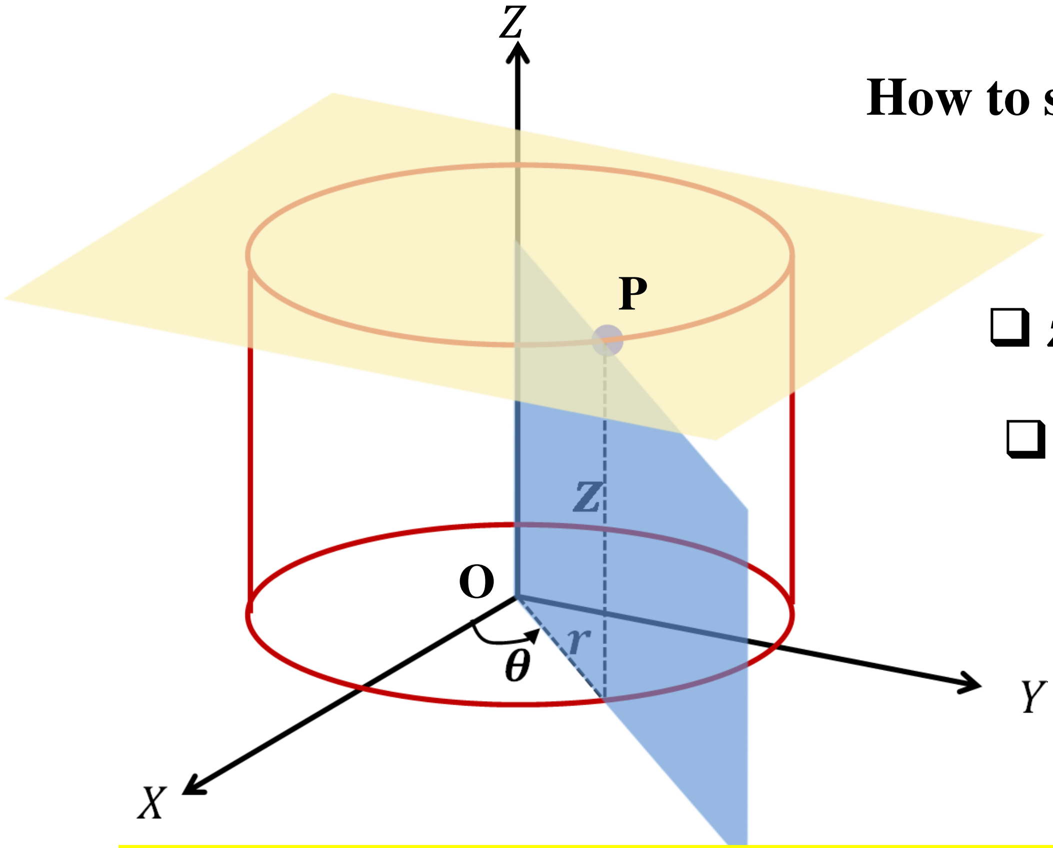
PH101: PHYSICS1

Lecture 3

II. Cylindrical coordinate system (r, θ, z)



How to specify a point 'P' in space ?

$$(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{z})$$

- ❑ $\boldsymbol{z}$-Height from the XY plane
- ❑ (r, θ) Coordinate of the foot of the point in XY plane

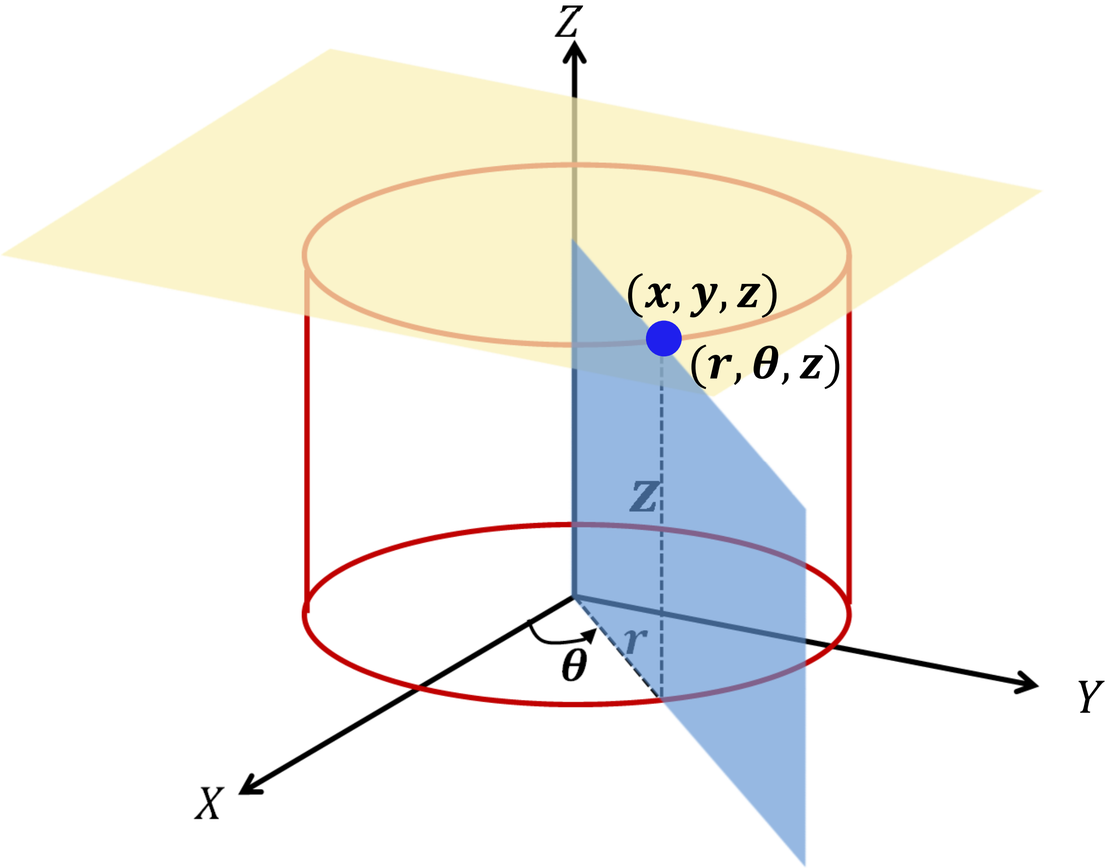
❑ $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{z})$ coordinates system is known as cylindrical coordinate system

Why the name cylindrical?

❑ Point 'P' is the intersection of three surfaces: A cylindrical surface $\boldsymbol{r} = \boldsymbol{constant}$; A half plane containing z-axis with **$\boldsymbol{\theta}$=constant** and a plane **$\boldsymbol{z}$=constant**.

Coordinate transformation: Cartesian to cylindrical

Transformation equation is very similar to polar coordinate with additional *z*-coordinate.



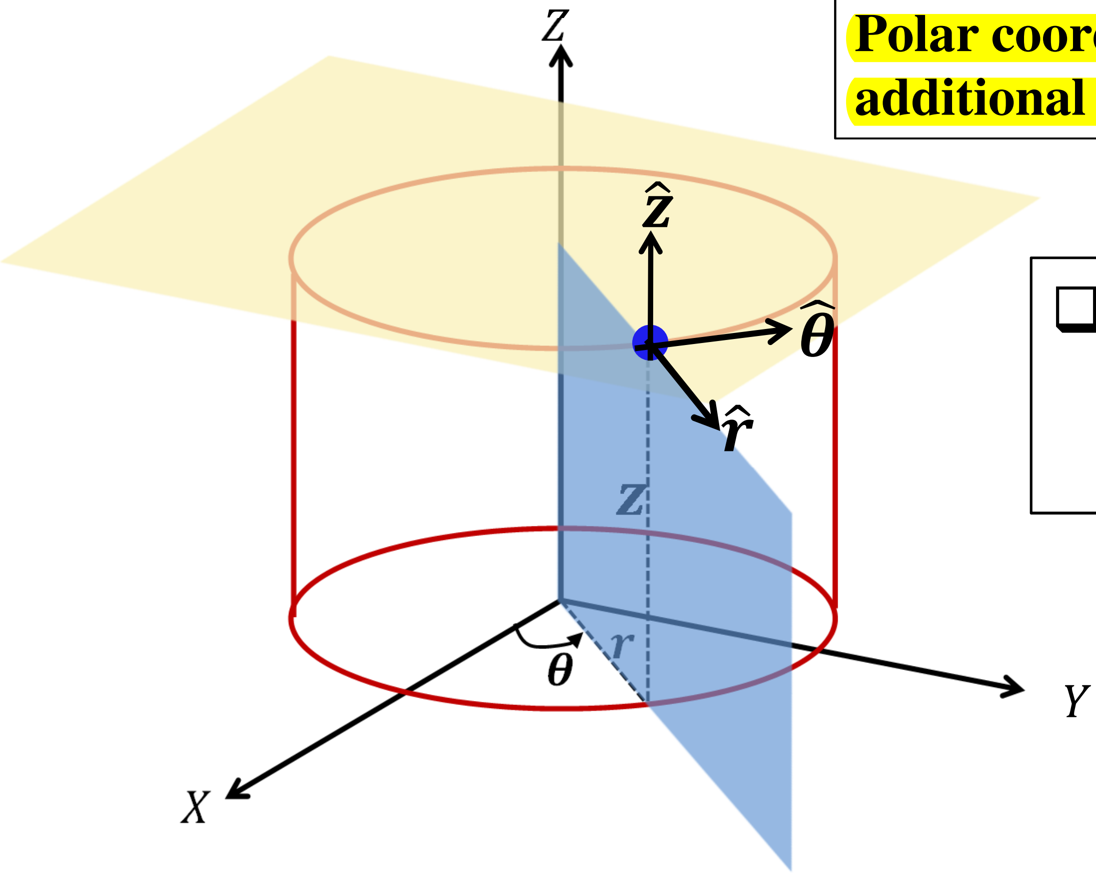
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Reverse transformation

$$r = \left(x^2 + y^2\right)^{1/2}$$
$$\theta = \tan^{-1}\frac{y}{x}$$
$$z = z$$

Note: Instead of (r, θ) many books use notation (ρ, φ).

Unit vectors in cylindrical coordinate system



Polar coordinate unit vectors $(\hat{r}, \hat{\theta})$ + additional unit vector in the z −direction.

- $\hat{r}, \hat{\theta}$ **and** $\hat{z}$ are unit vectors along increasing direction of coordinates r, θ **and** z.

$$\hat{r} = \cos\theta\ \hat{x} + \sin\theta\ \hat{y}$$
$$\hat{\theta} = -\sin\theta\ \hat{x} + \cos\theta\ \hat{y}$$
$$\hat{z} = \hat{z}$$

$\hat{r}$ and $\hat{\theta}$ are **orthogonal** but their directions depend on location.

Position, Velocity, Acceleration, Newton's law in cylindrical coordinate system

Vector components are very similar to polar coordinate+ z −component

Position vector

$$\overrightarrow{OP} = \vec{R} = r\hat{r} + z\hat{z}$$

Velocity

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$$

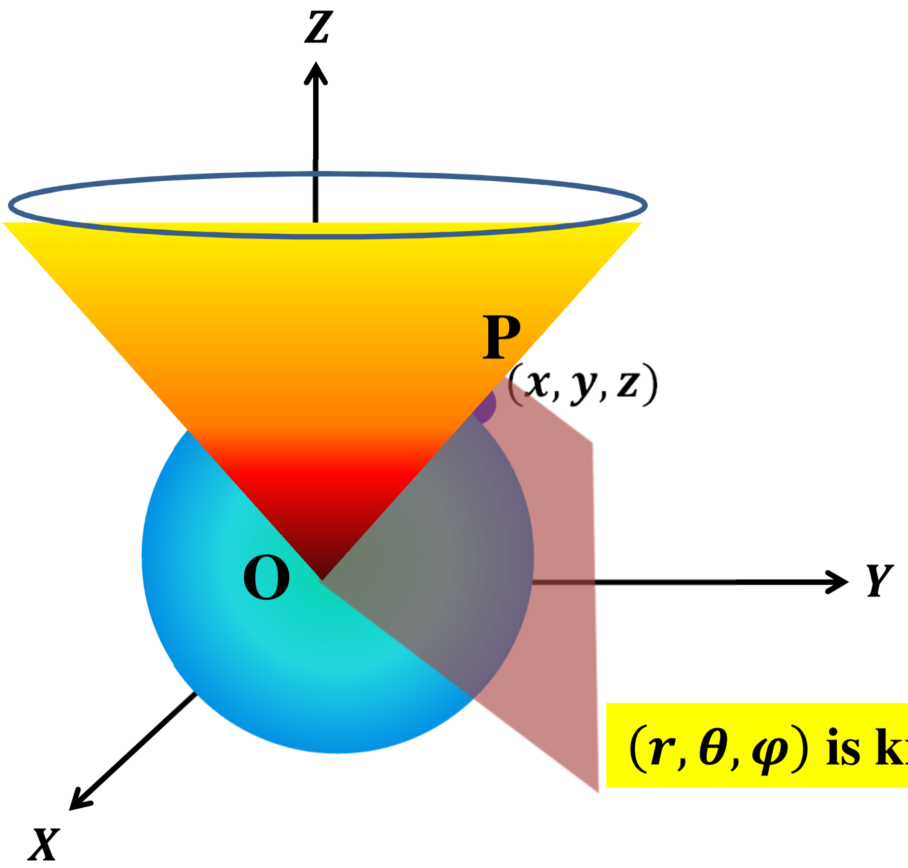
Acceleration

$$\vec{a}=(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{z}\hat{z}$$

Newton's law

$$\vec{F} = F_r\hat{r} + F_\theta\hat{\theta} + F_z\hat{z} = m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{z}\hat{z}]$$

III. Spherical polar coordinate system



r → Radial distance from origin

θ →Angle of radial vector with z-axis.

φ → Angle between X-axis and the projection of radial vector in XY plane

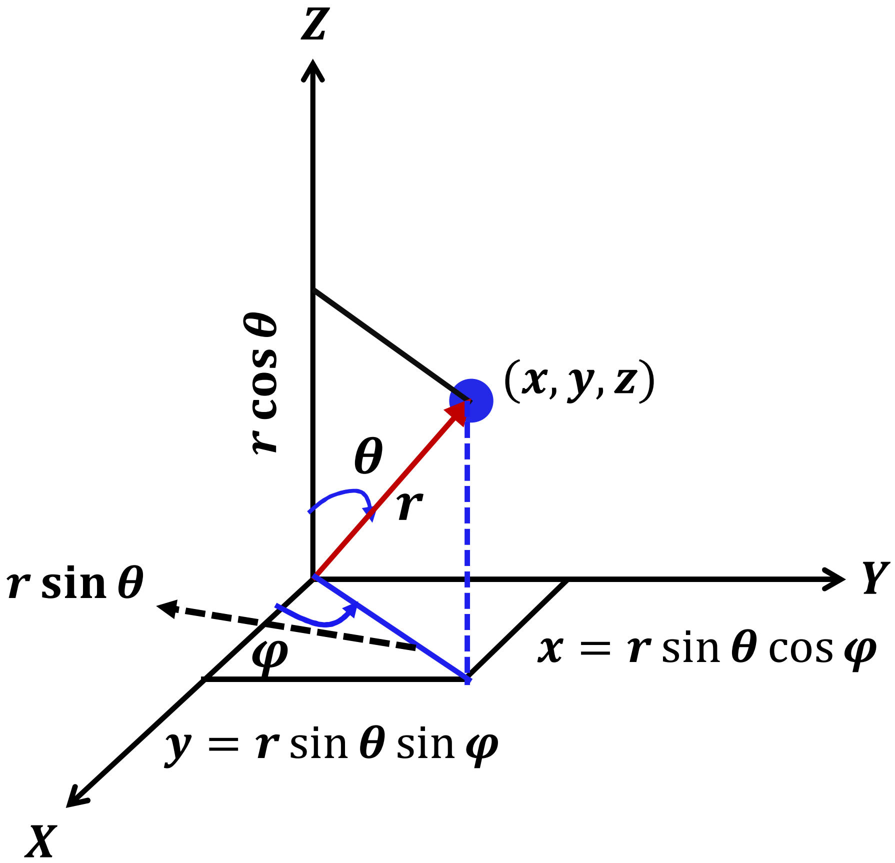
(r, θ, φ) is known as spherical polar coordinate

Note that point (r, θ, φ) is at the intersection of three surfaces

- A sphere where r =**Constant**
- A cone about z=axis with **θ=constant**.
- A half plane containing z-axis and **φ= constant**

Be careful, notations are different.
r and θ are not planer coordinate.

Connection of spherical polar with cartesian



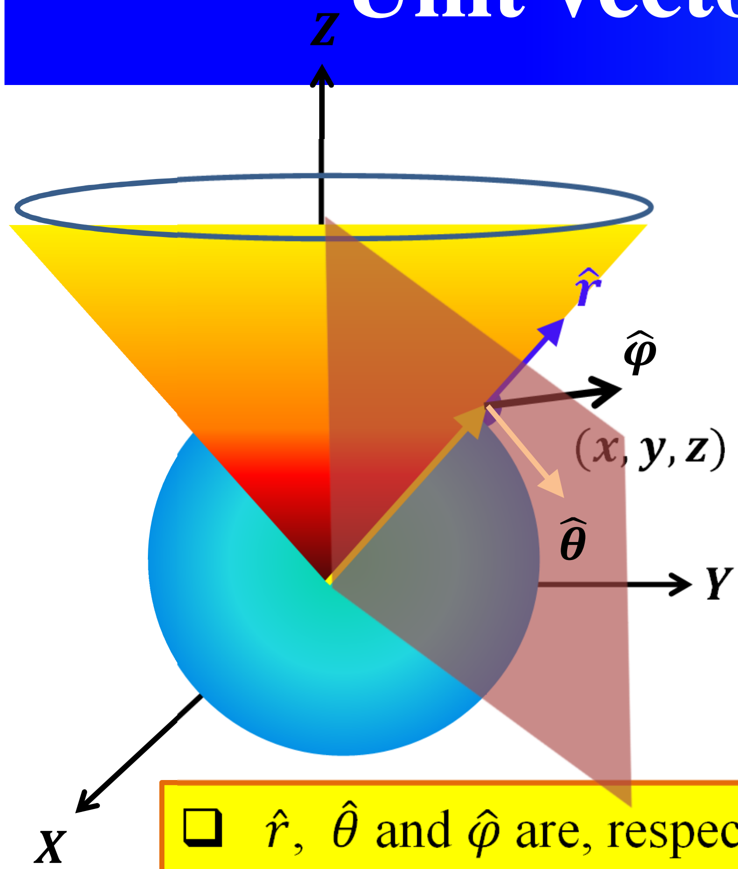
Transformation relations

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

Hence

$$r = \left(x^2 + y^2 + z^2\right)^{1/2}$$
$$\theta = \tan^{-1}\frac{\left(x^2 + y^2\right)^{1/2}}{z}$$
$$\varphi = \tan^{-1}\frac{y}{x}$$

Unit vectors in spherical polar



- Position vector
  $\vec{r} = r\sin\theta\cos\varphi\,\hat{x} + r\sin\theta\sin\varphi\,\hat{y} + r\cos\theta\,\hat{z}$

$$\hat{\boldsymbol{r}} = \frac{\vec{\boldsymbol{r}}}{\boldsymbol{r}}$$

$$\hat{\boldsymbol{r}} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$$

- $\hat{\boldsymbol{\varphi}}$ is the unit vector perpendicular to φ =constant plane (***rz plane***),

I,e, perpendicular to unit vectors $\hat{r}$ and $\hat{z}$

Thus $\hat{\varphi} = \frac{\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}}}{|\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}}|}$

$$= \frac{(-\hat{x}\sin\theta\sin\varphi + \hat{y}\sin\theta\cos\varphi)}{\sin\boldsymbol{\theta}}$$

$$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$$

- $\hat{r}$, $\hat{\theta}$ and $\hat{\varphi}$ are, respectively perpendicular to $r = const.$ $\theta = const.$ $\varphi = const.$

- $\hat{\boldsymbol{\theta}} = \frac{(\hat{\boldsymbol{\varphi}}\times\hat{\boldsymbol{r}})}{|\hat{\boldsymbol{\varphi}}\times\hat{\boldsymbol{r}}|}$

$$= \hat{x}\cos\varphi\cos\theta + \hat{y}\sin\varphi\cos\theta - \hat{z}\sin\theta$$

Unit vectors in spherical polar

$$\vec{r} = r\,sin\,\theta\,cos\,\varphi\,\hat{x} + r\,sin\,\theta\,sin\,\varphi\,\hat{y} + r\cos\theta\,\hat{z}$$

$$\hat{r} = \frac{\vec{r}}{r} = \hat{x}\cos\varphi\sin\theta + \hat{y}\sin\varphi\sin\theta + \hat{z}\cos\theta$$

$$\hat{\theta} = \frac{\partial\hat{r}}{\partial\theta} = \hat{x}\cos\varphi\cos\theta + \hat{y}\sin\varphi\cos\theta - \hat{z}\sin\theta$$

$$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi \quad (\equiv \hat{\theta}\ of\ Plane\ Polar\ \theta \rightarrow \varphi!)$$

Partial differential of unit vectors in spherical polar

Unit vectors in spherical polar coordinate are function of θ and φ only.

$$\frac{\partial \hat{r}}{\partial \theta} = \frac{\partial}{\partial \theta}(\hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta)$$
$$= (\hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta) = \hat{\theta}$$

$$\frac{\partial \hat{r}}{\partial \varphi} = \frac{\partial}{\partial \varphi}(\hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta)$$
$$= (-\hat{x}\sin\theta\sin\varphi + \hat{y}\sin\theta\cos\varphi) = \sin\theta\,\hat{\varphi}$$

Additionally, you may verify:

$$\frac{\partial \hat{\theta}}{\partial \theta} = \frac{\partial}{\partial \theta}(\hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta)$$
$$= (-\hat{x}\sin\theta\cos\varphi - \hat{y}\sin\theta\cos\varphi - \hat{z}\cos\theta) = -\hat{r}$$

$$\frac{\partial \hat{\theta}}{\partial \varphi} = \frac{\partial}{\partial \varphi}(\hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta)$$
$$= (-\hat{x}\cos\theta\sin\varphi + \hat{y}\cos\theta\cos\varphi) = -\cos\theta\,\hat{r}$$

Velocity in spherical polar coordinate

$$\boxed{\vec{r} = r\sin\theta\cos\varphi\,\hat{x} + r\sin\theta\sin\varphi\,\hat{y} + r\cos\theta\,\hat{z}}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r})$$

$$= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\hat{r} + r\left(\frac{\partial\hat{r}}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\hat{r}}{\partial\varphi}\frac{d\varphi}{dt}\right)$$

Chain rule

$$= \dot{r}\hat{r} + r\left(\dot{\theta}\hat{\theta} + \sin\theta\dot{\varphi}\,\hat{\varphi}\right)$$

$$\boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\,\hat{\varphi}}$$

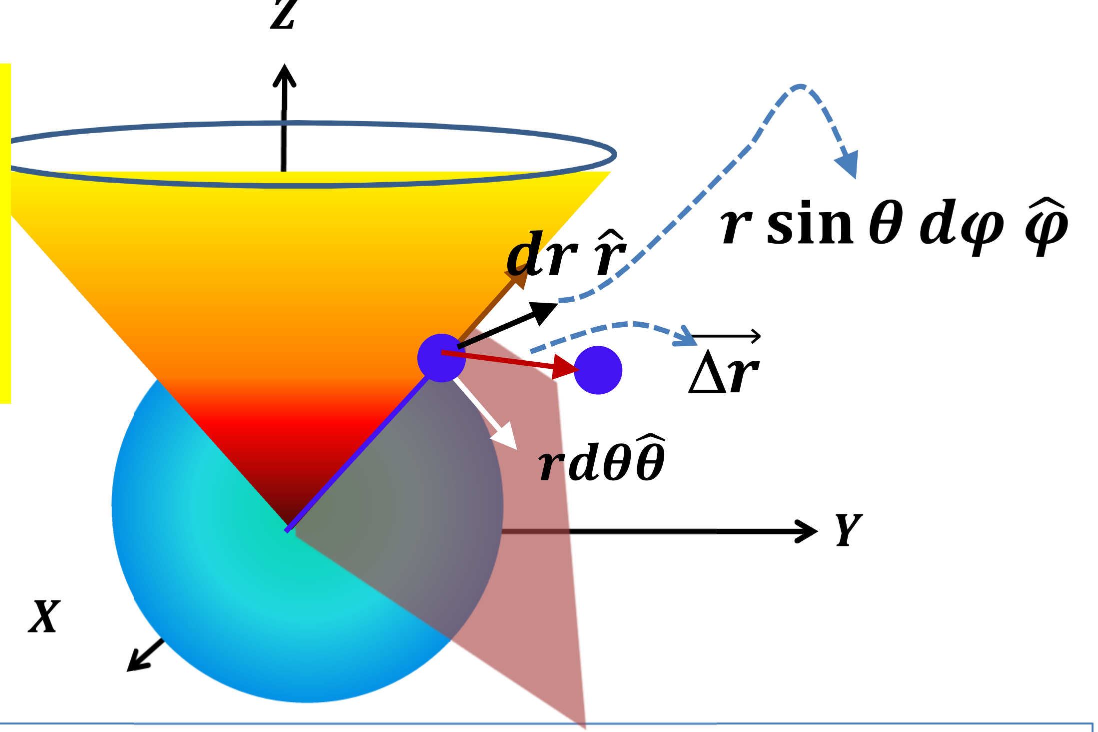
Acceleration

You must try to prove this

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 sin^2\theta\right)\hat{r} + (r\ddot{\theta} - 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\hat{\theta} + \left(r\ddot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + 2\dot{r}\dot{\varphi}\sin\theta\right)\hat{\varphi}$$

Velocity –to remember!

Elementary displacement in arbitrary direction $\overrightarrow{\Delta r}$ in Δt



$$\overrightarrow{\Delta \boldsymbol{r}} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta\, d\varphi\, \hat{\boldsymbol{\varphi}}$$

$$\vec{v} = \frac{\overrightarrow{\Delta \boldsymbol{r}}}{\Delta \boldsymbol{t}} = \frac{dr}{\Delta t}\hat{r} + \frac{rd\theta}{\Delta t}\hat{\theta} + \frac{r\sin\theta\, d\varphi}{\Delta t}\hat{\varphi}$$

$$\vec{\boldsymbol{v}} = \dot{\boldsymbol{r}}\hat{\boldsymbol{r}} + \boldsymbol{r}\dot{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \boldsymbol{r}\sin\boldsymbol{\theta}\dot{\boldsymbol{\varphi}}\,\hat{\boldsymbol{\varphi}}$$

Done!

Well, We are done with the necessary mathematical concepts!

Ok, Now in to Physics!