Partial derivatives

1. From our experience with real-valued functions of one-real variable, the definition for differentiability of a function at a point should give us the following:
   (a) Differentiability implies continuity at that point.
   (b) Chain rule holds for composition of functions.

2. We understand derivative as instantaneous rate of change of output with respect to the input. When functions of two independent variables are considered, there are two such rates that can be asked, one with respect to each variable.

3. *Definition:* Consider $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ and treat it as a function of two independent variables x and y which vary in A. Suppose that $(a, b) \in A$ and an ϵ-ball around the point (a, b), is also contained in A. The limit $\lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$, if it exists, is called the partial derivative of f w.r.t. the first variable x at (a, b) and is denoted by $\frac{\partial f}{\partial x}(a, b)$. Likewise, the limit $\lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$, if it exists, is called the partial derivative of f w.r.t. the second variable y at (a, b) and is denoted by $\frac{\partial f}{\partial y}(a, b)$.
   (a) Geometric meaning of these derivatives: The partial derivative of f w.r.t. x at the point (x_0, y_0) is the derivative of the single variable function $h(x) := f(x, y_0)$ at the point x_0 *i.e.* it is $h'(x_0)$. In the domain, $A \subset \mathbb{R}^2$, $h \to 0$ can be seen as the points approaching (x_0, y_0) along a horizontal line $y = y_0$. So, when appropriate, one can find the partial derivative of f w.r.t. x also by substituting $y = y_0$ and taking the derivative w.r.t. x of the resultant single variable function. Analogous remarks apply to partial derivative w.r.t y.
   (b) Example: Find the partial derivatives of $f(x, y) = x^2 + y^2$, $(x, y) \in \mathbb{R}^2$ at the points, $(0, 0)$ and $(1, 3)$.
   (c) Example: Find the partial derivatives at the point $(0, 0)$ of $f(x, y) = x^2 - y^2$, $(x, y) \in \mathbb{R}^2$.
   (d) Example: Find the partial derivatives at the point $(0, 0)$ of $f(x, y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$

      This is an example where both the partial derivatives exist, but the function is not even continuous at the point $(0, 0)$.
      This shows that even if both the partial derivatives of a function of two variables exist, we cannot call it "differentiable".

4. By analogy, we define the partial derivatives of real-valued functions of n variables. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a function. Suppose that $\mathbf{a} = (a_1, a_2, \dots, a_n) \in A$ and an ϵ-ball around the point $\mathbf{a}$, is also contained in A. The limit $\lim_{h\to 0} \frac{f(a_1,a_2,\dots,a_i+h,\dots,a_n)-f(a_1,a_2,\dots,a_i,\dots,a_n)}{h}$, if it exists, is called the partial derivative of f w.r.t. the i^{th} variable at $\mathbf{a}$ and is denoted by $\frac{\partial f}{\partial x_i}(\mathbf{a})$.

5. Example: Calculate the partial derivative of the function $f(x, y, z) = e^{x^2 y} + \frac{x+z}{y}$, $(x, y, z) \in \mathbb{R}^3, y \neq 0$ w.r.t second variable, at the point $(1, \pi, 1)$.

6. Rules for partial derivatives: Suppose $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and $g : A \subset \mathbb{R}^n \to \mathbb{R}$ are two functions whose partial derivatives w.r.t. the i^{th} variable exists at a point $\mathbf{a} \in A$. Then:
   (a) $\frac{\partial(f+g)}{\partial x_i}(\mathbf{a}) = \frac{\partial f}{\partial x_i}(\mathbf{a}) + \frac{\partial g}{\partial x_i}(\mathbf{a})$.
   (b) $\frac{\partial(fg)}{\partial x_i}(\mathbf{a}) = g(\mathbf{a})\frac{\partial f}{\partial x_i}(\mathbf{a}) + f(\mathbf{a})\frac{\partial g}{\partial x_i}(\mathbf{a})$ *i.e.*we say the Leibnitz's-rule for derivatives is satisfied by the product function fg.