Application of System of Linear Equations

1 Allocation of Resources

A great many applications of system of linear equations involve allocating limited resouces subject to set of conditions.

Example (D. Poole, Exercises 2.4, Problem 1):
A biologist has placed three strains of bacteria (denoted I, II and III) in a test tube where they will feed on three different food sources (A, B and C). Each day 400 units of food A, 600 units of food B, and 600 units of food C are placed in the test tube. The data on daily food consumption by the bacteria (in units per day) are as shown in the table given below. How many bacteria of each strain coexist in the test tube and consume all of the food?

	Bacteria Strain I	Bacteria Strain II	Bacteria Strain III
Food A	1	2	0
Food B	2	1	1
Food C	1	1	2

Solution:
Let x_1, x_2, and x_3 be the numbers of bacteria of strains I, II, and III respectively.
Writing Equation for Food A:
Since each bacteria of strain I consumes 1 unit of Food A, x_1 numbers of bacteria of strain I consumes x_1 units of Food A. Similarly, each bacteria of strain II consumes 2 units of Food A and hence x_2 numbers of bacteria of strain II consumes $2x_2$ units of Food A. Each bacteria of strain III consumes 0 units of Food A and hence x_3 numbers of bacteria of strain III consumes $0 \cdots x_3$ units of Food A. Since we want to use up all of 400 units of Food A, we have the equation

$$x_1 + 2x_2 \;=\; 400 \;.$$

Writing Equation for Food B:
Since each bacteria of strain I consumes 2 units of Food B, x_1 numbers of bacteria of strain I consumes $2x_1$ units of Food B. Similarly, each bacteria of strain II consumes 1 unit of Food B and hence x_2 numbers of bacteria of strain II consumes x_2 units of Food B. Each bacteria of strain III consumes 1 unit of Food B and hence x_3 numbers of bacteria of strain III consumes x_3 units of Food B. Since we want to use up all of 600 units of Food B, we have the equation

$$2x_1 + x_2 + x_3 \;=\; 600 \;.$$

Writing Equation for Food C:
Since each bacteria of strain I consumes 1 unit of Food C, x_1 numbers of bacteria of strain

I consumes x_1 units of Food C. Similarly, each bacteria of strain II consumes 1 unit of Food C and hence x_2 numbers of bacteria of strain II consumes x_2 units of Food C. Each bacteria of strain III consumes 2 units of Food C and hence x_3 numbers of bacteria of strain III consumes $2x_3$ units of Food C. Since we want to use up all of 600 units of Food C, we have the equation

$$x_1 + x_2 + 2x_3 \;=\; 600 \ .$$

Thus, the linear system of equations is

$$\begin{aligned} x_1 + 2x_2 \;&=\; 400 \\ 2x_1 + x_2 + x_3 \;&=\; 600 \\ x_1 + x_2 + 2x_3 \;&=\; 600 \end{aligned}$$

Solving the above system of linear equations, we get $x_1 = 160$, $x_2 = 120$ and $x_3 = 160$. Therefore, the biologist should put 160 numbers of bacterial of strain I, 120 numbers of bacteria of strain II and 160 numbers of bacteria of strain III in the test tube if all the food to be consumed.

Example (D. Poole, Exercises 2.4, Problem 5):
A coffee merchant sells three blends of coffee, namely, house blend, special blend and gourmet blend. A bag of house blend contains 300 grams of Colombian beans and 200 grams of French roast beans, 200 grams of Kenyan beans. A bag of special blend contains 200 grams of Colombian beans, 100 grams of French roast beans and 200 grams of Kenyan beans. A bag of gourmet blend contains 100 grams of Colombian beans, 200 grams of French roast beans and 200 grams of Kenyan beans. The merchant has on hand 30 kilograms of Colombian beans, 25 kilograms of French roast beans, and 15 kilograms of Kenyan beans. If he wishes to use up all of the beans, how many bags of each type of blend can be made?
Solution:
Step 1: Modeling the Problem into a System of Linear Equations
Let x_1, x_2, x_3 be the number of bags of house blend, special blend and gourmet blend respectively.
Since each house blend bag contains 300 grams of Colombian beans, x_1 number of house blend bags contains $300x_1$ grams of Colombian beans. Similarly, x_2 number of special blend bags contains $200x_2$ grams of Colombian beans and x_3 number of gourment blend bags contains $100x_3$ grams of Colombian beans. Since the merchant wants to use all 30 kilograms (30000 grams), we get an equation

$$300x_1 + 200x_2 + 100x_3 = 30000 \qquad \Longrightarrow \qquad 3x_1 + 2x_2 + x_3 = 300 \ .$$

Likewise we obtain equation for French roast beans as

$$200x_1 + 100x_2 + 200x_3 = 25000 \qquad \Longrightarrow \qquad 2x_1 + x_2 + 2x_3 = 250 \ .$$

Likewise we obtain equation for Kenyan beans as

$$0x_1 + 200x_2 + 200x_3 = 15000 \qquad \Longrightarrow \qquad 2x_2 + 2x_3 = 150\ .$$

Now we have the following system of linear equations

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 300 \\ 2x_1 + x_2 + 2x_3 &= 250 \\ 2x_2 + 2x_3 &= 150 \end{aligned}$$

Step 2: Solving the Linear System of Equations
Solving the above system of linear equations, we get $x_1 = 65$, $x_2 = 30$ and $x_3 = 45$. He can make 65 bags of house blend, 30 bags of special blend and 45 bags of gourmet blend.

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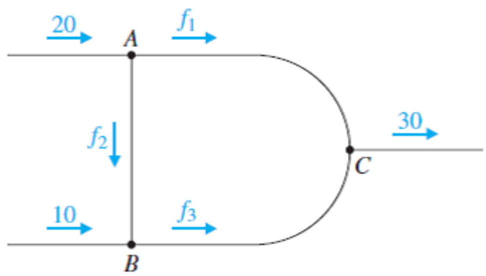
2 Network Analysis

Many practical situations give rise to networks: transportation networks, communications networks, and economic networks, to name a few. Of particular interest are the possible flows through networks.

A network consists of a finite number of nodes (also called junctions or vertices) connected by a series of directed edges known as branches (or arcs). Each branch will be labeled with a flow (or flow rate) that represents the amount of some commodities that can flow along or through that branch in the indicated direction. Many of the most important important kinds of networks have the following three properties:

- **One Dimensional Flow:** At any instant, the flow in a branch is in one direction only.
- **Flow Conservation at a Node:** The rate of flow into a node is equal to the rate of flow out of the node.
- **Flow Conservation in Network:** The rate of flow into the network is equal to the rate of flow out of the network.

A common problem in network analysis is use known flow rates in certain branches to find the flow rates in all of the branches.
Example (D. Poole, Exercises 2.4, Problem 15):
The following figure shows a network of water pipes with flows measured in liters per minute.



(a) Set up and solve a system of linear equations to find the possible flows.

(b) If the flow through AB is restricted to 5 L/min, what will the flows through the other two branches be?

(c) What are the minimum and maximum possible flows through each branch?

(d) We have been assuming that flow is always positive or non-negative. What would negative flow mean, assuming we allowed it? Give illustration for this example.

Solution:
(a): By flow conservation at each node, the rate of flow into a node is equal to the rate of flow out of the node.

$$\begin{aligned} &\text{At node } A: && 20 = f_1 + f_2 \\ &\text{At node } B: && 10 + f_2 = f_3 \\ &\text{At node } C: && f_1 + f_3 = 30 \end{aligned}$$

So the system of linear equations is as follows:

$$\begin{aligned} f_1 + f_2 &= 20 \\ f_2 - f_3 &= -10 \\ f_1 + f_3 &= 30 \end{aligned}$$

Solving the above system of linear equations, we get the possible flows as $f_1 = 30 - t$, $f_2 = -10 + t$ and $f_3 = t$ for all $t \in \mathbb{R}$.

(b): If $f_2 = 5$ then $t = 15$ and hence $f_1 = 15$ and $f_3 = 15$.

(c): Flows at all branches are to be non-negative real numbers. So, the minimum and maximum possible flows through each branch are given by

$$\begin{aligned} 0 &\leq f_1 \leq 20 \\ 0 &\leq f_2 \leq 20 \\ 10 &\leq f_3 \leq 30 \end{aligned}$$

(d): Negative flow would mean that water was flowing backward, against the direction of the arrow.

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3 Balancing Chemical Equations

When a chemical reaction occurs, certain molecules (the reactants) combine to form new molecules (the products). Chemical reactions are indicated by the chemical equations. The equation is usually written with reactants on the left, the products on the right, and an arrow in between to show the direction of the reaction.
A chemical equation is said to be a balanced chemical equation if for each type of atom in the reaction, the same number of atoms appears on each side of the arrow of the chemical equation.
For balancing a chemical equation, you need to formulate a system of linear equations and find the smallest positive integer values for the unknowns from the solutions set.
For example, the combustion of ammonia in oxygen produces nitrogen and water. The chemical equation for this chemical reaction is given by $NH_3 \ + \ O_2 \ \longrightarrow \ N_2 \ + \ H_20$. After balancing, it becomes $4NH_3 \ + \ 3O_2 \ \longrightarrow \ 2N_2 \ + \ 6H_20$.
Example (D. Poole, Exercises 2.4, Problem 9):
When butane (C_4H_{10}) burns in the presence of oxygen (O_2), it yields carbon dioxide (CO_2) and water (H_20). Write the chemical equation for this reaction and balance it.
Solution: The chemical equation for this reaction is given by

$$C_4H_{10} \ + \ O_2 \ \longrightarrow \ CO_2 \ + \ H_2O \ .$$

For balancing this chemical equation, we need to find the smallest positive integer values for the unknowns w, x, y and z such that

$$w \, C_4H_{10} \ + \ x \, O_2 \ \longrightarrow \ y \, CO_2 \ + \ z \, H_2O \ .$$

Formulating System of Linear Equations:

$$\begin{aligned}
\text{For Carbon Atom:} \quad & 4w \ = \ y \quad && \Longrightarrow \quad 4w - y \ = \ 0 \\
\text{For Hydrogen Atom:} \quad & 10w \ = \ 2z \quad && \Longrightarrow \quad 10w - 2z = 0 \\
\text{For Oxygen Atom:} \quad & 2x \ = \ 2y \quad && \Longrightarrow \quad 2x - 2y = 0
\end{aligned}$$

Solving the above system of linear equations, we get $w = 2$, $x = 13$, $y = 8$ and $z = 10$. So, the balanced equation for this reaction is given by

$$2 \, C_4H_{10} \ + \ 13 \, O_2 \ \longrightarrow \ 8 \, CO_2 \ + \ 10 \, H_2O \ .$$

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4 Electrical Circuits

A typical electrical network will have multiple batteries and resistors joined by some configuration of wires. A point at which three or more wires in a network are joined is called a node. A branch is a wire connecting two nodes, and a closed loop is a succession of connected branches that begin and end at the same node.

In a circuit with multiple loops, there is usually no way to tell in advnace which way currents are flowing. We will always take this direction to be clockwise. After calculating currents, those currents whose directions were assigned correctly will have positive values and those whose directions were assigned incorrectly will have negative values.

Following information/ Law and conventions are needed to solve the problem:
Ohm's Law: If a current of I amperes passes through a resistor with a resistance of R ohms, then there is a resulting drop of E volts in electrical potential that is the product of the current and resistance. That is, $E = IR$.

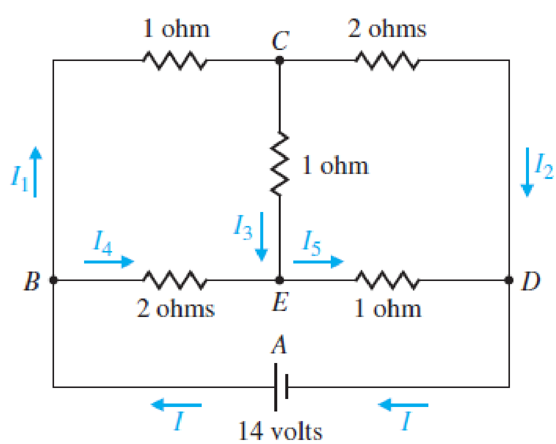
Kirchoff's Current Law: The sum of currents flowing into any node is equal to the sum of the currents flowing out.

Voltage Drop & Rise at a Resistor: A voltage drop occurs at a resistor if the direction assigned to the current through the resistor is the same as the direction assigned to the loop, and a voltage rise occurs at a resistor if the direction assigned to the current through the resistor is the opposite to that the assigned to the loop.

Voltage Drop & Rise at a Battery: A voltage drop occurs at a battery if the direction assigned to the loop is from $+$ to $-$ through the battery, and a voltage rise occurs at a battery if the direction assigned to the loop is from $-$ to $+$ through the battery.

Kirchoff's Voltage Law: In one traversal of any closed loop, the sum of the voltage rises equals the sum of the voltage drops.

Example (D. Poole, Exercises 2.4, Problem 21):
Conside the following bridge circuit.



(a) Find the currents I, I_1, I_2, I_3, I_4, I_5 in the above given bridge circuit.

(b) Find the effective resistance of this network.

(c) Can you change the resistance in branch BC (but leave everything else unchanged) so that the current through branch CE becomes 0?

Solution:
(a):
Formulating Linear Equations:
Kirchoff's current law provides

Node	Current In = Current Out
B	$I = I_1 + I_4$
C	$I_1 = I_2 + I_3$
D	$I_2 + I_5 = I$
E	$I_3 + I_4 = I_5$

The given electrical circuit has three closed loops, namely, $ABEDA$, $BCEB$ and $CDEC$. The loops have **clockwise** direction.
Kirchoff's voltage law provides

Loop	Voltage Rises = Voltage Drops
$ABEDA$	$14 = 2I_4 + I_5$
$BCEB$	$2I_4 = I_1 + I_3$
$CDEC$	$I_5 + I_3 = 2I_2$

Therefore, we get the following system of linear equations:

$$\begin{aligned}
I - I_1 - I_4 &= 0 \\
I_1 - I_2 - I_3 &= 0 \\
I - I_2 - I_5 &= 0 \\
I_3 + I_4 - I_5 &= 0 \\
2I_4 + I_5 &= 14 \\
I_1 + I_3 - 2I_4 &= 0 \\
2I_2 - I_3 - I_5 &= 0
\end{aligned}$$

Solving the above system of linear equations, we get $I = 10$, $I_1 = I_5 = 6$, $I_2 = I_4 = 4$ and $I_3 = 2$.

(b):
Using Ohm's law, we get

$$V = IR,$$

where $V = 14$ volts and $I = 10$ Ampreres. Therefore, the effective resistance in the circuit is given by

$$R_{eff} = V/I \qquad \Longrightarrow \qquad R_{eff} = 14/10 = 7/5 \text{ ohms .}$$

(C):
Yes. If we can change the resistance in BC as 4 ohms, then the current through branch CE becomes 0.

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5 Polynomial Interpolation

Given any n distinct points in the two dimensional xy-plane, there is a unique polynomial of degree $(n-1)$ or less who graph passes through these points.

Example:
Find a cubic polynomial whose graph passes through the points $(1, 3)$, $(2, -2)$, $(3, -5)$ and $(4, 0)$.
Solution:
Let $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be the cubic polynomial whose graph passes through the given points.
Formulating Linear Equations:

$$\begin{array}{ccrcl} P(1) = 3 & \text{gives that} & a_0 + a_1 + a_2 + a_3 & = & 3 \\ P(2) = -2 & \text{gives that} & a_0 + 2a_1 + 4a_2 + 8a_3 & = & -2 \\ P(3) = -5 & \text{gives that} & a_0 + 3a_1 + 9a_2 + 27a_3 & = & -5 \\ P(4) = 0 & \text{gives that} & a_0 + 4a_1 + 16a_2 + 64a_3 & = & 0 \end{array}$$

Solving the Linear System of Equations:
Solving the above system of linear equations, we get $a_0 = 4$, $a_1 = 3$, $a_2 = -5$ and $a_3 = 1$. Therefore,

$$P(x) = 4 + 3x - 5x^2 + x^3 \ .$$

Finding a Parabola/ Circle which passes through given 3 noncollinear points in the plane:
From elementary geometry we know that there is a unique parabola $y = ax^2 + bx + c$ passes through any three noncollinear points in a plane. Similarly, there is a unique circle $x^2 + y^2 + ax + by + c = 0$ passes through any three noncollinear points in a plane.

Example:
Find a parabola with an equation of the form $y = ax^2 + bx + c$ that passes through $(0, 1)$, $(-1, 4)$ and $(2, 1)$.
Solution:
Let $y = P(x) = ax^2 + bx + c$ be the equation of the parabola which passes through the given points.

Formulating Linear Equations:

$$\begin{array}{rcrcl} (0,1) & \text{gives that} & c & = & 1 \\ (-1,4) & \text{gives that} & a-b+c & = & 4 \\ (2,1) & \text{gives that} & 4a+2b+c & = & 1 \end{array}$$

Solving the Linear System of Equations:
Solving the above system of linear equations, we get $a = 1$, $b = -2$ and $c = 1$.
Therefore, the equation of the parabola passing through the given three points is

$$y = x^2 - 2x + 1\ .$$

Example:
Find a circle with an equation of the form $x^2 + y^2 + ax + by + c = 0$ that passes through $(0, 1)$, $(-1, 4)$ and $(2, 1)$.
Solution:
Let $x^2 + y^2 + ax + by + c = 0$ be the equation of the circle which passes through the given points.
Formulating Linear Equations:

$$\begin{array}{rcrcl} (0,1) & \text{gives that} & b+c & = & -1 \\ (-1,4) & \text{gives that} & -a+4b+c & = & -17 \\ (2,1) & \text{gives that} & 2a+b+c & = & -5 \end{array}$$

Solving the Linear System of Equations:
Solving the above system of linear equations, we get $a = -2$, $b = -6$ and $c = 5$.
Therefore, the equation of the circle passing through the given three points is

$$x^2 + y^2 - 2x - 6y + 5 = 0\ .$$

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6 Finite Linear Games

Modular Arithmetic and Finite Field

We know that computers represent data in terms of 0s and 1s, which can be interpreted as off/on, yes/no, false/true. Consider the set $S = \{0,\ 1\}$. Define addition $\bigoplus$ and multiplication $\bigotimes$ on F as follows:

$$\begin{array}{llll} 0 \bigoplus 0 = 0, & 0 \bigoplus 1 = 1, & 1 \bigoplus 0 = 1, & 1 \bigoplus 1 = 0. \\ 0 \bigotimes 0 = 0, & 0 \bigotimes 1 = 0, & 1 \bigotimes 0 = 0, & 1 \bigotimes 1 = 1. \end{array}$$

Here, the addition and multiplication are performed with integers modulo 2. Then $(S, \bigoplus, \bigotimes)$ is a field and it is denoted by $\mathbb{Z}_2$ or $GF(2)$.
Let

$$V = \{\mathbf{x} = (x_1, x_2, \cdots, x_n) \ : \ x_i \in \{0, 1\} \text{ for each } i = 1, 2, \ldots, n\}$$

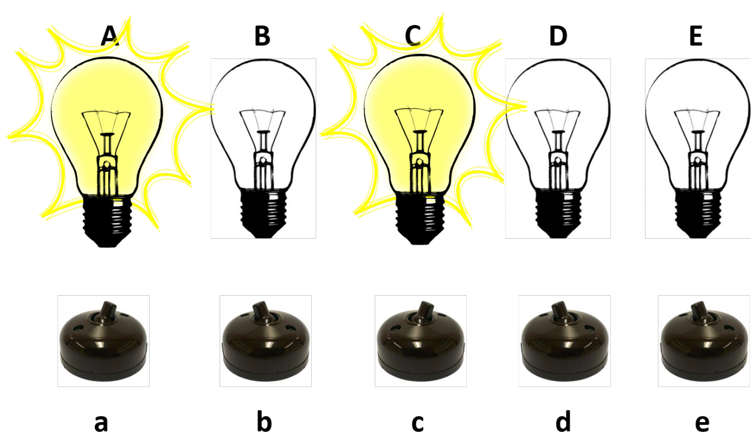
denote the vector space over the field $\mathbb{Z}_2$. The vectors in V are called binary vectors of length n. The vector space V is deonted by $\mathbb{Z}_2^n$ and on this vector space, the arithmetic is performed with modulo 2.

Finite Linear Games

There are many situations in which we must consider a physical system that has only finite number of states. These states can be changed by applying certain processes, each of which produces finitely many outcomes. The finiteness of such situations is perfectly suited to analysis using modular arithmetic, and often linear systems over some $\mathbb{Z}_p$, play a role.

A finite linear game is a problem that involves a situation which a physical system has only a finite number states which can be altered by applying certain processes. For example, the light bulbs are the physical system with a finite number of states: on or off. These states are changed by flipping a switch.

Puzzle invovling Bulbs (D. Poole, Example 2.35):
A row of five lights A, B, C, D, E is controlled by five switches a, b, c, d, e.
There are two states for the lights: 1-on or 0-off.



States Changing Rule: Each switch changes the state (1-on or 0-off) of the light directly above it and the states of the lights immediately adjacent to the left and right.
For example, suppse that the light **A** and **C** are on. See Figure given above. That is, the current stage is (1, 0, 1, 0, 0).

- If we push the switch **a**, then the light **A** changes its state, that is, from on to off and its immediately adjacent light **B** also changes its state, that is, from off to on.

This change in the states of bulbs can be written as (1, 0, 1, 0, 0) $\to$ (0, 1, 1, 0, 0) by pushing the switch **a**.

- If we push the switch **b**, then the light **B** changes its state, that is, from off to on and its immediately adjacent lights **A** and **C** also change their states as the light **A** changes from on to off and the light **C** changes from on to off. This change in the states of bulbs can be written as (1, 0, 1, 0, 0) $\to$ (0, 1, 0, 0, 0) by pushing the switch **b**.

- If we push the switch **c**, then the light **C** changes its state, that is, from on to off and its immediately adjacent lights **B** and **D** also change their states as the light **B** changes from off to on and the light **D** changes from off to on. This change in the states of bulbs can be written as (1, 0, 1, 0, 0) $\to$ (1, 1, 0, 1, 0) by pushing the switch **c**.

- If we push the switch **e**, then the light **E** changes its state, that is, from off to on and its immediately adjacent light **D** also changes its state, that is, from off to on. This change in the states of bulbs can be written as (1, 0, 1, 0, 0) $\to$ (1, 0, 1, 1, 1) by pushing the switch **e**.

Action of Switches in Vector Form: The actions of the five switches are given by

$$\mathbf{a} = \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}.$$

Sequence of Actions in Vector Addition Form: The initial configuration/vector **s** is given by

$$\mathbf{s} = \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix}.$$

By pushing the swith **a** changes to

$$\mathbf{s}+\mathbf{a} = \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}.$$

Starting with any initial configuration **s**, suppose we push the switches in the order **a**, **c**, **d**, **a**, **c**, **b**. This corresponds to the following vector sums in $\mathbb{Z}_2^5$, addition is commutative, so we have

$$\mathbf{s}+\mathbf{a}+\mathbf{c}+\mathbf{d}+\mathbf{a}+\mathbf{c}+\mathbf{b}=\mathbf{s}+\mathbf{2a}+\mathbf{b}+\mathbf{2c}+\mathbf{d}=\mathbf{s}+\mathbf{b}+\mathbf{d}$$

because by the fact that $2=0$ in $\mathbb{Z}_2$. Thus, we would achieve the same result by pushing only $\mathbf{b}$ and $\mathbf{d}$ and the order does not matter. Hence, we do NOT need to push any switch more than once.

Game/ Problem: Let the initial configuration $\mathbf{s}$ and the target configuration $\mathbf{t}$ be given. By performing actions on switches, is it possible to change from the initial configuration $\mathbf{s}$ to the target configuration $\mathbf{t}$?

That is, we need to determine whether there are scalars x_1, x_2, x_3, x_4, x_5 in $\mathbb{Z}_2$ such that

$$\mathbf{s}+x_1\,\mathbf{a}+x_2\mathbf{b}+x_3\,\mathbf{c}+x_4\,\mathbf{d}+x_5\,\mathbf{e} \;=\; \mathbf{t}\,.$$

That is, we need to solve (if possible) the linear system over $\mathbb{Z}_2$ that corresponds to the vector equation

$$x_1\,\mathbf{a}+x_2\mathbf{b}+x_3\,\mathbf{c}+x_4\,\mathbf{d}+x_5\,\mathbf{e} \;=\; \mathbf{t}-\mathbf{s}\,.$$

Example (D. Poole, Exercises 2.4, Problem 29):
Suppose all the lights are initially off.

(a) Can we push the switches in some order so that only the light **B** and the light **D** will be on?

(b) Can we push the switches in some order so that only the light **B** will be on?

Solution:
(a): Given that

$$\mathbf{s} \;=\; (0,\,0,\,0,\,0,\,0)^T \qquad \text{and} \qquad \mathbf{t} \;=\; (0,\,1,\,0,\,1,\,0)^T\,.$$

Now, we need to find the solution over $\mathbb{Z}_2$, if it exists, to the linear system that corresponds to the vector equation

$$x_1\,\mathbf{a}+x_2\mathbf{b}+x_3\,\mathbf{c}+x_4\,\mathbf{d}+x_5\,\mathbf{e} \;=\; \mathbf{t}-\mathbf{s}\,.$$

$$x_1\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}+x_2\begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}+x_3\begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix}+x_5\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}-\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}.$$

$$\begin{bmatrix}1&1&0&0&0\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\\0&0&0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}.$$

By Gaussian Elimination, the above system can be reduced to row echelon form of matrix system as

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} .$$

It gives that $x_3 = 1$, $x_1 + x_2 = 0$, $1 + x_2 + x_4 = 0$, $x_4 + x_5 = 0$.
If $x_5 = 0$ then $x_4 = 0$, $x_2 = 1$, $x_1 = 1$. That is, we need to push switches **a**, **b** and **c** which will make the lights **B** and **D** are on.
Alternatively, if $x_1 = 0$ then $x_2 = 0$, $x_4 = 1$, $x_5 = 1$. That is, altenatively we need to push switches **c**, **d** and **e** which will make the lights **B** and **D** are on.
(b): Given that

$$\mathbf{s} \;=\; (0,\,0,\,0,\,0,\,0)^T \qquad \text{and} \qquad \mathbf{t} \;=\; (0,\,1,\,0,\,0,\,0)^T \;.$$

Now, we need to find the solution over $\mathbb{Z}_2$, if it exists, to the linear system that corresponds to the vector equation

$$x_1\,\mathbf{a} + x_2\mathbf{b} + x_3\,\mathbf{c} + x_4\,\mathbf{d} + x_5\,\mathbf{e} \;=\; \mathbf{t} - \mathbf{s}\;.$$

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

By Gaussian Elimination, the above system can be reduced to row echelon form of matrix system as

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} .$$

The last row leads to $0 = 1$ and hence the above system has no solution.
Therefore, we can not make only the light **B** is on, by pushing switches in any order.

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7 Other/ Miscellaneous Applications

Example (D. Poole, Exercises 2.4, Problem 39):
There are two fields whose total area is 1800 square yards. One field produces grain at the rate of (2/3) bushel per square yard; the other field produces grain at the rate of (1/3) bushel per square yard. If the total yield is 1100 bushels, what is the size of each field?
Solution:
Let x be the area of the Field-1 and y be the area of the Field-2. Then,

$$x + y = 1800 .$$

Given that one field (Field-1) produces grain at the rate of (2/3) bushel per square yard. Therefore, x square yard Field-1 prodcues $((2x)/3)$ bushels as yield.
Given that other field (Field-2) produces grain at the rate of (1/2) bushel per square yard. Therefore, y square yard Field-2 prodcues $(y/2)$ bushels as yield.
Since the total yield is 1100 bushels, we get

$$\frac{2x}{3} + \frac{y}{3} = 1100 \qquad \Longrightarrow \qquad 4x + 3y = 6600 .$$

Therefore, the linear system of equations is

$$\begin{aligned} x + y &= 1800 \\ 4x + 3y &= 6600 \end{aligned}$$

Solving above system of linear equations, we get $x = 1200$ and $y = 600$. Thus, the size of Field-1 is 1200 square yards and the size of Field-2 is 600 square yards.
