1.

$$P\left(X_1 < X_2 \mid \min(X_1, X_2) = t\right)$$

$$= \frac{P\left(X_1 < X_2, \min(X_1, X_2) = t\right)}{P\left(\min(X_1, X_2) = t\right)}$$

$$= \frac{P\left(X_1 = t, X_2 > t\right)}{P\left(X_1 = t, X_2 > t\right) + P\left(X_2 = t, X_1 > t\right)}$$

$$= \frac{f_1(t)\left(1 - F_2(t)\right)}{f_1(t)\left(1 - F_2(t)\right) + f_2(t)\left(1 - F_1(t)\right)}$$

$$= \frac{r_1(t)}{r_1(t) + r_2(t)}.$$

2a.

$$E\left[MX \mid M = X\right] = E\left[X^2 \mid M = X\right] = \frac{E\left[X^2 1_{\{M = X\}}\right]}{P(M = X)}$$

$$= \frac{\lambda\mu \int_0^\infty \int_x^\infty x^2 e^{-\lambda x} e^{-\mu y} \, dy \, dx}{\frac{\lambda}{\lambda + \mu}} = \frac{\lambda \int_0^\infty x^2 e^{-(\lambda+\mu)x} \, dx}{\frac{\lambda}{\lambda + \mu}}$$

$$= (\lambda+\mu) \int_0^\infty x^2 e^{-(\lambda+\mu)x} \, dx = \frac{2}{(\lambda+\mu)^2}.$$

b) $E[MX \mid M=Y] = E[XY \mid M=Y] = \dfrac{E[XY 1_{\{M=Y\}}]}{P(M=Y)}$

$$= \frac{\lambda\mu \int_0^\infty \int_0^x xy\, e^{-\lambda x} e^{-\mu y}\, dy\, dx}{\frac{\mu}{\lambda+\mu}}$$

$$= \frac{\lambda\mu \int_0^\infty x \left\{ \frac{1-e^{-\mu x}}{\mu^2} - \frac{x e^{-\mu x}}{\mu} \right\} e^{-\lambda x}\, dx}{\frac{\mu}{\lambda+\mu}}$$

$$= \frac{\lambda \left[\frac{1}{\mu} \int_0^\infty x e^{-\lambda x}\, dx - \frac{1}{\mu} \int_0^\infty x e^{-(\mu+\lambda)x}\, dx - \int_0^\infty x^2 e^{-(\mu+\lambda)x}\, dx \right]}{\frac{\mu}{\lambda+\mu}}$$

$$= \frac{\lambda(\lambda+\mu)}{\mu} \left[\frac{1}{\mu\lambda^2} - \frac{1}{\mu(\mu+\lambda)^2} - \frac{2}{(\mu+\lambda)^3} \right]$$

$$= \frac{\lambda+\mu}{\mu^2\lambda} - \frac{\lambda}{\mu^2(\mu+\lambda)} - \frac{2\lambda}{\mu(\lambda+\mu)^2}.$$

$$= \frac{(\mu+\lambda)^3 - \lambda^2(\mu+\lambda) - 2\lambda^2\mu}{\mu^2\lambda(\mu+\lambda)^2}$$

$$= \frac{\mu^3 + \cancel{\lambda^3} + 3\cancel{\lambda^2\mu} + 3\mu\lambda^2 - \cancel{\lambda^2\mu} - \cancel{\lambda^3} - 2\cancel{\lambda^2\mu}}{\mu^2\lambda(\mu+\lambda)^2}$$

$$= \frac{\mu^2 + 3\lambda^2}{\mu\lambda(\mu+\lambda)^2}.$$

b) $E[MX \mid M=Y] = E[XY \mid M=Y] = \dfrac{E[XY \mathbb{1}_{\{M=Y\}}]}{P(M=Y)}$

$$= \frac{\lambda\mu \int_0^\infty \int_y^\infty xy\, e^{-\lambda x} e^{-\mu y}\, dx\, dy}{\frac{\mu}{\lambda+\mu}} = \frac{\mu \int_0^\infty y e^{-\mu y}\left[y e^{-\lambda y} + \frac{e^{-\lambda y}}{\lambda}\right] dy}{\frac{\mu}{\lambda+\mu}}$$

$$= (\lambda+\mu) \int_0^\infty y^2 e^{-(\lambda+\mu) y}\, dy + \frac{(\lambda+\mu)}{\lambda} \int_0^\infty y e^{-(\lambda+\mu) y}\, dy$$

$$= \frac{2}{(\lambda+\mu)^2} + \frac{1}{\lambda(\lambda+\mu)}.$$

3. Let us define

T_1 = My waiting time to enter the server 1

= The residual time of service at server 1 to previous customer after I join the queue.

T_2 = My service time at server 1.

T_3 = My waiting time at server 1 befor server 2 is done with previous customer

= The residual time of service at server 2 to the previous customer after server 1 is done with my service

T_4 = My service time at server 2.

Total time that I spend in the system is $T = T_1 + T_2 + T_3 + T_4$.

where $T_1 \sim Exp(\mu_1)$, $T_2 \sim Exp(\mu_1)$, $T_4 \sim Exp(\mu_2)$

$$E(T_3) = E(T_3 \mid T_2 < T_5)\, P(T_2 < T_5) + E(T_3 \mid T_5 < T_2)\, P(T_5 < T_2).$$

where T_5 = service time of the previous customers at server 2.

$\Rightarrow T_5 \sim Exp(\mu_2)$ and T_2 & T_5 are independent.

$\Rightarrow P(T_2 < T_5) = \dfrac{\mu_1}{\mu_1 + \mu_2}$.

Also $T_3 \mid T_2 < T_5 \sim Exp(\mu_2)$ and under $T_5 < T_2$, $T_3 = 0$

Hence $E(T_3) = \dfrac{\mu_1}{\mu_2(\mu_1 + \mu_2)}$.

$$\Rightarrow E(T) = E(T_1) + E(T_2) + E(T_3) + E(T_4)$$
$$= \frac{2}{\mu_1} + \frac{\mu_1}{\mu_2(\mu_1+\mu_2)} + \frac{1}{\mu_2}. \qquad \square$$

4. Easy to see that $N(t)$ is a counting process with $N(0) = 0$.

~~Since~~ $\cancel{N(t-s) = N_1(t-s) + N_2(t-s) = 1}$
~~Stationary and~~

Since $N(t) - N(s) = N_1(t) - N_1(s) + N_2(t) - N_2(s)$

Stationary and independent increments follow from the stationary and independent increments of N_1 & N_2 and their independence.

$N(t) = N_1(t) + N_2(t) \sim Poi(\lambda_1 + \lambda_2)t$.
since sum of two independent Poisson is again Poisson with parameters getting added.

5. Let

$$X = N(1)$$
$$Y = N(2) - N(1)$$
$$Z = N(4) - N(2).$$

Thus X, Y, Z are independent.

Then we want,

$$P(X+Y=2,\ Y+Z=3)$$

$$= P(X+Y=2, Y+Z=3 \mid Y=0)P(Y=0) + P(X+Y=2, Y+Z=3 \mid Y=1)P(Y=1) + P(X+Y=2, Y+Z=3 \mid Y=2)P(Y=2)$$

$$= P(X=2)P(Y=0)P(Z=3) + P(X=1)P(Y=1)P(Z=2) + P(X=0)P(Y=2)P(Z=1)$$

$$= \frac{e^{-4\lambda}\lambda^2(2\lambda)^3}{2!\,3!} + \frac{e^{-4\lambda}\lambda^2(2\lambda)^2}{2!} + \frac{e^{-4\lambda}\lambda^2\,2\lambda}{2!}.$$

6. By prob. 4, $N(t)$ is a Poisson process with rate 3.

a) $P(N(1)=2, N(2)=5) = P(N(1)=2, N(2)-N(1)=3)$

$$= P(N(1)=2)P(N(1)=3) = \frac{e^{-3}3^2}{2!}\,\frac{e^{-3}3^3}{3!}.$$

b) $P(N_1(1)=1 \mid N(1)=2)$

$$= \frac{P(N_1(1)=1,\ N_2(1)=1)}{P(N(1)=2)} = \frac{e^{-1}\, e^{-2}\ 2\times 2}{e^{-3}\ 3^2} = \frac{4}{9}.$$

7. Note that every arrival is an arrival from process 1 with probability $\frac{1}{3}$ and an arrival from process 2 with probability $\frac{2}{3}$.

[If X & Y are two iid exponential RVs with parameters λ & μ respectively then $P(X<Y)=\frac{\lambda}{\lambda+\mu}$ and $P(Y<X)=\frac{\mu}{\lambda+\mu}$]

Thus the required probability is

P(of two arrivals from the 1st process before the third arrival of the 2nd process)

$= P$(of atleast two arrivals from the 1st process among the first 4 arrivals)

$$= \sum_{k=2}^{4} \binom{4}{k} \left(\frac{1}{3}\right)^{k} \left(\frac{2}{3}\right)^{4-k}.$$

8.
a) Given that there has been two arrivals in the first hour, the arrival times are distributed according to the order statistics of two independent $U(0,1)$ random variables.

Thus P(both arrived during the first 20 minutes)

$= P(\max \text{ of two i.i.d. } U(0,1) \leq \frac{1}{3})$

$= \left(\frac{1}{3}\right)^2 = \frac{1}{9}$.

b) P(at least one arrived during first 20 minutes)

$= 1 - P$(no arrival in first 20 minutes)

$= 1 - P(\text{min. of two i.i.d. } U(0,1) > \frac{1}{3})$

$= 1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9}$.

$\boxed{9.}$ Let us define the RVs

X = Failure time of the machine.

T = Additional time until repair after failure of the machine.

We need to find $E(X+T) = E(X) + E(T)$

$X \sim Exp(\mu)$. $T \sim Exp(\lambda)$, as T can be considered as the first arrival time of a Poisson process with rate λ that starts at the time of failure.

Therefore, the required expectation is $\frac{1}{\mu} + \frac{1}{\lambda}$.

Alt solⁿ:

Let $N(t)$ be the number of inspection by the time t.
Then the time of the 1^{st} replacement is

$$T = \sum_{i=1}^{N(x)+1} T_i,$$

where T_i's are the inter arrival times of $N(t)$.

$\Rightarrow T_i \overset{i.i.d}{\sim} Exp(\lambda)$.

$$\text{Now, } E(T) = E\left(\sum_{i=1}^{N(x)+1} T_i\right) = E\left[E\left(\sum_{i=1}^{N(x)+1} T_i \,\Big|\, N(x)\right)\right]$$

$$= E\left(\frac{1}{\lambda}(N(x)+1)\right) = \frac{1}{\lambda} E(N(x)) + \frac{1}{\lambda}.$$

$$= \frac{1}{\lambda} E\left[E(N(x) \mid x)\right] + \frac{1}{\lambda}$$

$$= \frac{1}{\lambda} E(\lambda x) + \frac{1}{\lambda}$$

$$= \frac{1}{\mu} + \frac{1}{\lambda}$$

$\square$.

(10) $$E\left[e^{-T} X(T)\right] = \mu \int_0^\infty e^{-t} E\, X(t)\, e^{-\mu t}\, dt$$

$$= \mu \int_0^\infty e^{-(\mu+1)t} \lambda t (EY_1)\, dt$$

$$= \lambda\mu (EY_1) \int_0^\infty t\, e^{-(\mu+1)t}\, dt$$

$$= \frac{\lambda\mu}{(\mu+1)^2}(EY_1).$$