Continuous-time Markov Chain: Birth Death Process

SIDNEY CHAPMAN

ANDREY KOLMOGOROV

- **Backward Kolmogorov Equation**

$$p_{i,j}'(t) = \sum_k q_{i,k} p_{k,j}(t)$$

transition probability rate

$$q_{i,j} = v_i P_{i,j} \quad j \neq i$$

$$q_{i,i} = -v_i$$

- **Forward Kolmogorov Equation**

$$p_{i,j}'(t) = \sum_k p_{i,k}(t) q_{k,j}$$

The long-term behaviour of a CTMC depends on the transition probability matrix **P** of the corresponding embedded DTMC. A state i is transient/recurrent if the corresponding state of the embedded DTMC is transient/recurrent. For a null recurrent CTMC, the steady-state probability distribution of states exist.

If $\lim_{t \to \infty} p_{i,j}(t)$ exists, then

$\lim_{t \to \infty} p_{i,j}(t) = \pi_j$ independent of *i* where π_j is the probability of the state *j* at the steady state

Birth-death processes

- The Birth-Death process is the well-known example of continuous time MC.
- The process has the state space $V = \{0, 1, \ldots\}$.
- state transitions can occur only between neighbouring states. If the process is at state *i*, it can move only to the state *i*+1 (single birth) or *i*-1 (single death) at some random times.

Examples of Birth-death processes

- Poisson process is an example of a pure birth process.
- Total number of customers in a queuing system
- The population of a rare animal in a wildlife park

State holding time

We associate two times:

B_i= random time till the next birth. $B_i \sim \exp(\lambda_i)$

D_i=random time till the next death. $D_i \sim \exp(\mu_i)$

B_i s are independent of D_i

State holding time T_i at a state $i \neq 0$ is given by $T_i = \min(B_i, D_i)$.

Theorem: The state holding time for a Birth-death process at a state $i \neq 0$ is exponentially distributed with the rate parameter $(\lambda_i + \mu_i)$.

Proof

$$
\begin{aligned}
P(T_i > t) &= P(\min(B_i, D_i) > t) \\
&= P(B_i > t, D_i > t) \\
&= P(B_i > t)P(D_i > t) \\
&= e^{-(\lambda_i + \mu_i)t}
\end{aligned}
$$

$$\therefore 1 - F_{T_i}(t) = e^{-(\lambda_i + \mu_i)t}$$

$$\Rightarrow f_{T_i}(t) = (\lambda_i + \mu_i)te^{-(\lambda_i + \mu_i)t}$$

and $\nu_i = \lambda_i + \mu_i$

At state $i = 0$, $T_0 \sim \exp(\lambda_0)$

Transition probabilities of the embedded MC.

For $i \neq 0$,

$$P_{i,i+1} = P(B_i < D_i)$$

$$= \int_0^\infty \int_u^\infty \lambda_i e^{-\lambda_i u} \mu_i e^{-\mu_i v} dv du$$

$$= \frac{\lambda_i}{\lambda_i + \mu_i}$$

Similarly, $P_{i,i-1} = \dfrac{\mu_i}{\lambda_i + \mu_i}$

At $i = 0$, $\nu_0 = \lambda_0$ and $P_{01} = 1$

Transition rates

For $i \neq 0$,

$$q_{i,i+1} = \nu_i P_{i,i+1}$$

$$= (\lambda_i + \mu_i)\frac{\lambda_i}{\lambda_i + \mu_i} = \lambda_i$$

and

$$q_{i,i-1} = \nu_i P_{i,i-1}$$

$$= (\lambda_i + \mu_i)\frac{\mu_i}{\lambda_i + \mu_i} = \mu_i$$

$$\therefore q_{i,j} = \left\{ \vphantom{\begin{matrix} a \\ b \\ c \\ d \end{matrix}} \right.$$

For $i = 0$, $q_{0,1} = \lambda_0$

Thus the TPM for the embedded MC is given by

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & \ldots. & & & \\ \frac{\mu_1}{\lambda_1+\mu_1} & 0 & \frac{\lambda_1}{\lambda_1+\mu_1} & \ldots, & & & \\ \ldots. & & & & & & \\ 0 \ \ldots & & & & \frac{\mu_i}{\lambda_i+\mu_i} & 0 & \frac{\lambda_i}{\lambda_i+\mu_i} \ \ldots \\ \ldots & & & & & & \end{bmatrix}$$

The generator matrix is given by

$$\mathbf{Q} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0\ldots. & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0\ldots, & \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \ldots, \\ \ldots & & & & \\ \ldots & & & & \end{bmatrix}$$

The forward Kolmogorov equation is given by

$$p_{i,j}'(t) = -v_j p_{i,j}(t) + \sum_{k\neq j} p_{i,k}(t) q_{kj}$$

$$\therefore \frac{dp_{i,j}(t)}{dt} = -\left(\lambda_j + \mu_j\right) p_{i,j}(t) + \lambda_{j-1} p_{i,j-1}(t) + \mu_{j+1} p_{i,j+1}(t)$$

The backward Kolmogorov equation is given by

$$\frac{dp_{i,j}(t)}{dt} = -\left(\lambda_i + \mu_i\right) p_{i,i}(t) + \lambda_i p_{i+1,j}(t) + \mu_i p_{i-1,j}(t)$$

Because of the state varying parameters λ_i and μ_i, the solution of Kolmogorv equations is difficult.

Global Balance(GB) equations

We consider the special case when the steady state solution exists. Then as $t \to \infty$, $\lim_{t\to\infty} \frac{dp_{i,j}(t)}{dt} = 0$, $\lim_{t\to\infty} p_{i,j}(t) = \pi_j$ independent of *i*. We put the above results in the forward Kolmogorv equation

$$\frac{dp_{i,j}(t)}{dt} = -\left(\lambda_j + \mu_j\right) \quad p_{i,j}(t) + \lambda_{j-1} \quad p_{i,j-1}(t) + \mu_{j+1} \quad p_{i,j+1}(t)$$

$$\pi_{j-1}\lambda_{j-1} + \pi_{j+1}\mu_{j+1} - \left(\lambda_j + \mu_j\right)\pi_j = 0$$

Or $\pi_{j-1}\lambda_{j-1} + \pi_{j+1}\mu_{j+1} = \left(\lambda_j + \mu_j\right)\pi_j$

Solution of GB equation

At a state $j \neq 0$,

$$\pi_{j-1}\lambda_{j-1} + \pi_{j+1}\mu_{j+1} = \left(\lambda_j + \mu_j\right)\pi_j$$

At a state

At j=0, there cannot be further death. Therefor GB equation becomes

$\lambda_0\pi_0 = \mu_1\pi_1$ (1) $\sum_{j=0}^{\infty}\pi_j = 1$

To Summarise…

Birth-death process: A CTMC with

B_i = random time till the next birth. $B_i \sim \exp(\lambda_i)$

D_i =random time till the next death. $D_i \sim \exp(\mu_i)$

- The state holding time for a Birth-death process at a state $i \neq 0$ is exponentially distributed with the rate parameter $(\lambda_i + \mu_i)$.The transition probabilities for the embedded MC,

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i} \quad P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}$$

THANK YOU