Linear Algebra

Department of Mathematics
Indian Institute of Technology Guwahati

January – May 2019

MA 102 (RA, RKS, MGPP, KVK)

Matrix Operations

Topics:

- Matrix operations
- Invertible matrices
- Elementary matrices and reduction to rref
- Gauss-Jordan elimination for computing in inverse of a matrix
- LU factorization

Matrices

Recall that an $m \times n$ matrix A with entries a_{ij} has m rows and n columns and can be written as

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = [a_{ij}]_{m\times n}, \end{aligned}$$

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where $\mathbf{A}_i := [a_{i1}, a_{i2}, \ldots, a_{in}]$ is the i-th row of A for $i = 1 : m$ and $\mathbf{a}_j := [a_{1j}, \ldots, a_{mj}]^\top$ is the j-th column of A for $j = 1 : n$.

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Two matrices $A = [a_{ij}]$ and $B := [b_{ij}]$ are said to be equal (i.e, $A = B$) if A and B have the same size and $a_{ij} = b_{ij}$ for all i and j.

Special matrices

We denote the set of complex numbers by $\mathbb{C}$.

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Conjugate transpose: The conjugate transpose of an $m \times n$ complex matrix $A = [a_{ij}]_{m \times n}$ is the $n \times m$ matrix denoted by A^* and is given by

$$A^* = [\bar{a}_{ji}]_{n \times m} = ([\bar{a}_{ij}]_{m \times n})^\top = (\bar{A})^\top,$$

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Definition: Let A be an $n \times n$ matrix. Then A is said to be

1. symmetric if $A^\top = A$,
2. skew-symmetric if $A^\top = -A$,
3. Hermitian if $A^* = A$,
4. skew-Hermitian if $A^* = -A$.

Special matrices (recall)

Let A be an $m \times n$ matrix with (i, j)-th entry a_{ij}. Set $p := \min(m, n)$. Then

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- A is said to be a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$;
- A is said to be an upper triangular if $a_{ij} = 0$ for all $i > j$;
- A is said to be a lower triangular if $a_{ij} = 0$ for all $i < j$;

Identity matrix: An $n \times n$ diagonal matrix with all diagonal entries equal to 1 is called the identity matrix and is denoted by I_n or I.

Zero matrix: An $m \times n$ matrix with all entries 0 is called the zero matrix and is denoted by $\mathbf{O}_{m\times n}$ or simply by $\mathbf{O}$.

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2. Multiplication by a scalar: $\alpha A := [\alpha\, a_{ij}] \in \mathcal{M}_{m\times n}$.
   ($\alpha \in \mathbb{R}$ when A and B are real matrices, and $\alpha \in \mathbb{C}$ when A and B are complex matrices)

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3. $A + \mathbf{O} = A$, where $\mathbf{O} = \mathbf{O}_{m\times n} \in \mathcal{M}_{m\times n}$.
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7. $\alpha\, (\beta\, A) = (\alpha\beta)A$.
8. $1A = A$.

Matrix-vector multiplication revisited

Let $A := \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $\mathbf{x} := [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$. Recall the matrix-vector multiplication

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n. \in \mathbb{R}^m.$$

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- $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.
- $A(\alpha\mathbf{x}) = \alpha A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

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We refer to $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \mathbf{x} \longmapsto A\mathbf{x}$, as a linear mapping.

Matrix-matrix multiplication

Let $A \in \mathcal{M}_{m\times n}$ and $B := \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_p \end{bmatrix} \in \mathcal{M}_{n\times p}$.

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Then the linear map $AB : \mathbb{R}^p \longrightarrow \mathbb{R}^m$ is given by

$$\begin{aligned} AB\mathbf{y} &:= A(B\mathbf{y}) = A(y_1\mathbf{b}_1 + \cdots + y_p\mathbf{b}_p) \\ &= y_1 A\mathbf{b}_1 + \cdots + y_p A\mathbf{b}_p = \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_p \end{bmatrix} \mathbf{y} \end{aligned}$$

for all $\mathbf{y} \in \mathbb{R}^p$

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Thus if $A := [a_{ij}]_{m\times n}, B := [b_{ij}]_{n\times p}$ and $C := AB = [c_{ij}]_{m\times p}$ then

$$c_{ij} = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

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Note:

$$\begin{aligned} \mathbf{e}_i^\top A &= \mathbf{A}_i = i\text{-th row of } A, \text{ where } \mathbf{e}_i \in \mathbb{R}^m \\ A\mathbf{e}_j &= \mathbf{a}_j = j\text{-th column of } A, \text{ where } \mathbf{e}_j \in \mathbb{R}^n. \end{aligned}$$

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3. $\alpha(AB) = (\alpha A)B = A(\alpha B)$, if the respective matrix products are defined.

4. $I_m A = A = AI_n$, if A is of size $m \times n$.

Block matrix

Definition: An $m \times n$ block matrix (or a partition matrix) is a matrix of the form

$$A := \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ldots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$

where each A_{ij} is a $p_i \times q_j$ matrix for $i = 1 : m$ and $j = 1 : n$.

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Then $\begin{bmatrix} A_{i1} & \cdots & A_{in} \end{bmatrix}$ is the i-th block row of A and $\begin{bmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{bmatrix}$ is the j-th block column of A.

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Example: $\left[\begin{array}{cc|ccc|c} 1 & 2 & 2 & 0 & 1 & 4 \\ 3 & 4 & 1 & 2 & 3 & 5 \\ \hline 5 & 7 & 2 & 7 & 8 & 8 \\ 3 & 4 & 1 & 9 & 2 & 2 \end{array}\right]$ has 2 block rows and 3 block columns.

Block matrix operations

Block matrix addition: Let $A := [A_{ij}]_{m\times n}$ and $B := [B_{ij}]_{m\times n}$ be block matrices such that size of A_{ij} = size of B_{ij} for $i = 1 : m$ and $j = 1 : n$. Then $A + B := [A_{ij} + B_{ij}]_{m\times n}$.

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Block matrix multiplication: Let $A := [A_{ij}]_{m\times n}$ and $B := [B_{ij}]_{n\times p}$ be block matrices. If the matrix multiplication $C_{ij} := \sum_{k=1}^{n} A_{ik}B_{kj}$ is well defined for $i = 1 : m$ and $j = 1 : p$ then AB is an $m \times p$ block matrix given by $AB = [C_{ij}]_{m\times p}$.

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Example: $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix} =$

$$\begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} & A_{11}B_{13} + A_{12}B_{23} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & A_{21}B_{13} + A_{22}B_{23} \end{bmatrix}.$$

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Definition: An $n \times n$ matrix A is said to be invertible if there exists a matrix B satisfying $AB = I_n = BA$. The matrix B is called an inverse of A.

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- The zero matrix $\mathbf{O}$ is not invertible.
- If A has a zero row, then A is not invertible.

Properties of invertible matrices

Fact: Let A and B be two invertible matrices of the same size.

1. If $c \neq 0$ then cA is also invertible, and $(cA)^{-1} = \frac{1}{c}A^{-1}$.
2. The matrix AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.
3. The matrix A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$.
4. For any non-negative integer k, the matrix A^k is invertible, and $(A^k)^{-1} = (A^{-1})^k$.

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Let $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$ then A is invertible, and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

for any matrix there exist a inverse iff the determinant of the matrix is not zero.

If $ad - bc = 0$ then A is not invertible.

Elementary matrices

Type	Operation	Inverse operation
I	$R_i \longleftarrow \alpha\, R_i$	$R_i \longleftarrow \frac{1}{\alpha}\, R_i$
II	$R_j \longleftarrow cR_i + R_j$	$R_j \longleftarrow -cR_i + R_j$
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$$\text{Type I}: E_2(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \alpha x_2 \\ x_3 \end{bmatrix}.$$

$$E_2(\alpha)A = \begin{bmatrix} \mathrm{row}_1(A) \\ \alpha\, \mathrm{row}_2(A) \\ \mathrm{row}_3(A) \end{bmatrix} = \text{ multiply 2nd row of } A \text{ by } \alpha.$$

$$(E_2(\alpha))^{-1} = E_2(\frac{1}{\alpha}).$$

Type II elementary matrices

$$\text{Type II}: E_{13}(2) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = I_3 + 2e_3e_1^T$$

The matrix E_{13} is obtained by performing $R_3 \leftarrow 2R_1 + R_3$ on I_3.

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$$E_{13}(2)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + 2x_1 \end{bmatrix} \Rightarrow E_{13}A = \begin{bmatrix} \mathrm{row}_1(A) \\ \mathrm{row}_2(A) \\ \mathrm{row}_3(A) + 2\,\mathrm{row}_1(A) \end{bmatrix}.$$

$(E_{13}(2))^{-1} = E_{13}(-2)$ corresponds to $R_3 \leftarrow -2R_1 + R_3$ on I_3.

Type III elementary matrices

Type III : E_{ij} is obtained by performing $R_i \leftrightarrow R_j$ on I.

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow E_{23} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} \Rightarrow E_{23}A = \begin{bmatrix} \mathrm{row}_1(A) \\ \mathrm{row}_3(A) \\ \mathrm{row}_2(A) \end{bmatrix}.$$

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$(E_{ij})^{-1} = E_{ij}$ corresponds to row operation $R_i \leftrightarrow R_j$ on I.

Observation: Inverse of an elementary matrix is also an elementary matrix of same type.

Row operation via elementary matrices

Crux of the matter:

- Type I: Multiplying $E_i(c)$ to A giving $E_i(c)A$ amounts to performing the row operation $R_i \leftarrow cR_i$ on A.
- Type II: Multiplying $E_{ij}(c)$ to A giving $E_{ij}(c)A$ amounts to performing the row operation $R_j \leftarrow cR_i + R_j$ on A..
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Recall that two matrices A and B are said to be row equivalent if A can be transformed to B by elementary row operations.

Theorem: The matrices A and B are row equivalent $\iff$ $B = E_k \cdots E_2 E_1 A$ for some elementary matrices $E_1, E_2, \cdots, E_k$.

Elementary matrices and echelon form

Forward GE: $m \times n$ matrix $A \longrightarrow$ row echelon form $\mathrm{ref}(A)$

$$\Downarrow$$

$\mathrm{ref}(A) = E_p \cdots E_2 E_1 A$ for some elementary matrices $E_1, \ldots, E_p$.

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Fact: Let $[A \mid b] \longrightarrow \mathrm{ref}([A \mid b]) =: [U \mid d]$. Then the system $Ax = b$ and $Ux = d$ are equivalent.

Example

$$\text{Find rref of } A = \begin{bmatrix} 0 & 2 & -4 & 4 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 1 & 7 \\ 2 & 1 & 0 & -3 \end{bmatrix}.$$

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So to find A^{-1}, use GJE to $\left[A \mid I_n\right]$.

Gauss-Jordan method:

$$[A \mid I_n] \longrightarrow [I_n \mid X] \Rightarrow A \text{ is invertible and } A^{-1} = X.$$

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$$\Rightarrow A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Characterization of invertibility

Theorem: Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. A is invertible.
2. $Ax = b$ has a unique solution for every b in $\mathbb{R}^n$.
3. $Ax = 0$ has only the trivial solution.
4. The reduced row echelon form of A is I_n.
5. A is a product of elementary matrices.
6. $\mathrm{rank}(A) = n$

LU Factorization

An $n \times n$ matrix A has an LU factorization if $A = LU$, where U is upper triangular and L is unit lower triangular (diagonals are 1).

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- $E_p \dots E_2 E_1 A = \mathrm{ref}(A) \Rightarrow A = LU$.
- $L := E_1^{-1} E_2^{-1} \dots E_p^{-1}$ and $U := \mathrm{ref}(A)$.
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Solution of $Ax = b$ via LU factorization (if exists):

- Compute $A = LU$.
- Solve $Ly = b$ for y - forward substitution.
- Solve $Ux = y$ for x - back substitution.

LU factorization (cont.)

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Set $m_{21} := a_{21}/a_{11}$ and $m_{31} := a_{31}/a_{11}$

when $a_{11} \neq 0$ (pivot) and define

$$E_1 := \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix}.$$

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$$E_2 E_1 A = E_2 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}.$$

Set $m_{32} := a_{32}^{(1)}/a_{22}^{(1)}$ if $a_{22}^{(1)} \neq 0$ (pivot) and define

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$E_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix}$. Then we have

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$$\text{Hence } A = LU, \text{ where } L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}.$$

Examples: LU factorization

Let $A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. Then $A = LU$, where

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Three theorems with similar proof structure

Theorem: rref of a matrix is unique. Equivalently, if R_1 and R_2 are in rref and are row equivalent, then $R_1 = R_2$.

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Theorem: Let A and B be $m \times n$ matrices. Suppose that the systems $Ax = b$ and $Bx = c$ are consistent. Then the two systems are equivalent if and only if the matrices $[\ A \mid b\ ]$ and $[\ B \mid c\ ]$ are row equivalent.

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$$R_1\mathbf{e}_{k+1} = \alpha_1 R_1\mathbf{e}_1 + \cdots + \alpha_k R_1\mathbf{e}_k \Longrightarrow R_1\mathbf{x} = \mathbf{0}, \tag{2}$$

where $\mathbf{x} := [\alpha_1, \ldots, \alpha_k, -1, 0, \ldots, 0]^\top$.

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Hence we have $R_1 = R_2$. ■

*** End ***