Lecture 31 Continuous time Markov process 2

CTMP

- The probabilistic evolution of a CTMP $X(t)$ is in terms of a partial differential equation (PDE), known as the Fokker Planck (FP) equations.

$$\frac{\partial f(x,t/x_0,t_0)}{\partial t}=-\mu(x,t)\frac{\partial f(x,t/x_0,t_0)}{\partial x}+\frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f(x,t/x_0,t_0)}{\partial x^2}$$

where $\mu(x,t)=\lim_{\Delta t\to 0}\frac{E((X(t+\Delta t)-X(t))/X(t)=x)}{\Delta t}$ and

$$\sigma^2(x,t)=\lim_{\Delta t\to 0}\frac{E((X(t+\Delta t)-X(t))^2/X(t)=x)}{\Delta t}$$

- When $\mu(x,t)$ and $\sigma^2(x,t)$ are constants, the FP equation simplifies to the *diffusion equation* given by:

$$\frac{\partial f(x,t/x_0,t_0)}{\partial t}=-\mu\frac{\partial f(x,t/x_0,t_0)}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 f(x,t/x_0,t_0)}{\partial x^2}$$

with μ and σ^2 respectively known as the drift and the diffusion coefficients. For the Wiener process, the transition pdf follows the above PDE.

The solution to the diffusion equation

$$\frac{\partial f(x,t/x_0,t_0)}{\partial t}=\frac{1}{2}\sigma^2\frac{\partial^2 f(x,t/x_0,t_0)}{\partial x^2}$$

under the initial condition $X(0)=0$ with probability 1, is given by

$$f(x,t/x_0=0,t_0=0)=\frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2 t}\right)}$$

. The CTMP $X(t)$ with

$$f(x,t)=\frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2 t}\right)}$$

Is called the Brownian motion process.

If $\mu(x,t)=\mu\neq 0$, then

$$f(x,t)=\frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{1}{2}\left(\frac{(x-\mu t)^2}{\sigma^2 t}\right)}$$

Wiener process or Brownian motion process

Definition: The random process $\{X(t), t \geq 0\}$ is called a ***Wiener process or the Brownian motion process*** if it satisfies the following conditions:

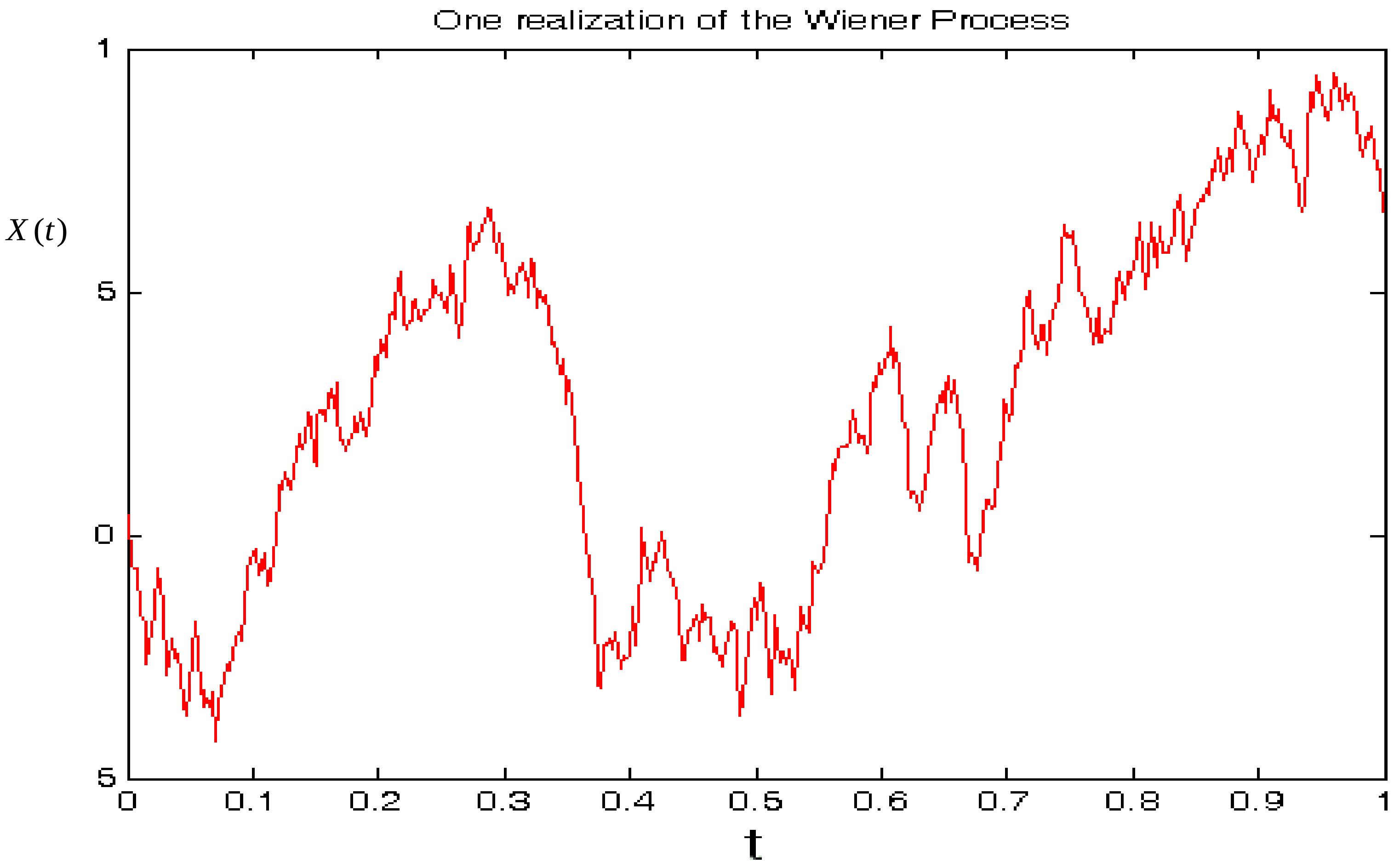
(1) $X(0) = 0$ with probability 1.

(2) $X(t)$ is an independent increment process.

(3) For each $t_0 \geq 0, t \geq 0$ $X(t+t_0) - X(t_0)$ has the normal distribution with mean 0 and variance $\sigma^2 t$.

$$f_{X(t+t_0)-X(t_0)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 t}}$$

A realization of the Wiener process



Properties of the Wiener process

- We have $f_{X(t)}(x) = \dfrac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 t}}$

- The conditional CDF

$$\begin{aligned}
F(x,t / x_0,t_0) &= P(X(t) \le x / X(t_0) = x_0) \\
&= P(X(t) - X(t_0) \le x - x_0 / X(t_0) = x_0) \\
&= P(X(t) - X(t_0) \le x - x_0) \\
&= F_{X(t)-X(t_0)}(x - x_0)
\end{aligned}$$

Taking the partial derivative w.r.t. x, we get

$$\therefore\ f(x,t / x_0,t_0) = f_{X(t)-X(t_0)}(x - x_0) = \frac{1}{\sqrt{2\pi\sigma^2 (t - t_0)}} e^{-\frac{1}{2}\frac{(x-x_0)^2}{\sigma^2 (t-t_0)}}$$

$$E(X(t) \mid X(t_0) = x_0) = x_0$$

Example Let $\{X(t), t \geq 0\}$ be a standard Brownian motion.

(a) Find $P(1 < X(1) < 2)$

(b) Find $P(X(2) < 3 \mid X(1) = 1)$

Hint:

(a) From $f_{X(t)}(x) = \dfrac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 t}}$, we get

$$f_{X(1)}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$\therefore P(1 < X(1) < 2) = \int_1^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2}{2}} dx$$

(b) From $f(x, t / x_0, t_0) = f_{X(t)-X(t_0)}(x - x_0) = \dfrac{1}{\sqrt{2\pi\sigma^2(t - t_0)}} e^{-\frac{1}{2}\frac{(x-x_0)^2}{\sigma^2(t-t_0)}}$, we get

$$f_{X(2)|X(1)=1}(x) = f_{X(t)-X(t_0)}(x - x_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2}$$

Autocorrelation and autocovariance function of the Wiener process

$$\begin{aligned} R_X(t_1,t_2) &= EX(t_1)X(t_2) \\ &= EX(t_1)\{X(t_2)-X(t_1)+X(t_1)\} \quad \text{Assuming } t_2>t_1 \\ &= EX(t_1)E\{X(t_2)-X(t_1)\}+EX^2(t_1) \\ &= EX^2(t_1) \\ &= \sigma^2 t_1 \end{aligned}$$

Similarly if $t_1>t_2,\ R_X(t_1,t_2)=\sigma^2 t_2$

$$\therefore R_X(t_1,t_2)=\sigma^2\min(t_1,t_2)$$

Thus the Wiener process is not stationary. Since the process is zero-mean,

$$C_X(t_1,t_2)=\sigma^2\min(t_1,t_2)$$

Continuity of the Differentiability of the Wiener process:

The Wiener process is m.s. continuous every where

For a Wiener process $\left\{X\left(t\right)\right\}$,

$$R_X\left(t_1,t_2\right)=\sigma^2\min\left(t_1,t_2\right)$$

$$\therefore R_X\left(t,t\right)=\sigma^2\min\left(t,t\right)=\sigma^2 t=\lim_{t_1\to t,t_2\to t} R_X\left(t_1,t_2\right)$$

Thus the autocorrelation function of the Wiener process is continuous everywhere implying that the process is m.s. continuous everywhere.

A Wiener process is m.s. differentiable nowhere.

We have,

$$R_X(t_1,t_2) = \sigma^2 \min(t_1,t_2)$$

$$\therefore R_X(t,t_2) = \begin{cases} \sigma^2 t_2 & \text{if } t_2 < t \\ t & \text{other wise} \end{cases}$$

$$\therefore \frac{\partial R_X(t,t_2)}{\partial t_2} = \begin{cases} \sigma^2 & \text{if } t_2 < t \\ 0 & \text{if } t_2 > t \\ does\ not\ exist\, at\ t_2 = t \end{cases}$$

$$\therefore \frac{\partial R_X(t,t_2)}{\partial t_2} \text{ does not exist at } t_2 = t$$

$$\therefore \frac{\partial^2 R_X(t_1,t_2)}{\partial t_1 \partial t_2} \text{ does not exist at } (t_1 = t_2 = t)$$

Thus a Wiener process is m.s. differentiable nowhere.

Wiener process as limit of a symmetrical random walk process

Recall the random walk(RW) *process* $\{X_n\}_{n=0}^{\infty}$ given by

$$X_n = \sum_{i=1}^{n} Z_i = X_{n-1} + Z_n$$

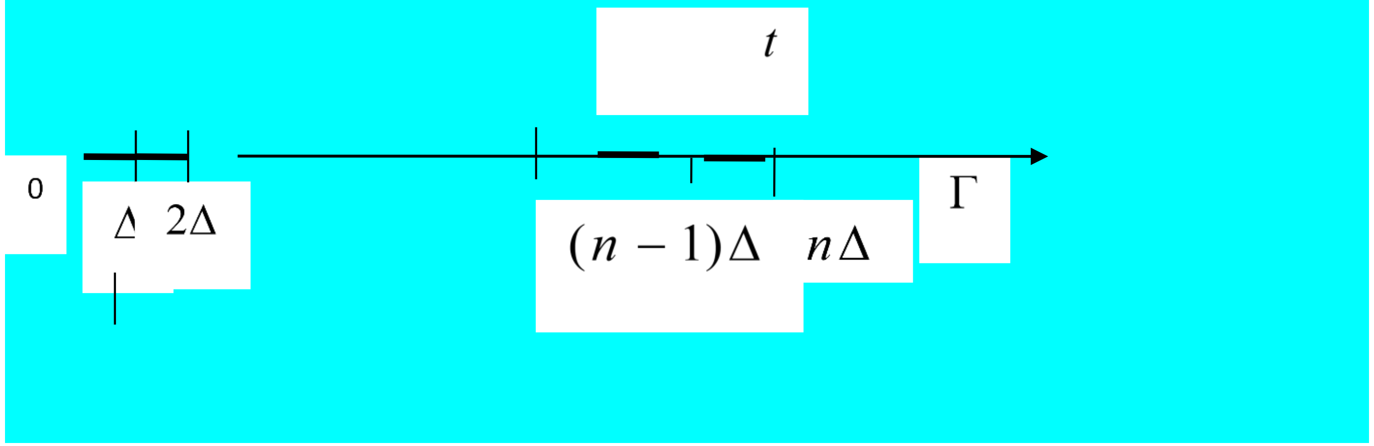
where $n \geq 1$, $\{Z_n\}$ is a sequence of i.i.d. random variables and $X_0 = 0$. If Z_n takes two values on the real line $Z_1 = \begin{cases} 1 \; with \;\; probability \; \frac{1}{2} \\ -1 \; with \;\; probability \; \frac{1}{2} \end{cases}$ then $\{X_n\}$ is called a *symmetrical random walk*.

Suppose a continuous-time process $\{W_n(t)\}$ defined over $\Gamma = [0, \infty)$ by

$$W_n(t) = s\sum_{i=1}^{n} Z_i \quad n\Delta = t$$

where the discrete instants in the time axis are separated by Δ and *s* and –*s* are the RW step .

Assume Δ to be infinitesimally small.



Clearly, $EW_n(t) = 0$ and $\operatorname{var}(W_n(t)) = s^2 4n \frac{1}{2} \times \frac{1}{2} = ns^2 = \frac{t}{\Delta} s^2$

Suppose $\Delta \to 0$ and $n \to \infty$ constrained by the condition that $\lim_{\Delta \to 0} \frac{s^2}{\Delta} = \sigma^2$.

According to the central-limit theorem, the distribution of $W_n(t)$ converges in distribution $W(t) \sim N(0, \sigma^2 t)$.

$$\therefore\ f_{W(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 t}}.$$

Thus the Wiener process can be derived as a limit of the symmetric rando walk process.