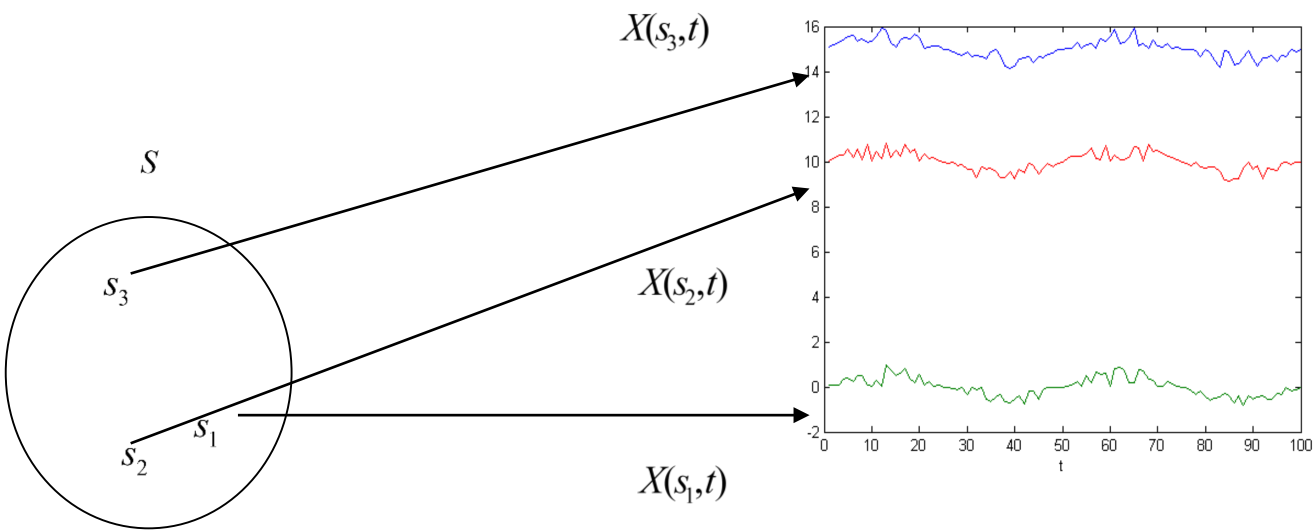
Lecture 3 Random Processes

Random Process (RP)

- **A random process maps each sample point to a waveform.**
- Consider a probability space $\{S, \mathbb{F}, P\}$. A *random process* can be defined on $\{S, \mathbb{F}, P\}$ as an indexed family of RVs $\{X(s,t) \mid s \in S, t \in \Gamma\}$. Γ is an index set usually denoting time.
- **For a fixed** $s_0 \in S$, $X(s_0, t)$ ***is a single realization of the random process and is a deterministic function.***
- *The random process* $\{X(s,t)\}$ *is normally denoted by* $\{X(t)\}$*.*



Continuous-time and Discrete-time random processes

- When the index set Γ is uncountable, then the process $\{X(t)\}$ is a continuous-time random process. Otherwise we have the discrete-time random process.
- If $\Gamma \subseteq \mathbb{Z}$ and the process is dented by $\{X(n)\}$ or $\{X_n\}$
- The family of random variables $\{X_n \mid n = 0, 1, \ldots\}$ is called a random sequence

Continuous-state and Discrete-state random processes

- **The values taken by a random process is known as *states*** and the set *V* of all states is called state space
- **If** *V* **is countable, then the corresponding process is known as a continuous- state process**
- **For example, a sampled and quantized speech waveform is modelled as a discrete-time discrete-state random process. RP**

Probability structure of a random process

- Consider a random process $\{X(t), t \in \Gamma\}$
- To describe $\{X(t)\}$ we have to use joint CDF of the RVs at all possible instants t.
- ***For any positive integer*** n***, the collection*** $\{X(t_1), X(t_2), ..., X(t_n)\}$ ***represents*** n ***jointly distributed random variables. Thus a random process*** $\{X(t), t \in \Gamma\}$ ***at these*** n ***instants*** $t_1, t_2, ..., t_n \in \Gamma$ ***can thus be described by specifying the*** n-th order ***joint distribution function***

$$F_{X(t_1), X(t_2), ..., X(t_n)}(x_1, x_2, ..., x_n) = P(X(t_1) \le x_1, X(t_2) \le x_2, ..., X(t_n) \le x_n)$$

Moments of a Random Process

We can define various moments and central moments.

- **Mean of the random process**

$$E(X(t)) = \mu_X(t),\ \forall t$$

- **Autocorrelation function** $R_X(t_1,t_2) = E(X(t_1)X(t_2)),\ \forall t_1, t_2$

- **Auto-covariance function**
$$\begin{aligned} C_X(t_1,t_2) &= E(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2)) \\ &= R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2) \end{aligned}$$

and so on

Example Gaussian Random Process

.The process $\{X(t)\}$ is called Gaussian if for any $k \in \mathbb{N}$ and any time points $t_1, t_2, \ldots, t_k$ the random vector $\mathbf{X} = [X(t_1)\ X(t_2), \ldots\ldots X(t_k)]'$ is jointly Gaussian with the joint CDF

$$f_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1, x_2, \ldots x_k) = \frac{e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_X)'\mathbf{C}_X^{-1}(\mathbf{x}-\boldsymbol{\mu}_X)}}{\left(\sqrt{2\pi}\right)^n \sqrt{\det(\mathbf{C}_\mathbf{X})}}$$

where $\mathbf{C}_\mathbf{X} = E(\mathbf{X} - \boldsymbol{\mu}_\mathbf{X})(\mathbf{X} - \boldsymbol{\mu}_\mathbf{X})'$

and $\boldsymbol{\mu}_\mathbf{X} = E(\mathbf{X}) = \left[E(X_{t_1})\ \ E(X_{t_2})\ldots\ldots E(X_{t_k})\right]'$

Stationary Random Process

- **An RP $\{X(t)\}$ is called *strict-sense stationary* (SSS) if its probability structure is invariant with time. In terms of the joint CDF**

$$F_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1,x_2,\ldots,x_k) = F_{X(t_1+\tau),X(t_2+\tau),\ldots,X(t_k+\tau)}(x_1,x_2,\ldots,x_k), \quad \forall k \in \mathbb{N} \ \text{ and } \ \forall \tau, t_1,\ldots,t_k \in \Gamma$$

- Analysing an SSS random process is highly complex. We look for a weak form of stationarity

An RP $\{X(t)\}$ is called wide sense stationary process (WSS) if $\forall h, t, t_1, t_2$

$$1.\ EX(t) = EX(t+h) = \text{constant and}$$
$$2.\ R_X(t_1,t_2) = R_X(t_1+h, t_2+h)$$

If we put $h = -t_1$, then

$R_X(t_1,t_2) = R_X(0, t_2 - t_1) \quad \forall t_1, t_2$ is a function of lag $\tau = t_2 - t_1$ only.

Example of a WSS processSinusoid with random phase

Consider the random process $X(n) = A\cos(\omega_0 t + \phi)$, A and ω_0 are constants and $\phi \sim U(0, 2\pi)$

Note $$f_\Phi(\phi) = \begin{cases} \dfrac{1}{2\pi} & 0 \le \phi \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore EX(t) = EA\cos(\omega_0 t + \phi)$$

$$= \int_0^{2\pi} A\cos(\omega_0 t + \phi)\frac{1}{2\pi}d\phi = \; 0$$

$$R_X(t_1, t_2) = EX(t_1)X(t_2)$$

$$= EA\cos(\omega_0 t_1 + \phi)A\cos(\omega_0 t_2 + \phi)$$

$$= \frac{A^2}{2}E[\cos(\omega_0(t_1 + t_2) + 2\phi) + \cos(\omega_0(t_1 - t_2))]$$

$$= \frac{A^2}{2}c\,\mathrm{os}(\omega_0(t_1 - t_2))$$

Hence $\{X(t)\}$ is *WSS.*

Important Properties of $R_X(\tau) = EX(t)X(t+\tau)$

- $$R_X(0) = EX^2(t) = \text{Mean-square value}$$

(Average power)

- $$R_X(-\tau) = R_X(\tau)$$
- $$|R_X(\tau)| \leq R_X(0)$$

This follows from the Cauchy Schwartz inequality

$$|< X(t), X(t+\tau) >| \leq \|X(t)\| \|X(t+\tau)\|$$

$$\therefore |EX(t)X(t+\tau)| \leq \sqrt{EX^2(t)}\sqrt{EX^2(t+\tau)}$$

$$\Rightarrow |R_X(\tau)| \leq \sqrt{R_X(0)}\sqrt{R_X(0)} = R_X(0)$$

Cross-correlation Function

- The *cross-correlation function* two random processes $X(t)$ and $Y(t)$ is defined as

$$R_{XY}(t,\tau) = EX(t)Y(t+\tau)$$

- $X(t)$ and $Y(t)$ are called *jointly WSS* if they are individually WSS and

$R_{XY}(t,\tau)$ is a function of time lag τ only . Thus we can write

$$R_{XY}(t,\tau) = R_{XY}(\tau)$$

Frequency-domain Analysis of a WSS signal

- Recall that a deterministic signal $\{g(t)\}$ has the frequency-domain representation in terms of its Fourier transform (FT)

$$G(\omega) = FT(g(t)) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}d\varpi$$

- The inverse is given by

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(w)e^{j\omega t}dw$$

- $G(\omega)$ exists if $\int_{-\infty}^{\infty} |g(t)|dt < \infty$
- Such a frequency-domain representation is not possible for a WSS random process.

Power spectral density (PSD)

- Average power of an RP $\{X(t)\}$ is given by

$$R_X(0) = E X^2(t)$$

- PSD $S_X(\omega)$ indicates the contribution of each component frequency ω to the average power.

- For a WSS RP $\{X(t)\}$, $S_X(\omega)$ exists and given by

$$S_X(\omega) = \lim_{T \to \infty} \frac{1}{2T} E \left| \int_{-T}^{T} X(t) e^{-j\omega t} dt \right|^2$$

Wiener-Khinchin theorem

- $R_X(\tau)$ and $S_X(w)$ form a FT pair

$$S_X(w) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau$$

Taking the invers FT

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(w) e^{j\omega\tau} dw$$

- $S_X(w)$ is a real and even function of w

White noise process

A white noise process $\{W(t)\}$ has a constant PSD

$$S_W(\omega) = \frac{N_0}{2} \qquad -\infty < \omega < \infty$$

Corresponding autocorrelation function $R_W(\tau) = \frac{N}{2}\delta(\tau)$

The average power of white noise

$$P_{avg} = EW^2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{N}{2}d\omega \to \infty$$ (Not realizable!)