PH102: Electrodynamics
Quiz I

Total marks: 10 Time: 50 mins.

Answer all the questions.

1. Consider a triangular lamina S with its vertices at $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 1, 1)$, oriented in the "positive" direction in the first octant.

(a) Determine the equation of the plane of which this lamina is a part (Hint: Take the equation of the plane in the form $z = ax + by + c$, for $a, b, c \in \mathbb{R}$).

(b) Calculate the flux of the <u>curl</u> of the vector field $\vec{F} = xyz(\hat{x} + \hat{y})$ through the lamina S. [$1 + 6 = 7$ **Marks**]

2. Find the value of $\nabla^2 \left(\frac{1}{r}\right)$, where r is the magnitude of the position vector. [2 **Mark**]

3. Calculate $\int_0^\infty dx f(x) \delta(x^2 - a^2)$, where a is a positive real number and $f(x)$ is a continuous function within the limits of integration. [1 **Mark**]

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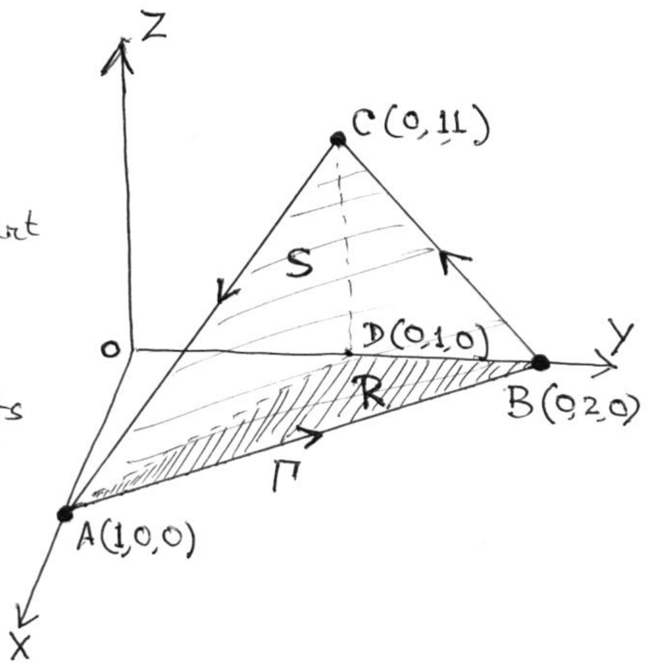
ANSWER-1

Q1. Given vector field $\vec{F} = xyz(\hat{x}+\hat{y})$



(a) Equation of the plane of which S is a part is of the form

$$z = ax + by + c$$

Since the plane passes through points (1,0,0), (0,2,0) and (0,1,1) it must satisfy

$$\left.\begin{aligned} 0 &= a+c \\ 0 &= 2b+c \\ 1 &= b+c \end{aligned}\right\} \quad \begin{aligned} &\text{Solving yields} \\ &a = -2 \\ &b = -1 \\ &c = 2 \end{aligned}$$

$\Rightarrow$ Equation of the plane is $\boxed{z = -2x - y + 2}$ [1 mark]

(b) **Info. for the grader**: This part may be done in two ways:

(1) Directly calculating the surface integral of $\vec{\nabla}\times\vec{F}$ over S with R as the projection plane.

(2) By using Stokes Law and determining the contour integral about the boundary Γ in the counter-clockwise direction.

METHOD 1 (tedious)

We need to calculate $I = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ xyz & xyz & 0 \end{vmatrix} = -(xy)\hat{x} + (xy)\hat{y} + z(y-x)\hat{z}$$ [1 Mark]

The outward normal (positive orientation) unit vector to S is

$$\hat{n} = \frac{\vec{\nabla}(z+2x+y-2)}{|\vec{\nabla}(z+2x+y-2)|} = \frac{2\hat{x}+\hat{y}+\hat{z}}{\sqrt{6}}$$

Also, the normal to the projection region R on the xy-plane is $\hat{z}$, which implies $|\hat{n}.\hat{z}| = \frac{1}{\sqrt{6}}$

$\therefore\ dS = \frac{dx\,dy}{|\hat{n}.\hat{z}|} = \frac{dx\,dy}{1/\sqrt{6}}$ [1 Mark]

$$\text{Thus, } I = \iint_S (\vec{\nabla}\times\vec{F}).\hat{n}\,dS$$

$$= \iint_R \left[(-xy)\hat{x} + (xy)\hat{y} + z(y-x)\hat{z}\right].\frac{2\hat{x}+\hat{y}+\hat{z}}{\sqrt{6}}\left[\frac{dx\,dy}{1/\sqrt{6}}\right]$$

$$= \iint_R \left[-2xy + xy + z(y-x)\right]dx\,dy$$

$$= \iint_R \left[-xy + (y-x)(-2x-y+2)\right]dx\,dy$$

where we substituted $z = -2x-y+2$, the equation of the plane. [½ Mark]

Limits in R

Note that the equation of the line AD is $y = 1-x$ and that of the line AB is $y = 2-2x = 2(1-x)$.

$$\boxed{\begin{array}{ll} \text{Limits of } y: & 1-x \le y \le 2(1-x) \\ \text{Limits of } x: & 0 \le x \le 1 \end{array}}$$

[½ Mark]

Therefore, $I = \int_0^1 dx \int_{1-x}^{2(1-x)} dy \left(2x^2 - 2x - y^2 + 2y - 2xy\right)$

$$= \int_0^1 dx \left[(2x^2 - 2x)y - \frac{1}{3}y^3 + y^2(1-x) \right]_{(1-x)}^{2(1-x)}$$

$$= \int_0^1 dx \left[-2x(1-x)^2 - \frac{7}{3}(1-x)^3 + 3(1-x)^3 \right]$$

$$= \int_0^1 dx \left[-2x(1-x)^2 + \frac{2}{3}(1-x)^3 \right]$$

$$= \int_0^1 dx \left[\frac{2}{3}(1-x) - 2x \right](1-x)^2$$

$$= \frac{1}{3}\int_0^1 dx\, (1-x)^2 (2-8x)$$

$$= \frac{2}{3}\int_0^1 dx\, (1-x)^2 (4-4x-3)$$

$$= \frac{8}{3}\int_0^1 dx\, (1-x)^3 - 2\int_0^1 dx\, (1-x)^2$$

$$= \left[-\frac{2}{3}(1-x)^4 + \frac{2}{3}(1-x)^3 \right]_0^1 = \frac{2}{3} - \frac{2}{3} = 0$$ [3 Marks]

METHOD 2 (Smart Method)

Using Stokes' Theorem

$$I = \iint_S (\vec{\nabla} \times \vec{F})\, d = \oint_\Gamma \vec{F} \cdot d\vec{l}$$ [$\frac{1}{2}$ mark]

Here we calculate the line integral

$$\oint_\Gamma \vec{F} \cdot d\vec{l} = \left[\int_{AB} + \int_{BC} + \int_{CA} \right] \vec{F} \cdot d\vec{l} = \left[\int_{AB} + \int_{BC} + \int_{CA} \right] xyz\,(dx + dy)$$

[$\frac{1}{2}$ Mark]

Since along AB, $z=0$ and along BC, $x=0$, in either case the line integral vanishes, i.e.,

$$\int_{AB} \vec{F}.d\vec{l} = 0 \ \& \ \int_{BC} \vec{F}.d\vec{l} = 0$$ [1 marks]

$$\text{So, } I = \oint_{\Gamma} \vec{F}.d\vec{l} = \int_{CA} \vec{F}.d\vec{l} = \int_{CA} xyz\,(dx+dy)$$

Now, the parametric equation of the line CA is

$$(x, y, z) = (t, 1-t, 1-t); \quad 0 \leqq t \leqq 1$$ [2 Marks]

$$\therefore I = \oint_{\Gamma} \vec{F}.d\vec{l} = \int_0^1 dt\, t(1-t)^2 (dt - dt) = 0$$ [2 marks]

$\Rightarrow$ The total flux of $\vec{\nabla} \times \vec{F}$ through S is zero.

Q2,

$$\nabla^2 \left(\frac{1}{r}\right) = \vec{\nabla} \cdot \vec{\nabla} \left(\frac{1}{r}\right)$$

Now, $\vec{r} = (x-x_0)\hat{x} + (y-y_0)\hat{y} + (z-z_0)\hat{z}$, is the position vector with respect to (x_0, y_0, z_0).

$$\therefore \quad r = |\vec{r}| = \left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{1/2}.$$

Calculate $\vec{\nabla}\left(\frac{1}{r}\right)$ first :

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\left[\hat{x}\frac{\partial r}{\partial x} + \hat{y}\frac{\partial r}{\partial y} + \hat{z}\frac{\partial r}{\partial z}\right]$$

$$= -\frac{1}{r^2}\left[\hat{x}\frac{\partial r}{\partial x} + \hat{y}\frac{\partial r}{\partial y} + \hat{z}\frac{\partial r}{\partial z}\right]$$

Now, $$\frac{\partial r}{\partial x} = \frac{1}{2}\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{-1/2} \cdot 2(x-x_0)$$

$$= \frac{x-x_0}{r}$$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y-y_0}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z-z_0}{r}$.

$$\therefore \quad \vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^3}\left[(x-x_0)\hat{x} + (y-y_0)\hat{y} + (z-z_0)\hat{z}\right]$$

$$= -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}. \longrightarrow \left[\begin{array}{c}\text{Marks} \\ 1+\end{array}\right]$$

Finally, $$\nabla^2\left(\frac{1}{r}\right) = -\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = -4\pi\,\delta^3(\vec{r}) \rightarrow \left[\begin{array}{c}\text{Marks} \\ 1+\end{array}\right]$$

Q3.

Use of the formular $\delta(y(x)) = \sum_i \dfrac{\delta(x - x_i)}{\left|\frac{dy}{dx}\right|_{x = x_i}}$

where x_i's are the roots of equⁿ. $y(x) = 0$,

yields :

$$\delta(x^2 - a^2) = \frac{1}{2|a|}\left(\delta(x+a) + \delta(x-a)\right)$$

$$= \frac{1}{2a}\left[\delta(x+a) + \delta(x-a)\right] \rightarrow \left[\begin{array}{c}\text{Marks} \\ \tfrac{1}{2}+\end{array}\right]$$

(As a is $+ve$)

$$\therefore \int_0^\infty f(x)\,\delta(x^2 - a^2)\,dx$$

$$= \frac{1}{2a}\underbrace{\int_0^\infty f(x)\,\delta(x+a)\,dx}_{\substack{x = -a \text{ is outside} \\ \text{the limit of integration.} \\ \text{(vanishes)}}} + \frac{1}{2a}\underbrace{\int_0^\infty f(x)\,\delta(x-a)\,dx}_{\substack{x = a \text{ is inside} \\ \text{the limit of} \\ \text{integration.}}}$$

$$= \frac{1}{2a} f(a) \longrightarrow \left[\begin{array}{c}\text{Marks} \\ \tfrac{1}{2}+\end{array}\right]$$