\right]$, where S is the surface bounding the volume V. Take as your volume the cylindrical tube from radius $a < R$ out to radius $b > R$.

Solution:
(a) The magnetic field inside a solenoid is $B = \mu_0 nI$. The magnetic flux through a single turn is $\Phi_1 = \mu_0 nI\pi R^2$. For length l, there are nl number of turns and hence the total flux is $\Phi = \mu_0 n^2\pi R^2 Il$. Since the flux is related to the self-inductance as $\Phi = LI$, therefore, $L = \mu_0 n^2 \pi R^2 l$. The energy stored is

$$W = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2 \pi R^2 l I^2.$$

(b) The vector potential inside a solenoid can be calculated as

$$\oint \vec{A}\cdot d\vec{l} = A(2\pi r) = \int \vec{B}\cdot d\vec{a} = \mu_0 nI(\pi r^2)$$

$$\implies \vec{A} = \frac{\mu_0 nI}{2} r\hat{\phi}$$

At the surface of the solenoid, $\vec{A} = \frac{\mu_0 nI}{2}R\hat{\phi}$. The energy stored for one turn is

$$W_1 = \frac{1}{2}\oint(\vec{A}\cdot\vec{I})dl = \frac{1}{2}\frac{\mu_0 nI}{2}RI(2\pi R).$$

For length l, there are nl number of turns and hence the stored energy is

$$W = \frac{1}{2}\frac{\mu_0 nI}{2}RI(2\pi R)(nl) = \frac{1}{2}\mu_0 n^2 \pi R^2 l I^2.$$

(c) The magnetic field inside is $B = \mu_0 n I$ and zero outside. The stored energy is

$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau = \frac{1}{2\mu_0}(\mu_0 n I)^2 \int d\tau = \frac{1}{2\mu_0}(\mu_0 n I)^2(\pi R^2 l) = \frac{1}{2}\mu_0 n^2 \pi R^2 l I^2.$$

(d) $W = \frac{1}{2\mu_0}\left[\int_V B^2 d\tau - \oint_S (\vec{A}\times\vec{B})\cdot d\vec{a}\right]$ where S is the surface bounding the volume V. Taking the volume as the cylindrical tube from radius $a < R$ out to radius $b > R$ and noting that $B = 0$ outside the solenoid, we have

$$\int_V B^2 d\tau = \mu_0^2 n^2 I^2 \pi (R^2 - a^2) l.$$

For the surface enclosing this volume $\vec{A}\times\vec{B} = 0$ at $s = b$ surface lying outside the solenoid. On the inside surface at $s = a$, the vector potential and magnetic field are $\vec{A} = \frac{\mu_0 n I}{2} a\hat{\phi}, \vec{B} = \mu_0 n I \hat{z}$. Therefore, $\vec{A}\times\vec{B} = \frac{1}{2}\mu_0^2 n^2 I^2 a(\hat{\phi}\times\hat{z}) = \frac{1}{2}\mu_0^2 n^2 I^2 a\hat{s}$. The surface integral of this will get contribution only from the inner surface at $s = a$ with area vector pointing in the $-\hat{s}$ direction. Therefore,

$$\oint (\vec{A}\times\vec{B})\cdot d\vec{a} = \int \left(\frac{1}{2}\mu_0^2 n^2 I^2 a\hat{s}\right)[a d\phi dz(-\hat{s})] = -\frac{1}{2}\mu_0^2 n^2 I^2 a^2 (2\pi l).$$

The stored energy is therefore,

$$W = \frac{1}{2\mu_0}\left[\mu_0^2 n^2 I^2 \pi (R^2 - a^2) l + \mu_0^2 n^2 I^2 a^2 \pi l\right] = \frac{1}{2}\mu_0 n^2 I^2 R^2 \pi l.$$

5. Consider a magnetic field given by

$$\begin{aligned} \vec{B} &= B_0(t)\hat{z} \qquad s < a \\ &= 0 \qquad\quad s > a \end{aligned}$$

Calculate the induced electric field.

Solution:
Here one can use the analogy between the equations representing magnetic field produced by current distributions and electric field produced by changing magnetic field:

$$\vec{\nabla}\cdot\vec{B} = 0,\ \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla}\cdot\vec{E} = 0,\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

They suggest that the electric field distribution produced by changing magnetic field is same as the magnetic field produced by the corresponding current distribution. In the given problem we have

$$\frac{\partial \vec{B}}{\partial t} = \frac{dB_0}{dt}\hat{z} \qquad s < a$$
$$= 0 \qquad s > a$$

A corresponding current distribution would be a uniform current density $\vec{J}$ along the z direction within radius a and zero outside. This is identical to the situation of a cylindrical wire of radius a carrying uniform current across its cross section. The magnetic field for such a current distribution can be found by Ampere's law. The corresponding electric field can be found by using Faraday's law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{s}$$

For $s < a$, we have

$$\implies E(2\pi s) = -\pi s^2 \frac{\partial B}{\partial t} = -\pi s^2 \frac{dB_0}{dt}$$
$$\implies \vec{E} = -\frac{s}{2}\frac{dB_0}{dt}\hat{\phi}$$

For $s > a$, we have

$$\implies E(2\pi s) = -\pi a^2 \frac{\partial B}{\partial t} = -\pi a^2 \frac{dB_0}{dt}$$
$$\implies \vec{E} = -\frac{a^2}{2s}\frac{dB_0}{dt}\hat{\phi}$$