MA 102 (Mathematics II)
IIT Guwahati

Tutorial Sheet No. 4 | Linear Algebra | February 7, 2019

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1. Determine whether the following are vector spaces (under the usual operations of addition and scalar multiplication of functions) over $\mathbb{R}$.
   (a) $\{f : (a,b) \to \mathbb{R} \mid f(c) = 0\}$, where $c \in (a,b)$.
   (b) $\{f : (a,b) \to \mathbb{R} \mid f(c) \neq 0$ for any $c \in (a,b)\}$.
   (c) $\{f : (a,b) \to \mathbb{R} \mid f$ is continuous in $(a,b)\}$.
   (d) $\{f : (a,b) \to \mathbb{R} \mid f$ is continuous everywhere except at c, where $c \in (a,b)$. $\}$
   (e) $\{f : (a,b) \to \mathbb{R} \mid f$ is a one-one function $\}$.
   (f) $\{f : (a,b) \to \mathbb{R} \mid$ range of f is a finite set $\}$.
   (g) $\{f : (a,b) \to \mathbb{R} \mid f' = 0\}$.
   (h) $\{f : (a,b) \to \mathbb{R} \mid f'' - 3f' + 7f = 0\}$.

2. (a) If $\mathbb{U} := \left\{ \begin{bmatrix} a & b \\ -b & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$, then find $\mathbb{V}$ such that $\mathbb{U} \oplus \mathbb{V} = \mathcal{M}_2(R)$.

   (b) Let $\mathcal{C}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f$ is continuous $\}$ and $\mathbb{U} = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is continuous and $f(-x) = f(x)$ for all $x \in \mathbb{R}\}$. Then find $\mathbb{V}$ such that $\mathbb{U} \oplus \mathbb{V} = \mathcal{C}(\mathbb{R})$.

3. (a) Let $\mathbb{V}$ is a vector space over $\mathbb{R}$ and let $A := [a_{ij}] \in \mathcal{M}_k(\mathbb{R})$ be invertible. Show that $\mathbf{u}_1, \cdots, \mathbf{u}_k \in \mathbb{V}$ are linearly independent if and only if $\sum_{i=1}^{k} a_{i1}\mathbf{u}_i, \cdots, \sum_{i=1}^{k} a_{ik}\mathbf{u}_i$ are linearly independent.

   (b) Show that $\{\mathbf{u}, \mathbf{v}\} \subseteq \mathbb{V}$ is linearly independent iff $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

4. Let $\mathbb{W}, \mathbb{U}$ be subspaces of $\mathbb{V}$. Show that $\mathbb{W} \cup \mathbb{U}$ is a subspace iff either $\mathbb{W} \subseteq \mathbb{U}$ or $\mathbb{U} \subseteq \mathbb{W}$. What about union of three subspaces?

5. Let $\mathbb{V}$ be a finite dimensional vector space. Let U and W be subspaces of $\mathbb{V}$. Show that $\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap W)$.

6. If $\mathcal{W}_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 = 2x_3 + x_2 \right\}$ and $\mathcal{W}_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid 2x_1 = 2x_3 + 3x_2 \right\}$ then determine $\mathcal{W}_1 \cap \mathcal{W}_2$ and $\mathcal{W}_1 + \mathcal{W}_2$?

7. Extend
$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$
to a basis of $\mathbb{R}^6$ using GJE.

8. Find a basis for each of the following subspaces.
   (a) $U := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 2a - c - d = 0, a + 3b = 0, a, b, c, d \in \mathbb{R} \right\}$.

(b) $V := \{p(x) \in \mathbb{R}[x] : \deg(p(x)) \leq 4 \text{ and } p(-2) = 0\}$.

9. Let $\mathbb{V}$ be a vector space and S be a subset of $\mathbb{V}$. Let $L = \{\mathbb{U} | \ \mathbb{U} \preccurlyeq \mathbb{V},\ S \subseteq \mathbb{U}\}$. Then show that $\operatorname{span}(S) = \bigcap_{\mathbb{U} \in L} \mathbb{U}$ = the smallest subspace containing S.

10. Consider $\mathbb{W} = \{v \in \mathbb{R}^6 \,|\, v_1 + v_2 + v_3 = 0,\ v_2 + v_3 + v_4 = 0,\ v_5 + v_6 = 0\}$. Find a basis of $\mathbb{W}$ and extend it to a basis of $\mathbb{R}^6$.

11. Consider $S := \{1 + x, (1 + x)^2, 1 - x^2, 10\} \subseteq \mathbb{R}[x]$. Describe $\operatorname{span}(S)$ and find its dimension.

12. Find a basis for $\operatorname{span}(1 - 2x, 2x - x^2, 1 - x^2, 1 + x^2)$ in $\mathbb{R}_2[x]$.

13. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of a vector space $\mathbb{V}$. Show that $\{\mathbf{v}_1, \mathbf{v}_1+\mathbf{v}_2, \mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3, \ldots, \mathbf{v}_1+\cdots+\mathbf{v}_n\}$ is also a basis of $\mathbb{V}$.

14. Determine whether the set $\mathcal{B}$ given below is a basis for $\mathcal{M}_2(\mathbb{R})$.

 (a) $\mathcal{B} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$.

 (b) $\mathcal{B} := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$.

15. Find a basis for each of the following subspaces.

 a) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a - d = 0, a, b, c, d \in \mathbb{R} \right\}$.

 b) $\{a + bx + cx^3 : a, b, c \in \mathbb{R}, a - 2b + c = 0\}$.

 c) $\{A \in \mathcal{M}_{m \times n}(\mathbb{R}) : \text{row sums of } A \text{ are zero}\}$.

16. Let $U := \{A \in \mathcal{M}_3(\mathbb{R}) : A^\top = A \text{ and } \operatorname{Tr}(A) = 0\}$. Find two bases of U and extend these bases to bases of the real symmetric matrices of size 3×3.