PH102: Tutorial Problem set

Tutorial 3

2018-10-24

3.01. Compute the divergence of the vector field $\vec{v} = (r\cos\theta)\hat{r} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}$. Check the divergence theorem for this vector field, using as your volume the inverted hemispherical bowl of radius R, resting on xy plane and centered at the origin (see Figure 1).

3.02. Compute the line integral of $\vec{F} = r\cos^2\theta\hat{r} - r\cos\theta\sin\theta\hat{\theta} + 3r\hat{\phi}$ around the path shown in Figure 2. Do it either in cylindrical or in spherical coordinates. Check your answer using Stokes' theorem.

3.03. In PH 101 you encountered the momentum operator in quantum mechanics. Recall that the momentum operator had the form $p = \frac{\hbar}{i}\frac{d}{dx}$ in one dimension. Now that we have discussed everything in general in three dimensions, $\vec{p} = \frac{\hbar}{i}\vec{\nabla}$. Hence the angular momentum operator $\vec{L} = \vec{r}\times\vec{p} = \frac{\hbar}{i}(\vec{r}\times\vec{\nabla})$. Show that the angular momentum operator in spherical polar coordinate is of the form

$$\vec{L} = \frac{\hbar}{i}\left(-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \hat{\phi}\frac{\partial}{\partial\theta}\right) .$$

3.04. Evaluate the following integrals:

$$(a)\ \int_{-1}^{+1} 9x^2\delta(3x+1)dx$$

$$(b)\ \int_V \left[r^4 + r^2(\vec{r}\cdot\vec{c}) + c^4\right]\delta^3(\vec{r}-\vec{c})dV,$$

where V is a sphere of radius 6 about the origin and $\vec{c} = 5\hat{x} + 3\hat{y} + 2\hat{z}$.

3.05. Prove the identities:
(a)

$$\delta\Big(y(x)\Big) = \sum_i \frac{\delta(x-x_i)}{|\frac{dy}{dx}|_{x=x_i}} ,$$

where x_i's are roots of the equation $y(x) = 0$.
(b)

$$\frac{d\theta}{dx} = \delta(x) ,$$

where $\theta(x)$ be the step function, defined as

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{for } x \leq 0 . \end{cases}$$

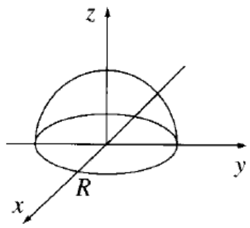

Figure 1: Problem 3.01

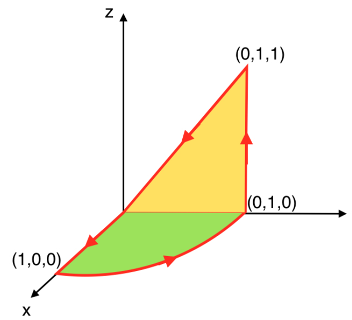

Figure 2: Problem 3.02

Take home problems

H3.01. Evaluate the following integrals:

$$(a)\ \int_{-2}^{+2}(2x+3)\delta(3x)dx$$

$$(b)\ \int_V |\vec{r}-\vec{b}|\delta^3(5\vec{r})dV,$$ where V is the volume of a cube of side 2, centered on the origin.

H3.02. Prove the following identities:

$$(a)\ x\frac{d}{dx}\delta(x) = -\delta(x)\ .$$

$$(b)\ \delta(kx) = \frac{\delta(x)}{|k|}\ ,$$

where k is a constant.