Lecture 32 Martingale 1

For a Markov process,

- Given $X(t_n)$, the random variable $X(t_{n+1})$ is conditionally independent of $X(t_1), X(t_2), \ldots, X(t_{n-1})$.
- The probability of the future state depends on the current state.

Another important class of RP is the Martingale process.

For a Martingale process,

- Given $X(t_n)$, the conditional expected value $X(t_{n+1})$ is $X(t_n)$ itself.
- The best prediction of the future value is the current value itself!

Conditional Expectation

The conditional expectation of Y given $X = x$ is defined by

$$E(Y/X = x) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} y f_{Y/X}(y/x)\,dy, & X \text{ and } Y \text{ are continuous} \\ \displaystyle\sum_{y \in R_Y} y p_{Y/X}(y/x), & X \text{ and } Y \text{ are discrete} \end{cases}$$

We can similarly define $E(X/Y = y)$

Conditional Expectation as a random variable

- Note that $E(Y/X=x)$ is a function of x.

$g(x)$

- Using this function, we may define a random $g(X$

$g(X) = EY/X$ RV X

Thus we may consider *EY/X* as a function of the random va

E(Y/X=x) as the value of *E(Y/X)* at *X=x*

- We can similarly define the conditional expectation

$E(X_{n+1}/X_n, X_{n-1}, X_{n-2})$ etc.

Total expectation theorem

$$EE(Y/X) = EY$$

x y

Proof

$$EE(Y/X) = \int_{-\infty}^{\infty} E(Y/X = x) f_X(x)dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{Y/X}(y/x)dy\, f_X(x)dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_X(x)\, f_{Y/X}(y/x)dydx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)dydx$$

$$= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy \rightarrow f_Y(y)$$

$$= \int_{-\infty}^{\infty} y f_Y(y)dy = EY$$

Theorem $EE(Y/Z,X)/X = EY/X$

$$E(Y/Z=z,X=x)=\int_{-\infty}^{\infty} y f_{Y/Z,X}(y)dy=\int_{-\infty}^{\infty} y\frac{f_{Y,Z,X}(y,z,x)}{f_{Z,X}(z,x)}dy$$

$$\therefore EE(Y/Z=z,X=x)/X=x)$$

$$=\int_{-\infty}^{\infty}(E(Y/Z=z,X=x)/X=x\ )f_{Z,X/X=x}(z,x)dz$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\frac{f_{Y,Z,X}(y,z,x)}{f_{Z,X}(z,x)}dy\quad f_{Z,X/X=x}(z,x)dz$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\frac{f_{Y,Z,X}(y,z,x)}{f_{Z,X}(z,x)}\frac{f_{Z,X}(z,x)}{f_X(x)}dy\quad dz$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\frac{f_{Y,Z,X}(y,z,x)}{f_X(x)}dy\quad dz=\int_{-\infty}^{\infty} y\int_{-\infty}^{\infty}\frac{f_{Y,Z,X}(y,z,x)}{f_X(x)}dz\quad dy$$

$$=\int_{-\infty}^{\infty} y\frac{f_{Y,X}(y,x)}{f_X(x)}\ dy=EY/X=x$$

$f(y|x)$ $= E\,y|x = x$

Conditional Expectation and prediction

Theorem: $E(Y - EY/X)^2 \leq E(Y - g(X))^2$

$g_1(x)$

$$E(Y - g(X))^2 = E\left(Y - g_1(X) + g_1(X) - g(X)\right)^2$$

$$= E(Y - g_1(X))^2 + E(g_1(X) - g(X))^2$$

$$+ 2E(Y - g_1(X))(g_1(X) - g(X))$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y - g_1(x))(g_1(x) - g(x)) f(y,x)\,dy\,dx$$

$f(x) f(y|x)$

$$= \int_{-\infty}^{\infty} (g_1(x) - g(x)) f(x) \int_{-\infty}^{\infty} (y - g_1(x)) f(y|x)\,dy$$

$EY|X=x - EY|X=x$

$$E(Y - g(X))^2 \geq E(Y - g_1(X))^2$$

$E(Y - EY|X)^2 \leq E(Y - g(Y))^2$

MARTINGALE

- A martingale is a random process in which the best estimate of future value conditioned on past including present values is equal to the present value itself.
- Is an abstract model of a fair game: *the expected fortune after a bet should be equal to the present fortune itself*.

Widely used in engineering and stochastic finance

Definition A discrete-time random process $\{X_n, n \geq 0\}$ is called martingale process if for all $n \geq 1$,

(i) $E|X_n| < \infty$, and

(ii) $E(X_{n+1} / X_0, X_1, ..., X_n) = X_n$

If the equality sign in(ii) aboe is replaced by $\leq$, then $\{X_n, n \geq 0\}$ is called a *supermartingale* and if it is replaced by $\geq$, then $\{X_n, n \geq 0\}$ is a *submartingale*.

Example 1: Consider the *sum process* $\{X_n\}_{n=0}^{\infty}$ given by

$$X_n = \sum_{i=1}^{n} Z_i, \quad n \geq 1$$

where $\{Z_n\}$ is a sequence of *i.i.d.* random variables with $EZ_n = 0$ and $X_0 = 0$. Then $\{X_n\}_{n=0}^{\infty}$ is a martingale.

$E|X_n| < \infty$

Proof: We have

$$X_{n+1} = \sum_{i=1}^{n+1} Z_i = X[n] \cancel{+ Z[n+1]}$$

$X_n + Z_{n+1}$

$$\therefore E(X_{n+1} / X_0, X_1, \ldots, X_n) = E(X_n + Z_{n+1}) / X_0, X_1, \ldots, X_n$$

$$= EX_n / X_0, X_1, \ldots, X_n + EZ_{n+1} / X_0, X_1, \ldots, X_n$$

$= EZ_{n+1} - 0$

$$= X_n + EZ_{n+1}$$

$$= X_n$$

Example 2 Consider the symmetrical random walk(RW) *process* $\{X_n\}_{n=0}^{\infty}$ given by

$$X_n = \sum_{i=1}^{n} Z_i = X_{n-1} + Z_n$$

where $n \geq 1$, $\{Z_n\}$ is a sequence of i.i.d. random variables with

$$Z_1 = \begin{cases} 1 \ with \ \ probability \ \dfrac{1}{2} \\ -1 \ with \ \ probability \dfrac{1}{2} \end{cases}$$ and $X_0 = 0$. Then $\{X_n\}$ is a Martingale process.

Example 3: Gambler's ruin problem: A player has an initial capital of X_0 amount of money. He will gain 1 unit of money with probability p and lose 1 unit of money with probability $1-p$. Let the gain of each stage be denoted as Z_i and the cumulative gain at n^{th} stage be $X_n = \sum_{i=1}^{n} Z_i,\ \ n \geq 1$. Then $\{X_n\}_{n=0}^{\infty}$ is a martingale if $p = 1/2$.

Proof: Here

$$EZ[n] = 1 \times \frac{1}{2} - 1 \times \frac{1}{2}$$
$$= 0$$

Thus the problem reduces to the problem in Example 1.

If $p \neq 1/2$, then $EZ[n] = 2p - 1 \neq 0$.

$\to X_{n+1}$ Z_{n+1}

$$\therefore E(X_{n+1} / X_0, X_1, \ldots, X_n) = X[n] + EZ[n+1]$$
$$= X[n] + 2p - 1$$

X_n

$1 \times p + (-1) \times (1-p)$

$= 2p - 1$

Thus $\{X_n, n \geq 0\}$ is a submartingale if $p \geq 1/2$.

Example 3: Consider the *product process* $\{X_n\}_{n=0}^{\infty}$ given by

$$X_n = \prod_{i=0}^{n} Z_i, \quad n \geq 1$$

Where $\{Z_n\}$ is a sequence of *i.i.d.* random variables with $EZ_n = 1$ and $X_0 = 1$. Then $\{X_n\}_{n=0}^{\infty}$ is a martingale.

$\prod_{i=0}^{n} Z_i \quad X_n \qquad Z_{n+1}$

Proof: We have

$$X_{n+1} = \prod_{i=0}^{n+1} Z_i = X_n Z_{n+1}$$

$$\therefore E(X_{n+1} / X_0, X_1, \ldots, X_n) = E(X_n Z_{n+1}) / X_0, X_1, \ldots, X_n$$

$$= EX_n / X_0, X_1, \ldots, X_n \times \underline{EZ_{n+1} / X_0, X_1, \ldots, X_n}$$

$$= X_n \times EZ_{n+1}$$

$$= X_n$$

$\{X_n\}$ is a martingale process.