Physics II: Electromagnetism

PH 102

Lecture 6

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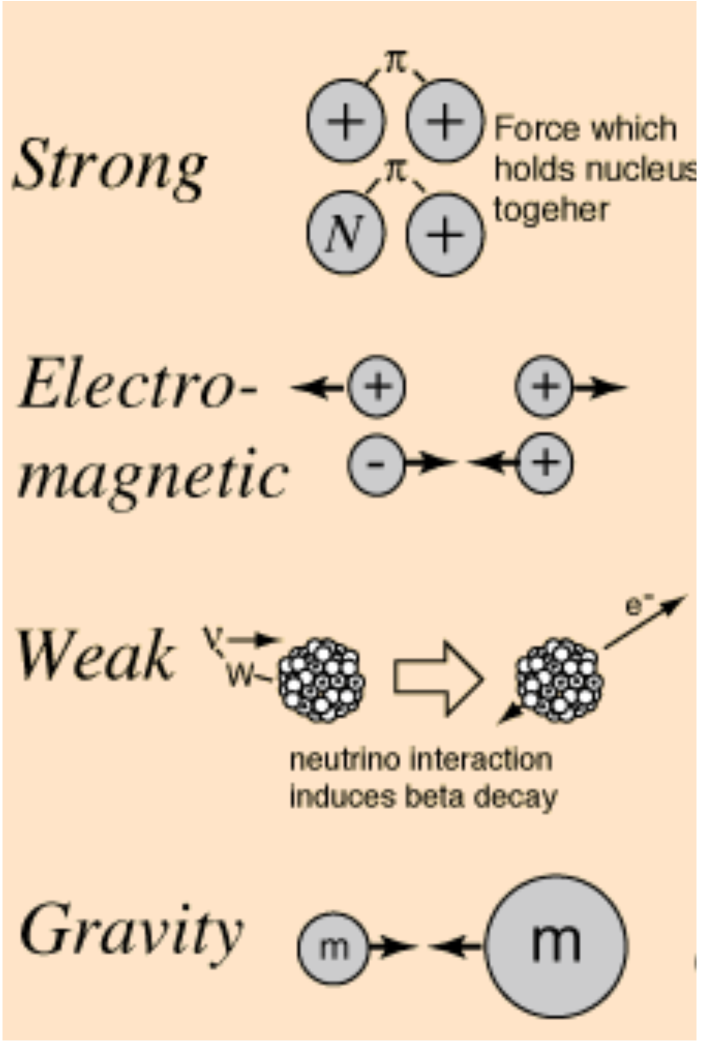
January-May 2019

Fundamental forces in nature

- Gravitational
- Electromagnetic
- Weak
- Strong



We will study the nature of electromagnetic forces in this course

Primary Goal of the Course

To understand a set of 4 equations known as the Maxwell's equations:

$$
\begin{aligned}
\vec{\nabla}.\vec{E} &= \frac{\rho}{\epsilon_0} \\
\vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla}.\vec{B} &= 0 \\
\vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned}
$$

Situations described by these eqns can be extremely complicated and to start with we will simplify life by assuming that nothing depends on time - "**static** case"

Electrostatics

$$
\boxed{
\begin{aligned}
\vec{\nabla}.\vec{E} &= \frac{\rho}{\epsilon_0} \\
\vec{\nabla} \times \vec{E} &= 0
\end{aligned}
}
$$

Magnetostatics

$$
\boxed{
\begin{aligned}
\vec{\nabla}.\vec{B} &= 0 \\
\vec{\nabla} \times \vec{B} &= \mu_0 \vec{J}
\end{aligned}
}
$$

Electricity and magnetism are distinct phenomena so long as charges and currents are static. Independence of $\vec{E}$ and $\vec{B}$ does not appear until there are charges or currents. Only when there are sufficiently rapid changes in the charges and currents with time, will $\vec{E}$ and $\vec{B}$ depend on each other!

Some important points to note before we start

Each particle in the Universe carries with it a number of properties. These determine how the particle interacts with each of the four forces. For the force of gravity, this property is mass. For the force of electromagnetism, the property is called electric charge.

For the purposes of this course, we can think of electric charge as a real number, $q \in \mathbb{R}$. Importantly, charge can be positive or negative. It can also be zero, in which case the particle is unaffected by the force of electromagnetism.

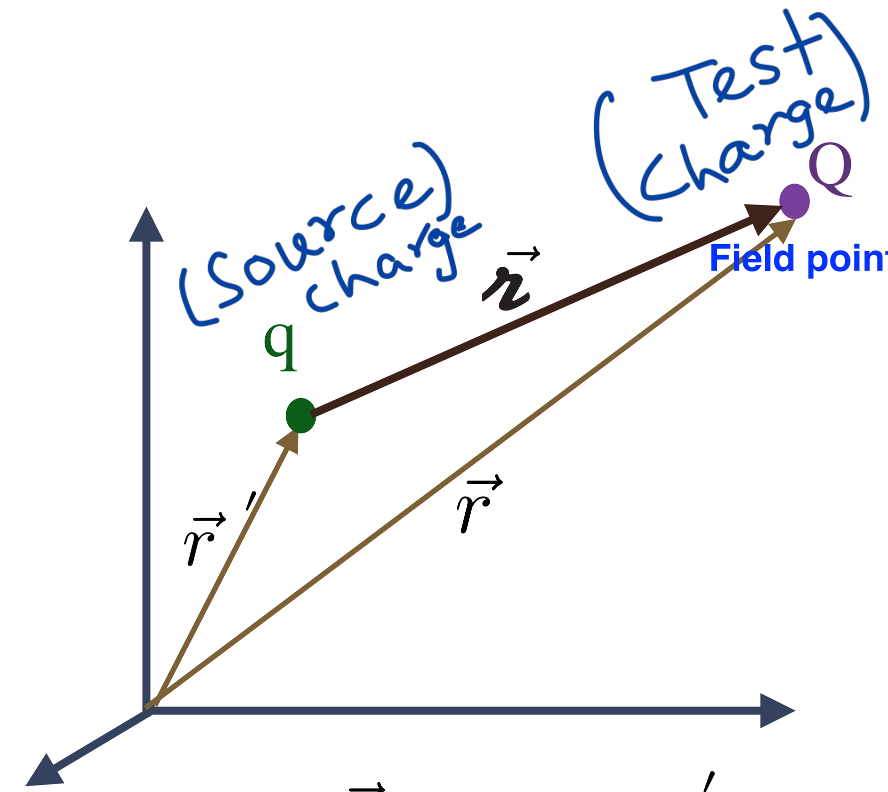
The SI unit of charge is the Coulomb, denoted by C. At a fundamental level, Nature provides us with a better unit of charge. This follows from the fact that charge is quantised: the charge of any particle is an integer multiple of the charge carried by the electron: $e = 1.60217657 \times 10^{-19}$ C. i.e. $q = n\,e$.

An aside: the charge of quarks is actually $q = -e/3$ and $q = 2e/3$. This doesn't change the spirit of the above discussion since we could just change the basic unit. But, apart from in extreme circumstances, quarks are confined inside protons and neutrons so we rarely have to worry about this

Electrostatics: Coulomb's law

Suppose, we have some electric charges, what force do they exert on another charge ?

Coulomb's law:



Between two charges at rest there is a force directly proportional to the product of the charges and inversely proportional to the square of the distance between. The force is along the straight line from one charge to another

$$\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}}$$ (based on experiments)

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$
(separation vector)

$\epsilon_0 = 8.85 \times 10^{-12}\ \frac{C^2}{N\ m^2}$ is called permittivity of free space

Force direction is from source charge to test charge; F is repulsive if charges have same sign whereas F is attractive if their signs are opposite.

What happens when there are many point charges?

Electrostatics: Superposition Principle

When there are more than two charges present, we must supplement Coulomb's law with another fact of nature: the linear superposition principle

The force on any charge is the vector sum of the Coulomb forces from each of the other charges present.

The force $\vec{F}_{12}$ on a charge q_1 due to another charge, say q_2, is independent of the presence of a third charge, say q_3.

Total force on charge q_1 due to presence of q_2 and q_3 is given by

$$\vec{F} = \underbrace{\vec{F}_{12}}_{\text{Force on } q_1 \text{ due to } q_2} + \underbrace{\vec{F}_{13}}_{\text{Force on } q_1 \text{ due to } q_3}$$

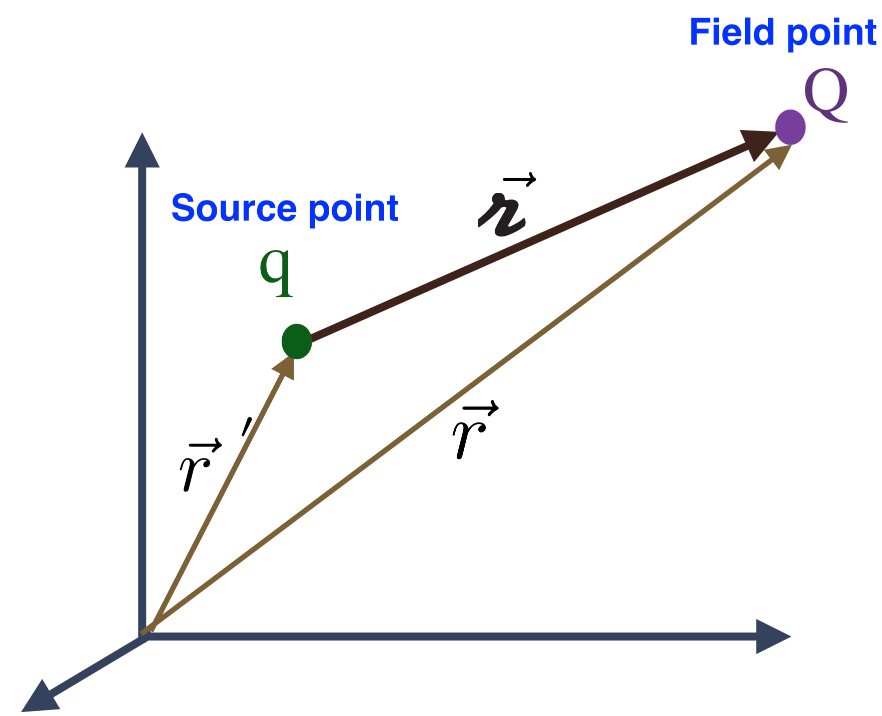
Can easily be generalised for any number of charges

This is possible as the force is proportional to the value of the source charge

Coulomb's law and the Principle of Superposition constitute the physical input for electrostatics. There is nothing more that is required to understand electrostatics!

Electric field

Let us write the force on Q due to q as follows:



$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}Q \equiv \vec{E}(\vec{r})Q$$

- $\vec{E}(\vec{r})$ is called the electric field of the source charge q.

- The field is a function of the position $\vec{r}$, since the separation vector $\vec{\mathscr{r}}$ depend on the location of the charge Q (field point).

- While it takes two charges to feel a force, it takes only one charge to produce a field. A charge at the origin produces the field $\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$ at point $\vec{r}$.

- The field due to charge q is non-zero everywhere, not just where there is another charge to feel the field.

- Think of the field $\vec{E}(\vec{r})$ as a condition in space, produced by the presence of q. With q present, any charge placed at $\vec{r}$ will feel a force; while without it, it will just sit there.

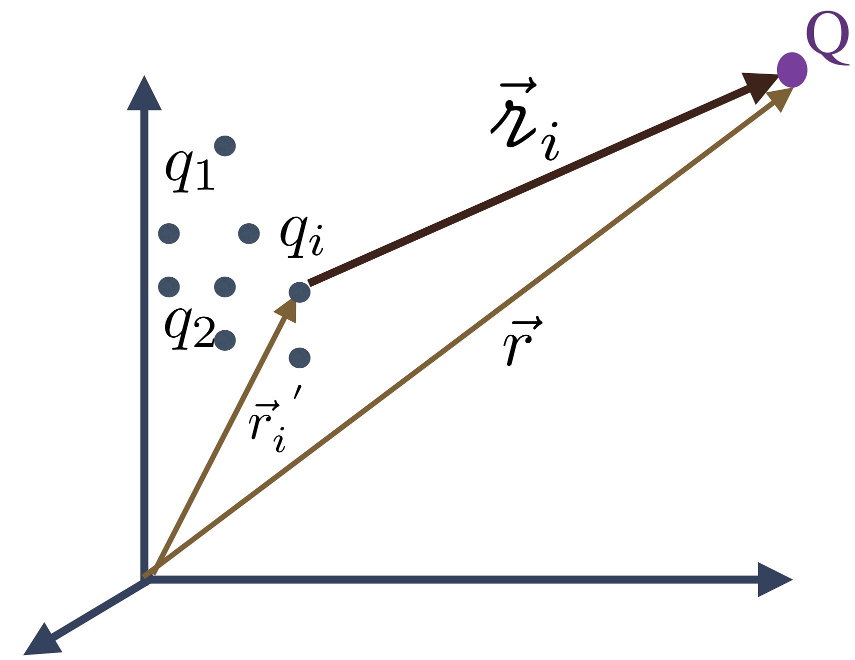
Electric field

• If there are many charges,invoke the superposition principle: the field at some $\vec{r}$ due to many charges will be the (vector) sum of the fields due to each one.

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \cdots\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \cdots\right)$$

$$= Q\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i \quad = Q\vec{E}(\vec{r})$$



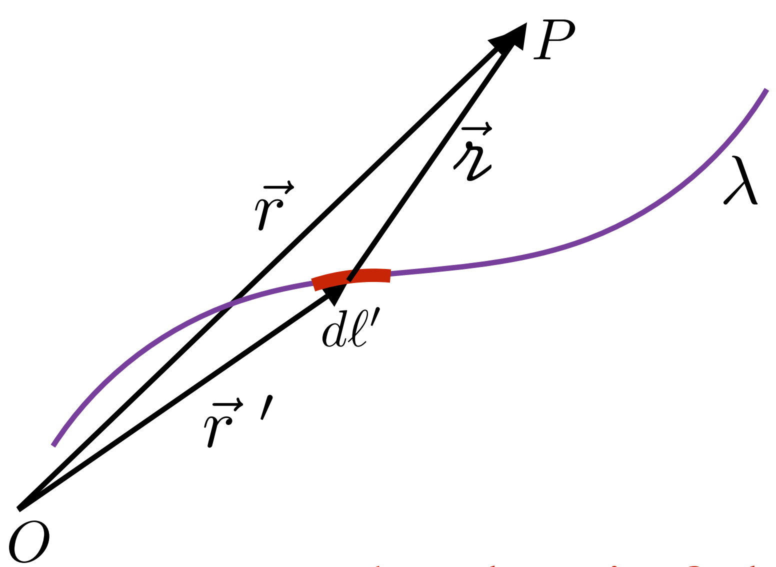
• Note that $\vec{E}(\vec{r})$, which is a vector quantity, depends on the location of the field point and is determined by the configurations of the source charges q_i.

• To measure a field is easier: put a known test charge q at $\vec{r}$, equate the force it experiences to $q\vec{E}$. If $q = 1C$, the force and $\vec{E}$ are numerically equal but dimensionally different. That is why "field is the force on a unit charge".

Continuous charge distributions: Line, Surface, Volume

Our definition of electric field assumes that the source of the field is a collection of discrete point charges q_i. However, charge can be distributed continuously over a region also.

Line charge distribution:



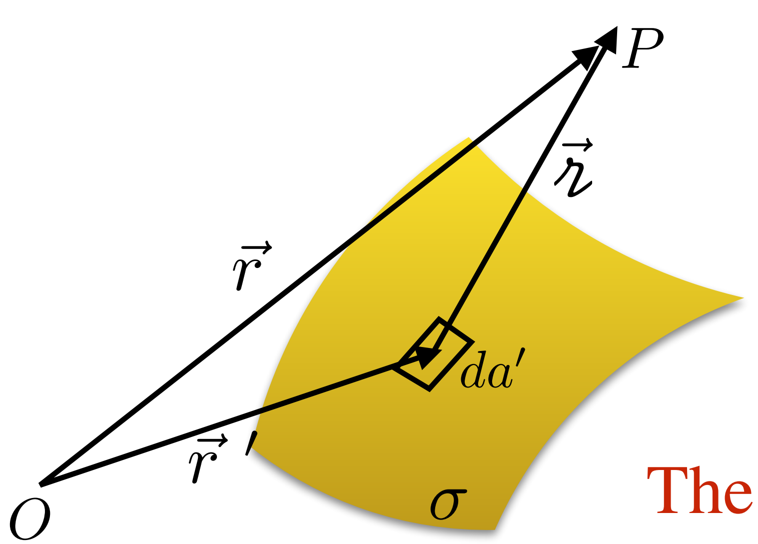
Charge is distributed over an arbitrary curve. On an infinitesimal length element $d\ell'$ along the curve, the amount of charge is $\lambda d\ell'$. λ: charge per unit length or line charge density.

$$\lambda = \lim_{\Delta\ell' \to 0} \frac{\Delta q}{\Delta \ell'}$$

The electric field of a line charge:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r}\,')}{\mathscr{r}^2} \hat{\mathscr{r}} d\ell'$$

Surface charge distribution:



Charge is smeared over a surface with charge per unit area (surface charge density) σ. The total charge contained in a surface is then given by $q = \int \sigma da'$

$$\sigma = \lim_{\Delta a' \to 0} \frac{\Delta q}{\Delta a'}$$

The electric field of a surface charge:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r}\,')}{\mathscr{r}^2} \hat{\mathscr{r}} da'$$

Continuous charge distributions:

Example:

Suppose we have a hemispherical surface of radius R with charge density $\sigma(\theta, \phi) = \sigma_0 \cos\theta$. What is the total charge present on the hemisphere?

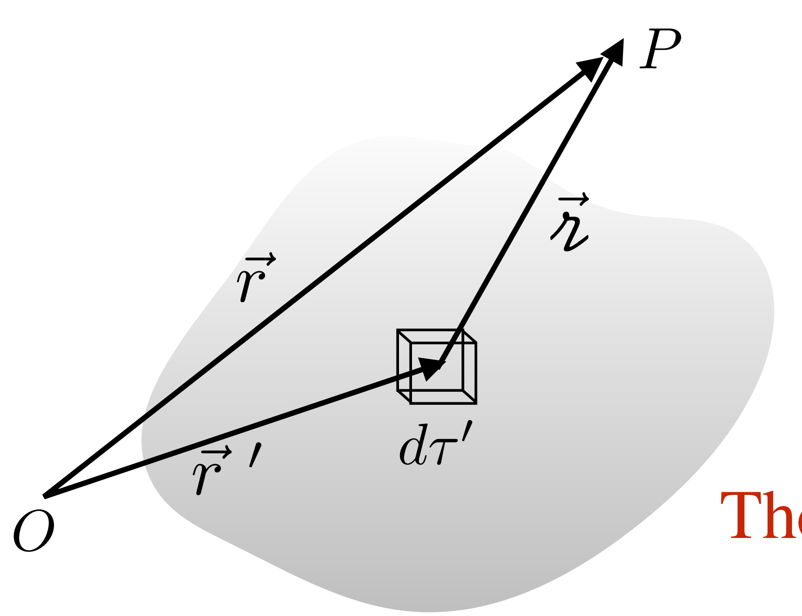
Total charge on the hemispherical surface a' : $Q = \int_{a'} \sigma(\vec{r}\,')da'$.

Recall that for spherical symmetry, the elementary area (for constant R surface) is $da' = R^2 \sin\theta d\theta d\phi$. Then

$$Q = \int_{a'} \sigma_0 \cos\theta R^2 \sin\theta d\theta d\phi = \frac{\sigma_0 R^2}{2} \int_{\theta=0}^{\pi/2} \sin 2\theta d\theta \int_{\phi=0}^{2\pi} d\phi = \pi\sigma_0 R^2$$

Volume charge distribution: If the charge fills a volume, with charge per unit volume (volume charge density) ρ, then $dq = \rho d\tau'$.

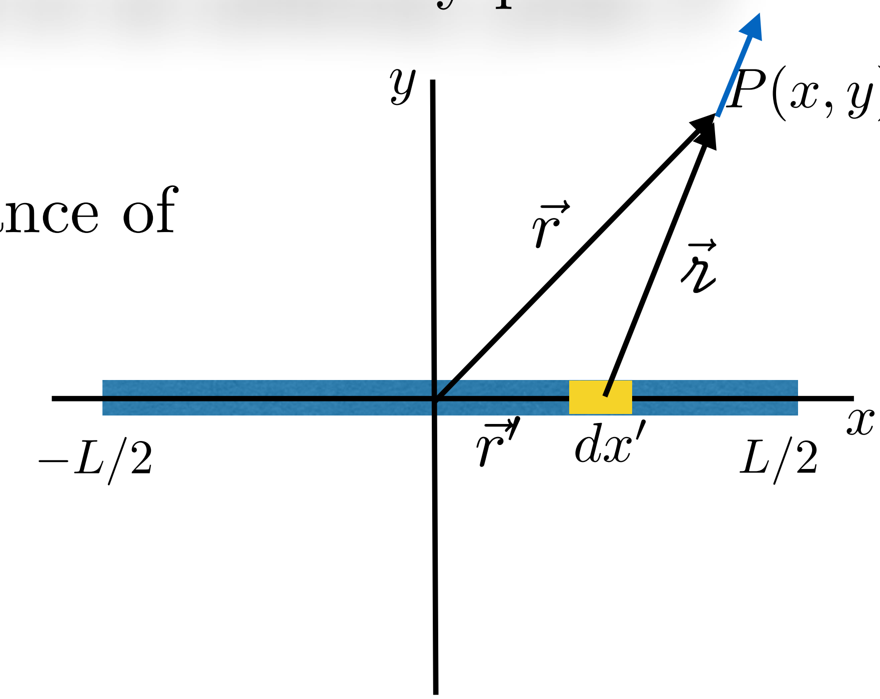
$$\rho = \lim_{\Delta\tau' \to 0} \frac{\Delta q}{\Delta\tau'}$$



The electric field of a volume charge: $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int \frac{\rho(\vec{r}\,')}{\mathscr{r}^2} \hat{\mathscr{r}} d\tau'$

Electric fields for continuous charge distribution: Example:

Consider a charged line of length L having uniform line charge density λ placed along the x-axis. Obtain an expression for electric field at an arbitrary point P in the xy plane.



Consider an element of line of width dx' at x'. The distance of the point $P(x,y)$ from the charge element is

$$\mathscr{r}^2 = (x - x')^2 + y^2$$

$$\text{and } \vec{\mathscr{r}} = (x - x')\hat{x} + y\hat{y}$$

The field at $P(x,y)$ due to element dx' at $(x',0)$ is:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx'}{\mathscr{r}^3}\vec{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx'}{[(x-x')^2 + y^2]^{3/2}}[(x-x')\hat{x} + y\hat{y}]$$

x and y component of the net field $\vec{E}$ at P:

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{(x-x')dx'}{[(x-x')^2 + y^2]^{3/2}}$$

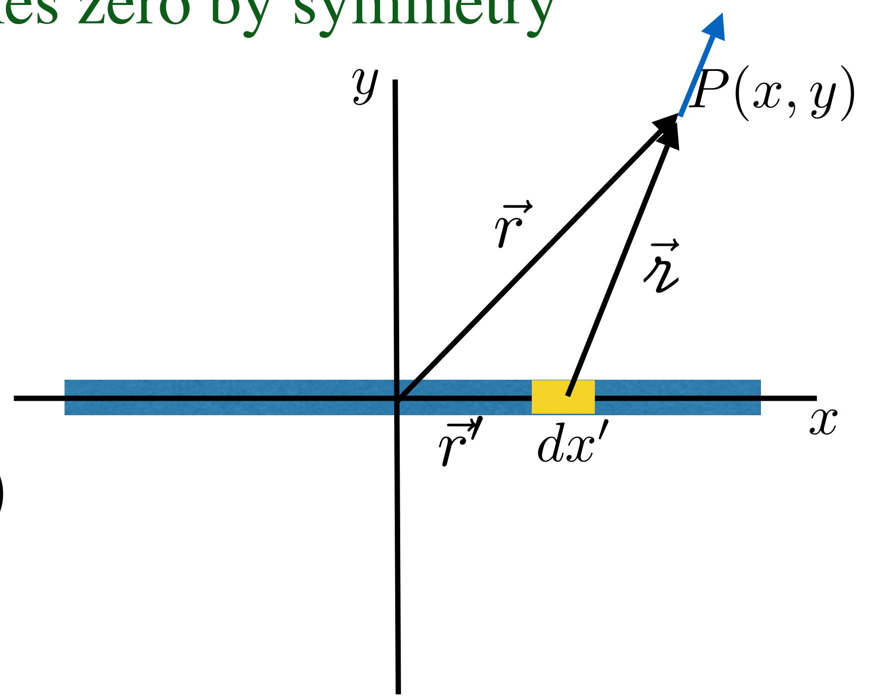
$$E_y = \frac{\lambda}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{y\,dx'}{[(x-x')^2 + y^2]^{3/2}}$$

Can not be evaluated in a closed form

Electric fields for continuous charge distribution: Example:

However, we can do the integrals for an infinitely long line charge

In this case however, the x-component of the field becomes zero by symmetry



Explicitly

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{(x-x')dx'}{[(x-x')^2+y^2]^{3/2}} = +\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{zdz}{(z^2+y^2)^{3/2}}$$

$$= 0$$

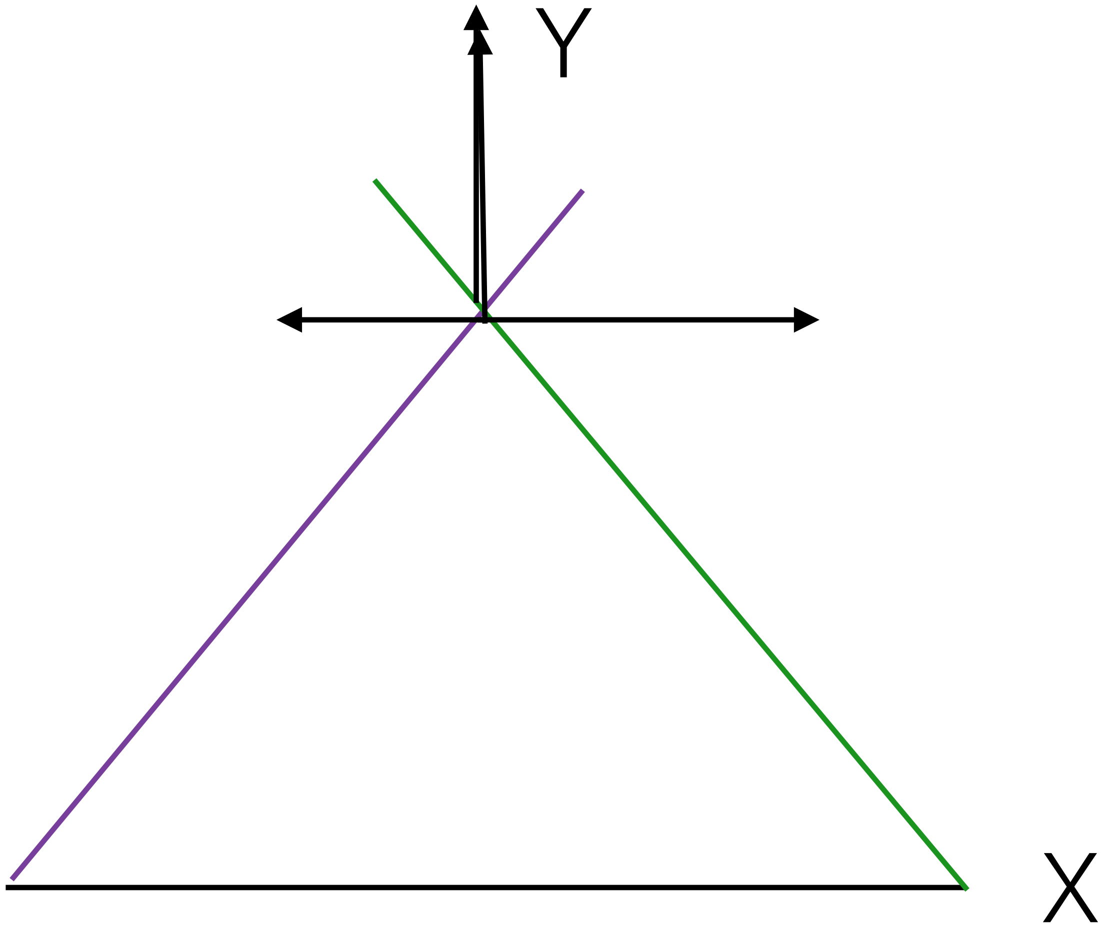
(where $z = x - x'$)

$$E_y = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{ydx'}{[(x-x')^2+y^2]^{3/2}} = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{ydz}{(z^2+y^2)^{3/2}}$$

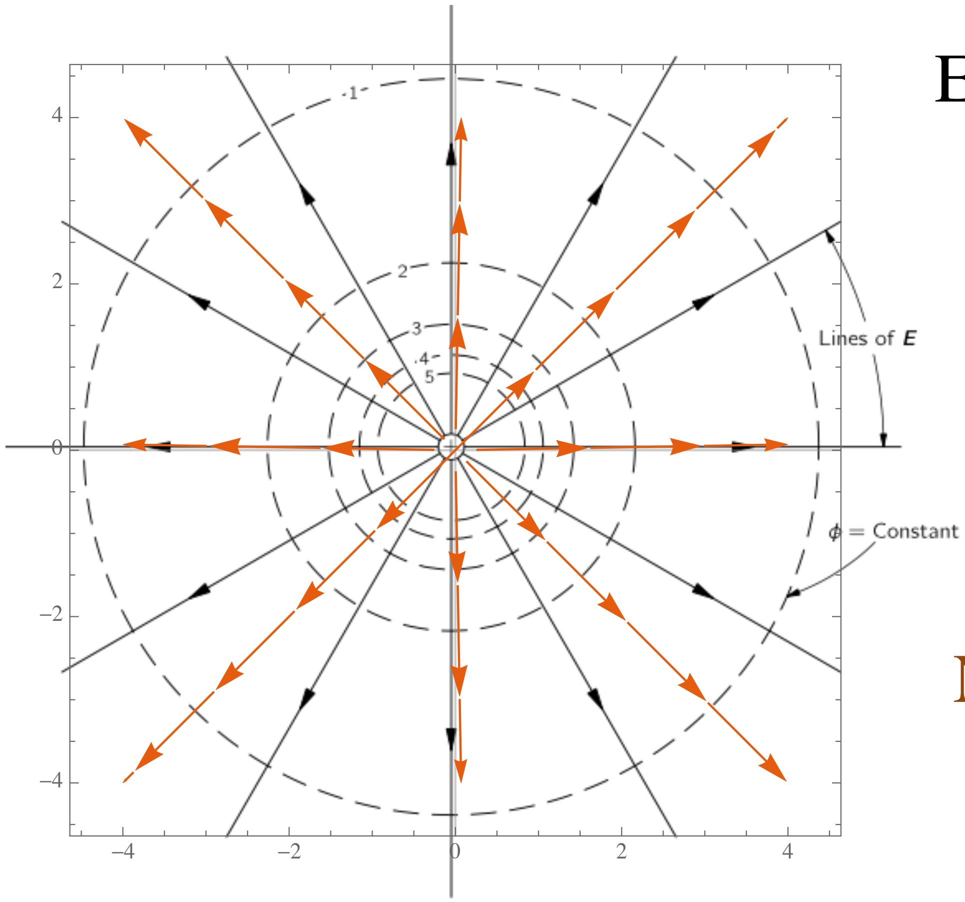
Substitute $z = y\tan\theta$:

$$= \frac{\lambda}{4\pi\epsilon_0}\int_{-\pi/2}^{\pi/2}\frac{y^2\sec^2\theta}{y^3\sec^3\theta}d\theta = \frac{\lambda}{4\pi\epsilon_0}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta \boxed{= \frac{\lambda}{2\pi\epsilon_0 y}}$$

Field of an infinite line charge placed on the x-axis: $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 y}\hat{y}$


Y
X

How to “see” the electric field:



Electric field at a distance r due to a charge +q at origin:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

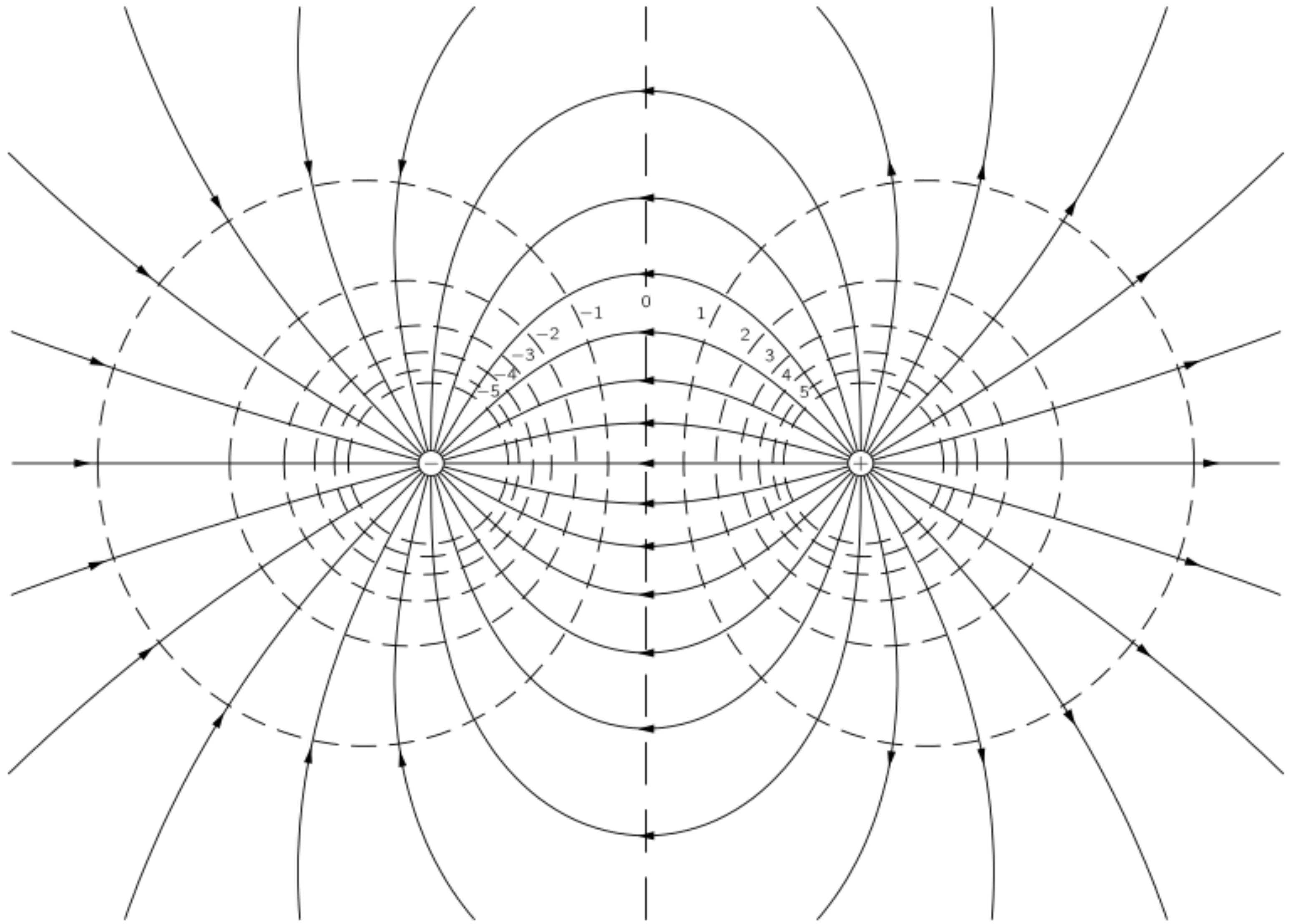
The field points radially outward and strength decreases~$1/r^2$

Nicer way to represent the field is to connect these arrows to form field lines.

In doing so, did we throw away information about the strength of the electric field?

NO! Magnitude of the field is indicated by the density of the field lines: strong near centre where field lines are close together and weak farther out, where they are relatively far apart.

Electric Field Lines:



Field lines begin on +ve charges and end on negative ones. Can't terminate midair.

Field lines can never cross because if they cross then at the intersection the field would have two different directions at once!

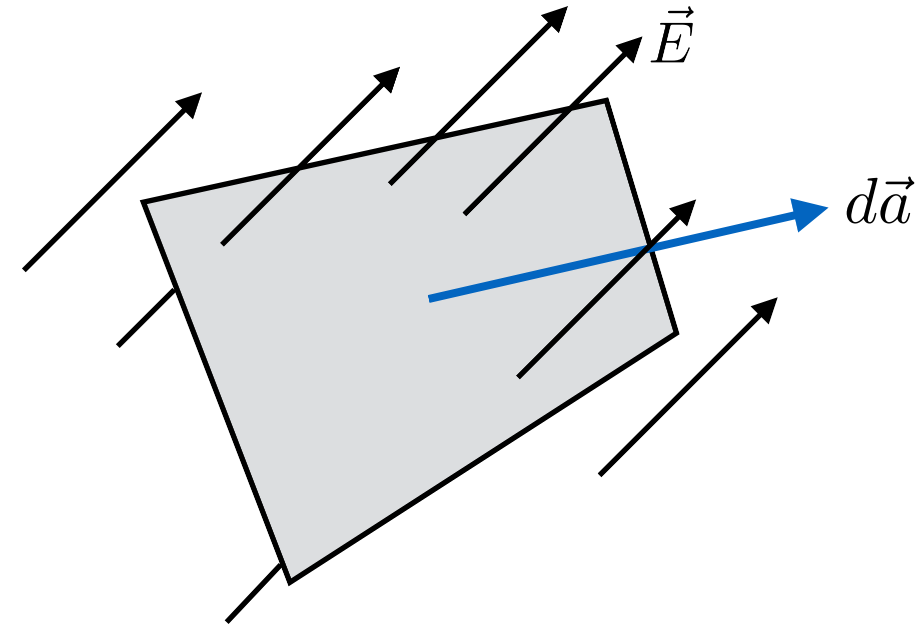
In this model the flux of vector field $\vec{E}$ through a surface S: $\boxed{\Phi_E \equiv \int_S \vec{E}.d\vec{a}}$

"Number of field lines" crossing through S

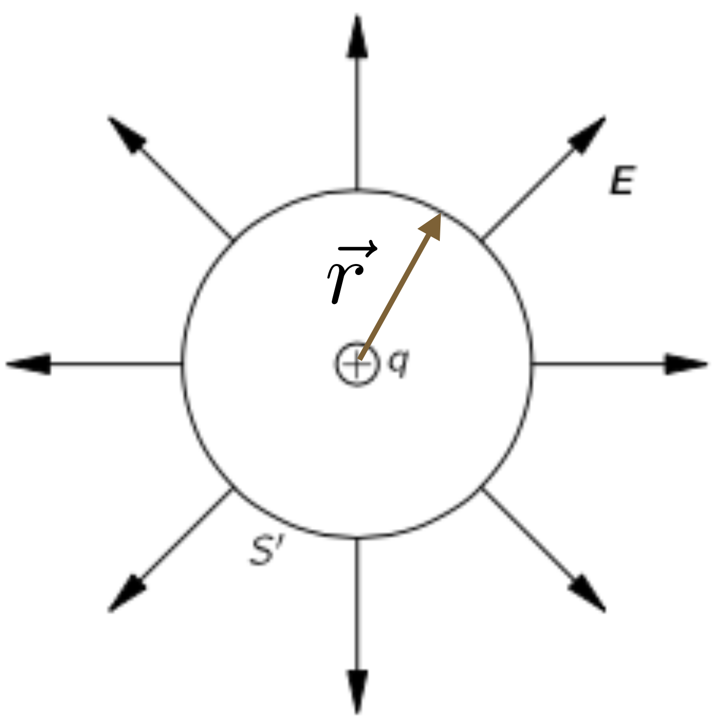
Flux of the electric field

$$\boxed{\Phi_E \equiv \int_S \vec{E}.d\vec{a}}$$

The field strength was proportional to the density of field lines (the number per unit area), hence $\vec{E}.d\vec{a}$ is proportional to the number of lines passing through the infinitesimal area $d\vec{a}$



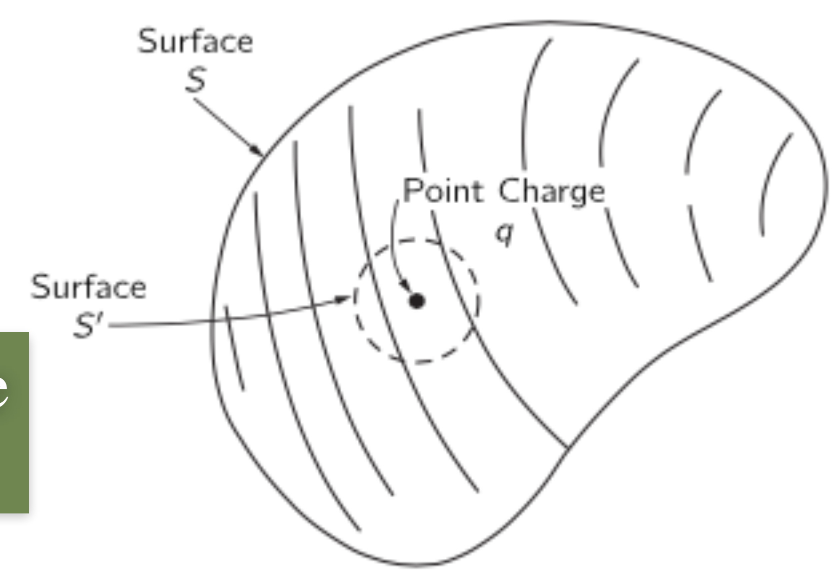
The dot product picks out the component of $d\vec{a}$ along the direction of $\vec{E}$. It is the area perpendicular to $\vec{E}$ that we are thinking of when we say that the density of field lines is the number per unit area.



For a point charge at the origin the flux of $\vec{E}$ through a sphere of radius r:

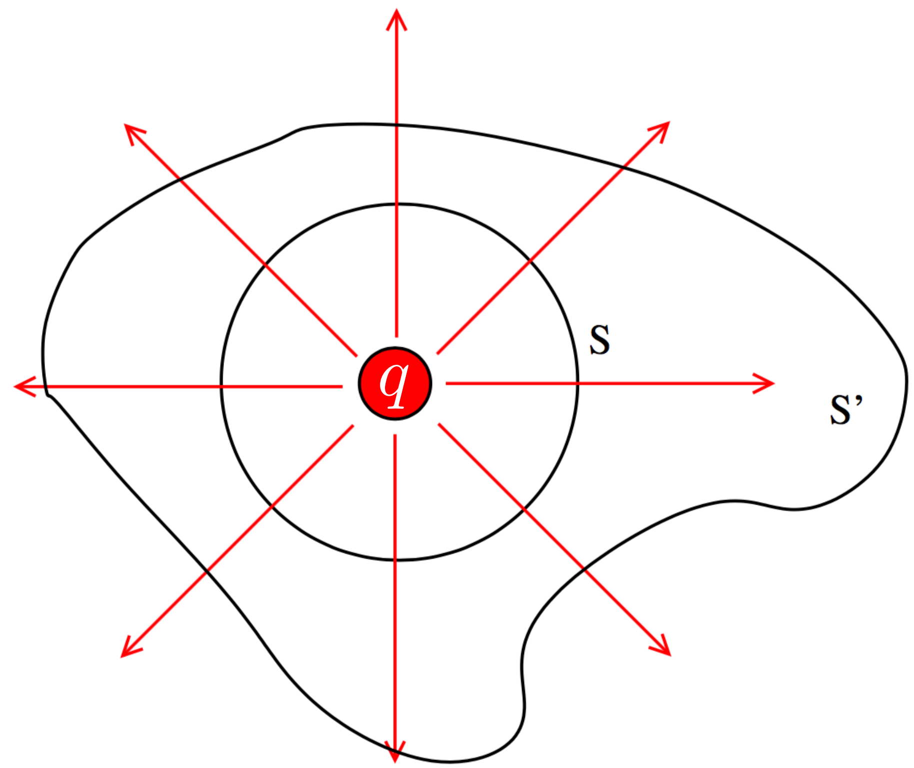
$$\oint \vec{E}.d\vec{a} = \frac{1}{4\pi\epsilon_0} \oint \frac{q}{r^2} \hat{r}.(r^2 \sin\theta d\theta d\phi)\hat{r} = \frac{q}{\epsilon_0}$$

Independent of r !!

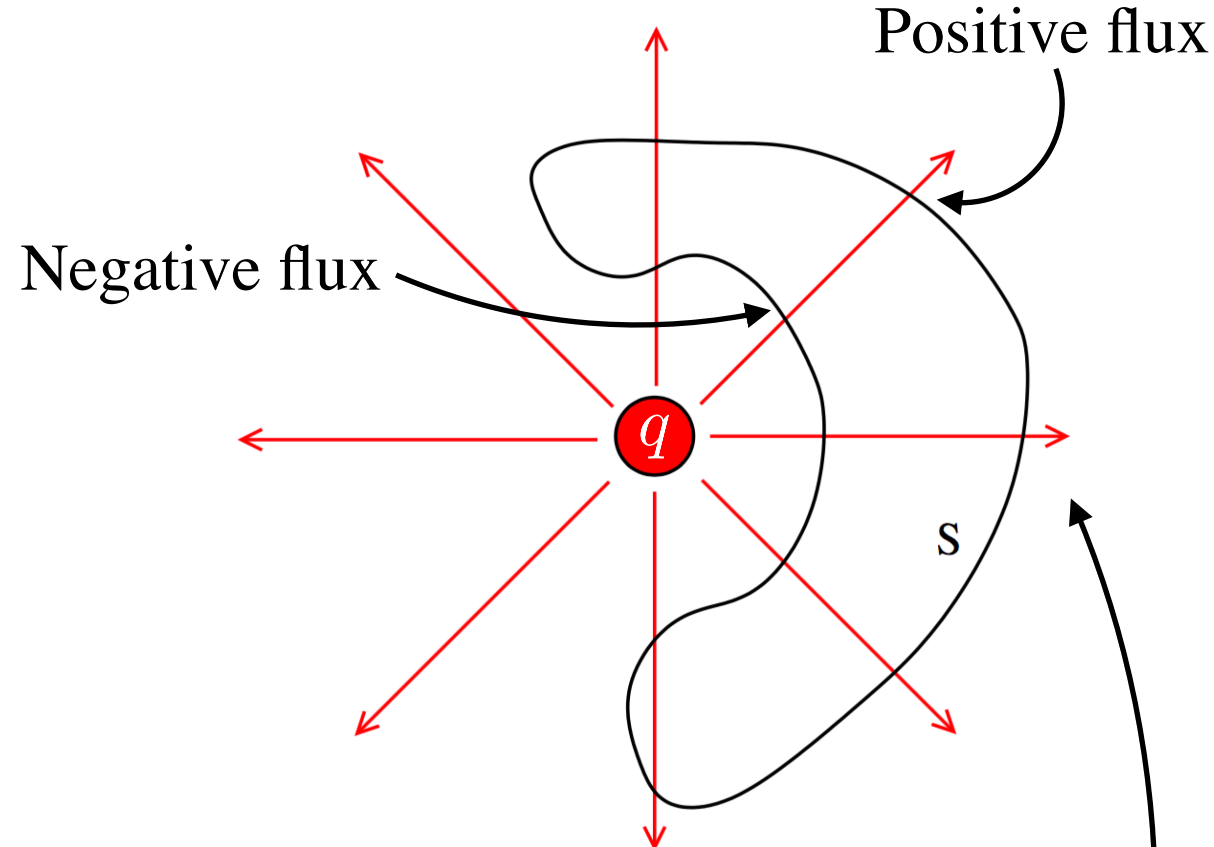


However, the surface need not be a spherical one, any surface enclosing charge q will have the same flux q/ϵ_0

Flux of the electric field



The flux through S and S$'$ is the same: $\frac{q}{\epsilon_0}$



The flux through S vanishes (why?)

$= 0$

What if the surface encloses a bunch of charges instead of just one?

Flux through any surface that encloses bunch of charges

$$\oint \vec{E}.\vec{da} = \sum_{i=1}^{n} \left(\oint \vec{E}_i.\vec{da} \right) = \sum_{i=1}^{n} \frac{q_i}{\epsilon_0} = \frac{Q_{\rm enc}}{\epsilon_0}$$

Charge enclosed by the surface

Gauss's Law:
Flux through an enclosed surface is proportional to the charge enclosed by the surface

The surface S is called a Gaussian surface.

Gauss's Law

$$\oint_{\mathcal{S}} \vec{E}.d\vec{a} = \frac{Q_{\rm enc}}{\epsilon_0}$$ (Integral form)

Using Divergence Theorem, we can convert the integral form to a differential form:

$$\oint_{\mathcal{S}} \vec{E}.d\vec{a} = \int_{\mathcal{V}} (\vec{\nabla}.\vec{E})d\tau$$

Volume charge density.

Rewriting $Q_{\rm enc}$ in terms of the charge density ρ: $Q_{\rm enc} = \int_{\mathcal{V}} \rho d\tau$

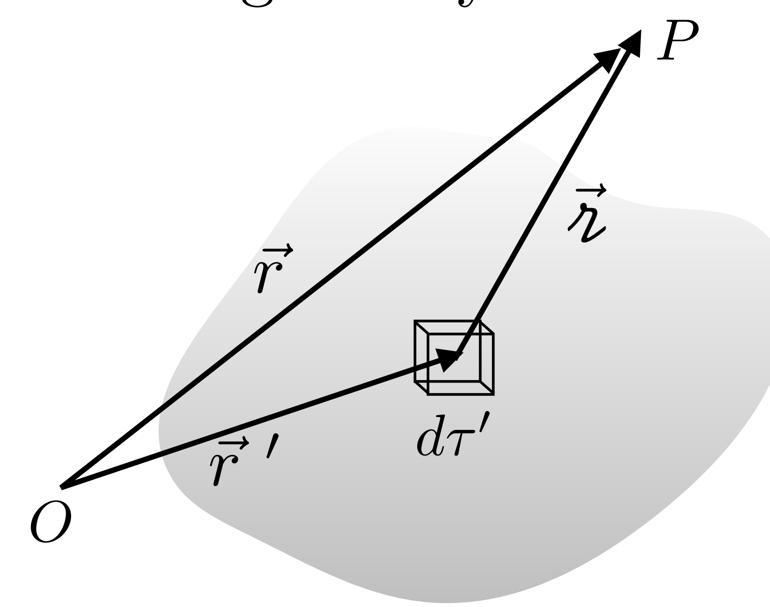
$$\int_{\mathcal{V}} (\vec{\nabla}.\vec{E})d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau \implies \boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}}$$ (Differential form)

$\mathcal{V}$ is an arbitrary volume enclosed by a closed surface $\mathcal{S}$

Integral form is applicable to any type of charge distribution, whereas differential is valid for only volume charge distribution

Divergence of the electric field: Direct calculation

Recall that the electric field for a charge distribution was given by:



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\vec{r}\,')d\tau'$$

$\vec{r}$-dependence is contained in $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$\rho = 0$ in the exterior anyway, so include all space

Hence: $$\vec{\nabla}.\vec{E} = \frac{1}{4\pi\epsilon_0}\int \vec{\nabla}.\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\rho(\vec{r}\,')d\tau'$$

w.r.t. $\vec{r}$ (Not $\vec{r}\,'$)!!

But, we have seen $$\vec{\nabla}.\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}}) = 4\pi\delta^3(\vec{r} - \vec{r}\,')$$

$$\therefore \quad \vec{\nabla}.\vec{E} = \frac{1}{4\pi\epsilon_0}\int 4\pi\delta^3(\vec{r} - \vec{r}\,')\rho(\vec{r}\,')d\tau'$$

$$= \frac{1}{\epsilon_0}\int \delta^3(\vec{r} - \vec{r}\,')\rho(\vec{r}\,')d\tau' = \frac{1}{\epsilon_0}\rho(\vec{r})$$

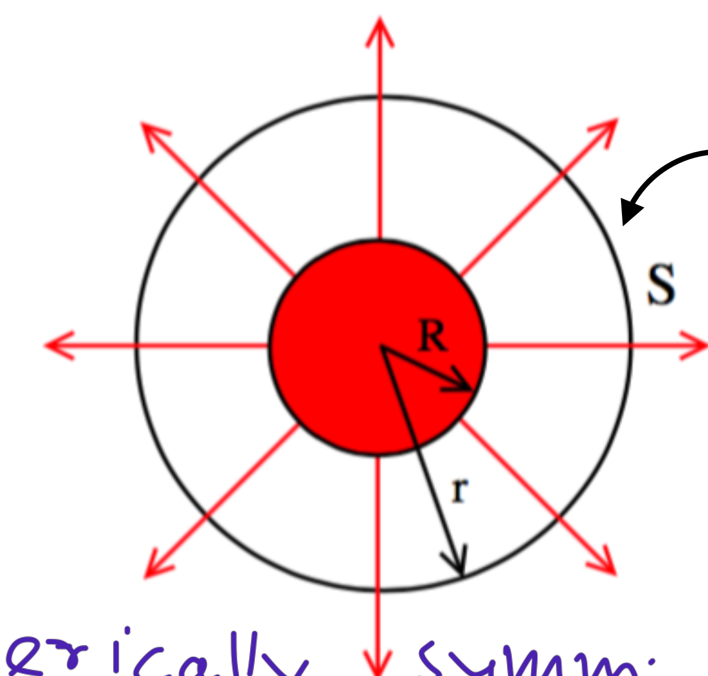
Gauss's Law in differential Form

Integral form can be obtained by inverse procedure.

Applications of Gauss's Law

Find the field outside a uniformly charged solid sphere of radius R and total charge q.

Imagine a spherical surface $\mathcal{S}$ at $r > R$ (Gaussian Surface).



Gauss's Law $\Longrightarrow \oint_{\mathcal{S}} \vec{E}.d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$

Here $Q_{\text{enc}} = q$.

$\vec{E}$ and $d\vec{a}$ both points radially outward!

• Spherically symm.
$\Rightarrow \phi$ independent

$$\therefore \int_{\mathcal{S}} \vec{E}.d\vec{a} = \int_{\mathcal{S}} |\vec{E}|da$$

Magnitude of $\vec{E}$ is constant over the Gaussian Surface.

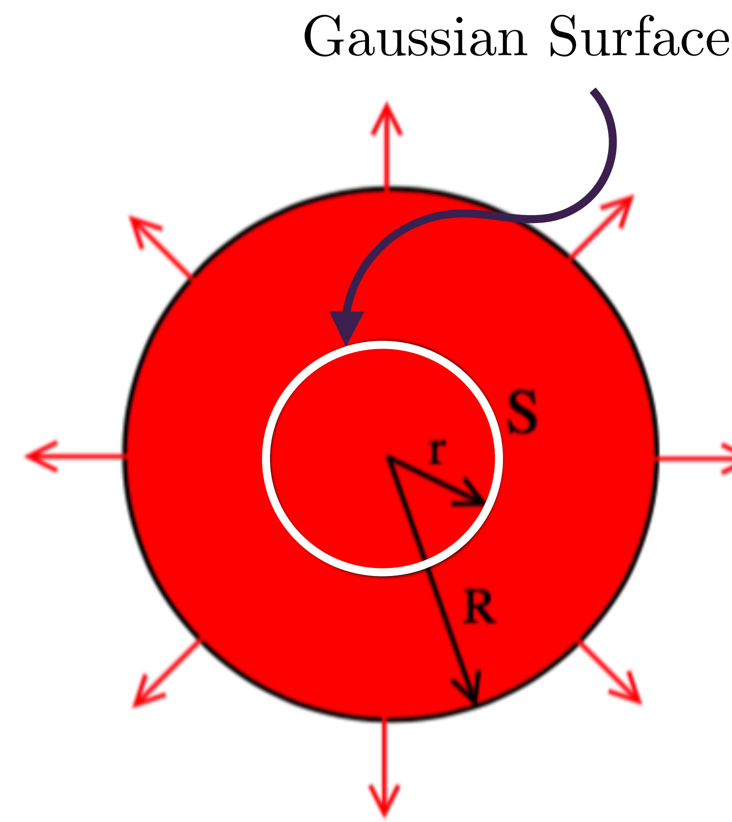
• Also θ indep.
as for any θ
the charge dist.
does not change.

$$\therefore \int_{\mathcal{S}} |\vec{E}|da = |\vec{E}| \int_{\mathcal{S}} da = |\vec{E}|4\pi r^2$$

Thus : $|\vec{E}|4\pi r^2 = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{q}{\epsilon_0} \Longrightarrow \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

The field outside is same as it would have been if all the charge had been concentrated at the centre of the sphere.

What about the field inside the sphere?



Field inside depends on how the charge is distributed.

If the distribution is uniform: $q = \frac{4}{3}\pi R^3 \rho$

Let us pick Gaussian Surface as a sphere of radius $r < R$ centered at the origin.

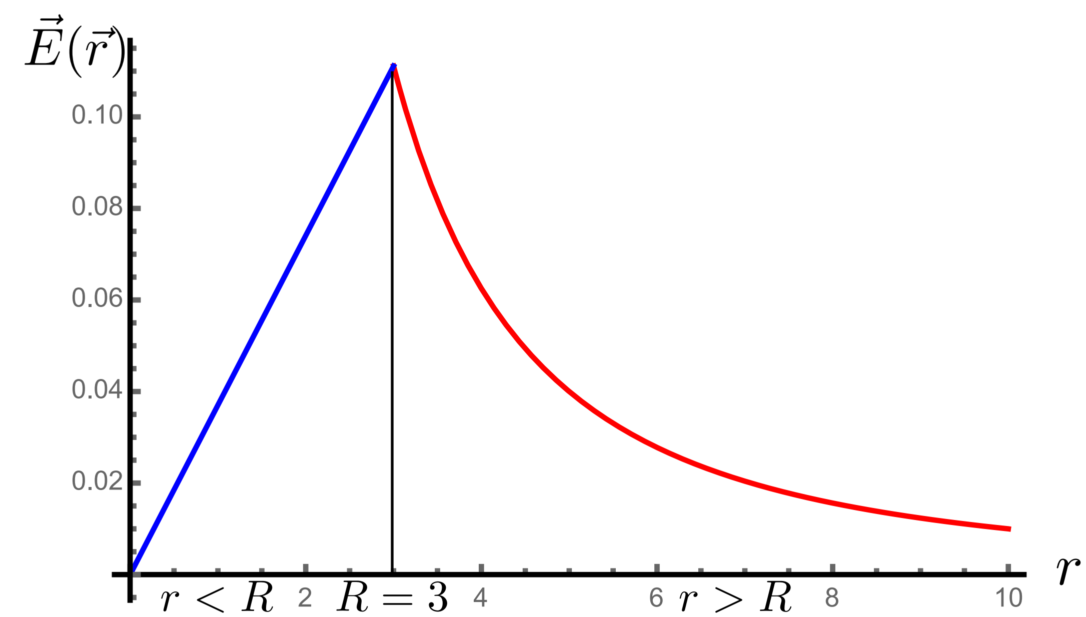
Charge enclosed by this sphere $\frac{4}{3}\pi r^3 \rho = q\frac{r^3}{R^3}$

Gauss's Law $\implies \int_{\mathcal{S}} \vec{E}.d\vec{a} = \frac{1}{\epsilon_0}\frac{qr^3}{R^3}$

Again from the symmetry argument:

$$\int_{\mathcal{S}} \vec{E}.d\vec{a} = |\vec{E}| \int_{\mathcal{S}} da = |\vec{E}| 4\pi r^2 = \frac{1}{\epsilon_0}\frac{qr^3}{R^3}$$

$$\therefore \vec{E}(\vec{r}) = \frac{qr}{4\pi\epsilon_0 R^3}\hat{r}$$



Outside the sphere we revert to the inverse-square form. At the surface, $r = R$, the electric field is continuous but the derivative dE/dr is not.