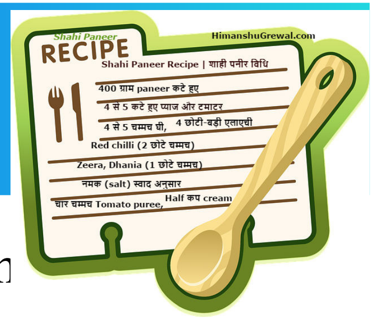
PH101

Lecture 7

More examples on Lagrange's equation;
generalized momentum,
Cyclic coordinates
Conservation of Momentum.

Let's recall: The recipe of Lagrangian!



I. (a) Recognize, & obtain the **constraint relations**, (b) determine th **DOF**, and (c) choose appropriate **generalized coordinates**!

II. Write down the **total** kinetic energy T and potential energy V of the **whole system** in terms of the **Cartesian** coordinates, **to begin with**!

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + z_i^2) \qquad \& \qquad V = V(x_i, y_i, z_i) \qquad | \, i = 1, N$$

III. Obtain appropriate transformation equations (Cartesian -→ generalized coordinates) using constraint relations:

$$x_i = x_i\,(q_1, \ldots, q_n,\; t)$$
$$y_i = y_i\,(q_1, \ldots, q_n,\; t)$$
$$z_i = z_i\,(q_1, \ldots, q_n,\; t)$$

IV. Convert T and V from Cartesian to suitable **generalized -coordinates** ($\boldsymbol{q_j}$) and **generalized velocities** ($\dot{\boldsymbol{q}}_{\boldsymbol{j}}$) to write L as,

$$\boldsymbol{L(q_j, \dot{q}_j, t) = T(q_j, \dot{q}_j, t) - \mathrm{V}(q_j)} \qquad j = 1, \mathrm{n}$$

V. Now Apply E-L equations:

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \qquad \boldsymbol{for\ each\ \mathrm{j} = 1, \mathrm{n}!}$$

Lagrange's equations (constraint-free motion)

Before going further let's see the Lagrange's equations recover Newton's 2nd Law, if there are NO constraints!

Let a particle of mass, m, in 3-D motion under a potential, $V(x, y, z)$

If No constraints, then its, DOF=3; Generalized coordinates: (x, y, z)

Now,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

Corresponding E-L equations are,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

3-such!

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = m\dot{x} = p_x$$ - the x-component of linear momentum!

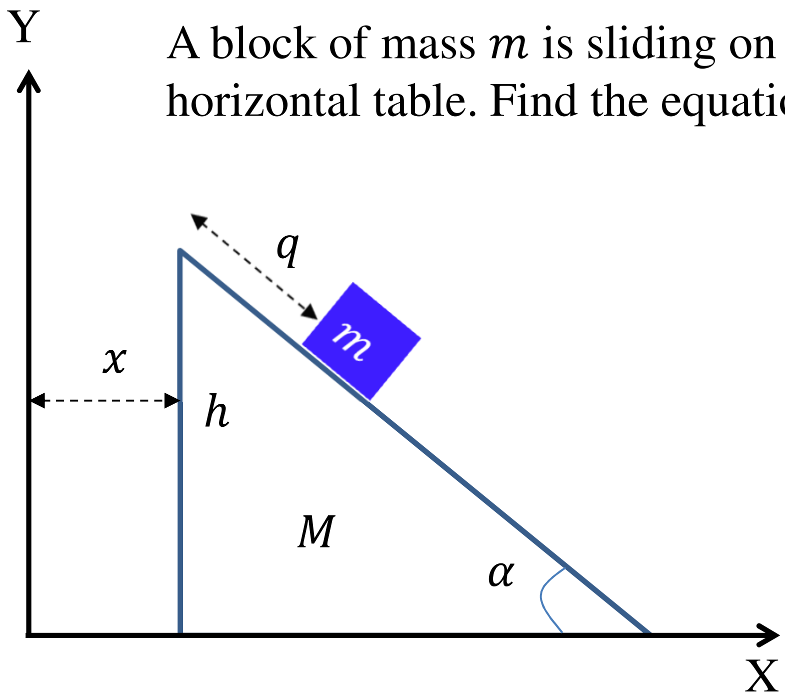
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{dp_x}{dt} \quad \& \quad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = \boldsymbol{Fx}$$ -the x-component of force!

$$\boxed{F_x = \frac{dp_x}{dt}}$$

Newton's 2nd Law!

Lagrange's equation: Example 3

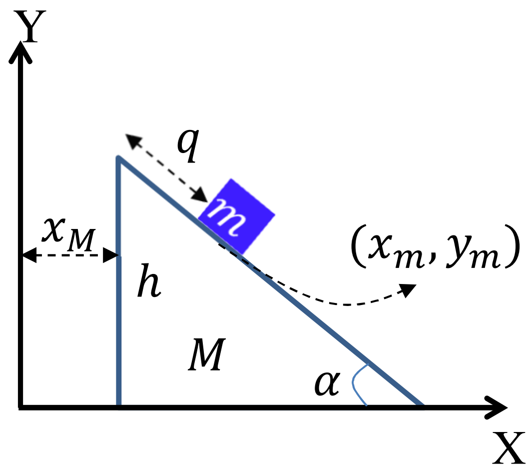
A block of mass m is sliding on a wedge of mass M. Wedge can slide on the horizontal table. Find the equation of motion.



Initial conditions!

At time t=0: the wedge is stationary and at a distance l from the origin, and the mass m is gently placed at the top point of the Wedge!

Lagrange's equation: Example 3



Four constrains: $z_M = 0$; $y_M = 0$; $z_m = 0$;
$$\frac{h - y_m}{x_m - x_M} = \tan\alpha = constant$$

$$x_m = x_M + q\cos\alpha;$$
$$y_m = h - q\sin\alpha$$

Step-1: *Find the degrees of freedom and choose suitable generalized coordinates*

One particle $N = 2$, $no. of\ constrains\ (k) = 4$;
So DOF$= 3 \times 2 - 4 = 2$.
The distance of the wedge from origin (x_M) and distance slipped by the block (q) can serve as generalized coordinates of the system.

Only translation of the given rigid bodies are considered, thus for the calculation of degrees of freedom both of them are considered as point particles.

Lagrange's equation: Example 3

Step-2: *Find out transformation relations*

$$T = \frac{1}{2}m\left(\dot{x}_m{}^2 + \dot{y}_m{}^2\right) + \frac{1}{2}M\dot{x}_M{}^2 \; ; \quad V = mgy_m$$

Step-3: *Write T and U in Cartesian*

$$x_m = x_M + q\cos\alpha; \qquad y_m = h - q\sin\alpha$$
$$\dot{x}_m = \dot{x}_M + \dot{q}\cos\alpha; \qquad \dot{y}_m = -\dot{q}\sin\alpha$$

Step-4: Convert T *and* V *in generalized coordinate using transformations*

$$T = \frac{1}{2}m[\dot{x}_M{}^2 + \dot{q}^2 + 2\dot{x}_M\dot{q}\cos\alpha] + \frac{1}{2}M\dot{x}_M{}^2;$$
$$V = mg(h - q\sin\alpha)$$

Lagrange's equation: Example 3

Step-5: *Write down Lagrangian*

$$L = T - V$$

$$L = \frac{1}{2}m\left[\dot{x}_M{}^2 + \dot{q}^2 + 2\dot{x}_M\dot{q}\cos\alpha\right] + \frac{1}{2}M\dot{x}_M{}^2 - mg(h - q\sin\alpha)$$

Step-5: *Write down Lagrange's equation for each generalized coordinates*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_M}\right) - \frac{\partial L}{\partial x_M} = 0; \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

From eqn 1

$$\frac{d}{dt}\boxed{[m\dot{x}_M + m\dot{q}\cos\alpha + M\dot{x}_M]} = 0 \qquad \text{----(1)}$$

$$(m + M)\ddot{x}_M + m\ddot{q}\cos\alpha = 0 \qquad \text{----(2)}$$

From eqn 2

$$\frac{d}{dt}[m\dot{q} + m\dot{x}_M\cos\alpha] - [mg\sin\alpha] = 0$$

$$m(\ddot{q} + \ddot{x}_M\ \cos\alpha) - mg\sin\alpha = 0 \qquad \text{----(3)}$$

An Interesting point: Example 3

$$\frac{\partial L}{\partial \dot{x}_M} = m\dot{x}_M + m\dot{q}\cos\alpha + M\dot{x}_M = \text{constant!}$$ (from eq (1))

But what's this quantity? The total linear momentum, say $\boldsymbol{P_x}$!

So Lagrange's equation tells us that the **total linear momentum is conserved**! We didn't have to **impose** it to solve!

From the Initial conditions given: $x_M = l\ \ \dot{x}_M = 0;\ q = 0;\ \dot{q} = 0$

Initial Px = 0; So it any other time later!

$$\dot{x}_M = \frac{-m\dot{q}\cos\alpha}{(M+m)} \Rightarrow \ddot{x}_M = \frac{-m\ddot{q}\cos\alpha}{(M+m)}$$

This shall be substituted in eq (3): $(\ddot{q} + \ddot{x}_M\cos\alpha) = g\sin\alpha$

And, Solve the problem completely!

(It's left to you to verify with the Newtonian Scheme!)

In some cases further time derivative (such as equation (2)) may not be unnecessary!

Generalized momentum: A few points

Generalized velocity is the rate of charge of generalized coordinate $\dot{q}_j = \frac{dq_j}{dt}$

Generalized momentum is not the mass multiplied by generalized velocity.

$$\cancel{p_j = m\dot{q}_j}$$

In specific cases, this relation may be true but it is not the general case.

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Definition of **generalized momentum**

Unit/dimension of the generalized momentum depends on generalized coordinate under consideration.

Generalized definition of momentum allows to consider non-mechanical systems, for example EM field. Example: charged particle in EM field $\vec{p} = m\vec{v} + e\vec{A}$

Generalized momentum

Lagrangian of a free particle

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Thus $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$; now $m\dot{x} \rightarrow x$ component of linear momentum (p_x)

$p_x = m\dot{x} = \frac{\partial L}{\partial \dot{x}}$; Similarly, $p_y = \frac{\partial L}{\partial \dot{y}}$ and $p_z = \frac{\partial L}{\partial \dot{z}}$

Lagrangian of a freely rotating wheel with moment of inertia I is

$$L = \frac{1}{2}I\dot{\theta}^2$$

And $\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \rightarrow$ Angular momentum

In both the examples, momentum was the derivative of the Lagrangian with respect to generalized velocity.

Generalized momentum *associated with generalized coordinate q_j* by

$$\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j}}$$

$\rightarrow$ Also known as conjugate momentum or canonical momentum

Cyclic coordinates

If a particular coordinate does not appear in the Lagrangian, it is called '**Cyclic**' or '**Ignorable**' coordinate.

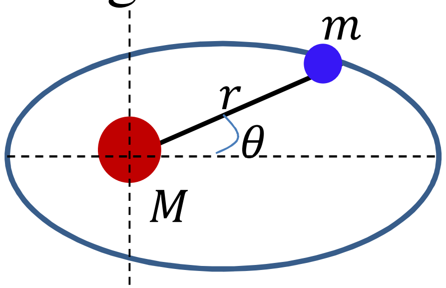
Example 1: Lagrangian of a point mass under gravity,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Since neither x nor y appear in the Lagrangian, they are cyclic.
Hence P_x & P_y will be conserved!

Example 2: Lagrangian for a planet of mass m orbiting around the sum (mass M):

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$$



Since θ does not appear in the Lagrangian, it is cyclic coordinate.

Hence $P_\theta = \frac{\partial L}{\partial \dot{\theta}_j} = mr^2\dot{\theta}$ (≡ **Ang. Momentum!** -will be **conserved!**)

Cyclic coordinates and conservation of conjugate momentum

- If there is no explicit dependence of L on generalized coordinate q_j, then

$$\frac{\partial L}{\partial q_j} = 0$$

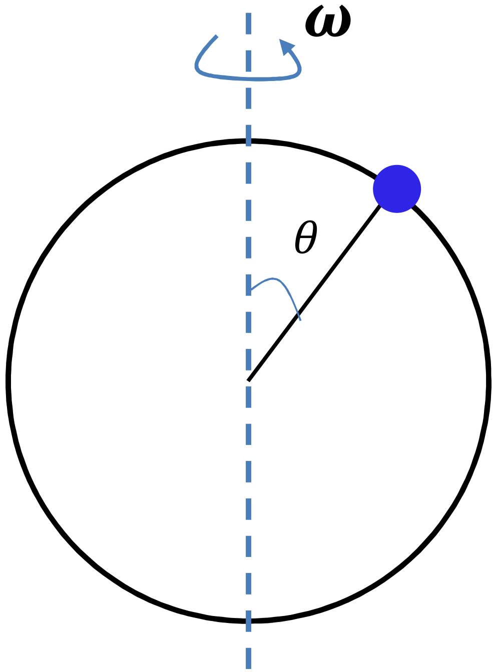
Thus Lagrange's equation corresponding to cyclic coordinate become,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad \Rightarrow \frac{dp_j}{dt} = 0$$

Hence, $\boldsymbol{p_j}$ **=constant**

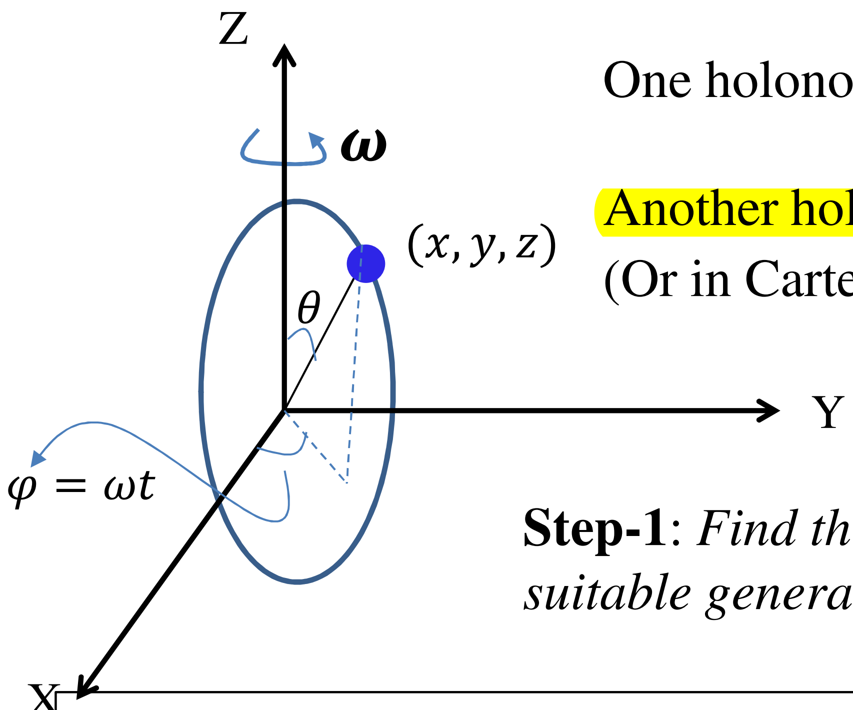
Generalized momentum conjugate to a cyclic coordinate is a constant

Lagrange's equation: Example 4



A bead is free to slide along a frictionless hoop of radius R. The hoop rotates with constant angular speed ω around a vertical diameter. Find the equation of motion for the position of the bead.

Lagrange's equation: Example 4



One holonomic constraint relation

$$x^2 + y^2 + z^2 = R^2$$

Another holonomic constraint $\dot{\varphi} = \omega, i, e. \varphi = \omega t$

(Or in Cartesian this constrain is $\tan^{-1}\frac{y}{x} = \omega t$)

Step-1: *Find the degrees of freedom and choose suitable generalized coordinates*

One particle $N = 1,\ \ no. of\ constrains\ (k) = 2$

$$thus\ degrees\ of\ freedom = 3 \times 1 - 2 = 1$$

Hence number of generalized coordinates must be one.

Choice of Generalized coordinate: 'θ' , which the angle of particle with rotation axis (z-axis) of hoop.

Lagrange's equation: Example 4

Step-2: *Find out transformation relations*

$$x = R\sin\theta\cos\omega t\ ;y = R\sin\theta\sin\omega t\ ; z = R\cos\theta$$

$$\dot{x} = R\cos\theta\cos\omega t\ \ \dot{\theta} - R\,\omega\sin\theta\sin\omega t$$
$$\dot{y} = R\cos\theta\sin\omega t\ \ \dot{\theta} + R\omega\sin\theta\cos\omega t$$
$$\dot{z} = -R\sin\theta\,\dot{\theta}$$

Step-3: *Write T and V in Cartesian*

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2); \qquad \& \quad V = mgz$$

Step-4:Convert T *and* V *to generalized coordinate , either using,*

(a) transformation at Step#2 Or,

(b) in this case employing spherical polar equations.

$$T = \frac{1}{2}m[R^2\dot{\theta}^2 + R^2\omega^2 sin^2\theta];$$
V$= mgR\cos\theta$

Example 4: continue

Step-5: *Write down Lagrangian*

$$L = T - V$$

$$L = \frac{1}{2}m\left[R^2\dot{\theta}^2 + R^2\omega^2 sin^2\theta\right] - mgR\cos\theta$$

Step-5: *Lagrange's equation*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

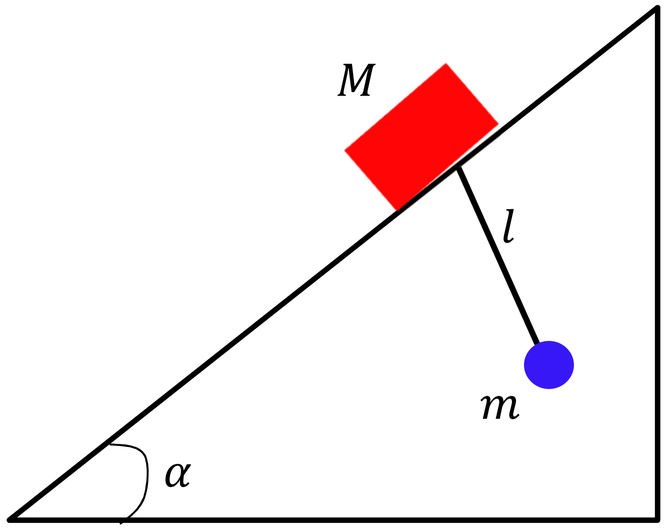
$$\frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta} \qquad \& \qquad \frac{\partial L}{\partial\theta} = mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta$$

$$\frac{d}{dt}\left[mR^2\dot{\theta}\right] - \left[mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta\right] = 0$$

$$mR^2\ddot{\theta} - \left[mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta\right] = 0$$

Example-5

A mass M slides down a frictionless plane inclined at angle α. A pendulum, with length l, and mass m, is attached to M. Find the equations of motion.



Questions?