Markov Processes

- For the concepts like the LLN and the CLT, the independence of the sequence played an important role.
- The Markov processes consider the evolution and the steady-state behavior of a stochastic process with a simple dependence model
- They are widely used in diverse applications like population studies, queueing systems and the restoration of a degraded photograph.

Conditional Independence

The Markovian model is built around the notion of conditional independence of events.

Consider three events A, B and C in $(S, \mathbb{F}, P)$. The joint probability of A, B and C is given by

$$P\left(A \cap B \cap C\right) = P(A)P\left(B \cap C \mid A\right)$$

Applying the chain rule:

$$P(A \cap B \cap C) = P(A)P(B / A)P(C / A \cap B)$$

Conditional Independence….

Given A, the events B and C are called *conditionally independent* if

$$P(B \cap C / A) = P(B / A) P(C / A) \qquad (1)$$

Or

$$P(C / A \cap B) = P(C / A) \qquad (2)$$

(1) and (2) are equivalent, because

$$P(C / A \cap B) = \frac{P(A \cap B \cap C)}{P(A \cap B)}$$

$$= \frac{P(A) P(B \cap C / A)}{P(A \cap B)}$$

$$= \frac{P(A) P(B / A) P(C / A)}{P(A) P(B / A)} = P(C / A)$$

Thus,if B and C are conditionally independent given *A*, the joint probability is given by

$$P(A \cap B \cap C) = P(A) P(B | A) P(C | A)$$

Example- A box contains two coins: a fair coin and one fake two-headed coin (P(H)=1

A coin is chosen at random and tossed twice. Define the following events.

A= First coin toss results in an H

B= Second coin toss results in an H

C= Fair coin has been selected.

Examine if A and B are conditionally independent given C and also if A and B are (unconditionally) independent.

Solution: We have $P(A|C)=P(B|C)=1/2$. Also, given that the fair coin is selected, we have $P(A \cap B|C)=1/2 X 1/2=1/4$.

Thus A and B are conditionally independent given C

Example- Cntd..

To find $P(A)$,$P(B)$, and $P(A \cap B)$, we use the law of total probability:

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$$P(A) = P(C)P(A/C) + P(C')P(A/C')$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 = \frac{3}{4}$$

Similarly, $P(B) = \dfrac{3}{4}$

To find $P(A \cap B)$, we have

$$P(A \cap B) = P(C)P(A \cap B/C) + P(C')P(A \cap B/C')$$

$$= P(C)P(A/C)(B/C) + P(C')P(A/C')(B/C')$$

$$\left(by\ conditional\ independence\ of\ A\, and\ B\right)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 \times 1 = \frac{5}{8}$$

$$\therefore P(A \cap B) \neq P(A)P(B)$$

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Conditionally Independent RVs

Given X, the RVs Y and Z are called *conditionally independent* if

$$F(y,z/X=x)=F(y/X=x)F(z/X=x)$$

Equivalently, if

$$F(y/Z=z,X=x)=F(y/X=x)$$

Markov process

A random process $\{X(t), t \in \Gamma\}$ defined on $(S, \mathbb{F}, P)$ is called a Markov process if for any sequence of time $t_1 < t_2 < ..... < t_n < t_{n+1} \in \Gamma$,

$$P(X(t_{n+1}) \leq x | X(t_1) = x_1, X(t_2) = x_2, ..., X(t_n) = x_n)$$
$$= P(\{X(t_{n+1}) \leq x | X(t_n) = x_n)$$

For a Markov process, given $X(t_n)$, the random variable $X(t_{n+1})$ is conditionally independent of $X(t_1), X(t_2), ..., X(t_{n-1})$

This property is known as the *Markovian property.*

Markov Chain

$X(t_n)$ takes values from a set V called the *state space*. The elements of V are called the *states* of the process $\{X(t_n)\}$.

Suppose V is countable. Since *V* has one-to-one correspondence with some subset of $\mathbb{Z}$, we can assume V as a set consisting of integers. Thus $\{X(t_n) = i\}$ means the event that $X(t_n)$ takes the ith state. For such a process, the Markovian property in can be expressed in terms of the probability mass function and the process is called a *Markov chain* (MC). A Markov chain may be a *continuous-time Markov chain (CTMC*) or a *discrete-time Markov chain (DTMC)*.

CTMC

Suppose $\{X(t)\}$ takes values from a discrete state space $V=\{0,1,2,\cdots\}$. Then $\{X(t)\}$ is called a CTMC if for any $n\geq 1$ and

$t_1 < t_2 < \ldots\ldots < t_n < t_{n+1} \in \Gamma$,

$$P(X(t_{n+1})=j|X(t_0)=i_0, X(t_1)=i_1, \ldots, X(t_n)=i)$$

$$=P(X(t_{n+1})=j|X(t_n)=i)$$

Example Independent increment process

For any $n > 1$ and $t_0 < t_1 < \ldots < t_{n+1} \in \Gamma$, we have

$$P(X(t_{n+1}) = j / X(t_0) = i_0, X(t_1) = i_1, \ldots, X(t_n) = i)$$
$$= P(X(t_{n+1}) - X(t_n) = j - i / X(t_0) = i_0, X(t_1) = i_1, \ldots, X(t_n) = i)$$
$$= P(X(t_{n+1}) - X(t_n) = j - i) \text{ (Using the independent increment property)}$$

Similarly,

$$P(X(t_{n+1}) = j / X(t_n) = i)$$
$$= P((t_{n+1}) - X(t_n) = j - i / X(t_n) = i)$$
$$= P(X(t_{n+1}) - X(t_n) = j - i)$$
$$\therefore P(X(t_{n+1}) = j / X(t_0) = i_0, X(t_1) = i_1, \ldots, X(t_n) = i) = P(X(t_{n+1}) = j / X(t_n) = i)$$

Thus $\{X(t)\}$ is a CTMC.

DTMC

Consider a *discrete-time random process* $\{X_n, n \geq 0\}$ taking values from a countable set V. $\{X_n, n \geq 0\}$ is said to be a DTMC if

$$P\left(X_{n+1} = j | X_0 = i_0, X_1 = i_1, \cdots, X_n = i\right) = P\left(X_{n+1} = j | X_n = i\right).$$

__**Example**__

Suppose $\{X_n, n \geq 0\}$ is a sequence of iid and integer-valued random variables. Then

$$P\left(X_{n+1} = j | X_0 = i_0, X_1 = i_1, \ldots, X_n = i\right) = P\left(X_{n+1} = j\right)$$
$$= P\left(X_{n+1} = j / X_n = i\right)$$

Therefore, a sequence of integer –valued iid random variables is trivially a DTMC.

Example

Suppose $\{Z_n, n \geq 0\}$ is a sequence of iid and integer-valued random variables and $X_n = \sum_{i=0}^{n} Z_i$. Then $\{X_n, n \geq 0\}$ is an MC.

Solution:

We have $X_{n+1} = \sum_{i=0}^{n+1} Z_i = \sum_{i=0}^{n} Z_i + Z_{n+1} = X_n + Z_{n+1}$

$$\therefore P\left(X_{n+1} = j \mid X_n = i\right) = \frac{P\left(X_{n+1} = j, X_n = i\right)}{P\left(X_n = i\right)}$$

$$= \frac{P\left(Z_{n+1} = j - i, X_n = i\right)}{P\left(X_n = i\right)}$$

$$= \frac{P\left(Z_{n+1} = j - i\right) P\left(X_n = i\right)}{P\left(X_n = i\right)}$$

$$= P\left(Z_{n+1} = j - i\right)$$

Arguing in the similar manner, we can show that $P\left(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \cdots, X_n = i\right) = P\left(Z_n = j - i\right)$

$\therefore P\left(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \cdots, X_n = i\right) = P\left(X_{n+1} = j \mid X_n = i\right)$

$\therefore \{X_n, n \geq 0\}$ is an MC.

Example contd..

When Z_n takes values from $\{-1,1\}$, then $\left\{ X_n = \sum_{i=0}^{n} Z_i , n \geq 0 \right\}$ is the simple random walk process. Thus the simple random walk process is an MC.

To Summarise

- Conditional Independence

$$P(B \cap C / A) = P(B/A)P(C/A)$$

- Markov Process

For a Markov process $\{X(t), t \in \Gamma\}$

$$P(X(t_{n+1}) \leq x | X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_n) = x_n)$$
$$= P(\{X(t_{n+1}) \leq x | X(t_n) = x_n)$$

- $X(t)$ takes values from a state space *V* whose elements are called states.

To Summarise..

- **CTMC**

For a CTMC $\left\{ X(t), t \geq 0 \right\}$

$$P(X(t_{n+1}) = j \mid X(t_0) = i_0, X(t_1) = i_1, \ldots, X(t_n) = i)$$
$$= P(X(t_{n+1}) = j \mid X(t_n) = i)$$

- **DTMC**

For a DTMC $\left\{ X_n, n \geq 0 \right\}$ taking values from a countable set V.

$$P\left(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \cdots, X_n = i \right) = P\left(X_{n+1} = j \mid X_n = i \right)$$

Thank You