MA 102 (Mathematics II)
IIT Guwahati

Tutorial Sheet No. 6 | Linear Algebra | March 14, 2019

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1. (a) If A is an $n \times n$ matrix with $nullity(A) = k$, then show that there exists an invertible matrix P such that $P^{-1}AP = \begin{bmatrix} \mathbf{0} & B \\ \mathbf{0} & D \end{bmatrix}$, where D is an $(n-k) \times (n-k)$ matrix.

   (b) Hence show that for any eigenvalue λ of A, the algebraic multiplicity of $\lambda \geq$ the geometric multiplicity of λ.

   (c) Deduce that $rank(A) \geq$ the number of nonzero eigenvalues of A.

2. Find the eigenvalues of the $n \times n$ matrix A which has all diagonal entries equal to 3 and all other entries equal to 2. Find two eigenvectors $\mathbf{x}, \mathbf{y}$ corresponding to two distinct eigenvalues of A.
   Hint: $A = B+I$, where B is the matrix having all entries equal to 2. Check that $\mathbf{x}$ is an eigenvector of B corresponding to an eigenvalue λ if and only if $\mathbf{x}$ is an eigenvector of A corresponding to eigenvalue $\lambda + 1$.

3. If $A = \mathbf{u}\mathbf{u}^T$ where $\mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n$, then find the eigenvalues of A and show that A is diagonalizable.
   Hint: Check that $rank(A) = 1$ and $\mathbf{u}^T\mathbf{u}$ is the nonzero eigenvalue of A.

4. True or False? Give justifications.

   (a) If both A and A^{-1} has only integer entries, then $det(A) = +1$ or -1.

   (b) If A and B are non square matrices such that both AB and BA are defined then either AB or BA has a zero eigenvalue.

   (c) If A is a 3×3 matrix with eigenvalues $0, 3, 4$ and D is a 2×2 matrix with eigenvalues $0, 3$ then the matrix $C = \begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix}$ has eigenvalues $0, 3, 4$ with algebraic multiplicities $2, 2, 1$, respectively.
   Hint: $det(C - \lambda I) = det(A - \lambda I)det(D - \lambda I)$.

   (d) An upper triangular matrix with all diagonal entries equal to a is diagonalizable only if A is a diagonal matrix.
   Hint: Look at $null(A - aI)$.

   (e) Eigenvalues of real matrices occur in conjugate pairs (that is if $a + ib$ is an eigenvalue of A then $a - ib$ is also an eigenvalue of A).
   Hint: $\overline{det(A - cI)} = det(\overline{A - cI})$.

5. Let $A \in \mathcal{M}_n(\mathbb{C})$. If $\langle A\mathbf{x}, \mathbf{x} \rangle$ is real for all $\mathbf{x} \in \mathbb{C}^n$ then show that A is Hermitian.

6. Use Gram Schmidt procedure to transform the basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$ to an orthogonal basis. What happens if the order in which the vectors are taken changes, does the elements of the orthogonal basis remain the same?

7. Let $\mathbb{V}$ be a finite dimensional inner product space and $B := \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be an orthonormal subset of $\mathbb{V}$.

(a) Let $\mathbf{x}, \mathbf{y} \in \mathbb{V}$. Show that $\mathbf{x} = \mathbf{y} \Leftrightarrow \langle \mathbf{x}, \mathbf{w} \rangle = \langle \mathbf{y}, \mathbf{w} \rangle$ for all $\mathbf{w} \in \mathbb{V}$.

(b) Show that $|\langle \mathbf{v}, \mathbf{u}_1 \rangle|^2 + \cdots + |\langle \mathbf{v}, \mathbf{u}_m \rangle|^2 \leq \|\mathbf{v}\|^2$ for all $\mathbf{v} \in \mathbb{V}$.

(c) Show that $|\langle \mathbf{v}, \mathbf{u}_1 \rangle|^2 + \cdots + |\langle \mathbf{v}, \mathbf{u}_m \rangle|^2 = \|\mathbf{v}\|^2$ for all $\mathbf{v} \in \mathbb{V} \Leftrightarrow B$ is an orthonormal basis of $\mathbb{V}$.

(d) Suppose that B is an orthonormal basis of $\mathbb{V}$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ prove that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{u}_1 \rangle \overline{\langle \mathbf{y}, \mathbf{u}_1 \rangle} + \cdots + \langle \mathbf{x}, \mathbf{u}_m \rangle \overline{\langle \mathbf{y}, \mathbf{u}_m \rangle}.$$

8. Let $\mathcal{M}_2(\mathbb{R})$ be the inner product space of all real 2×2 matrices with respect to the inner product $\langle A, B \rangle = \text{trace}(B^T A)$ for all $A, B \in \mathcal{M}_2(\mathbb{R})$. Find the orthogonal complement of the set $\mathcal{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ in $\mathcal{M}_2(\mathbb{R})$.

9. Let $\mathbb{V}$ be a finite dimensional complex inner product space. Fix a vector $\mathbf{v} \in \mathbb{V}$ and define $T : \mathbb{V} \to \mathbb{C}$ by $T(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u} \in \mathbb{V}$. Determine T^*, that is, determine $T^*\mathbf{w}$ for all $\mathbf{w} \in \mathbb{C}$.

10. Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional inner product spaces. Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation. Prove that:
(i) T is injective if and only if T^* is surjective.
(ii) T is surjective if and only if T^* is injective.
(iii) $dim(null(T^*)) = dim(null(T)) + dim(\mathbb{W}) - dim(V)$.
(iv) $dim(range(T^*)) = dim(range(T))$.

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