PH 102 : Electromagnetism
Tutorial 9
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1 Solutions to Selective Take Home Problems

1. A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.
(a) What is the magnetic dipole moment of the sphere?
(b) Find the magnetic field at a point (r, θ) inside the sphere.
(c) Using the results of (b) find the average magnetic field within the sphere. Hint: Average magnetic field is defined as

$$\vec{B}_{\text{avg}} = \frac{1}{\frac{4}{3}\pi R^3} \int \vec{B} d\tau$$

Compare this result with the result of (a) and show that the average magnetic field is related to the magnetic dipole moment as

$$\vec{B}_{\text{avg}} = \frac{\mu_0}{4\pi} \frac{2\vec{m}}{R^3}$$

Solution:
The vector potential for a charged spinning spherical shell, as discussed in the class, is

$$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_0 R\sigma}{3}(\vec{\omega} \times \vec{r}) = \frac{\mu_0 R\omega\sigma}{3} r \sin\theta \hat{\phi} & \text{for } r \leq R \\ \frac{\mu_0 R^4 \sigma}{3r^3}(\vec{\omega} \times \vec{r}) = \frac{\mu_0 R^4 \omega\sigma}{3} \frac{\sin\theta}{r^2} \hat{\phi} & \text{for } r \geq R. \end{cases}$$

For $\vec{\omega}$ along the z-axis, the vector potential outside the sphere can be written as

$$\vec{A} = \frac{\mu_0 R^4 \omega\sigma}{3} \frac{\sin\theta}{r^2} \hat{\phi} = \frac{\mu_0}{4\pi} \frac{1}{r^2} \frac{4\pi R^4 \omega\sigma}{3} (-\sin\theta \hat{\theta} \times \hat{r}) \quad (r \geq R)$$

(a) Compering it with the dipole contribution to vector potential $\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$, the dipole moment can be written as

$$\vec{m} = \frac{4\pi R^4 \omega\sigma}{3} \hat{z}$$

To find the dipole moment for a charged spinning solid sphere, we can first consider a shell at a distance r from the centre having a thickness dr. If ρ is the volume charge density of the sphere, then the shell of radius r and thickness dr has surface charge density $\sigma = \rho dr$. The total magnetic dipole moment of the solid sphere can then be found by integrating over all such shells from radius $r = 0$ to $r = R$. This is given by

$$\vec{m} = \frac{4\pi}{3}\omega\rho\hat{z}\int_0^R r^4 dr = \frac{4\pi}{3}\omega\rho\frac{R^5}{5}\hat{z} = \frac{1}{5}Q\omega R^2\hat{z}$$

where, in the last step, we have used $\rho = \frac{Q}{(4/3)\pi R^3}$.

(b) To find the magnetic field at a point (r,θ) inside the sphere, we first find the vector potential $\vec{A}$. Consider a spherical shell of radius r' and thickness dr' inside the solid sphere of radius R. The surface charge density of this shell is $\sigma = \rho dr'$, where ρ is the volume charge density. The vector potential outside and inside this shell are given by the above expressions. To find the net potential at a radial distance r inside the sphere, we integrate over all possible such shells of infinitesimal thickness inside the radius $r' < r$ and outside the radius $r < r' < R$. The infinitesimal vector potential is

$$d\vec{A}(r,\theta) = \frac{\mu_0 (r')^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi} + \frac{\mu_0 r'\omega\sigma}{3} r\sin\theta\hat{\phi}$$

. The net vector potential can be found by substituting $\sigma = \rho dr'$ and integrating over r':

$$\vec{A}(r,\theta) = \frac{\mu_0\omega\rho}{3}\frac{\sin\theta}{r^2}\hat{\phi}\int_0^r (r')^4 dr' + \frac{\mu_0\omega\rho}{3} r\sin\theta\hat{\phi}\int_r^R r' dr'$$

$$\implies \vec{A}(r,\theta) = \frac{\mu_0\omega\rho}{3}\sin\theta\left[\frac{1}{r^2}\frac{r^5}{5} + \frac{r}{2}(R^2 - r^2)\right]\hat{\phi} = \frac{\mu_0\omega\rho}{2} r\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\hat{\phi}$$

. Now the magnetic field can be found by taking the curl of $\vec{A}$:

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0\omega\rho}{2}\frac{1}{r^2\sin\theta}\left[\hat{r}\frac{\partial}{\partial\theta}\left(r\sin\theta r\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\right) - r\hat{\theta}\frac{\partial}{\partial r}\left(r\sin\theta r\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\right)\right]$$

$$\implies \vec{B} = \frac{\mu_0\omega\rho}{2}\left[\hat{r}2\cos\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right) - \hat{\theta}\sin\theta 2\left(\frac{R^2}{3} - \frac{2r^2}{5}\right)\right]$$

$$\implies \vec{B} = \mu_0\omega\rho\left[\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\cos\theta\hat{r} - \left(\frac{R^2}{3} - \frac{2r^2}{5}\right)\sin\theta\hat{\theta}\right]$$

Using $\rho = \frac{Q}{(4/3)\pi R^3}$,

$$\vec{B} = \frac{\mu_0\omega Q}{4\pi R}\left[\left(1 - \frac{3r^2}{5R^2}\right)\cos\theta\hat{r} - \left(1 - \frac{6r^2}{5R^2}\right)\sin\theta\hat{\theta}\right]$$

(c) Due to the symmetry of the problem, the average magnetic field will be in the z direction. Therefore, we can take out only the z components of the magnetic field found in part (b). Writing $\hat{r},\hat{\theta}$ in terms of $\hat{x},\hat{y},\hat{z}$ that is $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$, $\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$ and considering only the z-component (that is, take the z component of $\hat{r}(\cos\theta)$ and $\hat{\theta}(-\sin\theta)$, we can write down the average magnetic field as

$$B_{\text{avg}} = \frac{1}{\frac{4}{3}\pi R^3}\int B_z d\tau$$

$$\implies B_{\text{avg}} = \frac{\mu_0\omega Q}{4\pi R}\frac{1}{\frac{4}{3}\pi R^3}\int\left[\left(1-\frac{3r^2}{5R^2}\right)\cos^2\theta + \left(1-\frac{6r^2}{5R^2}\right)\sin^2\theta\right] r^2\sin\theta dr d\theta d\phi$$

$$\implies B_{\text{avg}} = \frac{3\mu_0\omega Q}{(4\pi R^2)^2}2\pi\int_0^\pi\left[\left(\frac{R^3}{3}-\frac{3}{5}\frac{R^5}{5R^2}\right)\cos^2\theta + \left(\frac{R^3}{3}-\frac{6}{5}\frac{R^5}{5R^2}\right)\sin^2\theta\right]\sin\theta d\theta$$

$$\implies B_{\text{avg}} = \frac{3\mu_0\omega Q}{8\pi R^4}R^3\int_0^\pi\left(\frac{16}{75}\cos^2\theta + \frac{7}{75}\sin^2\theta\right)\sin\theta d\theta = \frac{3\mu_0\omega Q}{8\pi R}\frac{1}{75}\int_0^\pi(7+9\cos^2\theta)\sin\theta d\theta$$

$$\implies B_{\text{avg}} = \frac{\mu_0\omega Q}{200\pi R}(-7\cos\theta - 3\cos^3\theta)|_0^\pi = \frac{\mu_0\omega Q}{200\pi R}(20) = \frac{\mu_0\omega Q}{10\pi R}$$

Using the expression for magnetic dipole moment obtained in part (a) that is, $\vec{m} = \frac{1}{5}Q\omega R^2\hat{z}$ it is straightforward to show that

$$\frac{\mu_0}{4\pi}\frac{2\vec{m}}{R^3} = \frac{\mu_0\omega Q}{10\pi R}\hat{z} = \vec{B}_{\text{avg}}$$

Therefore, the average magnetic field, over a sphere of radius R, due to steady current within the sphere, is

$$\vec{B}_{\text{avg}} = \frac{\mu_0}{4\pi}\frac{2\vec{m}}{R^3}$$

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2. A thin glass rod of radius R and length L carries a uniform charge σ. It is spinning about its axis, at an angular velocity ω. Find the magnetic field at a distance $s \gg R$ from the center of the rod (see figure 1).

   Solution:

If the dipole is at the origin and the field is measured at a point in the xz plane ($\phi = 0$) then,
$\vec{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(2cos\theta\hat{r} + sin\theta\hat{\theta}) = \frac{\mu_0}{4\pi}\frac{m}{r^3}[2cos\theta(sin\theta\hat{x} + cos\theta\hat{z}) + sin\theta(cos\theta\hat{x} - sin\theta\hat{z})] = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left[3sin\theta cos\theta\hat{x} + (2cos^2\theta - sin^2\theta)\hat{z}\right].$
The glass rod is assumed to be a stack of such dipoles from $z = -L/2$ to $z = L/2$.

The coordinate system is chosen such that the field point is on the x axis at a distance s (See figure 2). Symmetry of this coordinate system implies dipoles above and below $z = 0$ will cancel the $\hat{x}$ components, thus $\vec{B} = \frac{\mu_0}{4\pi}2M\hat{z}\int_0^{L/2}\frac{3cos^2\theta - 1}{r^3}dz$.
Here, dipole moment per unit length $M = \frac{m}{h} = \frac{I\pi R^2}{h} = \frac{(\sigma v h)\pi R^2}{h} = \frac{(\sigma\omega R h)\pi R^2}{h} = \pi\sigma\omega R^3$. The dipole moment in elemental length dz is therefore, taken to be Mdz which is then integrated over. We can also write

$$sin\theta = \frac{s}{r} \implies \frac{1}{r^3} = \frac{sin^3\theta}{s^3}$$

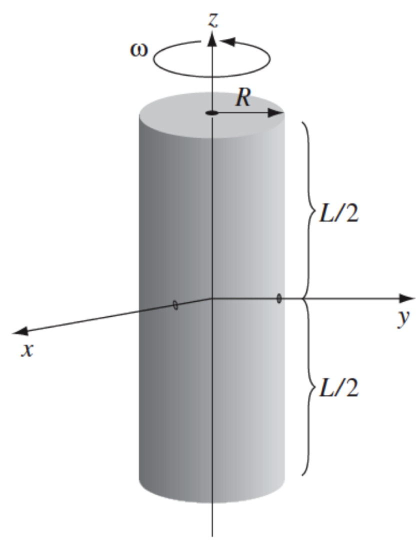


Figure 1: Figure for take home problem 2.

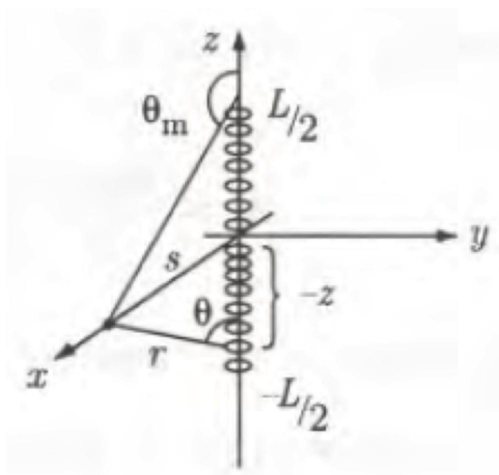


Figure 2: Figure for solution to take home problem 2.

$$z = -s\ cot\theta \implies dz = \frac{s}{sin^2\theta}d\theta$$

Total field,

$$\vec{B} = \frac{\mu_0}{2\pi}\pi\sigma\omega R^3\hat{z}\int_{\pi/2}^{\theta_m}(3cos^2\theta - 1)\frac{sin^3\theta}{s^3}\frac{s}{sin^2\theta}d\theta = \frac{\mu_0\sigma\omega R^3}{2s^2}\hat{z}\int_{\pi/2}^{\theta_m}(3cos^2\theta - 1)sin\theta d\theta$$

$$= \frac{\mu_0\sigma\omega R^3}{2s^2}\hat{z}(-cos^3\theta + cos\theta)\Big|_{\pi/2}^{\theta_m} = \frac{\mu_0\sigma\omega R^3}{2s^2}cos\theta_m sin^2\theta_m\hat{z}.$$

Also, $\sin\theta_m = \frac{s}{\sqrt{s^2+(L/2)^2}}$ and $\cos\theta_m = \frac{-L/2}{\sqrt{s^2+(L/2)^2}}$. Hence, $\vec{B} = -\frac{\mu_0\sigma\omega R^3 L}{4(s^2+(L/2)^2)^{3/2}}\hat{z}$.

3. Suppose the field inside a large piece of magnetic material is $\vec{B}_0$, so that $\vec{H}_0 = \vec{B}_0/\mu_0 - \vec{M}$.
(a) Now a small spherical cavity is hollowed out of the material (as shown in figure 3). Find the field at the centre of the cavity, in terms of $\vec{B}_0, \vec{M}$. Also find $\vec{H}$ at the centre of the cavity in terms of $\vec{H}_0, \vec{M}$.
(b) Do the same for a long needle-shaped cavity running parallel to $\vec{M}$.
(c) Do the same for a thin wafer-shaped cavity perpendicular to $\vec{M}$.

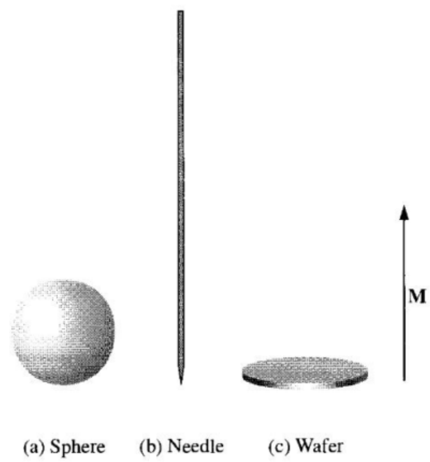
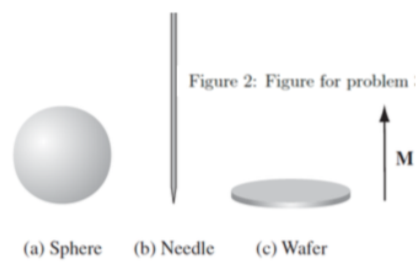


Figure 3: Figure for take home problem 3.

Solution:

(a) D. J. Griffiths, Example 6.1, the field inside a magnetized sphere is (2/3) μ_0**M.** The field after the removal of the sphere is $\mathbf{B} = \mathbf{B}_0 - (2/3)\ \mu_0 \mathbf{M}$.



Thus in the cavity $\mathbf{H} = \mathbf{B}/\mu_0 = [\mathbf{B}_0 - (2/3)\ \mu_0 \mathbf{M}]/\mu_0 = \mathbf{H}_0 + \mathbf{M} - (2/3)\,\mathbf{M} = \mathbf{H}_0 + 1/3\ \mathbf{M}$.

(b) The field inside a long solenoid is $\mu_0 K = \mu_0 M$. Thus the field of the bound charge on the inside surface of the needle shaped cavity is μ_0M, but *pointing down.* Thus, $\mathbf{B} = \mathbf{B}_0 - \mu_0 \mathbf{M}$ and $\mathbf{H} = \mathbf{B}/\mu_0 = [\mathbf{B}_0 - \mu_0 \mathbf{M}]/\mu_0 = \mathbf{H}_0 + \mathbf{M} - \mathbf{M} = \mathbf{H}_0$.

(c) For the thin wafer, the bound currents are very small and also far away from the center. Hence, $\mathbf{B} = \mathbf{B}_0$ and the $\mathbf{H} = \mathbf{B}/\mu_0$
$= \mathbf{B}_0/\mu_0 = \mathbf{H}_0 + \mathbf{M}$.

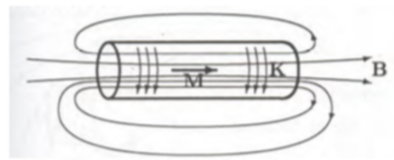
4. A short circular cylinder of radius a and length L carries a "frozen-in" uniform magnetisation $\vec{M}$ parallel to its axis. Find the bound current and sketch the magnetic field of the cylinder: one for $L \gg a$, one for $L \ll a$ and one for $L \approx a$.

Solution: The magnetization being uniform, the volume current, $\mathbf{J}_b = \nabla \times \mathbf{M} = \mathbf{0}$,

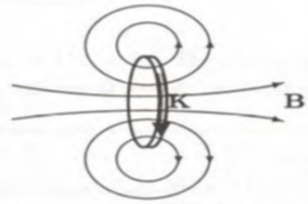
The surface current, $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M\hat{\boldsymbol{\phi}}$.

Here, **M** is in the z direction and surface vector **n** is in the radial direction.

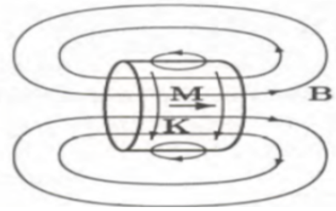
L >> a, This situation is similar to a long solenoid with the surface current flowing in the azimuth direction and uniform field inside.



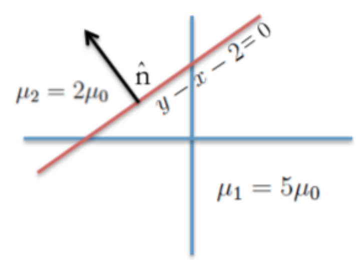
L << a, This situation is similar to a physical dipole.



L ≈ a, This is the intermediate case of the above two.



5. Given that $\vec{H_1} = -2\hat{i} + 6\hat{j} + 4\hat{k}$ A/m in the region $y - x - 2 \leq 0$, where $\mu_1 = 5\mu_0$. Calculate
   (a) $\vec{M_1}$ and $\vec{B_1}$.
   (b) $\vec{M_2}$ and $\vec{B_2}$ in the region $y - x - 2 \geq 0$, where $\mu_2 = 2\mu_0$.



Solution: $\hat{\mathbf{n}} = (\hat{j} - \hat{i})/\sqrt{2}$

(a) $\mathbf{M}_1 = \chi_m \mathbf{H}_1 = (\mu_{r1} - 1)\,\mathbf{H}_1 = 4\,\mathbf{H}_1$ and $\mathbf{B}_1 = \mu_1 \mathbf{H}_1 = 5\mu_0 \mathbf{H}_1$

(b) $\mathbf{H}_{1n} = (\mathbf{H}_1 \cdot \hat{\mathbf{n}})\,\hat{\mathbf{n}} = 4\,(-\hat{i} + \hat{j})$, $\mathbf{H}_{1t} = \mathbf{H}_1 - \mathbf{H}_{1n} = 2\,(\hat{i} + \hat{j} + \hat{k})$.

Using the boundary conditions, $\mathbf{B}_{2n} = \mathbf{B}_{1n}$ and $\mu_2 \mathbf{H}_{2n} = \mu_1 \mathbf{H}_{1n} \Rightarrow 2\mu_0 \mathbf{H}_{2n} = 5\mu_0 \mathbf{H}_{1n}$

Thus, $\mathbf{H}_{2n} = 10\,(-\hat{i} + \hat{j})$ and $\mathbf{H}_{2t} = \mathbf{H}_{1t} = 2(\hat{i} + \hat{j} + \hat{k})$, $\mathbf{H}_2 = \mathbf{H}_{2n} + \mathbf{H}_{2t} = -8\,\hat{i} + 12\,\hat{j} + 4\,\hat{k}$

and $\mathbf{B}_2 = \mu_2 \mathbf{H}_2$