Tutorial 2 on Relativity

Q.1 In the file MichelsonMorley.pdf, the authors discuss an analysis of the expected results of the experiment if Galilean relativity was valid - which means the speed of light depends on the reference frame. They find that the total change in the time taken between the rotated and non-rotated apparatus for the beam of light to reach the eye is non-zero. Re-analyze this experiment in a similar way except now assume that the speed of light is independent of the reference frame (special relativity). You should find that the above total change in the time taken is zero.

A. Since the speed of light is assumed to be a universal constant in this new analysis, this means there is no 'ether wind'. The time t_1 (in the notation of this pdf file) now simply becomes,

$$t_1 = \frac{2L_1}{c} \tag{1}$$

Similarly since there is no ether wind, there is no need for light to compensate for the ether wind while traveling perpendicular to the wind by developing a transverse velocity as was the case in the original analysis. Hence,

$$t_2 = \frac{2L_2}{c} \tag{2}$$

This means the change in the time taken to traverse these two paths is,

$$\Delta t = t_1 - t_2 = \frac{2L_1}{c} - \frac{2L_2}{c} \tag{3}$$

Now when the apparatus is rotated by 90^0, we get

$$t'_2 = \frac{2L_2}{c}; \quad t'_1 = \frac{2L_1}{c} \tag{4}$$

Hence,

$$\Delta t' = t'_1 - t'_2 = \frac{2L_1}{c} - \frac{2L_2}{c} \tag{5}$$

Hence the difference between the time increment between the original and the rotated configuration ism

$$\Delta t - \Delta t' = (\frac{2L_1}{c} - \frac{2L_2}{c}) - (\frac{2L_1}{c} - \frac{2L_2}{c}) = 0 \tag{6}$$

which is precisely the result found by Michelson when he actually performed the experiment.

Q.2 On earth, every person that is born at some place at some time, finally dies at some other (or rarely, the same) place at a later time. Show that there is no reference frame in which it is going to appear that the person (called P) died before she was born. This is a non-issue in Galilean relativity since time is absolute, but requires a proof in Special Relativity.

A. Consider two events, birth B: $(x = 0, y = 0, z = 0, t = 0)$ and death D: $(x, y = 0, z = 0, t > 0)$ as recorded by reference frame S (person on earth or the individual herself called reference frame S). These two events are recorded as birth B: $(x' = 0, y' = 0, z' = 0, t' = 0)$ and death D: $(x', y' = 0, z' = 0, t')$ by a spaceship S' moving relative to S with speed $v, |v| < c$ along the x-axis and that the person moves around only along the x-axis during her lifetime (not required but keeps things simple). Let us assume for simplicity that the spaceship passes by the place where she was born at the moment of birth so that B is also $(x' = 0, y' = 0, z' = 0, t' = 0)$ (not really needed but keeps calculations simple). We have to show that the time of death as seen by the spaceship also occurs after birth ie. $t' > 0$.

To do this, note that the average speed of the person P calculated through her lifetime [everyone travels around to many places during their lifetime] has to be always less than the speed of light. This average speed u as seen by reference frame S is,

$$u = \frac{x}{t};\ so\ that\ \ -c < u < c\ so\ that\ \ -c\,t < x < c\,t\ as\ t > 0 \tag{7}$$

Now according to Lorentz transformation,

$$x' = \gamma(x - v\,t);\ \ t' = \gamma(t - \frac{vx}{c^2}) \tag{8}$$

Since $-c\,t < x < c\,t$ it follows that $-\frac{v}{c}\,t < \frac{vx}{c^2} < \frac{v}{c}\,t$, or $t - \frac{v}{c}\,t < t - \frac{vx}{c^2} < t + \frac{v}{c}\,t$. Now since $-1 < \frac{v}{c} < 1$ and $t > 0$, it follows that $0 < t - \frac{v}{c}\,t < 2t$.

This means $t - \frac{vx}{c^2} > t - \frac{v}{c}\,t > 0$. Hence $t' = \gamma(t - \frac{vx}{c^2}) > 0$. In other words, the person P will never appear to anyone to have died before she was born (even in Special Relativity).

Those unhappy with the simplifying assumptions made, are invited to do this in the most general case (spaceship moving in a general direction starting at some general point etc.) keeping in mind the central idea that the average speed of the person during her lifetime is less than c.

Q.3 In 1962, the famous Indian Physicist E.C.G. Sudarshan (1931-2018) along with his collaborators Bilaniuk and Deshpande proposed that there may be particles that travel faster than light which do not violate Einstein's special relativity [American Journal of Physics 30, 718 (1962); doi: 10.1119/1.1941773]. These particles were later called "tachyons". In fact, the following classification is standard. With respect to some reference frame S,

a) 'Bradyons' are particles with speed $|u| < c$ ('bradys' in Greek means 'slow')

b) 'Luxons' are particles that move with the speed of light $|u| = c$ (eg. photons, gluons and gravitons) ('lux' means light in Latin)

c) 'Tachyons' are particles that move with the speed $|u| > c$ ('tachus' means 'speedy' in Greek)

In Special Relativity, only reference frames have to move with (constant) velocities less than c, particles that live in them can do whatever they want.

With respect to another frame S' moving with speed v, $|v| < c$ relative to S in the x-direction, show that **A bradyon in S will also be a bradyon in S', a luxon in S will also be a luxon in S' and finally a tachyon in S (if you can find one) will also be a tachyon in S', no matter how small or large $|v| < c$ is.**

Note that the boost velocity v has to be less than the speed of light – you are not allowed to sit on a tachyon and observe the world. This is because Lorentz transformations don't make sense if $|v| > c$ since $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$ will become imaginary and $x^{'} = \gamma(x - vt)$ etc. will not make sense etc.

A.

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} \tag{9}$$

Hence [show this],

$$\frac{c^2 - u^{'2}}{c^2} = \frac{(c^2 - u^2)(c^2 - v^2)}{(c^2 - uv)^2} \tag{10}$$

Bradyons: Since $-c < v < c$ [always] and $-c < u < c$ [only for bradyons]

$$(c^2 - u^2) > 0 \; ; \; (c^2 - v^2) > 0 \tag{11}$$

This means,

$$\frac{c^2 - u^{'2}}{c^2} = \frac{(c^2 - u^2)(c^2 - v^2)}{(c^2 - uv)^2} > 0 \tag{12}$$

or,

$$-c < u' < c \tag{13}$$

This means even in S' the particle is a bradyon.

Luxons: $c^2 = u^2$, so that $c^2 = u^{'2}$

Tachyons: $c^2 - u^2 < 0$ and $c^2 - v^2 > 0$. This means $c^2 - u^{'2} < 0$. Hence even in S' the particle is a tachyon.

Q.4 *Throwing: The International Cricket Council (ICC) declares a bowling action illegal when a player is throwing rather than bowling the ball. This is defined by the ICC as being when the arm's angle at the elbow is less than 165 degrees at time of release of ball (180 degrees would be when the arm is perfectly straight).*
Two brothers who are umpires are watching an India vs Pakistan match where the Pakistani bowler Saeed Ajmal is seen bowling to Sachin Tendulkar. One of the umpires, Gary Gaia is the TV umpire. His brother, John Ouranos is an astronaut in a spaceship moving with speed $0.95c$ parallel to the cricket pitch in the direction of the ball being bowled. Gary notices that the Ajmal has bowled illegally since at the time of the release of the ball, the upper half of his arm was horizontal but the lower half was making an angle of 20^0 with the horizontal. Gary awards a no-ball to India for throwing. Does John who is watching the match from his spaceship agree or disagree with his brother? What angle does John measure? Answer these questions if John's spaceship is moving perpendicular (instead of parallel) to the pitch with the same speed ie. $0.95c$.

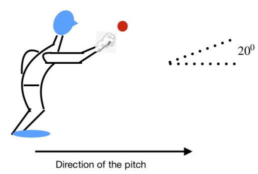

Figure 1: Saeed Ajmal

A. The reference frame of the TV umpire and Ajmal are the same since Ajmal is moving very slowly compared to the speed of light relative to the TV umpire. In other words angles seen by Ajmal are the same as those seen by the TV umpire. Reference frame S is that of the TV umpire. Now the scale factor $\gamma = 3.2$ in the problem.

$$tan(\theta = 20^0) = \frac{\Delta y}{\Delta x} \tag{14}$$

The goal is to find the angle $\theta^{'}$ seen by John in his spaceship.

$$tan(\theta^{'}) = \frac{\Delta y^{'}}{\Delta x^{'}} \tag{15}$$

Case I: Spaceship is travelling parallel to the pitch. We showed in class that

$$tan(\theta') = \gamma \; tan(\theta = 20^0) = 3.2 \; tan(\theta = 20^0); \; hence \; \theta^{'} = 49.35^0 \tag{16}$$

John will agree with his brother.

Case II: Spaceship is travelling perpendicular to the pitch. It is easy to show that

$$cot(\theta') = \gamma \; cot(\theta = 20^0) = 3.2 \; cot(\theta = 20^0); \; hence \; \theta^{'} = 6.49^0 \tag{17}$$

Now John will disagree with his brother.

Q.5 A particle with (rest) mass m moving with velocity $\mathbf{u}$ in a reference frame S is defined to have a (relativistic) momentum $\mathbf{p}$ and (relativistic) energy E given by

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}}; \quad E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} \tag{18}$$

These unusual definitions ensure that in collisions, if momentum is conserved before and after collision in one reference frame, it is also seen to be conserved in a reference frame moving relative to the earlier one. The usual non-relativistic definitions $\mathbf{p} = m\mathbf{u}$ and $E = \frac{1}{2}mu^2$ do not have this property [under Lorentz transformations, but under Galilean transformation these simple older formulas are sufficient to ensure conservation laws are applicable in all inertial reference frames]. If $|\mathbf{u}| \ll c$ show by binomial expansion that,

$$\mathbf{p} = m\mathbf{u} + m\mathbf{u}\frac{u^2}{2c^2} + m\mathbf{u}\frac{3u^4}{8c^4} + ... \tag{19}$$

$$E = mc^2 + \frac{1}{2}mu^2 + \frac{1}{2}mu^2\ \frac{3u^2}{4c^2} + ... \tag{20}$$

Show that in a reference frame S' moving with a speed v in the x-direction, the energy and momentum of the particle becomes,

$$E^{'} = \gamma_v(E - vp_x); \quad p^{'}_x = \gamma_v(p_x - \frac{v}{c^2}E); \quad p^{'}_y = p_y; \quad p^{'}_z = p_z; \quad where\ \gamma_v = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \tag{21}$$

Using this, show that in a two-body collision, if the (relativistic) energy and (relativistic) momentum are conserved in one reference frame S they are also conserved in reference frame S'.

Not to be discussed in tutorial but important to do at home: Show that energy and momentum as defined by the usual formulas $E = \frac{1}{2}mu^2$, $\mathbf{p} = m\mathbf{u}$ ARE NOT CONSERVED in a two body collision if you assume *Lorentz transformations* are correct. Also show that energy and momentum as defined by the usual formulas $E = \frac{1}{2}mu^2$, $\mathbf{p} = m\mathbf{u}$ ARE CONSERVED in a two body collision if you assume *Galilean transformations* are correct.

A. The binomial expansion question is trivial (just expand γ_u in powers of u). The transformation of momentum and energy can be derived using the law of addition of velocities derived in class.

$$u^{'}_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}; \ u^{'}_y = \frac{u_y}{\gamma_v(1 - \frac{u_x v}{c^2})}; \ u^{'}_z = \frac{u_z}{\gamma_v(1 - \frac{u_x v}{c^2})} \tag{22}$$

The momentum in S' is,

$$p^{'}_x = \frac{mu^{'}_x}{\sqrt{1-\frac{u'^2}{c^2}}}; \ p^{'}_y = \frac{mu^{'}_y}{\sqrt{1-\frac{u'^2}{c^2}}}; \ p^{'}_z = \frac{mu^{'}_z}{\sqrt{1-\frac{u'^2}{c^2}}}; \ E^{'} = \frac{mc^2}{\sqrt{1-\frac{u'^2}{c^2}}} \tag{23}$$

where $u^{'2} = u^{'2}_x + u^{'2}_y + u^{'2}_z$. Insert Eq.(22) into Eq.(23) to get,

$$p^{'}_x = \frac{mc^2\ (u_x - v)}{\sqrt{(c^2-u^2)\,(c^2-v^2)}}; \ p^{'}_y = \frac{c\ mu_y}{\sqrt{c^2-u^2}}; \ p^{'}_z = \frac{c\ mu_z}{\sqrt{c^2-u^2}}; \ E^{'} = \frac{mc^2\ \left(c^2 - u_x v\right)}{\sqrt{(c^2-u^2)\,(c^2-v^2)}} \tag{24}$$

This is nothing but,

$$E^{'} = \gamma_v(E - vp_x); \quad p^{'}_x = \gamma_v(p_x - \frac{v}{c^2}E); \quad p^{'}_y = p_y; \quad p^{'}_z = p_z \tag{25}$$

Two-body collisions: Imagine a particle with momentum and energy $\mathbf{p}_1, E_1$ collides with another particle with momentum and energy $\mathbf{p}_2, E_2$. Let us assume that the total momentum before collision is same as total momentum after collision and total energy before collision is same as total energy after collision in some reference frame called S. This means,

$$\mathbf{p}_{1,B} + \mathbf{p}_{2,B} = \mathbf{p}_{1,A} + \mathbf{p}_{2,A}; \; E_{1,B} + E_{2,B} = E_{1,A} + E_{2,A} \tag{26}$$

Here 'B' means before and 'A' means after. The big question now is - are these conservation laws also applicable when this collision is seen from reference frame S'? In other words, is this also true?

$$\mathbf{p}^{'}_{1,B} + \mathbf{p}^{'}_{2,B} = \mathbf{p}^{'}_{1,A} + \mathbf{p}^{'}_{2,A}; \; E^{'}_{1,B} + E^{'}_{2,B} = E^{'}_{1,A} + E^{'}_{2,A} \tag{27}$$

It is easy to show using the transformation laws in Eq.(25) that Eq.(27) is valid if Eq.(26) is valid. I will only do it for x- component of momentum and energy the y-z components are trivial. Examine the difference,

$$D^{'}_{p,f} = p^{'}_{x,1,B} + p^{'}_{x,2,B} - p^{'}_{x,1,A} - p^{'}_{x,2,A} \tag{28}$$

Note that we are assuming,

$$D_{p,f} = p_{x,1,B} + p_{x,2,B} - p_{x,1,A} - p_{x,2,A} = 0; \quad D_{E,f} = E_{1,B} + E_{2,B} - E_{1,A} - E_{2,A} = 0 \tag{29}$$

Inserting Eq.(25) into Eq.(28) this difference is

$$D^{'}_{p,f} = \gamma_v(p_{x,1,B} - \frac{v}{c^2}E_{1,B}) + \gamma_v(p_{x,2,B} - \frac{v}{c^2}E_{2,B}) - \gamma_v(p_{x,1,A} - \frac{v}{c^2}E_{1,A}) - \gamma_v(p_{x,2,B} - \frac{v}{c^2}E_{2,A}) = \gamma_v(D_{p,f} - \frac{v}{c^2}D_{E,f}) = 0 \tag{30}$$

Similarly,

$$D^{'}_{E,f} = E^{'}_{1,B} + E^{'}_{2,B} - E^{'}_{1,A} - E^{'}_{2,A} = \gamma_v(E_{1,B} - v\; p_{x,1,B}) + \gamma_v(E_{2,B} - v\; p_{x,2,B}) - \gamma_v(E_{1,A} - v\; p_{x,1,A}) - \gamma_v(E_{2,B} - v\; p_{x,2,A})$$

$$= \gamma_v(D_{E,f} - v\; D_{p,f}) = 0 \tag{31}$$

Q.6 Imagine a particle moving in the x-y plane in reference frame S. Now S' is a spaceship that moves in the x direction with speed v relative to S. If u_x, u_y, u_z are the instantaneous velocity components of the particle and a_x, a_y, a_z are the components of acceleration, find the corresponding components as seen from S'.

A. Start with the already derived formulas from class:

$$u^{'}_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}; \; u^{'}_y = \frac{u_y}{\gamma_v(1 - \frac{u_x v}{c^2})}; \; u^{'}_z = \frac{u_z}{\gamma_v(1 - \frac{u_x v}{c^2})} \tag{32}$$

This means,

$$du_x^{'} = \frac{du_x}{1-\frac{u_x v}{c^2}} + \frac{(u_x - v)}{(1-\frac{u_x v}{c^2})^2}\ \frac{v}{c^2}\ du_x \tag{33}$$

$$du_y^{'} = \frac{du_y}{\gamma_v(1-\frac{u_x v}{c^2})} + \frac{u_y}{\gamma_v(1-\frac{u_x v}{c^2})^2}\ \frac{v}{c^2}\ du_x \tag{34}$$

$$du_z^{'} = \frac{du_z}{\gamma_v(1-\frac{u_x v}{c^2})} + \frac{u_z}{\gamma_v(1-\frac{u_x v}{c^2})^2}\ \frac{v}{c^2}\ du_x \tag{35}$$

Also,

$$dt^{'} = \gamma_v(dt - \frac{v}{c^2}dx) \tag{36}$$

Dividing one by the other we get (this result below is when the boost direction is in some general direction $\hat{v}$) ,

$$\mathbf{a}^{'} = \frac{\mathbf{a}}{\gamma_v^2\left(1-\frac{\mathbf{v.u}}{c^2}\right)^2} - \frac{(\mathbf{a.v})\mathbf{v}(\gamma_v - 1)}{v^2\gamma_v^3\left(1-\frac{\mathbf{v.u}}{c^2}\right)^3} + \frac{(\mathbf{a.v})\mathbf{u}}{c^2\gamma_v^2\left(1-\frac{\mathbf{v.u}}{c^2}\right)^3}$$

In this particular question you may set, $\mathbf{v} = v\ \hat{x}$.