Variation of Parameters, Use of a Known Solution to Find Another and Cauchy-Euler Equation

Department of Mathematics
IIT Guwahati

RA/RKS/MGPP/KVK

Variation of Parameters

- The variation of parameter is a more general method for finding a particular solution (y_p). The method applies even when the coefficients of the differential equation are functions of x.

Consider $L(y) = g(x)$, where

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y,$$

where $p_{n-1}(x), \ldots, p_0(x) \in C((a, b))$. We know the general solution to $L(y) = g$ is given by

$$y(x) = y_h(x) + y_p(x),$$

where y_h is the general solution to $Ly = 0$ and $y_p(x)$ is a particular solution to $L(y) = g$.

Suppose we know a fundamental solution set $\{y_1, \ldots, y_n\}$ for $L(y) = 0$. Then

$$y_h(x) = C_1 y_1(x) + \cdots + C_n y_n(x).$$

In this method, seek a particular solution (y_p) of the form

$$y_p(x) = v_1(x) y_1(x) + \cdots + v_n(x) y_n(x),$$

and try to determine the functions $v_1, \ldots, v_n$.

Differentiating y_p,

$$y_p' = \sum_{i=1}^{n} v_i y_i' + \sum_{i=1}^{n} v_i' y_i.$$

To avoid second and higher-order derivatives of v_i's, we impose the condition

$$\sum_{i=1}^{n} v_i' y_i = 0. \qquad (1)$$

Therefore,

$$y_p' = \sum_{i=1}^{n} v_i y_i'.$$

Again, differentiating y_p', we obatin

$$y_p'' = \sum_{i=1}^{n} v_i y_i'' + \sum_{i=1}^{n} v_i' y_i'.$$

For the same reason, we impose the condition

$$\sum_{i=1}^{n} v_i' y_i' = 0. \tag{2}$$

Likewise, on computing $y_p''', \ldots, y_p^{(n-1)}$, we impose $(n-3)$ additional conditions involving v_i''s as

$$\sum_{i=1}^{n} v_i' y_i'' = 0, \quad \cdots, \quad \sum_{i=1}^{n} v_i' y_i^{(n-2)} = 0. \tag{3}$$

Finally, for the nth condition, use the derived conditions and the fact that $L(y_h) = 0$ and $L(y_p) = g$ to obtain

$$\sum_{i=1}^{n} v_i' y_i^{(n-1)} = g. \tag{4}$$

Therefore, we seek $v_1', \ldots, v_n'$ that satisfy the system

$$\begin{array}{rcl}
y_1 v_1' + \cdots + y_n v_n' & = & 0, \\
y_1' v_1' + \cdots + y_n' v_n' & = & 0, \\
\vdots \quad + \quad \vdots \quad + \vdots & = & \vdots \\
y_1^{(n-1)} v_1' + \cdots + y_n^{(n-1)} v_n' & = & g.
\end{array}$$

The existence of $v_1', \ldots, v_n'$ requires

$$\begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W(y_1, \ldots, y_n)(x) \neq 0$$

on (a, b), which is true as $\{y_1, \ldots, y_n\}$ is a fundamental solution set.

On solving for $v_1', \ldots, v_n'$, we find that

$$v_k'(x) = \frac{g(x) W_k(x)}{W(y_1, \ldots, y_n)(x)}, \quad k = 1, \ldots, n,$$

where the determinant $W_k(x)$ is obtained from $W(y_1, \ldots, y_n)(x)$ by replacing kth column by $[0, \ldots, 0, 1]^T$.

We can express $W_k(x)$ as

$$W_k(x) = (-1)^{(n-k)} W(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)(x)$$

for $k = 1, \ldots, n$. Integrating $v_k'(x)$ yields

$$v_k(x) = \int \frac{g(x) W_k(x)}{W(y_1, \ldots, y_n)(x)} dx, \quad k = 1, \ldots, n.$$

Finally, substituting the v_k's back into y_p, we obtain

$$y_p(x) = v_1(x) y_1(x) + \cdots + v_n(x) y_n(x)$$

we obtain

$$y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{g(x) W_k(x)}{W(y_1, \ldots, y_n)(x)} dx.$$

For $n = 2$, v_1' and v_2' are given by

$$v_1'(x) = \frac{-g(x)y_2(x)}{W(y_1, y_2)(x)}, \quad v_2'(x) = \frac{g(x)y_1(x)}{W(y_1, y_2)(x)},$$

where $W(y_1, y_2)(x) \neq 0$. Integrating these equations, we obtain

$$v_1(x) = \int \frac{-g(x)y_2(x)}{W(y_1, y_2)(x)} dx, \quad v_2(x) = \int \frac{g(x)y_1(x)}{W(y_1, y_2)(x)} dx.$$

Thus, the particular solution is given by

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x).$$

Example: Consider $y'' + y = \ cosec\, x$.

$$y_h(x) = c_1 \sin x + c_2 \cos x.$$

The two linearly independent solutions are $y_1(x) = \sin x$ and $y_2(x) = \cos x$ and $W(y_1, y_2) = -1 \neq 0$.

$$v_1(x) = \int \frac{-g(x)y_2(x)}{W(y_1, y_2)(x)} dx = \int \frac{-\cos x \ cosec\, x}{-1} dx = \log(\sin x).$$

$$v_2(x) = \int \frac{g(x)y_1(x)}{W(y_1, y_2)(x)} dx = \int \frac{\sin x \ cosec\, x}{-1} dx = -x.$$

$$y_p = \sin x \log(\sin x) - x \cos x.$$

The general solution is
$y(x) = c_1 \sin x + c_2 \cos x + \sin x \log(\sin x) - x \cos x.$

Use of a known solution to find another

Assume that $y_1(x) \neq 0$ is a known solution of $L(y) = 0$, where

$$L(y) = y'' + p(x)y' + q(x)y.$$

We know $L((cy_1)) = 0$, where c is any arbitrary constant. The basic idea is to replace c by an unknown function $v(x)$ so that $L(y_2) = 0$, where $y_2 = v(x)y_1(x)$.

Suppose $L(y_2) = L(vy_1) = 0$. Then, we have

$$v(y_1'' + py_1' + qy_1) + v''y_1 + v'(2y_1' + py_1) = 0.$$

Since $L(y_1) = 0$, we have

$$v''y_1 + v'(2y_1' + py_1) = 0 \Rightarrow \frac{v''}{v'} = -2\frac{y_1'}{y_1} - p.$$

Integrating

$$v' = \frac{1}{y_1^2} e^{-\int p dx} \Rightarrow v(x) = \int \frac{1}{y_1^2} e^{-\int p dx} dx.$$

Thus, the second solution is $y_2(x) = v(x) y_1(x)$.

Example: Given that $y_1 = e^x$ is a solution to $y'' - 2y' + y = 0$. Determine the second linear independent solution y_2.

Note that $v(x) = x$. The second linearly dependent solution is

$$y_2(x) = vy_1 = xe^x.$$

Cauchy-Euler Equation

An equation of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = g(x),$$

where a_i's are constants is called Cauchy-Euler equation.

The substitution $x = e^t$ transform the above equation into an equation with constant coefficients. For simplicity, take $n = 2$.

Assume that $x > 0$ and let $x = e^t$. By the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t = x\frac{dy}{dx},$$

hence

$$x\frac{dy}{dx} = \frac{dy}{dt}.$$

Differentiating $x\frac{dy}{dx} = \frac{dy}{dt}$ with respect to t, we find that

$$\begin{aligned}
\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{dx}{dt}\frac{dy}{dx} + x\frac{d}{dt}\left(\frac{dy}{dx}\right) \\
&= \frac{dy}{dt} + x\frac{d^2y}{dx^2}\frac{dx}{dt} = \frac{dy}{dt} + x\frac{d^2y}{dx^2}e^t \\
&= \frac{dy}{dt} + x^2\frac{d^2y}{dx^2}.
\end{aligned}$$

Thus

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

Substituting into the equation we obtain the constant coefficient ODE

$$a_2\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+a_1\frac{dy}{dt}+a_0y=g(e^t),$$

which may be written as

$$a_2\frac{d^2y}{dt^2}+(a_1-a_2)\frac{dy}{dt}+a_0y=g(e^t).$$

Note: Observe that in the proof it is assumed that $x>0$. If $x<0$, the substitution $x=-e^t$ will reduced the Cauchy-Euler equation to constant coefficients ODE. The method can be applied to higher-order Cauchy-Euler equation.

Example: Consider $x^2y'' - 2xy' + 2y = x^3, \quad x > 0.$
Setting $x = e^t$, we obtain

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = e^{3t},$$

or

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}.$$

The GS to the homogeneous equation is

$$y_h(x) = c_1e^t + c_2e^{2t} = c_1x + c_2x^2.$$

To find a particular solution, let $y_p = Ae^{3t}$. Then, $A = \frac{1}{2}$ hence, $y_p = \frac{1}{2}e^{3t} = \frac{1}{2}x^3$. The GS is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1x + c_2x^2 + \frac{1}{2}x^3, \quad x > 0. \end{aligned}$$

*** End ***