### **Continuous-time Markov Chain 2**



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#### We discussed

For a CTMC,

$$P(X(s+t)=j|X(s)=i,X(t_k)=i_k,k=1,2,..,n)=P(X(s+t)=j|X(s)=i)$$

For a homogeneous CTMC,

$$p_{i,j}(t) = P(X(s+t) = j|X(s) = i) = P(X(t) = j|X(0) = i)$$

When a CTMC enters a state i, it spends a random duration  $T_i$  called the state holding time

Distributed as 
$$f_{T_i}(t) = v_i e^{-v_i t}$$
  $v_i > 0$ 

Once the CTMC leaves state i, it jumps to one of the state j. The jumping process is an embedded MC with the transition probability  $P_{i,j}$ ,  $j \neq i$  such that  $\sum P_{ij} = 1$ .

#### We discussed

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Short-time behavior a later for the property of t

$$\lim_{\Delta t \to 0} \frac{1 - P_{ii}(\Delta t)}{\Delta t} = v_i \quad \text{and}$$

$$\lim_{\Delta t \to 0} \frac{P_{ij}(\Delta t)}{\Delta t} = q_{ij} \quad = \text{posterior}$$

where 
$$q_{ij} = v_i P_i$$

## **Chapman Kolmogorov equation for CTMC**

Chapman Kolmogorov Equation:

$$p_{ij}(s+t) = \sum_{k} p_{ik}(s) p_{kj}(t)$$

- ➤ The above transition probabilities are function of timeduration and not the number of steps.
- Use of this difference equation is difficult.
  The dynamics is better studied in terms of two differential equations:

Kolmogorov backward equation and Kolmogorov forward equation

## **Kolmogorov Backward Equation**

$$p_{ij}(t + \Delta t) = P(X(t + \Delta t) = j \mid X(0) = i)$$

$$= \sum_{k} p_{ik}(\Delta t) p_{kj}(t)$$

$$= p_{ii}(\Delta t) p_{ij}(t) + \sum_{k \neq i} p_{ik}(\Delta t) p_{kj}(t)$$

$$= (1 - v_i \Delta t + o(\Delta t)) p_{ij}(t) + \sum_{k \neq i} q_{ik} \Delta t p_{kj}(t)$$

$$\therefore \lim_{\Delta t \to 0} \frac{p_{ij}(t + \Delta t) - p_{ij}(t)}{\Delta t} = -v_i p_{ij}(t) + \sum_{k \neq i} q_{ik} p_{kj}(t)$$

$$\therefore p_{ij}'(t) = -v_i p_{ij}(t) + \sum_{k \neq i} q_{ik} p_{kj}(t)$$

Substituting  $-v_i = q_{i,i}$ , we get

$$p_{ij}'(t) = \sum_{k} q_{ik} p_{kj}(t)$$

### Forward Kolmogorov Equation

$$t + \Delta t$$

Consider the figure as shown above. Here,

$$p_{ij}(t + \Delta t) = \sum_{k} p_{ik}(t) p_{kj}(\Delta t) = p_{ij}(t) p_{jj}(\Delta t) + \sum_{k \neq j} p_{ik}(t) p_{kj}(\Delta t)$$

$$= (1 - v_j \Delta t + o(\Delta t)) p_{ij}(t) + \sum_{k \neq j} p_{ik}(t) (q_{ik} \Delta t + o(\Delta t))$$

$$p_{ij}(t) = -v_j p_{ij}(t) + \sum_{k \neq j} p_{ik}(t) q_{kj}$$

Putting  $q_{ij} = -v_j$ , we can rewrite the above differencial equations as:

Forward Kolmogorov Equation  $p_{ij}'(t) = \sum_{k} p_{ik}(t)q_{kj}$ 

## **Matrix Form of Kolmogorov Equations**

We can define the matrices

$$\mathbf{P}(t) = \begin{bmatrix} p_{00}(t) & p_{01}(t) & \dots \\ p_{10}(t) & p_{11}(t) & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}, \ \mathbf{P'}(t) = \begin{bmatrix} p_{00}'(t) & p_{01}'(t) & \dots \\ p_{10}'(t) & p_{11}'(t) & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$
and 
$$\mathbf{Q} = \begin{bmatrix} q_{00} & q_{01} & \dots \\ q_{10} & q_{11} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\mathbf{Q}_{i,j} = \lambda_{i} \cdot \mathbf{P}_{i,j}$$

## **Matrix Form of Kolmogorov Equations**

73n matrix form, the Kolmogorov backward and forward equations can be written as

$$\mathbf{P'}(t) = \mathbf{QP}(t)$$

and  $\mathbf{P'}(t) = \mathbf{P}(t)\mathbf{Q}$ 

**Example** A certain system has two states – under operation state 1 and under repair state 0. The duration of operation and repair are exponential RVs with rate parameters  $\lambda$  and  $\mu$  respectively.

Find 
$$P(t) = \begin{bmatrix} \rho_{00}(t) & \rho_{01}(t) \\ \rho_{10}(t) & \rho_{11}(t) \end{bmatrix}$$
 and analyse if  $q(t) = -3t$ .

Solution-
The rate matrix
$$Q = \begin{bmatrix} q_{00}(t) & p_{11}(t) \\ q_{10}(t) & p_{11}(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

$$\Rightarrow \lambda \cdot P_{00} = -3t$$

The forward kolmogrov equation is given by

$$P'(t) = P(t)Q$$

#### Solution-

The rate matrix 
$$\mathbf{Q} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

The forward kolmogrov equation is given by

$$P'(t) = P(t)Q$$

$$\Rightarrow \begin{bmatrix} p'_{00}(t) & p'_{01}(t) \\ p'_{10}(t) & p'_{11}(t) \end{bmatrix} = \begin{bmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{bmatrix} \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

$$\therefore p'_{00}(t) = -\mu p_{00}(t) + \lambda p_{01}(t) \\ = -\mu p_{00}(t) + \lambda (1 - p_{00}(t)) \quad \therefore p_{00}(t) + p_{01}(t) = 1$$

We have to solve the linear differential equation

$$p'_{00}(t) = -(\mu + \lambda)p_{00}(t) + \lambda$$
 with the initial condition

$$\therefore p_{00}(t) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

$$p(x(0) = 0 | x(0) = 0)$$
 $p_{00}(0) = 1$ 

Similarly, 
$$p_{1,1}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

Now

$$p_{01}(t) = 1 - p_{00}(t)$$

$$= 1 - \frac{\lambda}{\lambda + \mu} - \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

$$= \frac{\mu}{\lambda + \mu} - \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

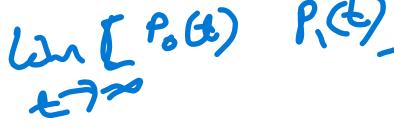
Similarly,

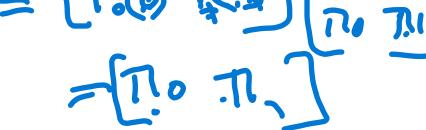
$$p_{10}(t) = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

$$\lim_{t \to \infty} \mathbf{P}(t) = \begin{bmatrix} \frac{\lambda}{\lambda + \mu} & \frac{\mu}{\lambda + \mu} \\ \frac{\lambda}{\lambda + \mu} & \frac{\mu}{\lambda + \mu} \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{bmatrix}$$

also observe that, irrespective of the values of the initial state probabilities  $[p_0(0) \quad p_1(0)]$ , the steady-state state probabilities are given by  $[\pi_0 \quad \pi_1]$ 

$$\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix}$$





The above example illustrates a remarkable property of the CTMC without proof:

If  $\lim_{t\to\infty} p_{i,j}(t)$  exists, then

 $\lim_{t\to\infty} p_{i,j}(t) = \pi_j$  independent of i where  $\pi_j$  is the probability of the state j at the steady state.

## To Summarise

- To characterize the transition probabilities dynamically, Kolmogorov backward and forward differential equations are used.
- Backward Kolmogorov Equation

$$p_{ij}'(t) = \sum_{k} q_{ik} p_{kj}(t)$$

• Forward Kolmogorov Equation  $p_{ij}'(t) = \sum_{k} p_{ik}(t) q_{kj}$ 

# **Summary**

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Backward equation

$$\mathbf{P'}(t) = \mathbf{QP}(t)$$

and forward equation

$$\mathbf{P'}(t) = \mathbf{P}(t)\mathbf{Q}$$

