Quality Factor

Fig. 10 shows two voltage response characteristics. The peak response occurs at the resonant frequency (ω_O). The resonant frequency is a function of the values of the inductor and the capacitor. The two characteristics differ from each other by having different bandwidths. A narrower bandwidth is preferred for applications such as tuners or frequency selective circuits. The bandwidth is a function of the half power frequencies. The sharpness of the response is measured through a parameter called quality factor.

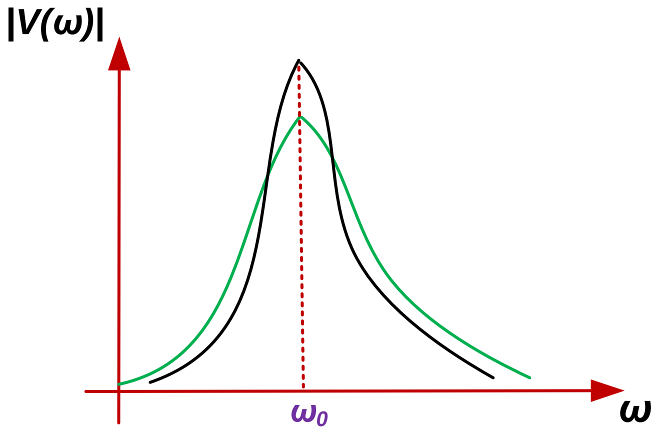

Fig. 10

The sharpness of the response is dependent on the maximum energy that can be stored in the circuit and the energy that is lost during one complete period of the response. Quality factor (***Q***) is defined as

$$\boldsymbol{Q} = \text{Quality factor} = 2\pi \frac{\textbf{Maximum Energy stored}}{\textbf{Total Energy Lost per Period}}$$

$$\boldsymbol{Q} = 2\pi \frac{[\boldsymbol{w_L(t)} + \boldsymbol{w_C(t)}]_{\max}}{\boldsymbol{P_R T}}$$

where $\boldsymbol{w_L(t)}$ and $\boldsymbol{w_C(t)}$ are the instantaneous energies stored with the inductor and the capacitor respectively. ***T*** is the period of the sinusoidal input. $\boldsymbol{P_R}$ is the power lost in the resistor. Let the input current at the resonant frequency $\boldsymbol{\omega_O}$ is ***i(t).***

$$\boldsymbol{i(t)} = \boldsymbol{I_m \cos\omega_0 t}$$

In a parallel ***RLC*** circuit, the complete input current will flow through the resistor. Hence, the voltage response at the resonance is

$$\boldsymbol{v(t)} = \boldsymbol{Ri(t)} = \boldsymbol{RI_m \cos\omega_0 t}$$

Energy stored in the capacitor is

$$w_C(t) = \frac{1}{2}Cv^2 = \frac{I_m^2 R^2 C}{2}\cos^2\omega_0 t$$

Instantaneous energy stored in the inductor is

$$w_L(t) = \frac{1}{2}Li_L^2 = \frac{1}{2}L\left(\frac{1}{L}\int_0^t v\,dt'\right)^2$$

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$$= \frac{1}{2}L\left(\frac{1}{L}\int_0^t RI_m cos\omega_0 t' dt'\right)^2$$

$$= \frac{I_m^2 R^2 C}{2}\sin^2\omega_0 t$$

The total instantaneous stored energy is constant which is given as

$$w(t) = w_L(t) + w_C(t) = \frac{I_m^2 R^2 C}{2}$$

This constant value is also the maximum value. The average power absorbed by the resistor is

$$P_R = \frac{1}{2}I_m^2 R$$

$$\Rightarrow P_R T = \frac{1}{2f_0}I_m^2 R$$

The quality factor at resonance is

$$Q_0 = 2\pi\frac{I_m^2 R^2 C/2}{I_m^2 R/2f_0} = \omega_0 RC$$

This result holds only for simple parallel ***RLC*** circuit

$$Q_0 = \omega_0 RC = \frac{1}{\sqrt{LC}}RC = R\sqrt{\frac{C}{L}}$$

$$Q_0 = \omega_0 RC = \frac{R}{x_{CO}} = \frac{R}{x_{\mathrm{LO}}}$$

where X_{CO} and X_{LO} are the capacitive and the inductive reactances at resonance. The bandwidth is the difference between the two half power frequency values. We can relate the bandwidth with the quality factor and the resonant frequency. The admittance of the parallel ***RLC*** circuit is

$$Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$
$$= \frac{1}{R} + j\frac{1}{R}\left(\frac{\omega\omega_0 CR}{\omega_0} - \frac{\omega_0 R}{\omega\omega_0 L}\right)$$
$$= \frac{1}{R}\left[1 + jQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

The magnitude of the admittance at resonance is ***1/R***. The magnitude of the admittance is $\sqrt{2}$ times this magnitude at half tower frequency points. Hence it is $\sqrt{2}$***/R*** at half power frequency points. In this case the imaginary part of the admittance will have a magnitude of unity.

$$Q_0\left(\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1}\right) = 1$$

$$Q_0\left(\frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2}\right) = -1$$

$$\omega_1 = \omega_0\left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} - \frac{1}{2Q_0}\right]$$

$$\omega_2 = \omega_0\left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} + \frac{1}{2Q_0}\right]$$

$$B = \omega_2 - \omega_1 = \frac{\omega_0}{Q_0}$$

$$\omega_1\omega_2 = \omega_0^2\left[1 + \left(\frac{1}{2Q_0}\right)^2 - \left(\frac{1}{2Q_0}\right)^2\right]$$
$$\Rightarrow \omega_0^2 = \omega_1\omega_2$$
$$=\rangle \omega_0 = \sqrt{\omega_1\omega_2}$$

Circuits with higher Q_0 have narrower bandwidth or sharper response. They have greater frequency selectivity.

In case of a series resonant circuit, the input impedance has a value ***R*** at the resonance. At half power frequencies, the impedance value will be equal to $\sqrt{2}$***R.***

$$Q_0 = \frac{\omega_0 L}{R}$$

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$