Logic.

We argue based on statements.
In mathematics, we are concern with two types of statements. – true or false.
It can only take these two value.

- For example:

1) 2 is greater than 1 – true
2) There are finite numbers of integers – false
3) This sentence is false. (neither true or false.)

If the above sentence is true – then the sentence can't be false.
If the statement is false – the sentence is true. – Therefore, it can is neither true ~~or~~ nor false.

4) $1+1=2$ (always true) – this statement is always true.

5) $x^2=1$, true when $x=1$ or $x=-1$.
It is false when $x=5$.

So the last sentence is true based on context. We need to specify the context.

Based on the simple sentences which are true or false we can form complex statements. This is through connectives.

Logical connectives:

We ~~do~~ denote sentences through A and B.

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1. negation. A can take two truth values: truth, false.

A	∼ A
True	False
False	True.

~~If A is~~ When A is true, negation of A is false. And when A is false negation of A is true.

④ So

A	$\sim(\sim A)$
True	True
False	False

double negation is same as the orginal statement.

② Conjunction of statements A, B is denoted as A and B

In symbol $A \wedge B$

Truth value of $A \wedge B$

A	B	$A \wedge B$
True (T)	True	T
T	False (F)	F
F	T	F
F	F	F

A statement A and B ($A \wedge B$) is true when both A and B are true. When any one of them is false the statement is false.

3) Disjunction of A, B is denoted as A or B, in symbol $A \vee B$

The truth table is:

A	B	A ∨ B
T	T	T
T	F	T
F	T	T
F	F	F

A ∨ B / A or B is a true statement when both A and B are true, any one of them (A, B) is true. It is false when both A, B are false.

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3) Implication of statements A, B,

If A then B , $A \to B$.

Truth table is:

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

The statement $A \to B$ is true when B is true and A is true. Whenever A is false, it is always true. This statement is false when A is true and B is false.

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4) Equivalence of statements A, B

A if and only if B , $A \leftrightarrow B$.

Truth table is:

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

The statement $A \leftrightarrow B$ (A if and only if B) is true when both A and B are true, and when both A and B are false.

The statement is false when one of them A or B is false..

$$(A \rightarrow B) \leftrightarrow (\sim B \rightarrow \sim A)$$

A	B	$A \rightarrow B$	$\sim A$	$\sim B$	$\sim B \rightarrow \sim A$	$(A \rightarrow B) \leftrightarrow (\sim B \rightarrow \sim A)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

always true irrespective of truth value of A & B

so tautology.

~~$(A \wedge B) \Leftrightarrow (\sim A \vee \sim B)$~~ ~~$\sim(A \vee B)$~~

$\sim(\sim A \wedge \sim B)$

A	B	$A \wedge B$	$\sim A$	$\sim B$	$\sim A \vee \sim B$	$\sim(\sim A \vee \sim B)$	$(A \to B) \to \sim(\sim A \vee \sim$
T	T	T	F	F	F	T	T
T	F	F	F	T	T	F	T
F	T	F	T	F	T	F	T
F	F	F	T	T	T	F	T

$(A \wedge B) \Leftrightarrow \sim(\sim A \vee \sim B)$ / tautology.

$(A \to B) \Leftrightarrow \sim(A \wedge \sim B)$

A	B	$A \to B$	$\sim B$	$A \wedge \sim B$	$\sim(A \wedge \sim B)$	$(A \to B) \Leftrightarrow \sim(A \wedge \sim B)$
T	T	T	F	F	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$(A \to B) \Leftrightarrow \sim(A \wedge \sim B)$ is a tautology.

$\sim(A \wedge B) \Leftrightarrow \sim A \vee \sim B$

A	B	$A \wedge B$	$\sim(A \wedge B)$	$\sim A$	$\sim B$	$\sim A \vee \sim B$	$\sim(A \wedge B) \Leftrightarrow \sim A \vee \sim B$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

$\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$

It is a tautology.

Try $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$

Consider a set of statements like

~~A, B, C~~ A, B, C, D, E, F

① $A \rightarrow B$

② $\sim B \rightarrow C$

③ $D \rightarrow E$

④ $E \wedge C \rightarrow F$

⑤ $B \rightarrow \sim E$

⑥ D

$\sim A \wedge F$.

The above set of statements are connected in the given way. Based on these statement we get the implication that $\sim A \wedge F$.

We will argue in the following way:

Assume each of the 6 statements

~~are~~ is true.

6 D is true and 3 is true

so E is true.

~~From~~ For statement 5 to be true, B is

~~true~~, ~~so~~ false.

so for statement 1 to be true

A is false..

~~so~~ Since B is false ~~so~~ so ~B is

true. So statement 2 is true

when C is true.

C is true and E is true so F is

true, since statement 4 is true

~~so~~ ~~we can conclude th~~

Since A is false so ~A is true.

so ~~~A and~~ $\sim A \wedge F$ is true.

Another Example

① $M \wedge A \rightarrow (S \rightarrow L)$

② $M \wedge \sim A \rightarrow \sim L$

③ $C \rightarrow (B \leftrightarrow L)$

$M \wedge C \rightarrow (A \wedge B) \vee (\sim A \wedge \sim B)$

–

~~MA~~ $M \wedge C$ is true

and $(A \wedge B) \vee (\sim A \wedge \sim B)$ is true

– $M \wedge C \rightarrow (A \wedge B) \vee (\sim A \wedge \sim B)$ is true.

—

C is true ~~&~~ ~~B~~, it implies

$(B \leftrightarrow L)$ is true for $C \rightarrow (B \leftrightarrow L)$ to be true.

$B \leftrightarrow L$ is true when

B and L is true.

Or B and L both false

- ~~If~~ when L is true, then $\sim L$ is false

This implies. $M \wedge \sim A$ must be false

so for $M \wedge \sim A \to \sim L$.

If A is true, ~~then~~ it implies $M \wedge \sim A$ is false.

M is true and ~~sup~~ also A then $M \wedge A$ is true. It ~~means~~ implies $S \to L$ must be true.

$S \to L$ is always when L is true.

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when A, M, B, L, C are true all the three statement are true.

~~Ma~~ $M \wedge C$ is true.

A and B is true so

$(A \wedge B) \vee (\sim A \wedge \sim B)$ is also true.

Quantifiers and Context

For example $x^2=1$, is either statement true for all integers.

we need to specify the set. (context).

~~∃, for all values.~~

∃: there exists, for some (existential quantifier)

∀: for all, for every. (universal quantifier).

There exists an integer x such that $x^2=1$.

$x^2=1$ is true for $x=1$ and $x=-1$ both of them are integer.

$(\exists x)(x \in I)(x^2=1)$ (In symbols)

Examples of quantifiers:

For every integer x, there exists an integer y such that $x+y=0$.

$(\forall x)(\exists y)(x+y=0)$.

Explanation: Take any integer x. 10,

~~then~~, $x=10$, $y=-10$, $x+y=10_2$

This is true statement.

~~For~~ There exists an integer y, such that for every integer x, then $x+y=0$

$$(\exists y)(\forall x)(x+y=0)$$

Take any two integers 10, 20, $x=10$, $x=20$. ~~and fix~~ can we have a same y such that $10+y=0$, and $20+y=0$

- Not possible. So the above statement is false.

- The sequence of quantifiers ~~are~~ is very important. It changes the meaning.

Individual Choice:

- Suppose there are three alternatives x, y, z. An individual has to choose from this set of three alternatives.
- Alternative we ask the individual to rank or order the alternatives.
- The choice or ranking is based on preference of the individual over the alternatives.
- We assume that the preferences are given. Taking a given type of preferences of the individuals whether the individual can rank or order all the alternatives or make choices.
- We do not study the way preferences are formed.

- Preferences over alternatives are based on binary relation.
- What is a binary relation? It is relation between the elements of a set. For example "is brother of" is a binary relation. Suppose the set is family members. $S = \{a, b, c, d\}$ is the set of family members. a= father, b= mother, c= elder son, d = younger son.
- In this example we have c is brother of d , and d is brother of c. a is not brother of d, a is not brother of b, a is not brother of c, b is not brother of a, b is not brother of d, b is not brother of c.
- We define it technically. We denote the binary relation by R. Let the set be $S = \{A, B, C, D\}$. We get cRd and dRc, $\sim aRb$ etc. In this example the binary relation can be represented as $R = \{(c, d), (d, c)\}$. A binary relation is a set of ordered pairs (x, y) such that x and y belongs to S where S is the set of alternatives.

Another example of binary relation " is taller than ". Suppose the set of alternative is $S = \{p1, p2, p3, p4\}$ these are the heights of four peaks, p1= 1000ft, p2=2000 ft, p3= 3000 ft, p4= 1500 ft. R is taller than so we have
$R = \{(p2, p1), (p3, p2), (p3, p4), (p3, p1), (p4, p1), (p2, p4)\}$.
We define the individual choice based on binary relation. The binary relation we take is R : atleast as good as .
$S = \{x, y, z, w\}$,
$xRy,\ yRx,\ xRz,\ xRw,\ xRx,\ yRy,\ yRw,\ zRz,\ wRy,\ zRz.$
We can write it as

$$R = \{(x, y),\ (y, x), (x, z),\ (x, w),\ (x, x),\\ (y, y),\ (y,)w,\ (z, z),\ (w, y),\ (w, w)\}$$

Properties of binary relation

$S = \{x, y, z, w\}$.
Reflexivity
A binary relation R is reflexive, for all $x, \in S,\ xRx$.
R : is taller than is not reflexive, p1 is taller than p1 is not possible.
R : is atleast as good as is reflexive when
$(x, x), (y, y), (z, z), (w, w) \in R$.

Completeness

A binary relation R is complete, for all $x, y \in S$, if $(x \neq y)$ then xRy or yRx.

R is taller than satisfies completeness,

$R = \{(p2, p1), (p3, p2), (p3, p4), (p3, p1), (p4, p1), (p2, p4)\}$

R : is atleast as good as satisfies completeness, if atleast these ordered pairs $(x, y), (y, z), (z, w) \in R$.

Suppose $S^{'} = \{p1, p2, p3, p4, p5\}$ and p5=3000. We cannot have $p3Rp5$ and $p3Rp5$. So R as is taller than is not complete when the set is $S^{'}$.

Transitivity

For all $x, y, z \in S$, if xRy and yRz then xRz.

If x is atleast as good as y, and y is atleast as good as z, then x is atleast as good as z.

R atleast as good as satisfies transitivity for the set S , if $(x, y), (y, z) \in S$ then $(x, z) \in R$, if $(z, w), (x, z) \in R$ then $(x, w) \in R$.

Suppose $R = \{(x, y), (y, z), (x, z), (z, w), (x, w)\}$. This is transitive but not complete.

R : is taller than is transitive. When $(p2, p1)$, $(p3, p2) \in R$ then $(p3, p1) \in R$. Check for all the combinations.

Suppose $X = \{x, y, z\}$ and x = grand father, y= father, z=son.
R; is son of. It is binary relation.
$R = \{(z, y), (y, x)\}$.
$(z, x) \notin R$. Therefore, this is not transitive.

Anti-symmetry:
For all $x, y \in S$, if xRy and yRx then $x = y$.
$X = \{2, 3, 4, 5, 2\}$.
R : is greater than equal to
$R = \{(3, 2), (2, 2), (2, 2), (5, 4)(4, 3), (4, 2), (4, 2)\}$.
It satisfies anti- symmetry.
R; is taller than
It also satisfies anti-symmetry. It is trivially true.

R; at as good as . This may not satisfy anti-symmetry.
We will see later than xRy and yRx implies xIy. If x is at least as good as y and y is atleast as good as x then x is indifferent to y. They may not be same object.

Asymmetry:

For all $x, y\imath S$, if xRY then $\sim yRx$ or yRx is not possible.

R; is taller than . It satisfies asymmetry.

R : is brother of $X = \{x, y, z\}$ x= daughter, y= elder son, z= younger son.

$R = \{(y, x), (z, x), (y, z), (z, y)\}$. It violates asymmetry because $(y, z), (z, y) \in R$.

Symmetry:
For all $x, y \in S$, if xRy the yRx.
R : is atleast as good as
$R = \{(x, y), (y, x), (y, z), (z, x), (x, z), (z, x)\}$. The above R is symmetric.
$X = \{a, b, c\}$, a= elder son, middle son, younger son. R : is brother of. It satisfies symmetry.
R : is taller than violates symmetry.

Completeness

A binary relation R is complete, for all $x, y \in S$, if $(x \neq y)$ then xRy or yRx.

R is taller than satisfies completeness,

$R = \{(p2, p1), (p3, p2), (p3, p4), (p3, p1), (p4, p1), (p2, p4)\}$

R : is atleast as good as satisfies completeness, if atleast these ordered pairs $(x, y), (y, z), (z, w), (x, z) \in R$.

Suppose $S^{'} = \{p1, p2, p3, p4, p5\}$ and p5=3000. We cannot have $p3Rp5$ and $p3Rp5$. So R as is taller than is not complete when the set is $S^{'}$.

Transitivity

For all $x, y, z \in S$, if xRy and yRz then xRz.

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We will see later than xRy and yRx implies xIy. If x is at least as good as y and y is atleast as good as x then x is indifferent to y. They may not be same object.

Asymmetry:
For all $x, y \in S$, if xRY then $\sim yRx$ or yRx is not possible.
R; is taller than . It satisfies asymmetry.
R : is brother of $X = \{x, y, z\}$ x= daughter, y= elder son, z= younger son.
$R = \{(y, x), (z, x), (y, z), (z, y)\}$. It violates asymmetry because $(y, z), (z, y) \in R$.

Symmetry:
For all $x, y \in S$, if xRy the yRx.
R : is atleast as good as
$R = \{(x, y), (y, x), (y, z), (z, x), (x, z), (z, x)\}$. The above R is symmetric.
$X = \{a, b, c\}$, a= elder son, middle son, younger son. R : is brother of. It satisfies symmetry.
R : is taller than violates symmetry.

A binary relation satisfying reflexivity, completeness and transitivity is called ordering.
If a binary relation is ordering then based on that binary relation we can rank or order the alternatives.
$X = \{x, y, z\}$. R : is atleast as good as.
$R = \{(x, x), (y, y), (z, z), (x, y), (y, z), (x, z)\}$. This is an ordering, it satisfies all the three properties.
Now suppose $R = \{(x, x), (y, y), (z, z), (x, y), (x, z)\}$. This is not an ordering because it is not complete .
Now suppose $R = \{(x, x), (y, y), (z, z), (x, y), (y, z), (z, x)\}$. This is not an ordering.

A binary relation satisfying reflexivity and transitivity is called quasi ordering.

R : is atleast as good as is also weak preference. From this weak preference, we define strict preference (P) and indifference (I) relation.

xPy if and only if $[xRy$ and $\sim yRx]$.

x is preferred to y if and only if x is atleast as good y and y is not atleast as good as x.

$R = \{(x,x),(y,y),(z,z),(x,y),(y,z),(x,z)\}$. Here xPy, yPz and xPz.

xIy if and only if [xRy and yRx].
x is indifferent to y if and only if x is atleast as good as y and y is atleast as good as x.
$R = \{(x,x),(y,y),(z,z),(x,y),(y,x),(y,z),(z,y),(x,z),(z,x)\}$.
Here we have xIy, yIz and xIz.
Suppose $S = \{x, y, z\}$ and R : is atleast a s good as.
An element x in S is a maximal element of S with respect to binary relation R if and only if there does not exist any y such that $y \in S$ and y is preferred to x.
In symbols
An element x in S is a maximal element of S with respect to binary relation R if and only if $\sim [\exists y : (y \in S \& yPx)]$.

An element of a set is maximal if it is not dominated or preferred by any other element of that set.
The set of maximal elements in S is called its maximal set and is denoted by $M(S, R)$.
When $R = \{(x, x), (y, y), (z, z), (x, y), (y, z), (x, z)\}$, $x \in M(S, R)$.
When $R = \{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y), (x, z), (z, x)\}$, $x, y, z \in M(S, R)$.

An element x in S is a best element of S with respect to a binary relation R if and only if for all $y \in S$ x is atleast as good as y.
In symbols
An element x in S is a best element of S with respect to a binary relation R if and only if
$\forall y : (y \in S \rightarrow xRy)$.
The set of best elements in S is called its choice set and is denoted $C(S, R)$.
When $R = \{(x, x), (y, y), (z, z), (x, y), (y, z), (x, z)\}$, $x \in C(S, R)$.
When
$R = \{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y), (x, z), (z, x)\}$,
$x, y, z \in C(S, R)$.

From the definition of maximal element and best element, it is clear that if an element is a best element of a set, then it is also maximal element. So, the choice set is a subset of maximal set. $C(S, R) \subset M(S, R)$.

A maximal element may not be best element. Suppose $S = \{x, y, z\}$ and $R = \{(x, y), (z, y)\}$. In this example we have xPy and zPy. We do not have xRz and zRx. So The set of best element is empty. We have maximal elements $x, z \in M(S, R)$.

This implies that $M(S, R)$ is not a subset of $C(S, R)$.
We can have situations where both choice set and maximal set is empty.
Suppose $S = \{x, y, z\}$ and $R = \{(x, y), (y, z), (z, x)\}$. We have xPy, yPz and zPx. We dont have any $x \in S$ such that xRy for all $y \in S$. There is always an x such that xPz for all y. So maximal set is empty.

Indifference relation
for all $x, y \in S$, xRy and yRx then xIy

Preference relation
for all $x, y \in S$, xRy and $\sim yRx$ then xPy

Some results on strict preference and indifference

If R is an ordering, then for all $x, y, z \in S$

1) xIy & $yIz \rightarrow xIz$.
2) xPy & $yIz \rightarrow xPz$.
3) xIy & $yPz \rightarrow xPz$.
4) xPy & $yPz \rightarrow xPz$.

Proof of 1). xIy & $yIz \rightarrow$ $(xRy$ & $yRx)$&$(yRz$ &$zRy)$.
$\rightarrow (xRy$ & $yRz)$&$(zRy$ &$yRz)$
$\rightarrow xRz$ & zRx from transitivity
$\rightarrow xIz$.

Proof of 2. xPy & $yIz \rightarrow (xRy$ & $\sim yRx)$ & $(yRz$ & $zRy)$.
We have xRy & yRz implying xRz from transitivity.
Suppose zRx it implies xIz. We have yIz. Therefore using the first result we get xIy. We are given xPz. A contradiction. Therefore zRx is not true. We have xRz and $\sim zRx$, it implies xPz.

The proof of 3 is similar to the proof of 2.

Proof of 4. xPy & $yPz \rightarrow (xRy$ & $\sim yRx)$ & $(yRz$ & $\sim zRy)$. It implies $(xRy$ & $yRz)$. It implies xRz from transitivity. Suppose zRx. We get zIx. Take yPz and zIx. Using the second result we have yPx. It is a contradiction because we have xpy. Thus, we cannot have zRx. Therefore, we have xRz and $\sim zRx$. It implies xPz.

Result:
Any finite quasi ordered set has at least one maximal element.

Proof. Suppose the set of maximal set is empty. It implies that that for every $y \in S$ there exist one x such that xPy. Take for example $x_1, x_2, x_3 \in S$. From above we have $x_1 P x_2$, $x_2 P x_3$ and $x_3 P x_1$. This implies that it violates transitivity. Therefore, R is not quasi ordered. This implies that set of maximal element is not empty.

We define choice function.
A choice function $C(S, R)$ defined over S is a functional relation such that the choice set $C(S, R)$ is non-empty for every non empty subset s of S.
It means that there is a non-empty choice set or there are best elements for every non-empty subset of S.
We have seen that whenever completeness is violated, choice set is empty. It implies that choice function does not exist.
If reflexivity is violated then also choice set is empty . For single element set, reflexivity is required for non-empty choice set.

If transitivity is violated, choice set can be empty. Example xPy, yPz and zPx. In this situation choice set is empty for $\{x, y, z\}$. We cannot have a choice function for this subset of S.

If R is an ordering defined over a finite set S, then a choice function $C(S, R)$ is defined over S.

Proof: Suppose the choice set over S is empty. This means that choice function is not defined. If choice set is empty it implies that it violates any one of the following property;
completeness, reflexivity, or transitivity.
This implies that R is not an ordering if anyone of the above property is violated. This implies that choice set is not empty since R is an ordering. Thus, choice function defined fo S.

We have proved in the last class
If R is an ordering, then for all $x, y, z \in S$
xPy & $yPz \rightarrow xPz$
This is all the definition of quasi-transitivity.
If for all $x, y, z \in S$, xPy and $yPz \;\rightarrow\; xPz$, then R is quasi transitive.
We have shown that reflexivity, completeness and transitivity of R over S implies that there is a choice function $C(S, R)$.
We take a weaker condition quasi transitivity of R.

Result:
If R is reflexive, complete and quasi-transitive over a finite set X, then a choice function $C(S,R)$ is defined over X.

Proof: Suppose there are n elements in S and $S \subset X$. The elements are $x_1, x_2, x_3 ..., x_n$. Take the first pair (x_1, x_2), by reflexivity and completeness of R, there is a best element in this pair (x_1, x_2).
Consider a set of j elements $(x_1, x_2, ...x_j)$. Suppose a_j is a best element of for the set $(x_1, x_2, ...x_j)$. This implies that $a_j R x_k$ for $k = 1, 2, 3...j$.
Now consider the set $(x_1, x_2 ..., x_j, x_{j+1})$, in this set we can have $a_j R x_{j+1}$ or $x_{j+1} P a_j$. If we have $a_j R x_{j+1}$ then a_j is a best element for the set $(x_1, x_2 ..., x_j, x_{j+1})$. So, there exist a best element for this set. It implies that choice function exist.

Suppose $x_{j+1}Pa_j$. In this x_{j+1} is not a best element only when x_kPx_{j+1} for some $k = 1, 2, 3...j$. Let this be x_k. We have x_kPx_{j+1} and $x_{j+1}Pa_j$. By quasi transitivity we have x_kPa_j. But we have a_jRx_k since a_j is the best element of the set $(x_1, x_2, ...x_j)$. A contradiction. So we cannot have x_kPx_{j+1} for some $k = 1, 2, 3...j$. Thus, x_{j+1} is a best element when $x_{j+1}Pa_j$.

Thus, we get that reflexivity, completeness and quasi transitivity implies that choice function can be defined.

We define Acyclicity

R is acyclical over S if and only the following holds:

for all $x_1, x_2..., x_j \in S$, if $x_1 P x_2$ & $x_2 P x_3$ & ....&$x_{j-1} P x_j$ then $x_1 R x_j$.

It is much weaker than quasi transitivity condition.

Result:
If R is reflexive and complete, then a necessary and sufficient condition for $C(S, R)$ to be defined over finite S is that R be acyclical over S.

Proof: We first proof there necessary part, that is if $C(S, R)$ is defined over S then R must be acyclical over S given reflexivity and completeness are satisfied by R.
Suppose R is not acyclical. This implies that there is a subset of of j alternatives in S such that $x_1 P x_2, ... x_{j-1} P x_j$ and $x_j P x_1$. This implies that there is no best element in this subset, so choice function is not defined over S.

Sufficiency part. Suppose all the alternatives are indifferent to each other. It implies all the elements are in the best element. Suppose there is only one pair satisfying the strict preference that is $x_2 P x_1$. We have x_2 is the best element. But x_2 cannot be best element of S if there is some element say x_3 in S such that $x_3 P x_2$. If $x_1 P x_3$ then by acyclicity we have $x_1 R x_2$. It contradicts $x_2 P x_1$. So x_3 is the best element of the set x_1, x_2, x_3. If we continue in this way, we can exhaust all the elements and have best choice set being non-empty. We get the acyclicaity is a sufficient condition for the choice function to be defined when reflexivity and completeness are given.

The existence of a choice function explains rational choice. By specifying certain properties on choice function we define rational choice.
For example suppose we choose x from the set x, y, z and choose y from the set x, y. We cannot rationalize this choice outcome. We define certain properties. They are consistency properties.

Property α: $x \in S_1 \subset S_2 \rightarrow [x \in C(S_2) \rightarrow x \in C(S_1)]$ for all x.
If some element of a subset S_1 of S_2 is best in the set S_2 then it must be best in S_1.
It means if x is best in $\{x, y, z\}$ then x must be best in $\{x, z\}$ subset of $\{x, y, z\}$.

Property β :
$[x, y \in C(S_1) \& S_1 \subset S_2] \rightarrow [x \in C(S_2)$ if and only if $\; y \in C(S_2)]$.

If x, Y are best element of a set S_1 which is a subset of S_2, then if x is a best element of S_2 implies y is also a best element of S_2 and if y is a best element of S_2 it implies x is also a best element of S_2.

Result:
Every choice function $C(S, R)$ generated by a binary relation R satisfies property α but not necessarily property β.

Proof: If x belongs to $C(S, R)$ then xRy for all $y \in S$. Therefore xRy for all y in all the subsets of S. This is property α.
Suppose we have xIy, xPz and zPy for a triple x, y, z. Note that acyclicity is satisfied. $C([x, y]) = [x, y]$ and $C([x, y, z]) = [x]$. It violates property β.

We have seen in the last class that property α is always satisfied by choice function when it is defined. It means when that the binary relation satisfies reflexivity, completeness, and acyclicity.

Consider an example $xIy,\ xPz,\ zPy$, then the choice sets are $C([x,y],R) = [x,y]$ and $C([x,y,z],R) = [x]$. It violates property β.

We have done that if R is an ordering then, for all $x, y, z \in S$, xPy & $yIz \rightarrow xPz$.
When the above is satisfied by a binary relation R we call R to satisfy PI transitivity.

Relation R is *PI* transitive over S if and only if for all $x, y, z \in S$, xPy & $yIz \rightarrow xPz$.

Result:
A choice function $C(S, R)$ generated by a binary relation R satisfies property β if and only if R is PI transitive.

Proof: R must satisfy reflexivity and completeness then only choice function can be defined. We have seen it. We need one more property that is acyclicity to be able to define choice function. We have seen that reflexivity, completeness and acyclicity does not satisfy property β.
Suppose PI transitivity is not satisfied. It implies that here is a triple $x, y, z \in S$ such that xpy, yIz and zRx. From this we $C([y, z], R) = [y, z]$. And $C([x, y, z], R) = [z]$ and $y \in C([x, y, z], R)$ is not true. It violated property β. If R is not PI transitive then property β is violated. We get that PI transitivity is a necessary condition.

We proof the sufficiency part. Suppose property β is violated. Then we have a pair $x, y \in S_1$ such that $x, y \in C(S_1, R)$ and $x \in C(S_2, R)$, $y \notin C(S_2, R)$ when $S_1 \subset S_2$. This implies that there exist a $z \in S_2$ such that zPy and xRz. Since $x, y \in C(S_1, R)$. It implies that xIy. Suppose PI transitivity holds, so we have zPy and yIx implying zPx. We already have xRz . A contradiction. Therefore, PI transitivity cannot hold. Thus, we get PI implies that property β is satisfied.
We get that PI is a stronger condition then acyclicity.
We have when R satisfies reflexivity, completeness and PI transitivity, the choice function defined gives rational outcomes. It means that consistent outcomes.

If R_i is an ordering and it satisfies continuity and monotonicity then we can define a real valued function called utility function $U(x)$.

We move to social choice . We assume that X is the set of social sates. There are n individuals. The preference relation of the ith individual is $R_i, \; i = 1, 2, 3, ...n$. R denotes the social preference relation.
We assume that each individual i has an ordering over the elements in X. Each R_i satisfies reflexivity, completeness and transitivity. We dont assume that R social preference relation has to satisfy these three properties. We will find when R is going to be an ordering.

A collective choice rule is a functional relation F such that for any set of n individual orderings $R_1, R_2, ...R_n$ one and only social preference relation R is determined,

$R = f(R_1, R_2, ...R_n)$.

Suppose x, y, z. There are three individual. The preference relation are:

1	2	3
x	x	x
y	z	yz
z	y	

The collective choice rule is $f\begin{pmatrix} x & x & x \\ y & z & yz \\ z & y & \end{pmatrix}$

A collective choice rule is decisive if and only if its range is restricted to complete preference relations R.
For all $x, y \in X$
$x\bar{R}y$ if and only if for all $i : xR_i y$.
$x\bar{P}y$ if and only if $x\bar{R}y$ & $\sim (y\bar{R}x)$. Pareto preference.
$x\bar{I}y$ if and only if $x\bar{R}y$ & $y\bar{R}x$. Indifference.

From these Pareto relations we define collective choice rule under certain condition.

Result:
Relation $\bar{R}$ is a quasi -ordering for every logically possible combination of individual preferences.

Proof: $\bar{R}$ is reflexive as for all $x \in X : wehavexR_ix$. Since R_i are reflexive.
For all $x, y, z \in X$, $x\bar{R}y$ and $y\bar{R}z \rightarrow$ for all i xR_iy and yR_iz.
Since R_i are transitive.
It implies for all i : xR_iz. This implies $x\bar{R}z$. Thus, $\bar{R}$ is transitive.

Completeness can be violated. For example xP_1y and yP_2x then $x\bar{R}y$ and $y\bar{R}x$ are not defined.

Result:
A necessary and sufficient condition for $\bar{R}$ to be an ordering and for $R = \bar{R}$ to be decisive collective choice rule is that for all x,$x, y \in X$ if there exist i such that xP_iy then xR_jy for all j.

Proof: For any pair x, y if xI_iy for all i, then the condition is trivially true. If for i xP_iy then for all j we must have xR_jy. It implies $x\bar{R}y$.
If condition is violated than there exist i such that xP_iy and there exist j such that yP_jx. It implies that $x\bar{R}y$ is not possible and $y\bar{R}x$ is not possible. So $\bar{R}$ is not complete.

Completeness can be violated. For example xP_1y and yP_2x then $x\bar{R}y$ and $y\bar{R}x$ are not defined.

Result:
A necessary and sufficient condition for $\bar{R}$ to be an ordering and for $R = \bar{R}$ to be decisive collective choice rule is that for all $x, y \in X$ if there exist i such that xP_iy then xR_jy for all j.

Proof: For any pair x, y if xI_iy for all i, then the condition is trivially true. If for i xP_iy then for all j we must have xR_jy. It implies $x\bar{R}y$.
If condition is violated than there exist i such that xP_iy and there exist j such that yP_jx. It implies that $x\bar{R}y$ is not possible and $y\bar{R}x$ is not possible. So $\bar{R}$ is not complete.

We define Pareto Optimality:
For any n tuple of individual preferences $(R_1, R_2, ..., R_n)$, a state $x \in X$ is Pareto-optimal in X if and only if $\sim [\exists y \in X : y\bar{P}x]$.

A Pareto-optimal state is also called economically efficient.
It also implies that if an alternative or social state is preferred over other alternatives by all the individuals then society should also prefer it over other alternatives.
If xP_iy for all i then society should prefer x over y.
If x is not Pareto optimal it means that there exist an alternative Y which is preferred to x by all individuals.

Result:
For every set of individual preferences $(R_1, R_2, ...R_n)$ over any finite set of social states X, there is atleast one Pareto optimal state.

Proof: The binary relation $\bar{R}$ already defined is Pareto relation. We have already shown that $\bar{R}$ is quasi ordering. It is due to R_i being orderings. And Pareto optimal set is the maximal set of X with respect to $\bar{R}$. So set of Pareto optimal set is $M(X, \bar{R})$. We have also proved that Maximal set is non-empty when X is finite and the binary relation is quasi ordering. Since $\bar{R}$ is quasi ordering and X is finite, so $M(X, \bar{R})$ is non-empty.

Example: $X = \{x, y, z\}$.
Preference of the individuals:

1	2	3
x	x	x
z	zy	y
y		z

Here in this example x is only Pareto optimal state. y is not Pareto optimal because xPy for all individuals. Similarly z is not Pareto optimal, xPz for all individuals.

Another example

$X = \{x, y, z\}$.

Individual preferences:

1	2	3
x	y	x
y	x	z
z	z	y

Here in this example x and y are Pareto optimal state. z is not Pareto optimal because xPz for all individuals.

Assumptions

A social welfare function (SWF) is a collective choice rule f,, the range of which is restricted to the set of ordering over X.
We impose certain conditions on this social welfare function f.

Unrestricted domain (Condition U); The domain of the rule f must include all logically possible combinations of individual orderings.

f must have an image for all possible individual orderings over social states.

Example: $X = \{x, yz\}$. Examples of possible orderings are

1	2	3		1	2	3		1	2	3
x	y	z		z	xyz	y		y	x	zx
y	z	x		y		z		z	z	y
z	x	y		x		x		x	y	

Take the example of Pareto principle, we have $\bar{R}$ based on R_i. This $\bar{R}$ is defined if it is valid for all individuals. If we have

$$\begin{array}{ccc} 1 & 2 & 3 \\ x & y & z \\ y & z & x \\ z & x & y \end{array}.$$

We dont have $x\bar{R}y$, $x\bar{R}y$, $z\bar{R}y$, $y\bar{R}z$, $x\bar{R}z$ $z\bar{R}x$.
$\bar{R}$ is not complete, in this case.

Majority decision rule:
It is a collective choice rule where x is socially at least s good as y if and only if at least as many people prefer x to y as prefer y to x that $N(xPy) \geq N(yPx)$.

It violates unrestricted domain.
Take example , 6 individuals, xP_iy for 4 individuals, yP_iz for 4 individuals and zp_ix for 3 individuals. The social welfare violates transitivity.

Pareto Principle (Condition P): For any $x, y \in X$, for all i
$xP_iy \rightarrow xPy$.

If everyone prefers x to y then society must also prefer x to y. The social welfare function violates Pareto principle when everybody prefer x to y and the social relation gives y is preferred to x.

Independence of irrelevant alternatives (Condition I): let R and $r^{'}$ be the social binary relations determined by f corresponding respectively to two sets of indidividual preferences, $(R_1, R_2....R_n)$ and $R_1^{'}, R_2^{'}, ..., R_n^{'})$. If for all pairs of alternatives x, y in a subset of S of X, $xR_iy \leftrightarrow xR_i^{'}y$, for all i then $C(S, R)$ and $C(S, R^{'})$ are the same.

If for any two alternative x and y, the two binary relations R_i and $R_i^{'}$ have same relations then social preference must be same over x and y.

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Example

1	2	3
x	y	x
y	z	y
z	x	z

$$f\begin{pmatrix} x & y & y \\ y & z & y \\ z & x & z \end{pmatrix} = \begin{pmatrix} x \\ yz \end{pmatrix}$$

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If for any two alternatives x and y, the two binary relations R_i and $R_i^{'}$ have same relations then social preference must be same over x and y.

Borda Rule:
The preferences of 3 individuals over $\{x, y, z\}$ are;

1	2	3
x	y	x
y	z	z
z	x	y

In Borda rule rule assign number integers to each alternatives of individuals. The indiividuals have report their full preference ordering. Since there are three alternatives, so 2 points is given to the best alternatives, 1 is given to the second position and 0 is given to the last position. We aggregate the scores of each alternatives based on the individual preference ordering. The social preference relation is based on the aggregate score.

1	2	3
x 2	y 2	x 2
y 1	z 1	z 1
z 0	x 0	y 0

The aggregate score of x is $2 + 0 + 2 = 4$

$y: \ 1 + 2 + 0 = 3$

$z: 0 + 1 + 1 = 2$

The social preference relation based on these scores is $xPy, \ yPz$ and xPz. This is Borda outcome.

$$f \begin{pmatrix} x & y & x \\ y & z & z \\ z & x & y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Take another preference profile of three individuals, name is B and previous one is A

1	2	3
x	y	x
y	z	zy
z	x	

If we consider preference profile of A and B over the alternative x, y, we see that they are same.
$xP_1y,\ yP_2x,\ xP_3y$ in A.
$xP_1^{'}y,\ yP_2^{'}x,\ xP_3^{'}y$ in B.
Only difference between A and B is between z, and y for individual 3.
According to independence of irrelevant alternatives, the social preference relation between x and y should not be different in case of A and B.

Borda count of B is

1	2	3
x 2	y 2	x 2
y 1	z 1	zy 1
z 0	x 0	

So $x; 2+0+2=4$
$y : 1+2+1=4$
$z : 0+1+1=2.$
The social preference relation based on these scores is xIy, xPz and yPz.
We see that the social preference relation between x, y is not same in A and B. Therefore Borda rule violates independence of irrelevant alternative condition.

Majority rule: it satisfies independence of irrelevant alternatives.
Individual 1: xyz, Individual 2: zxy, Individual 3: yxz
$n(xpy) = 2$ and $n(yPx) = 1$.
Now change the position of z.
Like In Individual 1 $x(yz)$, zxy, $(zx)y$.
Individual 2 $x(yz)$, $(zx)y$, xzy.
Individual 3: yzx, zyx, $(yz)x$ and $y(xz)$. With any of these changes in the preferences of z, $n(xpy) = 2$ and $n(yPx) = 1$ remain same.

Non-dictatorship (Condition D); There is no individual i such that for every element in the domain of rule f, for all $x, y \in X$, $xP_iy \ \rightarrow xPy$.

There should not any individual such that the social preference is based on the preference of this person.
For example if xP_1y and yP_ix for all i except 1 and social preference is xPy. It violates non dictatorship.
Further we assume that there are atleast two individuals and atleast three alternatives.

A set of individuals V is almost decisive for x against y if xPy whenever xP_iy for every $i \in V$. and yP_ix for every $i \notin V$.
For example

1	2	3
x	y	x
y	z	z
z	x	y

$$f\begin{pmatrix} x & y & x \\ y & z & z \\ z & x & y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For x, y individual 1 and 3 forms a almost decisive set.

A set of individuals V is decisive for x against y if xPy when xP_iy for every $i \in V$.

1	2	3
x	z	x
y	yx	z
z		y

$$f\begin{pmatrix} x & z & x \\ y & yx & z \\ z & & y \end{pmatrix} = \begin{pmatrix} x \\ yz \\ \ \end{pmatrix}$$

Individual 1 and 3 forms a decisive set for x, y. It is almost decisive for x, z.

Suppose there is an individual J who is almost decisive over x and y and denote it by $D(x, y)$. $D(\bar{x}, y)$ denotes that J is decisive over x, y.

A set of individuals V is almost decisive for x against y if xPy whenever xP_iy for every $i \in V$ and yP_ix for every $i \notin V$.
A set of individuals V is almost decision for a pair $x, y \in X$, $[xP_iy \ \forall i \in V \ \& \ yP_ix \forall i \notin V] \ \rightarrow \ xPy$.

A set of individuals V is decisive for x against y if xPy when xP_iy for every $i \in V$.
A set of individuals V is decisive for a pair $x, y \in X$ $[xP_iy \forall i \in V] \ \rightarrow \ xPy$.

Notice that it is possible that a set V can be almost decisive but not decisive.
Suppose $xP_iy\forall i \in V$ is true and there exists $j \notin V$ such that xR_jy and the social preference is yRx. It satisfies almost decisive but not decisive.
Therefore, it is not always true that $D(x,y) \rightarrow \bar{D}(x,y)$.

But $\bar{D}(x,y) \rightarrow D(x,y)$.
If $xP_iy\forall i \in V$ and $j \notin V$, xR_jy, and xPy then if part of the definition of almost decisive set is is false so the state is trivially true. It is decisive.

Lemma 1
If there is some individual J who is almost decisive for any ordered pair of alternative, then an Social welfare function satisfying conditions U, P, and I implies that J must be a dictator.

Proof: Suppose that person J is almost decisive for some x against some y, it means $\exists\ x,\ y \in X : D(x, y)$. Let z be another alternative and let i refers to all individuals other than J. Assume $xP_jy\&yP_Jz$, and that yP_ix & yP_iz. We have not specified the preferences of the persons other than J between x and z.
Now $[D(x, y)\&xP_jy\&yP_ix] \rightarrow xPy$. Further $[yP_jz\&yP_iz] \rightarrow yPz$ from the condition P Pareto principle. But $[xPy\&yPz] \rightarrow xPz$, by the transitivity of the strict social preference relation.

We have xPz without any assumption on the preference relation over x and z on part of the individuals other than J. We have only assumed yP_iz and yP_ix. Now, these ranking of x and y and the ranking of y and z has any effect on the ranking of x and z it violates condition I , independence of irrelevant alternatives.. Hence xPz must be independent of the assumptions on x and y and y and z. Also, xPz is the consequence of xP_jz alone without having any effect from the ordering of i s. This means that J is decisive for x against z.
We get $D(x, y) \to \bar{D}(x, z)$ (1).

Now suppose $zP_jx\&xP_jy$, and $zP_ix\&yP_ix$. We have zPx from Pareto condition. And $D(x,y)\&xP_Jy\&yP_ix$, we have xPy. Using transitivity of strict social preference relation we have,
$zPx\&xPy \rightarrow zPy$. Again, we have got the social relation over z and y without specifying the individual preference relation of i s. Hence J is decisive for z against y.
We get $D(x,y) \rightarrow \bar{D}(z,y)$ (2).
We can also show $D(x,z) \rightarrow \bar{D}(y,z)$ (3)
by interchanging z with y in (2).

Again putting x in place of z, z in place of y, and y in place of x, we obtain from (1)
$D(y,z) \to \bar{D}(y,x)$ (4).
We have $D(x,y) \to \bar{D}(x,z)$ from (1)
$\to D(x,z)$ using the definition of decisive and almost decisive
$\to \bar{D}(y,z)$ from (3)
$\to D(y,z)$ from definition
$\to \bar{D}(y,x)$ from (4).
We get that $D(x,y) \to \bar{D}(y,x)$ (5).
By interchanging x and y in (1), (2) and (5), we get
$D(x,y) \to [\bar{D}(y,z) \& \bar{D}(z,x) \& \bar{D}(x,y)]$ (6).

Now, $D(x, y) \rightarrow \bar{D}(y, x)$ from (5)
$\rightarrow D(y, x)$.
Hence from (6) we have
$D(x, y) \rightarrow [\bar{D}(y, z) \& \bar{D}(z, x) \& \bar{D}(x, y)]$ (7).
Combining (1), (2), (5) and (7), it is seen that $D(x, y)$ implies that individual J is decisive for every ordered pair of alternatives from the set of three alternative $\{x, y, z\}$ given the condition U, P and I. Thus, J is a dictator over any set of three alternatives containing x and y.

Now, consider a larger number of alternatives. Take any two alternatives u and v out of the entire set of alternatives. If u and v are so chosen that they are same as x and y, then $\bar{D}(u, v)$ holds, as can be shown by taking a triple consisting of u, v and any other alternative z. If one of u and v is same as one of x and y, say u and x are same but not v and y, then take the triple consisting of x(*or* u), y, and v. Since $D(x, y)$, holds it again follows that $\bar{D}(u, v)$ and also $\bar{D}(v, u)$.

Let both u and v be different from x and y. Now first take $\{x, y, u\}$ and we get $\bar{D}(x, u)$ which implies $D(x, u)$. Now take the triple $\{x, u, v\}$. Since $D(x, u)$, it follows from previous argument that $\bar{D}(u, v)$ and also $\bar{D}(v, u)$. Thus, $D(x, y)$ for some x and y, implies $\bar{D}(u, v)$ for all possible ordered pairs (u, v). Therefore, individual J is a dictator.

Example

1	2	3
x	y	y
y	z	x
z	x	z

Suppose individual 1 is the almost decisive individual over x, y. We have yP_2x and yP_3x. So social preference over x, y is xPy. We have $yP_iz, i = 1, 2, 3$. So from Pareto principle, we have yPz. Now using transitivity we have $xPy \& yPz \rightarrow xPz$.

1	2	3
x	y	z
y	z	y
z	x	x

Here in this example individual 1 is almost decisive over x, y. So we have xPy. For other pair (x, z) and (y, z), we cannot say anything by simply using the properties U, P, I. Thus, in this case we do not have a social preference relation.

Here, the if part in the second part of the statement of lemma is false. So lemma is trivially true when there is an almost decisive individual.

Theorem: There is no social welfare function satisfying conditions U, P, I and D.

Proof: For any pair of alternatives, there is atleast one decisive set, the set of all individuals. It is due to Pareto principle. Thus, for every pair of alternatives there is also at least one almost decisive set, since a decisive set is also almost decisive. Lets compare all the sets of individuals that are almost decisive for some pair, it may not be same pair of alternatives. From these set of almost decisive sets chose the smallest decisive set. Let this set be called V and let it be almost decisive for x against y.

If V contains only one individual, then the lemma we get that individual is decisive over all pairs of alternative, thus dictator.
Suppose V contains two or more individuals. We divide V into two parts, V_1 containing a single individual, and V_2 contains the rest of V. V_3 contains all individuals not contain in V.
Due to U we can take any logically possible combination of individual orderings. Suppose we have
For all $i \in V_1$, $xP_iy \& yP_iz$.
For all $j \in V_2$, $zP_jx \& xP_jy$.
For all $k \in V_3$, $yP_kx \& zP_kx$.
Since V is almost decisive for x against y and since X is preferred to y for all individuals in V and y is preferred to x for all in V_3, so xPy.

Note that zP_iy for $jinV_2$ and for the rest we have y is preferred to z. It implies that V_2 is almost decisive for y, z , if we have zPy. Then we get a contradiction because V_2 is smaller than V. We have assume that V is the smallest almost decisive set. Therefore, we cannot have zPy. It implies that for y and z pair we have yRz. Now $xPy\&yRz \rightarrow xPz$. Now only individual in V_1 prefers x to z and all others prefer z to x. Thus, we have V_1 as almost decisive set for the pair x and z. Thus, we get a contraction since we assume V is the smallest almost decisive set. Thus, our assumption that V contains two or more is not true. Thus, V contains only one individual. Using the lemma we get that this individual is decisive for all the pairs of alternatives. Thus, there is a dictator whenever a social welfare function satisfies U, P and I.

Example:

$$\begin{array}{ccc} 1 & 2 & 3 \\ x & z & x \\ z & x & y \\ y & y & z \end{array}$$ What can we say about the social preference?

From Pareto principle we can say xPy. We cannot say anything further based on the four conditions of Arrow's Theorem.

Suppose We have one more additional information that yPz. Then using transitivity we have xPz. Thus the social preference is

$$f\begin{pmatrix} x & z & x \\ z & x & y \\ y & y & z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Note here that individual 3 has become the dictator.
The additional information is giving that individual 3 is almost decisive for the pair y, z.

Instead if we had the additional information zPy. In this we have xPy and zPy. We cannot say anything about z and x.
We need further information, this can be either zPx or xPz . If zPx then we individual 2 as dictator. If xPz the we have individual 1 as dictator.
If we have xIz. We dont have any dictator. In this case also we need additional restriction. Here it is not in the form of almost decisive set.

Example:

1	2	3
x	z	x
y	x	z
z	y	y
t	t	t

From the Pareto principle we get
xPy, yPt, zPt, and xPt.
We need additional information on y and z pair. If yPz, then by transitivity we have xPz. Thus, the social preference ordering is

$$f\begin{pmatrix} 1 & 2 & 3 \\ x & z & x \\ y & x & z \\ z & y & y \\ t & t & t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$

Note Individual 1 is decisive, so the dictator.

Condorcet Paradox:
There exists profiles of individual preferences such that method of majority decision does not constitute a social welfare function.

Proof: Suppose there are three individual, The preference profile is

1	2	3
x	y	z
z	x	y
y	z	x

The method of majority decision gives social preference relation in the following way

$xRy \leftrightarrow N(i \in N, xP_iy) \geq N(i \in N, yP_ix)$.

$xRy \leftrightarrow N(i \in N, xR_iy) \geq N(i \in N, yR_ix)$.

Here, $N(yP_ix) = 2 > N(xP_iy) = 1$ it implies yPx.
$N(zP_iy) = 2 > N(yP_iz) = 1$, it implies zPy.
$N(xP_iz) = 2 > N(zP_ix) = 1$, it implies xPz.
We have $zPy \& yPx \ \rightarrow zPx$, from transitivity. But we have xPz. It violates transitivity. Therefore, it is not a social welfare function.

Voting Methods:

- Based on scoring method:
  - Plurality Method
  - Borda Count
- Pairwise comparison
  - Simple Majority
  - Absolute Majority
  - Two-third Majority
- Voting procedure
  - Instant run-off

Scoring Method

Based on preferences of the individual scores are given to the alternative. Based on aggregate score we derive the social preference.

Suppose individual preferences are

1	2	3
x	y	y
y	z	x
z	x	z

Plurality Method: scoring rule is $(1, 0, 0)$. Give one point to the most preferred alternative and zero to all other alternatives. So, the scores in above preference profile is

1	2	3
x 1	y 1	y 1
y 0	z 0	x 0
z 0	x 0	z 0

Aggregate scores are $x : 1$, $y : 2$, $z : 0$.

So the social preference is

$$\begin{pmatrix} y \\ x \\ z \end{pmatrix}$$

The winner in the election using plurality voting method is y.

Borda Count: scoring rule is $(2, 1, 0)$. If there are four alternatives, the scoring rule is $(3, 2, 1, 0)$. The scores in the given individual preference profile is

1	2	3
x 2	y 2	y 2
y 1	z 1	x 1
z 0	x 0	z 0

The aggregate scores are

$x : 3$, $y : 5$, $z : 1$.

The social preference ordering is

$$\begin{pmatrix} y \\ x \\ z \end{pmatrix}.$$

The winner is y, if this method is used.

The social ranking may not same in the case of Plurality and Borda count.

1	2	3
x	y	y
z	x	xz
y	z	

Plurality rule gives social preference as $\begin{pmatrix} y \\ x \\ z \end{pmatrix}$.

Borda Count method gives social preference as $\begin{pmatrix} xy \\ z \end{pmatrix}$.

Pairwise comparison

Simple majority : $xRy \leftrightarrow N(xP_iy) \geq N(yP_ix)$.

1	2	3
x	y	y
y	z	x
z	x	z

Simple majority rule: $N(xP_iy) = 1 < N(yP_ix) = 2$, $N(yP_iz) = 3 > N(zP_iy) = 0$, $N(xP_iz) = 2 < N(zP_ix) = 1$.

Social preference: yPx, yPz, xPz.

Absolute majority: $xRy \leftrightarrow N(xP_iy) \geq \frac{N}{2} + 1$. If we have xRy based on absolute majority , we will also have xPy since yRx is not possible.

1	2	3
x	y	y
y	z	x
z	x	z

Absolute majority: $N(xP_iy) = 1 < N(yP_ix) = 2$,
$N(yP_iz) = 2 > N(zP_iy) = 1$, $N(xP_iz) = 2 < N(zP_ix) = 1$.
Social preference : yPx, yPz, xPz.

Two-third majority: $xRy \leftrightarrow N(xP_iy) \geq \frac{2N}{3}$.

1	2	3
x	y	y
y	z	x
z	x	z

Two-third majority: $N(xP_iy) = 1 < N(yP_ix) = 2$,
$N(yP_iz) = 3 > N(zP_iy) = 0$, $N(xP_iz) = 2 < N(zP_ix) = 1$.
Social preference : yPx, yPz, xPz.

Example:

1	2	3
x	x	yz
y	z	x
z	y	

.

Simple majority: xPy, yIz and xPz.

Absolute majority: xPy and xPz, not possible to compare y and z. It is not complete.

Two-third majority: xPy and xPz, not possible to compare y and z. It is not complete.

Possibility result:
Simple majority violates transitivity. If we move away from social welfare function which requires transitivity, can we generate a reflexive, complete and acyclic social ordering. So that we have social choice function .

A social decision function is a collective choice rule f, the range of which is restricted to those preference relations$R$, each of which generates a choice function $C(S, R)$ over the whole set of alternatives X.

Result: There is an Social decision function satisfying condition U, P, I and D.

Proof: We provide one example. Since we have to show existence, it proofs the statement.
Suppose $xRy \leftrightarrow \sim [(\forall i : yR_ix)\&(\exists i : yP_ix)]$.
This R is reflexive and complete. It is easy to see that P is satisfied. For pair x, y, the social relation is based on individual preferences over x and y, there is no role any other alternative , say z. So it satisfies I.
Suppose xP_1y and yP_2x and xP_3y, here we have both xRy and yRx, so social preference is xIy. The condition D is satisfied.
We have to show that quasi-ordering is satisfied.
$[xPy\&yPz] \rightarrow [\{\forall i : xR_iy\&\exists i : xP_iy\}\&\forall i : yR_iz]$
$\rightarrow [\forall i : xR_iz\&\exists i : xP_iz]$.
$\rightarrow xPz$.

Thus, R is quasi-transitive. We know that if R is quasi transitive, complete and reflexive, there is going to be best element for each non-empty subset S.Thus, there exist a social decision function.

A collective choice rule gives x is preferred to y if it is Pareto-superior to y. xRy if y is not Pareto superior to x.
1: xyz , 2: yzx, 3: zxy.
We have y is not Pareto superior to x, x is not Pareto superior to y, y is not Pareto superior to z, z is not Pareto superior to y. So We have xIy and yIz. For x, z also, we have xIz.

- Aggregation of individual preference to reach a social preference relation.
- Arrow's Theorem says that we cannot have a social welfare function satisfying unrestricted domain, Pareto principle, Independence of irrelevant alternatives, and non-dictatorship.
- We know that simple majority violates transitivity.
- Borda count violates independence of irrelevant alternatives.
- Instead of social welfare function, if we consider social decision rule - a function for which choice set is defined, we get possibility result.
- Social preference relation based on Pareto superiority and Pareo non-superiority.

- May's Theorem- it characterizes simple majority rule as a decisive collective choice rule.
- By restricting the domain, number of possibility results can be generated.

Paretian Liberal Paradox

Liberal value means that for some alternatives which are personal to some individuals, those relevant individuals should be free to do what they like. If relevant person is allowed to do whatever he likes in case of his personal things, keeping everything else constant, the social state will be at a better state.

Liberal values is defined as (condition L): For each person i there is at least one pair of distinct alternatives (x, y) such that he is decisive in the social choice between them in either order that is $xP_iy \rightarrow xPy$ and $yP_ix \rightarrow yPx$.

Decisiveness is taken for every individual for a particular distinct pair of alternatives. The weaker version of this definition can be to restrict the number of decisive individual.

Minimal Liberalism (condition L^*): there are at least two persons k and j and two pairs of distinct alternatives (x, y) and (z, w) such that k and j are decisive over (x, y) and (z, w) respectively, each pair taken in either order.
It is obvious that condition L implies L^*.

Theorem: There is no social decision function satisfying conditions U, P and L^*.

Proof: When (x, y) and (z, w) are same pair then condition L^* does not hold. Suppose the pair has one of the elements in common, $x = z$ then let $xP_k y,\ wP_j x$ and for all i $yP_i w$. By condition L^* we have xPy and wPx. By condition P we have yPw. This violates acyclicity. We have $xPy \& yPw$ and wPx. Therefore, we dont any best element. Thus, choice set is empty. We cannot have social decision function.
Suppose all four alternatives x, y and z, w are distinct. We assume that $xP_k y$, $zP_j w$ and for all i we have $wP_i x \& yP_i z$.

These orderings are possible for example

k	j	l
w	y	w
x	z	y
y	w	x
z	x	z

By condition L^* we have xPy and zPw. By condition P we have wPx and yPz.

We have xPy & yPz & zPw and wPx. It violates acyclicity. Therefore, there is no best element. Thus, choice set is empty. We cannot have a social decision function.

In this theorem we dont have independence of irrelevant alternatives and transitivity of social preference relation and still we are getting a negative result.

The negative result implies that the four conditions U, P, L^* and SDF are not compatible together.
If we relax these conditions we may get possibility result.

We may drop acyclicity condition on social preference. If we drop acyclicity on the social preference relation R, then R may not satisfy condition α. We may have a situation like x is chosen from (x, y), y is chosen from (y, z) and z is chosen from (x, z) and x is chosen from (x, y, z). This social choice is not consistent is the way we have defined consistent choice based on property α and β.

Dropping Pareto principle means we have a social preference relation which gives outcome oposite of what everybody likes. It is very natural to have such assumption we want the social choice to be based on individual preference ordering.

We may weaken the definition of minimal liberalism by reducing the number of decisive individual. Already we have only two individuals. If we make it one, then we may have that individual as dictator. It will be against liberal values.

We may restrict the domain. It is often done in the literature. We have to be careful in doing so.

Example:

k	j	l
x	y	y
y	x	z
w	z	x
z	w	w

Individual k is decisive over x, y and Individual j is decisive over z, w. We get xPy and zPw. From Pareto we have yPz and xPw. Thus, we have xPy, yPz, zPw and xPw. We have x as the best element in this social preference.

A game is played between players, where each player has to choose some strategies or strategy and each player gets payoff.
Example chess, tic-tac-toe,
Actions of players are : moving their pieces at their turn. The turns come sequentially. Payoff: the satisfaction of winning a game of chess.
Actions of players are: marking different positions at their turn. The turn comes sequentially. Payoff: satisfaction of winning a game of tic-tac-toe.

Three friends can jointly invest and put effort in a project and get a profit from this project. How to share the profit among the three friends. It may happen only two form a coalition out of the three. Action: to form coalition. Payoff: share of the profit.
There are two sellers selling an object and one buyer who wants to buy only one unit of the object. Both the seller wants to sell to the buyer. The buyer can buy from seller 1 or seller 2.
Action: seller 1 and seller 2 to sell and buy to buy. It can be understood as coalition $\{seller1, buyer\}$, $\{seller2, buyer\}$, $\{seller1, seller2, buyer\}$.
In the first two coalition, if they form then transaction happens. In the third coalition, their may be sharing between seller 1 and seller 2 in the transaction with the buyer. The payoffs are gain from the trade or transaction to the buyer and seller.

We study strategic situations using game. While choosing action or strategy a player always consider how others are going to react or behave. We use Nash equilibrium to study such situation.

We are going to study only coalition formation game. We expand the profit sharing game.

Suppose there are three students $\{A, B, C\}$. This is the set of players. Jointly they can produce the following values or profits.

Coalitions	Value or worth of coalitions
$\{A\}$	10
$\{B\}$	10
$\{C\}$	10
$\{A, B\}$	20
$\{A, C\}$	20
$\{B, C\}$	30
$\{A, B, C\}$	50

Here the question are
How to divide the worth or value of the coalition and which one of the coalitions are going to the be formed?

Suppose one person A owns an old car which values nothing to him. There are two potential buyers, Buyer B values it at 1000 and buyer C values it at 1050. The trade between these people can be analysed based on coalition formation.

Coalitions	Value or worth of coalitions
$\{A\}$	0
$\{B\}$	0
$\{C\}$	0
$\{A, B\}$	1000
$\{A, C\}$	1050
$\{B, C\}$	0
$\{A, B, C\}$	1050

Suppose seller sells at p_B to B, then B will buy if $p_B \leq 1000$. Thus, the value of A is p_B and value of B is $1000 - p_B$. So $v(\{A, B\}) = p_B + 1000 - p_B = 1000$.

Suppose seller sells at p_C to C, then C will buy if $p_C \leq 1050$. Thus, the value of A is p_C and value of C is $1050 - p_C$. So $v(\{A, C\}) = p_C + 1050 - p_C = 1050$.

Suppose seller A sells at p_C to C and C offers p_c to seller and $1000 - p_c$ to buyer B. So the values are P_c to A, $1000 - P_c$ to buyer B and $1050 - p_c - (1000 - p_c)$ to buyer C. The worth of the coalition is
$v(\{A, B, c\}) = p_C + (1000 - p_c) + (1050 - p_c - (1000 - p_c) = 1050$.

Suppose seller A sells at p_C to C and C offers p_c to seller and buyer B gets zero. So the values are P_c to A, 0 to buyer B and $1050 - p_c$ to buyer C. The worth of the coalition is $v(\{A, B, c\}) = p_C + (1000 - p_c) + 0 = 1050.$
What is going to be the outcome?

There are going to be n players, $N = \{A_1, A_2, A_3 ... A_N\}$ a finite set of players and $n \geq 2$.
A subset S of N is called a coalition of N.
The entire set is called the grand coalition.
If there are N players then we have 2^N possible coalitions.
If we have 2 players, 8 coalitions are possible;
$\{\emptyset\}, \{A_1\}, \{A_2\}, \{A_3\}, \{A_1, A_2\}, \{A_1, A_3\}, \{A_2, A_3\}, \{A_1, A_2, A_3\}$
Suppose $S \subset N$, it means S is coalition then complement of S is $N \backslash S$ is the set of all players not in S but in N.
$|S|$ denotes the number of players in the coalition S.

The worth of a coalition is defined based on a function called characteristic function.

A function $v : 2^N \rightarrow R_+$ such that $v(\emptyset) = 0$, where R_+ denotes non negative real numbers.

An example

Coalitions	Value or worth of coalitions
$\{A\}$	0
$\{B\}$	0
$\{C\}$	0
$\{A, B\}$	1000
$\{A, C\}$	1050
$\{B, C\}$	0
$\{A, B, C\}$	1050
$\{\emptyset\}$	0

A function $v : 2^N \to R_+$ such that $v(\emptyset) = 0$, where R_+ denotes non negative real numbers.
$v()$ is a characteristic function. There are games of the following nature

$v(N) = 0$
$v(S) = -v(N \setminus S)$ for all subsets S of N.
$v(\emptyset) = 0$.
$v(S \cup T) \geq v(S) + v(T)$ where S and T are disjoint subsets of N.

$v()$ is a characteristic function of zero -sum game.
Here in this case, the range of the characteristic function is all real number
$v : 2^N \to R$ such that $v(\emptyset) = 0$ and R denotes real numbers.

G^N denotes the set of all characteristic form games with transferable utility.

What do we mean by transferable utility.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 0$
$\{2\}$	$v(\{2\}) = 0$
$\{3\}$	$v(\{3\}) = 0$
$\{1,2\}$	$v(\{1,2\}) = 1$
$\{1,3\}$	$v(\{1,3\}) = 1$
$\{2,3\}$	$v(\{2,3\}) = 1$
$\{1,2,3\}$	$v(\{1,2,3\}) = 3$

Here, the value of coalition can be shared in any way like $v(\{1,2\}) = 1$ can be shared as $(x_1, x_2) = (0.2, 0.8)$, $(x_1, x_2) = (0.5, 0.5)$. Each allocation gives a specific utility to each player. When we represent the game based only on characteristic function, we assume that payoffs or sharing of values can be done in any way and we dont need to specify the utility functions of each player.

v a particular type of characteristic function denotes a specific coalitional game, it is an element of G^N.

Given a game $v \in G^N$, for any player $A_i \in N$, for any coalition $S \subseteq N \setminus \{A_i\}$, the marginal contribution A_i makes to the expanded coalition $S \cup \{A_i\}$ by joining is $v(S \cup \{A_i\}) - v(S)$.
The marginal contribution of player A_i to the coalition S, if it joins.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 0$
$\{2\}$	$v(\{2\}) = 0$
$\{3\}$	$v(\{3\}) = 0$
$\{1,2\}$	$v(\{1,2\}) = 1$
$\{1,3\}$	$v(\{1,3\}) = 1$
$\{2,3\}$	$v(\{2,3\}) = 1$
$\{1,2,3\}$	$v(\{1,2,3\}) = 3$

$v(\{1,2,3\}) - v(\{1,2\}) = 3 - 1 = 2$ is the contribution of player 3 in the coalition $\{1,2,3\}$.
$v(\{1,2,3\}) - v(\{1,3\}) = 3 - 1 = 2$ is the contribution of player 2 in the coalition $\{1,2,3\}$.

A game $v \in G^N$ is called super additive if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subset N$ such that $S \cap T = \emptyset$. The worth of a coalition is higher than the worth of the individual players. The worth of a bigger coalition is higher than the worth of a smaller coalition.

A super additive game is cohesive if $v(N) \geq \sum_{j=1}^{k} v(S_j)$ for every partition $\{S_1, S_2, S_3 ... S_k\}$ of N.
A partition $\{S_1, S_2, S_3 ... S_k\}$ of set N means $\cup_{j=1}^{k} S_j = N$ and $S_i \cap S_j = \emptyset$ when $i \neq j$.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 0$
$\{2\}$	$v(\{2\}) = 0$
$\{3\}$	$v(\{3\}) = 0$
$\{1,2\}$	$v(\{1,2\}) = 1$
$\{1,3\}$	$v(\{1,3\}) = 1$
$\{2,3\}$	$v(\{2,3\}) = 1$
$\{1,2,3\}$	$v(\{1,2,3\}) = 3$

It is super additive game.

A game $v \in G^N$ is called sub-additive if $v(s \cup T) \leq v(S) + v(T)$ for all $S, T \subset N$ such that $S \cap T = \emptyset$.
Cost sharing, as the size of the coalition increases, the aggregate cost decreases.

A game is additive if $v(s \cup T) = v(S) + v(T)$ for all $S, T \subset N$ such that $S \cap T = \emptyset$.

A game $vinG^N$ is called inessential if $v(N) = \sum_{A_i \in N} v(\{A_i\})$.

A game $vinG^N$ is called essential if $v(N) > \sum_{A_i \in N} v(\{A_i\})$.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 3$
$\{2\}$	$v(\{2\}) = 3$
$\{3\}$	$v(\{3\}) = 3$
$\{1,2\}$	$v(\{1,2\}) = 2$
$\{1,3\}$	$v(\{1,3\}) = 2$
$\{2,3\}$	$v(\{2,3\}) = 2$
$\{1,2,3\}$	$v(\{1,2,3\}) = 1$

Sub-additive game.

A game $v \in G^N$ is called symmetric if for any $s, T \subseteq N$ with $|S| = |T|$, we have $v(S) = v(T)$.

A game $v \in G^N$ is monotonic if $S \subseteq T \subseteq N$ implies that $v(S) \leq v(T)$.

A game $v \in G^N$ is constant sum if for all $S \subseteq N, \; v(S) + v(N \setminus S) = v(N)$.

Suppose there are seven players $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$, suppose there are two disjoint set L and R of these players. $R = \{A_1, A_4, A_7\}$ and $L = \{A_2, A_3, A_5, A_6\}$. Players of set R has right shoe and players in set L has left shoe. A pair of shoe contains two shoes - left and right. A pair of shoe worth 1 and if there is only left or only right, it has no value. Consider set $S = \{A_1, A_3, A_6\}$, so one right shoe. $S \cap R = \{A_1\}$ and $L \cap S = \{A_3, A_6\}$, two left shoes. If there is cooperation between the players in S, then it will have one pair of shoe. The minimum number of left or right shoes determine the number of pairs. So the worth of coalition is $\min\{|S \cap R|, |S \cap L|\}$, when $|S| \geq 2$. We get the following characteristic function.

$$v(s) = \begin{cases} 0, \ \ if \ \ |S| \in \{0, 1\}, \\ \min\{|S \cap R|, |S \cap L|\}, \ \ if \ \ |S| \geq 2. \end{cases}$$

$V(N) = \min\{|R|, |L|\}$.

Example:
Suppose there are two sellers and two buyers of a product. Each seller has one unit of the good and the worth of the object to the sellers is 150. Each buyer wants one unit and values it 200. What is the characteristic function of this game?

Set of players is $\{A_1, A_2, A_3, A_4\}$, $\{A_1, A_2\}$ is the set of sellers and $\{A_3, A_4\}$ is the set of buyers.
We have $v(\{A_1\}) = 150$, $v(\{A_2\}) = 150$, $v(\{A_3\}) = 0$, $v(\{A_4\}) = 0$.
$v(\{A_1, A_2\}) = 300$,
$v(\{A_3, A_4\}) = 0$, coalition of only buyers cannot generate any value.

$$v(\{A_1, A_3\}) = v(\{A_1, A_4\}) = v(\{A_2, A_3\}) = v(\{A_2, A_4\}) = 200.$$

Suppose seller 1 sells at p_1 to buyer 3. This $p_1 \geq 150$ and $p_1 \leq 200$, the valuation of the seller 1 is 150 and the valuation of the buyer is 200. If this coalition forms then seller 1 gets p_1 and buyer 3 gets $200 - p_1$. So $v(\{A_1, A_3\}) = p_1 + (200 - p_1) = 200$.

$$v(\{A_1, A_2, A_3\}) = v(\{A_1, A_2, A_4\}) = 350.$$

Suppose seller 1 and 2 want to sell to buyer 3. Seller 1 can give p_2 to seller 2 for not selling to buyer 3 and sell at p_3 to buyer 3. Gain to seller 1 is $p_3 - p_2$, so we want $p_3 > p_2$. The gain to seller 2 is $150 + p_2$ and to buyer 3 is $200 - p_3$. The value of the coalition is $v(\{A_1, A_2, A_3\}) = (p_3 - p_2) + (150 + p_2) + (200 - p_3) = 350$.

$v(\{A_1, A_3, A_4\}) = v(\{A_2, A_3, A_4\}) = 200$

Suppose buyer 3 and 4 want to buy from seller 1. Buyer 3 can bribe Buyer 4 by paying p_4 so that buyer 4 does not make any offer to seller 1. Buyer 3 will offer p_5 to seller 1 to get the object. The gain to seller 1 is p_5, gain to buyer 4 is p_4 and gain to buyer 3 is $200 - p_4 - p_5$. The value of the coalition is $v(\{A_1, A_3, A_4\}) = p_5 + p_4 + (200 - p_4 - p_5) = 200$.

$v(\{A_1, A_2, A_3, A_4\} = 400$.

Seller 1 sells to buyer 3 and seller 2 sells to buyer 4. Seller 1 sells at p_6 to buyer 3 and seller 2 sells at p_7 to buyer 4. The gain are, seller 1 gains p_6, seller 2 gains p_7, buyer 3 gains $200 - p_6$, buyer 4 gains $200 - p_7$. The value of the coalition is $v(\{A_1, A_2, A_3, A_4\} = p_6 + p_7 + (200 - p_6) + (200 - p_7) = 400$.

The outcome of a coalition game is payoffs to each player, an allocation of the value of the coalition.

Given a game $v \in G^N$, an outcome of the game or allocation (a payoff vector) is an $n-$ coordinated vector $x = (x_{A_1}, x_{A_2}, x_{A_3}, ...x_{A_n})$.

x_{A_i} is the $i-$ th coordinate of allocation vector $x = (x_{A_1}, x_{A_2}, x_{A_3}, ...x_{A_n})$ denotes the amount received by player i.

$x(S) = \sum_{A_i \in S} x_{A_i}$ for any subset S of N. It denotes the sum of the allocations or pay-offs received by players in coalition S.

While choosing which coalition to be part of each player is going to compare the payoffs received that coalition and what it receives if it remains on his own or stay alone.

Given a game $v \in G^N$, a payoff vector or allocation x is called individually rational if $x_{A_i} \geq v(\{A_i\})$ for all $A_i \in N$.
Each player should get in a coalition whatever he gets by staying alone.

Given a game $v \in G^N$, a pay-off vector or allocation x is called totally rational or Pareto efficient if $x(N) = v(N)$.
The aggregate amount generated under a payoff vector or allocation must be equal to the amount earned by the grand coalition.

$x(N) = v(N)$ is taken as two inequalities

$x(N) \leq v(N)$, sum of payoff vector or allocation must be feasible.
$x(N) \geq v(N)$, the grand coalition cannot value or earn more than $x(N)$.

Given a game $v \in G^N$, an imputation in v is a payoff vector or allocation which is individually rational and Pareto efficient (totally rational).
The set of all imputations associated with $v \in G^N$ is denoted by $I(v)$.
Imputation is payoff vector or allocation that assign each players at least as much as he can earn on his own and assign all the players together the maximum value they can create when the grand coalition is formed.

Core:

Given a game $v \in G^N$, the core of v is the set of all imputations x in $I(c)$ such that $x(S) \geq v(s)$ for all non-empty coalitions $S \subset N$. The core of a game $v \in G^N$ is denoted by $C(v)$.

If $x(S) < v(S)$ then by dividing $\frac{v(s)-x(S)}{|S|}$ among the members of S, each player in S can be made better off. So, an allocation x such that $x(S) < v(S)$ is not in core.

Further, in an allocation in core each player should receive what it gets when it stays alone. If an allocation is core allocation then to make one player better - off, another player has o be made worst-off.

Example: A bargaining problem between two person. They want to divide a cake of size 1 between them. If they fail to divide each get zero.

$x_{A_1} \geq 0, x_{A_2} \geq 0$ and $x_{A_1} + x_{A_2} = 1$. These are the core allocation.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 0$
$\{2\}$	$v(\{2\}) = 0$
$\{3\}$	$v(\{3\}) = 0$
$\{1,2\}$	$v(\{1,2\}) = 1$
$\{1,3\}$	$v(\{1,3\}) = 1$
$\{2,3\}$	$v(\{2,3\}) = 1$
$\{1,2,3\}$	$v(\{1,2,3\}) = 3$

See figure 2

Core:

Given a game $v \in G^N$, the core of v is the set of all imputations x in $I(c)$ such that $x(S) \geq v(s)$ for all non-empty coalitions $S \subset N$. The core of a game $v \in G^N$ is denoted by $C(v)$.

Example 1

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 0$
$\{2\}$	$v(\{2\}) = 0$
$\{3\}$	$v(\{3\}) = 0$
$\{1,2\}$	$v(\{1,2\}) = 1$
$\{1,3\}$	$v(\{1,3\}) = 1$
$\{2,3\}$	$v(\{2,3\}) = 1$
$\{1,2,3\}$	$v(\{1,2,3\}) = 3$

We want to find the core allocation of the above coalition game.
We have $v(\{1\}) = 0 \leq x_{A_1},\ v(\{2\}) = 0 \leq x_{A_2},\ v(\{3\}) = 0 \leq x_{A_3}$.
$v(\{1,2\}) = 1 \leq x_{A_1} + x_{A_2},\ v(\{1,3\}) = 1 \leq x_{A_1} + x_{A_3},\ v(\{2,3\}) = 1 \leq x_{A_2} + x_{A_3}$.
$v(\{1,2,3\}) = 3 \leq x_{A_1} + x_{A_2} + x_{A_3}$.

Substituting $v(\{2,3\}) = 1 \leq x_{A_2} + x_{A_3}$ in $v(\{1,2,3\}) = 3 = x_{A_1} + x_{A_2} + x_{A_3}$. We have $2 \geq x_{A_1}$, similarly we get $2 \geq x_{A_2}$ and $2 \geq x_{A_3}$.
Therefore, core allocations are $2 \geq x_{A_1} \geq 0$, $2 \geq x_{A_2} \geq 0$, $2 \geq x_{A_3} \geq 0$ and $3 = x_{A_1} + x_{A_2} + x_{A_3}$.
It is shown in figure 1 and 2.

Example 2

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 1$
$\{2\}$	$v(\{2\}) = 1$
$\{3\}$	$v(\{3\}) = 1$
$\{1,2\}$	$v(\{1,2\}) = 2$
$\{1,3\}$	$v(\{1,3\}) = 2$
$\{2,3\}$	$v(\{2,3\}) = 2$
$\{1,2,3\}$	$v(\{1,2,3\}) = 3$

We want to find the core allocation of the above coalition game.
We have $v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.
$v(\{1,2\}) = 2 \leq x_{A_1} + x_{A_2},\ v(\{1,3\}) = 2 \leq x_{A_1} + x_{A_3},\ v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$.
$v(\{1,2,3\}) = 3 \leq x_{A_1} + x_{A_2} + x_{A_3}$.

Substituting $v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$ in $v(\{1,2,3\}) = 3 = x_{A_1} + x_{A_2} + x_{A_3}$. We have $1 \geq x_{A_1}$, similarly we get $1 \geq x_{A_2}$ and $1 \geq x_{A_3}$.
And we have
$v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.

Therefore, core allocation is $(x_{A_1}, x_{A_2}, x_{A_3}) = (1,1,1)$.
See figure 3 and 4.

Example 3

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 1$
$\{2\}$	$v(\{2\}) = 1$
$\{3\}$	$v(\{3\}) = 1$
$\{1,2\}$	$v(\{1,2\}) = 2$
$\{1,3\}$	$v(\{1,3\}) = 2$
$\{2,3\}$	$v(\{2,3\}) = 2$
$\{1,2,3\}$	$v(\{1,2,3\}) = 4$

We want to find the core allocation of the above coalition game.
We have $v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.
$v(\{1,2\}) = 2 \leq x_{A_1} + x_{A_2},\ v(\{1,3\}) = 2 \leq x_{A_1} + x_{A_3},\ v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$.
$v(\{1,2,3\}) = 4 \leq x_{A_1} + x_{A_2} + x_{A_3}$.

Substituting $v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$ in $v(\{1,2,3\}) = 4 = x_{A_1} + x_{A_2} + x_{A_3}$. We have $2 \geq x_{A_1}$, similarly we get $2 \geq x_{A_2}$ and $2 \geq x_{A_3}$.
And we have
$v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.
Therefore, core allocation are
$2 \geq x_{A_1} \geq 1$, $2 \geq x_{A_2} \geq 1$, $2 \geq x_{A_3} \geq 1$ and $4 = x_{A_1} + x_{A_2} + x_{A_3}$
It is shown in figure 5 and 6.

Example 4

Suppose one person A owns an old car which values nothing to him. There are two potential buyers, Buyer B values it at 1000 and buyer C values it at 1050. The trade between these people can be analysed based on coalition formation.

Coalitions	Value or worth of coalitions
$\{A\}$	0
$\{B\}$	0
$\{C\}$	0
$\{A, B\}$	1000
$\{A, C\}$	1050
$\{B, C\}$	0
$\{A, B, C\}$	1050
$\{\emptyset\}$	0

What is the core allocation?

We have $x_A \geq 0$, $x_B \geq 0$, $x_C \geq 0$,
$x_A + x_B \geq 1000, \quad x_A + x_C \geq 1050, \quad x_B + x_C \geq 0$
$x_A + x_B + x_C \geq 1050.$

Substituting $x_A + x_C \geq 1050$ in $x_A + x_B + x_C = 1050$, we get $x_B \leq 0$. We have $x_B \geq 0$, thus $x_B = 0$.
Substituting $x_A + x_B \geq 1000$ in $x_A + x_B + x_C = 1050$, we get $x_C \leq 50$. We have $x_C \geq 0$, thus $50 \geq x_C \geq 0$.
From this we get the core allocation as
$C(v) = \{(x_A, x_B, x_c) = (1050 - d, 0, d) | 0 \leq d \leq 50\}$.

Example 5

In a three player game v let player A_1 be a firm that uses an input in its production and each of the players A_2 and A_3 is supplier of its input. Some output having value 1 is created if ownership of input is transferred from either or both of A_2 and A_3 to A_1. No value is created if this interaction does not take place. What is the core allocation?

$v(\{A_1\}) = v(\{A_2\}) = v(\{A_3\}) = 0.$
$v(\{A_1, A_2\}) = v(\{A_1, A_3\}) = v(\{A_1, A_2, A_3\}) = 1$ and
$v(\{A_2, A_3\}) = 0.$

Substituting $x_{A_1} + x_{A_2} \geq 1$ in $x_{A_1} + x_{A_2} + X_{A_3} = 1$, we get $x_{A_3} \leq 0$.
We have $x_{A_3} \geq 0$, thus $x_{A_3} = 0$.
Substituting $x_{A_1} + x_{A_3} \geq 1$ in $x_{A_1} + x_{A_2} + X_{A_3} = 1$, we get $x_{A_2} \leq 0$.
We have $x_{A_2} \geq 0$, thus $x_{A_2} = 0$.
From above we get $X_1 = 1$. Thus core allocation is
$C(v) = ((x_{A_1}, x_{A_2}, x_{A_3}) = (1, 0, 0)\}$.

Consider a game $v \in G^N$; the payoff vectors x and y and an arbitrary coalition $S \subseteq N$. x dominates y via coalition S , if $x_{A_i} > y_{A_i}$ for all $A_i \in S$ and $x(S) \leq v(S)$.

In figure 1, we see the allocations which are dominated. The allocations which are not dominated. In the bargaining game where $v(A_1) = 0, v(A_2) = 0$, and $v(A_1, A_2) = 1$. The allocations given by the line $x_{A_1} + x_{A_2} = 1$, $x_{A_1} \geq 0$, and $x_{A_2} \geq 0$ are not dominated by any other allocation.

Any two allocations which are imputation one cannot dominate the other.
Result:
Consider a game $v \in G^N$ and suppose $x \in I(v)$ is an arbitrary imputation. Then the following statement are equivalent.
$i)$ $x \in C(v)$.
$ii)$ There is no payoff vector that dominates x.

Proof

First we prove $i) \to ii)$. Suppose there exists an allocation y which dominates x. This implies $y_{A_i} > x_{A_i}$ for all $A_i \in S$. This implies $x(S) = \sum_{A_i \in S} x_{A_i} < \sum_{A_i \in S} y_{A_i} = y(S) \leq v(S)$.

We have $x \in C(v)$. It implies that $\sum_{A_i \in S} x_{A_i} \geq v(S)$. A contradiction. Therefore, there exists no allocation y that dominates x.

Now we prove $ii) \to i)$.

Suppose x is not a core allocation. This implies that $x(S) < v(S)$ for some coalition S. This implies that there can be an allocation y of the following nature,

$$y_{A_i} = \begin{cases} x_{A_i} + \frac{v(S) - x(S)}{|S|}, & \text{if} A_i \in S \\ 0, A_i & \notin S \end{cases}$$

We have $y_{A_i} > x_{A_i}$ for all $A_i \in S$ and $y(S) = x(S) + |S|\frac{v(S)-x(S)}{|S|} = v(S)$. We get that y dominates x. This implies that if x is not in core then there exists an allocation which dominates x. This proves $ii) \to i)$.
This is another way to find core allocations.

Is the set of core allocations always non-empty?

Consider the following coalition game.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 1$
$\{2\}$	$v(\{2\}) = 1$
$\{3\}$	$v(\{3\}) = 1$
$\{1,2\}$	$v(\{1,2\}) = 3$
$\{1,3\}$	$v(\{1,3\}) = 3$
$\{2,3\}$	$v(\{2,3\}) = 3$
$\{1,2,3\}$	$v(\{1,2,3\}) = 4$

We have $v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.
$v(\{1,2\}) = 3 \leq x_{A_1} + x_{A_2},\ v(\{1,3\}) = 3 \leq x_{A_1} + x_{A_3},\ v(\{2,3\}) = 3 \leq x_{A_2} + x_{A_3}$.
$v(\{1,2,3\}) = 4 \leq x_{A_1} + x_{A_2} + x_{A_3}$.

Substituting $v(\{2,3\}) = 3 \leq x_{A_2} + x_{A_3}$ in $v(\{1,2,3\}) = 4 = x_{A_1} + x_{A_2} + x_{A_3}$. We have $1 \geq x_{A_1}$, similarly we get $1 \geq x_{A_2}$ and $1 \geq x_{A_3}$.
We $v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$. This implies $x_{A_1} = 1,\ x_{A_2} = 1,\ x_{A_3} = 1$.
$x_{A_1} + x_{A_2} + x_{A_3} = 3 < v(\{1,2,3\}) = 4$. Therefore, $x_{A_1} = 1,\ x_{A_2} = 1,\ x_{A_3} = 1$ is not a core allocation.
It is shown in figure 2 and 3.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 1$
$\{2\}$	$v(\{2\}) = 1$
$\{3\}$	$v(\{3\}) = 1$
$\{1,2\}$	$v(\{1,2\}) = 2$
$\{1,3\}$	$v(\{1,3\}) = 2$
$\{2,3\}$	$v(\{2,3\}) = 2$
$\{1,2,3\}$	$v(\{1,2,3\}) = 3$

This game is inessential though a constant sum game. We have a core allocation.
We have $v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.
$v(\{1,2\}) = 2 \leq x_{A_1} + x_{A_2},\ v(\{1,3\}) = 2 \leq x_{A_1} + x_{A_3},\ v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$.
$v(\{1,2,3\}) = 3 \leq x_{A_1} + x_{A_2} + x_{A_3}$.

Substituting $v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$ in $v(\{1,2,3\}) = 3 = x_{A_1} + x_{A_2} + x_{A_3}$. We have $1 \geq x_{A_1}$, similarly we get $1 \geq x_{A_2}$ and $1 \geq x_{A_3}$.
And we have
$v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.

Therefore, core allocation is $(x_{A_1}, x_{A_2}, x_{A_3}) = (1, 1, 1)$.
See figure 4 and 5.

Result

Let the game $v \in G^N$ be constant-sum and essential. Then $C(v)$ (set of core allocation) is empty.

Proof

Suppose there exists a constant - sum essential game $v \in G^N$ such that $C(v)$ is non-empty. Suppose $x \in C(v)$. It implies that $x_{A_i} \geq v(\{A_i\})$ and $\sum_{A_j \in N \setminus A_i} x_{A_j} \geq v(N \setminus A_i)$. We have $x(N) = x_{A_i} + \sum_{A_j \in N \setminus \{A_i\}} x_{A_j} = v(N)$.
We have $-\sum_{A_j \in N \setminus A_i} x_{A_j} \leq -v(N \setminus A_i)$. This implies $x_{A_i} \leq v(N) - v(N \setminus A_i)$. Since v is constant sum game, we have $v(\{A_i\}) + v(N \setminus \{A_i\}) = v(N)$. This together with the above we get $v(N) - x_{A_i} \geq v(N) - V(\{A_i\})$. This implies that $x_{A_i} \leq v(\{A_i\})$. Thus, we have $x_{A_i} = v(\{A_i\})$. Since x is in core, so we have $\sum_{A_i \in N} x_{A_i} = v(N)$. Since v is essential game, so $\sum_{A_i \in N} v(\{A_i\}) < v(N)$. Thus we have $v(N) < v(N)$. A contradiction. Therefore, $x \notin C(v)$. Core is empty.

The set of core allocation is not always non-empty.
There is a theorem which characterizes the non-emptiness of core for based on weights assigned to coalitions. We will not do that.

Problem:
Suppose a building worth 2000 per month to its owner. A cloth merchant is ready to pay a monthly rent of 2500, whereas a bank offers to pay 3000 per month. Find the core allocation of this game.

Solution:
First we have to formulate the characteristic function.
Suppose players are A, B and C, where A is owner, C is cloth merchant and B is the bank. $v_A = 2000$, $v_B = 0$, $v_C = 0$,
$v_{AC} = 2500$, $v_{AB} = 3000$, $v_{BC} = 0$ and $v_{ABC} = 3000$.
$x_A \geq 2000$, $x_B \geq 0$, $x_C \geq 0$.
$x_A + x_C \geq 2500$, $x_A + x_B \geq 3000$, $x_B + x_C \geq 0$

$x_A + x_C + x_B \geq 3000.$

Substituting $x_B + x_C \geq 0$ in $x_A + x_C + x_B = 3000$, since core allocation is an imputation we have, $x_A \leq 3000$ And we have $x_A \geq 2000$. So we get that
$2000 \leq x_A \leq 3000.$

Again we have $x_B \leq 500$, by substituting $x_A + x_C \geq 2500$ in $x_A + x_C + x_B = 3000$. This implies that $0 \leq x_B \leq 500$.

We have $x_C \leq 0$, by substituting $x_A + x_B \geq 3000$ in $x_A + x_C + x_B = 3000$. This implies that $0 = x_C$.
Thus, core allocations are
$2500 \leq x_A \leq 3000$, $0 \leq x_B \leq 500$, and $x_C = 0$.

Problem:
Suppose there are four players $\{A_1, A_2, A_3, A_4\}$, suppose there are two disjoint set L and R of these players. $R = \{A_1, A_2, \}$ and $L = \{A_3, A_4, \}$. Players of set R has right shoe and players in set L has left shoe. A pair of shoe contains two shoes - left and right. A pair of shoe worth 1 and if there is only left or only right, it has no value. Consider set $S = \{A_1, A_3, A_4\}$, so one right shoe. $S \cap R = \{A_1\}$ and $L \cap S = \{A_3, A_4\}$, two left shoes. If there is cooperation between the players in S, then it will have one pair of shoe. The minimum number of left or right shoes determine the number of pairs. So the worth of coalition is $\min\{|S \cap R|, |S \cap L|\}$, when $|S| \geq 2$.
We get the following characteristic function.

$$v(s) = \begin{cases} 0, \ \ if \ \ |S| \in \{0,1\}, \\ \min\{|S \cap R|, |S \cap L|\}, \ \ if \ \ |S| \geq 2. \end{cases}$$

$v(N) = \min\{|R|, |L|\}$.

We need to find core allocation.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 0$
$\{2\}$	$v(\{2\}) = 0$
$\{3\}$	$v(\{3\}) = 0$
$\{4\}$	$v(\{4\}) = 0$
$\{1,2\}$	$v(\{1,2\}) = 0$
$\{1,3\}$	$v(\{1,3\}) = 1$
$\{1,4\}$	$v(\{2,3\}) = 1$
$\{2,3\}$	$v(\{2,3\}) = 1$
$\{2,4\}$	$v(\{2,4\}) = 1$
$\{4,3\}$	$v(\{4,3\}) = 0$
$\{1,2,3\}$	$v(\{1,2,3\}) = 1$
$\{1,2,4\}$	$v(\{1,2,4\}) = 1$

coalitions	v()
$\{1,3,4\}$	$v(\{1,4,3\}) = 1$
$\{2,3,4\}$	$v(\{4,2,3\}) = 1$
$\{1,2,3,4\}$	$v(\{1,4,2,3\}) = 2$

$x_i \geq 0,\ i = 1,2,3,4.$
$x_1 + x_3 \geq 1,\ x_1 + x_4 \geq 1,\ x_2 + x_3 \geq 1,\ x_2 + x_4 \geq 1,$
$x_1 + x_2 \geq 0,\ x_4 + x_3 \geq 0.$
$x_1 + x_2 + x_3 \geq 1,\ x_1 + x_2 + x_4 \geq 1,\ x_1 + x_4 + x_3 \geq 1,$
$x_4 + x_2 + x_3 \geq 1.$
$x_1 + x_2 + x_3 + x_4 \geq 2$

We have $x_1 + x_3 \leq 1$, by substituting $x_2 + x_4 \geq 1$ in $x_1 + x_2 + x_3 + x_4 = 2$. And we have $x_1 + x_3 \geq 1$, so $x_1 + x_3 = 1$
Similarly we have $x_2 + x_4 \leq 1$ and we have $x_2 + x_4 \geq 1$, so $x_2 + x_4 = 1$.

We also have $x_1 + x_4 = 1$ and $x_2 + x_3 = 1$.
We also get $x_1 \leq 1$ by substituting $x_4 + x_2 + x_3 \geq 1$ in $x_1 + x_2 + x_3 + x_4 = 2$. Similarly we have $x_2 \leq 1$, $x_3 \leq 1$, $x_4 \leq 1$.

From $x_1 + x_3 = 1$, $x_2 + x_4 = 1$, $x_1 + x_4 = 1$ $x_2 + x_3 = 1$ and $x_1 + x_2 + x_3 + x_4 = 2$. We have $x_1 = x_2$ and $x_3 = x_4$.

The core allocations are
$0 \leq x_i \leq 1, i = 1, 2, 3, 4$, $x_1 = x_2$, $x_3 = x_4$, and
$x_1 + x_2 + x_3 + x_4 = 2$.