MA 102 (Mathematics II)
IIT Guwahati

Tutorial Sheet No. 2 | Linear Algebra | January 24, 2019

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1. Let A be 4×3 matrix such that $\text{rank}(A) = 3$. Then show that there exists a 3×4 matrix B such that $BA = I_3$.

> **Solution:** The rref of A is of the form $[I_3, \mathbf{0}]^T$. Hence there exists a invertible P such that $PA = [I_3, \mathbf{0}]^T$. Take $B = [I_3, \mathbf{0}]P$, then $BA = I_3$.

2. Find all the solutions of the linear system with the augmented matrix $[A|\mathbf{b}]$ as given below:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 2 \\ 5 & 6 & 7 & 8 & 5 \\ 9 & 10 & 11 & 12 & 8 \end{array}\right]$$

   (a) Find $\mathbf{b}'$ such that $A\mathbf{x} = \mathbf{b}'$ does not have a solution.

   (b) By changing exactly one entry of A, find an A' such that $A'\mathbf{x} = \mathbf{b}$ will be consistent for all $\mathbf{b} \in \mathbb{R}^3$.

> **Solution:** Solution set$= \{ \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{4} \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} | \alpha, \beta \in \mathbb{R}\}$.
> a) Since $R_3 = 2R_2 - R_1$, where R_i is the ith row of A, take $\mathbf{b}'$ such that $b_3' \neq 2b_2' - b_1'$.
> b) Since $R_3 = 2R_2 - R_1$, and no two rows are LD, change any one entry of A then the rows of A will be LI or $rank(A) = 3$.

3. Let $A \in \mathcal{M}_5(\mathbb{R})$ be invertible with row sums 1. Show that the sum of all the elements of A^{-1} is 5.

> **Solution:** Let $\mathbf{1} = [1, 1, 1, 1, 1]^T$. Then $A\mathbf{1} = [1, 1, 1, 1, 1]^T = \mathbf{1}$, which gives $A^{-1}\mathbf{1} = \mathbf{1}$, i.e., A^{-1} has row sums 1 and the result follows.

4. True or False? Give justifications.

   (a) If for all $A \in \mathcal{M}_n(R)$, $AB = A$ then $B = I_n$.

   (b) If A and B are square matrices of order n with $AB = I_n$ then A and B are invertible and $BA = I_n$.
   Hint: If P is invertible then $\text{rank}(P) = n$. $AB = I$ implies there exists an invertible P such that $PAB = P$, where PA is in rref.

   (c) If A is an $m \times n$ matrix with at least one nonzero row (at least one entry of this row is nonzero) then A is row equivalent to a matrix B, with all nonzero rows.

(d) If all the columns of an $n \times m$ nonzero matrix (it has at least one nonzero entry) A are equal then $rank(A) = 1$.

(e) If A is an $m \times n$ matrix with a zero column (all entries of the column is zero) then the rref of A will again have a zero column.

(f) If P is any invertible matrix such that PA is defined then, $Ax = b$ and $PAx = Pb$ are equivalent.

Solution:

(a) True, take $A = I_n$.

(b) True. Observation: If P is invertible then $\text{rank}(P) = n$.
$AB = I$ implies there exists an invertible P such that $PAB = P$, where PA is in rref. Since P is invertible, PAB cannot have a zero row, hence PA cannot have a zero row. So $PA = I_n$ or $A = P^{-1}$ and $B = P$. $AB = I$ implies $B(AB)B^{-1} = I = BA(BB^{-1}) = BA$.

(c) True. If the rref of A has a zero row, say $\tilde{a}_i$, then replace $\tilde{a}_i$ with $\tilde{a}_i + \tilde{a}_j$, where $\tilde{a}_j$ is some nonzero row of the rref.

(d) True. (Each row of A is a multiple of some nonzero row of A.)

(e) True.

(f) True.

5. Using Gauss Jordan elimination prove that

$$\left\{ \alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\} + \left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} : \alpha \in \mathbb{R} \right\} + \left\{ \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\} = \mathbb{R}^3.$$

Solution: Check that the rref of A is I_3. Therefore, for any $\mathbf{b} \in \mathbb{R}^3$, the system $A\mathbf{x} = \mathbf{b}$ is consistent, where the columns of A are given by $[2, 1, 1]^T, [1, 1, 0]^T$ and $[0, 1, 1]^T$. Thus, $\mathbf{b}$ is a linear combination of $[2, 1, 1]^T, [1, 1, 0]^T$ and $[0, 1, 1]^T$, and therefore, $\mathbb{R}^3$ is a subset of the set in the left. That the set in the left is a subset of $\mathbb{R}^3$ is obvious.

6. If A is upper triangular and B is any matrix such that $AB = I$, then show that each diagonal entry of A is nonzero.

Solution: Note that A is square, suppose of order n. Suppose $R = \text{RREF}(A) = PA$, where P is invertible. Now, if A has at least one zero diagonal entry, consider the least i such that $a_{ii} = 0$, then the corresponding column of R is a nonleading column. Thus, R has less than n leading columns, and so has a zero row. Consequently, $RB = PAB = PI = P$ has a zero row, which is not possible because P is invertible.

7. Show that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 = 2x_3 + x_2 \right\}$ is a subspace of $\mathbb{R}^3$.

   (a) Find $\{\mathbf{u}, \mathbf{v}\}$ such that $span\{\mathbf{u}, \mathbf{v}\} = S$.

   (b) Find a $\mathbf{v}'$ such that $span\{\mathbf{u}, \mathbf{v}'\} = span\{\mathbf{v}, \mathbf{v}'\} = S$.

   (c) Find an $\mathbf{u}'$ such that $span\{\mathbf{u}', \mathbf{v}'\}$ is not a subspace of S. Geometrically what will be the picture of S and $span\{\mathbf{u}', \mathbf{v}'\}$?

Solution: a) Since $S = \{\alpha \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} |\alpha, \beta \in \mathbb{R}\}$, one choice can be $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

b)Take any $\mathbf{v}' \in S$ but not in $span\{\mathbf{u}\}$ or $span\{\mathbf{v}\}$. For example take $\mathbf{v}' = \mathbf{u} + \mathbf{v}$.

c) Take $\mathbf{u}'$ not in S, then $span\{\mathbf{u}', \mathbf{v}'\}$ will correspond to a plane in $\mathbb{R}^3$ and will intersect the plane associated with S in a line given by $span\{\mathbf{v}'\}$.

8. By using Gauss Jordan elimination find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 12 \end{bmatrix}.$$

9. Using LU factorization of the matrix A solve the system of linear equations with the augmented matrix $[A|\mathbf{b}]$ as given below:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 1 & 2 & 3 & 4 & 30 \\ 1 & 4 & 8 & 15 & 93 \\ 1 & 3 & 6 & 10 & 65 \end{array}\right].$$

10. Show that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 = 2x_3 - x_2,\ 2x_2 = x_3 \right\}$ is a subspace of $\mathbb{R}^3$.
    Find an $\mathbf{u}$ such that $span\{\mathbf{u}\} = S$. Find an $\mathbf{u}'$ such that $span\{\mathbf{u}, \mathbf{u}'\}$ gives a plane in $\mathbb{R}^3$. Find a $\mathbf{v}$ such that $span\{\mathbf{v}\}$ is not a subspace of $span\{\mathbf{u}, \mathbf{u}'\}$. What will be the $span\{\mathbf{u}, \mathbf{u}', \mathbf{v}\}$?

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