PH 102: Physics II

Lecture 16 (Spring 2019)

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LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Lectures	Division
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	I, II (4-4:55 pm)
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	III, IV (9-9:55 am)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	I, II (3-3:55 pm)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	III, IV (10-10:55 am)
Tut 8	19-3-2019	Lec 15, 16			
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole	5.4	1	I, II (4-4:55 pm)
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole	5.4	1	III, IV (9-9:55 am)
Lec 18	22-3-2019	Magnetic Materials, Magnetization	6.1	1	I, II (3-3:55 pm)
Lec 18	22-3-2019	Magnetic Materials, Magnetization	6.1	1	III, IV (10-10:55 am)
Tut 9	26-3-2019	Lec 17, 18			
Lec 19	27-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	I, II (4-4:55 pm)
Lec 19	27-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	III, IV (9-9:55 am)
Lec 20	28-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	I, II (3-3:55 pm)
Lec 20	28-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	III, IV (10-10:55 am)
Tut 10	2-4-2019	Lec 19, 20			
Lec 21	3-4-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	I, II (4-4:55 pm)
Lec 21	3-4-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	III, IV (9-9:55 am)
Lec 22	4-4-2019	Maxwell's equations, Discussions, problem solving	7.3	1	I, II (3-3:55 pm)
Lec 22	4-4-2019	Maxwell's equations, Discussions, problem solving	7.3	1	III, IV (10-10:55 am)
Tut 11	9-4-2019	Quiz II			

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	I, II (4-4:55 pm)
Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	III, IV (9-9:55 am)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	I, II (3-3:55 pm)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	III, IV (10-10:55 am)
Tut 12	23-4-2019	Lec 23, 24			
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	I, II (4-4:55 pm)
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	III, IV (9-9:55 am)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	I, II (3-3:55 pm)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	III, IV (10-10:55 am)
Tut 13	30-4-2019	Lec 25, 26			
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	I, II (4-4:55 pm)
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	III, IV (9-9:55 am)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	I, II (3-3:55 pm)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	III, IV (10-10:55 am)

Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\mathrm{enc}}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

In the integral form of Ampere's law, the integration is done over a closed loop (of any arbitrary shape) which encloses the current (See Lecture 15)

Application of Ampere's Law

- The role of Ampere's law in the context of Biot-Savart law of magnetostatics is equivalent to that of Gauss's law in the context of Coulomb's law in electrostatics.

- For currents with appropriate symmetry (infinite straight lines, infinite planes, infinite solenoids, toroids), Ampere's law in integral form can be applied to simplify the calculation of magnetic field.

- See solved examples 5.7-5.10 (Introduction to Electrodynamics, D. J. Griffiths)

Example 5.7 (Introduction to Electrodynamics, D J Griffiths): Find the magnetic field a distance s from a long straight wire, carrying a steady current I.

Since the problem has a symmetry, it is obvious that the magnitude of magnetic field is constant around an amperian loop of radius s, centred on the wire. Using Ampere's law:

$$\oint \vec{B} \cdot d\vec{l} = B \oint dl = (2\pi s)B = \mu_0 I$$

$$\implies B = \frac{\mu_0 I}{2\pi s}$$

Same answer as Ex 5.5 (Lecture 1), but obtained in a much simpler way

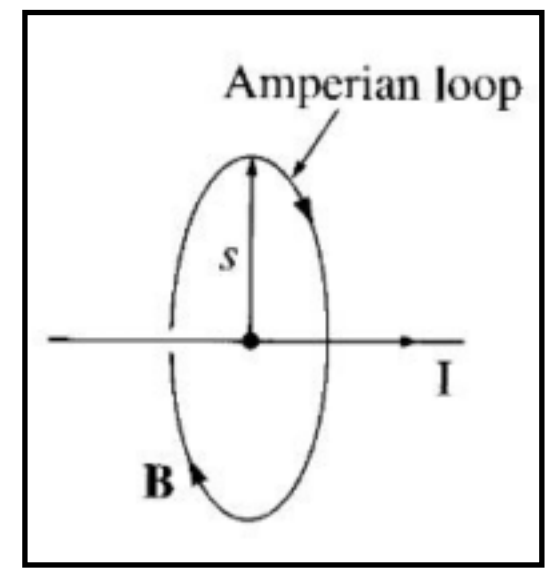


Fig. 5.32 (Introduction to Electrodynamics, D. J. Griffiths)

Magnetic field of an infinite surface current:

Surface current is given by: $\vec{K} = K\hat{x}$

By symmetry, $\vec{B}$ should be along $\pm\hat{y}$

For the Amperian loop:

$$\oint \vec{B} \cdot d\vec{l} = 2Bl = \mu_0 Kl$$

$$\vec{B} = \begin{cases} +\frac{\mu_0}{2}K\hat{y} & \text{for } z < 0, \\ -\frac{\mu_0}{2}K\hat{y} & \text{for } z > 0. \end{cases}$$

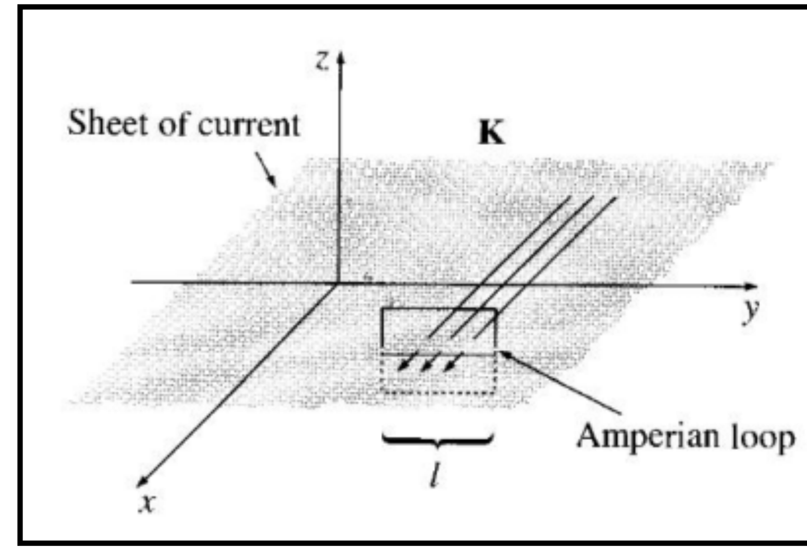
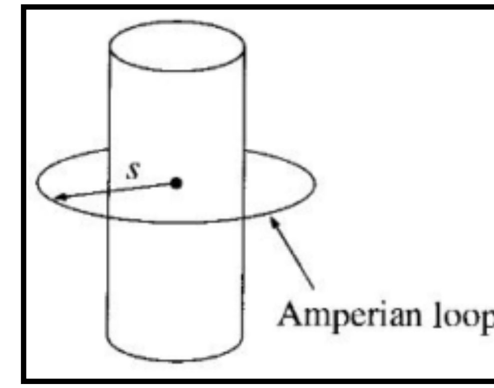


Fig. 5.33 (Introduction to Electrodynamics, D. J. Griffiths)

Why not along x, z?

Linear solenoid:



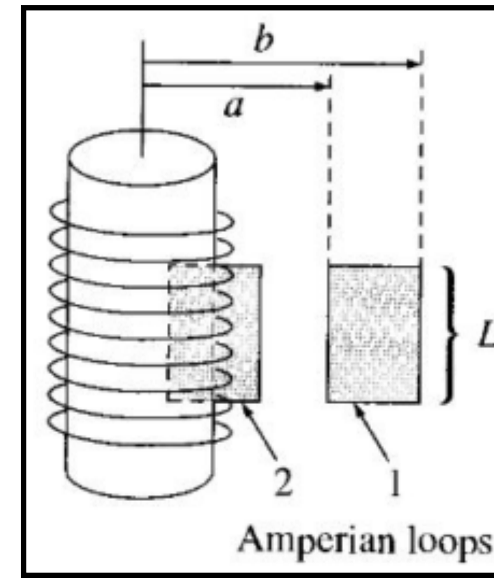


Fig. 5.36, 5.37 (Introduction to Electrodynamics, D. J. Griffiths)

For the circular loop, enclosed current = 0

$$\oint \vec{B} \cdot d\vec{l} = 0 = B_\phi (2\pi s) \quad \Longrightarrow \quad B_\phi = 0$$

Hence, for rectangular loop 1:

$$\oint \vec{B} \cdot d\vec{l} = [B(a) - B(b)]L = \mu_0 I = 0 \quad \Longrightarrow \quad B(a) = B(b)$$

Thus, the field outside is same everywhere.

Since field has to vanish at infinity it means $B_{\text{outside}} = 0$ (everywhere)

For rectangular loop 2: $\oint \vec{B} \cdot d\vec{l} = BL = \mu_0 (nL) I \Longrightarrow \vec{B} = \mu_0 n I \hat{z}$ (inside)

(Much easier than using Biot-Savart law, Problem 5.44, Lecture 15)

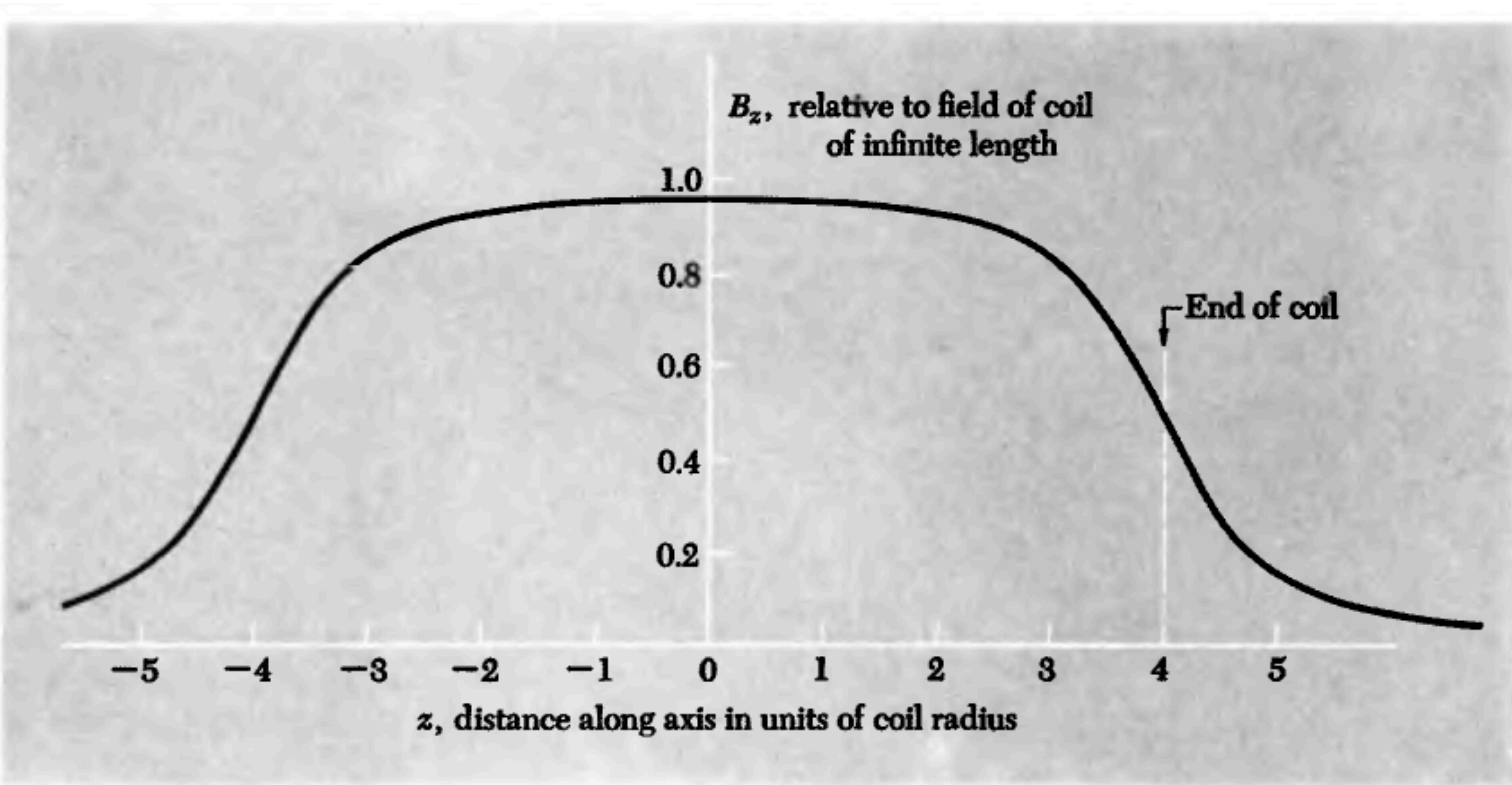
Linear Solenoid (Finite)

Exercise: Find the magnetic field at the axis of a finite solenoid of length L, Radius R, n number of turns per unit length with its centre coinciding with the origin.

Consider an elemental length of the solenoid and use the expression for magnetic field on the axis of a current carrying loop. Integrate over the length of the solenoid and show that the field at the axis is

$$B_z = \frac{\mu_0 n I}{2}\left[\frac{L/2 - z}{\sqrt{(z-L/2)^2 + R^2}} + \frac{L/2 + z}{\sqrt{(z+L/2)^2 + R^2}}\right]$$

Field of a Finite Linear Solenoid



Credit: E M Purcell

Toroidal Solenoid:

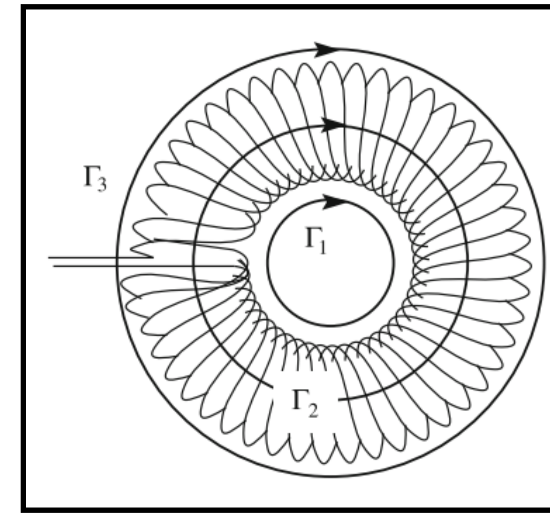


Image credit: Springer

Applying Ampere's law for circle Γ_1

$$\oint \vec{B} \cdot d\vec{l} = B(2\pi s) = 0 \implies B = 0 \qquad s < R_1$$

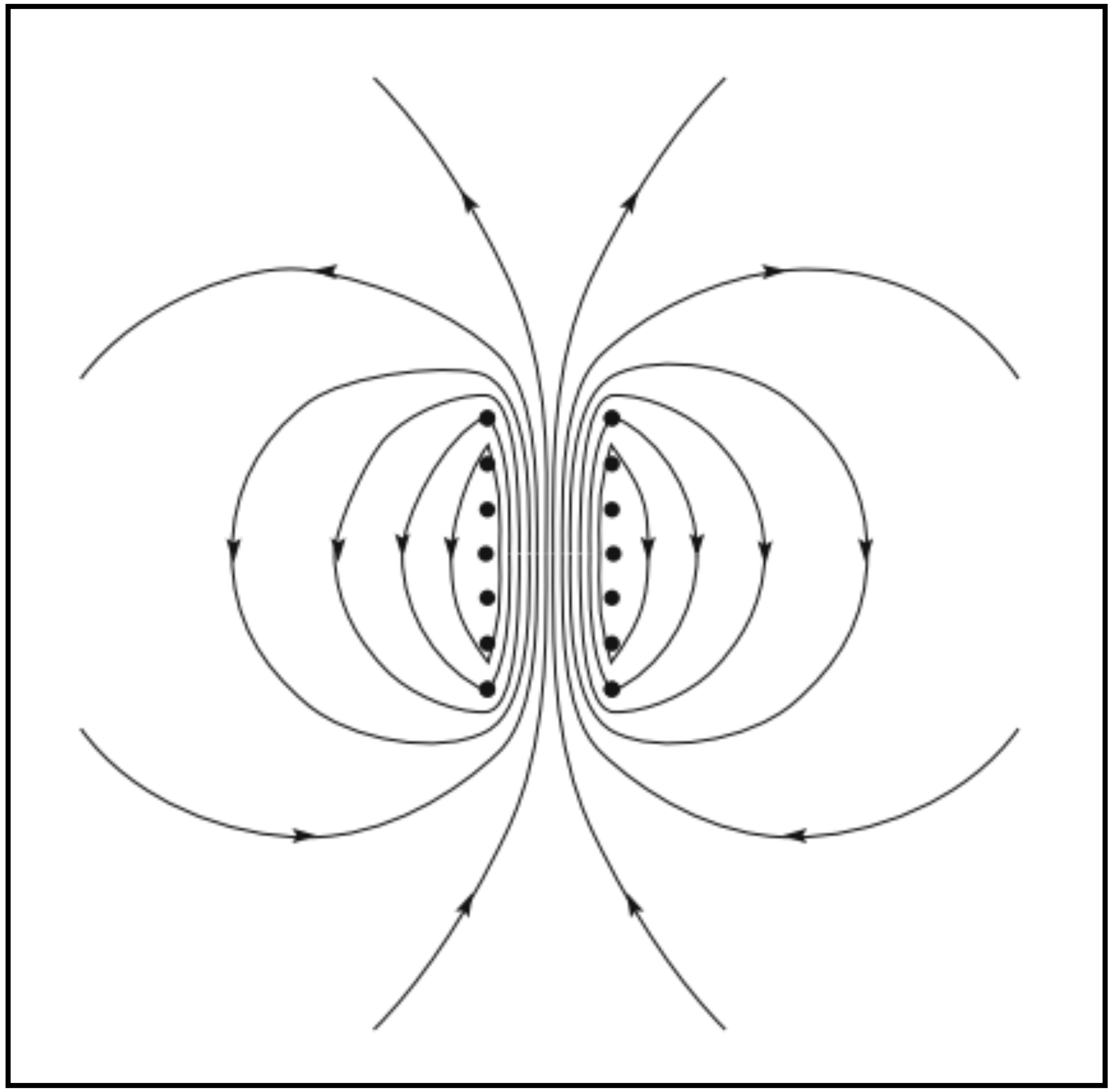
For circle Γ_3 also, the enclosed current is zero due to equal ad opposite currents and hence B=0 (outside).

For circle Γ_2 : $\oint \vec{B} \cdot d\vec{l} = B(2\pi s) = \mu_0 NI \implies B = \frac{\mu_0 NI}{2\pi s}, \quad \vec{B}(\vec{r}) = \frac{\mu_0 NI}{2\pi r}\hat{\phi}, \quad R_1 < r < R_2$

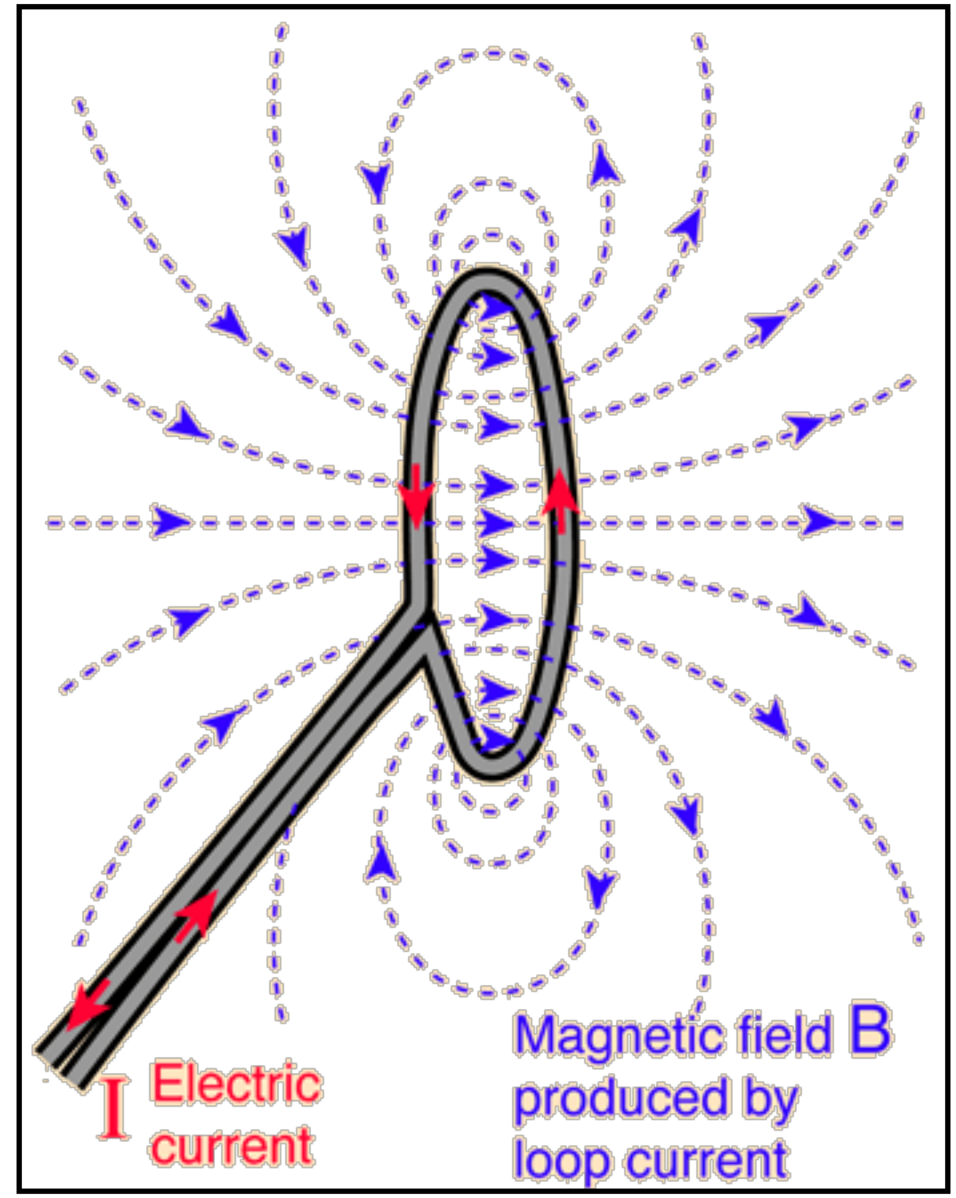
For $R_2 - R_1 \ll R_1$ one can assume the length of the solenoid to be $2\pi r$ so that $n = N/(2\pi r)$ is the no. of turns per unit length.

$$B = \mu_0 n I \qquad R_1 < r < R_2$$

See example 5.10 (Introduction to Electrodynamics, D J Griffiths): Show that the magnetic field of the toroid is circumferential!



Field lines for solenoid



Field lines for a current carrying loop

Image credit: Springer, hyperphysics

Exercise: Calculate the magnetic field inside and outside of an infinitely long hollow cylinder (of radius R) carrying uniform surface current $\vec{K} = K\hat{z}$ along the axis of the cylinder, compare it with the result for a linear solenoid having axis along the same direction.

Use Ampere's law and show that the field (which is in the circumferential direction) is given by:

$$B_\phi = \begin{cases} \frac{\mu_0 K R}{r}, & r > R \\ 0, & r < R \end{cases}$$

Exercise: Calculate magnetic field inside and outside of an infinitely long solid cylinder (of radius R) having uniform volume current $\vec{J} = J\hat{z}$

Use Ampere's law and show that the field (which is in the circumferential direction) is given by:

$$B_\phi = \begin{cases} \frac{\mu_0 J R^2}{2r}, & r > R \\ \frac{\mu_0 J r}{2}, & r < R \end{cases}$$

Magnetic Vector Potential

Electrostatics $\vec{\nabla} \times \vec{E} = 0 \implies \vec{E} = -\vec{\nabla} V$

Magnetostatics $\vec{\nabla} \cdot \vec{B} = 0 \implies \vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

Both electric and magnetic potentials have built-in ambiguities. For example,

$$V \to V + C, \vec{\nabla} C = 0; \vec{A} \to \vec{A} + \vec{\nabla} \lambda$$

Such redefinitions of the potentials do not change the fields.

One can always use this freedom to eliminate the divergence of magnetic vector potential (To simplify Ampere's law written above in terms of vector potential)!

Magnetic Vector Potential

- Let the original vector potential is not divergenceless.

$$\vec{A}_0 \to \vec{A} = \vec{A}_0 + \vec{\nabla}\lambda \quad \Longrightarrow \vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}_0 + \nabla^2 \lambda$$

- For the final vector potential to be divergenceless, we can choose the scalar function in such a way that

$$\nabla^2 \lambda = -\vec{\nabla} \cdot \vec{A}_0$$

- The above equation is similar to the Poisson's equation in electrostatics:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \Longrightarrow V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{\mathfrak{r}} d\tau'$$

(If $\rho \to 0$ as $r' \to \infty$)

Magnetic Vector Potential

- Similarly, if $\vec{\nabla} \cdot \vec{A}_0$ goes to zero at infinity, one can always find a scalar function as

$$\lambda = \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \vec{A}_0}{\mathfrak{r}} d\tau'$$

- Therefore, it is always possible to make the magnetic vector potential divergenceless. Coulomb Gauge!

- For such a case, the Ampere's law in terms of vector potential simply becomes similar to the Poisson's equation $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ (Using $\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$)

whose solution (assuming the current goes to zero at infinity) is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'})}{\mathfrak{r}} d\tau'$$

Advantages!

Though it is a vector, but it is still simpler in many cases, to find the vector potential than the magnetic field itself.

The vector potential, typically, is parallel to the direction of the given current.

The freedom in choosing vector potential $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Phi$ without affecting magnetic field simplifies many calculations in electrodynamics: *Gauge Symmetry!*

It also enables to understand the deep relations between electric and magnetic phenomena/fields. It also has tremendous use for time-varying fields as well as electromagnetic radiation.

Magnetic vector potential can also have observable consequences instead of just being a mathematical tool: *Aharanov-Bohm Effect!*

Magnetic Vector Potential

- For line and surface currents, the vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{\mathfrak{r}}dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathfrak{r}}d\vec{l'}; \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{\mathfrak{r}}da'$$

- Example 5.11 (Introduction to Electrodynamics, D. J. Griffiths): A spherical shell, or radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω .Find the vector potential it produces at point r.

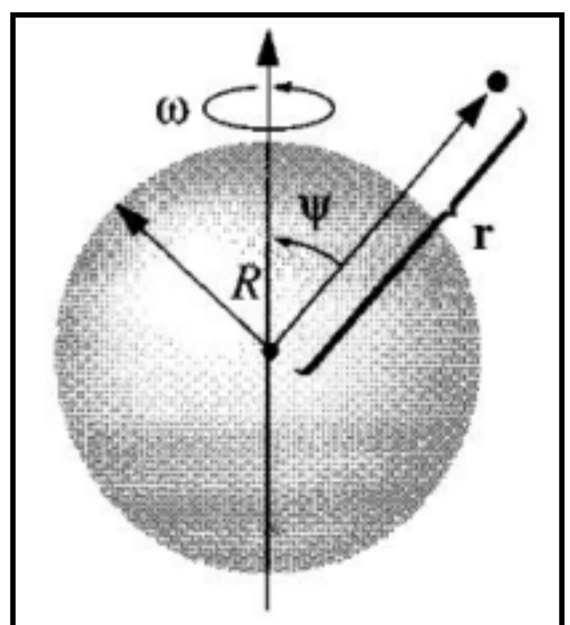


Figure 5.45, Introduction to Electrodynamics, D. J. Griffiths

The surface current for an elemental area of the spinning charged spherical shell is $\vec{K}(\vec{r'}) = \sigma\vec{v} = \sigma(\vec{\omega} \times \vec{r'})$

$$\vec{\omega} = \omega \sin\psi \hat{x} + \omega \cos\psi \hat{z}$$

$$\vec{\omega} \times \vec{r'} = R\omega[-(\cos\psi \sin\theta' \sin\phi')\hat{x} + (\cos\psi \sin\theta' \cos\phi' - \sin\psi \cos\theta')\hat{y} + (\sin\psi \sin\theta' \sin\phi')\hat{z}]$$

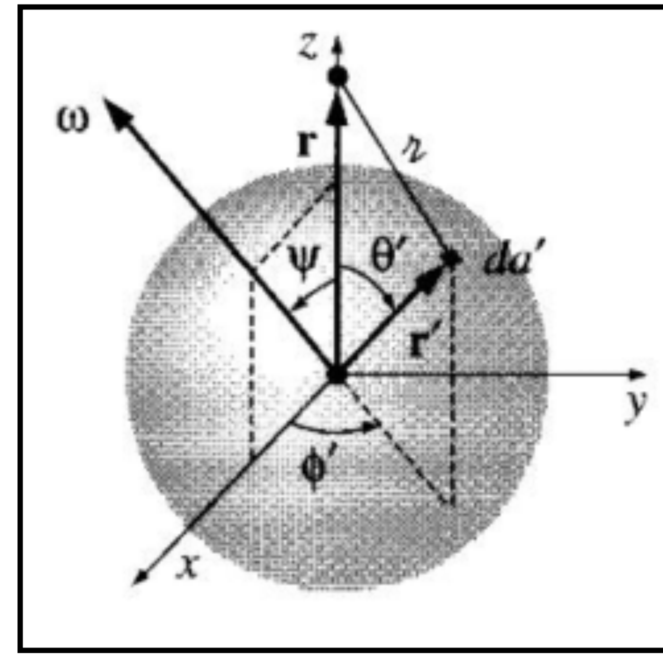


Figure 5.45, Introduction to Electrodynamics, D. J. Griffiths

Vector potential: $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r'})}{\mathfrak{r}} (R^2 \sin\theta' d\theta' d\phi')$

Ignoring the terms in $\vec{\omega} \times \vec{r'}$ which have $\sin\phi', \cos\phi'$ (as they identically vanish after integration), we get:

$$\vec{A}(\vec{r}) = -\frac{\mu_0 R^3 \sigma\omega \sin\psi}{2} \int_0^{\pi} \frac{\cos\theta' \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}} d\theta' \hat{y}$$

$$\mathfrak{r} = |\vec{r} - \vec{r'}| = \sqrt{R^2 + r^2 - 2Rr\cos\theta}$$

Using $u = \cos\theta'$, the above integral becomes

$$\int_{-1}^{+1} \frac{u du}{\sqrt{R^2 + r^2 - 2Rru}} = -\frac{(R^2 + r^2 + Rru)}{3R^2 r^2} \sqrt{R^2 + r^2 - 2Rru}\Big|_{-1}^{+1}$$

$$= -\frac{1}{3R^2 r^2}\left[(R^2 + r^2 + Rr)|R - r| - (R^2 + r^2 - Rr)(R + r)\right]$$

The integral has the following possible values:

$$-\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r|-(R^2+r^2-Rr)(R+r)\right]=\begin{cases}\frac{2r}{3R^2} & \text{for } r<R\\ \frac{2R}{3r^2} & \text{for } r>R\end{cases}$$

Also, using the fact that $-\omega r\sin\psi\hat{y}=(\vec{\omega}\times\vec{r})$, the vector potential is

$$\vec{A}(\vec{r})=\begin{cases}\frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}) & \text{for } r<R\\ \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}) & \text{for } r>R\end{cases}$$

If the shell spins about the z axis then,

$$\vec{A}(r,\theta,\phi)=\begin{cases}\frac{\mu_0 R\sigma\omega}{3}r\sin\theta\hat{\phi} & \text{for } r<R\\ \frac{\mu_0 R^4\sigma\omega}{3r^3}r\sin\theta\hat{\phi} & \text{for } r>R\end{cases}$$

Calculate the magnetic field using this potential!

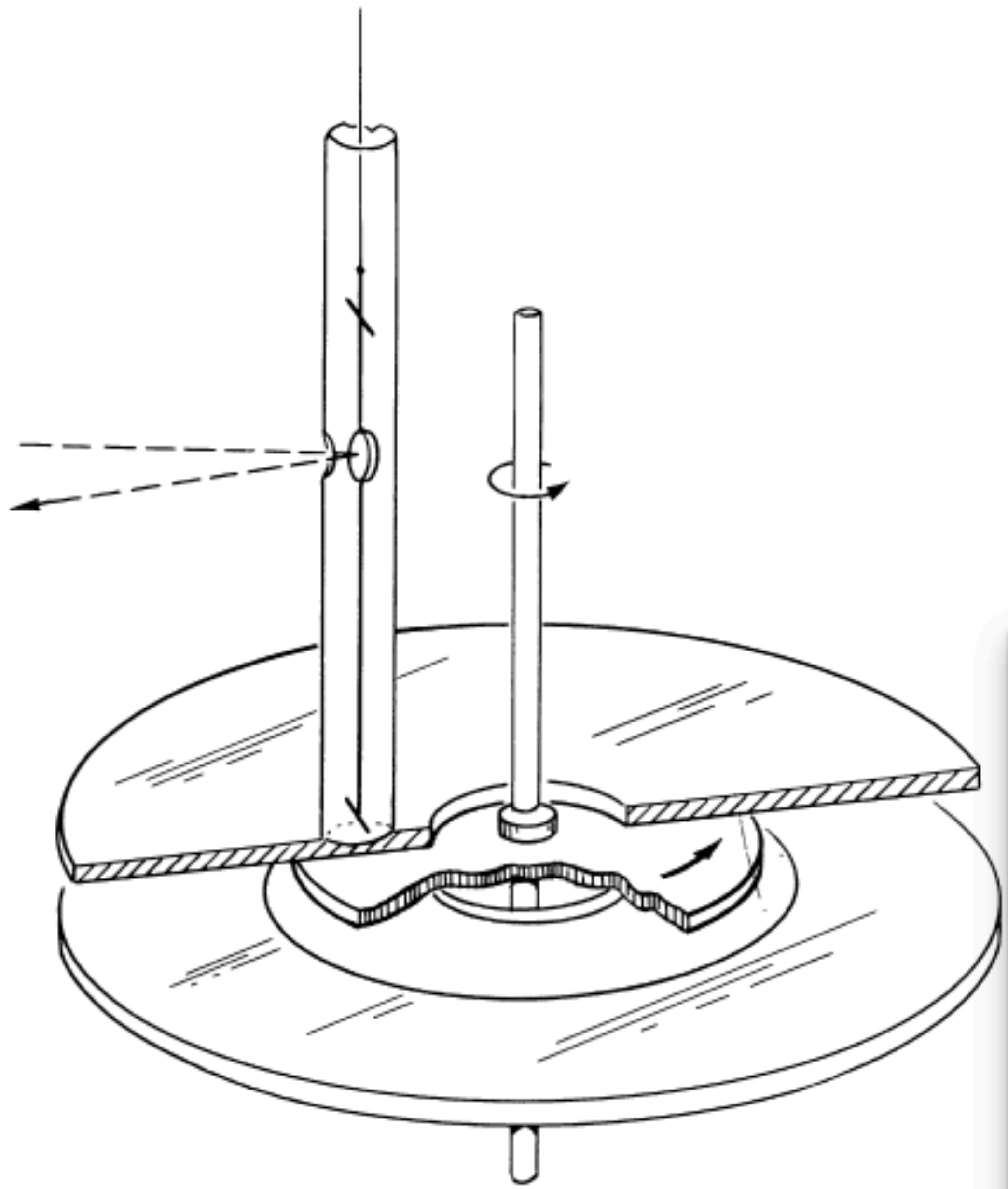
Tutorial 8

- This exercise indicates that a current flowing in a wire and a moving electrically charged object are essentially alike as sources of magnetic field (a fact which was not obvious in 19th century).

- Maxwell, through his unified theory of electricity of magnetism suggested this, but no experimental proof was available at that time.

- Henry Rowland devised an experiment to measure magnetic field of a rotating charged disk (that produces a field five order of magnitudes smaller than earth's magnetic field)

Rowland's apparatus



Credit: E M Purcell

ON THE MAGNETIC EFFECT OF ELECTRIC CONVECTION[1]

[*American Journal of Science* [3], *XV*, 30-38, 1878]

The experiments described in this paper were made with a view of determining whether or not an electrified body in motion produces magnetic effects. There seems to be no theoretical ground upon which we can settle the question, seeing that the magnetic action of a conducted electric current may be ascribed to some mutual action between the conductor and the current. Hence an experiment is of value. Professor Maxwell, in his 'Treatise on Electricity,' Art. 770, has computed the magnetic action of a moving electrified surface, but that the action exists has not yet been proved experimentally or theoretically.

The apparatus employed consisted of a vulcanite disc 21·1 centimetres in diameter and ·5 centimetre thick which could be made to revolve around a vertical axis with a velocity of 61· turns per second. On either side of the disc at a distance of ·6 cm. were fixed glass plates having a diameter of 38·9 cm. and a hole in the centre of 7·8 cm. The vulcanite disc was gilded on both sides and the glass plates had an annular ring of gilt on one side, the outside and inside diameters being 24·0 cm. and 8·9 cm. respectively. The gilt sides could be turned toward or from the revolving disc but were usually turned toward it so that the problem might be calculated more readily and there should be no uncertainty as to the electrification. The outside plates were usually connected with the earth; and the inside disc with an electric battery, by means of a point which approached within one-third of a millimetre of the edge and turned toward it. As the edge was broad, the point would not discharge unless there was a difference of potential between it and the edge. Between the electric battery and the disc,

[1] The experiments described were made in the laboratory of the Berlin University through the kindness of Professor Helmholtz, to whose advice they are greatly indebted for their completeness. The idea of the experiment first occurred to me in 1868 and was recorded in a note book of that date.

Magnetic Vector Potential

- Example 5.12 (Introduction to Electrodynamics, D. J. Griffiths): Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

- Since the current extends to infinity*, the simple expressions mentioned in the last two slides are no longer applicable. $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I}}{\mathfrak{r}} dl'$

- Typically, the direction of vector potential will match the direction of the current.

*Note that use of vector potential to find magnetic field can be complementary to using Ampere's law. For finite (asymmetric) current configurations, where Ampere's law may not be useful, the simple formula for vector potential may help.

Example 5.12: Although the formula for vector potential can not be used, we can find its line integral around a closed amperian loop around the axis of the solenoid.

$$\oint \vec{A} \cdot d\vec{l} = A(2\pi s)$$

A and I are typically in same direction i.e. circumferential

We know that: $$\oint \vec{A} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a}$$

Using the known values of B: $$\vec{B}_{\text{out}} = 0, \vec{B}_{\text{in}} = \mu_0 n I \hat{z}$$

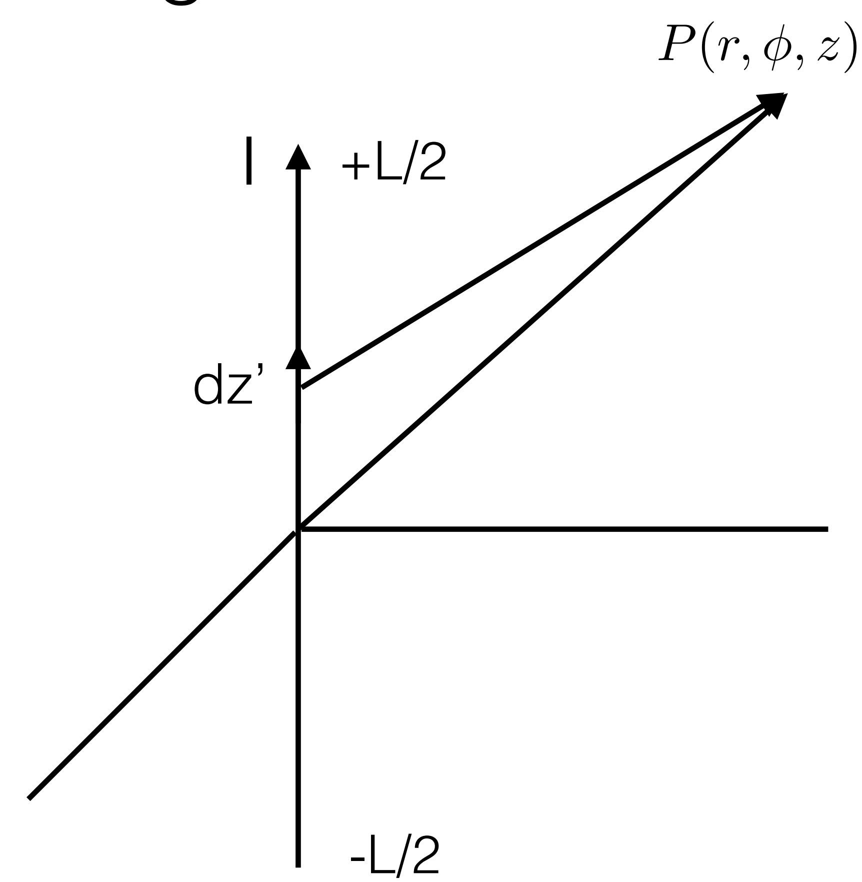
We can find A to be:

$$\vec{A}(s) = \begin{cases} \frac{\mu_0 n I(\pi s^2)}{2\pi s}\hat{\phi} = \frac{\mu_0 n I}{2} s\hat{\phi} \quad \text{for} \quad s < R \\ \frac{\mu_0 n I(\pi R^2)}{2\pi s}\hat{\phi} = \frac{\mu_0 n I}{2}\frac{R^2}{s}\hat{\phi} \quad \text{for} \quad s > R \end{cases}$$

Take curl of $\vec{A}$ & verify the known results for $\vec{B}$ in case of solenoid

Calculate magnetic vector potential for a current (I) carrying wire of length L. Using the answer, find the corresponding magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Verify your answer with the expression for field obtained using Biot-Savart law (Lecture 15).

Assuming current to be in z direction, it is straightforward to show that A is also in z direction.

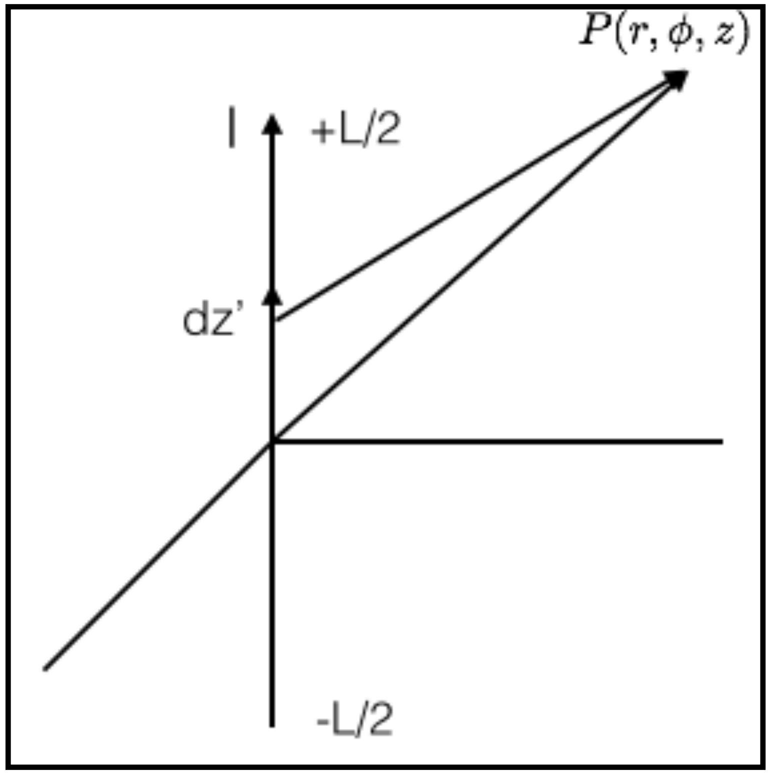


$$
\begin{aligned}
A_z &= \frac{\mu_0 I}{4\pi} \int_{-L/2}^{L/2} \frac{dz'}{\mathfrak{r}} \\
&= \frac{\mu_0 I}{4\pi} \int_{-L/2}^{L/2} \frac{dz'}{[(z - z')^2 + r^2]^{1/2}} \\
&= \frac{\mu_0 I}{4\pi} \left(\sinh^{-1}\left(\frac{-(z - L/2)}{r} \right) + \sinh^{-1}\left(\frac{z + L/2}{r} \right) \right)
\end{aligned}
$$

One can now find the magnetic field using the curl of vector potential

$$
\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
&= \frac{1}{r}\left[\left(\frac{\partial A_z}{\partial \phi} - \frac{\partial (r A_\phi)}{\partial z} \right) \hat{e_r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) r\hat{e_\phi} + \left(\frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right) \hat{e_z} \right] \\
&= -\frac{\partial A_z}{\partial r}\hat{e_\phi} \\
&= \frac{\mu_0 I}{4\pi r}\left[\frac{z + L/2}{[r^2 + (z + L/2)^2]^{1/2}} - \frac{z - L/2}{[r^2 + (z - L/2)^2]^{1/2}} \right] \hat{e_\phi} \\
&= \frac{\mu_0 I}{4\pi r}(\sin\theta_2 - \sin\theta_1)\hat{e_\phi}
\end{aligned}
$$

Which is same as the result obtained in Lecture 15!



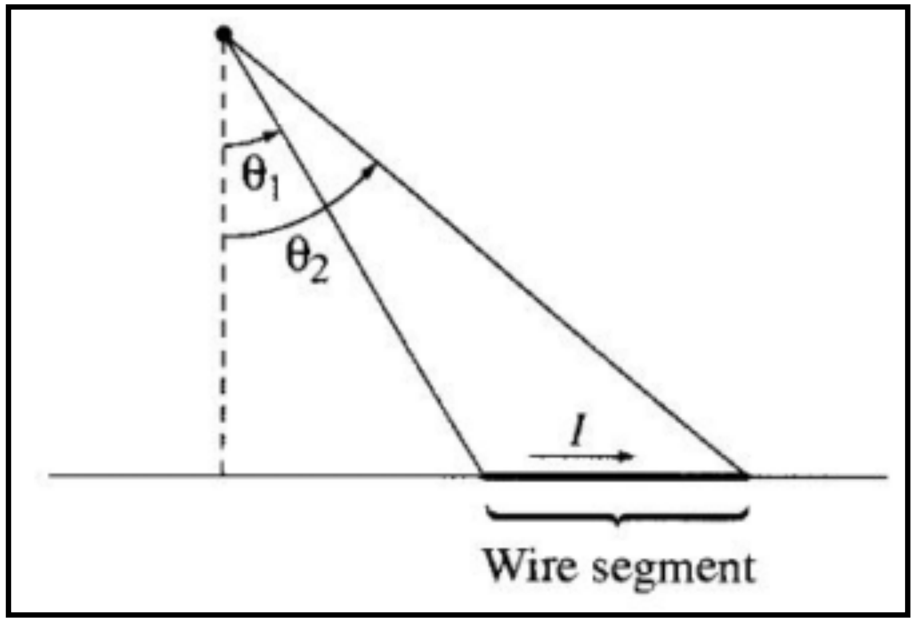


Fig. 5.19 (Introduction to Electrodynamics, D. J. Griffiths)

Exercise: Calculate the magnetic vector potential for a surface current $\vec{K} = K\hat{z}$ of finite width w. Use the result to find the corresponding magnetic field. Hint: Use the results for a wire obtained in the previous example.

Exercise: Find a suitable vector potential for an infinitely long current (I) carrying wire. Once again, the simple expression for A proportional to current can not be used. Compare the results with the scalar potential for an infinite line charge.

To find the current density if the vector potential is given

Problem 5.23 (Introduction to Electrodynamics, D J Griffiths): What current density would produce the vector potential $\vec{A} = k\hat{\phi}$ in cylindrical coordinates?

Since $\vec{J} = (\vec{\nabla} \times \vec{B})/\mu_0$, we first find the magnetic field:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{s}\left[-\frac{\partial A_\phi}{\partial z}\hat{s} + \frac{\partial (sA_\phi)}{\partial s}\hat{z}\right] = \frac{1}{s}\frac{\partial (sk)}{\partial s}\hat{z} = \frac{k}{s}\hat{z}$$

Therefore,

$$\vec{J} = \frac{1}{\mu_0}\frac{1}{s}\left(-s\hat{\phi}\frac{\partial (k/s)}{\partial s}\right) = \frac{k}{\mu_0 s^2}\hat{\phi}$$

Summary: Magnetostatics

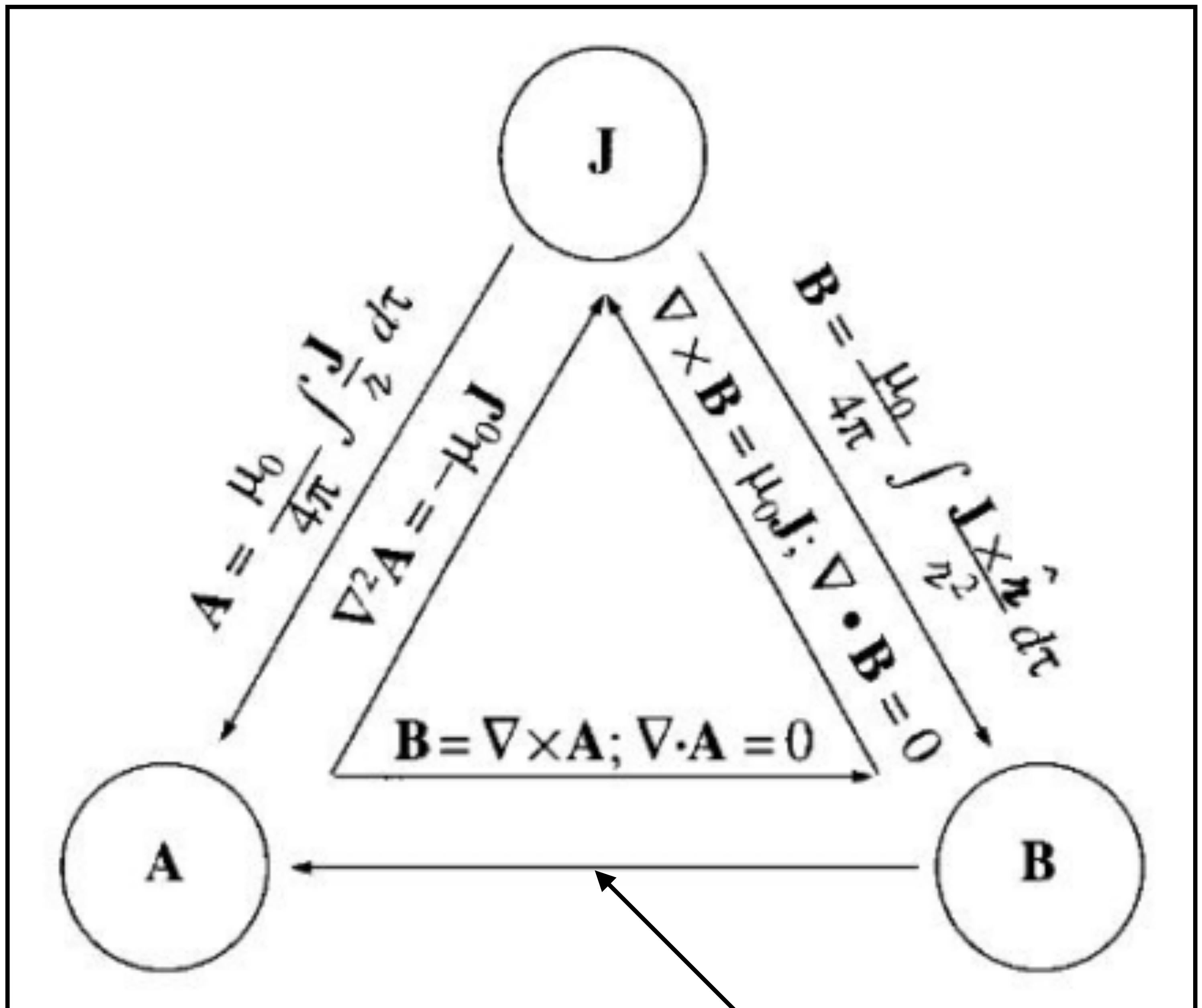


Figure 5.48, Introduction to Electrodynamics, D. J. Griffiths

Missing Link: No equation for A in terms of B

Magnetostatic Boundary Conditions

- Just like electric field suffers a discontinuity at a surface charge, so the magnetic field is discontinuous at a surface current.

- Using the integral form of $\vec{\nabla} \cdot \vec{B} = 0$ that is,

$$\oint \vec{B} \cdot d\vec{a} = 0$$

to a thin pillbox straddling the surface, we get

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

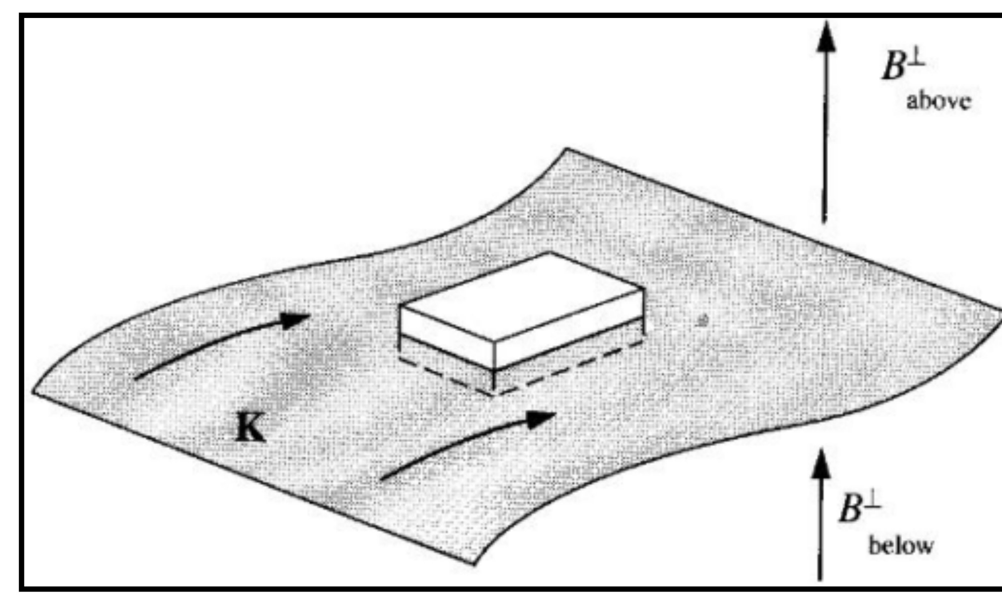


Figure 5.49, Introduction to Electrodynamics, D. J. Griffiths

Magnetostatic Boundary Conditions

- The boundary conditions for tangential components can be found by taking an Amperian loop running perpendicular to the current which gives

$$\oint \vec{B} \cdot d\vec{l} = (B^{\|}_{\mathrm{above}} - B^{\|}_{\mathrm{below}})l = \mu_0 I_{\mathrm{enc}} = \mu_0 K l$$

$$\implies B^{\|}_{\mathrm{above}} - B^{\|}_{\mathrm{below}} = \mu_0 K$$

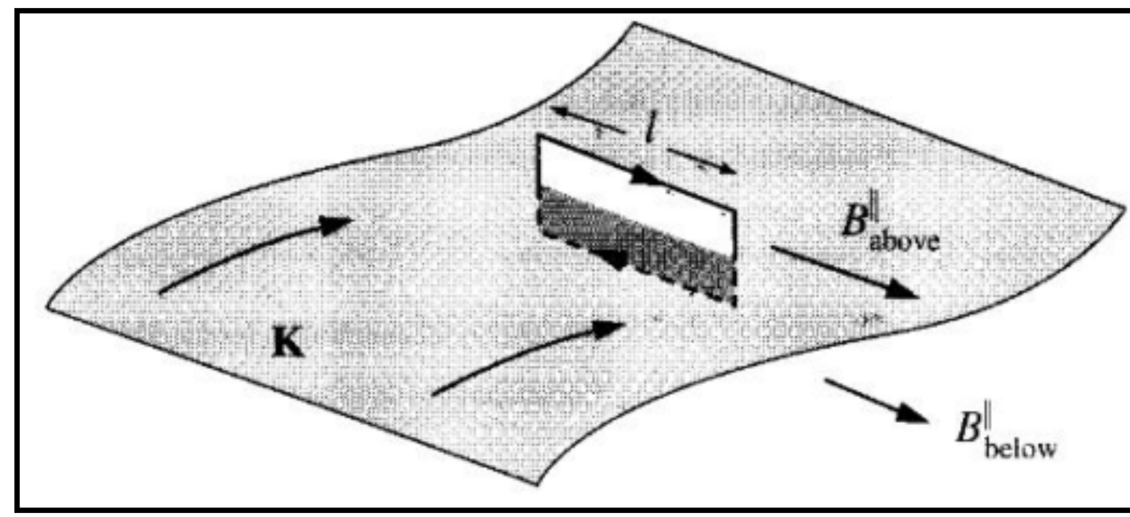


Figure 5.50, Introduction to Electrodynamics, D. J. Griffiths

- In general,

$$\vec{B}_{\mathrm{above}} - \vec{B}_{\mathrm{below}} = \mu_0(\vec{K} \times \hat{n})$$

where $\hat{n}$ is a unit vector perpendicular to the surface, pointing upward

Magnetostatic Boundary Conditions

- Magnetic vector potential is continuous across any boundary.

- Continuity of normal components is guaranteed by

$$\vec{\nabla} \cdot \vec{A} = 0 \implies \oint \vec{A} \cdot d\vec{a} = 0$$

- For tangential components, we can calculate

$$\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a} = \Phi$$

which is zero for an Amperian loop of vanishing thickness. Thus, tangential components are continuos.

- The derivative of vector potential however, is discontinuous

$$\frac{\partial \vec{A}_{\text{above}}}{\partial n} - \frac{\partial \vec{A}_{\text{below}}}{\partial n} = -\mu_0 \vec{K}$$

Since A is continuous across the boundary we have, at all points on the surface: $\vec{A}_{\text{above}} = \vec{A}_{\text{below}}$

If the boundary is the x-y plane, the above condition means

$\dfrac{\partial A}{\partial x}, \dfrac{\partial A}{\partial y}$ are same above and below.

Only normal derivatives can be discontinuous

Why?

From the boundary condition on magnetic field:

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0(\vec{K} \times \hat{n})$$

The parallel components of B are $\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y}$

Using the continuity of x,y derivatives, we get:

$$\left(-\frac{\partial A_{y\text{above}}}{\partial z} + \frac{\partial A_{y\text{below}}}{\partial z}\right)\hat{x} + \left(\frac{\partial A_{x\text{above}}}{\partial z} - \frac{\partial A_{x\text{below}}}{\partial z}\right)\hat{y} = \mu_0(\vec{K} \times \hat{n})$$

Considering the surface current to be in x direction, the right hand side of the previous relation is $-\mu_0 K\hat{y}$

Equating x and y components on both sides:

$$\left(-\frac{\partial A_{y\text{above}}}{\partial z}+\frac{\partial A_{y\text{below}}}{\partial z}\right)=0,\ \left(\frac{\partial A_{x\text{above}}}{\partial z}-\frac{\partial A_{x\text{below}}}{\partial z}\right)=-\mu_0 K$$

Therefore, in general

$$\frac{\partial \vec{A}_{\text{above}}}{\partial n}-\frac{\partial \vec{A}_{\text{below}}}{\partial n}=-\mu_0\vec{K}$$