PH 102: Physics II

Lecture 21 (Spring 2019)

IIT Guwahati
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LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Lectures	Division
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	I, II (4-4:55 pm)
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	III, IV (9-9:55 am)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	I, II (3-3:55 pm)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	III, IV (10-10:55 am)
Tut 8	19-3-2019	Lec 15, 16			
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic Materials, Magnetization	5.4, 6.1	1	I, II (4-4:55 pm)
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic Materials, Magnetization	5.4, 6.1	1	III, IV (9-9:55 am)
Lec 18	22-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	I, II (3-3:55 pm)
Lec 18	22-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	III, IV (10-10:55 am)
Tut 9	26-3-2019	Lec 17, 18			
Lec 19	27-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	I, II (4-4:55 pm)
Lec 19	27-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	III, IV (9-9:55 am)
Lec 20	28-3-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	I, II (3-3:55 pm)
Lec 20	28-3-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	III, IV (10-10:55 am)
Tut 10	2-4-2019	Lec 19, 20			
Lec 21	3-4-2019	Maxwell's equations	7.3	1	I, II (4-4:55 pm)
Lec 21	3-4-2019	Maxwell's equations	7.3	1	III, IV (9-9:55 am)
Lec 22	4-4-2019	Discussions, problem solving	7.3	1	I, II (3-3:55 pm)
Lec 22	4-4-2019	Discussions, problem solving	7.3	1	III, IV (10-10:55 am)
Tut 11	9-4-2019	Quiz II			

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	I, II (4-4:55 pm)
Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	III, IV (9-9:55 am)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	I, II (3-3:55 pm)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	III, IV (10-10:55 am)
Tut 12	23-4-2019	Lec 23, 24			
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	I, II (4-4:55 pm)
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	III, IV (9-9:55 am)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	I, II (3-3:55 pm)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	III, IV (10-10:55 am)
Tut 13	30-4-2019	Lec 25, 26			
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	I, II (4-4:55 pm)
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	III, IV (9-9:55 am)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	I, II (3-3:55 pm)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	III, IV (10-10:55 am)

Maxwell's Equations

The equations involving electric and magnetic fields so far:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss's (Coulomb's) Law}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{Faraday's Law}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \text{Ampere's Law}$$

These four equations used to represent electromagnetic theory before Maxwell.

But there is an inconsistency in this set of equations! This is related to the known rule of vector calculus when applied to the above equations: divergence of curl is zero

Inconsistency in Ampere's Law

Taking divergence on both sides of the last two equations gives:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \cdot \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B})$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{J})$$

While both sides of the first equation are zero, the right hand side of the second equation need not be true, in general.

Continuity equation: $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

Lecture 19

For non-steady current $\vec{\nabla} \cdot \vec{J} \neq 0$ and hence Ampere's law can not be correct if we go beyond magnetostatics!

Inconsistency in Ampere's Law

Ampere's law in integral form: $$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

Consider two different amperian loops as shown in the circuit for charging a capacitor.

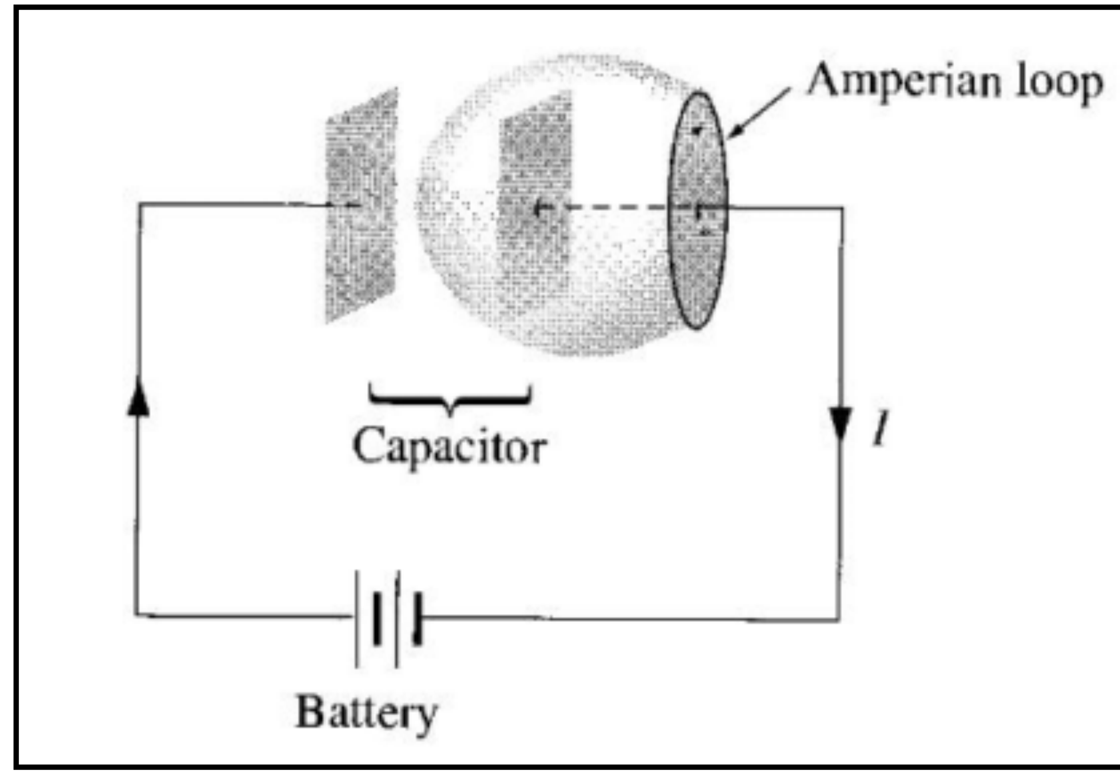


Figure 7.42, Introduction to Electrodynamics, D. J. Griffiths

For the loop with the surface in the plane of the loop,

$$I_{\text{enc}} = I$$

whereas for the loop with balloon shaped surface

$$I_{\text{enc}} = 0$$

This ambiguity is coming because of non-steady current: Deviation from magnetostatics & hence the usual Ampere's law discussed earlier!

How did Maxwell Fix Ampere's Law?

The inconsistency in Ampere's law comes from the fact that $\vec{\nabla} \cdot \vec{J} = 0$ need not be true if we go beyond magnetostatics.

Using the continuity equation:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \vec{\nabla} \cdot \vec{E}) = -\vec{\nabla} \cdot \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$\implies \vec{\nabla} \cdot \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = 0$$

Therefore, Ampere's law can be corrected if we do the following replacement:

$$\vec{J} \to \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Ampere’s Law (Modified)

The corrected form of Ampere’s law (after Maxwell):

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The second term on the right hand side not only rescues the continuity equation, but also has an interesting implication:

A changing electric field induces a magnetic field

Validity of the second term was confirmed much later in 1888 with Hertz’s experiments on electromagnetic waves.

The second term is known as displacement current $\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

So that the Ampere’s law can be written as

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_d)$$

Charging of a capacitor & $\vec{J}_d$

The electric field inside the capacitor plates (assuming the separation to be tiny):

$$E = \frac{\sigma}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{Q}{A}$$

$$\implies \frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

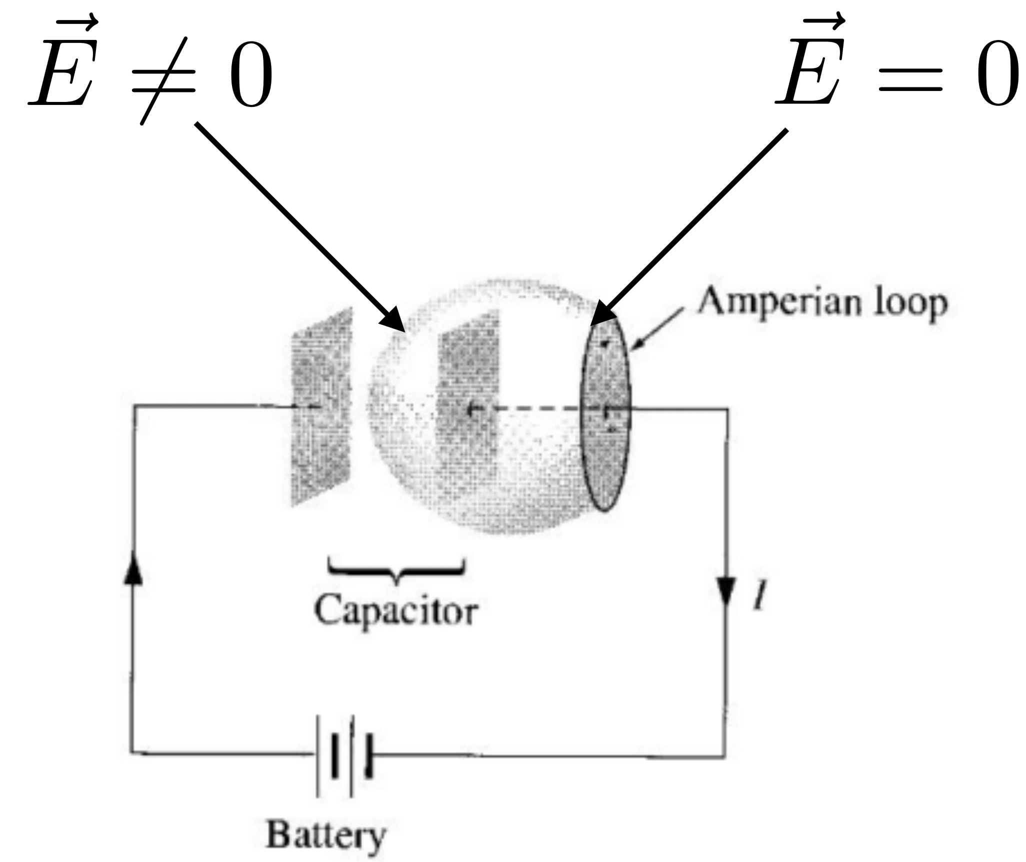


Figure 7.42, Introduction to Electrodynamics, D. J. Griffiths

Now, for the flat amperian loop:

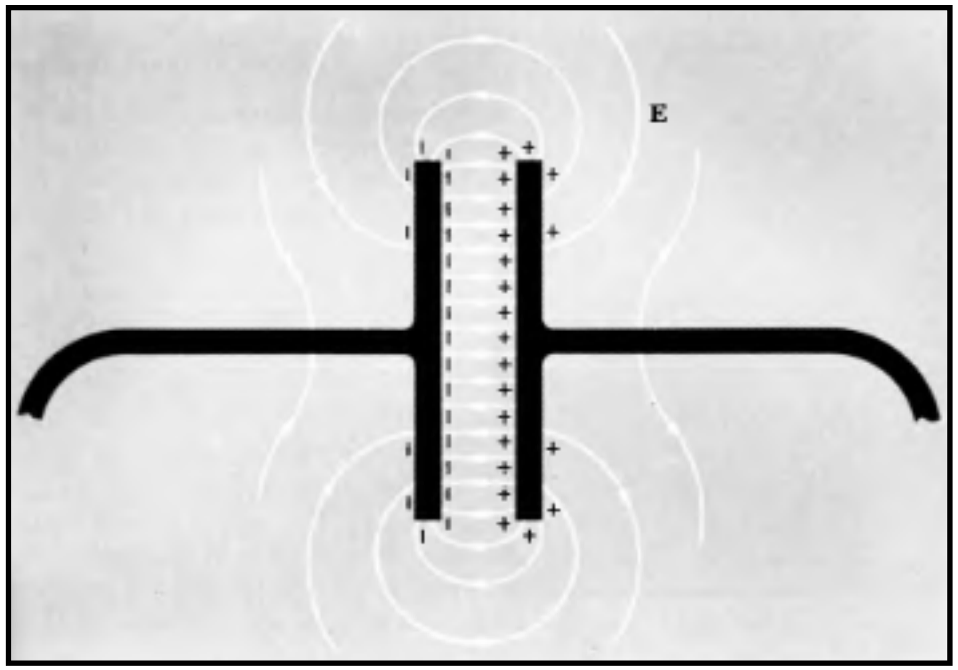
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} = \mu_0 I$$

For the balloon shaped loop:

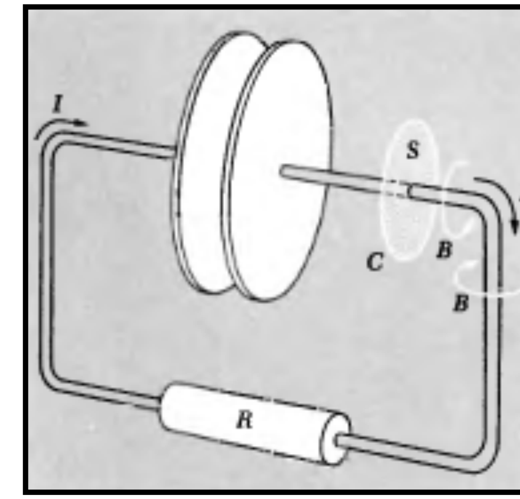
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \mu_0 I$$

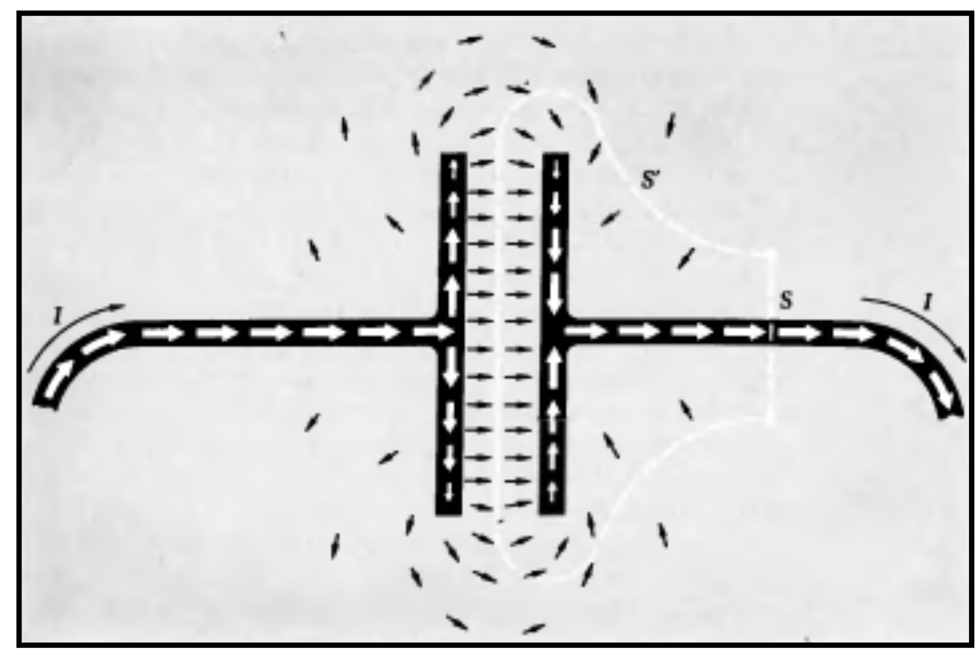
The displacement current is not a current at all. It is, in fact, associated with the generation of magnetic fields by time-varying electric fields.

For a charged capacitor connected to a resistor, the field configurations look like:



E is decreasing everywhere with time as C gets discharged



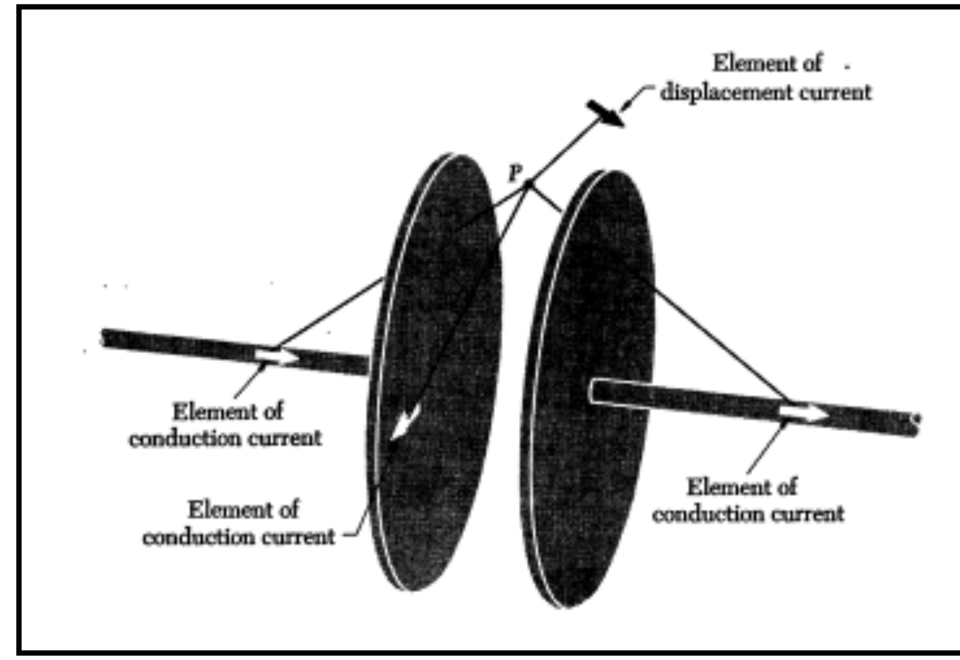


Conduction current (white arrows)
Displacement current (black arrows)

Image credit: E & M, Purcell
McGraw Hill

Why didn't Faraday discover the displacement current?

Consider the point P in the space between the discharging capacitor plates. Naively, both conduction and displacement currents should contribute to the field at P.



However, $\vec{J}_d$ has the same form as $\vec{E}$.

E is mostly electrostatic, except that it is dying.

Taking curl of $\vec{J}_d$ and using Faraday's law:

$$\vec{\nabla} \times \vec{J}_d = \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) = -\epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

For slowly changing fields (quasi-static), this must be negligible.

See chapter 9, E & M, Purcell, McGraw Hill

Image credit: E & M, Purcell McGraw Hill

Why didn't Faraday discover the displacement current?

A curl-less vector field can be constructed in a way similar to electric field in electrostatics.

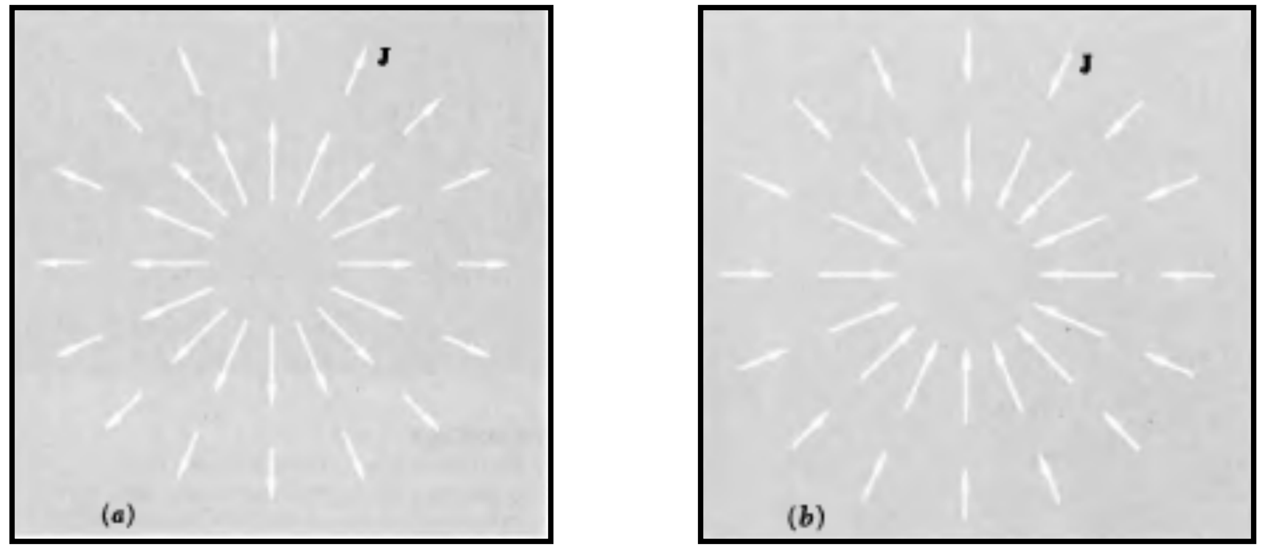


Image credit: E & M, Purcell McGraw Hill

However, the magnetic field due to such symmetrical current distribution must vanish!

Therefore, in the quasi-static limit, conduction currents are the only sources needed to account for the magnetic field.

If Faraday had tried to measure B at point P using a compass needle, he would not have needed to invent a displacement current to explain it.

To see the effect of displacement currents, rapidly changing fields are required, needing changes to occur in time it takes light to cross the apparatus ($c\tau \approx d$). Hertz demonstrated it in 1888.

Why didn't Faraday discover the displacement current?

For a parallel plate capacitor with 1 cm spacing, charged upto 100 volts, the electric field is 10000 volts per metre. If the capacitor is discharged in 0.1 s, then the induced magnetic field is:

$$\oint \vec{B} \cdot d\vec{l} = \epsilon_0 \mu_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{a} \implies lB \approx \epsilon_0 \mu_0 \frac{El^2}{t}$$

For a square loop of dimension l ~ 0.1 m, the magnetic field turns out to be $B \approx 10^{-9}$ Gauss.

This field is too tiny for Faraday to detect around 200 years back!

For FM signal that oscillates 10^9 Hz, the time t in the above example becomes 10^{-9} s from 0.1 s. This gives rise to a magnetic field of 0.1 Gauss, that can easily be detected with modern day devices.

Perform a similar exercise to convince that it was easier for Faraday, on the contrary, to detect the induced electric field produced by change in magnetic field!

Problem 7.31 (Introduction to Electrodynamics, D J Griffiths) A fat wire, radius a, carries a current I, uniformly distributed over its cross section. A narrow gap in the wire of width w, forms a parallel plate capacitor. Find the magnetic field in the gap, at a distance s<a from the axis. (Assume $w \ll a$).

Solution: The displacement current density is:

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{I}{A}\hat{z} = \frac{I}{\pi a^2}\hat{z}$$

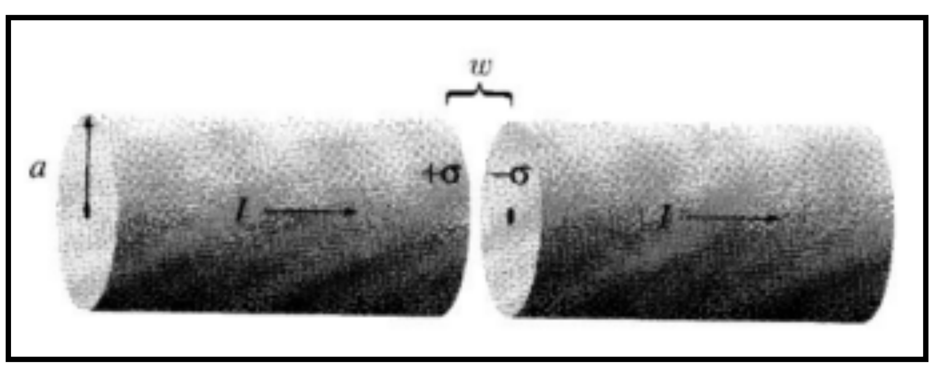
For an amperian loop of radius s<a

$$\oint \vec{B} \cdot d\vec{l} = B(2\pi s) = \mu_0 I_d$$

$$\implies B = \frac{\mu_0}{2\pi s}\frac{I}{\pi a^2}(\pi s^2)$$

$$\implies B = \frac{\mu_0 I s^2}{2\pi s a^2}$$

$$\implies \vec{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}$$



Maxwell's Equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$ Gauss's (Coulomb's) Law

$$\vec{\nabla} \cdot \vec{B} = 0$$ No name

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ Faraday's Law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$ Ampere's Law with Maxwell's Correction

Rearranging them with fields on left and source on the right

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad \vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

Maxwell's equations together with the force law $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ summarises the theoretical content of classical electrodynamics.

Maxwell's Equations in Matter

In the presence of matter, we have bound charges and bound currents

$$\rho_b = -\vec{\nabla} \cdot \vec{P}, \quad \vec{J}_b = \vec{\nabla} \times \vec{M}$$

Polarisation introduces surface charge density $\sigma_b = P$ at one end and $-\sigma_b$ at the other end. If P increases with time, it changes the charges at the ends, giving rise to a current

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp \implies \vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

Polarisation Current

If P points to the right and increasing, then each plus (minus) charge moves a bit to the right (left) giving rise to the polarisation current satisfying the continuity equation

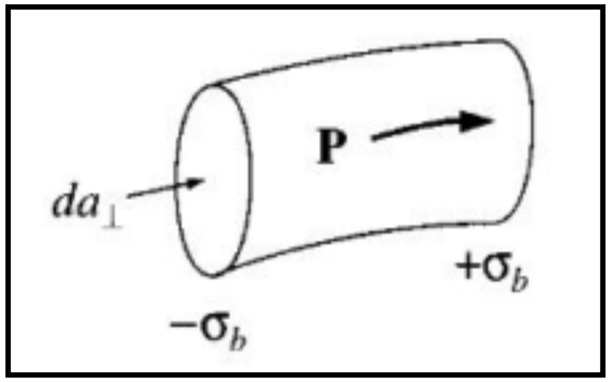


Figure 7.45, Introduction to Electrodynamics, D. J. Griffiths

$$\vec{\nabla} \cdot \vec{J}_p = \vec{\nabla} \cdot \frac{\partial \vec{P}}{\partial t} = \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{P}) = -\frac{\partial \rho_b}{\partial t}$$

$\longrightarrow$ Conservation of bound charges

Maxwell's Equations in Matter

Total charge density can therefore, be split into free and bound ones: $\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla} \cdot \vec{P}$

The current density can be similarly split into three parts: free current, bound current and the polarisation current: $\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = \vec{J}_f + \vec{\nabla} \times \vec{M} + \dfrac{\partial \vec{P}}{\partial t}$

Gauss's law can now be written as

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}(\rho_f - \vec{\nabla} \cdot \vec{P})$$

$$\implies \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \vec{\nabla} \cdot \vec{D} = \rho_f$$

Where $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ is the electric displacement.

Maxwell's Equations in Matter

Ampere's law (with Maxwell's correction term) can be written as

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\implies \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) = \vec{J}_f + \frac{\partial}{\partial t} (\epsilon_0 \vec{E} + \vec{P})$$

$$\implies \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Therefore, Maxwell's equations in terms of free charge and free current are:

$$\vec{\nabla} \cdot \vec{D} = \rho_f, \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Maxwell's Equations in Matter

Instead of using both E and D, both B and H, we can use the known relations between them to write the equations as

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\epsilon}, \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{B} = \mu \vec{J}_f + \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

where the information about the material is contained in the definitions of

$$\epsilon = \epsilon_0(1 + \chi_e), \mu = \mu_0(1 + \chi_m)$$

Boundary Conditions

The Maxwell's equations in integral form are:

$$\oint_{\mathcal{S}} \vec{D} \cdot d\vec{a} = Q_{f_{\rm enc}}, \quad \oint_{\mathcal{S}} \vec{B} \cdot d\vec{a} = 0$$

$$\oint_{\mathcal{P}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_{\mathcal{S}} \vec{B} \cdot d\vec{a}, \quad \oint_{\mathcal{P}} \vec{H} \cdot d\vec{l} = I_{f_{\rm enc}} + \frac{d}{dt} \oint_{\mathcal{S}} \vec{D} \cdot d\vec{a}$$

where S is a closed surface in the first two equations and P is a closed loop that bounds a surface S in last two equations.

Applying the 1st equation to a thin Gaussian pillbox shown in figure, we get

$$\vec{D}_1 \cdot \vec{a} - \vec{D}_2 \cdot \vec{a} = \sigma_f a$$

$$\Longrightarrow \; D_1^{\perp} - D_2^{\perp} = \sigma_f$$

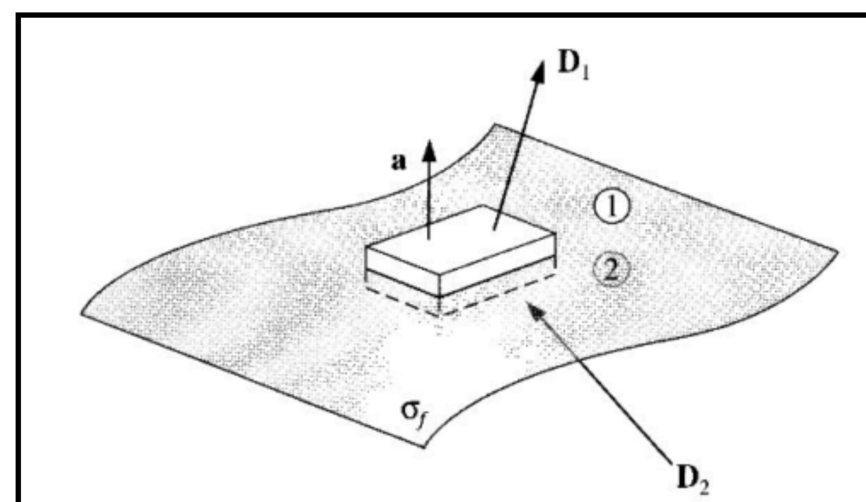


Figure 7.46, Introduction to Electrodynamics, D. J. Griffiths

Boundary Conditions

Similarly, using the second equation: $$B_1^\perp - B_2^\perp = 0$$

Using the third equation to the amperian loop shown in figure, we get

$$\vec{E}_1 \cdot \vec{l} - \vec{E}_2 \cdot \vec{l} = -\frac{d}{dt} \oint_{\mathcal{S}} \vec{B} \cdot d\vec{a}$$

Considering the amperian loop to have negligible width so that the flux through it vanishes,

$$E_1^{||} - E_2^{||} = 0$$

Thus, the components of B (E) perpendicular (parallel) to the interface are continuous.

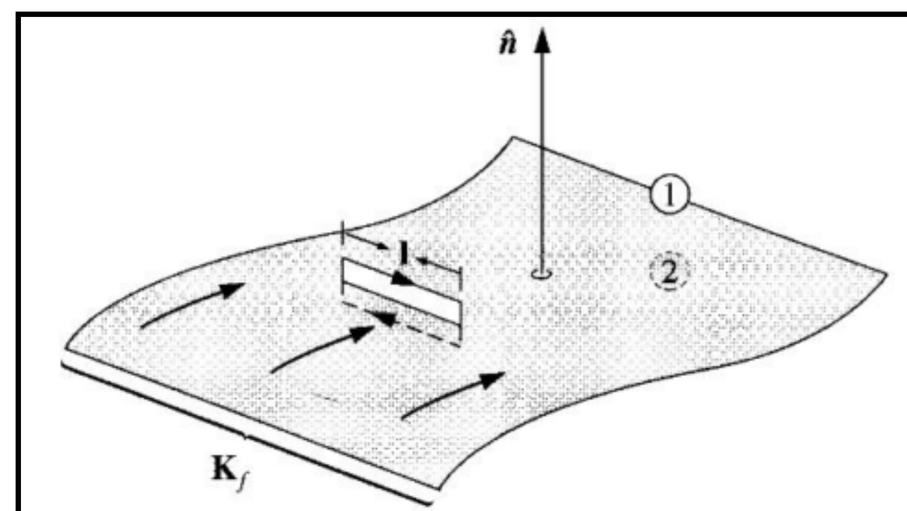


Figure 7.47, Introduction to Electrodynamics, D. J. Griffiths

Boundary Conditions

Similarly, using the fourth equation $\vec{H}_1 \cdot \vec{l} - \vec{H}_2 \cdot \vec{l} = I_{f_{\text{enc}}}$ where $I_{f_{\text{enc}}}$ is the free current passing through the amperian loop.

In the limit of infinitesimal width of the loop, volume current and surface integral contributions can be ignored like before. However, a surface current can contribute even in this limit.

$$I_{f_{\text{enc}}} = \vec{K}_f \cdot (\hat{n} \times \vec{l}) = (\vec{K}_f \times \hat{n}) \cdot \vec{l}$$

Therefore, $$H_1^{\|} - H_2^{\|} = \vec{K}_f \times \hat{n}$$

The parallel components of H are discontinuous by an amount proportional to the free surface current density.

Boundary Conditions: Summary

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, \quad B_1^{\perp} - B_2^{\perp} = 0$$

$$E_1^{\parallel} - E_2^{\parallel} = 0, \quad \frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = \vec{K}_f \times \hat{n}$$

In the absence of any free charge or free current:

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0, \quad B_1^{\perp} - B_2^{\perp} = 0$$

$$E_1^{\parallel} - E_2^{\parallel} = 0, \quad \frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = 0$$

Vector & Scalar Potentials in Electrodynamics

We know that $\vec{B} = \vec{\nabla} \times \vec{A}$. Using it in Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A})$$

$$\implies \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\implies \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\Phi$$

$$\implies \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

Electrostatic field

Faraday's field (Tutorial 10)

Vector & Scalar Potentials in Electrodynamics

Changing vector potential as $\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$

$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\Lambda)$$

$$\implies \vec{B}' = \vec{\nabla} \times \vec{A} = \vec{B}$$

$$\vec{E}' = \vec{\nabla}\Phi - \frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\Lambda) = -\vec{\nabla}\left(\Phi + \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \vec{A}}{\partial t}$$

Demanding $\vec{E}' = \vec{E}$, we must have $\Phi \to \Phi' = \Phi - \dfrac{\partial \Lambda}{\partial t}$

Thus, the electric and magnetic fields do not change if

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda, \;\; \Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

Gauge transformations

Vector & Scalar Potentials in Electrodynamics

Maxwell's equations in free space:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Using the fields in terms of potentials, we can write the 1st and the 3rd equations as

$$\vec{\nabla} \cdot \left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\epsilon_0}$$

$$\implies \nabla^2 \Phi + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}$$

Vector & Scalar Potentials in Electrodynamics

The 3rd equation becomes

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \right)$$

$$\implies \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \mu_0 \epsilon_0 \vec{\nabla} \left(\frac{\partial \Phi}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

Thus, the equations in terms of potentials are in general coupled differential equations.

Solving them, in general, can be difficult.

Can we simplify (uncouple) them using the freedom we have in choosing potentials?

Vector & Scalar Potentials in Electrodynamics

If we use the condition $\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \Phi}{\partial t} = 0$ then the equations in terms of potentials can be written as

$$\nabla^2 \Phi - \mu_0 \epsilon_0 \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

which are now decoupled and similar to the familiar Poisson's equations.

One can solve them easily for any source and find the resulting electromagnetic fields.

Vector & Scalar Potentials in Electrodynamics

The condition on the potentials we have used here to simplify the coupled differential equations is

$$\vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \Phi}{\partial t} = 0$$

This is known as the **Lorentz Gauge** condition.

Compare it with **Coulomb Gauge** condition that we had used in magnetostatics to obtain a definition of vector potential in terms of current:

$$\vec{\nabla} \cdot \vec{A} = 0$$