Lecture 34 MARTINGALE 3

$\{X_n\}$ is a martingale iff

$E|X_n| < \infty$

and $E(X_{n+1} | X_0, X_1, \dots X_n) = X_n$

X_n is the optimal estimator for X_{n+1} given X_n

$\to$ past data.

- For a Martingale process $\{X_n\}$,

$$E\left(X_n\right) = \text{constant}$$

$$E\left(X_n X_{n+m}\right) = EX_n^2,\ m \geq 0$$

- A martingale $\{X_n\}$ is an orthogonal increment process, i.e. fo

$$n_1 < n_2 < n_3 < n_4$$

$$E(X_{n_2} - X_{n_1})(X_{n_4} - X_{n_3}) = 0$$

- For a martingale process $\{X_n\}$, EX_n^2 is a monotonically inc
sequence.

Martingale convergence theorem

Let $\{X_n, n \geq 0\}$ be a martingale and $EX_n^2 \leq M \leq \infty$ for all n. Then $\{X_n\}$ converges in the m.s. sense as $n \to \infty$ to a random variable X.

Proof:

$$E\left(X_{n+m} - X_n\right)^2$$

$$= EX_{n+m}^2 + EX_n^2 - 2EX_n X_{n+m}$$

$$= EX_{n+m}^2 + EX_n^2 - 2EX_n^2$$

$$= EX_{n+m}^2 - EX_n^2$$

EX_n^2 is a bounded monotonically increasing sequence and hence converegent

$$\therefore \lim_{n\to\infty} E\left(X_{n+m} - X_n\right)^2 = \lim_{n\to\infty}(EX_{n+m}^2 - EX_n^2)$$

$$= 0$$

$\Rightarrow \{X_n\}$ ~~is~~ converges in M.S

Thus, there exists an RV X such that

$$\{X_n\} \xrightarrow{m.s.} X$$

The theorem has a stronger version. Under the conditions of the martingale convergence theorem, it can be shown that

$$\{X_n\} \xrightarrow{a.s.} X$$

Continuous Time Martingale

The process $\{X(t), t \in T\}$ is called a martingale if

1. $|EX(t)| < \infty \quad \forall t$ and

$E|X(t)| < \infty$

2. $EX(t) \mid X(t_1), X(t_2), \ldots, X(t_n) = X(t_n)$

   for any $t_1 < t_2 < \ldots < t_n < t$.

The condition (2) is conveniently written as

$E(X(t) \mid X(u), u \le s) = X(s), s < t$.

Martingal property of the Wiener process

Let $X(t)$ be a standard Weiner process.

Examine if

(i) $X(t)$ and *(ii)* $Y(t) = X^2(t) - t$ and

(iii) $z(t) = e^{aX(t) - \frac{a^2 t}{2}}$, $a \in R$ are martingale.

$X(t) \sim N(0, t)$

Solution:

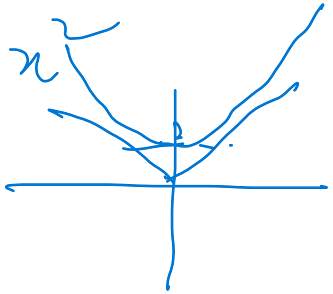
Standard Weiner process

$X(t) \sim N(0,t)$

$f_{X(t)}(x) = \dfrac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$

$$E|X(t)| = \int_{-\infty}^{\infty} |u| \frac{e^{-\frac{u^2}{2t}}}{\sqrt{2\pi}} t,\quad du < \infty$$

$\therefore E|X(t)| \le E(1+x^2) = 1+t < \infty \qquad |x| < 1+x^2$



$E(X(t)\,|\,X(u); u \le s)$

$= E(X(t) - X(s) + X(s)\,|\,X(u), u \le s)$

$= E(X(t) - X(s)\,|\,X(u), u \le s) + E(X(s)\,|\,X(u), u \le s)$

$= 0 + X(s)$ $\Rightarrow EX(t) - EX(s) + X(s)$

$= X(s)$

Therefore, $X(t)$ is a martingale.

$E X(t) \,|\, X(u), u \le s$

$Y(t) = X^2(t) - t$

$E|Y(t)| \quad E|X^2(t) - t| \le EX^2(t) = t < \infty$

$E\, Y(t) \,|\, X(u), u \le s$

$= E(X^2(t) - t) \,|\, X(u), u \le s)$

$= EX^2(t) - t$

$= E(X(t) - X(s) + X(s))^2 - t$

$= E(X(t) - X(s))^2 + X^2(s) + 2X(s)(X(t) - X(s)) - t$

$= t - s + X^2(s) - t$

(iii) $Z(t) = e^{aX(t) - \frac{a^2 t}{2}}$

$$E(Z(t) \mid X(u), u \le s)$$

$$= E(e^{aX(t) - \frac{a^2 t}{2}} \mid X(u), u \le s)$$

$$= e^{\frac{-a^2 t}{2}} E\, e^{a(X(t) - X(s))} E\, e^{aX(s)} \mid X(u), u \le s$$

MGF at $s=a$

$X(t) - X(s) \sim N(0, t-s)$

$$= e^{-\frac{a^2 t}{2}} e^{\frac{a^2 (t-s)}{2} + aX(s)}$$

$$= e^{aX(s) - \frac{a^2 s}{2}}$$

Therefore, $Z(t)$ is a Martingale wrt $X(t)$.

$= X^2(s) - s$

$= Y(s)$

$\therefore Y(t)$ is a martingale process.

$X(s) E(X(t) - X(s))$

$= 0$

$E|Z(t)| = E\, e^{aX(t) - \frac{a^2 t}{2}}$

$= e^{-\frac{a^2 t}{2}} E\, e^{aX(t)}$

$E\, e^{sX(t)} \Big|_{s=a}$ MGF

$E|Z(t)| < \infty$

Poisson process $\{N(t)\}$ is not a martingale