A Circuit Paradox

Problem

- A simple yet deceptive circuit is shown in **Fig.1**.

- In **Fig.1**, we have a loop having two resistors, R1 and R2,that surrounds a solenoid magnet (**Fig.2**) that is excited by a time dependent current.

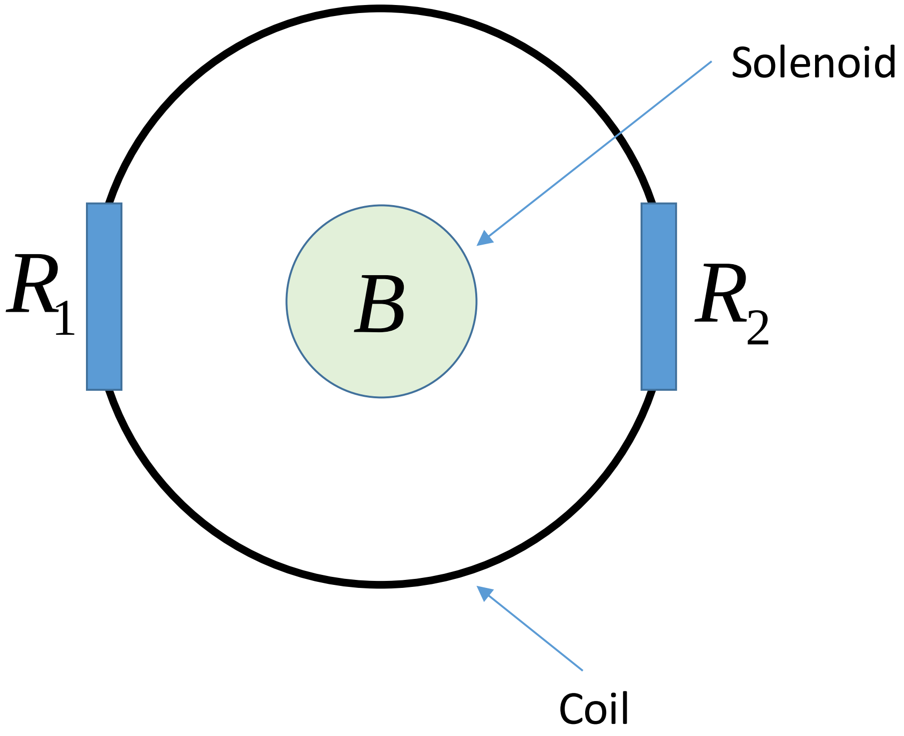

Fig1: The circuit

Problem

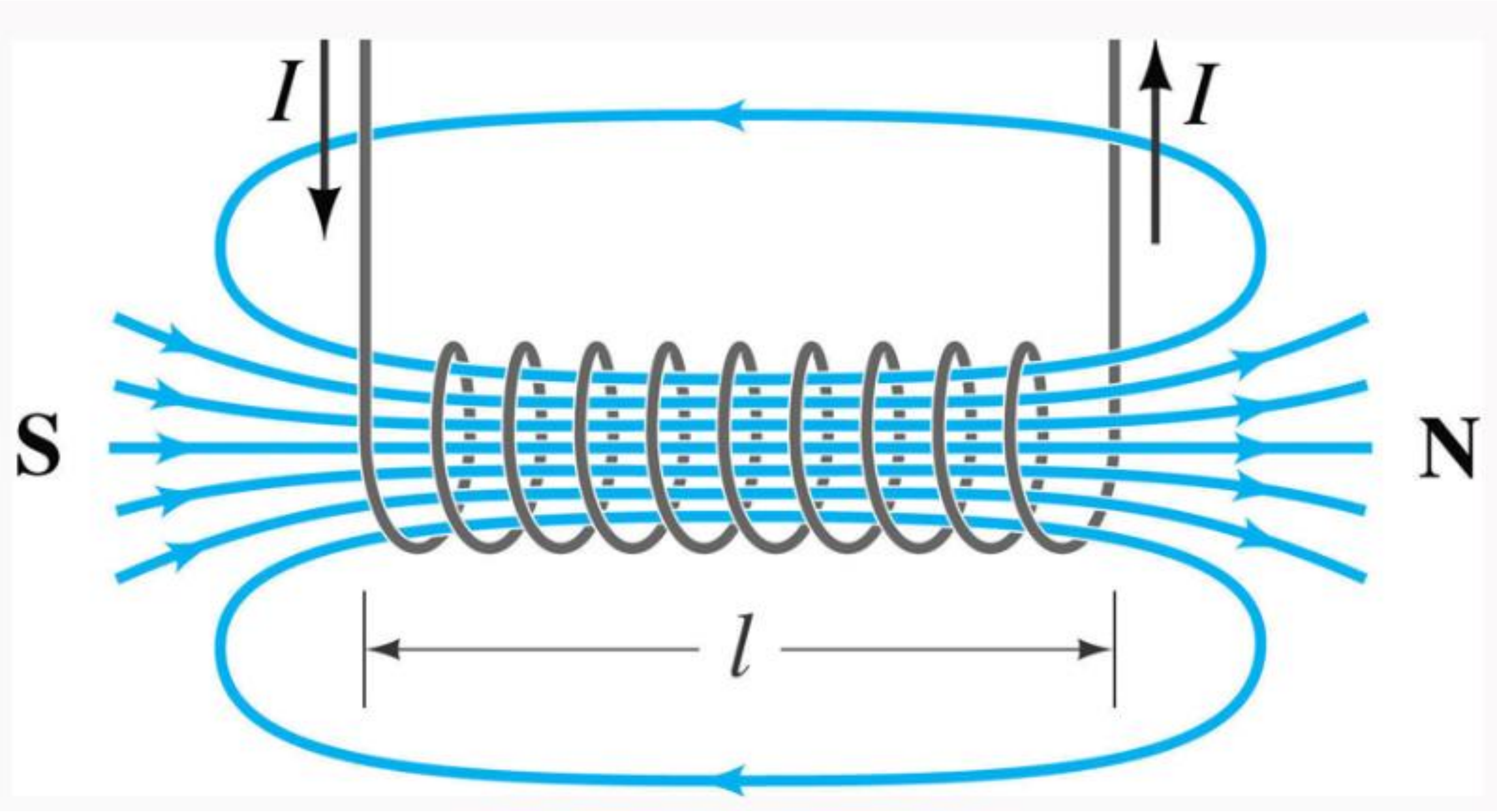

Fig12 Solenoid

Problem

- This circuit is probed by two voltmeters and an ammeter as shown in **Fig.3**.
- The positive leads of the voltmeters are connected to points ***a*** and ***c***, such that the directions of currents I1 and I2 are as shown in **Fig.3**.
- Both the voltmeters have same resistance ***R*** and it satisfies $R >> R_1, R_2$

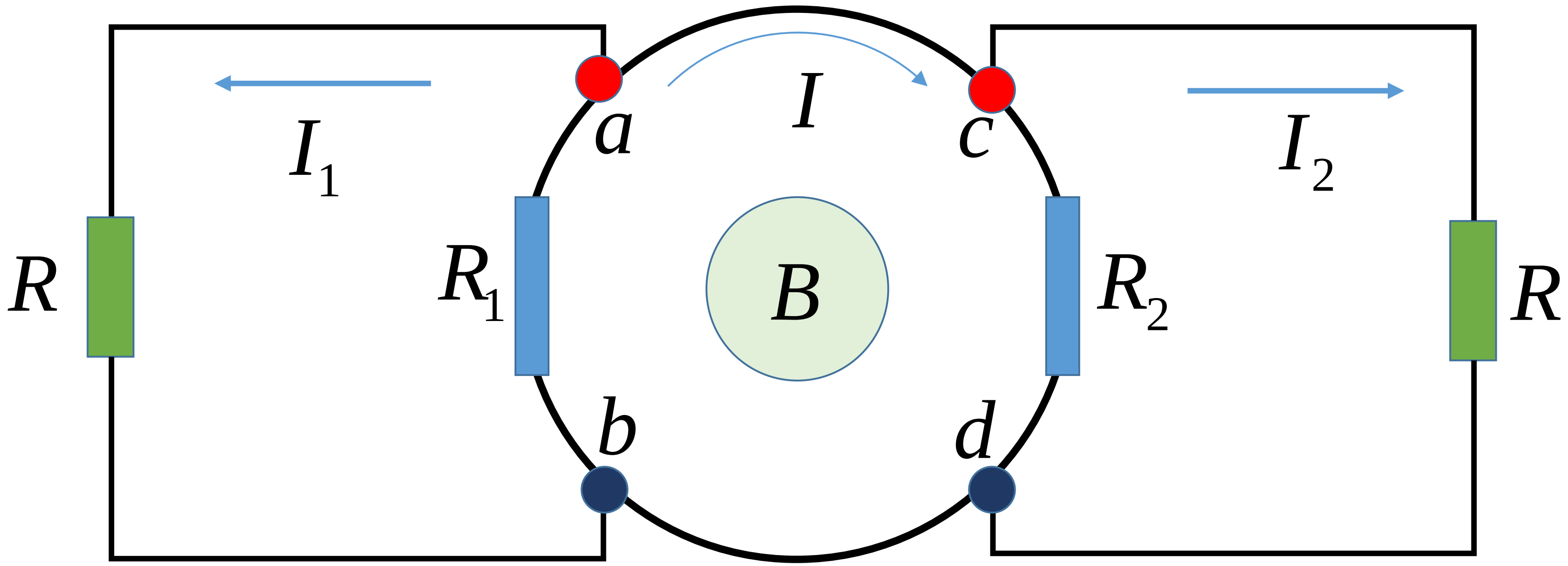

Fig3: The connection of voltmeters and ammeters

The paradox

- The paradox is that both the voltmeters do not measure the same magnitude of voltage.
- This is surprising because the points ***a*** and ***c***, and points ***b*** and ***d*** are the sme.

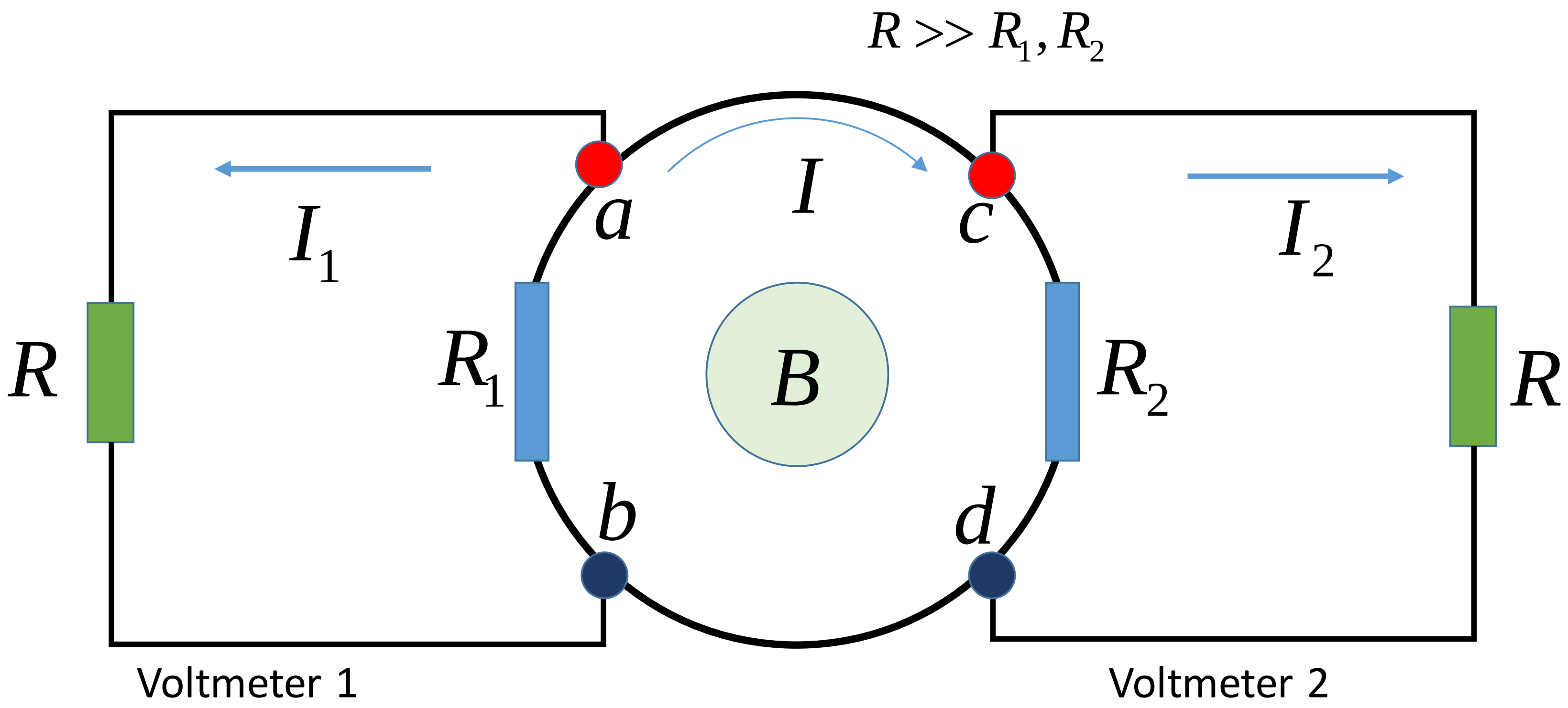

Fig.3: The connection of voltmeters and ammeters

Analysis using Kirchhoff's Laws

- To solve the circuit, we make an equivalent circuit of the original circuit.
- The equivalent circuit is shown in **Fig.4**.

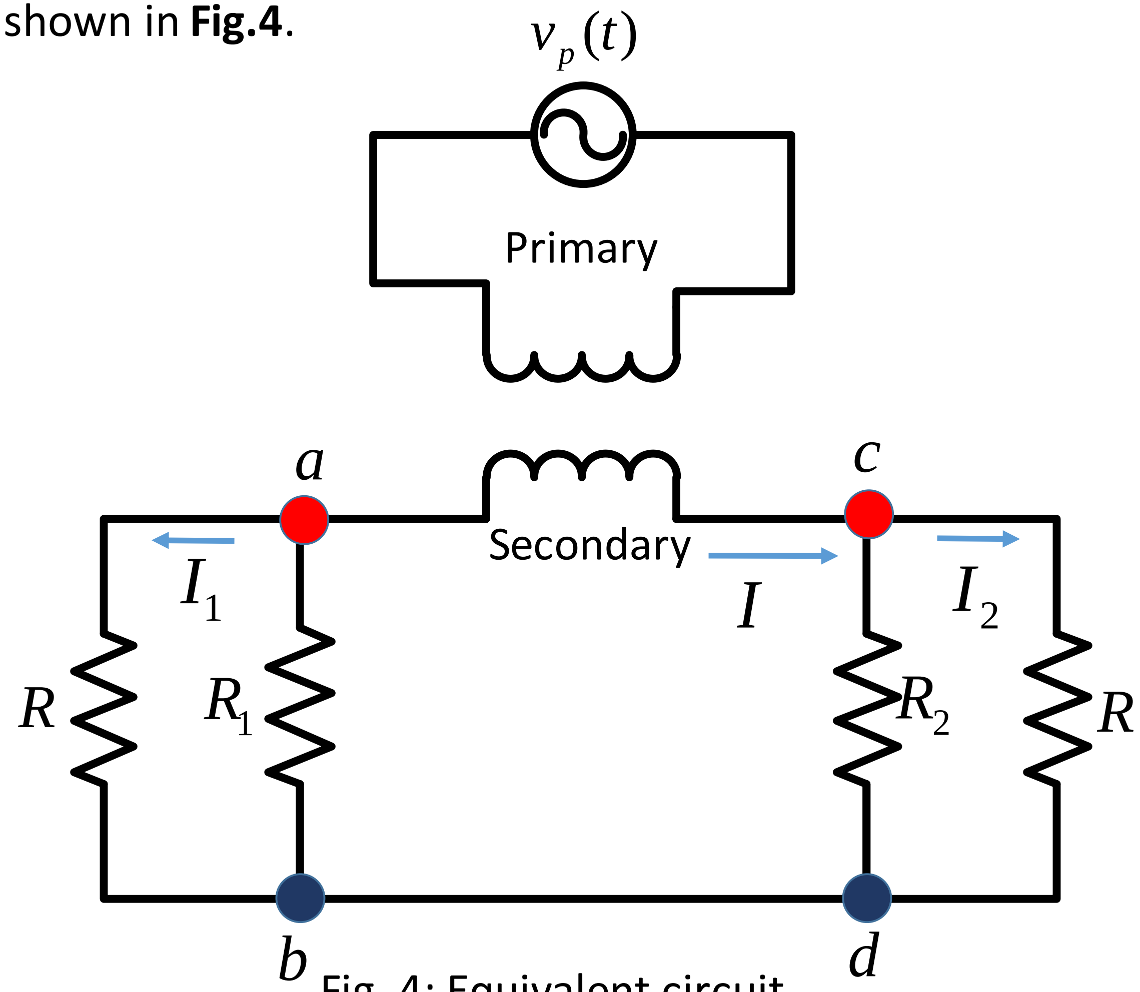

Fig. 4: Equivalent circuit

Analysis using Kirchhoff’s Laws

- In **Fig.4**, the loop with resistors R_1 and R_2 is the secondary of the transformer.

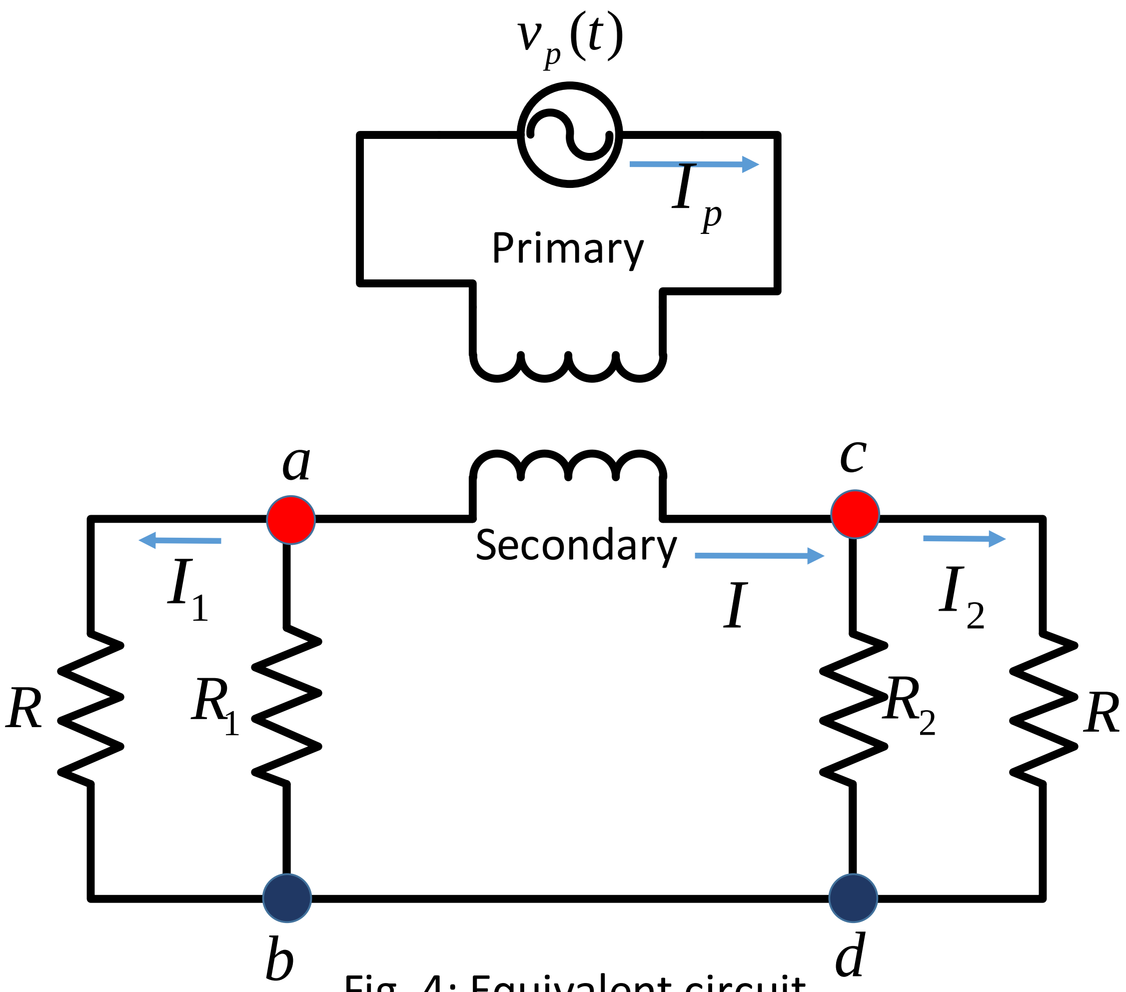

Fig. 4: Equivalent circuit

Analysis using Kirchhoff's Laws

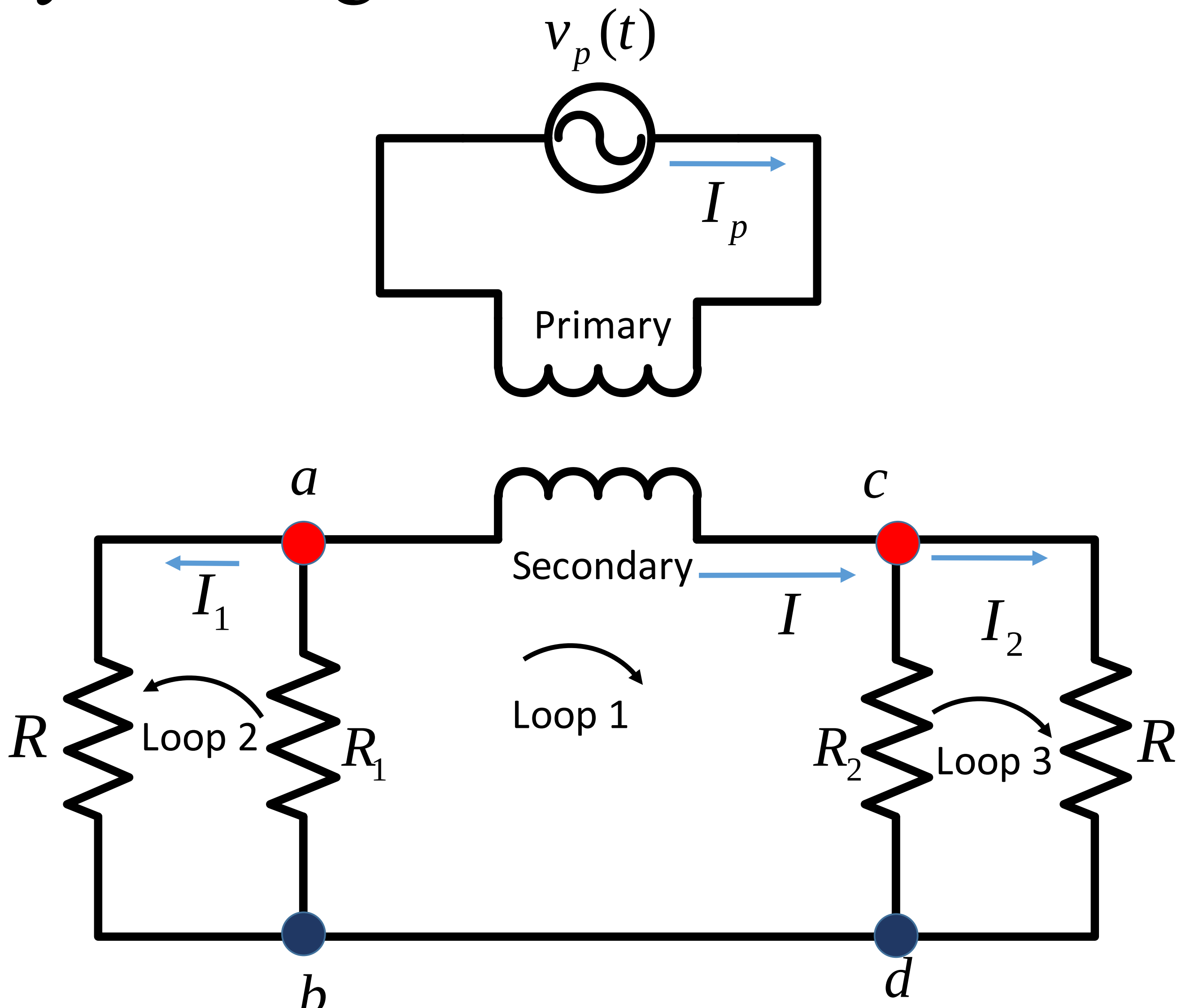

Fig. 5: Loops in equivalent circuit

Analysis using Kirchhoff's Laws

- The self inductance of the secondary coil is very small, that is

$$L_s \approx 0 \tag{1}$$

- The Kirchhoff's loop equation for the secondary loop (Loop1 in **Fig. 5**) is

$$(I + I_1)R_1 + (I - I_2)R_2 + M\frac{dI_p}{dt} = 0 \tag{2}$$

- Since I_p is known, we can write

$$\varepsilon = -M\frac{dI_p}{dt} = -\frac{d\Phi_p}{dt} \tag{3}$$

where Φ_p is the magnetic flux from the primary through the secondary loop.

- The **eq.2** can thus be written as:

$$\varepsilon = (R_1 + R_2)I + R_1 I_1 - R_2 I_2 \tag{4}$$

Analysis using Kirchhoff's Laws

- The Kirchhoff's equations for loops 2 and 3 are

$$I_1 R + \left(I_1 + I\right) R_1 = 0 \qquad 5$$

$$I_2 R + \left(I_2 - I\right) R_2 = 0 \qquad 6$$

- Since, the resistance of the voltmeters is very high ($R \gg R_1, R_2$), the **eq.5** and **eq.6** can be reduced to

$$R_1 I + R I_1 = 0 \qquad 7$$

$$-R_2 I + R I_2 = 0 \qquad 8$$

- Solving the **eq.4, 7,** and **8** gives

$$I = \frac{\varepsilon}{R_1 + R_2},\ I_1 = -\varepsilon \frac{R_1}{R_1 + R_2},\ I_2 = \varepsilon \frac{R_2}{R_1 + R_2} \qquad 9$$

Analysis using Kirchhoff's Laws

- From **eq.9** we can get the readings across the two voltmeters and they are:

$$V_{m1} = I_1 R = -\varepsilon \frac{R_1}{R_1 + R_2} \qquad 10$$

$$V_{m2} = I_2 R = \varepsilon \frac{R_2}{R_1 + R_2} \qquad 11$$

- The two voltmeter read voltages that are ***opposite in sign***, however, their magnitudes satisfy

$$|V_{m1}| + |V_{m2}| = \varepsilon \qquad 12$$

Discussion

- The meter readings do not depend on the locations of points a, b, c and d, so long as a and c are both between resistors R1 and R2 on the upper wire between them, and b and d are both between resistors R1 and R2 on the lower wire between them, and the leads are connected in the sense of the **Fig.6.**

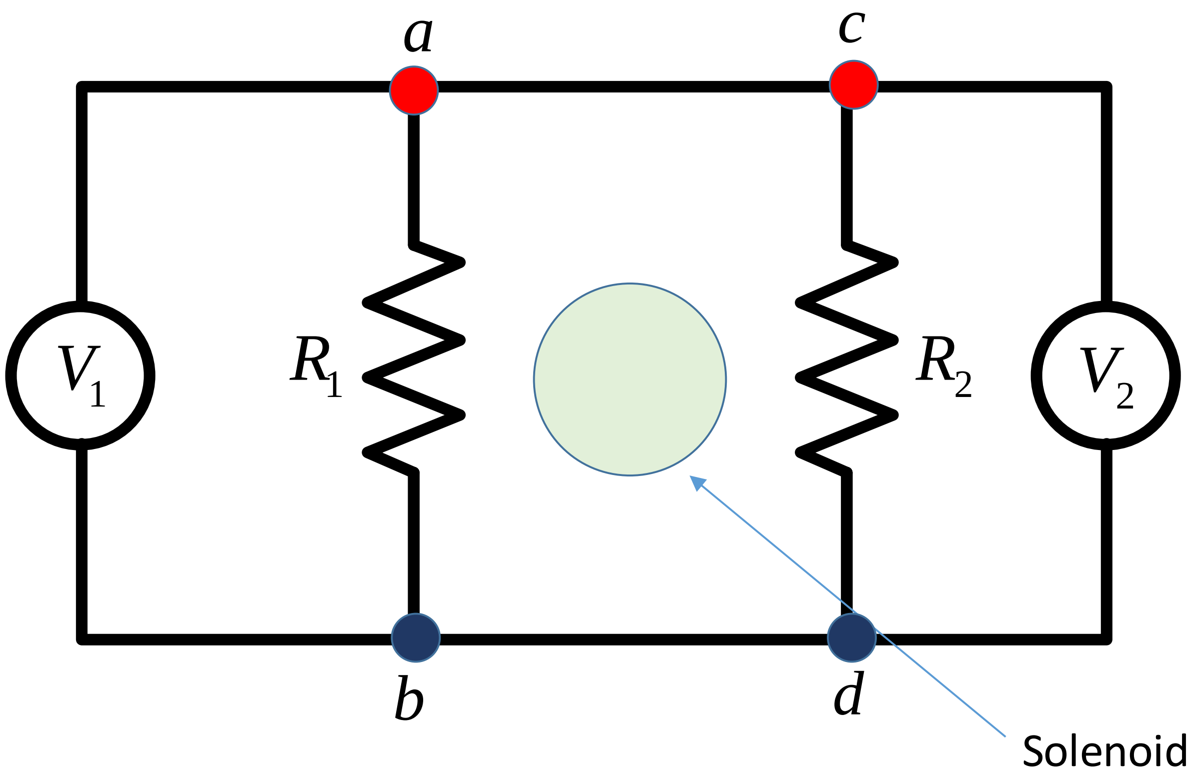

Fig. 6: Connection of two voltmeters

Discussion

- However, if both meters were outside the secondary loop and their leads both attached to that loop from its “right” side as shown in the **Fig.7**, **both the voltmeters would read**

$$V_{m1} = \varepsilon \frac{R_2}{R_1 + R_2} \quad 13$$

$$V_{m2} = \varepsilon \frac{R_2}{R_1 + R_2} \quad 14$$

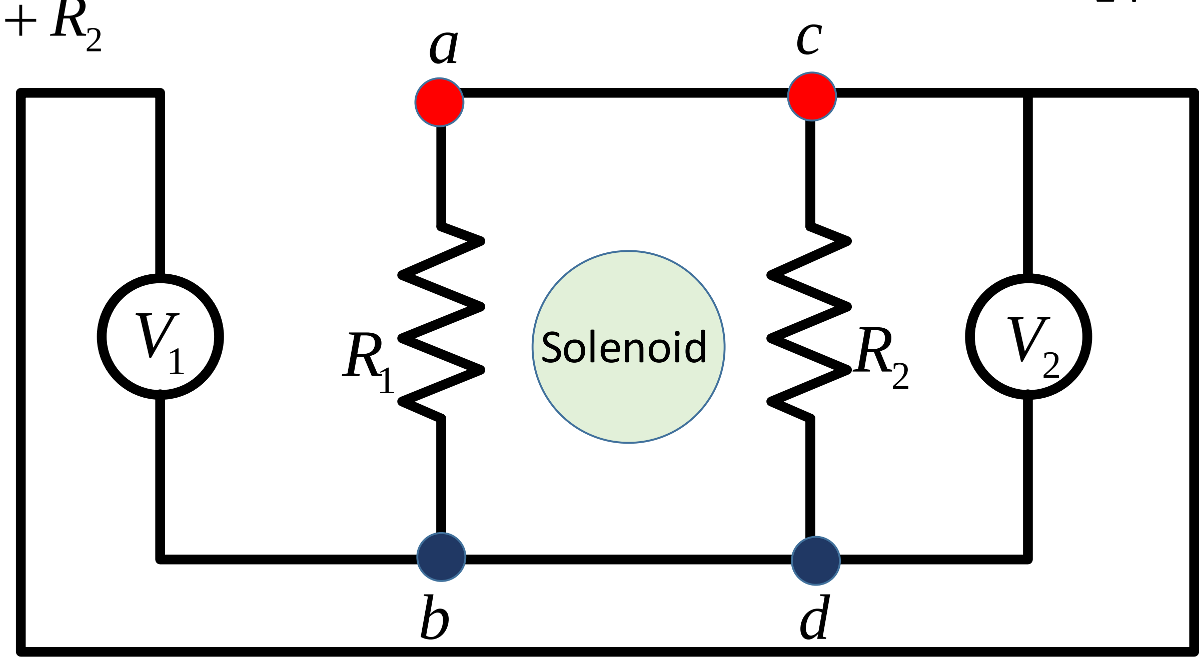

Fig. 7: Loops to the right

Discussion

- If both meters were outside the secondary loop and their leads both attached to that loop from its "left" side as shown in the **Fig.7**, **both the voltmeters would read**

$$V_{m1} = -\varepsilon \frac{R_1}{R_1 + R_2} \qquad 15$$

$$V_{m2} = -\varepsilon \frac{R_1}{R_1 + R_2} \qquad 16$$

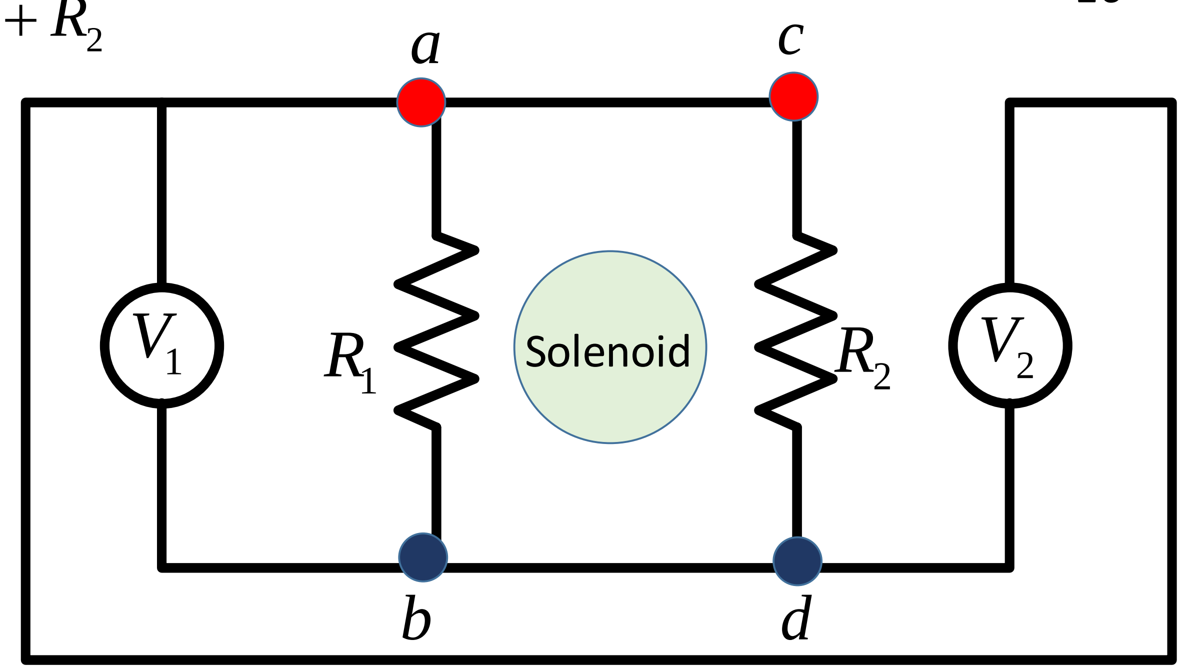

Fig. 7: Loops to the left

Connection of voltmeter

- Suppose the voltmeter leads cross the interior of the secondary loop, such that a fraction ***f*** of the magnetic flux of the solenoid passes through the voltmeter loop, as shown in **Fig.8**.

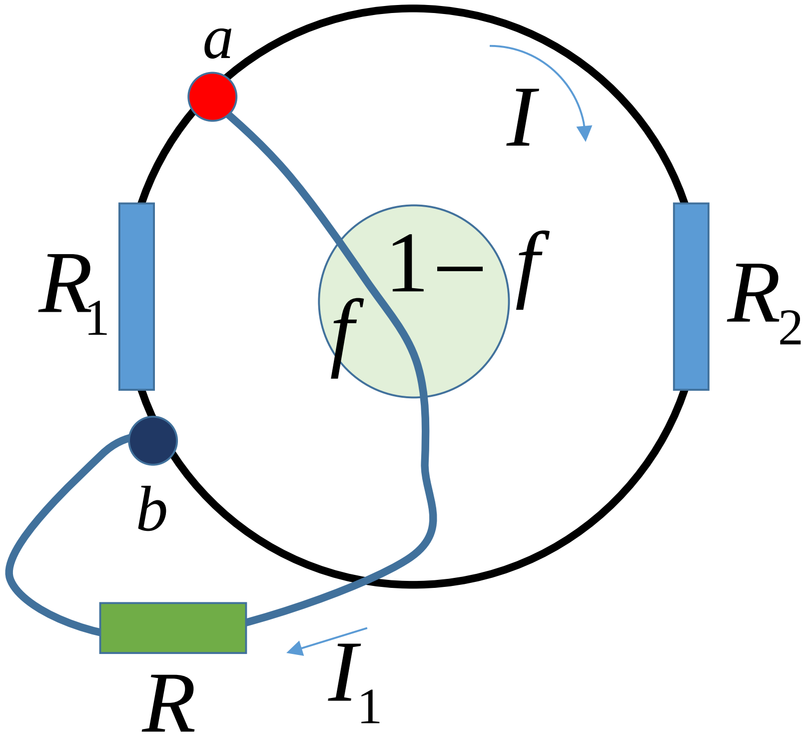

Fig. 8: The voltmeter leads enclose a fraction of magnetic field.

Connection of voltmeter

- Here R>>R1 (I1<,I), hence the Kirchhoff's loop equations for the secondary loop and voltmeter loops are

$$\varepsilon = (I + I_1)R_1 + IR_2 = (R_1 + R_2)I + R_1 I_1 \qquad 17$$

$$f\,\varepsilon = I_1 R + (I_1 + I)R_1 = IR_1 + I_1 R \qquad 18$$

- The sense of current I1 is the same as that of current I, thus the EMFs in both the loop have same sign.

- Solving **eq.17** and **eq18** gives

$$I = \frac{\varepsilon}{R_1 + R_2} \qquad 19$$

$$I_1 = -\frac{\varepsilon\left[R_1 - f(R_1 + R_2)\right]}{R(R_1 + R_2)} \qquad 20$$

Connection of voltmeter

- From **eq.19** and **eq.20** we can see that there is a continuum of possible readings of the voltmeter between f=0 and f=1.

- It is common practice to associate the terms of the loop equations with “circuit elements”, such as batteries, generators, resistors, capacitors and inductors. While use of Kirchhoff’s laws permits computation of the currents, identifying where the associated EMF’s are located is not always crisp, and interpreting measurements of currents by “voltmeters” can lead to misinterpretations of the results if one supposes that “voltmeters” measure “voltages”.