MA 102 (Mathematics II)
IIT Guwahati

Tutorial Sheet No. 5 | Linear Algebra | February 21, 2019

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1. True or False? Give justifications.
   (a) There exist distinct linear transformations $S, T : \mathbb{V} \to \mathbb{W}$ such that $ker(S) = ker(T)$ and $range(S) = range(T)$.
   (b) There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that none of T, T^2, T^3 is the identity transformation but $T^4 = I$ (identity transformation).
   (c) If $T : \mathbb{V} \to \mathbb{W}$ is a linear transformation then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is LI in $\mathbb{V}$ if and only if $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is LI in $\mathbb{W}$.
2. Determine a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ such that $range(T) = \{[x, y, z]^\top : x + 2y + z = 0\}$. If possible give two more such linear transformations with the same range.
3. If possible, find linear transformations $S : \mathbb{R}^2 \to \mathbb{R}_2[x]$ and $T : \mathbb{R}_2[x] \to \mathbb{R}^2$ such that
   (a) $S \circ T = I$.
   (b) $T \circ S = I$.
   (c) $range(T \circ S)$ is a line.
   (d) Neither S not T is the zero transformation but $S \circ T = \mathbf{0}$.
4. Let $\mathbb{V}, \mathbb{W}$ be finite dimensional vector spaces with ordered bases B and C, respectively. Let $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Show that $\text{rank}(T) = \text{rank}([T]_{C\leftarrow B})$ and $\text{nullity}(T) = \text{nullity}([T]_{C\leftarrow B})$.
5. True or False? Give justifications.
   (a) A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T([x, y]^\top) = [x, y]^\top$ for $x \neq 0$ and $T([0, y]^\top) = [0, 0]^\top$ satisfies $T(c[x, y]^\top) = cT([x, y]^\top)$ but is not a linear transformation.
   (b) Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. Then for any $\mathbf{v}_1, \mathbf{v}_2$ in $\mathbb{V}$ and $\mathbf{w}_1, \mathbf{w}_2$ in $\mathbb{W}$, there exists a linear transformation $T : \mathbb{V} \to \mathbb{W}$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$.
   (c) Let $\mathbb{V}$ and $\mathbb{W}$ be n-dimensional vector spaces and $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ be invertible. Then there exist ordered bases B and C of $\mathbb{V}$ and $\mathbb{W}$, respectively, such that $[T]_{C\leftarrow B} = I_n$.
6. Determine a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ such that $Ker(T) = \{[x, y]^\top : 2x + y = 0\}$.
7. Let $\mathbb{V}$ be a vector space and $\dim(\mathbb{V}) = n$. Show that there exists an LT $T : \mathbb{V} \to \mathbb{V}$ such that $T^j \neq \mathbf{0}$ for $j = 1, 2, \ldots, n-1$ but $T^n = \mathbf{0}$.
8. Let $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ be defined as $T(A) := A - A^\top$ for all $A \in \mathcal{M}_2(\mathbb{R})$. Find a basis of $\text{range}(T)$ and $\ker(T)$.
9. Find the change of basis matrices $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$ for the bases $B := [1, x, x^2]$ and $C := [1+x, x+x^2, 1+x^2]$ of $\mathbb{R}_2[x]$. Consider $p(x) := 1 + 2x - x^2$. Find $[p]_C$ using the change of basis matrix.
10. Let $\mathbb{V}$ be an n-dimensional vector space with an ordered basis $B := [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. Let $A \in \mathcal{M}_n(\mathbb{F})$ be an invertible matrix. Consider $C := BA$. Show that C is an ordered basis of $\mathbb{V}$ and that the change of basis matrix is given by $P_{B\leftarrow C} = A$.

11. True or False? Give justifications.

    (a) Let $\mathbf{x}$ be a nonzero vector. Then $\mathbf{x}$ is an eigenvector of A corresponding to an eigenvalue λ if and only if $\mathbf{x}$ is an eigenvector of A^2 corresponding to the eigenvalue λ^2.

    (b) Let A be a nonzero matrix such that $A^{31} = \mathbf{0}$. Then A has all eigenvalues equal to 0 and A is not diagonalizable.

    (c) If A is diagonalizable then $rank(A - cI) = rank(A - cI)^2$ for all $c \in \mathbb{C}$.

12. Let $\mathbb{V}, \mathbb{W}$ be n dimensional vector spaces with ordered bases B and C, respectively, and $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Show that T is invertible if and only if the matrix $[T]_{C \leftarrow B}$ is invertible. In such a case, show that

$$([T]_{C \leftarrow B})^{-1} = [T^{-1}]_{B \leftarrow C}.$$

13. Consider $\mathbb{U} := \mathbb{R}^3$, $\mathbb{V} := \mathcal{M}_2(\mathbb{R})$ and $\mathbb{W} := \mathbb{R}_2[x]$ with ordered bases $B := [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$, $C := \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]$ and $D := [1, x, x^2]$, respectively. Let $T : \mathbb{U} \to \mathbb{V}$ be given by $T[x, y, z]^\top = \begin{bmatrix} 0 & x \\ y & y + z \end{bmatrix}$ and $S : \mathbb{V} \to \mathbb{W}$ be given by $S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + (b + c)x + dx^2$. Then determine $[S \circ T]_{D \leftarrow B}$, $[S]_{D \leftarrow C}$ and $[T]_{C \leftarrow B}$ and verify that $[S \circ T]_{D \leftarrow B} = [S]_{D \leftarrow C}[T]_{C \leftarrow B}$.

14. For each LT T on $\mathbb{V}$, find the eigenvalues of T and an ordered basis B of $\mathbb{V}$ such that $[T]_B$ is a diagonal matrix.

    (a) $\mathbb{V} := \mathbb{R}_3[x]$ and $(Tp)(x) := xp'(x) + p''(x) - p(2)$.

    (b) $\mathbb{V} := \mathcal{M}_2(\mathbb{R})$ and $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) := \begin{bmatrix} d & b \\ c & a \end{bmatrix}$.

    (c) $\mathbb{V} := \mathcal{M}_2(\mathbb{R})$ and $T(A) := A^\top + 2\,\mathrm{Trace}(A) I_2$.

15. Let $T : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ be given by $(Tp)(x) := p(1) + p'(0)x + (p'(0) + p''(0))x^2$. Find eigenvalues and eigenvectors of T. Also, find an ordered basis B, if it exists, of $\mathbb{R}_2[x]$ such that $[T]_B$ is a diagonal matrix.

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