Linear Algebra

Department of Mathematics
Indian Institute of Technology Guwahati

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MA 102 (RA, RKS, MGPP, KVK)

Basis and dimension

Topics:

- Linear span
- Subspaces
- Linear independence
- Basis, Dimension & Rank

Linear combination

Definition: A vector $\mathbf{v}$ in $\mathbb{R}^n$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ if there exist real numbers $c_1, c_2, \ldots, c_k$ such that

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Theorem: A system of linear equations with augmented matrix $[A \mid \mathbf{b}]$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A.

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Exercise: Let $\mathbf{u} = [1, 2, 3]^\top$ and $\mathbf{v} = [-1, 1, -3]^\top$. Describe $\mathrm{span}(\mathbf{u}, \mathbf{v})$ geometrically.

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Exercise: Examine whether the sets
$S = \{[x, y, z]^\top \in \mathbb{R}^3 : x = y + 1\}$, $V = \{[x, y, z]^t \in \mathbb{R}^3 : x = 5y\}$
and $U = \{[x, y, z]^t \in \mathbb{R}^3 : x = z^2\}$ are subspaces of $\mathbb{R}^3$.

Direct sum of subspaces

Fact: Let A be an $m \times n$ matrix. Then $U := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called the nullspace of A.

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Definition: Let U and V be two subspaces of $\mathbb{R}^n$. If $U \cap V = \{\mathbf{0}\}$ then the sum $U + V$ is called the direct sum of U and V and is denoted by $U \oplus V$. Thus

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Fact: Let U and V be subspaces of $\mathbb{R}^n$. Then $U + V$ and $U \oplus V$ are subspaces of $\mathbb{R}^n$. If $\mathbf{z} \in U \oplus V$ then there exist unique $\mathbf{u} \in U$ and $\mathbf{v} \in V$ such that $\mathbf{z} = \mathbf{u} + \mathbf{v}$.

Linear dependence

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Exercise: Examine whether the sets
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Fact: Let $S := \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$. Consider the $n \times m$ matrix $A := [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_m]$. Then S is linearly dependent iff the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

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- The rows of A are linearly dependent iff $\mathbf{A}_1^\top, \ldots, \mathbf{A}_m^\top$ are linearly dependent in $\mathbb{R}^n$, i.e., the columns of $A^\top$ are linearly dependent.

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Suppose (6) holds. Then $EA^\top = \mathrm{rref}(A^\top)$ for some invertible matrix E. Now $\mathbf{e}_m^\top \mathrm{rref}(A^\top) = \mathbf{0} \Rightarrow A\mathbf{y} = 0$, where $\mathbf{y} := E^\top \mathbf{e}_m$.

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Exercise: Find a basis for the subspace $S := \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{bmatrix}.$$

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3. The null space of A, denoted $\mathrm{null}(A)$, is the subspace of $\mathbb{R}^n$ consisting of the solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. In other words, $\mathrm{null}(A) := \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$

Fundamental subspaces associated to a matrix

Definition: Let A be an $m \times n$ matrix.

1. The column space / range space of A, denoted $\mathrm{col}(A)$, is the subspace of $\mathbb{R}^m$ spanned by the columns of A.
   In other words, $\mathrm{col}(A) := \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$.
2. The row space of A, denoted $\mathrm{row}(A)$, is the subspace of $\mathbb{R}^n$ spanned by the rows of A. In other words,
   $\mathrm{row}(A) := \{\mathbf{x}^\top A \mid \mathbf{x} \in \mathbb{R}^m\}$

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4. The null space of $A^\top$: $\mathrm{null}(A^\top) = \{\mathbf{x} \in \mathbb{R}^m \mid A^\top\mathbf{x} = \mathbf{0}\}$.

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Theorem: If two matrices A and B are row equivalent, then

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Corollary: For any A, $\mathrm{row}(A) = \mathrm{row}(\mathrm{rref}(A))$.

Corollary: For any matrix A, the non-zero rows of $\mathrm{rref}(A)$ forms a basis of $\mathrm{row}(A)$.

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Theorem: Let P be an invertible matrix. Then a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ in $\mathbb{R}^n$ is linearly independent iff the set $\{P\mathbf{v}_1, P\mathbf{v}_2, \ldots, P\mathbf{v}_m\}$ is linearly independent.

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Corollary: Let $A := [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ and $\mathrm{rref}(A) = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_n]$. If are $\mathbf{b}_{j_1}, \mathbf{b}_{j_2}, \ldots, \mathbf{b}_{j_r}$ are pivot columns of $\mathrm{rref}(A)$, then $\{\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \ldots, \mathbf{a}_{j_r}\}$ is a basis of $\mathrm{col}(A)$.

Algorithm for computing bases of null spaces

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2. Suppose that R has p-nonzero rows. So it has p-pivot columns. Interchange columns of R (i.e., choose a permutation matrix P) so that

$$RP = \begin{bmatrix} I_p & F \\ 0 & 0 \end{bmatrix} = \text{column interchanged form of } R,$$

   where I_p is the identity matrix of size p.

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Then $\mathrm{rank}(X) = n - p$ and $RX = RPY = 0$. Thus columns of X span the null space of R and hence the null space of A.

Example

Compute bases of the null space, row space and the column space of the matrix

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- Solve $R\mathbf{x} = \mathbf{0}$ to find a basis of $\text{null}(R)$, or use the previous algorithm.

Example (cont.)

Interchanging 2nd and 3rd columns of R, we have

$$RP = \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & F \\ 0 & 0 \end{bmatrix}.$$

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Finally, interchange 2nd and 3rd row of Y to obtain X, that is,

$$X = PY = \begin{bmatrix} -3 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix},$$

which gives a basis of the null space of A.

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If $\mathbf{x}$ is a solution with $n - r$ free parameters, then setting all but one parameter to zero at a time results in $n - r$ linearly independent solutions. ■

The fundamental theorem of invertible matrices

Theorem: Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. A is invertible.
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