PH 102: Physics II

Lecture 15 (Spring 2019)

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IIT Guwahati

Course Details

- The details of the course is in moodle: https://intranet.iitg.ernet.in/moodle/.
- The course name is PH 102 (2019) with enrolment key: **ph102em**.
- All relevant materials related to the course will be uploaded in the same moodle page.
- The details of the tutorial groups and anything related to the pre-midsem part you may still access via http://www.iitg.ac.in/phy/ph102.php.
- For any queries, please write to dborah@iitg.ac.in

Texts:

1. D. J. Griffiths, *Introduction to Electrodynamics*, 3rd Ed., Prentice-Hall of India, 2005.
2. A.K.Ghatak, *Optics*, Tata Mcgraw Hill, 2007.

References:

1. N. Ida, *Engineering Electromagnetics*, Springer, 2005.
2. M. N. O. Sadiku, *Elements of Electromagnetics*, Oxford, 2006.
3. R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics, Vol.II*, Norosa Publishing House, 1998.
4. I. S. Grant and W. R. Phillips, *Electromagnetism*, John Wiley, 1990.

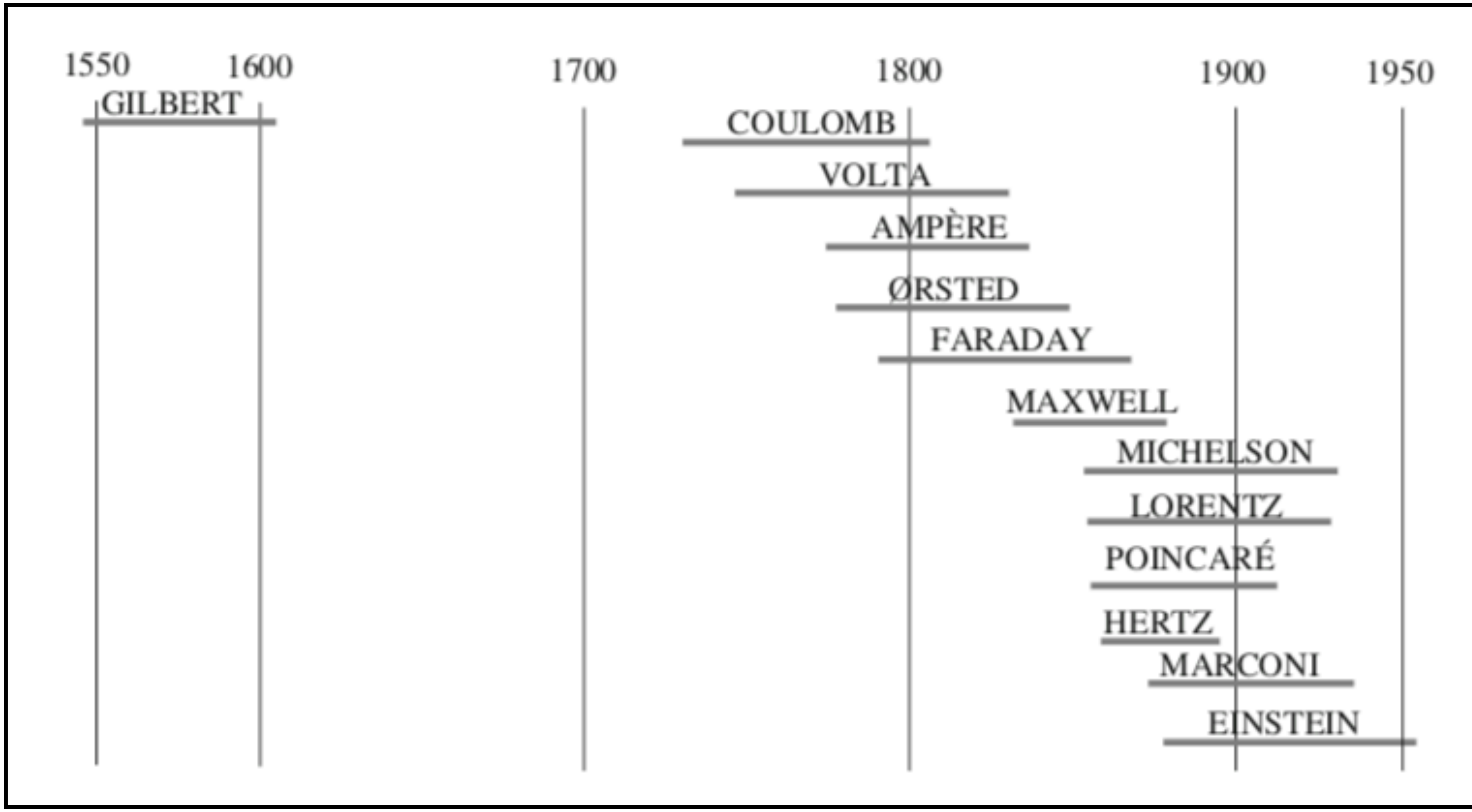
LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Lectures	Division
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	I, II (4-4:55 pm)
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	III, IV (9-9:55 am)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	I, II (3-3:55 pm)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	III, IV (10-10:55 am)
Tut 8	19-3-2019	Lec 15, 16			
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic Materials, Magnetization	5.4, 6.1	1	I, II (4-4:55 pm)
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic Materials, Magnetization	5.4, 6.1	1	III, IV (9-9:55 am)
Lec 18	22-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	I, II (3-3:55 pm)
Lec 18	22-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	III, IV (10-10:55 am)
Tut 9	26-3-2019	Lec 17, 18			
Lec 19	27-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	I, II (4-4:55 pm)
Lec 19	27-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	III, IV (9-9:55 am)
Lec 20	28-3-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	I, II (3-3:55 pm)
Lec 20	28-3-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	III, IV (10-10:55 am)
Tut 10	2-4-2019	Lec 19, 20			
Lec 21	3-4-2019	Maxwell's equations	7.3	1	I, II (4-4:55 pm)
Lec 21	3-4-2019	Maxwell's equations	7.3	1	III, IV (9-9:55 am)
Lec 22	4-4-2019	Discussions, problem solving	7.3	1	I, II (3-3:55 pm)
Lec 22	4-4-2019	Discussions, problem solving	7.3	1	III, IV (10-10:55 am)
Tut 11	9-4-2019	Quiz II			

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	I, II (4-4:55 pm)
Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	III, IV (9-9:55 am)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	I, II (3-3:55 pm)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	III, IV (10-10:55 am)
Tut 12	23-4-2019	Lec 23, 24			
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	I, II (4-4:55 pm)
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	III, IV (9-9:55 am)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	I, II (3-3:55 pm)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	III, IV (10-10:55 am)
Tut 13	30-4-2019	Lec 25, 26			
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	I, II (4-4:55 pm)
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	III, IV (9-9:55 am)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	I, II (3-3:55 pm)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	III, IV (10-10:55 am)

Electromagnetism: a brief history



credit: Springer

Static Charge

- In electrostatics, the field created by a charge q is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

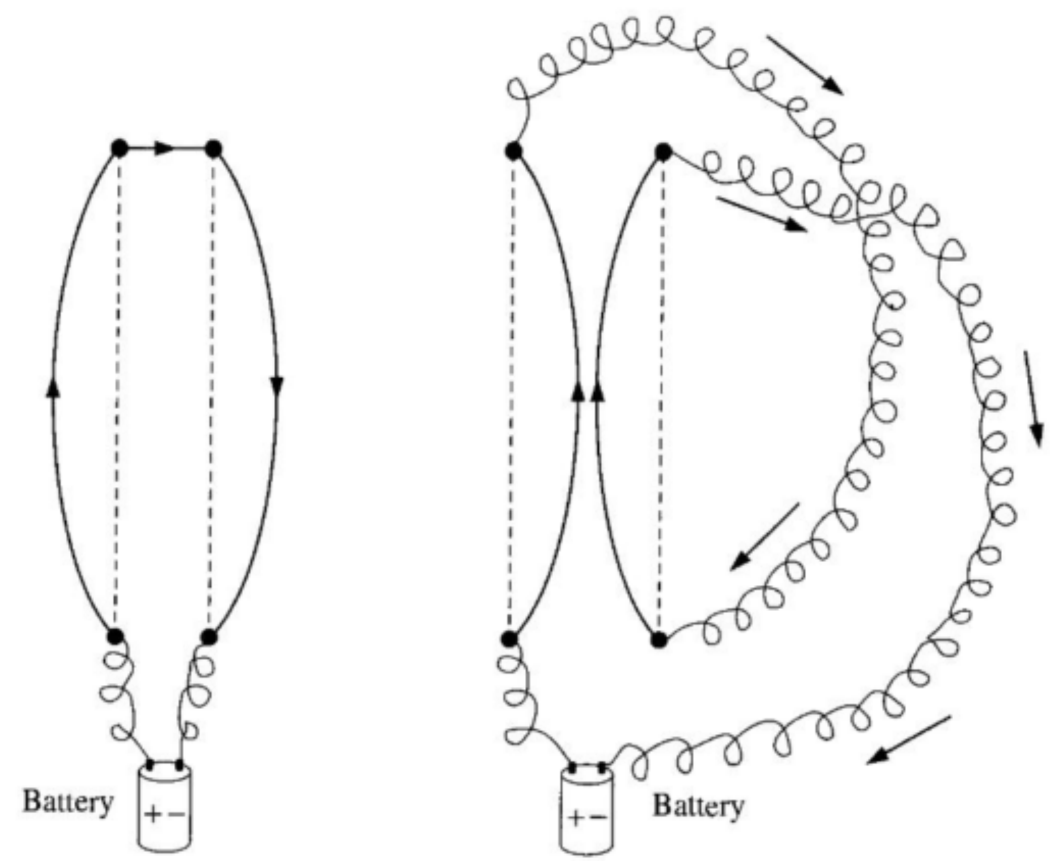
- The force felt by a test charge Q is $\vec{F} = Q\vec{E}$

- The torque on an electric dipole is $\vec{\tau} = \vec{p} \times \vec{E}$

- Force on the test charge due to the presence of several charges, in accordance with the principle of superposition is $\vec{F} = \sum_i Q\vec{E}_i$

Charges in Motion

- What if the charges are in motion?
- While static charges produce only electric field, moving ones produce a magnetic field too.
- Such magnetic fields can be easily detected by a magnetic needle (compass).
- A current carrying wire creates its own magnetic field. This can exert a magnetic force on a charge in motion or another current carrying wire.
- Wires carrying current (charges in motion) in opposite (same) directions repel (attract).

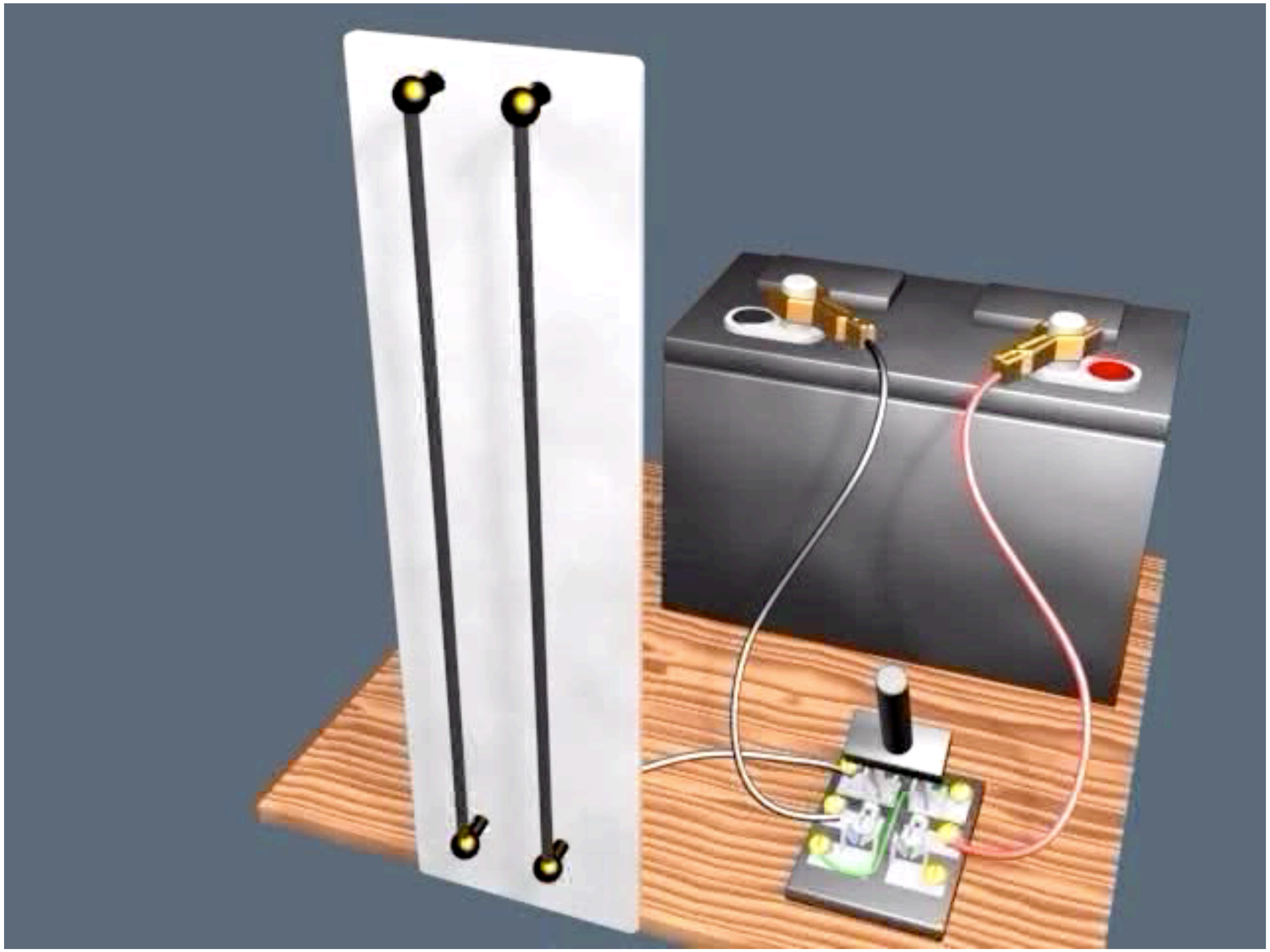


(a) Currents in opposite directions repel.

(b) Currents in same directions attract.

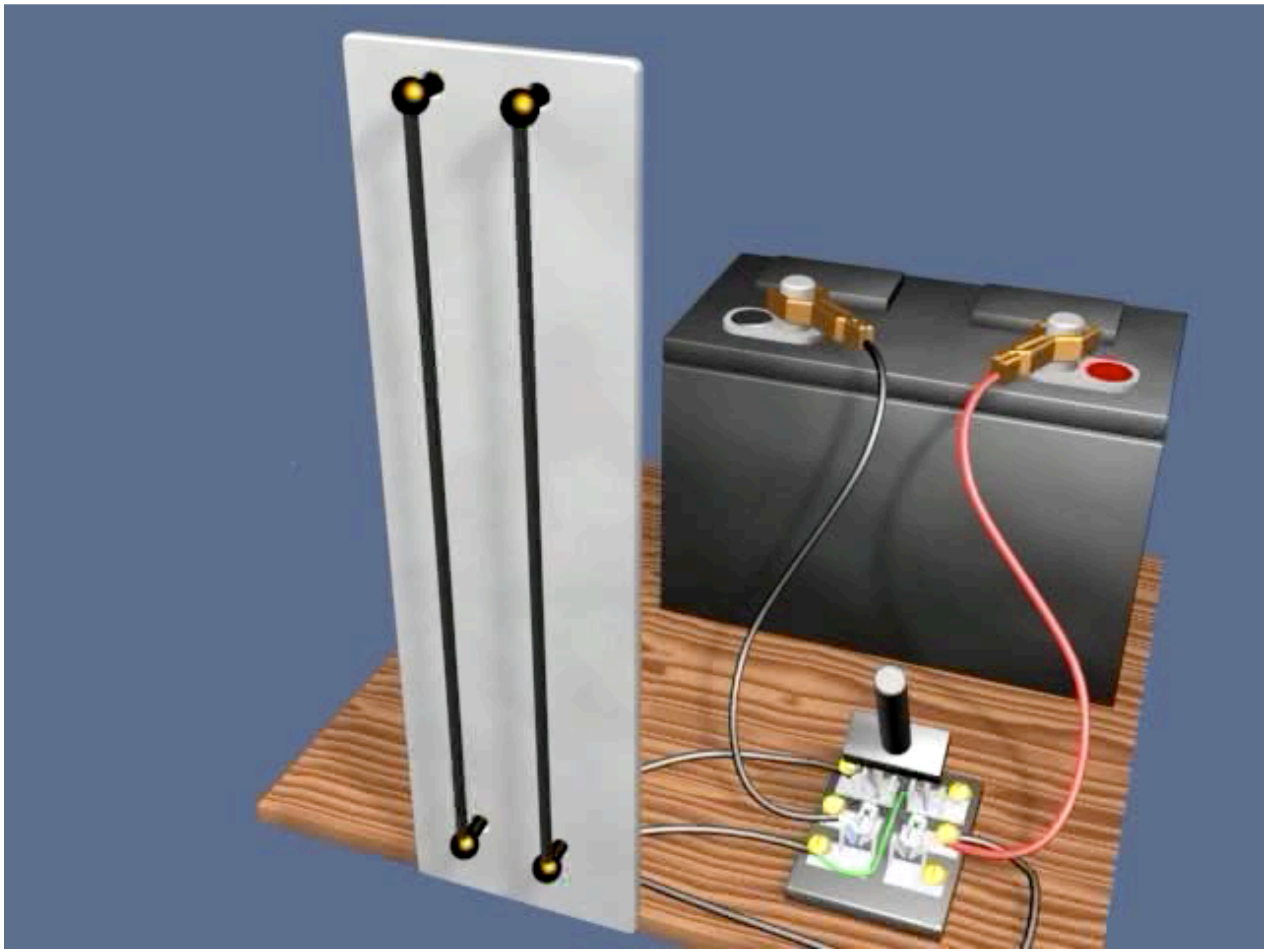
Fig. 5.2 (Introduction to Electrodynamics, D. J. Griffiths)

Parallel currents



Visualisation credit: MIT

Anti-parallel currents

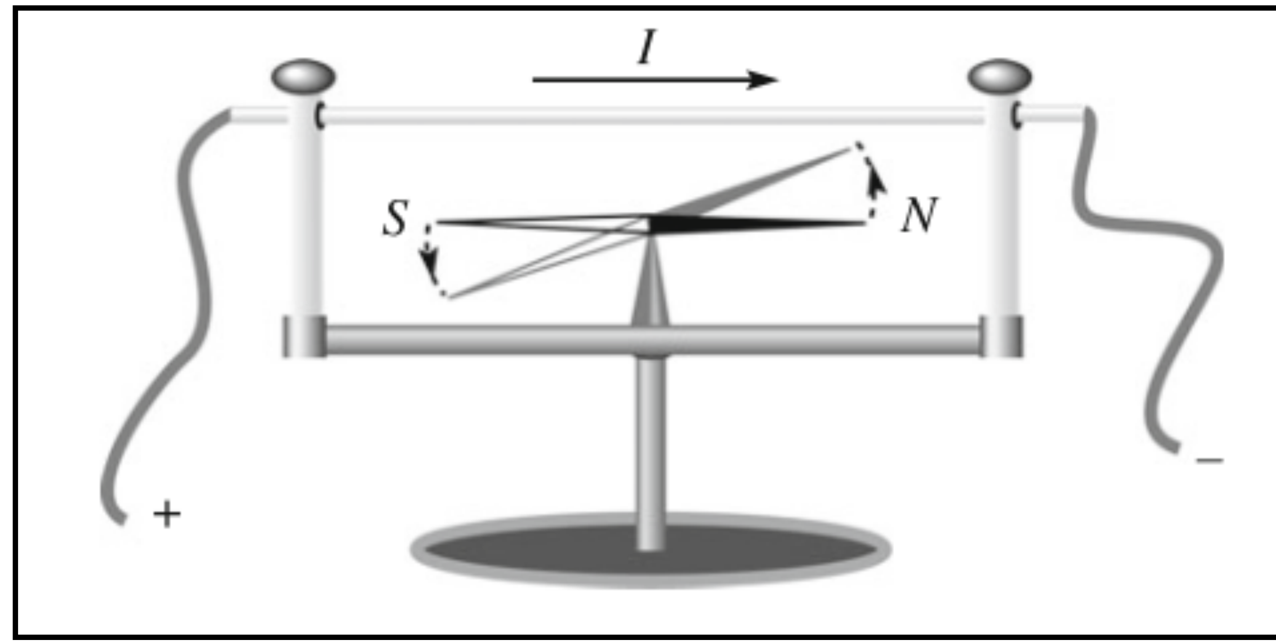


Visualisation credit: MIT

Orsted Experiment

Electric current produces magnetic field (H C Orsted, 1820).

Magnetic needle was found to change its equilibrium position when the current is turned on.



Orsted experiment, Credit: Springer

What would have happened if Orsted had kept the wire at right angle to the compass needle?

Magnetic Force

- Magnetic force on a charged particle having charge Q moving with velocity $\vec{v}$ in the presence of a magnetic field $\vec{B}$ is

$$\vec{F}_{\text{magnetic}} = Q(\vec{v} \times \vec{B})$$

- In the presence of both electric and magnetic field the net force is

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$ Lorentz Force Law

See Example 5.1, 5.2 (Introduction to Electrodynamics, D. J. Griffiths)

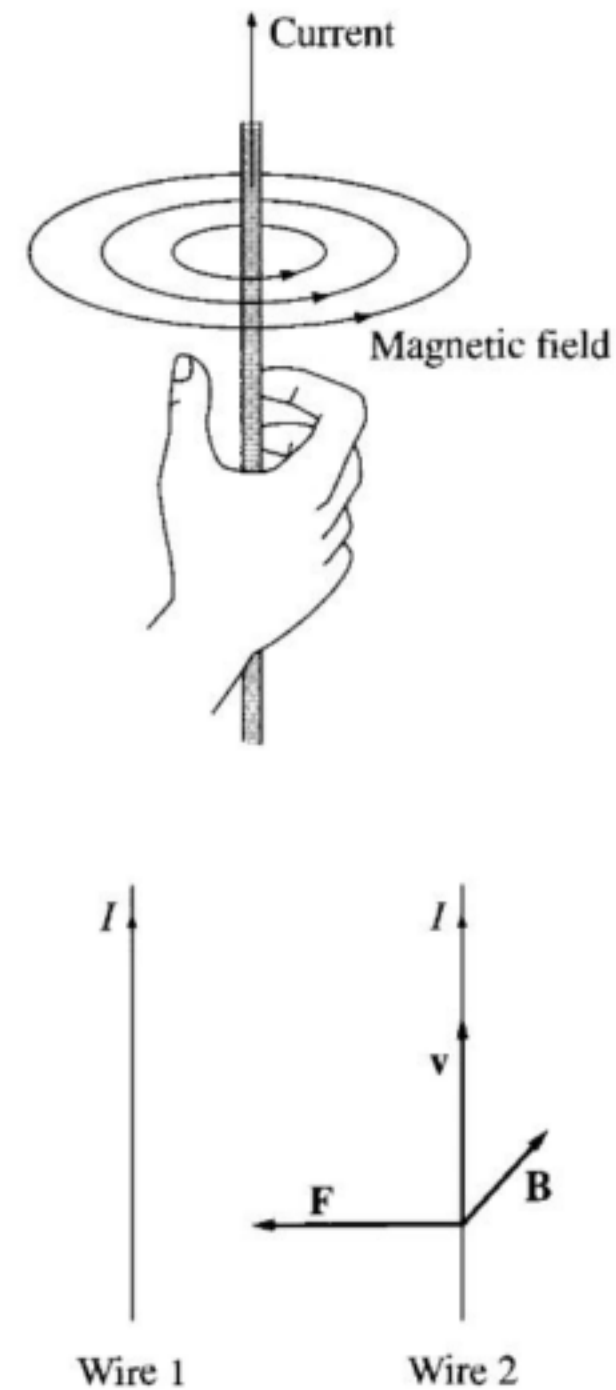
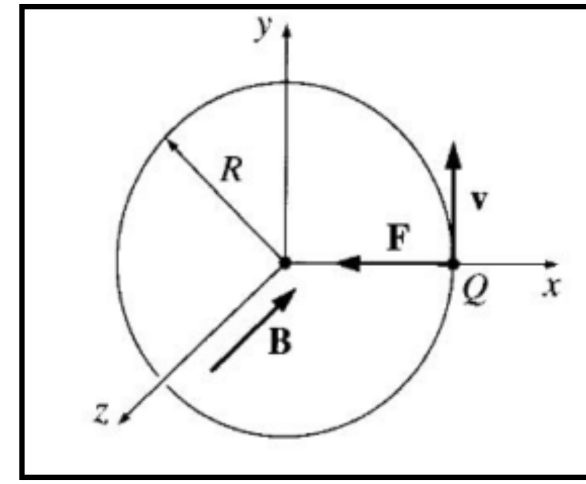


Fig. 5.3, 5.4 (Introduction to Electrodynamics, D. J. Griffiths)

Cyclotron: Motion of a charged particle in a magnetic field

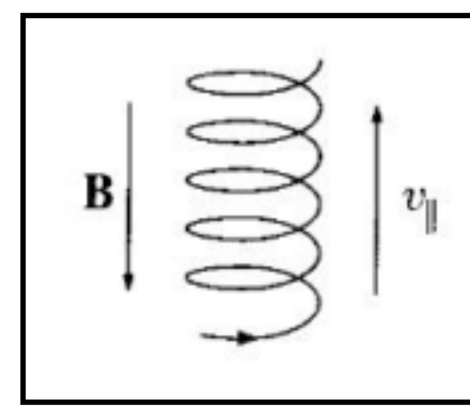


For $\vec{v} \perp \vec{B}$ we can write

$$QvB = m\frac{v^2}{R} \implies p = QBR$$

which is used for measuring particle momentum. If velocity has some component parallel to the magnetic field, then magnetic field does not affect the parallel motion.

The particle then moves in a helix.



$$Qv_\perp B = m\frac{v_\perp^2}{R}$$

The cyclotron is a charged particle accelerator invented by Ernest Orlando Lawrence (USA, 1901–1958) in 1932.

Fig. 5.5, 5.6 (Introduction to Electrodynamics, D. J. Griffiths)

Cyclotron

Charged particle can be accelerated to a very high speed as they complete a large number of round trips across the gap maintained at some fixed potential difference V.

Gain in speed in one trip across the gap is: $v = \sqrt{2qV/m}$

Gain in speed in n trips:

$$v_n = \sqrt{2nqV/m}$$

1 T magnetic field can accelerate a proton to a speed ~ 30% of c with just 100 volts peak voltage in fraction of a second. An electrostatic accelerator will need millions of volts to achieve the same.

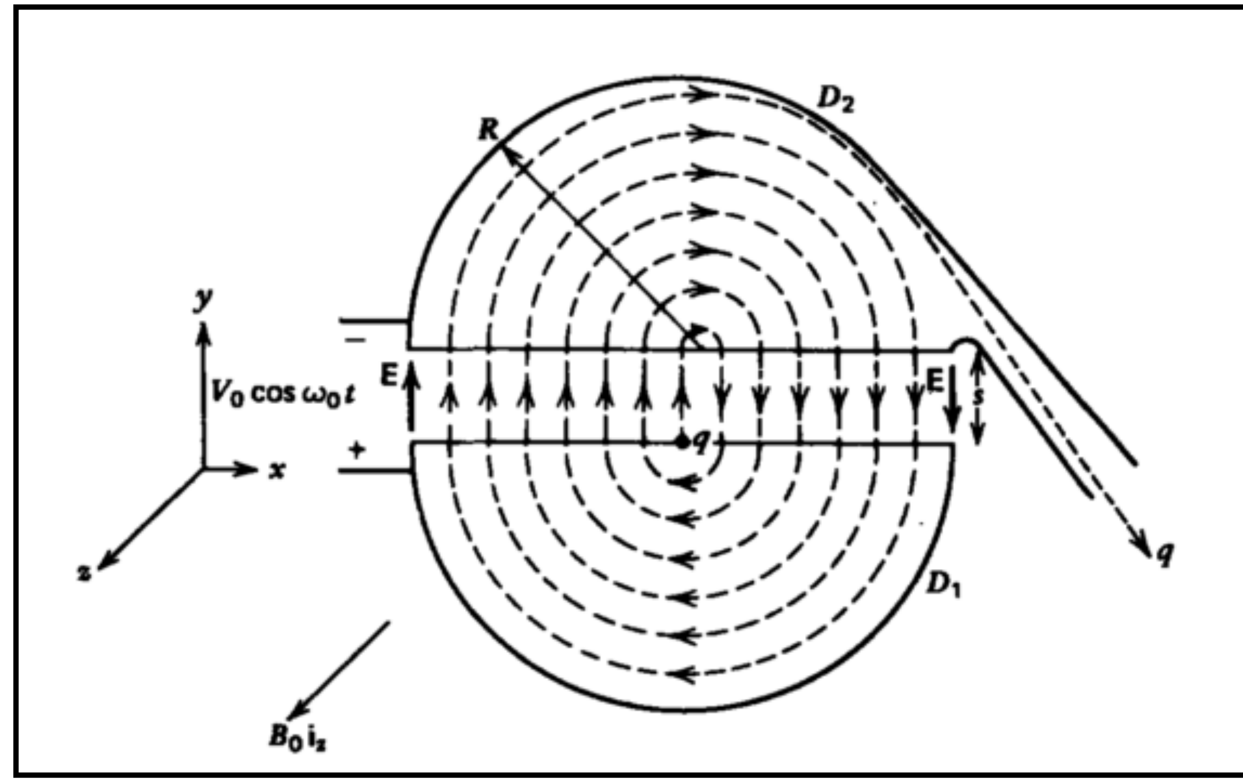


Image credit: Wiley

Another interesting application of Lorentz force law is Hall Effect: PH 110

Cycloid Motion:

Motion of charged particles in simultaneous presence of electric and magnetic fields at right angles to each other.

For $\vec{E} = E\hat{z}, \vec{B} = B\hat{x}$, the charged particle initially at the origin will move in the y-z plane and hence its velocity is $\vec{v} = \dot{y}\hat{y} + \dot{z}\hat{z}$

Why no motion along x-direction?

The Lorentz Force law says:

$$F = Q(\vec{E} + \vec{v} \times \vec{B}) = Q(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}) = m\vec{a} = m(\ddot{y}\hat{y} + \ddot{z}\hat{z})$$

which gives the equations of motion as: $m\ddot{y} = QB\dot{z}, \;\; m\ddot{z} = QE - QB\dot{y}$

Denoting $\omega = QB/m$, the equations are: $\ddot{y} = \omega\dot{z}, \;\; \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$

Differentiating the 1st and then using the second equation:

$$\dddot{y} = \omega^2\left(\frac{E}{B} - \dot{y}\right) \implies y(t) = C_1\cos\omega t + C_2\sin\omega t + (E/B)t + C_3$$

Using y(t) in the second equation:

$$\ddot{z} = \omega(E/B + C_1\omega \sin\omega t - C_2\omega\cos\omega t - E/B) = \omega^2(C_1 \sin\omega t - C_2\cos\omega t)$$

$$\implies z(t) = C_2\cos\omega t - C_1\sin\omega t + C_4$$

Using the initial conditions: $\dot{y}(0) = 0, \dot{z}(0) = 0, y(0) = 0, z(0) = 0$

one can find $C_3 = -C_1, C_2 = -\frac{E}{\omega B}, C_4 = -C_2, C_1 = 0$

The final solutions are: $y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t),\ \ z(t) = \frac{E}{\omega B}(1 - \cos\omega t)$

which is a circle $(y - R\omega t)^2 + (z - R)^2 = R^2, R = E/(\omega B)$ whose centre moves in the y-direction with speed $v = \omega R = E/B$

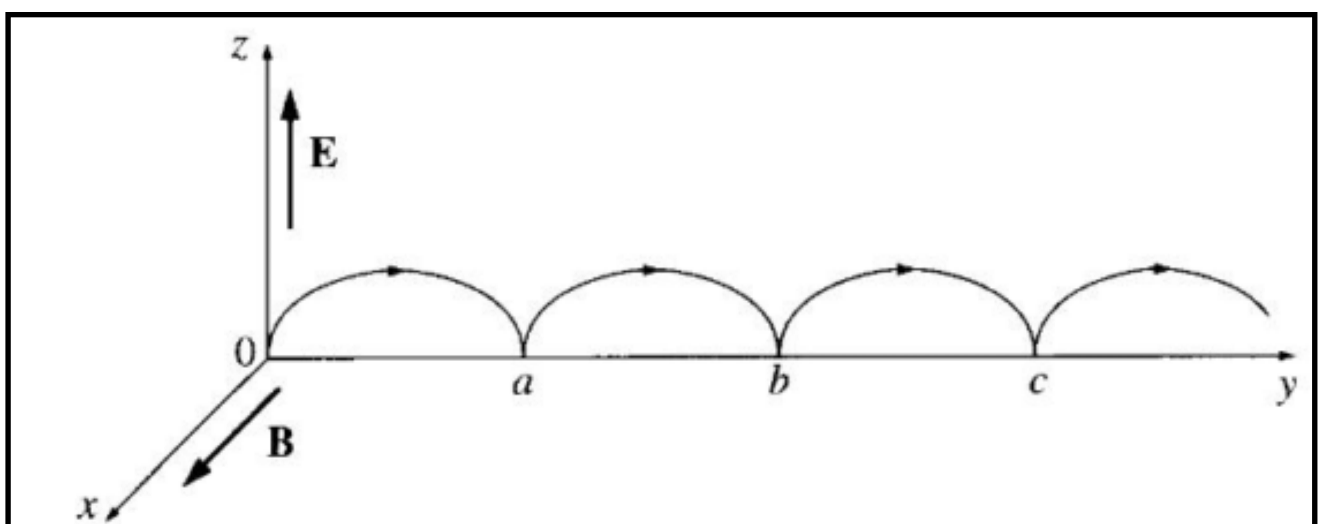


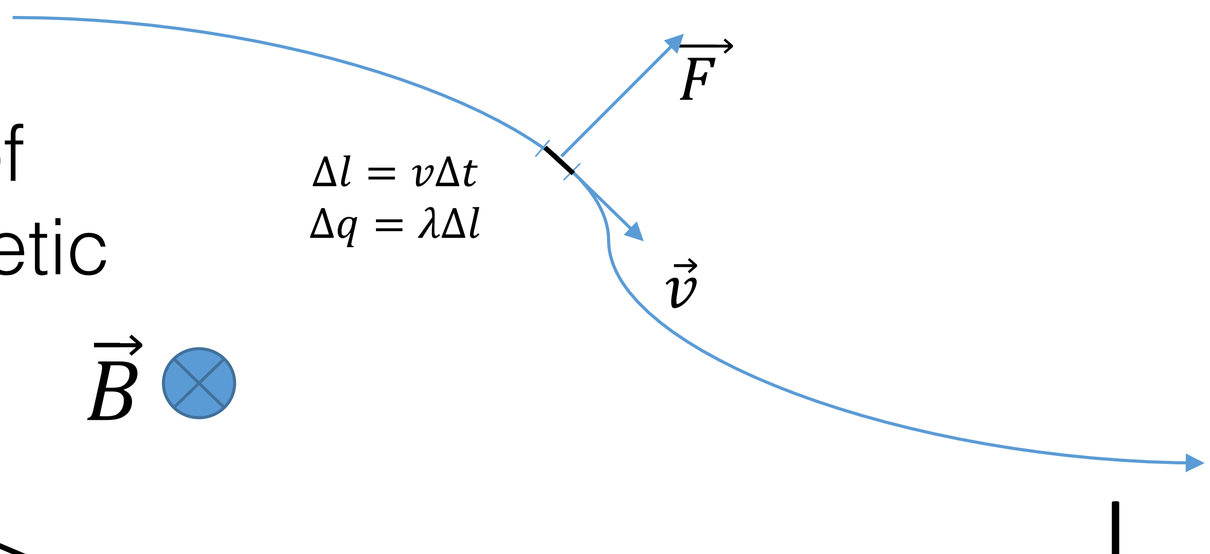
Fig. 5.7 (Introduction to Electrodynamics, D. J. Griffiths)

Motion is similar to that of a spot on the rim of a wheel rolling in y direction: **Cycloid motion**

Magnetic Force on current carrying wire

- Current passing through a wire can be written as $\vec{I} = \lambda\vec{v}$ where λ is the charge per unit length and $\vec{v}$ is the velocity.

- The magnetic force on a segment of this wire in the presence of a magnetic field B is



$$\vec{F} = \int(\vec{v}\times\vec{B})dq = \int(\vec{I}\times\vec{B})dl$$

(Using $dq = \lambda dl, \lambda\vec{v} = \vec{I}$)

- As I and dl point in the same direction we can write it as $\vec{F} = \int I(\vec{dl}\times\vec{B})$

- For constant current $\vec{F} = I\int(\vec{dl}\times\vec{B})$

Magnetic Force on surface & volume current

- Let the dI be the current in a ribbon of infinitesimal width $dl_{\perp}$. The surface current density is $\vec{K} \equiv \frac{d\vec{I}}{dl_{\perp}}$.
- If the surface charge density is σ and its velocity is $\vec{v}$ then the surface current density is $\vec{K} = \sigma\vec{v}$.
- The magnetic force on the surface current is given by $\vec{F} = \int(\vec{v} \times \vec{B})\sigma da = \int(\vec{K} \times \vec{B})da$
- If current in a tube of infinitesimal cross section $da_{\perp}$ is $d\vec{I}$, the volume current density is $\vec{J} \equiv \frac{d\vec{I}}{da_{\perp}}$
- If the volume charge density is ρ and the velocity is $\vec{v}$ then $\vec{J} = \rho\vec{v}$

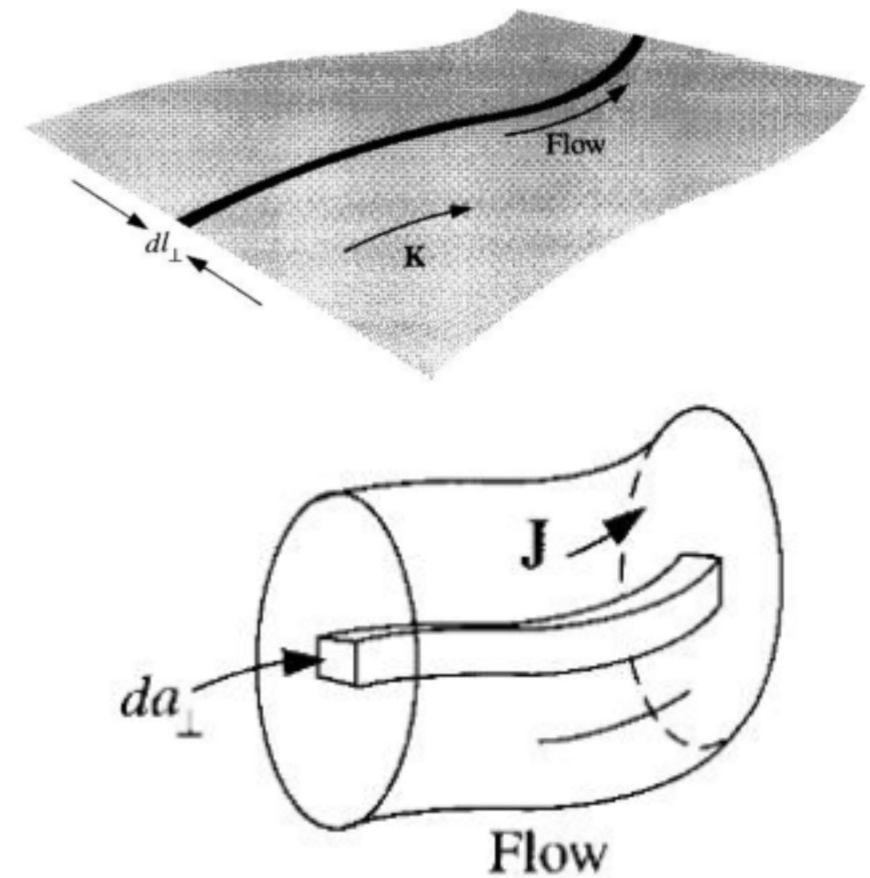
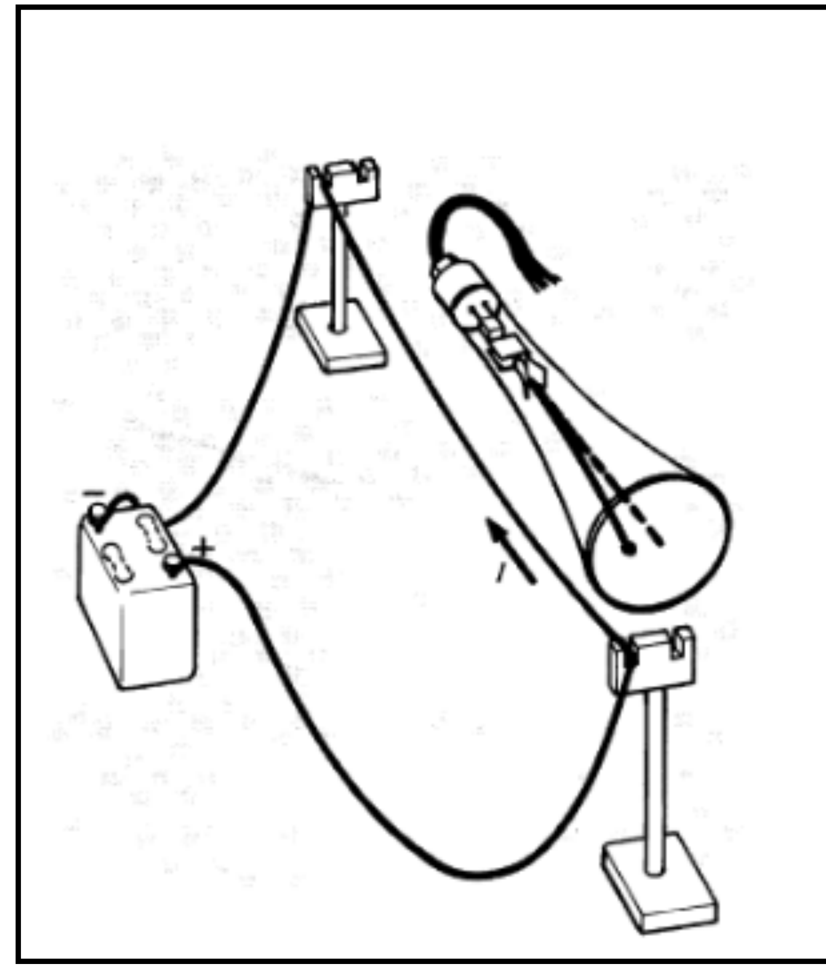


Fig. 5.13, 5.14 (Introduction to Electrodynamics, D. J. Griffiths)

- The magnetic force on a volume current is $\vec{F} = \int(\vec{v} \times \vec{B})\rho d\tau = \int(\vec{J} \times \vec{B})d\tau$

Cathode ray tube: It can be explained as attraction of currents in same direction or deflection of electrons in the beam by a magnetic field produced by current carrying wire. (PH 110)



Continuity Equation

- Since $\vec{J} \equiv \frac{d\vec{I}}{da_\perp}$, the current crossing a surface S can be written as $$I = \int_S J da_\perp = \int_S \vec{J} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{J}) d\tau$$
- Since electromagnetic charge is conserved, the net flow of current through the surface should be equal to the rate of change of charge density inside the volume enclosed by the surface that is,

$$\int_V (\vec{\nabla} \cdot \vec{J}) d\tau = -\frac{d}{dt} \int_V \rho d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

- Since this applies to any volume, we can write

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

Continuity Equation

Magnetic Forces do no work!

- Displacement of a moving charged particle in time dt is $d\vec{l} = \vec{v}dt$
- The work done is therefore

$$dW_{\text{magnetic}} = \vec{F}_{\text{magnetic}} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v}dt = 0$$

- Therefore, magnetic force may alter the direction in which the charged particle moves, but it can not alter its speed.

See example 5.3 (Introduction to Electrodynamics, D J Griffiths)

Magnetostatics

- Stationary charges $\longrightarrow$ Constant electric fields: Electrostatics. Steady current $\longrightarrow$ Constant magnetic fields: Magnetostatics.
- Steady current: continuous flow of current without change and without charge piling up anywhere.
- A moving point charge can not constitute a steady current, in strict sense.
- When a steady current flows through a wire, its magnitude I must be same all along the line so that there is no piling up of charges anywhere.
- In magnetostatics: $\dfrac{\partial \rho}{\partial t} = 0 \implies \vec{\nabla} \cdot \vec{J} = 0$

Magnetostatics

The magnetic field of a steady line current is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{\mathfrak{r}}}{\mathfrak{r}^2}dl' = \frac{\mu_0}{4\pi}I\int \frac{d\vec{l'}\times\hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

Biot-Savart Law

where the integration is along the current path and μ_0 is the permeability of free space

$$\vec{\mathfrak{r}} = \vec{r} - \vec{r'}$$

$$\mu_0 = 4\pi \times 10^{-7}\mathrm{N/A^2}$$

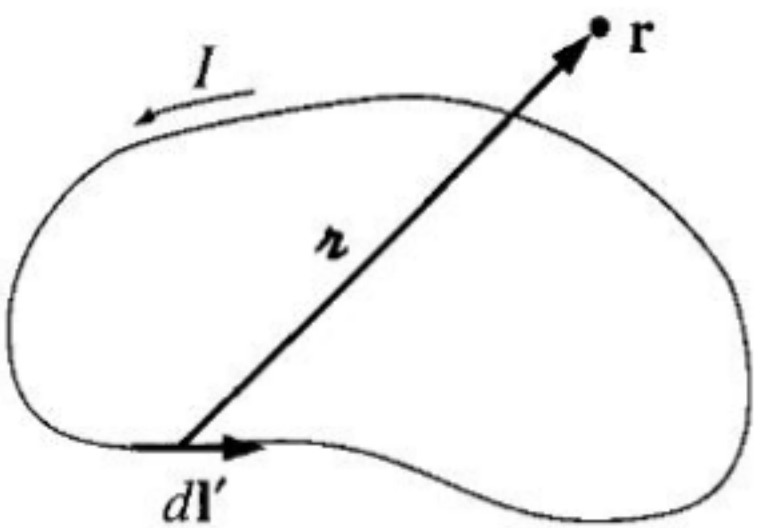


Fig. 5.17 (Introduction to Electrodynamics, D. J. Griffiths)

Unit of B (from Lorentz force law) is = Newton/(Coulomb metre/second)=Newton/(Ampere metre).

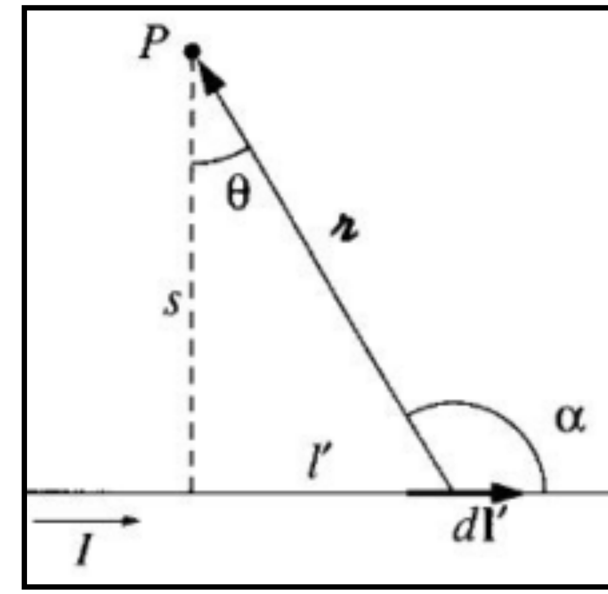
1 N/(A. m) = 1 Tesla (T) = 10000 Gauss (G)

$$[\mathrm{B}] = \left[\mathrm{FQ^{-1}V^{-1}}\right] = \left[\mathrm{MT^{-1}Q^{-1}}\right]$$

†Nikola Tesla (1856–1943) inventor and electrical engineer for whom the SI unit was named, invented the alternating-current induction motor and other useful electromagnetic devices. Gauss's work in magnetism was concerned mainly with the earth's magnetic field. Perhaps this will help you to remember which is the larger unit.

E M Purcell

Example 5.5 (Introduction to Electrodynamics, D. J. Griffiths):
Find the magnetic field at a distance s from a long straight wire carrying a steady current I.



Using Biot-Savart law: $B(\vec{s}) = \frac{\mu_0}{4\pi} I \int \frac{|d\vec{l'} \times \hat{\mathfrak{r}}|}{\mathfrak{r}^2}, \vec{\mathfrak{r}} = \vec{s} - \vec{l'}$

Using $$|d\vec{l'} \times \hat{\mathfrak{r}}| = dl' \sin\alpha = dl' \cos\theta$$

$$l' = s\tan\theta \implies dl' = \frac{s}{\cos^2\theta} d\theta \qquad s = \mathfrak{r}\cos\theta \implies \frac{1}{\mathfrak{r}^2} = \frac{\cos^2\theta}{s^2}$$

we can write,

$$B(\vec{s}) = \frac{\mu_0}{4\pi} I \int_{\theta_1}^{\theta_2} \left(\frac{s}{\cos^2\theta}\cos\theta d\theta\right)\left(\frac{\cos^2\theta}{s^2}\right)$$

$$\implies B(\vec{s}) = \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta d\theta$$

$$\implies B(\vec{s}) = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 - \sin\theta_1)$$

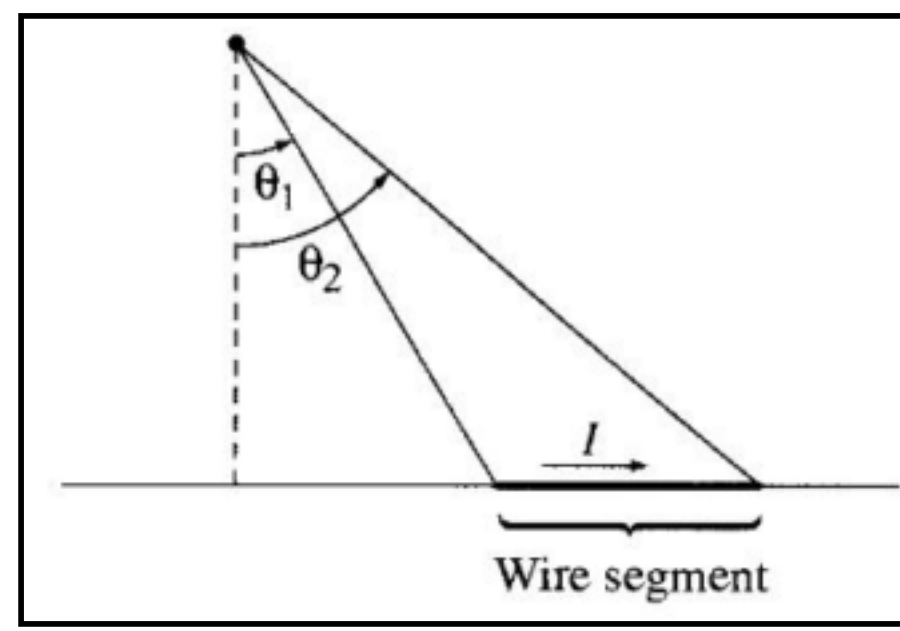


Fig. 5.18, 5.19 (Introduction to Electrodynamics, D. J. Griffiths)

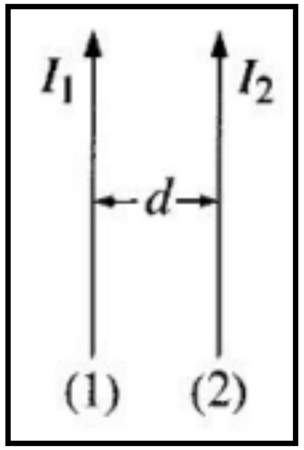
The magnetic field due to a straight wire carrying current I at a distance s is $B(\vec{s}) = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$

For an infinite wire, $\theta_1 = -\pi/2, \theta_2 = \pi/2$, therefore, $B(\vec{s}) = \frac{\mu_0 I}{2\pi s}$

Force between two parallel wires carrying current $I_{1,2}$ **and separation d**: (Assume infinite wires)

The field due to wire 1 at the location of wire 2: $B = \frac{\mu_0 I_1}{2\pi d}$

The force on the wire 2 in the presence of this field is:



$$F = I_2 \int |d\vec{l} \times \vec{B}| = I_2 \left(\frac{\mu_0 I_1}{2\pi d}\right) \int dl$$

Thus, the force per unit length on the wire 2 is:

$$f = \frac{\mu_0 I_1 I_2}{2\pi d}$$

which is (repulsive) attractive is the currents are (anti) parallel

Example 5.6 (Introduction to Electrodynamics, D. J. Griffiths): Find the magnetic field a distance z above the centre of a circular loop of radius R, which carries a steady current I.

Field due to an elemental current element is:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l'} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

Taking the vertical components* only and integrating:

$$B(z) = \frac{\mu_0 I}{4\pi} \int \frac{dl'}{\mathfrak{r}^2} \cos\theta = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{\mathfrak{r}^2} \int dl'$$

$$B(z) = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{\mathfrak{r}^2} 2\pi R = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$

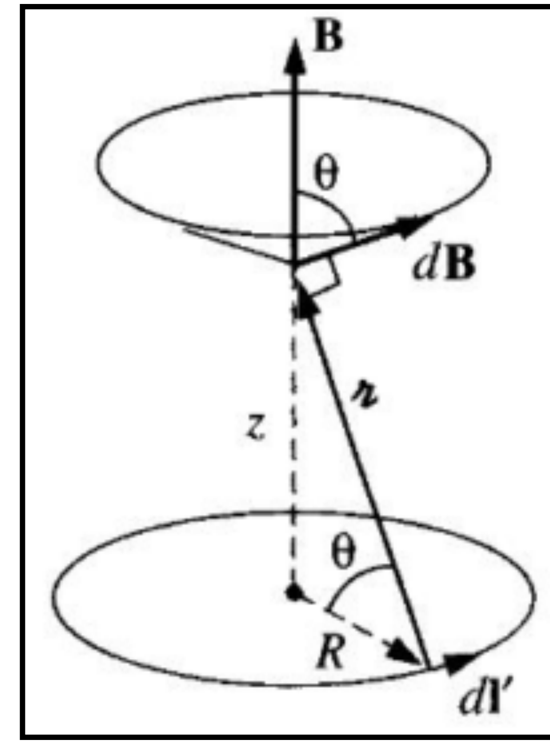


Fig. 5.21, (Introduction to Electrodynamics, D. J. Griffiths)

where we have used $\cos\theta = \dfrac{R}{\mathfrak{r}}, \;\; \mathfrak{r} = (R^2 + z^2)^{1/2}$

*The horizontal components cancel due to azimuthal symmetry.

Biot-Savart Law: Summary

- For line current: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} dl'$
- For surface current: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} da'$
- For volume current: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} d\tau'$

$$\vec{\mathfrak{r}} = \vec{r} - \vec{r'} \qquad \vec{I} = \lambda \vec{v}, \vec{K} = \sigma \vec{v}, \vec{J} = \rho \vec{v}$$

- **Superposition principle in magnetostatics**: For a collection of source currents, the net field is the vector sum of the fields due to each of them taken separately.

Apply to conductors with holes to find B everywhere

Problem 5.44 (Introduction to Electrodynamics, D. J. Griffiths): Use the Biot-Savart law for surface currents to find the field inside and outside an infinitely long solenoid of radius R, with n turns per unit length, carrying a steady current I.

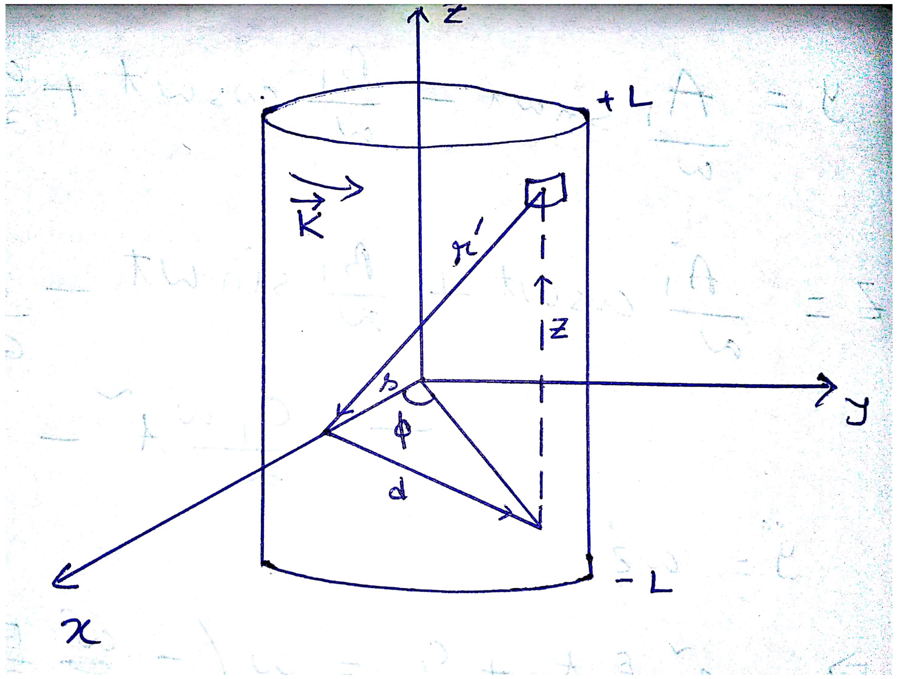
The surface current is $\vec{K} = K\hat{\phi} = K(-\sin\phi\hat{x} + \cos\phi\hat{y})$

Biot-Savart law says: $\vec{B} = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{\vec{K}\times\hat{r'}}{(r')^2}da$

$$da = Rd\phi dz$$

$$\vec{r'} = -\vec{d} - z\hat{z} = -(\vec{R}-\vec{s}) - z\hat{z}$$

$$\implies \vec{r'} = (s - R\cos\phi)\hat{x} - R\sin\phi\hat{y} - z\hat{z}$$



$$\vec{K}\times\vec{r'} = K\Big[(-z\cos\phi)\hat{x} + (-z\sin\phi)\hat{y} + (R - s\cos\phi)\hat{z}\Big]$$

$$(r')^2 = d^2 + z^2 = z^2 + R^2 + s^2 - 2Rs\cos\phi$$

$$B_z = \frac{\mu_0 KR}{4\pi}\int \frac{R - s\cos\phi}{(z^2+d^2)^{3/2}}d\phi dz$$

$$= \frac{\mu_0 KR}{4\pi}\int_0^{2\pi}(R - s\cos\phi)\left[\int_{-\infty}^{+\infty}\frac{dz}{(z^2+d^2)^{3/2}}\right]d\phi$$

x,y components vanish
due to symmetry about z axis

Using $\int_{-\infty}^{+\infty} \frac{dz}{(z^2+d^2)^{3/2}} = \frac{2}{d^2}$ we get

$$B_z = \frac{\mu_0 K R}{2\pi} \int_0^{2\pi} \frac{R - s\cos\phi}{(R^2+s^2-2Rs\cos\phi)} d\phi$$
$$= \frac{\mu_0 K R}{2\pi} \frac{1}{2R} \left[(R^2-s^2) \int_0^{2\pi} \frac{1}{(R^2+s^2-2Rs\cos\phi)} d\phi + \int_0^{2\pi} d\phi \right]$$

Using $\int_0^{2\pi} \frac{1}{a+b\cos\phi} d\phi = \frac{2\pi}{(a^2-b^2)^{1/2}}$ $\quad a = R^2+s^2, \quad b = -2Rs$

$$B_z = \frac{\mu_0 K}{4\pi} \left[\frac{R^2-s^2}{|R^2-s^2|} 2\pi + 2\pi \right] = \frac{\mu_0 K}{2} \left[\frac{R^2-s^2}{|R^2-s^2|} + 1 \right]$$

Inside the solenoid s<R, $B_z = \frac{\mu_0 K}{2}(1+1) = \mu_0 K = \mu_0 n I$

Outside the solenoid s>R, $B_z = \frac{\mu_0 K}{2}(-1+1) = 0$

The same can be found at one step using Ampere's law (to be discussed later).

Ampere's Law

- Magnetic field due to an infinitely long wire carrying current I is

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

- The field "circles" around the wire as shown in the figure.

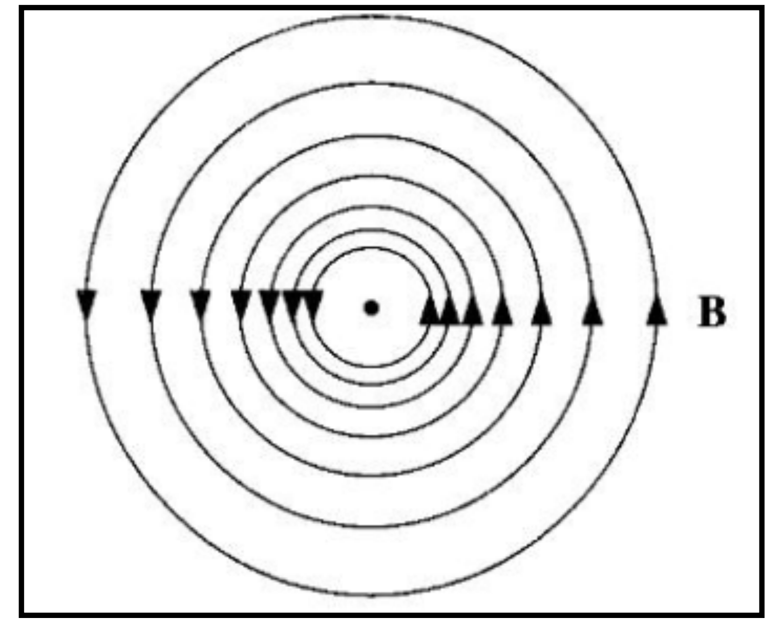


Fig. 5.27 (Introduction to Electrodynamics, D. J. Griffiths)

- The line integral of magnetic field around a circular path of radius r, entered at the wire is

$$\oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi r} dl = \frac{\mu_0 I}{2\pi r} \oint dl = \mu_0 I$$

Ampere's Law

- The line integral of B around a closed path does not depend upon the distance r of the path from the wire: B decreases at the same rate as the circumference increases.

Ampere's Law (loop of arbitrary shape)

- In cylindrical polar coordinates (r, ϕ, z) the magnetic field due to an infinite wire carrying current along the z axis is $$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$
- Using the cylindrical line element $$d\vec{l} = dr\hat{r} + r d\phi\hat{\phi} + dz\hat{z}$$
- The line integral of B around an arbitrary closed loop is

$$\oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\oint \frac{r d\phi}{r} = \frac{\mu_0 I}{2\pi}\oint d\phi = \mu_0 I$$

- If the loop does not enclose the wire at all, then ϕ will go from ϕ_1 to ϕ_2 and back again resulting in $\int d\phi = 0$

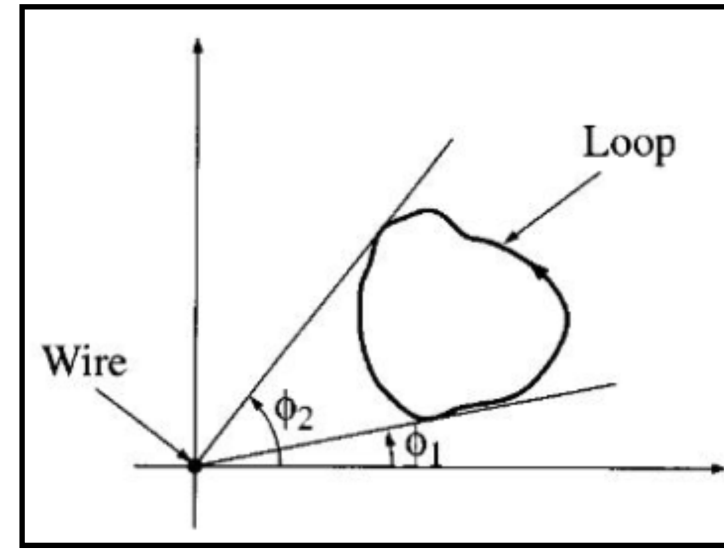


Fig. 5.28 (Introduction to Electrodynamics, D. J. Griffiths)

- If there is a bundle of straight wires, the ones passing through the loop contributes $\mu_0 I_{\text{enc}}$ to the magnetic field

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

Ampere's Law

For a loop of the specific shape (shown in figure)

$$\oint \vec{B}\cdot d\vec{l} = \int_{ab} \vec{B}\cdot d\vec{l} + \int_{bc} \vec{B}\cdot d\vec{l} + \int_{cd} \vec{B}\cdot d\vec{l} + \int_{da} \vec{B}\cdot d\vec{l}$$

$$= 0 + B_2(r_2\theta) + 0 + B_1 r_1(2\pi - \theta)$$

$$= \frac{\mu_0 I}{2\pi r_2}(r_2\theta) + \frac{\mu_0 I}{2\pi r_1} r_1(2\pi - \theta)$$

$$\Longrightarrow \oint \vec{B}\cdot d\vec{l} = \mu_0 I$$

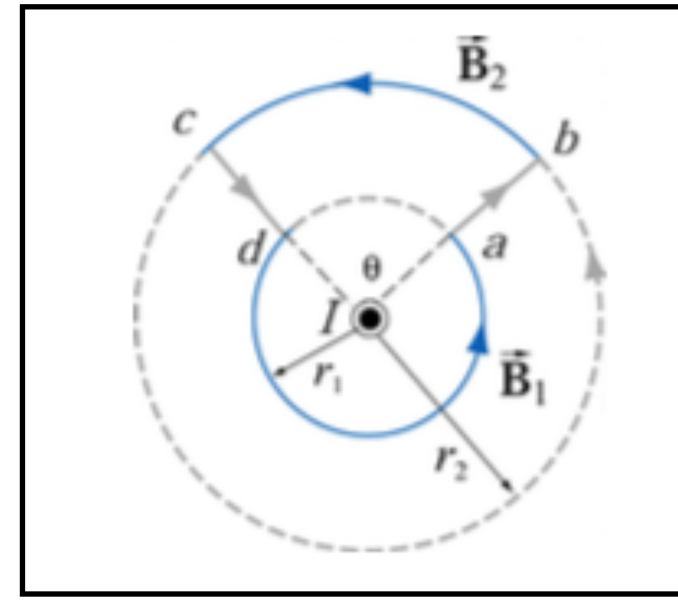


Image credit: MIT

Ampere's law is useful in calculating magnetic field for symmetric configurations such as:

Infinite long straight wire, infinite large 2D sheet, infinite solenoid, toroid carrying steady currents.

Divergence & Curl of Magnetic Field

- If J is the volume current density then the current enclosed by the integration path is $I_{\text{enc}} = \int \vec{J} \cdot d\vec{a}$ where the integral is taken over the surface bounded by the loop.

- Using Stoke's theorem

$$\oint \vec{B} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

- Since this should be true for any area, we have

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Ampere's Law

Divergence & Curl of Magnetic Field From Biot-Savart Law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} d\tau'$$ Biot-Savart Law

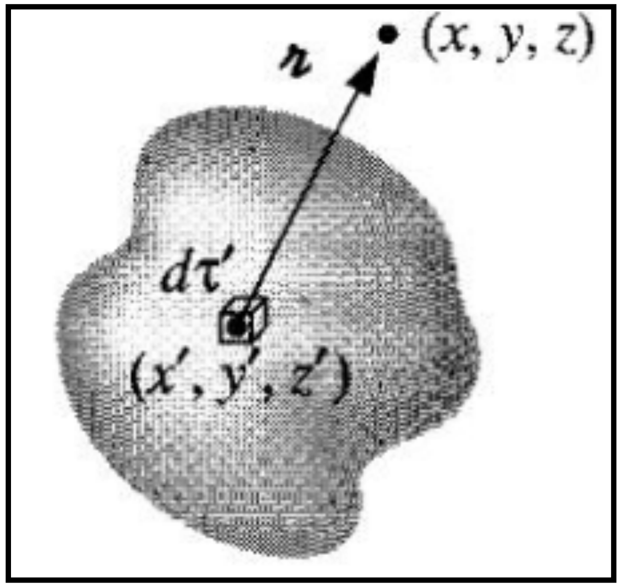


Fig. 5.30 (Introduction to Electrodynamics, D. J. Griffiths)

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{\nabla} \cdot \left(\frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) d\tau'$$

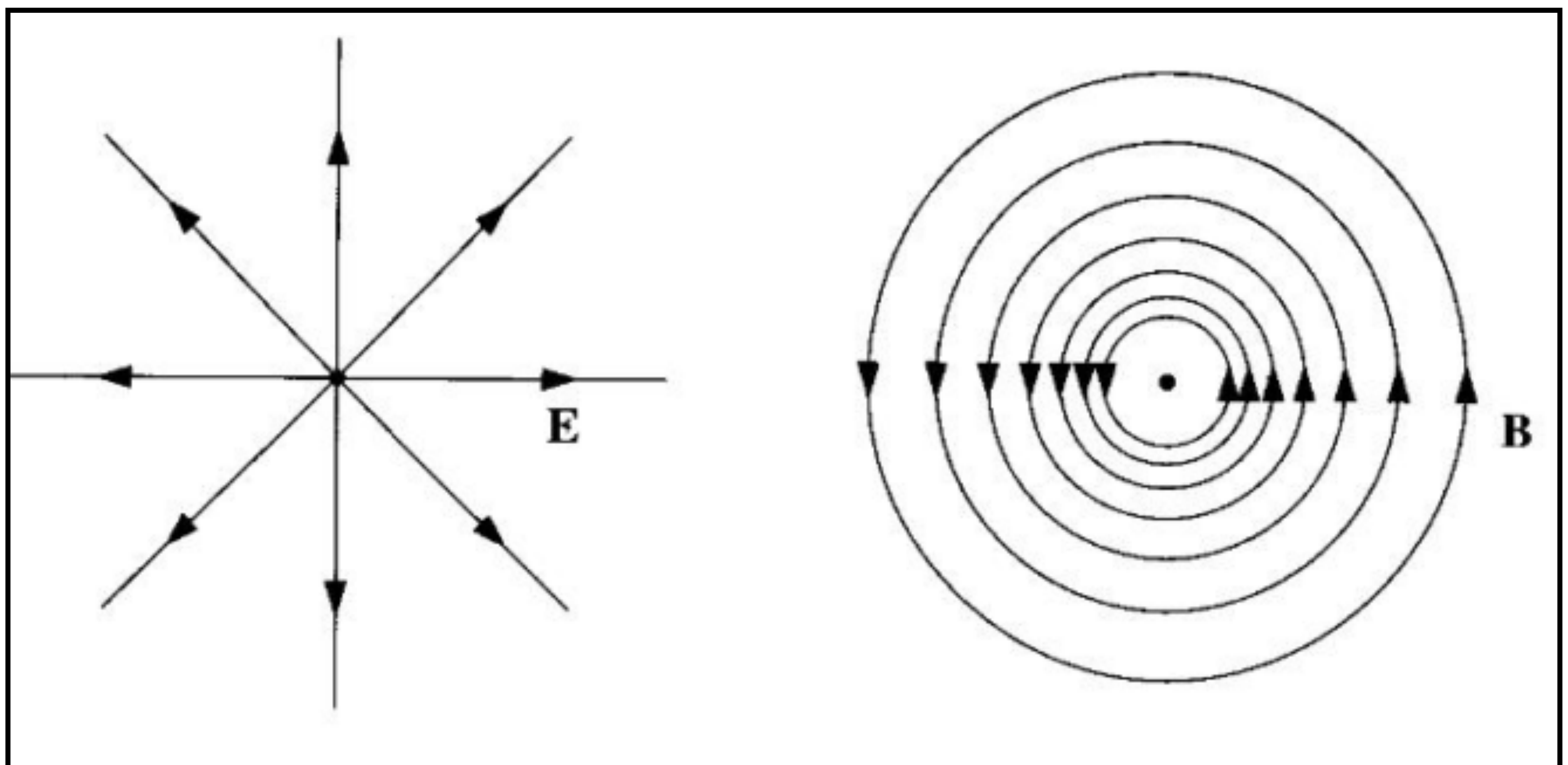
$$\vec{\mathfrak{r}} = \vec{r} - \vec{r'}$$

$$\vec{\nabla} \cdot \left(\vec{J}(\vec{r'}) \times \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) = \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \cdot (\vec{\nabla} \times \vec{J}(\vec{r'})) - \vec{J}(\vec{r'}) \cdot \left(\vec{\nabla} \times \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right)$$

Using $\quad \vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

Divergence & Curl of Magnetic Field

- We have $\vec{\nabla} \times \vec{J}(\vec{r'}) = 0, \vec{\nabla} \times \left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right) = 0$ which implies $\vec{\nabla} \cdot \vec{B} = 0$
- The magnetic field is divergence-less: no net outflow of magnetic lines of force through a closed surface. No magnetic analog to electric charge.

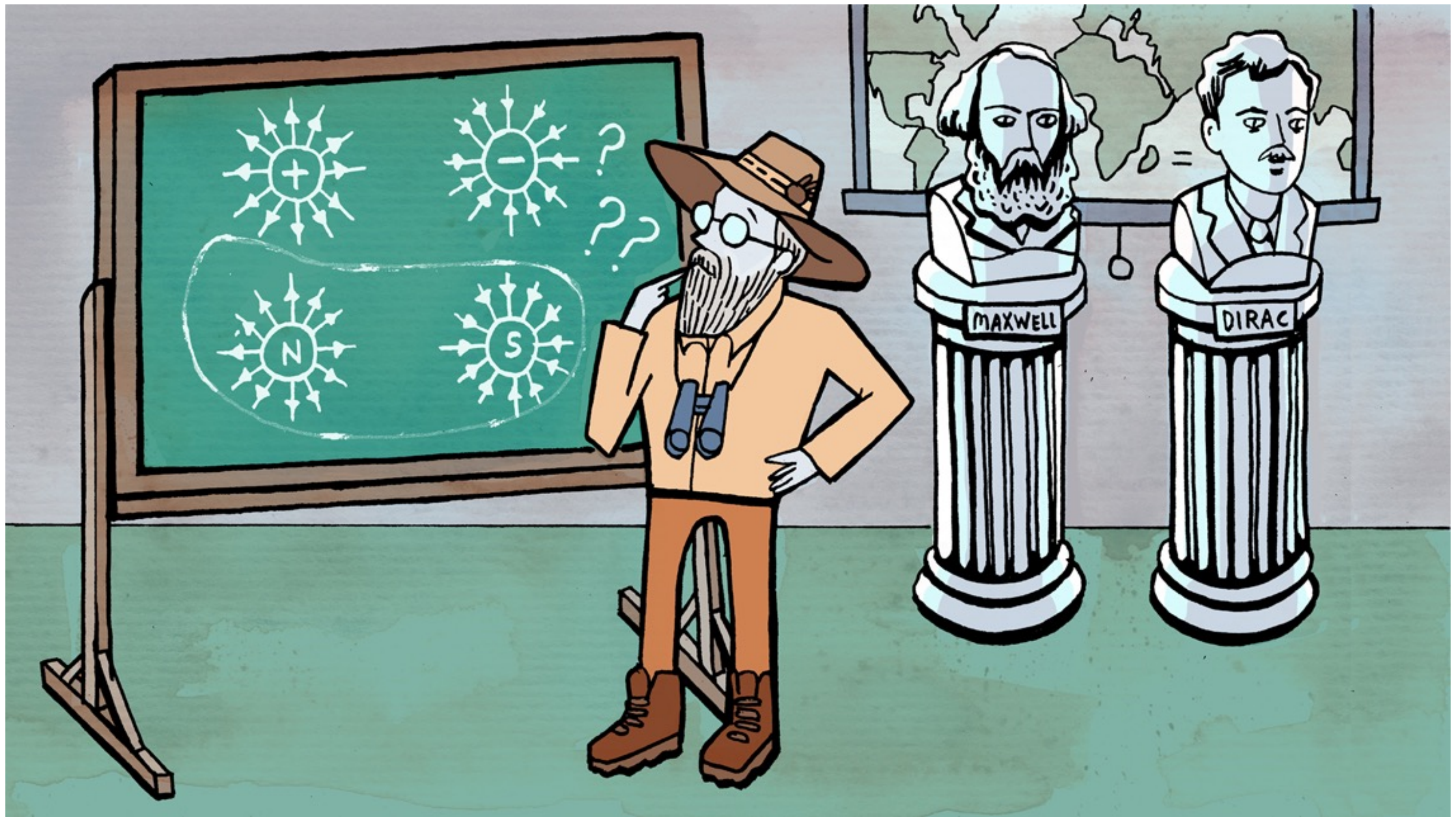


(a) Electrostatic field of a point charge

(b) Magnetostatic field of a long wire

Fig. 5.44 (Introduction to Electrodynamics, D. J. Griffiths)

No Magnetic Monopoles



$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\int_S \vec{B} \cdot d\vec{a} = 0$$

This is valid classically, in quantum as well as Grand Unified theories, monopoles can arise, in contrast to observations.

Null searches put bounds on monopole number densities in the observed Universe, to be around 40 order of magnitudes below the number density of photons.

Image Credit: Symmetrymagazine

Curl of Magnetic Field

- Similarly, $$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\frac{\vec{J}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) d\tau'$$
- Using the identity

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$$

we have

$$\vec{\nabla} \times \left(\vec{J}(\vec{r'}) \times \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) = \vec{J}(\vec{r'}) \left(\vec{\nabla} \cdot \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \right) - (\vec{J}(\vec{r'}) \cdot \vec{\nabla}) \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

The other 2 terms identically vanish $$\left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} \cdot \vec{\nabla} \right) \vec{J}(\vec{r'}) = 0, \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} (\vec{\nabla} \cdot \vec{J}(\vec{r'})) = 0$$

Curl of Magnetic Field

- We know the identity: $\vec{\nabla}\cdot\left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right) = 4\pi\delta^3(\vec{\mathfrak{r}})$
- Since $\vec{\mathfrak{r}} = \vec{r} - \vec{r'}$ and derivatives act only on $\hat{\mathfrak{r}}/\mathfrak{r}^2$ one can have the interchange $\vec{\nabla} \leftrightarrow \vec{\nabla}'$ just by incorporating an extra minus sign. Therefore,

$$-(\vec{J}(\vec{r'})\cdot\vec{\nabla})\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} = (\vec{J}(\vec{r'})\cdot\vec{\nabla}')\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

- Taking the x-component of the RHS of above expression

$$(\vec{J}(\vec{r'})\cdot\vec{\nabla}')\left(\frac{x-x'}{\mathfrak{r}^3}\right) = \vec{\nabla}'\cdot\left(\frac{x-x'}{\mathfrak{r}^3}\vec{J}(\vec{r'})\right) - \left(\frac{x-x'}{\mathfrak{r}^3}\right)(\vec{\nabla}'\cdot\vec{J}(\vec{r'}))$$

Using $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$

Curl of Magnetic Field

- For steady current $\vec{\nabla}' \cdot \vec{J}(\vec{r'}) = 0$ and hence

$$\left(-(\vec{J}(\vec{r'}) \cdot \vec{\nabla})\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right)_x = \vec{\nabla}' \cdot \left(\frac{x - x'}{\mathfrak{r}^3}\vec{J}(\vec{r'})\right)$$

- Therefore, the integral becomes

$$\int_V \vec{\nabla}' \cdot \left(\frac{x - x'}{\mathfrak{r}^3}\vec{J}(\vec{r'})\right) d\tau' = \oint_S \frac{x - x'}{\mathfrak{r}^3}\vec{J}(\vec{r'}) \cdot d\vec{a'}$$

which vanishes for J=0 at the surface S (Current is zero on the boundary, all current is inside).

- The curl of magnetic field is therefore, $\left(\vec{\nabla} \cdot \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} = 4\pi\delta^3(\vec{\mathfrak{r}})\right)$

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r'})4\pi\delta^3(\vec{r} - \vec{r'})d\tau' = \mu_0\vec{J}(\vec{r})$$

Ampere's Law

- Home work: Prove that $\vec{\nabla} \cdot \vec{B} = 0, \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ uniquely determine magnetic field. Hint: Use uniqueness theorem!