**MA 102 (Mathematics II)**
**IIT Guwahati**

Tutorial Sheet No. 1 | Linear Algebra | January 10, 2019

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1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Prove or disprove the following statements.
   (a) The equality $\langle \mathbf{u}, \mathbf{v}\rangle = \frac{1}{4}(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2)$ holds.
   (b) The equality $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$ holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   (c) There exist $\mathbf{u}$ and $\mathbf{v}$ such that $\|\mathbf{u}\| = 1, \|\mathbf{v}\| = 2$ and $\langle \mathbf{u}, \mathbf{v}\rangle = 3$.

2. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$. What does this say about parallelogram in $\mathbb{R}^2$? Further, show that $|\langle \mathbf{u}, \mathbf{v}\rangle| = \|\mathbf{u}\|\,\|\mathbf{v}\|$ if and only if $\mathbf{u} = \alpha\mathbf{v}$ for some scalar $\alpha$.

**Solution:** First part is easy. For the second part, suppose that $\|\mathbf{v}\| \neq 0$. Define $a := \langle \mathbf{u}, \mathbf{v}\rangle/\|\mathbf{v}\|^2$ and $\mathbf{w} = \mathbf{u} - a\mathbf{v}$. Then $\mathbf{v} \perp \mathbf{w}$ and $\mathbf{u} = \mathbf{w} + a\mathbf{v} \Rightarrow \|\mathbf{u}\|^2 = |a|^2\|\mathbf{v}\|^2 + \|w\|^2$. Since $|a| = \|\mathbf{u}\|/\|\mathbf{v}\|$, it follows that $\mathbf{w} = 0$.

3. Express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, where
   (a) $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$;
   (b) $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

**Solution:** (a) Put $\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \mathbf{w} = a\mathbf{u} + b\mathbf{v} = \begin{bmatrix} a+b \\ -a+b \end{bmatrix}$ which gives $a = -2, b = 4$, thus $\mathbf{w} = -2\mathbf{u} + 4\mathbf{v}$. Similarly, (b) $\mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$.

4. True or False? Give justifications.
   (a) If $\hat{A}$ is the matrix obtained from $A$ by replacing the $i$th column $\mathbf{a}_i$ of $A$ by $2\mathbf{a}_i$ then the systems $\hat{A}\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$ are equivalent.
   (b) If the rref of a $5 \times 5$ matrix $A$ has the third column as $[1, 2, 0, 0, 0]^\top$ then $[-1, -2, 1, 0, 0]^T$ is a solution of $A\mathbf{x} = \mathbf{0}$.
   (c) For an $n \times n$ matrix $A$, the systems $A\mathbf{x} = \mathbf{0}$ and $A^\top\mathbf{x} = \mathbf{0}$ are equivalent.

**Solution:** a) False. Take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ then $[1, -1]^T$ is a solution of $A\mathbf{x} = \mathbf{0}$ but not of $\hat{A}\mathbf{x} = \mathbf{0}$.
b)True. Observation: The first two columns are leading columns and the third is not. Hence the first three rows of the rref of A are $[1, 0, 1, *, *], [0, 1, 2, *, *], [0, 0, 0, *, *]$.
c)False. Consider $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

5. The *trace* of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of its diagonal entries and is denoted by $\mathrm{tr}(A)$, i.e. $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$.
Prove the following: if $A$ and $B$ are $n \times n$ matrices and $\alpha$ is scalar, then

   1. $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$;
   2. $\mathrm{tr}(\alpha A) = \alpha \mathrm{tr}(A)$;
   3. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

**Solution:** Easy.

6. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are two distinct solutions of the system $A\mathbf{x} = \mathbf{b}$. Prove that there are infinitely many solutions to this system. Interpret your findings geometrically.

**Solution:** Show that for each $\lambda \in \mathbb{R}$, $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ is also a solution. This means that for the case when $A$ is a $3 \times 3$ matrix the entire line passing through the points $\mathbf{x}$ and $\mathbf{y}$ is in the set of solutions.

7. Decide whether the following pairs are row-equivalent:

(a) $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 4 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \end{bmatrix}$

**Solution:** (a) No, first one has row-rank one and the other two.
(b) Both are row equivalent to the row reduced echelon form $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$. So they are row-equivalent.
(c) No, two matrices must have the same size, in order to be row equivalent.

8. Find all the solutions of the linear system with the augmented matrix $[A \mid \mathbf{b}]$ as given below:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 2 \\ 5 & 6 & 7 & 8 & 5 \\ 9 & 10 & 11 & 12 & 8 \end{array}\right]$$

   (a) Find $\hat{\mathbf{b}}$ such that $A\mathbf{x} = \hat{\mathbf{b}}$ does not have a solution.
   (b) By changing exactly one entry of $A$, find an $\hat{A}$ such that $\hat{A}\mathbf{x} = \mathbf{b}$ will be consistent for all $\mathbf{b} \in \mathbb{R}^3$.

**Solution:** Solution set $= \left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{4} \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} | \alpha, \beta \in \mathbb{R} \right\}$.
a) Since $R_3 = 2R_2 - R_1$, where $R_i$ is the $i$th row of $A$, take $\mathbf{b}'$ such that $b'_3 \neq 2b'_2 - b'_1$.
b) Since $R_3 = 2R_2 - R_1$, and no two rows are LD, change any one entry of $A$ then the rows of $A$ will be LI or $rank(A) = 3$.

9. Determine the reduced row echelon form and the rank of the following matrices

$$(a)\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad (b)\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad (c)\begin{bmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{bmatrix}$$

**Solution:** (c)$rref(A) = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

**Solution:** Find the values solving $a+b+c=1, -a+b-c=5, 4a+2b+c=1$ applying Gaussian elimination.

10. If $A$ and $B$ are $m \times n$ matrices such that $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are equivalent, then show that $A$ and $B$ are row equivalent.

**Solution:** Answer: Note that the above statement is true if and only if $R^A x = 0$ and $R^B x = 0$ are equivalent implies $R^A = R^B$, where $R^A$ and $R^B$ are the RREF's of $A$ and $B$ respectively.
Let us assume $R^A x = 0$ and $R^B x = 0$ are equivalent, to show $R^A = R^B$.
If the first column of $R^A$ is not equal to that of $R^B$, then one of $R^A$ or $R^B$, say $R^A$ must have the first column as the zero column and for the other it will be $[1, 0, ..., 0]^T$.
Then $[1, 0, ..., 0]^T$ will be a solution of $R^A x = 0$ but not of $R^B x = 0$, which is not possible. So let us assume that the first $k$ columns of $R^A$ and $R^B$ are equal and $R^A_{(k+1)} \neq R^B_{(k+1)}$ where $R^A_{(k+1)}$ and $R^B_{(k+1)}$ are the $(k+1)$ th columns of $R^A$ and $R^B$ respectively. Then both $R^A_{(k+1)}$ and $R^B_{(k+1)}$ cannot be leading columns, WLOG let $R^A_{(k+1)}$ not be a leading column.
Let $s$ be the number of leading columns in the first $k$ columns of $R^A$ and $R^B$. If the $(k+1)$th column of either $R^A$ or $R^B$ is the zero column, then by the previous argument, we get a contradiction. Hence assume that the $(k+1)$ th column is nonzero for both $R^A$ and $R^B$. Then there exists an $i \in \{1, 2, .., s\}$ such that $R^A_{(i,k+1)} \neq R^B_{(i,k+1)}$.

Note that $-R^A_{(1,k+1)}e_1 - R^A_{(2,k+1)}e_2 - \dots - R^A_{(s,k+1)}e_s + \begin{bmatrix} R^A_{(1,k+1)} \\ \dots \\ R^A_{(s,k+1)} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$, where $e_i$ is the $i$ th column of $I_m$.
Take $\mathbf{u} = [u_1, u_2, ..u_k, 1, 0, .., 0]^T$, where for $i = 1, 2, .., k$,
$u_i = 0$ if the $i$ th column of $R^A$ is not a leading column and
$u_i = -R^A_{(j,k+1)}$ if the $i$ th column has the leading entry of the $j$ th row of $R^A$.
Then check that $R^A\mathbf{u} = \mathbf{0}$ but $R^B\mathbf{u} \neq \mathbf{0}$.

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