MA 102 (Mathematics II)
IIT Guwahati

Tutorial Sheet No. 3 | Linear Algebra | January 31, 2019

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1. True or False? Give justifications.
   (a) Let A be an $m \times n$ matrix. Then there exist $\mathbf{b}$ and $\mathbf{b}'$ such that $A\mathbf{x} = \mathbf{b}$ has a unique solution but $A\mathbf{x} = \mathbf{b}'$ has infinitely many solutions.
   (b) Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$ such that $\mathbf{x}^T\mathbf{y} = 0$. Then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent (LI).
   (c) Let S_1, S_2 and S_3 be distinct subsets of $\mathbb{R}^n$ such that $\text{span}(S_1 \cup S_2) = \text{span}(S_1 \cup S_3)$. Then $\text{span}(S_2) = \text{span}(S_3)$.
   (d) The column spaces of A and $\text{rref}(A)$ are equal.

Solution: a) False. Let $S_H = \{x | A\mathbf{x} = \mathbf{0}\}$. If $A\mathbf{x} = \mathbf{b}$ is consistent then $S = \{x | A\mathbf{x} = \mathbf{b}\} = S_H + y$ for some $y \in S$.
b) True. Since $\mathbf{x}, \mathbf{y}$ are nonzero vectors and not LI implies $\mathbf{x} = \alpha\mathbf{y}$ and $\mathbf{y} = \beta\mathbf{x}$ for nonzero scalars α and β.
c) False. For example consider $n = 2$, $S_1 = \{[1,0]^T\}$, $S_2 = \{[1,1]^T\}$ and $S_3 = \{[1,2]^T\}$.
d) False. Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

2. Check whether the set $S = \left\{ \begin{bmatrix} 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is LI.

Solution: Yes it is LI.

3. Let S be a subspace of $\mathbb{R}^4$ and $\mathbf{x}, \mathbf{y} \in S$ be LI.
   (a) Show that if $\mathbf{u} \in \mathbb{R}^4 \setminus S$ then $\{\mathbf{x}, \mathbf{y}, \mathbf{u}\}$ is LI.
   (b) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4 \setminus S$ are LI then does it imply that $\{\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}\}$ is LI?

Solution: a) If there exist c_1, c_2, c_3 not all zeros, such that $c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{u} = \mathbf{0}$, then $c_3 \neq 0$, otherwise it will contradict that $\mathbf{x}, \mathbf{y} \in S$ are LI. But then $\mathbf{u} = (-\frac{c_1}{c_3})\mathbf{x} + (-\frac{c_2}{c_3})\mathbf{y}$ which will contradict $\mathbf{u} \in \mathbb{R}^4 \setminus S$.
b) No. For example take $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $S = \text{span}\{\mathbf{x}, \mathbf{y}\}$.

4. Let $A \in \mathcal{M}_n(\mathbb{R})$. Show that $\text{row}(A^T A) = \text{row}(A)$, that is, $A^T A$ and A are row equivalent.

Solution: $A\mathbf{x} = \mathbf{0} \Rightarrow A^T A\mathbf{x} = \mathbf{0}$. Also $A^T A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x}^T A^T A\mathbf{x} = \mathbf{y}^T\mathbf{y} = 0$ which implies $\mathbf{y} = A\mathbf{x} = \mathbf{0}$.

Remark: The above result is also true for any $m \times n$ matrix A.
If $m < n$ then add $(n - m)$ zero rows to A to obtain $B = \begin{bmatrix} A \\ \mathbf{0} \end{bmatrix}$, then $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are equivalent and by the above argument $B\mathbf{x} = \mathbf{0}$ and $A^T A\mathbf{x} = \mathbf{0}$ are equivalent. Hence B and $A^T A$ are row equivalent, which will imply, $\text{row}(A) = \text{row}(A^T A)$.
If $n < m$ then add $(m - n)$ zero rows to $A^T A$ to obtain $C = \begin{bmatrix} A^T A \\ \mathbf{0} \end{bmatrix}$.
Then $C\mathbf{x} = \mathbf{0}$ and $A^T A\mathbf{x} = \mathbf{0}$ are equivalent, and $C\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$ are equivalent. Hence A and C are row equivalent and $\text{row}(A) = \text{row}(A^T A)$.

5. Show that the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ are LI if and only if $P\mathbf{x}, P\mathbf{y}, P\mathbf{z}$ are LI for any $n \times n$ invertible matrix P.

Solution: For any c_1, c_2, c_3, $c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z} = \mathbf{0}$ obviously implies $P(c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z}) = \mathbf{0}$. Conversely $P(c_1\mathbf{x}+c_2\mathbf{y}+c_3\mathbf{z}) = \mathbf{0}$ implies $c_1\mathbf{x}+c_2\mathbf{y}+c_3\mathbf{z} = \mathbf{0}$, (since for an invertible P, $P\mathbf{x} = \mathbf{0}$ has the trivial solution). Since $P(c_1\mathbf{x} + c_2\mathbf{y} + c_3\mathbf{z}) = c_1P\mathbf{x} + c_2P\mathbf{y} + c_3P\mathbf{z}$, the result follows.

6. Let $A \in \mathcal{M}_5(\mathbb{R})$ be such that $\text{rref}(A)$ has the 1^{st}, 3^{rd} and the 5^{th} column as the only pivot columns.
   (a) Find two LI solutions of $A\mathbf{x} = \mathbf{0}$.
   (b) Show that the columns $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_5$ (the 1^{st}, 3^{rd} and the 5^{th} column of A) are LI and spans the column space of A.
   (c) Can the sets $\{\mathbf{a}_1, \mathbf{a}_2\}$, $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ and $\{\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ be LI?

Solution: a) $\mathbf{u} = \begin{bmatrix} -\tilde{a}_{12} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -\tilde{a}_{14} \\ 0 \\ -\tilde{a}_{24} \\ 1 \\ 0 \end{bmatrix}$, where $\tilde{a}_{ij}$ is the (i, j)th entry of $\tilde{A}$.
b) That the columns $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_5$ are LI follows from Question 5.
By inspection one can check that $\tilde{\mathbf{a}}_2 = \tilde{a}_{12}\tilde{\mathbf{a}}_1$ and $\tilde{\mathbf{a}}_4 = \tilde{a}_{14}\tilde{\mathbf{a}}_1 + \tilde{a}_{24}\tilde{\mathbf{a}}_3$. Hence again by the result of Question 5, $\mathbf{a}_2 = \tilde{a}_{12}\mathbf{a}_1$ and $\mathbf{a}_4 = \tilde{a}_{14}\mathbf{a}_1 + \tilde{a}_{24}\mathbf{a}_3$ and the result follows.
c) The sets $\{\mathbf{a}_1, \mathbf{a}_2\}$, $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ will not be LI follows from part b).
$\{\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ will be LI if $\tilde{a}_{14} \neq \mathbf{0}$.

7. True or False? Give justifications.
   (a) If $\{\mathbf{x}, \mathbf{y}\}$ and $\{\mathbf{u}, \mathbf{v}\}$ are two different LI subsets of $\mathbb{R}^2$, then $\{\mathbf{x}, \mathbf{u}\}$ and $\{\mathbf{y}, \mathbf{v}\}$ are also LI sets.

(b) If $\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right\}$ is LI in $\mathbb{R}^3$ then $\left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right\}$ is LI in $\mathbb{R}^2$.

(c) If S is a subspace of $\mathbb{R}^n$ then $\mathbf{x} + S$ is a subspace if and only if $\mathbf{x} \in S$.

(d) If the diagonal entries of a 4×4 upper triangular matrix A are $1, 2, 3$ and 4 then $S_1 = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = 2\mathbf{x}\}$ is a subspace of $\mathbb{R}^4$ but $S_2 = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} = 5\mathbf{x}\}$ is not.

8. Let $S = \left\{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} a \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}\right\}$. Find the values of a for which $\text{span}(S) \neq \mathbb{R}^3$.

9. If a diagonal entry of a 3×3 upper triangular matrix is zero, then show that the columns are linearly dependent.

10. True or False? Give justifications.

(a) If S is a subspace of $\mathbb{R}^n$ of dimension n, then $S = \mathbb{R}^n$.

(b) For any two matrices A and B for which AB is defined, $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.

(c) If $C = [A \mid B]$, then $\text{rank}(C) \leq \text{rank}(A) + \text{rank}(B)$.

(d) If $C = \begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix}$, then $\text{rank}(C) \geq \text{rank}(A) + \text{rank}(D)$.

Solution:

(a) True. If $y \in \mathbb{R}^n$ but not in S, then for any basis $\mathbb{B}$ of S, $\mathbb{B} \cup \{y\}$ is LI.

(b) True. $\text{row}(AB) \subseteq \text{row}(B)$ and $\text{col}(AB) \subseteq \text{col}(A)$.

(c) True. Let $\text{rank}(A) = k$, $\text{rank}(B) = r$ and let the columns $a_{i_1}, \ldots, a_{i_k}$ of A form a basis of $\text{col}(A)$ and the columns $b_{j_1}, \ldots, b_{j_r}$ of B form a basis of $\text{col}(B)$. Then $a_{i_1}, \ldots, a_{i_k}, b_{j_1}, \ldots, b_{j_r}$ spans $\text{col}[A|B]$. Hence $\text{rank}[A|B] \leq r + k$.

(d) True. If $\text{rank}(A) = k$ and if the columns $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of A forms a basis of $\text{col}(A)$ then the corresponding columns $(i_1, i_2, \ldots, i_k)$ in C are LI.
If $\text{rank}(D) = r$ and the columns $d_{j_1}, d_{j_2}, \ldots, d_{j_r}$ of D forms a basis of $\text{col}(D)$ then the corresponding columns $(m + j_1, m + j_2, \ldots, m + j_r)$ (if A has m columns) in C are LI.
It can be easily checked that the columns $i_1, i_2, \ldots, i_k, m + j_1, m + j_2, \ldots, m + j_r$ of C are LI. Hence $\text{rank}(C) \geq r + k$.

11. If $\text{rank}(A) = \text{rank}(A^2)$ then show that $\text{rank}(A^2) = \text{rank}(A^3)$. Is $\text{rank}(A^5) = \text{rank}(A^6)$?

Hint: Note that $\text{col}(A^2) \subseteq \text{col}(A)$, $\text{rank}(A^2) = \text{rank}(A)$ implies $\text{col}(A^2) = \text{col}(A)$. Again note that $\text{col}(A^3) \subseteq \text{col}(A^2)$, show $\text{col}(A^3) = \text{col}(A^2)$, and so on.

Solution: Note that $\text{col}(A^2) \subseteq \text{col}(A)$. $\text{rank}(A^2) = \text{rank}(A)$ implies $\text{col}(A^2) = \text{col}(A)$.
Again note that $\text{col}(A^3) \subseteq \text{col}(A^2)$. If $y \in \text{col}(A^2)$, then $y = A^2 z = A(Az)$ for some $z \in \mathbb{R}^n$. Since $\text{col}(A^2) = \text{col}(A)$, $Az = A^2 u$ for some $u \in \mathbb{R}^n$. Hence $y = A(A^2 u)$ for some $u \in \mathbb{R}^n$, that is $y \in \text{col}(A^3)$. Hence $\text{col}(A^3) = \text{col}(A^2)$.

By similar argument one can show that $\text{rank}(A^k) = \text{rank}(A^{k+1})$ for all $k \in \mathbb{N}$.
(One could have argued similarly by considering the row space.)

12. (a) Show that for any two $m \times n$ matrices A and B, $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.
Hint: $A + B = [A|B] \begin{bmatrix} I_n \\ I_n \end{bmatrix}$.

Solution: Since $A + B = [A|B] \begin{bmatrix} I_n \\ I_n \end{bmatrix}$, $\text{rank}(A+B) \leq rank[A|B] \leq \text{rank}(A) + \text{rank}(B)$.

(b) Hence show that if A is an $m \times n$ matrix and B is the matrix obtained by changing exactly k entries of A, then $\text{rank}(A) - k \leq \text{rank}(B) \leq \text{rank}(A) + k$.

Hint: $B = A + C$, where C has exactly k nonzero entries.

Solution: Since C has at most k nonzero rows, $\text{rank}(C) \leq k$, Hence $\text{rank}(A+C) \leq \text{rank}(A) + k$.
To show the other inequality, note that $A = B + (-C)$.

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