**MA 102 (Mathematics II)**
**IIT Guwahati**

Tutorial Sheet No. 3 | Linear Algebra | January 31, 2019

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1. True or False? Give justifications.
   (a) Let $A$ be an $m \times n$ matrix. Then there exist $\mathbf{b}$ and $\mathbf{b}'$ such that $A\mathbf{x} = \mathbf{b}$ has a unique solution but $A\mathbf{x} = \mathbf{b}'$ has infinitely many solutions.
   (b) Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$ such that $\mathbf{x}^T\mathbf{y} = 0$. Then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent (LI).
   (c) Let $S_1, S_2$ and $S_3$ be distinct subsets of $\mathbb{R}^n$ such that $\text{span}(S_1 \cup S_2) = \text{span}(S_1 \cup S_3)$. Then $\text{span}(S_2) = \text{span}(S_3)$.
   (d) The column spaces of $A$ and $\text{rref}(A)$ are equal.

2. Check whether the set $S = \left\{ \begin{bmatrix} 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is LI.

3. Let $S$ be a subspace of $\mathbb{R}^4$ and $\mathbf{x}, \mathbf{y} \in S$ be LI.
   (a) Show that if $\mathbf{u} \in \mathbb{R}^4 \setminus S$ then $\{\mathbf{x}, \mathbf{y}, \mathbf{u}\}$ is LI.
   (b) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4 \setminus S$ are LI then does it imply that $\{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$ is LI?

4. Let $A \in \mathcal{M}_n(\mathbb{R})$. Show that $\text{row}(A^TA) = \text{row}(A)$, that is, $A^TA$ and $A$ are row equivalent.

5. Show that the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ are LI if and only if $P\mathbf{x}, P\mathbf{y}, P\mathbf{z}$ are LI for any $n \times n$ invertible matrix $P$.

6. Let $A \in \mathcal{M}_5(\mathbb{R})$ be such that $\text{rref}(A)$ has the $1^{st}$, $3^{rd}$ and the $5^{th}$ column as the only pivot columns.
   (a) Find two LI solutions of $A\mathbf{x} = \mathbf{0}$.
   (b) Show that the columns $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_5$ ( the $1^{st}$, $3^{rd}$ and the $5^{th}$ column of $A$) are LI and spans the column space of $A$.
   (c) Can the sets $\{\mathbf{a}_1, \mathbf{a}_2\}$, $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ and $\{\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ be LI?

7. True or False? Give justifications.
   (a) If $\{\mathbf{x}, \mathbf{y}\}$ and $\{\mathbf{u}, \mathbf{v}\}$ are two different LI subsets of $\mathbb{R}^2$, then $\{\mathbf{x}, \mathbf{u}\}$ and $\{\mathbf{y}, \mathbf{v}\}$ are also LI sets.
   (b) If $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\}$ is LI in $\mathbb{R}^3$ then $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\}$ is LI in $\mathbb{R}^2$.
   (c) If $S$ is a subspace of $\mathbb{R}^n$ then $\mathbf{x} + S$ is a subspace if and only if $\mathbf{x} \in S$.
   (d) If the diagonal entries of a $4 \times 4$ upper triangular matrix $A$ are $1, 2, 3$ and $4$ then $S_1 = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = 2\mathbf{x}\}$ is a subspace of $\mathbb{R}^4$ but $S_2 = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = 5\mathbf{x}\}$ is not.

8. Let $S = \left\{ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} a \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \right\}$. Find the values of $a$ for which $\text{span}(S) \neq \mathbb{R}^3$.

9. If a diagonal entry of a $3 \times 3$ upper triangular matrix is zero, then show that the columns are linearly dependent.

10. True or False? Give justifications.

    (a) If $S$ is a subspace of $\mathbb{R}^n$ of dimension $n$, then $S = \mathbb{R}^n$.

    (b) For any two matrices $A$ and $B$ for which $AB$ is defined, $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.

    (c) If $C = [A \,|\, B]$, then $\text{rank}(C) \leq \text{rank}(A) + \text{rank}(B)$.

    (d) If $C = \begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix}$, then $\text{rank}(C) \geq \text{rank}(A) + \text{rank}(D)$.

11. If $\text{rank}(A) = \text{rank}(A^2)$ then show that $\text{rank}(A^2) = \text{rank}(A^3)$. Is $\text{rank}(A^5) = \text{rank}(A^6)$?

    Hint: Note that $\text{col}(A^2) \subseteq \text{col}(A)$, $\text{rank}(A^2) = \text{rank}(A)$ implies $\text{col}(A^2) = \text{col}(A)$. Again note that $\text{col}(A^3) \subseteq \text{col}(A^2)$, show $\text{col}(A^3) = \text{col}(A^2)$, and so on.

12. (a) Show that for any two $m \times n$ matrices $A$ and $B$, $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

    Hint: $A + B = [A|B] \begin{bmatrix} I_n \\ I_n \end{bmatrix}$.

    (b) Hence show that if $A$ is an $m \times n$ matrix and $B$ is the matrix obtained by changing exactly $k$ entries of $A$, then $\text{rank}(A) - k \leq \text{rank}(B) \leq \text{rank}(A) + k$.

    Hint: $B = A + C$, where $C$ has exactly $k$ nonzero entries.

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