PH 102 : Electromagnetism
Tutorial 10
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1. Two concentric metal spherical shells, of radius a and b, respectively, are separated by weakly conducting material of conductivity σ as shown in part (a) of figure 1.
(a) If they are maintained at a potential difference V, what current flows from one to the other?
(b) What is the resistance between the shells?
(c) Notice that if $b \gg a$ the outer radius b is irrelevant. How do you account for that? Exploit this observation to determine the current flowing between two metal spheres, each of radius a, immersed deep in the sea and held quite far apart (shown in part (b) of figure 1), if the potential difference between them is V. (This arrangement can be used to measure the conductivity of sea water.)

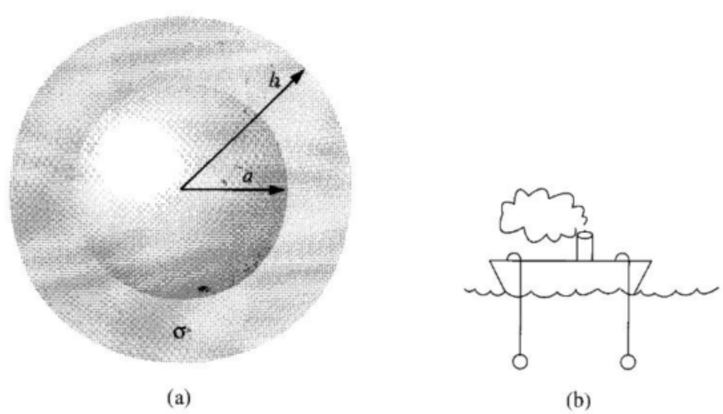

Figure 1: Figure for problem 1.

Solution:
(a) If Q is the charge on the inner shell, the electric field in the space between the two shells is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}.$$

The potential difference between the two shells can, therefore, be found as

$$V_a - V_b = -\int_b^a \vec{E}\cdot d\vec{r} = -\frac{1}{4\pi\epsilon_0}Q\int_b^a \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\implies Q = \frac{4\pi\epsilon_0(V_a - V_b)}{(1/a - 1/b)}.$$

The current that flows from one shell to the other is

$$I = \int \vec{J}\cdot d\vec{a} = \sigma \int \vec{E}\cdot d\vec{a} = \sigma \frac{Q}{\epsilon_0} = \frac{\sigma}{\epsilon} \frac{4\pi\epsilon_0 (V_a - V_b)}{(1/a - 1/b)} = 4\pi\sigma \frac{(V_a - V_b)}{(1/a - 1/b)} = 4\pi\sigma \frac{V}{(1/a - 1/b)}.$$

(b) The resistance between the shells is

$$R = \frac{V_a - V_b}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right).$$

(c) For large $b (b \gg a)$, the second term $1/b$ is negligible compared to $1/a$ and hence the outer radius become irrelevant. In this case, the resistance is $R = 1/(4\pi\sigma a)$. Since all the resistance is confined to a region around the inner sphere only, the resistance for the two submerged spheres (in part (b) of figure 1) is $R \approx 1/(4\pi\sigma a) + 1/(4\pi\sigma a) = 1/(2\pi\sigma a)$. Since the potential difference between them is V, the current flowing between them is

$$I = \frac{V}{R} = 2\pi\sigma a V.$$

2. A capacitor C is charged upto a potential V and connected to an inductor L, as shown schematically in figure 2. At time $t = 0$ the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor is included in series with C and L?

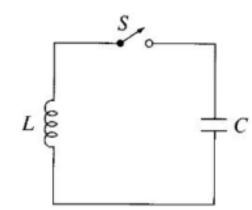

Figure 2: Figure for problem 5.

Solution:
The emf in the circuit is $\mathcal{E} = -L\frac{dI}{dt} = Q/C$ where Q is the charge on the capacitor. Using $I = dQ/dt$, this can be written as

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q = -\omega^2 Q, \quad \omega = \frac{1}{\sqrt{LC}}$$

The general solution of this

$$Q(t) = A\cos\omega t + B\sin\omega t$$

Using the initial condition: $t = 0, Q = CV$, we get $A = CV$.The current in the circuit is

$$I(t) = \frac{dQ}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

Using the initial condition: $t = 0, I = 0$, we get $B = 0$. Therefore,

$$I(t) = -A\omega \sin \omega t = -CV\omega \sin \omega t = -V\sqrt{\frac{C}{L}} \sin\left(\frac{t}{\sqrt{LC}}\right).$$

If a resistor is included in series with C and L, the equation will be

$$-L\frac{dI}{dt} = \frac{Q}{C} + IR \implies L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0.$$

This is similar to the equation of a damped harmonic oscillator. Assuming $Q = e^{\alpha t}$ and using it in the differential equation, we get

$$L\alpha^2 + R\alpha + \frac{1}{C} = 0 \implies \alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0$$

$$\implies \alpha = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4\frac{1}{LC}}}{2}$$

$$\implies -\frac{R}{2L} \pm i\omega, \quad \omega^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

Thus, the solution can be written as

$$Q(t) = e^{-\frac{R}{2L}t}(A\cos \omega t + B\sin \omega t)$$

Since the amplitude $e^{-\frac{R}{2L}t}$ gets damped with time, it is known as the damped harmonic oscillator.

3. (a) Use the analogy between Faraday's law and Ampere's law, together with the Biot-Savart law, to show that

$$\vec{E}(\vec{r}, t) = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{\vec{B}(\vec{r'}, t) \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

for Faraday-induced electric fields.
(b) Show that $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$, where $\vec{A}$ is the vector potential. Check this result by taking the curl of both sides.
(c) A spherical shell of radius R carries a uniform charge σ. It spins about a fixed axis at an angular velocity $\omega(t)$ that changes slowly with time. Find the electric field inside and outside the sphere. [*Hint*: There are two contributions here: the Coulomb field due to the charge, and the Faraday field due to the changing $\vec{B}$.]

Solution:
(a) In magnetostatics, we have

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \implies \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'}) \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'.$$

Therefore, for electric fields that are generated only due to the change in magnetic field that is,

$$\vec{\nabla} \cdot \vec{E} = 0, \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

we can write by substituting $\vec{J} \to -\frac{1}{\mu_0}\frac{\partial \vec{B}}{\partial t}$ in the expression for magnetic field above:

$$\vec{E}(\vec{r}, t) = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{\vec{B}(\vec{r'}, t) \times \hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} d\tau'.$$

(b) Following the same steps as above for $\vec{\nabla} \cdot \vec{A} = 0, \vec{\nabla} \times \vec{A} = \vec{B}$, we can show that

$$\vec{A}(\vec{r}, t) = \frac{1}{4\pi}\int \frac{\vec{B}(\vec{r'}, t) \times \hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} d\tau'$$

Comparing the expressions for $\vec{E}(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$, we get $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$. Taking curl on both sides of this gives Faraday's law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

(c) The Coulomb field is zero inside the sphere and

$$\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{\sigma 4\pi R^2}{r^2}\hat{r} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{r}$$

outside the sphere. As shown in part (b), the Faraday field is $-\frac{\partial \vec{A}}{\partial t}$. The vector potential in the quasi-static approximation (as discussed in class earlier) is given by

$$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_0 R\sigma}{3}(\vec{\omega} \times \vec{r}) = \frac{\mu_0 R\omega\sigma}{3} r\sin\theta\hat{\phi} & \text{for } r \leq R \\ \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega} \times \vec{r}) = \frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi} & \text{for } r \geq R. \end{cases}$$

where ω is a function of time now. Denoting $d\omega/dt \equiv \dot{\omega}$, the electric field can be written as

$$\vec{E}(r, \theta, \phi, t) = \begin{cases} -\frac{\mu_0 R\dot{\omega}\sigma}{3} r\sin\theta\hat{\phi} & \text{for } r \leq R \\ \frac{\sigma R^2}{\epsilon_0 r^2}\hat{r} - \frac{\mu_0 R^4\dot{\omega}\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi} & \text{for } r \geq R. \end{cases}$$

4. A rectangular closed loop of mass m and self inductance L is dropped with initial velocity $v_0\hat{i}_x$ between the pole faces of a magnet that has a concentrated uniform magnetic field $B_0\hat{i}_z$. Here $\hat{i}_n$ denotes unit vector along the n-axis, $(n \equiv x, y, z)$. Neglect the presence of gravity. The schematic diagram for the same is shown in figure 3 where s denotes the thickness of the field region whereas N, S denote north and south poles of the magnet respectively.
(a) What is the imposed flux through the loop as a function of the loop's position x $(0 < x < s)$ within the magnet?
(b) If the wire has conductivity σ and cross-sectional area A, what equation relates the induced current i in the loop and the loop's velocity?
(c) What is the force on the loop in terms of current i?

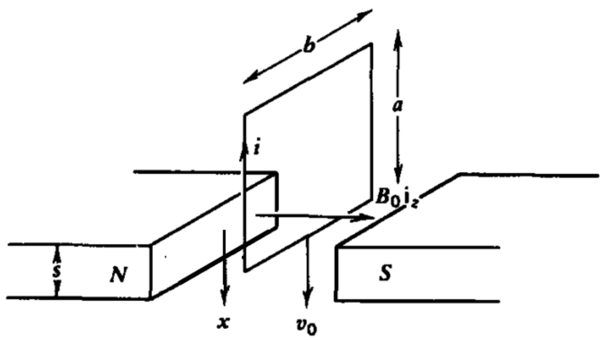

Figure 3: Figure for problem 4.

(d) Write down the second order differential equation for loop's velocity $v(t)$ in terms of $\omega_0^2 = \frac{B_0^2 b^2}{mL}, \alpha = \frac{2(a+b)}{\sigma AL}$.
(e) Find the loop's velocity at time $t = \frac{2\pi}{\beta}$ where $\beta = \sqrt{\omega_0^2 - (\alpha/2)^2}$ with ω_0, α are same as defined above. (*Hint*: This can be found by solving the second order differential equation for $v(t)$ in a way similar to solving for charge $q(t)$ in an LCR circuit without any emf source.)
(f) Find the induced current in the loop at time $t = \frac{2\pi}{\beta}$ where β is same as defined above.
(g) For $\sigma \to \infty$, what minimum initial velocity is necessary for the loop to pass through the magnetic field?

Solution:
(a) The flux is $\Phi = B_0 x b, 0 < x < s$.
(b) The induced emf in the loop of self inductance L is $-Ldi/dt$, and the potential drop across its resistance is $-iR$. The motional emf in the loop due to its motion is $d\Phi/dt$. Using these, we can write the emf equation as

$$\frac{d\Phi}{dt} - L\frac{di}{dt} - iR = 0 \implies L\frac{di}{dt} + iR = B_0 b\frac{dx}{dt} = B_0 bv$$

where, in the last step, we have used the result of part (a). Also, the resistance R of the loop can be found from its conductivity and dimensions as $R = 2(a+b)/(\sigma A)$.
(c) The force on the loop is $F = -iB_0 b\hat{x}$.
(d) The equation of motion follows from Newton's second law:

$$m\frac{dv}{dt} = F = -iB_0 b \implies i = -\frac{m}{B_0 b}\frac{dv}{dt}$$

Differentiating with respect to time:

$$m\frac{d^2 v}{dt^2} = -B_0 b\frac{di}{dt}$$

Using the results from part (b), we can substitute for di/dt and get

$$m\frac{d^2v}{dt^2} = -\frac{1}{L}B_0b(B_0bv - iR) = -\frac{1}{L}B_0b(B_0bv + -\frac{mR}{B_0b}\frac{dv}{dt})$$

$$\implies -\frac{mL}{B_0b}\frac{d^2v}{dt^2} - \frac{mR}{B_0b}\frac{dv}{dt} = B_0bv$$

In terms of ω_0^2, α, the above equation can simply be written as

$$\frac{d^2v}{dt^2} + \alpha\frac{dv}{dt} + \omega_0^2 v = 0$$

(e) Assuming the solution of the second order differential equation in part (d) as $v = Ae^{\mu t}$, we can substitute it in the equation to get $\mu^2 + \alpha\mu + \omega_0^2 = 0$ whose solutions are

$$\mu = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \omega_0^2} = -\frac{\alpha}{2} \pm i\beta,\ \beta = \sqrt{\omega_0^2 - \left(\frac{\alpha}{2}\right)^2}$$

Thus, the solution for $v(t)$ can be written as

$$v(t) = (A_1 \sin\beta t + A_2\cos\beta t)e^{-\frac{\alpha}{2}t}$$

The constants of integration can be found by using the initial conditions. Initial velocity was $v(t=0) = v_0$ and initial current was $i(t=0) = 0$. Using the first boundary condition, we can find $A_2 = v_0$. Now, before using the second boundary condition, we find current as a function of time as

$$i(t) = -\frac{m}{B_0b}\left[(A_1\beta\cos\beta t - A_2\beta\sin\beta t)e^{-\frac{\alpha}{2}t} - (A_1\sin\beta t + A_2\cos\beta t)e^{-\frac{\alpha}{2}t}\frac{\alpha}{2}\right]$$

which at $t = 0$ gives $i(t=0) = 0$ if

$$A_1\beta = A_2\frac{\alpha}{2} \implies A_1 = \frac{\alpha}{2\beta}v_0$$

Using these, the expression for velocity can be written as

$$v(t) = v_0\left(\frac{\alpha}{2\beta}\sin\beta t + \cos\beta t\right)e^{-\frac{\alpha}{2}t}$$

Now, at time $t = \frac{2\pi}{\beta} = t_0$, the velocity of the loop becomes

$$v(t_0) = -v_0 e^{-\frac{\pi\alpha}{\beta}}$$

(f) Using the integration constants evaluated above, the expression for current becomes

$$i(t) = \frac{m}{B_0b}v_0\left[\frac{\alpha^2}{4\beta} + \beta\right]\sin\beta t e^{-\frac{\alpha}{2}t} = \frac{mv_0}{B_0b\beta}\omega_0^2\sin\beta t e^{-\frac{\alpha}{2}t}$$

which at $t = \frac{2\pi}{\beta} = t_0$ will become $i(t_0) = 0$.
(g) For $\sigma \to \infty$, we have $R \to 0, \alpha \to 0, \beta = \omega_0$. The velocity is $v(t) = v_0 \cos\omega_0 t$. Current is given by

$$i(t) = \frac{mv_0\omega_0}{B_0 b}\sin\omega_0 t$$

The distance the loop passes through in time t is given by

$$x(t) = \int_0^t v(t)dt = \frac{v_0}{\omega_0}\sin\omega_0 t$$

For the loop to pass through the magnetic field completely

$$x_{\max} > s \implies \frac{v_0}{\omega_0} > s \implies v_0 > s\omega_0 = \frac{B_0 bs}{\sqrt{mL}}$$

.

5. Consider a solid cylindrical wire of radius R_1 surrounded by a thin long cylindrical coaxial shell of radius R_2. In the inner cylindrical solid wire, current I is distributed uniformly. In the outer cylindrical shell the same current flows, but in the opposite direction. Find the
(a) Magnetic energy stored in the cable per unit length of the cable.
(b) Self inductance per unit length of the cable.

Solution:
(a) To find the magnetic energy stored, one has to find the magnetic field in all the regions, using Ampere's law:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

In the inner solid cylinder of radius R_1, the current I is distributed uniformly and hence for a loop of radius $s < R_1$, the enclosed current is $I_{\text{enc}} = Is^2/R_1^2$. The enclosed current in the intermediate region between two wires $R_1 < s < R_2$ is same as I. The enclosed current for $s > R_2$ on the other hand, is zero as equal and opposite currents flow in the two wires. Using these, we can find the magnetic field as

$$\vec{B} = \frac{\mu_0 Is}{2\pi R_1^2}\hat{\phi},\ s < R_1$$

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi},\ R_1 < s < R_2$$
$$\vec{B} = 0,\ s > R_2$$

Thus the stored magnetic energy is

$$E_{\text{mag}} = \frac{1}{2\mu_0}\int B^2 d\tau = \frac{1}{2\mu_0}\left[\int_0^{R_1}\left(\frac{\mu_0 Is}{2\pi R_1^2}\right)^2 sdsd\phi dz + \int_{R_1}^{R_2}\left(\frac{\mu_0 I}{2\pi s}\right)^2 sdsd\phi dz\right]$$

After integrating the angular coordinate between $0 - 2\pi$, we can find the magnetic energy per unit length l as

$$\frac{E_{\text{mag}}}{l} = \frac{\mu_0 I^2}{4\pi}\left(\frac{1}{4} + \ln\frac{R_2}{R_1}\right)$$

(b) Self inductance L is related to the stored energy as $E_{\text{mag}} = LI^2/2$. Therefore, self inductance per unit length is

$$L/l = \frac{\mu_0}{4\pi}\left(\frac{1}{4} + \ln\frac{R_2}{R_1}\right)$$