Linear Algebra

Department of Mathematics
Indian Institute of Technology Guwahati

January - May 2019

MA 102 (RA, RKS, MGPP, KVK)

Eigenvalues and eigenvectors

Topics:

- Determinant of Matrices
- Eigenvalues and Eigenvectors

The determinant det(A) of 2×2 matrix A is defined by

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Proof: Use induction on the size of the matrix.

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Let A be a square matrix.

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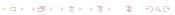
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Henceforth, we consider $\mathbb{F} = \mathbb{C}$.

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Algebraic and geometric multiplicities

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Exercise

- Let A be a matrix with eigenvalue λ and corresponding eigenvector \mathbf{x} .
 - For any positive integer n, show that λ^n is an eigenvalue of A^n with corresponding eigenvector \mathbf{x} .
 - If A is invertible, then show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector \mathbf{x} .
 - If A is invertible then show that for any integer n, λ^{-n} is an eigenvalue of A^{-n} with corresponding eigenvector \mathbf{x} .
- Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ be eigenvectors of a matrix A with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$, respectively. Let $\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_m \mathbf{v}_m$. Show that for any positive integer k,

$$A^k \mathbf{x} = c_1 \lambda_1^k \mathbf{v}_1 + c_2 \lambda_2^k \mathbf{v}_2 + \ldots + c_m \lambda_m^k \mathbf{v}_m.$$



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- Rank each page in the database, so that when a user does a search and the subset of pages in the database with the desired information has been found, the more important pages can be presented first.

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Eigenvalue problem
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By Perron-Frobenius theorem, there is a unique vector $\mathbf{v} = [x_1, \dots, x_n]^\top$ with $x_j > 0$ for j = 1 : n such that

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The rank of the web page P_j is given by the j-th component x_j of the eigenvector \mathbf{v} of the Google matrix G. Google sets m = 0.15.

References

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