→ Fibonacci Sequence :

* Cassini's Theorem :

For $n>0$, $F_{n+1} F_{n-1} - F_n^2 = (-1)^n$

Basis : $n=1$ $F_2 F_0 - F_1^2 = 1\times 0 - 1^2 = -1 = (-1)^1$ ✓

Step : $F_{n+2} F_n - F_{n+1}^2 = (F_{n+1} + F_n) F_n - (F_n + F_{n-1}) F_{n+1}$

$= F_n^2 - F_{n-1} F_{n+1} = -(-1)^n = (-1)^{n+1}$ Proved

* $a_0 = 1$

$a_r = 3a_{r-1} + 2$, $r \geq 1$

1, 5, 17, 53, 160

$a_r z^r = 3a_{r-1} z^r + 2 \cdot z^r$

$\sum_{r=1}^{\infty} a_r z^r = 3\sum_{r=1}^{\infty} a_{r-1} z^r + 2\sum_{r=1}^{\infty} z^n$

$A(z) - a_0 = 3z\ A(z) + \frac{2z}{1-z}$

$A(z) - 1 = 3z A(z) + \frac{2z}{1-z}$

~~$-2z = (3z-1) A(z)$~~

$A(z)(1-3z) = \frac{1+z}{1-z}$

$\Rightarrow A(z) = \frac{1+z}{(1-z)(1-3z)} = \frac{2}{1-3z} - \frac{1}{1-z}$

$\langle a_r = 2.3^r \rangle$ $\langle a_r = 1 \rangle$

$= \langle a_r = 2.3^r - 1 \rangle$

---

* Linear Recurrences with constt coeff (LRCC) :

$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \cdots\cdots + c_k a_{r-k} = f(r)$

$c_0, c_1, \cdots c_k \in \mathbb{R}$

If $c_0, c_k \neq 0$ then order is k.

Eg: i) $3a_r + 2a_{r-1} = r^2$ (1st order ~~power~~)

ii) $7a_r - a_{r-2} = 3$ (2nd order)

iii) $3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5$ (2nd order)

let $a_3 = 0$ & $a_4 = 1$

substituting for ~~b~~ a_5 :

$= 35$

$= 35/3$

substituting for a_5:

$-\frac{175}{3} + 2\cdot 1 = 41$

$= \frac{123 - 6 + 1765}{3}$

$a_6 = \frac{292}{9}$

going backwards:

$3 - 0 + 2a_2 = 21$

$a_2 = 9$

$0 - 45 + 2a_1 = 14$

$a_1 = 59/2$

---

k consecutive elements of LRCC of order k form a boundary condⁿ ⇒ Unique Solⁿ.

Let we have,

- $\langle a_r = P_r \rangle$ as a particular solⁿ
- we're considering the LRCC:

$c_0 a_r + c_1 a_{r-1} + \ldots + c_k a_{r-k} = f(r)$ — (1)

Consider a diff. LRCC:

$c_0 a_r + c_1 a_{r-1} + \ldots + c_k a_{r-k} = 0$ — (2)

(homogeneous LRCC (RHS = 0))

$\langle a_r = h_r \rangle$ is a solⁿ of (2).

$\langle a_r = P_r + h_r \rangle$ is also a solⁿ for (1).

* hLRCC of order k, (2)

$\langle a_r = A\alpha^r \rangle$ ← $A\alpha^r$ plug in

$A[c_0\alpha^r + c_1\alpha^{r-1} + \ldots c_k\alpha^{r-k}] = 0$ — (3) ← can safely cut

$\underbrace{c_0\alpha^k + c_1\alpha^{k-1} + \ldots + c_{k-1}\alpha + c_k = 0}_{\text{characteristic Eqⁿ of LRCC.}}$ — (4)

Say α_1 is a root of this eqⁿ

∴ solⁿ is of the form —

$A_1\alpha_1^r$ ← solⁿ of (2)

If α_1 is a root of the characteristic eqⁿ, then α_1 is a characteristic root of the LRCC.

$(A_1\alpha_1^r + A_2\alpha_2^r + \ldots A_k\alpha_k^r)$

All homogeneous solⁿ (if they're distinct) can be written like this.

Total solⁿ :—

$(A_1\alpha_1^r + A_2\alpha_2^r + \ldots A_k\alpha_k^r) + P_r$

α_1 is a root with multiplicity $m > 1$

α_1 is a root of (4)

$(\alpha - \alpha_1)$ is a factor of LHS (4)

$(\alpha - \alpha_1)$ " " " LHS (3)

$(\alpha - \alpha_1)\, g(\alpha)$

Derivative —

$(\alpha - \alpha_1) g'(\alpha) + g(\alpha) = \frac{d}{d\alpha}(\text{LHS } (3))$

$c_0 r\alpha_1^{r-1} + c_1(r-1)\alpha_1^{r-2} + \ldots + c_k(r-k)\alpha_1^{r-k-1} = 0$

Multiply by $A_2\alpha_1$:

$c_0 A_2 r\alpha_1^r + c_1 A_2(r-1)\alpha_1^{r-1} + \ldots + c_k A_2(r-k)\alpha_1^{r-k} = 0$

$\langle a_i = A_2 i\, \alpha_1^i \rangle$

* $\langle a_r = (A_1 + A_2 r + A_3 r^2 + \ldots + A_m r^{m-1})\alpha_1^r \rangle$ is a solⁿ of the hLRCC

$\alpha_1 \quad \alpha_2 \quad \alpha_3$ ———

$m_1 \quad m_2 \quad m_3$ ———

multiplicity

eg: $4\alpha^3 - 20\alpha^3 + 17\alpha - 4 = 0$

$\frac{1}{2}, \frac{1}{2}, 4$

$\langle a_r = (A_1 + A_2 r)(\frac{1}{2})^r + A_3(4)^r \rangle$

$a_r = 5a_{r-1} + 6a_{r-2} = 3r^2$

Assume a particular sol^n : $P_1 r^2 + P_2 r + P_3$

Plug it in : $P_1 r^2 + P_2 r + P_3 + 5(P_1(r-1)^2 + P_2(r-1) + P_3) + 6(P_1(r-2)^2 +$

$P_2(r-2) + P_3) = 3r^2$

Upon simplifying $\Rightarrow 12P_1 r^2 - (34P_1 - 12P_2)r + (29P_1 - 17P_2 + 12P_3)$

$= 3r^2$

$\Rightarrow 12P_1 = 3$ & $34P_1 - 12P_2 = 0$ & $29P_1 - 17P_2 + 12P_3 = 0$

$P_1 = \frac{1}{4}$ $P_2 = 17/24$ $P_3 = \frac{115}{288}$

$\therefore$ Particular sol^n = $\cancel{\frac{1}{4}r^2 +}$ $\langle P_n = \frac{1}{4}r^2 + \frac{17}{24}r + \frac{115}{288} \rangle$

* If $f(r)$ is a polynomial of degree t, then PS is also a polynomial of degree t.

eg: If $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$ ~~t deg~~

PS : $P_1 r^2 + P_2 r + P_3$ (deg = 2)

$12P_1 r^2 - (34P_1 - 12P_2)r + (29P_1 - 17P_2 + 12P_3) = 3r^2 - 2r + 1$

$P_1 = 1/4$ $34P_1 - 12P_2 = 2$ $P_3 = \frac{71}{288}$.

$\Rightarrow P_2 = 13/24$

eg: $a_r - 5a_{r-1} + 6a_{r-2} = 1$

PS = P (deg = 0)

$P - 5P + 6P = 2P = 1$

$\Rightarrow P = 1/2$ ~~const~~

So, PS is a constt seq. $(1/2, 1/2, 1/2 \dots)$

* If $f(r)$ is $c\beta^r$, where β is not a char. root, PS is of the form : $P\beta^r$

eg: $a_r + 5a_{r-1} + 6a_{r-2} = 42 \times 4^r$

PS : $P4^r$

HS :

$\alpha^2 + 5\alpha + 6 = 0$

$(\alpha+2)(\alpha+3) = 0$

$\alpha = -2$ or -3.

$P4^r + 5P4^{r-1} + 6P4^{r-2} = 42 \cdot 4^r$

$\Rightarrow 16P + 20P + 6P = 42 \times 16$

$\Rightarrow P = 16$

If $f(r)$ is of the form $(F_1 r^t + \ldots + F_{t+1})\beta^r$ [β is not char. root].

Then PS is also of the same form.

Eg: $a_r + a_{r-1} = 3r.2^r$

char. eq.^{n} = $\alpha + 1 = 0$

$[\alpha^r + \alpha^{r-1} = 0$

$\Rightarrow \alpha + 1 = 0]$ 2 is not a sol^n ✓

PS: $(P_1 r + P_2)\, 2^r$

$\Rightarrow (P_1 r + P_2)2^r + (P_1(r-1) + P_2)2^{r-1} = 3r.2^r$

$\Rightarrow (P_1 r + P_2)2 + P_1 r + P_2 - P_1 = \cancel{3}\, 6r$

$\Rightarrow 3P_1 r + 3P_2 - P_1 = 6r$

$3P_1 = 6$ $\quad P_2 = \cancel{1}\,2/3$

$P_1 = 2$

* If $f(r)$ is $(F_1 r^t + \ldots + F_{t+1})\beta^r$. β is a char. root of multiplicity (mux) m

then PS is of the form : $r^m(P_1 r^t + \ldots + P_{t+1})\beta^r$

Eg: $a_r - 2a_{r-1} = 3.2^r$
$[t=0 \quad \beta=2]$

Char. eq^n $\rightarrow$ $\alpha - 2 = 0$ $\quad \alpha = 2$ char root.

m = mux = 1

PS: $r.P_1 2^r$ $\quad \Rightarrow \quad Pr2^r - 2P(r-1)2^{r-1} = 3.2^r$

$\Rightarrow \quad 2Pr - 2Pr + 2P = 6$

$\Rightarrow \quad P = 3$

$\therefore$ $3r.2^r$ is a PS.

Eg: $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$

Char eq^n $-$ $\alpha^2 - 4\alpha + 4 = (\alpha-2)^2 = 0$

$\beta = 2$ $\quad$ mux = 2 $\quad$ $t = 1$ (poly. of deg 1)

PS: $r^2(P_1 r + P_2)\, 2^r$

Plug it in —DD—

$P_1 = 1/6$, $\quad P_2 = \cancel{1/2}\ 1$

$\langle P_r = r^2\left(\frac{r}{6} + 1\right) 2^r \rangle$

Eg: $a_r = a_{r-1} + 7$

$a_r - a_{r-1} = 7$

Char. eqn $\alpha - 1 = 0$

$\alpha = 1$

$\beta = 1 \quad m = 1$

— Do —

~~P = 7~~

$PS = r^1 (P) 1^r$

$P = 7$ (— Do —)

$\therefore PS = 7r$

Eg: $a_r - 2a_{r-1} + a_{r-2} = 7 \cdot 1^r$

Char. eqn : $\boxed{(\alpha^2 - 2\alpha + 1) = (\alpha - 1)^2 = 0}$

$\beta = 1 \quad m = 2$

PS : $r^2 (P) 1^r = Pr^2$

$Pr^2 - 2P(r-1)^2 + P(r-2)^2 = 7$

$P = 7/2 \quad \therefore PS = \frac{7}{2} r^2$

Eg: $a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$

$\langle p_r = 16 \times 4^r \rangle$

(Done in some pehle ka eg.)

$\alpha^2 + 5\alpha + 6 = 0$

$\alpha = -2$ or -3

$\boxed{\langle a_r = A_1(-2)^r + A_2(-3)^r + 16 \times 4^r \rangle}$ = Total soln.

Given : $a_2 = 278 \quad a_3 = 962$

substitute in total soln :

$278 = 4A_1 + 9A_2 + 256$ — (1)

$962 = -8A_1 - 27A_2 + 1024$ — (2)

— Solve —

$\Rightarrow$

$A_1 = 1$

$A_2 = 2$

$\therefore \langle a_r = (-2)^r + 2(-3)^r + 16 \times 4^r \rangle$

→ LRCC using generating f^n :

$c_0 a_r + c_1 a_{r-1} + \ldots + c_K a_{r-K} = f(r) \quad r \geq s$

$s \geq k$

Multiply both sides by z^r & sum.

$\underbrace{z^r (c_0 a_r + \ldots + c_K a_{r-K})}_{LHS} = f(r) z^r$

$\therefore \sum_{r=s}^{\infty} (LHS) z^r = \sum_{r=s}^{\infty} f(r) z^r$ — (0)

(1) — $\sum_{r=s}^{\infty} c_0 a_r z^r = c_0 [A(z) - (a_0 + a_1 z + \ldots + a_{s-1} z^{s-1})]$

(2) — $\sum_{r=s}^{\infty} c_1 a_{r-1} z^r = c_1 z [A(z) - (a_0 + a_1 z + \ldots + a_{s-2} z^{s-2})]$

⋮ so on

(k) — $\sum_{r=s}^{K} c_K a_{r-K} z^r = c_K z^K [A(z) - \overbrace{(a_0 + a_1 z + \ldots + a_{s-K-1} z^{s-K-1})}^{s-K \text{ terms (excluded)}}]$

Adding (1), (2), ... (k):

$= A(z)[c_0 + c_1 z + \ldots + c_K z^K] - \left\{ \begin{array}{l} c_0(a_0 + a_1 z + \ldots + a_{s-1} z^{s-1}) \\ + c_1 z (a_0 + \ldots + a_{s-2} z^{s-2}) \\ \vdots \\ + c_K z^K (a_0 + \ldots + a_{s-K-1} z^{s-K-1}) \end{array} \right.$

~~$= A(z)[c_0 + c_1 z + \ldots + c_K z^K]$~~

$= \sum_{r=s}^{\infty} f(r) z^r$ (From — (0))

$\therefore A(z) = \dfrac{\sum_{r=s}^{\infty} f(r) z^r + \{ \quad \}}{c_0 + c_1 z + \ldots + c_K z^K}$

Harmonic Numbers :

For $n > 0$: the n^{th} Harmonic no. H_n

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \qquad H_0 = 0$$

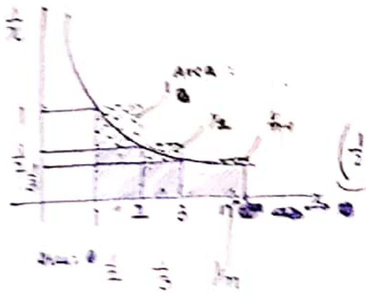
$0,\ 1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{3}, \ldots$

$H_2 = 1.5$
$H_3 = 1.8333$
$H_4 = 2.08333$
$H_5 = 2.28333$

$$H_n = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \left(\frac{1}{9} + \cdots \frac{1}{16}\right) + \cdots + \left(\cdots + \frac{1}{n}\right)$$

$\to \frac{1}{2} < (\ ) < 1 \quad \to \lceil \log n \rceil =$ Max. size

$\to H_n > \frac{\lceil \log n \rceil}{2}$ (crude approx.)



$$\underbrace{\left(\frac{1}{2} + \cdots \frac{1}{n}\right)}_{H_n - 1} < \underbrace{\int_1^n \frac{1}{x}\, dx}_{\ln n} < \underbrace{\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right)}_{H_n - 1/n}$$

$$\Rightarrow H_n - 1 < \ln n < H_n - \frac{1}{n} < H_n$$

* consider the sequence : $\langle a_r = 1 \rangle$ i.e. $1, 1, 1, \ldots$

$$A(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}$$

let $B(z) = \int A(z)\, dz = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$ (generating fn of) $\langle 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n} \rangle$

$$= \ln \frac{1}{1-z}$$

Harmonic seq = Prefix sum of.

Prefix sums $= \langle 0, 1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \ldots \rangle$

$= \langle a_r = H_r \rangle$

$$\boxed{H(z) = \frac{1}{1-z} \ln\left(\frac{1}{1-z}\right)}$$

$B z = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$; $z = 1/k$; $B\left(\frac{1}{k}\right) = \frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} + \cdots$

$= \ln\left(\frac{1}{1 - 1/k}\right) = \ln\left(k/k-1\right)$

* For $n \geq 0$, the n^{th} 2nd order Harmonic no. $H_n^{(2)}$

$$H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2} \quad \text{as } n \to \infty$$

(zeta of 2)

$$H_n^{(2)} \to \frac{\pi^2}{6} = \zeta(2)$$

For $n \geq 0$, the n^{th} 3rd order Harmonic no. $H_n^{(3)}$

$$H_n^{(3)} = \sum_{k=1}^{n} \frac{1}{k^3} \xrightarrow{n\to\infty} \zeta(3) \;(= 1.20205\ldots)$$

r^{th} order H. no. $H_n^{(r)} \to \zeta(r)$

$$\ln\left(\frac{k}{k-1}\right) = \frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} + \ldots$$

<u>LHS</u>:

$$\sum_{k=2}^{n} \ln\left(\frac{k}{k-1}\right) = \sum_{k=2}^{n} \ln k - \ln(k-1)$$

$$= \ln n - \cancel{\ln(n-1)} + \cancel{\ln(n-1)} - \cancel{\ln(n-2)} + \ldots + \cancel{\ln(3)} - \cancel{\ln 2} + \cancel{\ln 2} - \ln 1$$

(Telescopes)

$$= \ln n - \ln 1$$

$$= \ln n$$

<u>RHS</u>:

$$= \sum_{k=2}^{n} \frac{1}{k} + \frac{1}{2}\sum_{k=2}^{n} \frac{1}{k^2} + \frac{1}{3}\sum_{k=2}^{n} \frac{1}{k^3} + \ldots$$

$$= (H_n - 1) + \frac{1}{2}(H_n^{(2)} - 1) + \frac{1}{3}(H_n^{(3)} - 1) + \ldots = \ln n$$

LHS

$$\Rightarrow \cancel{\ln(n)}\; H_n - \ln n = 1 - \frac{1}{2}(H_n^{(2)} - 1) - \frac{1}{3}(H_n^{(3)} - 1) - \ldots$$

As $n \to \infty$

$$\text{RHS} = \underbrace{1 - \tfrac{1}{2}(\zeta(2) - 1) - \tfrac{1}{3}(\zeta(3) - 1) - \ldots}$$

~~LHS~~

$= \gamma$ (Euler's constt. Euler showed that converges to a constt γ)

$= 0.5772156\,55$ (It's still an open ques. whether γ is rational or not).

For large n,

$$H_n \sim (\ln n) + 0.58$$

<u>Textbook</u>: Concrete Mathematics by Graham, Knuth, Patashnik (for Harmonic Nos.)

→ Sterling Numbers : (2 kinds)

↳ of the second kind first :

A sterling no. $\left\{ {n \atop k} \right\}$ (n partition k) is the no. of ways to partition a set of size n into k non-empty subsets.

(k buckets (not numbered) in any order)

* For $n = 0$, $\left\{ {0 \atop 0} \right\} = 1$

* $\left\{ {n \atop n} \right\} = 1$ * $\left\{ {n \atop 0} \right\} = 0$ * $\left\{ {n \atop 1} \right\} = 1$ } Boundary cond^n s.

* $n > 0$ ~~times~~ items

Put x aside $1 < \underset{\text{items left}}{k} < n$

~~We can~~ Partition the remaining elements in $\left\{ {n-1 \atop k} \right\}$ ~~ways if one item is kept aied~~

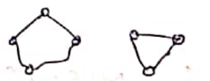
No. of ways $= k \times \left\{ {n-1 \atop k} \right\}$ ~~if k elements are kept aside.~~

$\boxed{x}$ $\left\{ {n-1 \atop k-1} \right\}$ ways. [the removed element 'x' is kept in it's own bucket]

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \cdot \left\{ {n-1 \atop k} \right\}$$

↳ of the first kind :

$\left[{n \atop k} \right]$ is the no. of ways to arrange n items into k non-empty cycles.

(n beads made into k garlands  ...)

(n cycle k)

* $\left[{n \atop n} \right] = 1$ * $\left[{n \atop 1} \right] = (n-1)!$ * $\left[{0 \atop 0} \right] = 1$ $\overset{n>0}{}$ * $\left[{n \atop 0} \right] = 1$

* $n > 0$ Pick one x to be kept aside.

Partition the rest ~~into~~ $(n-1)$ into k garlands.

The single bead can be inserted into any other cycle in no. of ways = no. of beads in that cycle (after every bead is a separate case in any fixed dir^n (say clockwise))

Scanned by CamScanner

So no. of ways = No. of beads in rest of the cycles
$= n-1 \begin{bmatrix} n-1 \\ k \end{bmatrix}$

If we don't insert the bead into any other cycle & keep it by itself, then no. of ways of permuting
$= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$

$$\therefore \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}$$

→ **Combinatorics** :

a of one kind
b of another kind
} a+b ways of chosing one of either kind.
ab " " " " both.

* ~~Permut~~ Permutation of a sequence of items is a re-ordering.
* Permutation of a set of items is an ordering.
* n distinct items & r permutations

items : $\underline{n}$ $\underline{n-1}$ $\underline{n-2}$ . . . $\underline{n-r+1}$
1 2 3 r

$P(n,r) = n(n-1)(n-2) \cdots (n-r+1)$
$= \dfrac{n!}{(n-r)!}$

Eg: i) MISSISSIPPI

M 4 I's 4 S's 2P's

No. of permutations $= \dfrac{11!}{2!\,4!\,4!}$

ii) a_i balls of colour i — All indistinguishable.

No. of balls $= a_1 + a_2 + \cdots + a_n$

No. of permutations $= \dfrac{(a_1+a_2+\cdots a_n)!}{a_1!\,a_2! \cdots a_n!}$

iii) r 0's and n-r 1's
No. of permutations $= \dfrac{n!}{\cdots} = \binom{n}{r}$

$$\binom{n}{r} = \frac{\ldots}{r!}$$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{1 \cdot 2 \cdot \ldots \cdot r}$$

Eg : (i) n course dinner. Each course has m choices.

No. of distinct dinners $= m^n$

(ii) n people, m-members' team

~~choose~~ ways of chosing $= \binom{n}{m}$

* If each member is fixed, ways $= P(n, m)$

(iii) Chose a team size r & captain

ways $= r \cdot \binom{n}{r}$

(iv) 17 girls, 10 boys. Choose a team of 8 girls & 2 boys.

ways $= \binom{17}{8} \cdot \binom{10}{2}$

(v) 17 people. Choose a team of size 3 or size 4

ways $= \binom{17}{3} + \binom{17}{4}$

* Symmetry identity : $\binom{n}{r} = \binom{n}{n-r}$

* Summation identity : $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$

with 'a'

without 'a'.

* Pascal's Triangle :

$$\binom{0}{0}$$

$$\binom{1}{0} \qquad \binom{1}{1}$$

$$\binom{2}{0} \qquad \binom{2}{1} \qquad \binom{2}{2}$$

$$\binom{3}{0} \qquad \binom{3}{1} \qquad \binom{3}{2} \qquad \binom{3}{3}$$

$\binom{n}{r} = \dfrac{P(n,r)}{r!}$

$\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!} = \dfrac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}$

eg: (i) n- course dinner. Each course has m choices.

No. of distinct dinners $= m^n$

(ii) n people. m- members' team
~~choos~~ ways of chosing $= \binom{n}{m}$

if each member is fixed, ways $= P(n, m)$

(iii) Chose a team, captain
size r &
ways $= r.\binom{n}{r}$

(iv) 17 girls, 10 boys. Choose a team of 8 girls & 2 boys.

ways $= \binom{17}{8} \cdot \binom{10}{2}$

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with 'a'
without 'a'.

* Pascal's Triangle :

$$\binom{0}{0}$$

$$\binom{1}{0} \qquad \binom{1}{1}$$

$$\binom{2}{0} \qquad \binom{2}{1} \qquad \binom{2}{2}$$

$$\binom{3}{0} \qquad \binom{3}{1} \qquad \binom{3}{2} \qquad \binom{3}{3}$$

* Pascal's Δ :

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

* $(x+y)^n = (x+y)(x+y)\ldots(x+y) = \sum_{r=0}^{n}\binom{n}{r} x^r y^{n-r}$

↳ Put $x=y=1$

$$2^n = \sum_{r=0}^{n}\binom{n}{r} = \binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{n}$$

~~$(1-1)^n$~~

↳ $x=1 \quad y=-1$

$$(1-1)^n = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}\ldots$$

$$0 = \text{''}$$

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots$$

* Set of size n

\# of subsets of size $r = \binom{n}{r}$

\# subsets $= \binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{r}+\ldots\binom{n}{n}$

$= 2^n$

Another way to see this — binary string. (each item can be either picked (1) or not (0))

* Thm :

$$1\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\ldots+n\binom{n}{n} = n.2^{n-1}$$

Algebraic proof : $\sum_{r=1}^{n} r.\binom{n}{r} = \sum_{r=1}^{n}\frac{r.n!}{r!(n-r)!} = \sum\frac{r\,n!}{r(r-1)!(n-r)!}$

$= n\sum\frac{(n-1)!}{(r-1)!(n-r)!} = n\sum\binom{n-1}{r-1} = n.2^{n-1}$

LHS = # of ways to chose a team of size ≥ 1 & pick a leader also.

RHS = # of ways to pick a person & pick a team of size ≥ 0 for him to lead.

Hockey stick identity :

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1\ \binom{2}{2} & & & & \\
 & & & 1 & & 3 & & 3\ \binom{3}{2} & & 1 & & & \\
 & & 1 & & 4 & & 6\ \binom{4}{2} & & 4 & & 1 & & \\
 & 1 & & 5 & & 10\ \binom{5}{2} & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20\ \binom{6}{3} & & 15 & & 6 & & 1
\end{array}
$$

$1+3+6+10 = 20$ ← Hockey stick identity.

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n-1}{r} + \cdots + \binom{r}{r}$$

Proof : Induction :

Basis : $\binom{r+1}{r+1} = \binom{r}{r} = 1$

Step : $\binom{n+1}{r+1} = \binom{n}{r} + \underbrace{\binom{n}{r+1}}_{\text{Hypothesis}}$

—— Do ——

* Eg a, b, c, d, e, f

$\binom{6}{3}$

starting with a $\binom{5}{2}$

b $\binom{4}{2}$

c $\binom{3}{2}$

d $\binom{2}{2}$

bcd	bde
bce	bdf
bcf	bef

* Eg # of diff. ordered triplets (a, b, c) of non-negative integers s.t $a+b+c = 50$

Soln 1 for non negative integer n

if $a+b = n$ # options (a, b) satisfying $= n+1$

$(0, n)\ (1, n-1) \ldots\ldots (n, 0)$

$a + b = 50 - c$

\# of options $= 51 - c$

$51, 50, 49, \ldots, 1$

$= \frac{51 \times 52}{2}$

Sol^n 2 :

$0 \quad 0 \quad 0 \mid \overbrace{0 \quad 0 \ldots}^{\text{ceptums } c(15)} \mid 0$

a   b   c

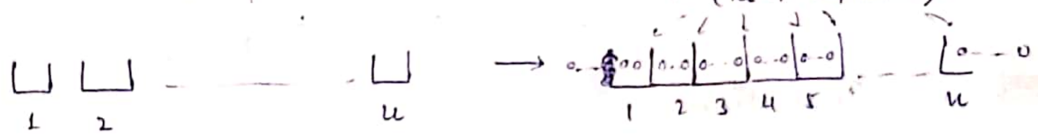
50 0's
2 1's

\# ways $= \frac{51 \times 52}{1 \times 2} = \frac{52!}{50!\,2!}$

Generalisable :

b indistinguishable balls
u distinguishable urns.

(u-1 ceptums)



Throw balls in (the 0s)

b 0's & $(u-1)$ 1's

\# ways $= \frac{(b+u-1)!}{b!\,(u-1)!}$

$= \binom{b+u-1}{b} = \binom{b+u-1}{u-1}$

* \# of binary strings of n with at least one 1.

Eg : (i) 10 children 31 flavours of ice cream.

\# of orders with at least 2 getting the same flavours?

\# of ways so everyone gets a distinct flavour $= P(31,10)$

$\therefore$ required no. of ways $= 31^{10} - P(31,10)$

(ii) The no. of elements without property A & without " B.

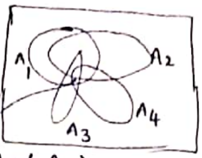
$= |\bar{A} \cap \bar{B}| =$

$= |\overline{A \cup B}| = N - |A \cup B| = N - (|A| + |B| - |A \cap B|)$

Inclusion Exclusion Principle :

$A_1, A_2 - - - - A_n$: Finite sets

$|A_1 \cup - - - \cup A_n|$



Consider $x \in A_1 \cup - - - \cup A_n$

x belongs to r of them

$x \in A_{i_1} \cap - - \cap A_{i_r}$

$S_1 = |A_1| + |A_2| + - - - + |A_n|$

x gets counted r times.

$S_2 = \sum_{i \neq j} |A_i \cap A_j|$ (x is counted $\binom{r}{2}$ times)

$S_3 = \sum_{\substack{i \neq j \\ j \neq k \\ k \neq i}} |A_i \cap A_j \cap A_k|$ (x is counted $\binom{r}{3}$ times)

$\vdots$

In S_r, x is counted $\binom{r}{r}$ times.

In S_{r+1}, x is counted 0 times. ($\because x$ doesn't belong here)

$S_1 - S_2 + S_3 - S_4 + ...$

x is counted $= \binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \binom{r}{4} . - - - - \binom{r}{r} + 0 + 0 + - - -$ times.

" $-\binom{r}{0} = \left[-\binom{r}{0}\right] + \binom{r}{1} - \binom{r}{2} + - - - -$

$= 0$

$\therefore$ No. of times x is counted $= \binom{r}{0} = 1$

$\therefore |A_1 \cup A_2 - - - A_n| = S_1 - S_2 + S_3 - - - - (-1)^{r+1} S_r$

$\therefore |A_1 \cap - - - A_n| = N_1 - |\bar{A}_1 \cup \bar{A}_2 \cup - - - \cup \bar{A}_n|$

→ A is finite

the unique natural no. equinumerous with A

$n = \{0, 1, - - - n-1\}$

is the cardinal no. of A.

↳ card A = card B iff $A \approx B$

↳ $\aleph_0$ (Aleph) : Cardinal no. of $\mathbb{N}$

~~$\aleph_0$~~ $2^{\aleph_0} = \aleph_1$ = cardinal no. of $\mathbb{R}$:
equinumerous to power set of natural nos.
(one-one mapping exists)

→ κ (kappa) & λ are 2 cardinal nos, sets K (letter K) & L (letter L)

$\kappa + \lambda$: card. no of $K \cup L$
where $K \cap L = \emptyset$
and κ = card (K) & λ = card (L)

$\kappa\lambda$: card. no. of $K \times L$

κ^{λ} : card. no of L_K

↳ Schröder Bernstein Theorem :

If $A \leq B$ and $B \leq A$ then $A \approx B$ (equinumerous)

$A \leq B$ if $\exists$ a 1-1 mapping from A into B.

↳ Continuum Hypothesis ~~Cantor Conjecture~~ :

Cantor conjectured that : There is no set of cardinality between $\aleph_0$ and $\aleph_1$
$(\aleph_1 > \aleph_0)$

~~1934 Gödel~~

This is a Gödel Proposition
(True but unprovable)
(i.e. CH or $\overline{CH}$ - neither is provable)

* $\vdash_s (con_s \rightarrow \nu)$ (consistent.)

ν : "I am not provable"

con_s : for any formula α, both α & $\neg\alpha$ cannot be proved in S.

[Hofstader Gödel, Escher, Bach : The Eternal Golden Braid]

$\vdash con_s$ implies $\vdash \nu_s$ But gödel proved it

$\therefore \nvdash \nu_s \Rightarrow \nvdash con_s$

S is incapable of proving it's own consistency.

$\vdash_{ZF} con_s$ eventually we get $\nvdash_{ZF} con_{ZF}$

But whole of mathematics is based on set theory.
Basically, mathematics cannot prove it's own consistency :).

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→ Cattle Numbers :

1. Triangulation of polygons : Partition of a polygon into Δs all of whose vertices are the vertices of the polygon.

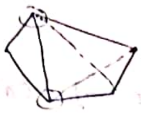
•) $t_3 = 1$

No. of ways to Δulate a Δ (3 vertices)

$\boxed{\begin{matrix} t_2 = 1 \\ t_1 = 1 \end{matrix}}$

•) $t_4 = 2$

•) $t_5 = 5$



(2 diagonals with one common vertex are drawn. There are 5 ways to chose this vertex $\therefore$ 5 ways to Δulate)

A Δ & a quad. are formed.

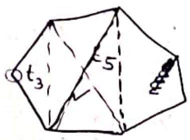
•) $t_6 =$

Consider a Δ involving the base − $t_3 t_4$ ways to Δulate further. (if blue wala Δ is picked).

If black wala Δ is picked, no. of ways to further Δulate $= t_5$

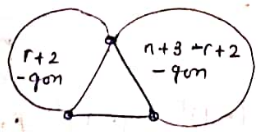
$$\therefore t_6 = \cancel{t_3 t_4}\ \underset{5}{t_5} + \underset{2}{t_4} + \underset{2}{t_4} + \underset{5}{t_5} = 14$$

•) t_7



$$t_2 t_6 + t_3 t_5 + t_4 t_4 + t_5 t_3 + t_6 t_2$$

•) In general



$$t_{r+2}\, t_{n-r-1}$$

Catalyn Numbers ↳

t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}
1	1	2	5	14	42	132	429	1430	4862
c_0	c_1	c_2	c_3	c_4	...				

$$c_r = t_{r+2}$$

→ <u>Catalyn Numbers</u> :

$$C_r = C_0 C_{r-1} + C_1 C_{r-2} + \text{———} + C_{r-1} C_0$$

$$f(z) = c_0 + c_1 z + c_2 z^2 + \text{-----}$$

$$f(z)^2 = f(z)\, f(z) = \text{PIO}$$

$$f(z)^2 = f(z)f(z)$$

$$= \underbrace{c_0^2}_{c_1} + \underbrace{(c_1c_0 + c_0c_1)}_{c_2}z + \underbrace{(c_2c_0 + c_1^2 + c_0c_2)}_{c_3}z^2 + \cdots$$

$$z\,f(z)^2 = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

$$= f(z) - c_0 = f(z) - 1$$

$$\Rightarrow \boxed{z f(z)^2 = f(z) - 1}$$

$$z f(z)^2 - f(z) + 1 = 0$$

$$\Rightarrow f(z) = \frac{1 \pm \sqrt{1-4z}}{2z} = \text{GF of Catalyn Nos.}$$

(Both + & − work)

How to find out which? — Take limit

$$\lim_{z\to 0} (c_0 + c_1 z + c_2 z^2 + \cdots) = c_0 = c_1$$

$$\lim_{z\to 0} \frac{1-\sqrt{1-4z}}{2z} = 1 \quad \text{(Pick this)}$$

$$\lim_{z\to 0} \frac{1+\sqrt{1-4z}}{2z} = \infty$$

Now,

$$(1+y)^\alpha = 1 + \sum_{n\geq 1} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} y^n$$

$$(1-4z)^{1/2} = 1 + \sum_{n\geq 1} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}(-4z)^n$$

$$= 1 + \sum_{n\geq 1} \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{(2n+3)}{2})}{n!}(-4)^n z^n$$

$$= 1 + \sum_{n\geq 1} \frac{(-1)^{n-1}(-1)^n[1.1.3.5\cdots(2n-3)]}{2^n\, n!} 4^n z^n$$

$$= 1 + \frac{2z}{n}\sum_{n\geq 1} \frac{(-1)^{n-1}(-1)^n[1.1.3\cdots(2n-3)]\, 2^{n-1} z^{n-1}}{[(n-1)!]^2} \times$$

$$= 1 - \frac{2z}{n}\sum_{n\geq 1} \frac{[1.2.3\cdots(2n-2)]\, z^{n-1}}{[(n-1)!]^2}$$

$$= 1 - \frac{2z}{n}\sum_{n\geq 1} \binom{2n-2}{n-1} z^{n-1}$$

$f(z) = \frac{1}{n} \sum_{n \geq 1} \binom{2n-2}{n-1} z^{n-1}$

$z^{(n-1)}$'s coeff. is : $\frac{1}{n+1} \binom{2n}{n}$

$(n-1)!$