Homogeneous Linear Systems with Repeated Eigenvalues and Nonhomogeneous Linear Systems

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Repeated real eigenvalues

Q. How to solve the IVP

$$\mathbf{x}'(t) = A\mathbf{x}(t), \ \ \mathbf{x}(0) = \mathbf{x}_0,$$

when A is not diagonalizable?

Definition: Let λ be an eigenvalue of A of multiplicity $m \leq n$. Then, for $k = 1, \ldots, m$, any nonzero solution $\mathbf{v}$ of

$$(A - \lambda I)^k \mathbf{v} = 0$$

is called a generalized eigenvector(GEV) of A.

Definition: An $n \times n$ matrix is said to be nilpotent of order k if $N^{k-1} \neq 0$ and $N^k = 0$.

Theorem: Let $\lambda_1, \ldots, \lambda_n$ be real eigenvalues of an $n \times n$ matrix A repeated according to their multiplicity. Then, there exists a basis of generalized eigenvectors for $\mathbb{R}^n$. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is any basis of generalized eigenvectors for $\mathbb{R}^n$, the matrix

$$P = [\mathbf{v}_1, \ldots, \mathbf{v}_n] \quad \text{is invertible},$$

$$A = S + N, \quad \text{where} \quad P^{-1}SP = \text{diag}[\lambda_j],$$

the matrix $N = A - S$ is nilpotent of order $k \leq n$, and $SN = NS$.

Using the above theorem, we have the following result.

Theorem: The IVP $\mathbf{x}'(t) = A\mathbf{x}(t), \ \mathbf{x}(0) = \mathbf{x}_0$ has the solution

$$\mathbf{x}(t) = P \, \text{diag}[e^{\lambda_j t}] \, P^{-1} \left[I + Nt + \cdots + \frac{N^{k-1} t^{k-1}}{(k-1)!} \right] \mathbf{x}_0.$$

Note: If λ is an eigenvalue of A with multiplicity n, then

$$S = \text{diag}\,[\lambda]$$

with respect to the usual basis for $\mathbb{R}^n$ and $N = A - S$. The solution to IVP is

$$\mathbf{x}(t) = e^{\lambda t}\left[I + Nt + \cdots + \frac{N^{k-1}t^{k-1}}{(k-1)!}\right]\mathbf{x}_0.$$

Example: Solve $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, where $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$.

The eigenvalues are $\lambda_1 = \lambda_2 = 2$. Thus, $S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and

$N = A - S = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. $N^2 = 0$ and

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{2t}[I + Nt]\mathbf{x}_0 = e^{2t}\begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}\mathbf{x}_0.$$

Example: Solve $\mathbf{x}' = A\mathbf{x}, \;\; \mathbf{x}(0) = \mathbf{x}_0$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

One GEV corresponding to $\lambda = 2$ and independent of $\mathbf{v}_2$ is obtained by solving

$$(A - 2I)^2\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \mathbf{v} = 0.$$

Choose $\mathbf{v}_3 = (0,1,0)^T$. The matrix P is then given by

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Then, determine S as

$$S = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix},$$

$$N = A - S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \text{ and } N^2 = 0.$$

The solution is then given by

$$
\begin{aligned}
\mathbf{x}(t) &= P \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} P^{-1}[I + Nt]\mathbf{x}_0 \\
&= \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ -2e^t + (2-t)e^{2t} & te^{2t} & e^{2t} \end{bmatrix} \mathbf{x}_0 .
\end{aligned}
$$

Repeated complex eigenvalues

Theorem: Let A be a real $2n \times 2n$ matrix with complex eigenvalues

$$\lambda_j = a_j + ib_j \ \text{ and } \ \lambda_j = a_j - ib_j, \ j = 1, \ldots, n.$$

Then there exists generalized complex eigenvectors

$$\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j \ \text{ and } \ \bar{\mathbf{w}}_j = \mathbf{u}_j - i\mathbf{v}_j \ j = 1, \ldots, n$$

such that $\{\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^{2n}$. The matrix

$$P = [\mathbf{v}_1 \ \ \mathbf{u}_1 \ \ \cdots \ \ \mathbf{v}_n \ \ \mathbf{u}_n] \ \text{ is invertible,}$$

$$A = S + N, \ \text{where} \ \ P^{-1}SP = \mathrm{diag} \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}.$$

The matrix $N = A - S$ is nilpotent of order $k \leq 2n$, and $SN = NS$.

The solution of IVP $\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = P\mathrm{diag}\, e^{a_j t} \begin{bmatrix} \cos(b_j t) & -\sin(b_j t) \\ \sin(b_j t) & \cos(b_j t) \end{bmatrix} P^{-1} \left[I + \cdots + \frac{N^{k-1} t^{k-1}}{(k-1)!} \right] \mathbf{x}_0.$$

Example: Solve the IVP $\mathbf{x}'(t) = A\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ where

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix A has eigenvalues $\lambda = i$ and $\bar{\lambda} = -i$ of multiplicity 2. To find eigenvectors, we need to solve the equations

$$(A - \lambda I)\mathbf{w} = 0, \quad (A - \lambda I)^2 \mathbf{w} = 0.$$

Now $(A-\lambda I)\mathbf{w}=0 \equiv\ z_1=z_2=0$ and $z_3=iz_4$. Thus, we have one eigenvector $\mathbf{w}_1=(0,0,i,1)^T$. The equation

$$(A-\lambda I)^2\mathbf{w}=\begin{bmatrix} -2 & 2i & 0 & 0 \\ -2i & -2 & 0 & 0 \\ -2 & 0 & -2 & 2i \\ -4i & -2 & -2i & -2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}=0$$

$$\Rightarrow z_1=iz_2 \text{ and } z_3=iz_4-z_1.$$

We now choose the GEV $\mathbf{w}_2=(i,1,0,1)^T$. Then $\mathbf{u}_1=(0,0,0,1)^T$, $\mathbf{v}_1=(0,0,1,0)^T$, $\mathbf{u}_2=(0,1,0,1)^T$, and $\mathbf{v}_2=(1,0,0,0)^T$. The matrix P and P^{-1} are given by

$$P=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad P^{-1}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

$$S = P\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

$$N = A - S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } N^2 = \mathbf{0}.$$

The solution to the IVP is given by

$$\begin{aligned}
\mathbf{x}(t) &= P \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{bmatrix} P^{-1}[I + Nt]\mathbf{x}_0 \\
&= \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ -t\sin t & \sin t - t\cos t & \cos t & -\sin t \\ \sin t + t\cos t & -t\sin t & \sin t & \cos t \end{bmatrix} \mathbf{x}_0.
\end{aligned}$$

Remark. The case when A has both real and complex repeated eigenvalues can be treated by combining of the above two theorems.

Nonhomogeneous linear systems

Recall the GS to the nonhomogeneous system

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t), \qquad (*)$$

is given by

$$\mathbf{x}(t) = \Phi(t)\mathbf{c} + \mathbf{x}_p(t),$$

where $\Phi(t)$ is fundamental matrix for the corresponding homogeneous system and $\mathbf{x}_p(t)$ is a particular solution to $(*)$.

We know $\Phi(t) = e^{At}$ is a fundamental matrix satisfies $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $\Phi(0) = I$. Further, any fundamental matrix $\Phi(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is given by $\Phi(t) = e^{At}\mathbf{C}$ for some nonsingular matrix $\mathbf{C}$.

We shall now attempt to find a particular solution $\mathbf{x}_p(t)$ by variation of parameters.

Theorem: If $\Phi(t)$ is a fundamental matrix of $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ on I, then the function

$$\mathbf{x}_p(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)ds$$

is the unique solution to $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$ on I satisfying the initial condition $\mathbf{x}_p(t_0) = 0$.

Proof. Let $\Phi(t)$ be a fundamental matrix of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ on I. We seek a particular solution $\mathbf{x}_p$ of the form

$$\mathbf{x}_p(t) = \Phi(t)\mathbf{v}(t),$$

where $\mathbf{v}(t)$ is a vector function to be determined.

Now

$$\begin{aligned} \mathbf{x}_p'(t) &= \Phi'(t)\mathbf{v}(t) + \Phi(t)\mathbf{v}'(t) \\ &= A(t)\Phi(t)\mathbf{v}(t) + \mathbf{f}(t). \end{aligned}$$

Since $\Phi'(t) = A(t)\Phi(t)$, we obtain

$$\Phi(t)\mathbf{v}'(t) = \mathbf{f}(t) \Longrightarrow \mathbf{v}(t) = \int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)ds, \quad t_0, t \in I.$$

Therefore,

$$\mathbf{x}_p(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)ds.$$

Notice that

$$\begin{aligned} \mathbf{x}_p'(t) &= \Phi'(t) \int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)ds + \Phi(t)\Phi^{-1}(t)\mathbf{f}(t) \\ &= A(t)\Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\mathbf{f}(s)ds + \mathbf{f}(t) \\ &= A(t)\mathbf{x}_p(t) + \mathbf{f}(t), \quad \forall t \in I, \end{aligned}$$

and $\mathbf{x}_p(t_0) = \mathbf{0}$.

Theorem: If $\Phi(t)$ is any fundamental matrix of $\mathbf{x}'(t) = A\mathbf{x}(t)$ then the solution of the IVP

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

is unique and is given by

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \int_0^t \Phi(t)\Phi^{-1}(s)\mathbf{f}(s)ds.$$

Proof: Differentiating, we obtain

$$\begin{aligned} \mathbf{x}'(t) &= \Phi'(t)\Phi^{-1}(0)\mathbf{x}_0 + \Phi(t)\Phi^{-1}(t)\mathbf{f}(t) \\ &\quad + \int_0^t \Phi'(t)\Phi^{-1}(s)\mathbf{f}(s)ds. \end{aligned}$$

Since $\Phi'(t) = A\Phi(t)$, it follows that

$$\begin{aligned} \mathbf{x}'(t) &= A\left[\Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \int_0^t \Phi(t)\Phi^{-1}(s)\mathbf{f}(s)ds\right] + \mathbf{f}(t) \\ &= A\mathbf{x}(t) + \mathbf{f}(t), \quad t \in \mathbb{R}. \end{aligned}$$

Remark. With $\Phi(t) = e^{At}$, the solution of the IVP

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

takes the form

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At} \int_0^t e^{-As}\mathbf{f}(s)ds.$$

Example: Solve $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$, where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } \mathbf{f}(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}.$$

In this case

$$e^{At} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = \Phi(t).$$

$$e^{-At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = \Phi(-t).$$

The solution of the IVP is

$$\begin{aligned} \mathbf{x}(t) &= e^{At}\mathbf{x}_0 + e^{At}\int_0^t e^{-As}\mathbf{f}(s)ds \\ &= \Phi(t)\mathbf{x}_0 + \Phi(t)\int_0^t \begin{bmatrix} f(s)\sin(s) \\ f(s)\cos(s) \end{bmatrix} ds. \end{aligned}$$

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