Department of Electronics & Electrical Engineering

Lecture 5

The Forced Solution

By

Dr. Praveen Kumar
Professor
Department of Electronics & Electrical Engineering

The response to sinusoidal source

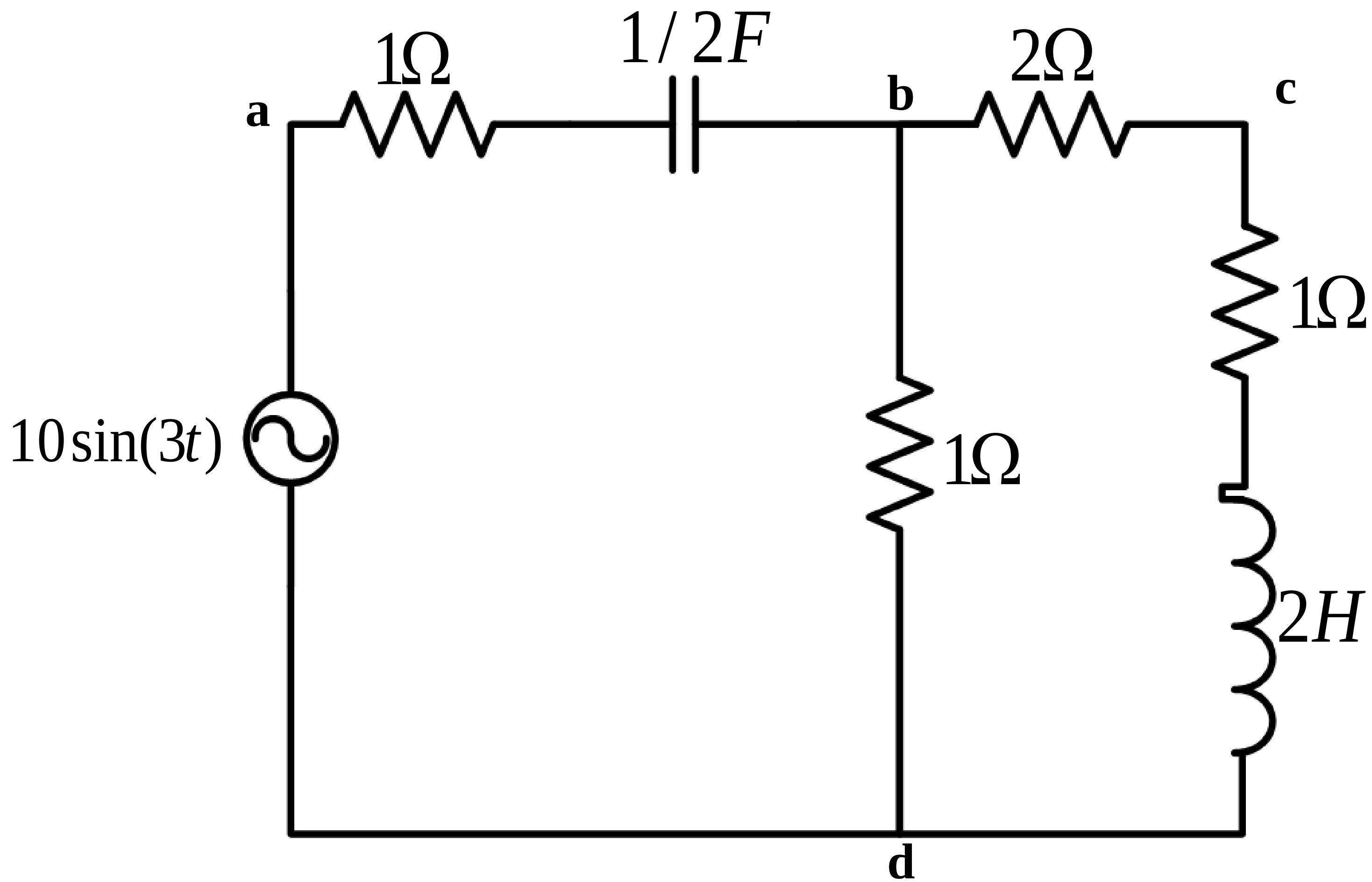

Figure 1: Network with sinusoidal excitation

The response to sinusoidal source

- In Fig.1, the excitation is

$$e_1(t) = 10\sin(3t) \tag{1}$$

- The equilibrium equation is found from the relationship

$$i_f(t) = \frac{1}{Z_{ad}} e_1(t) = \left(\frac{2p^2 + 4p}{4p^2 + 11p + 8} \right) 10\sin(3t) \tag{2}$$

- Rearranging the terms of eq.2 gives

$$(4p^2 + 11p + 8) i_f(t) = (2p^2 + 4p) 10\sin(3t) \tag{3}$$

- The right hand side of eq.3 is simplified as

$$(2p^2 + 4P) 10\sin(3t) = -180\sin(3t) + 120\cos(3t) \tag{4}$$

- The eq.4 clearly indicates that the assumed form of the forced solution must contain both a ***sine*** and a ***cosine*** component in order to have left hand side equal to the right hand side.

The response to sinusoidal source

- The assumed solution is

$$i_f(t) = A\sin(3t) + B\cos(3t) \tag{5}$$

$$pi_f(t) = \frac{di_f(t)}{dt} = 3A\cos(3t) - 3B\sin(3t) \tag{6}$$

$$p^2 i_f(t) = -9A\sin(3t) - 9B\cos(3t) \tag{7}$$

- Using eq.5 to 7 to substitutes the values of $\mathbf{i_f(t)}$, $\mathbf{pi_f(t)}$ and $\mathbf{p^2i_f(t)}$ into eq.4 gives

$$(-28A - 33B)\sin(3t) + (33A - 28B)\cos(3t) = -180\sin(3t) + 120\cos(3t) \tag{8}$$

- Equating the coefficients of **sin(3t)** and **cos(3t)** in eq.22 gives

$$28A + 33B = 180 \tag{9}$$

$$33A - 28B = 120 \tag{10}$$

$$\Rightarrow A = 4.8 \text{ and } B = 1.38$$

- The complete forced solution is

$$i_f(t) = 4.8\sin(3t) + 1.38\cos(3t) \tag{11}$$

The response to Polynomial Sources

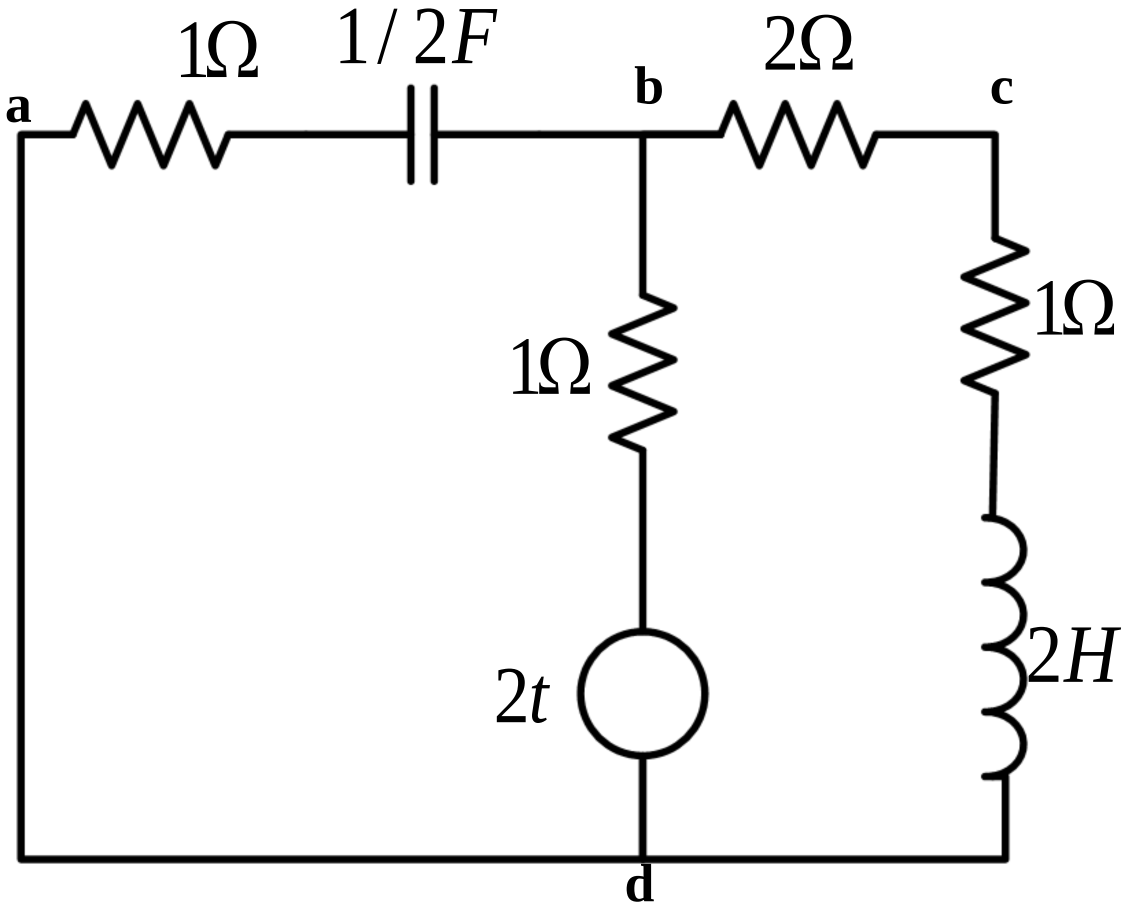

Fig.2: Network with sinusoidal excitation

The response to Polynomial Sources

- The excitation function in the Fig.2 is

$$f(t) = 2t \tag{12}$$

- The function in eq.12 varies linearly with time and has a slope of 2. Because of its sloping graphical representation, eq.12 is also referred to as ***ramp function***.

- The current $\mathbf{i_f(t)}$ is given by

$$i_f(t) = \frac{1}{Z}e(t) = \frac{2p^2 + 4p + 2}{4p^2 + 11p + 8}(2t) \tag{13}$$

- Rearranging the eq.13 gives

$$(4p^2 + 11p + 8)i_f(t) = (2p^2 + 4p + 2)(2t) \tag{14}$$

- Simplifying eq.14 gives

$$(4p^2 + 11p + 8)i_f(t) = 8 + 4t \tag{15}$$

The response to Polynomial Sources

- The forced response can be written as the sum of a constant term plus a ramp. Hence

$$i_f(t) = A + Bt \tag{16}$$

- Substituting $\mathbf{i_f(t)}$ from eq.16 into eq.15 gives

$$11B + 8A + 8Bt = 8 + 4t \tag{17}$$

- Equating the like coefficients and solving for the unknown quantities gives

$$A = \frac{5}{16}, B = \frac{1}{2} \tag{18}$$

- The complete expression for the forced solution becomes

$$i_f(t) = \frac{5}{16} + \frac{1}{2}t \tag{19}$$

The Natural Response (Transient Solution)

- The forced solution is the solution that is found to exist when the circuit has settled down in its response to the disturbing effect of an applied source function. This forced solution is also known as ***steady state solution***.

- The forced or the steady state solution always satisfies the defining differential equation but in general it ***is not a valid a solution over the entire time domain***.

- To illustrate the point consider the circuit shown in Fig.3. In this network it is desired to find the complete solution for the voltage ***v*** appearing across the resistor for all time ***t*** after the switch is closed.

- The equation that relates ***v*** to the source voltage ***E*** is

$$\frac{v}{E}=\frac{R}{R+pL}=\frac{1}{1+p(\frac{L}{R})} \qquad (20)$$

$$\Rightarrow \frac{L}{R}\frac{dv}{dt}+v=E \qquad (21)$$

The Natural Response (Transient Solution)

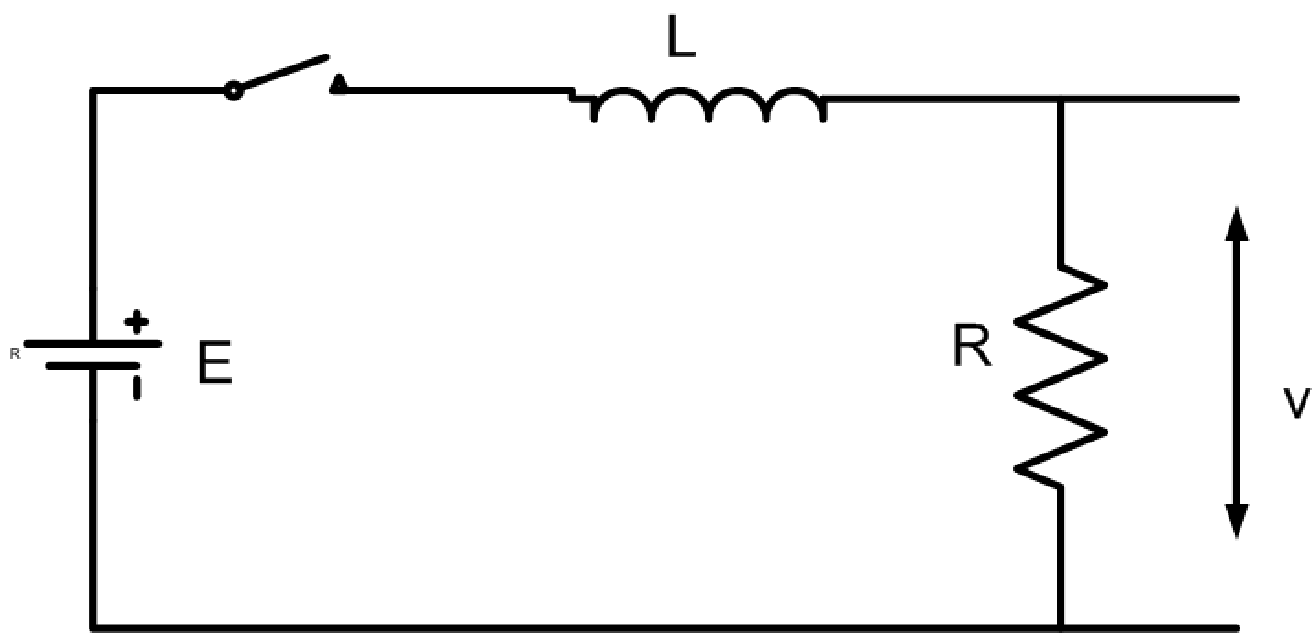

Fig.3: An R-L circuit

The Natural Response (Transient Solution)

- The forced solution of eq.21 is

$$v_f = E \qquad (22)$$

- The solution in eq.22 satisfies eq.21, however this solution does not qualify as a solution of the governing differential equation in the period immediately following the closing of the switch at time $t = 0^+$

- At time $t = 0^+$ the current is zero because of the presence of the inductor. Hence **v**, which is ***iR***, is also zero. Hence, the solution in eq.22 cannot be taken to be a complete description.

- In essence, then, there arises, in this time period, $t = 0^+$, immediately following the application of the source function, a need to add a ***complementary function*** to the forced solution.

- This complementary function will disappear as steady state is reached. It is the purpose of the ***complementary function*** to provide a smooth transition from the initial state of the response in the presence of ***energy storing*** elements to the final state.

The Natural Response (Transient Solution)

- In Fig.3, the complementary function (natural response) is obtained by solving the equation

$$\frac{L}{R}\frac{dv}{dt}+v=0 \tag{23}$$

- The solution to the eq.23 should satisfy that function, ***v*** and its derivative, ***dv/dt***, must be of the same form to make the left hand side equal to the right hand side.

- The plausible solution for eq.23 is

$$v=Ke^{st} \tag{24}$$

- Substituting ***v*** from eq.24 into eq.23 gives

$$\left(\frac{L}{R}s+1\right)Ke^{st}=0 \tag{25}$$

$$\Rightarrow s=-\frac{R}{L} \tag{26}$$

The Natural Response (Transient Solution)

- Hence, the complementary solution becomes

$$v = Ke^{-(R/L)t} \tag{27}$$

- The complete solution of the original governing differential equation (eq.23) is

$$v = E + Ke^{-(R/L)t} \tag{28}$$

- The quantity ***K*** is found from the initial condition which requires that ***v*** to be zero at $t=0^+$

$$0 = E + K \Rightarrow K = -E \tag{29}$$

- Hence, the complete solution is

$$v = E(1 - e^{-(R/L)t}) \tag{30}$$

- A graph of eq.30 is shown in Fig.4. The rapidity with which this transition takes place is entirely dependent upon the value ***s***. In eq.26 when ***R*** is large (or ***L*** is small) the transition occurs quickly because the transient term dies out in little time.

The Natural Response (Transient Solution)

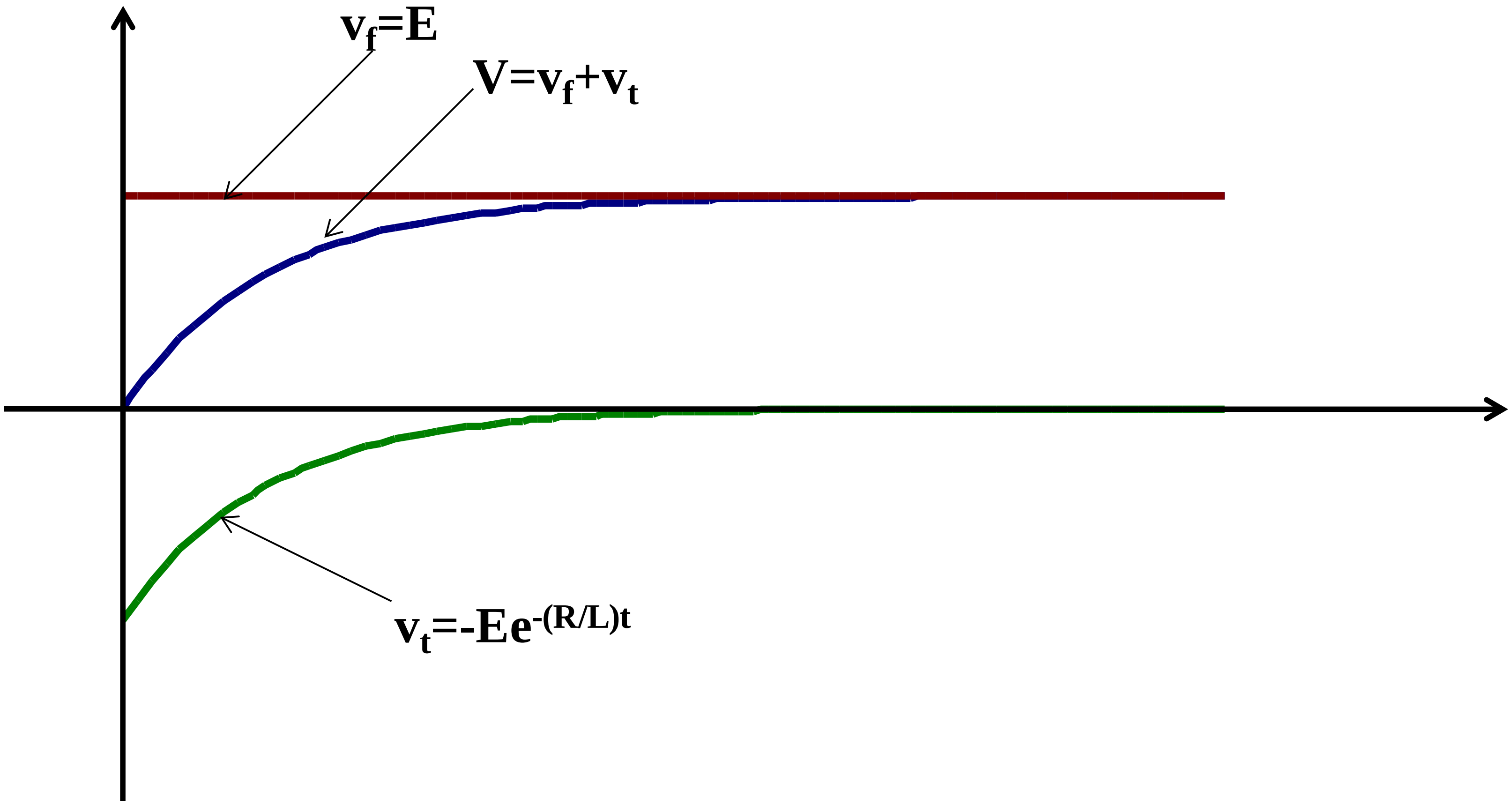

Fig.4: Complete solution of voltage across the resistor in network 3

Example 1

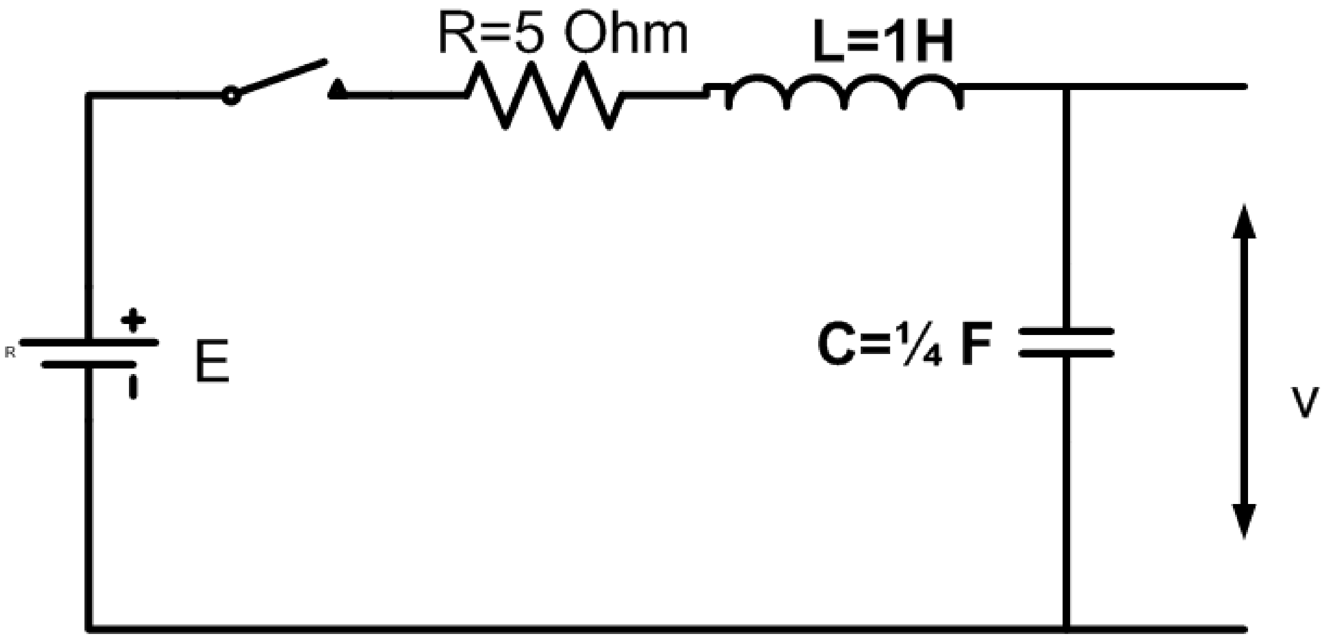

Fig.5: Network for the example

Example 1

- Consider the network shown in Fig.5. The differential equation governing the voltage **V** to the source **E**, is

$$v=\frac{\frac{1}{pC}}{R+pL+\frac{1}{pC}}=\frac{1}{p^2LC+pRC+1}E=\frac{4}{p^2+5p+4}E \qquad (31)$$

- The associated differential equation is

$$\frac{d^2v}{dt^2}+5\frac{dv}{dt}+4v=4E \qquad (323)$$

- The forced solution is

$$v_f=\left.\frac{4}{p^2+5p+4}\right|_{p=0}E=E\Rightarrow v_f=E \qquad (33)$$

- The characteristics equation is

$$D(p)=D(s)=s^2+5s+4=(s+4)(s+1)=0 \qquad (34)$$

Example 1

- Hence

$$s_1 = -4, s_2 = -1$$

Hence, the complementary solution is

$$v_t = K_1 e^{-4t} + K_2 e^{-t}$$

Hence, the complete solution is

$$v = v_f + v_t = E + K_1 e^{-4t} + K_2 e^{-t} \tag{35}$$

- Since there are two unknown coefficients, two initial conditions are required. The first is that **v** has a zero value at time $t=0^+$, then

$$v(0^+) = 0 = E + K_1 + K_2 \tag{36}$$

- To determine the second initial condition, differentiate the eq. 35

$$\frac{dv}{dt} = -4K_1 e^{-4t} - K_2 e^{-t} \tag{37}$$

Example 1

- The term dv/dt is the voltage drop across the capacitor. At time t=0+, the voltage across the capacitor is zero because the inductor does not allow the current to flow the network immediately. Hence,

$$\frac{dv}{dt} = -4K_1e^{-4t} - K_2e^{-t} = 0$$
$$-4K_1 - K_2 = 0 \tag{38}$$

- Solving eq.49 and eq.51 gives

$$K_1 = \frac{E}{3}, K_2 = -\frac{4}{3}E \tag{39}$$

- The complete solution is

$$v = E(1 + \frac{1}{3}e^{-4t} - \frac{4}{3}e^{-t}) \tag{40}$$

Example 1

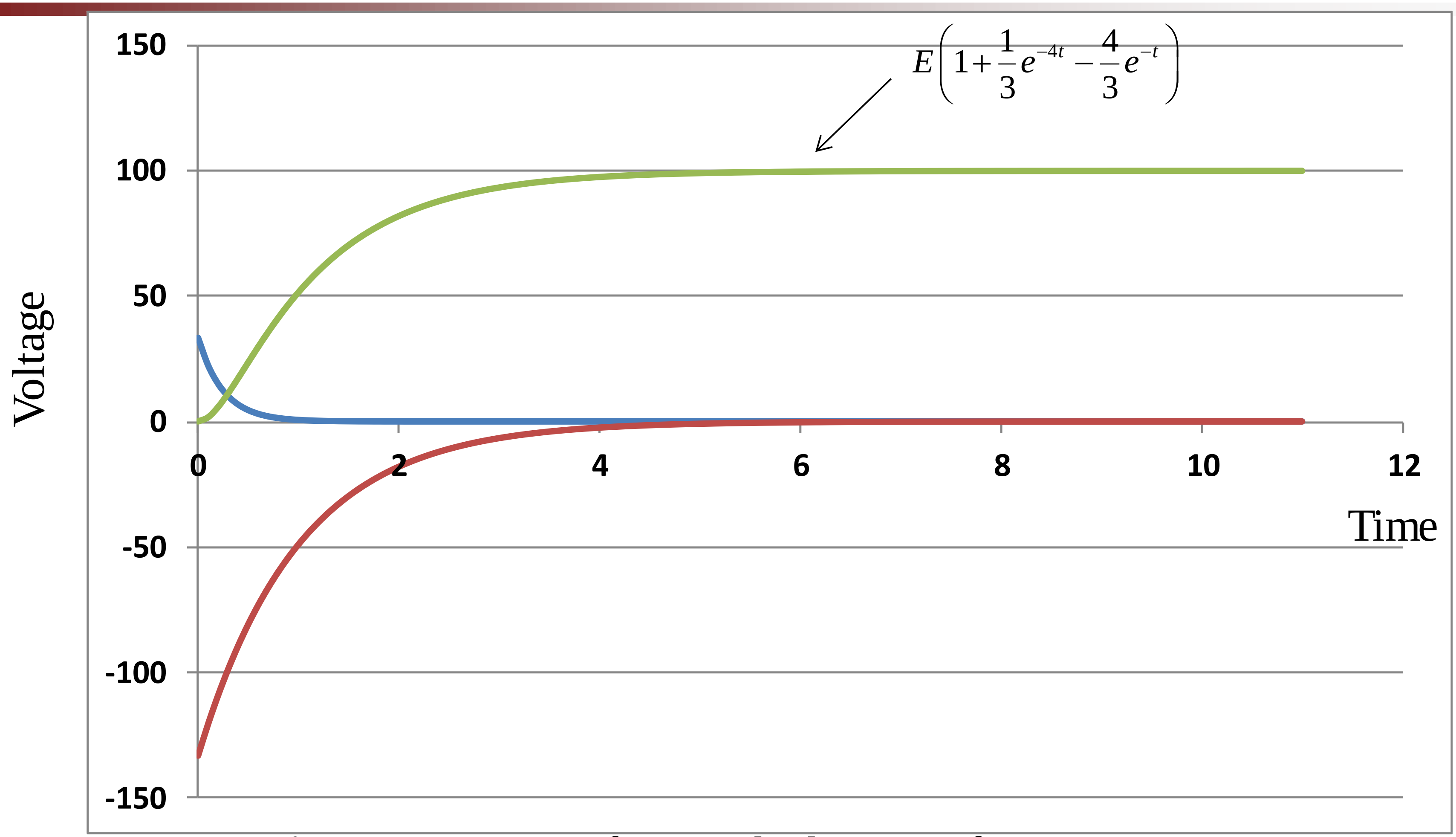

Fig.6: Response of network shown in fig.5

Example 2

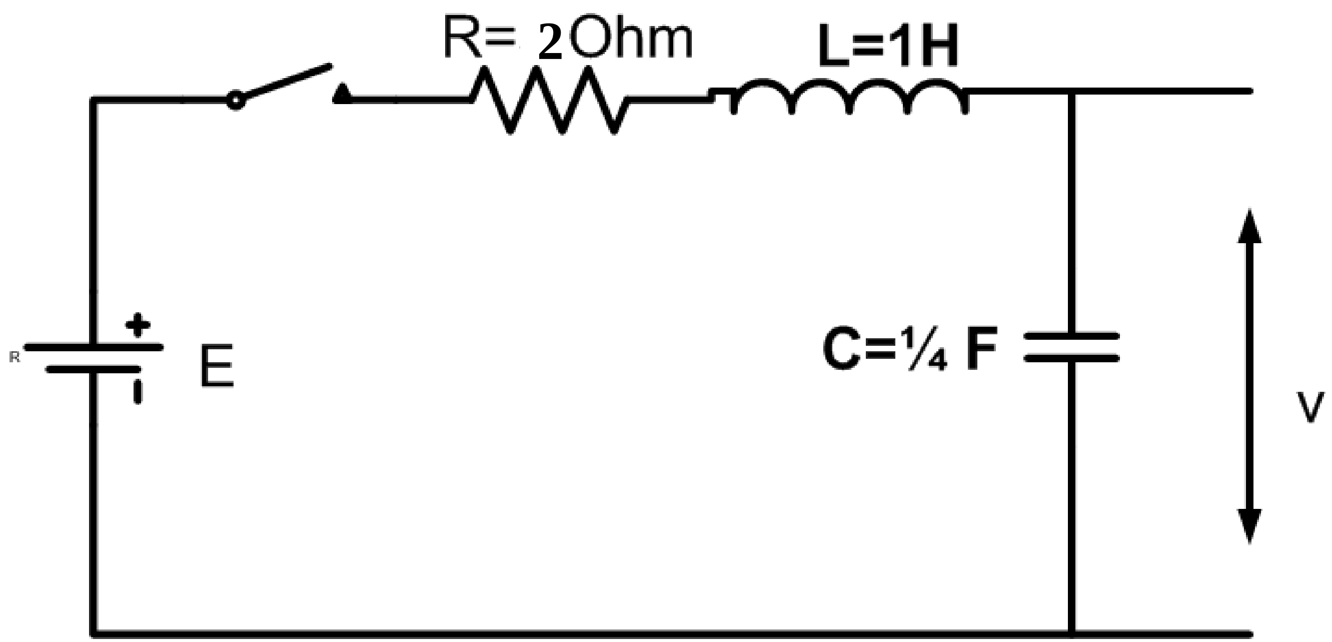

Fig.5: Network for the example

Example 2

- Consider the circuit shown in Fig. 7. The configuration of this circuit is same as in Fig.5 except for the fact that the resistance is reduced to 2 Ohms.

- The voltage **v** is given by

$$v = \frac{4}{p^2 + 2p + 4} E \tag{41}$$

- The forced component of the solution is obtained by

$$v_f = \left.\frac{4}{p^2 + 2p + 4}\right|_{p=0} E = E \Rightarrow v_f = E \tag{42}$$

- The characteristic equation and its roots are

$$D(p) = D(s) = s^2 + 2s + 4 \tag{43}$$

$$s_1 = -1 + j\sqrt{3}$$

$$s_2 = -1 - j\sqrt{3}$$

Example 2

- It is important to note that the roots are no longer real and distinct but are complex conjugates.

- This should lead us to expect some essential difference in the behaviour of the circuit as it reacts to the application of the source function.

- This difference is seen by writing the expression for the transient response:

$$v_f = K_1 e^{s_1 t} + K_2 e^{s_2 t} = 1/e^t \left(K_1 e^{i\sqrt{3}t} + K_2 e^{-j\sqrt{3}t} \right) \quad (44)$$

- Applying Euler's formula $e^{i\theta} = \cos\theta + j\sin\theta$, eq.44 can be written as

$$v_f = 1/e^t \left[\left(K_1 + K_2 \right) \cos\sqrt{3}t + j\left(K_1 - K_2 \right) \sin\sqrt{3}t \right] \quad (45)$$

- It should be kept in mind that the left hand side of **eq.45** is a real quantity. Moreover, vf is associated with a differential equation that has real coefficients.

Example 2

- Hence, the right hand side of eq.45 must also necessarily have real coefficients.
- This implies that the term **j(K1-K2**) merely means that **K1-K2** must assume such a value as to lead to a real coefficient.
- For ease of handling, the eq.45 can be rewritten as

$$v_f = 1/e^t\left(k_1 \cos\sqrt{3}t + k_2 \sin\sqrt{3}t\right) \qquad (46)$$

- The complete expression of the total solution is

$$v = E + v_f = E + 1/e^t\left(k_1 \cos\sqrt{3}t + k_2 \sin\sqrt{3}t\right) \qquad (47)$$

- Upon applying initial conditions to eq.47, the values of k1 and k2 can be obtained.

Example 2

- The first initial condition is

$$v(0^+) = 0 = E + k_1 \Rightarrow k_1 = -E \tag{48}$$

- To find k2, it is necessary to impose a second initial condition.

$$\frac{dv}{dt} = 0 = \sqrt{3}k_2 - k_1 \tag{49}$$

- Using eq.48, k2 is found as

$$k_2 = \frac{k_1}{\sqrt{3}} = -\frac{E}{\sqrt{3}} \tag{50}$$

- Substituting k1 and k2 into eq.47 gives

$$v = E\left[1 - \frac{1}{e^t}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right)\right] \tag{51}$$

- The response of the system is shown in Fig.7

Example 2

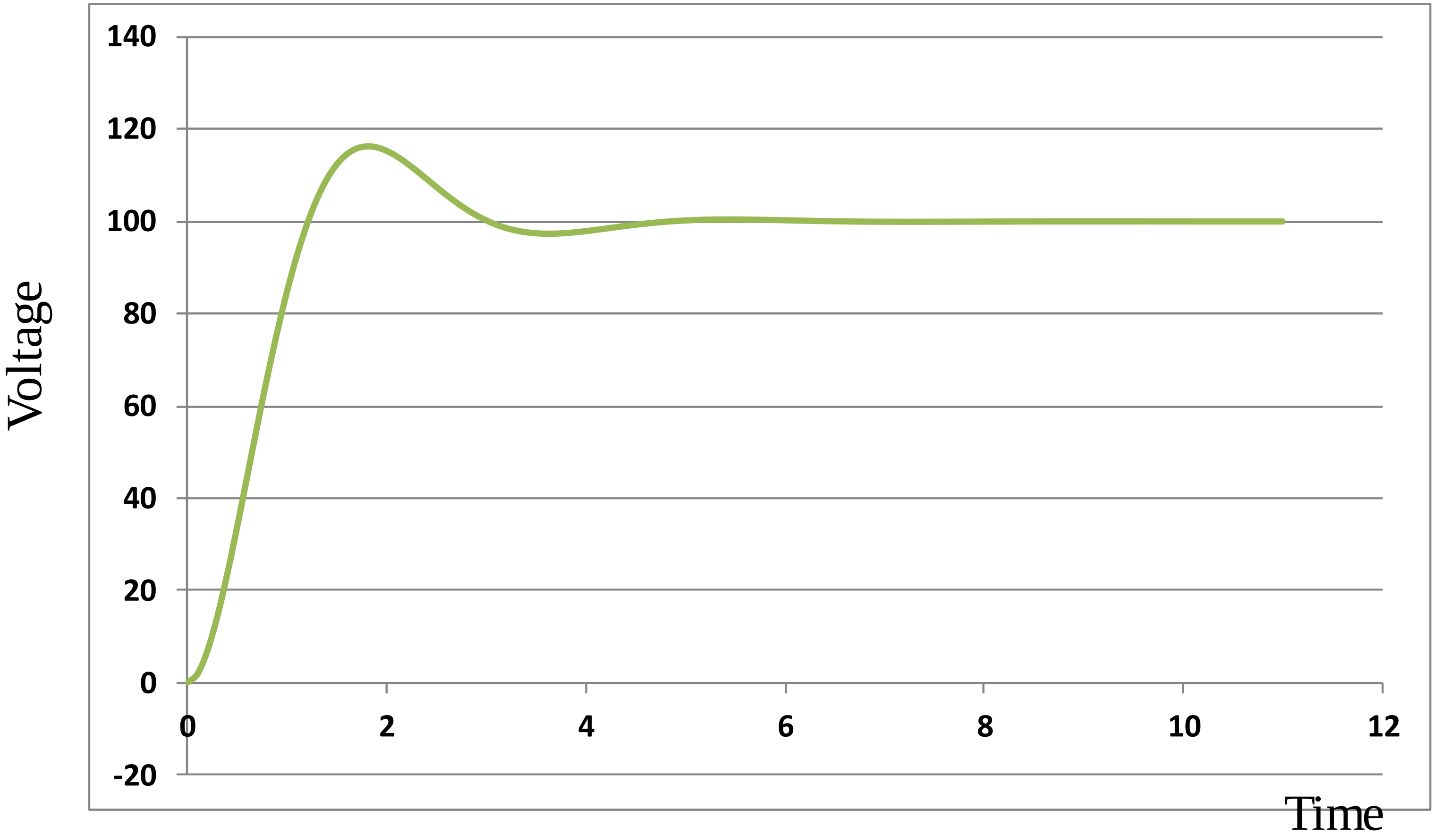

Fig.7: Response of network