Linear Algebra

Department of Mathematics
Indian Institute of Technology Guwahati

January – May 2019

MA 102 (RA, RKS, MGPP, KVK)

Vector spaces

Topics:

- Vector Spaces and Subspaces
- Linear Independence
- Basis and Dimension

Field axioms

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Fields axioms (cont.)

6. Distributivity: For all $\alpha, \beta, \gamma \in \mathbb{F}$,
   $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.

Remark: Elements of $\mathbb{F}$ are referred to as scalars. For vector spaces, the real field $\mathbb{R}$ can be replaced with any field $\mathbb{F}$.

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Remark

For any field, usually one writes ab instead of $a \cdot b$.

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For $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{V}$ and scalars α, β in $\mathbb{F}$

1. Closure: $\mathbf{u} + \mathbf{v} \in \mathbb{V}$ and $\alpha \cdot \mathbf{u} \in \mathbb{V}$
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The elements of $\mathbb{V}$ are vectors and the elements of $\mathbb{F}$ are scalars.

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A vector space $\mathbb{V}$ over $\mathbb{C}$ is called a complex vector space.

We mostly consider real and complex vector spaces.

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because, a polynomial is zero iff all of its coefficients are zero. The last system has nontrivial solutions.

Linear Dependence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a VS $\mathbb{V}$ over $\mathbb{F}$. Then S is linearly dependent (LD) if at least one of $v_i \in S$ is a linear combination of the rest of elements in S, i.e., if

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

for some $\mathbf{0} \neq [\alpha_1, \alpha_2, \ldots, \alpha_k]^T \in \mathbb{F}^k$.

Example

- Any finite set containing $\mathbf{0}$ is linearly dependent.
- In $\mathbb{R}_2[x]$, is $\{x^2,\ 1 - x^2,\ 1 + x^2\}$ linearly dependent?

$$\begin{aligned} & ax^2 + b(1 - x^2) + c(1 + x^2) = 0 \\ \Rightarrow \quad & (b + c) + (a - b + c)x^2 = 0 \\ \Rightarrow \quad & b + c = 0,\ a - b + c = 0, \end{aligned}$$

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- The set $\{1, x, x^2, \ldots\} \subseteq \mathbb{R}[x]$ is linearly independent.
- The set $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is linearly independent in $\mathcal{M}_2(\mathbb{R})$.

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- $\mathcal{C}^2((a,b),\mathbb{R}) := \{f : (a,b) \to \mathbb{R} \mid f''\text{is continuous}\}$ is a subspace of the VS $\mathcal{C}((a,b),\mathbb{R})$ over $\mathbb{R}$.

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  $(u_1, w_1) + (u_2, w_2) := (u_1 + u_2, w_1 + w_2), \alpha(u, w) := (\alpha u, \alpha w)$.

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- $\mathcal{C}^2((a,b),\mathbb{R}) := \{f : (a,b) \to \mathbb{R} \mid f''\text{is continuous}\}$ is a subspace of the VS $\mathcal{C}((a,b),\mathbb{R})$ over $\mathbb{R}$.
- The VS $\mathbb{R}$ over $\mathbb{R}$ has no nontrivial subspaces?
- If $\mathbb{U} \preccurlyeq \mathbb{W}$ and $\mathbb{W} \preccurlyeq \mathbb{V}$, then $\mathbb{U} \preccurlyeq \mathbb{V}$.
- Let $\{\mathbb{U}_i \mid \mathbb{U}_i \preccurlyeq \mathbb{V}\}$ be nonempty. Then $\cap_i \mathbb{U}_i \ \preccurlyeq \mathbb{V}$.
- Let $\mathbb{U}, \mathbb{W} \preccurlyeq \mathbb{V}$. Then $\mathbb{U} \cup \mathbb{W} \preccurlyeq \mathbb{V}$ iff $\mathbb{U} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{U}$.
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Prove the following statement:

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Let $\mathbb{V}$ be a vector space with $\dim \mathbb{V} = n$. Prove that

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- Any spanning set for $\mathbb{V}$ can be reduced to a basis for $\mathbb{V}$.