Department of Electronics & Electrical Engineering

Lecture 7

Sinusoidal Steady State Response

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Phasor Representation of Sinusoids

- Often in determining the sinusoidal steady state response of circuits it is necessary to perform algebraic operations such as addition, subtraction, multiplication and division of two or more sinusoidal quantities of the same frequency.

- Consider the two sinusoids

$$i_1 = I_{m1} \sin \omega t \qquad (1)$$

$$i_2 = I_{m2} \sin(\omega t + \theta_2) \qquad (2)$$

- The current resulting from addition of eq.1 and eq.2 can be written as

$$i_3 = i_1 + i_2 = I_{m1} \sin \omega t + I_{m2} \sin(\omega t + \theta_2) \qquad (3)$$

- One way of obtaining the result of eq.3. is to plot each sinusoid and then make a point by point summation of the two sine waves.

- The amplitude and the phase of the resulting sinusoid can then be measured and $\boldsymbol{i_3}$ can be written as

$$i_3 = I_{m3} \sin(\omega t + \theta_3) \qquad (4)$$

Phasor Representation of Sinusoids

- To solve eq.3 analytically, the following trigonometric identity is used

$$\sin(\omega t+\theta_2)=\sin\omega t\cos\theta_2+\cos\omega t\sin\theta_2 \tag{5}$$

- Hence, eq.3 can be written as

$$i_3=\left(I_{m1}+I_{m2}\cos\theta_2\right)\sin\omega t+\left(I_{m2}\sin\theta_2\right)\cos\omega t \tag{6}$$

- Using the identity in eq.5 in eq.4 gives

$$i_3=I_{m3}\cos\theta_3\sin\omega t+I_{m3}\sin\theta_3\cos\omega t \tag{7}$$

- Comparing the coefficients of eq.6 and eq.7 gives

$$I_{m3}\cos\theta_3=I_{m1}+I_{m2}\cos\theta_2 \tag{8}$$

$$I_{m3}\sin\theta_3=I_{m2}\sin\theta_2 \tag{9}$$

- From eq.8 and eq.9, the amplitude $\mathbf{I_{m3}}$ and the phase angle $\boldsymbol{\theta_3}$ can be obtained.

Phasor Representation of Sinusoids

- The method described in eq.1 to eq.9 gets very cumbersome when more than two sinusoids are used. In such a scenario it is convenient to use ***phasors***

- The current given in eq.1 can be expressed as

$$i_1 = I_{m1}\sin\omega t = \mathrm{Im}\left[I_{m1} e^{j\omega t} \right] \qquad (10)$$

- The exponential function $e^{j\omega t}$ can be treated as a rotational operator. The amplitude is always unity, but the cosine and sine components vary as time progresses, Fig.1.

- As ωt moves through one through one complete period of **2π radians** the line ***OA*** makes one complete traversal of the circle in an anticlockwise direction.

- Line ***OA*** is fixed in value to the amplitude of the sine function it represents. If a plot of the vertical components ***OA*** is taken over one complete revolution, the sine function shown in Fig.1b is generated.

Phasor Representation of Sinusoids

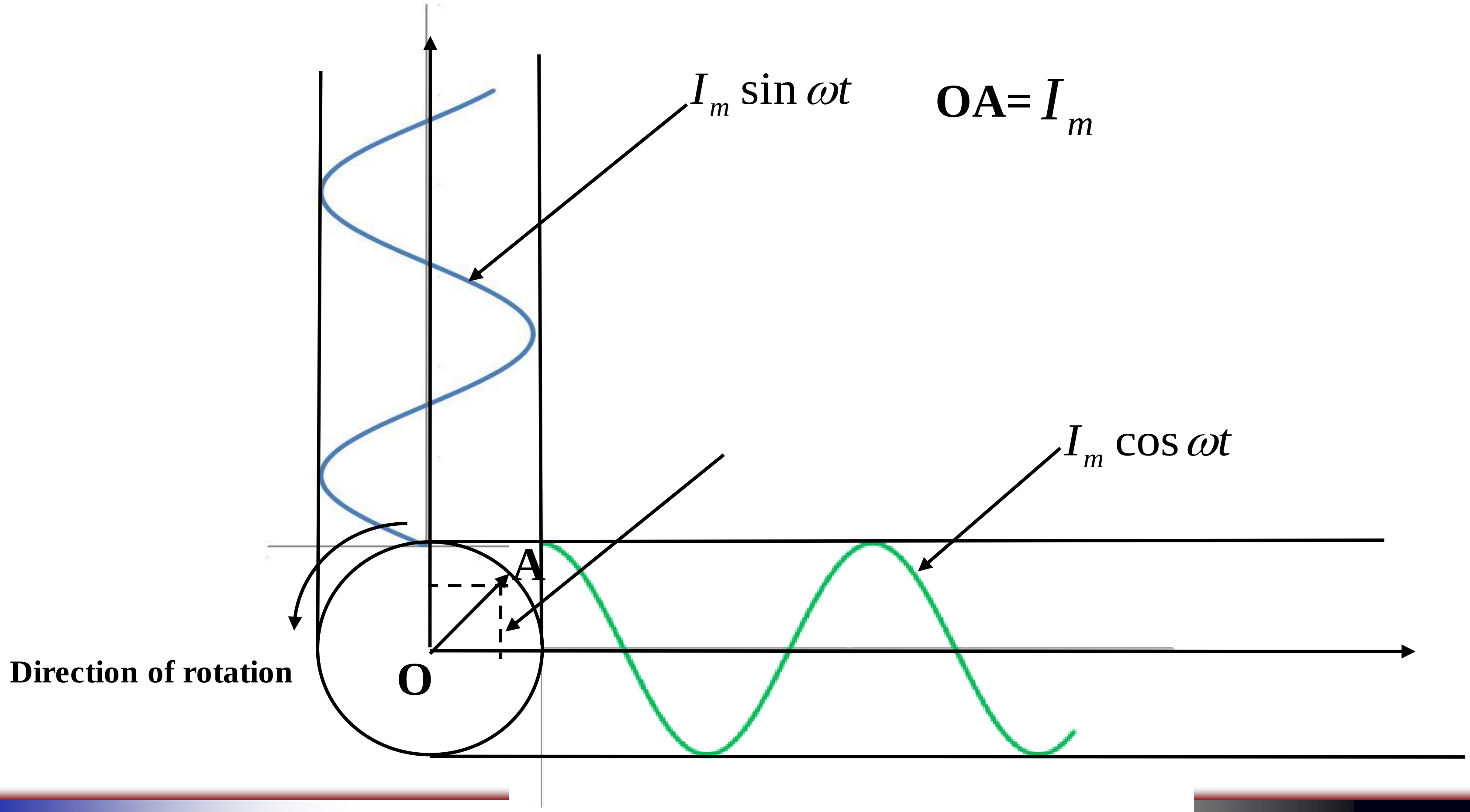


Figure 1

Phasor Representation of Sinusoids

- The current $\mathbf{i_2}$ as given in eq.2 can be represented in a similar fashion. Hence, in terms of the exponential notation

$$i_2 = I_{m2}\sin(\omega t + \theta_2) = \mathrm{Im}\left[I_{m2}e^{j(\omega t+\theta_2)}\right] = \mathrm{Im}\left[I_{m2}e^{j\theta_2}e^{j\omega t}\right] \qquad (11)$$

- To simplify the notation expressed in eq.11 a new quantity is defined

$$\bar{I}_{m2} = I_{m2}e^{j\theta_2} = I_{m2}\left(\cos\theta_2 + j\sin\theta_2\right) \qquad (12)$$

- The quantity $\bar{I}_{m2}$ is shown in Fig.2a and it is known as the ***phasor*** of the sinusoidal function of eq.11.

- The phasor ***OB*** can be located by specifying its magnitude, $\boldsymbol{I_{m2}}$ and its displacement from the horizontal $\boldsymbol{\theta_2}$.

- The eq,3 can be written in exponential form as

$$i_3 = i_1 + i_2 = \mathrm{Im}\left[\bar{I}_{m1}e^{j\omega t}\right] + \mathrm{Im}\left[\bar{I}_{m2}e^{j\theta}e^{j\omega t}\right] \qquad (13)$$

Phasor Representation of Sinusoids

- From complex algebra, the eq.13 can be written as

$$i_3 = \mathrm{Im}\left[\left(\overline{I}_{m1} + \overline{I}_{m2} e^{j\theta}\right) e^{j\omega t}\right] \quad (14)$$

- In eq.14, the term $e^{j\omega t}$ is factored out and this indicates that both the phasors are revolving at the same frequency ω.

- The eq.14 states that ***to add two sinusoidal quantities, it is enough to add the corresponding phasors***.

- If the resultant quantity of addition of two phasors is given by

$$\overline{I}_{m3} = \overline{I}_{m1} + \overline{I}_{m2} e^{j\theta} \quad (15)$$

$$\Rightarrow I_{m3} e^{j\theta_3} = I_{m1} + I_{m2} e^{j\theta_2} = (I_{m1} + I_{m2} \cos\theta_2) + jI_{m2} \sin\theta_2 \quad (16)$$

- The right hand side of eq.15 involves all the known quantities.

Phasor Representation of Sinusoids

- The magnitude of the complex entity shown given in eg.16 is

$$I_{m3} = \sqrt{\left(I_{m1} + I_{m2}\cos\theta_2\right)^2 + \left(I_{m2}\sin\theta_2\right)^2} \tag{17}$$

- The angle of the complex number given in eq.16 is

$$\theta_3 = \tan^{-1}\frac{I_{m2}\sin\theta_2}{I_{m1} + I_{m2}\cos\theta_2} \tag{18}$$

- The resultant phasor $\bar{I}_{m3}$ is shown in Fig.2 as line ***OD***. The phasors are added in the same manner as vectors in mechanics.

- The sinusoid $\boldsymbol{i_3}$ in Fig.2b can be considered as having been generated by the rotation of phasor $\bar{I}_{m3}$

- When the value of $\bar{I}_{m3}$ has been established, the corresponding time expression for $\boldsymbol{i_3}$ is

$$i_3 = \mathrm{Im}\left[\bar{I}_{m3}e^{j\omega t}\right] = \mathrm{Im}\left[I_{m3}e^{j\theta_3}e^{j\omega t}\right] = \mathrm{Im}\left[I_{m3}e^{j(\omega t+\theta_3)}\right]$$
$$= \mathrm{Im}\left[I_{m3}\cos(\omega t+\theta_3) + jI_{m3}\sin(\omega t+\theta_3)\right] = I_{m3}\sin(\omega t+\theta_3) \tag{19}$$

Phasor Representation of Sinusoids

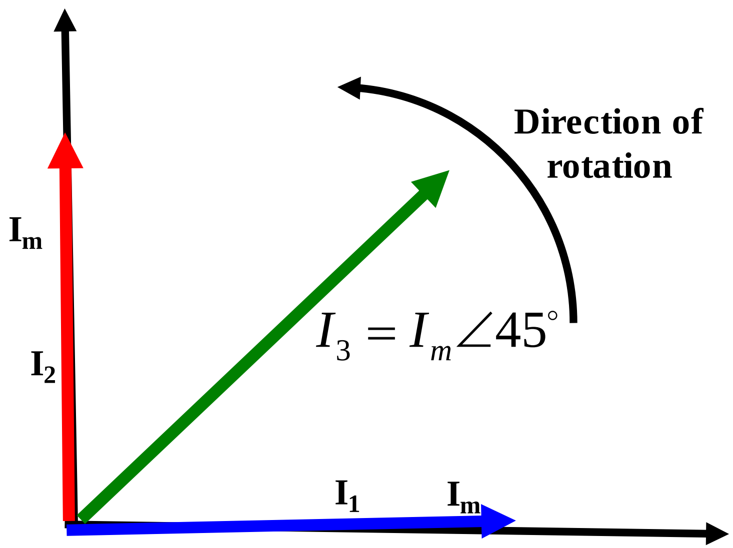


Figure 2a

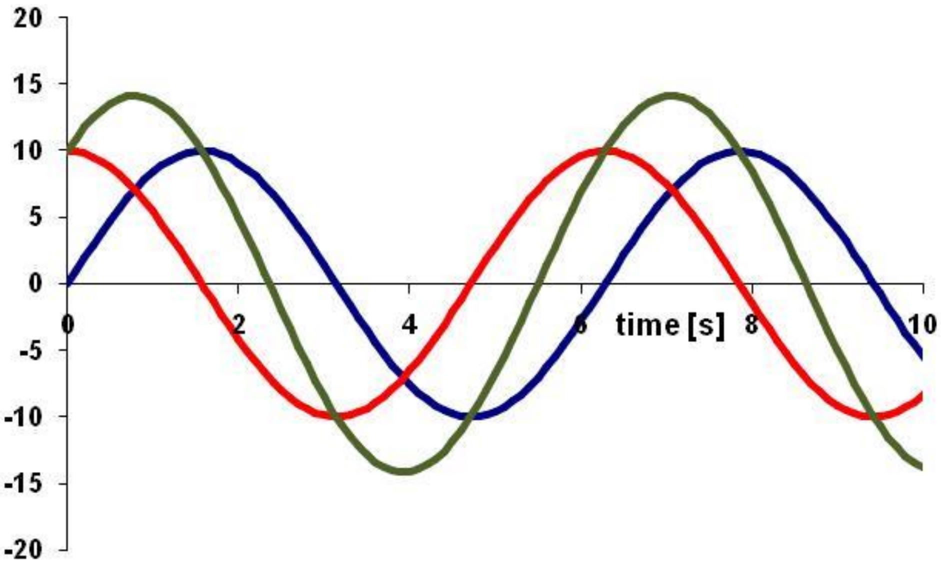


Figure 2b

Multiplication and Division of Complex Quantities

- In dealing with the sinusoidal steady state response of electric circuits the need frequently arises to multiply and divide complex numbers.

- As an illustration consider the that a phasor $\bar{I} = Ie^{j\theta}$ and $\bar{Z} = Ze^{j\phi}$. The product of these two phasors is

$$\overline{IZ} = Ie^{j\theta}Ze^{j\phi} = IZe^{j(\theta+\phi)} = IZ\angle\theta+\phi \qquad (20)$$

- Hence***, the product of two complex numbers is found by taking the product of their magnitudes and the sum of their angles.***

- To illustrate the division of the complex numbers consider the two phasors $\bar{V} = Ve^{j\theta}$ and $\bar{Z} = Ze^{j\phi}$ The division of these two phasors is given by

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{Ve^{j\theta}}{Ze^{j\phi}} = \frac{V}{Z}e^{j(\theta-\phi)} = \frac{V}{Z}\angle\theta-\phi \qquad (21)$$

- ***The division of one complex number by another involves the division of their magnitudes and difference in their phase angles.***

Power and Roots of Complex Quantities

- The ***nth*** power of the complex quantity $\bar{Z} = Ze^{j\phi}$ is obtained as

$$\bar{Z}^n = \left(Ze^{j\phi}\right)^n = Z^n e^{jn\phi} = Z^n \angle n\phi \qquad (22)$$

- ***The nth power of a complex number is a complex number whose magnitude is the nth power ot the magnitude of the original complex number and whose angle is n times as large as that of the original complex number.***

- To find the root of a complex number the exponent ***n*** is made a proper fraction in eq.23. The angle of the original complex number is increased by $\boldsymbol{2k\pi}$ in order to determine all the roots that satisfy eq.22.

- The fourth power of $\bar{Z} = Ze^{j\phi}$ is

$$\bar{Z}^{\frac{1}{4}} = \left(Ze^{j(\phi+2k\pi)}\right)^{\frac{1}{4}} = Z^{\frac{1}{4}} \angle \frac{\phi}{4} + \frac{k\pi}{2} \qquad (23)$$

Power and Roots of Complex Quantities

- The four distinct roots are:

$$\bar{Z}_1^{1/4} = Z^{1/4}\angle\frac{\phi}{4} \quad \text{for } k = 0 \tag{23}$$

$$\bar{Z}_2^{1/4} = Z^{1/4}\angle\frac{\phi}{4}+\frac{\pi}{2} \quad \text{for } k = 1 \tag{24}$$

$$\bar{Z}_3^{1/4} = Z^{1/4}\angle\frac{\phi}{4}+\pi \quad \text{for } k = 2 \tag{25}$$

$$\bar{Z}_4^{1/4} = Z^{1/4}\angle\frac{\phi}{4}+\frac{3\pi}{2} \quad \text{for } k = 3 \tag{26}$$

- ***Any further values assigned to k will yield results that are repetitions of those already listed in eq.23 to eq.26***

Complex Power

- Consider the ac load shown in Fig.3. The voltage and current in the network is

$$v = V_m \sin(\omega t) \tag{27}$$

$$i = I_m \sin(\omega t - \theta) \tag{28}$$

- The complex power ***S*** absorbed by the ac load is the product of the voltage and the complex conjugate of the current

$$S = \frac{1}{2}\mathbf{V}\mathbf{I}^* \tag{29}$$

- In terms of ***effective*** or ***rms*** values, the eq.40 can be written as

$$\mathbf{S} = \mathbf{V}_{\mathbf{rms}}\mathbf{I}^*_{\mathbf{rms}} \tag{30}$$

where

$$\mathbf{V}_{\mathbf{rms}} = V_{rms}\angle \omega t \tag{31}$$

$$\mathbf{I}_{rms} = I_{rms}\angle(\omega t - \theta) \tag{32}$$

Hence

$$\mathbf{S} = V_{rms} I_{rms} \angle\theta = V_{rms} I_{rms} \cos(\theta) + jV_{rms} I_{rms} \sin(\theta) \tag{33}$$

Complex Power

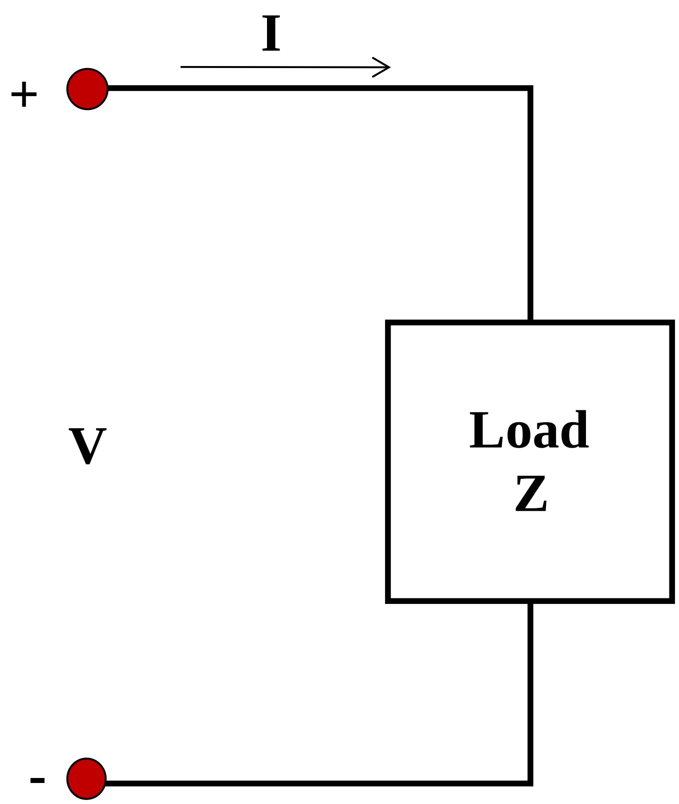


Fig.3: The load with a complex power

Complex Power

- The complex power may be expressed in terms of the load impedance ***Z***. The load impedance may be written as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V_{rms}}}{\mathbf{I_{rms}}} = \frac{V_{rms}}{I_{rms}} \angle \theta \qquad (34)$$

$$\Rightarrow \mathbf{V_{rms}} = \mathbf{Z I_{rms}} \qquad (35)$$

- Using eq.34 and eq.35 in eq.30 gives

$$\mathbf{S} = \mathbf{I}_{rms}^2 \mathbf{Z} = \frac{\mathbf{V}_{rms}^2}{\mathbf{Z}^*} = \mathbf{V_{rms}} \mathbf{I_{rms}} \qquad (36)$$

since

$$\mathbf{Z} = R + jX$$

$$\mathbf{S} = I_{rms}^2 (R + jX) = P + jQ \qquad (37)$$

where

$$P = I_{rms}^2 R = \text{Real power [watts]} \qquad (38)$$

$$Q = I_{rms}^2 X = \text{Reactive power [Volt-Ampere-Reactive]} \qquad (39)$$

Summary of Complex Power

Complex Power = $\mathbf{S}$ = P+jQ = $\frac{1}{2}\mathbf{VI}^{*} = V_{rms} I_{rms} \angle \theta$ (40)

Apparent Power = $S = |\mathbf{S}| = V_{rms} I_{rms}$ (41)

Real Power = $P = \text{Re}(\mathbf{S}) = V_{rms} I_{rms} \cos(\theta)$ (42)

Reactive Power = $Q = \text{Im}(\mathbf{S}) = V_{rms} I_{rms} \sin(\theta)$ (43)

Power Factor = $\frac{P}{S} = \cos(\theta)$ (44)

Example 1

- The voltage across a load is $v(t) = 60\cos(\omega t - 10^o)$ **V** and the current through the element in direction of voltage drop is $i(t) = 1.5\cos(\omega t + 50^o)$ **A** Find

(a) The complex power and the apparent power

(b) The real and reactive power

(c) The power factor and the load impedance

Solution

a. $V_{rms} = \frac{60}{\sqrt{2}}\angle -10^o, \quad I_{rms} = \frac{1.5}{\sqrt{2}}\angle 50^o$

The complex power is

$$\mathbf{S} = \mathbf{V}_{rms}\mathbf{I}^*_{rms} = \left(\frac{60}{\sqrt{2}}\angle -10^o\right)\left(\frac{1.5}{\sqrt{2}}\angle -50^o\right) = 45\angle -60^o$$

The apparent power is

$S = |\mathbf{S}| = 45$ Volt-Ampere [VA]

b. $\mathbf{S} = P + jQ = 45\cos(-60^o) + j\sin(-60^o) = 22.5 - j38.97$

The active power is

P=22.5 watts [w]

The reactive power is

$Q = -38.97$ Volt-Ampere-Reactive [var]

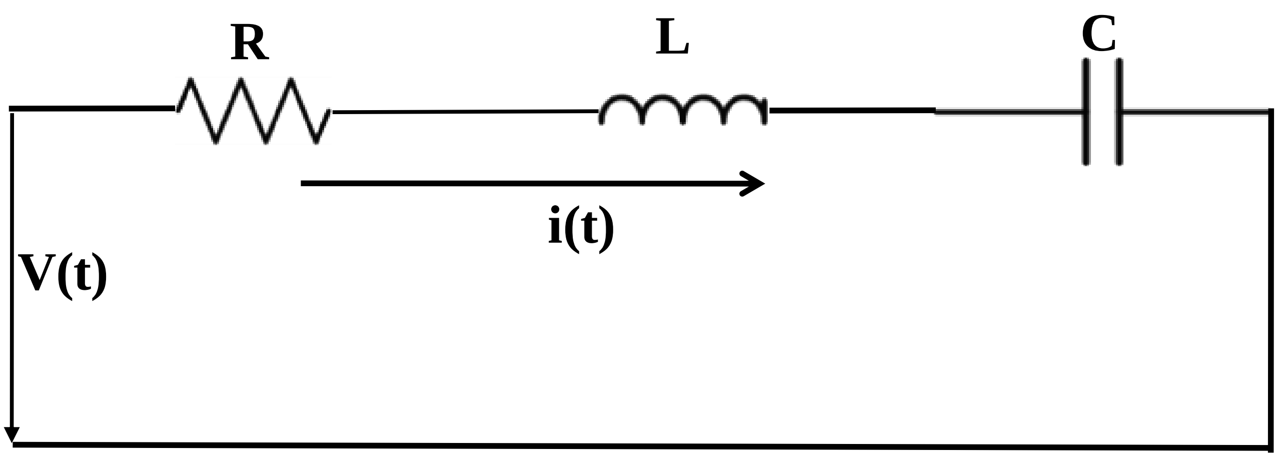
c. The power factor is

$pf = \cos(-60^o) = 0.5(leading)$ and the load impedance is $\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{60\angle -10^o}{1.5\angle 50^o} = 40\angle -60^o$

Example 2

Question: A voltage v(t)=220sin(ωt) Volts is applied to a series combination of a resistance of 15Ω, an inductive reactance 10Ω and a capacitive reactance of 5Ω.

i. Find the current in the circuit
ii. Sketch the phasor diagram



Solution

- The applied voltage is

$$V(t) = 220\sin(\omega t)$$

- The phasor representation of the voltage (rms)

$$\bar{V} = \frac{220}{\sqrt{2}}\angle 0^{\circ} = 155.56\angle 0^{\circ}$$

- The impedance of the circuit is

$$\bar{Z} = 15 + j10 - j5 = 15 + j5 = 15.81\angle 18.43^{\circ}$$

- The current in the circuit is

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{155.56\angle 0^{\circ}}{15.81\angle 18.43^{\circ}} = 9.84\angle -18.43^{\circ}$$

- The instantaneous current in the circuit is

$$i(t) = \sqrt{2} \times 9.84\sin(\omega t - 18.43 \times pi/180) = 13.92\sin(\omega t - 0.32)$$

Solution (The Phasor Diagram)

- The voltage across the resistor is

$$V_R = 15 \times \bar{I} = 15 \times 9.84\angle -18.43 = 147.6\angle -18.43$$

- The voltage across the inductor is

$$V_L = 15\angle 90^\circ \times \bar{I} = 15\angle 90^\circ \times 9.84\angle -18.43^\circ = 147.6\angle 71.57$$

- The voltage across the capacitor

$$V_C = 5\angle -90^\circ \times \bar{I} = 5\angle -90^\circ \times 9.84\angle -18.43^\circ = 49.2\angle -108.43^\circ$$

- The resultant voltage is

$$V = 155.56\angle 0^\circ$$

- The current is

$$I = 9.84\angle -18.43^\circ$$