**Question 1:** This question contains two parts; answer them in the corresponding space provided.

a) Consider a step function potential, defined as

$$V(x) = \begin{cases} -1, & -\pi \leq x \leq 0 \\ +1, & 0 < x \leq \pi \ , \end{cases}$$

whose representation in the Fourier space is $V(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_n \cos mx + b_n \sin mx\right)$. Using Fourier trick determine the coefficients $a_0, a_m$ and $b_m$. [4]

Fourier series expansion for $V(x)$ is given by

$$V(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos mx + b_m \sin mx\right)$$

Multiplying both sides by $\sin nx$ and integrating over $\mathcal{D}\,[-\pi, \pi]$:

$$\int_{-\pi}^{+\pi} V(x) \sin nx \, dx = \frac{a_0}{2} \underbrace{\int_{-\pi}^{+\pi} \sin nx \, dx}_{=0} + \sum_{m=1}^{\infty} a_m \underbrace{\int_{-\pi}^{+\pi} \cos mx \sin nx \, dx}_{=0}$$

$$+ \sum_{m=1}^{\infty} b_m \underbrace{\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx}_{2 \times \pi/2 \, \delta_{m,n} = \pi \, \delta_{m,n}}$$

$$\Rightarrow \int_{-\pi}^{0} (-1) \sin nx \, dx + \int_{0}^{\pi} \sin nx \, dx = \pi b_n$$

$$\Rightarrow \pi b_n = 2 \int_{0}^{\pi} \sin nx \, dx$$

$$\Rightarrow b_n = \frac{2}{\pi} \left[ - \frac{\cos nx}{n} \right]_0^{\pi} = \frac{2}{\pi n} (1 - \cos n\pi)$$

[Mark 2+]

$$\therefore \boxed{ b_n = \frac{2}{n\pi} (1 - \cos n\pi) = \begin{cases} \frac{4}{n\pi} & \text{for } n = \text{odd} \\ 0 & \text{for } n = \text{even} \end{cases} }$$

Next multiplying both sides by $\cos nx$ and integrating over $\mathcal{D}[-\pi, \pi]$:

$$\int_{-\pi}^{+\pi} dx \; V(x) \cos nx = \frac{a_0}{2} \underbrace{\int_{-\pi}^{+\pi} \cos nx \, dx}_{= 0} + \sum_{m=1}^{\infty} a_m \underbrace{\int_{-\pi}^{+\pi} \cos mx \cos nx \, dx}_{2 \times \frac{\pi}{2} \delta_{m,n}}$$

$$+ \sum_{m=1}^{\infty} b_m \underbrace{\int_{-\pi}^{+\pi} \sin mx \cos nx \, dx}_{= 0}$$

$$\Rightarrow -\int_{-\pi}^{0} dx \cos nx + \int_{0}^{\pi} \cos nx \, dx = \pi a_n$$

or, $\pi a_n = 0 \Rightarrow a_n = 0$

$$\therefore \boxed{a_n = 0}$$

[Mark 1+]

Finally, integrating bothsides over the domain $\mathcal{D}\ [-\pi, \pi]$ :

$$\int_{-\pi}^{+\pi} V(x)\, dx = \frac{a_0}{2} \int_{-\pi}^{+\pi} dx + \sum_{m=1}^{\infty} a_m \underbrace{\int_{-\pi}^{+\pi} \cos mx\, dx}_{\substack{\shortparallel \\ 0}}$$

$$+ \sum_{m=1}^{\infty} b_m \underbrace{\int_{-\pi}^{+\pi} \sin mx\, dx}_{\substack{\shortparallel \\ 0}}$$

$$\Rightarrow -\int_{-\pi}^{0} dx + \int_{0}^{\pi} dx = \frac{a_0}{2} \cdot 2\pi$$

or, $\pi a_0 = -\pi + \pi = 0$

$\Rightarrow \boxed{a_0 = 0}$ $\longrightarrow$ [ Mark 1+ ]

b) For a given vector field $\vec{V} = r\hat{r} + r\cos^2\theta\hat{\phi}$, find the volume charge density if it represents an electric field for a collection of static charges. [2]

The given vector field is $\vec{V} = r\hat{r} + r\cos^2\theta\,\hat{\phi}$.

First check if it represents the electric field for a collection of static charges.

So calculate $\vec{\nabla}\times\vec{V}$:

Here (In spherical polar coordinates)

$$\vec{\nabla}\times\vec{V} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\cdot r\cos^2\theta\right)\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(r) - \frac{\partial}{\partial r}\left(r^2\cos^2\theta\right)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[-\frac{\partial}{\partial\theta}(r)\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}\, r\left[\cos^3\theta - 2\sin^2\theta\cos\theta\right]\hat{r}$$

$$- \frac{1}{r}\cos^2\theta\cdot(2r)\ \hat{\theta}$$

$$= \left( \frac{\cos^3\theta - 2\sin^2\theta\cos\theta}{\sin\theta} \right)\hat{r} - 2\cos^2\theta\ \hat{\theta}.$$ $\rightarrow$ [Mark $1\frac{1}{2}$ +]

Since $\vec{\nabla} \times \vec{v} \neq 0$, $\vec{v}$ does not represent an electrostatic electric field.

$\Rightarrow$ The question of volume charge density does not arise here.

$\longrightarrow$ [Mark $\frac{1}{2}$]

**Question 2:** Consider a solid of uniform mass density $\rho$ that is bounded below by the $xy$ plane, above by the sphere $x^2 + y^2 + z^2 = 4a^2$, and laterally by the *half*-cylinder $s = 2a\cos\phi$ with $\phi \in [0, \pi/2]$. Here $(x, y, z)$ are in Cartesian coordinates, while $(s, \phi, z)$ are in cylindrical coordinates.

(a) Write down the expression for the differential moment of inertia $dI_z(x, y, z)$ about the $z$-axis in Cartesian coordinates.

(b) Express the above in cylindrical coordinates, i.e., $dI_z(s, \phi, z)$.

(c) Find the limits of the volume integration in cylindrical coordinates that you will need to obtain the total moment of inertia $I_z$ about $z$-axis.

(d) Determine the above moment of inertia of the system. $[\frac{1}{2} + \frac{1}{2} + 1 + 4 = 6]$

(a) Consider a small volume dV.

It contains $dm = \rho dV$ amount of mass.

$\therefore$ The differential form of moment of inertia of this dm mass about z-axis is (in Cartesian coordinates)

$$dI_z = dm(x^2+y^2)$$
$$= \rho\,(x^2+y^2)\,dV. \longrightarrow \left[\text{Mark } \tfrac{1}{2}+\right]$$

(b) In Cylindrical coordinates we have $dV = s\,ds\,d\phi\,dz$.

$$\therefore\ dI_z = \rho\,(x^2+y^2)\,s\,ds\,d\phi\,dz = \rho\,s^2 \cdot s\,ds\,d\phi\,dz$$
$$= \rho\,s^3\,ds\,d\phi\,dz$$

$\left[\text{Mark } \tfrac{1}{2}+\right]$

(c) To find the total moment of inertial, we have to integrate the above within the allowed region of mass. For the present case, the limits of integrations are:

$$\left.\begin{aligned} s &= 0 \longrightarrow 2a\cos\phi \\ z &= 0 \longrightarrow \sqrt{4a^2 - x^2 - y^2} = \sqrt{4a^2 - s^2} \\ \text{and } \phi &= 0 \longrightarrow +\pi/2 . \end{aligned}\right\}$$

$\left[\dfrac{\text{Mark}}{1+}\right]$

(d) Total moment of inertia:

$$I_z = \rho \int_{\phi=0}^{+\pi/2} \int_{s=0}^{2a\cos\phi} \int_{z=0}^{\sqrt{4a^2-s^2}} s^3\, ds\, d\phi\, dz$$

$$= \rho \int_{\phi=0}^{+\pi/2} d\phi \int_{s=0}^{2a\cos\phi} ds\; s^3 \sqrt{4a^2 - s^2}$$

Method I

$$\int_{s=0}^{2a\cos\phi} ds\; s^3 \sqrt{4a^2 - s^2}$$

$$\text{Put} \quad 4a^2 - s^2 = m^2$$

$$\therefore \quad s\,ds = -m\,dm$$

$$\text{and} \quad m = 2a \longrightarrow 2a \sin\phi$$

$$= \int_{m=2a}^{2a\sin\phi} (-m\,dm)\,(4a^2 - m^2)\cdot m$$

$$= \int_{2a}^{2a\sin\phi} (m^4 - 4a^2m^2)\,dm$$

$$= \left[\frac{m^5}{5} - \frac{4a^2m^3}{3}\right]_{2a}^{2a\sin\phi}$$

$$= \left(\frac{32a^5\sin^5\phi}{5} - \frac{32a^5\sin^3\phi}{3} - \frac{32a^5}{5} + \frac{32a^5}{3}\right)$$

$$= \frac{64}{15}a^5 + 32a^5\left(\frac{\sin^5\phi}{5} - \frac{\sin^3\phi}{3}\right)$$

$$\therefore I_z = \rho\int_{\phi=0}^{\pi/2} d\phi\left[\frac{64a^5}{15} + 32a^5\left(\frac{\sin^5\phi}{5} - \frac{\sin^3\phi}{3}\right)\right]$$

$$= \frac{32\pi a^5}{15}\rho + 32a^5\rho \int_0^{\pi/2} d\phi \left( \underbrace{\frac{\sin^5\phi}{5}}_{\frac{1}{5}\times\frac{8}{15}} - \underbrace{\frac{\sin^3\phi}{3}}_{\frac{1}{3}\times\frac{2}{3}} \right)$$

$$= \frac{32\pi a^5}{15}\rho + 32a^5\rho \underbrace{\left( \frac{8}{75} - \frac{2}{9} \right)}_{\frac{72-150}{75\times 9} = -\frac{78}{75\times 9}}$$

$$= \frac{32\pi a^5}{15}\rho - \frac{32\times 78}{75\times 9} a^5\rho$$

$$= \frac{32a^5\rho}{15}\left( \pi - \frac{78}{5\times 9} \right)$$

$$= \frac{32a^5\rho}{15}\left( \pi - \frac{26}{15} \right)$$

[ Marks 4 + ]

## METHOD-2

$$I_z = \rho \int_0^{\pi/2} d\phi \int_0^{2a\cos\phi} ds\; s^3\sqrt{(4a^2-s^2)}$$

Substitution Type $s = 2a\cos x$ $\quad\begin{cases} s: 0 \longrightarrow 2a\cos\phi \\ x: \pi/2 \rightarrow \phi \end{cases}$

$$I_z = -\rho(2a)^5 \int_0^{\pi/2} d\phi \int^{\phi} \cos^3 x \sin^2 x \, dx$$

$$= -\rho(2a)^5 \int_0^{\pi/2} d\phi \int_{\pi/2}^{\phi} (1-\sin^2 x)\sin^2 x \cos x \, dx$$

$$= -(2a)^5\rho \int_0^{\pi/2} d\phi \int_{\pi/2}^{\phi} (\sin^2 x - \sin^4 x)\cos x \, dx$$

$$= -(2a)^5\rho \int_0^{\pi/2} d\phi \left(\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x\right)\Bigg|_{\pi/2}^{\phi}$$

$$= -(2a)^5\rho \int_0^{\pi/2} \left[\tfrac{1}{3}\sin^3\phi - \frac{1}{5}\sin^5\phi - \frac{2}{15}\right] d\phi$$

$$= \frac{(2a)^5}{15}\rho\pi - \frac{(2a)^5}{1}\rho \int_0^{\pi/2} \left[\frac{1}{3}(1-\cos^2\phi)\sin\phi\right] d\phi + \frac{(2a)^5}{5}\rho \int_0^{\pi/2} \left[(1-\cos^2\phi)^2 \sin\phi\right] d\phi$$

$$= \frac{(2a)^5}{15}\rho\pi - \frac{(2a)^5}{3}\rho \int_0^{\pi/2} \left[\sin\phi - \cos^2\phi\sin\phi\right] d\phi + \frac{(2a)^5}{5}\rho \int_0^{\pi/2} \left[\sin\phi + \cos^4\phi\sin\phi - 2\cos^2\phi\sin\phi\right] d\phi$$

$$= \frac{(2a)^5}{15}\rho\pi - \frac{(2a)^5}{3}\rho \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/2} + \frac{(2a)^5}{5}\rho \left[-\cos\phi - \frac{1}{5}\cos^5\phi + \frac{2}{3}\cos^3\phi\right]_0^{\pi/2}$$

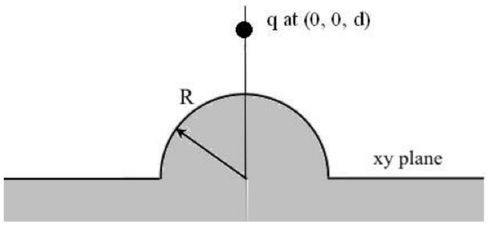
$$= \rho\frac{(2a)^5}{15}\pi - \rho(2a)^5\frac{2}{9} + \rho(2a)^5\frac{8}{75} = \rho\frac{(2a)^5}{15}\left[\pi - \frac{26}{15}\right]$$

**Question 3:** A conducting grounded sheet lies in the $xy$ plane with a hemispherical dome (radius $R$) centered at the origin. A point charge $q$ is placed at the distance $d(> R)$ on the $z$ axis.

a) Write down the image charges and their locations. (Don't derive, just guess on the basis of the classroom discussions and the tutorial problems.)

b) Find the potential at a point $\vec{r} = (x, y, z)$ for $z > 0$ and $r > R$.

c) Find the induced surface charge density and the corresponding total charge induced on the flat surface of the conductor, i.e., excluding the induced charges on the dome.




[2+1+3=6]

(a) Image charges:

$$q_1 = -q \quad \text{at } (0, 0, -d)$$

$$q_2 = -\frac{qR}{d} \quad \text{at } (0, 0, +R^2/d)$$

$$q_3 = +\frac{qR}{d} \quad \text{at } (0, 0, -R^2/d)$$

→ [Marks 2+]

(b) The potential at pt. $\vec{r} = (x, y, z)$ for $z > 0$ and $r > R$ is

$$V(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\{x^2+y^2+(z-d)^2\}^{1/2}} + \frac{(-q)}{\{x^2+y^2+(z+d)^2\}^{1/2}} + \frac{\left(-\frac{qR}{d}\right)}{\{x^2+y^2+(z-R^2/d)^2\}^{1/2}} + \frac{qR/d}{\{x^2+y^2+(z+R^2/d)^2\}^{1/2}}\right]$$

→ [Marks 1+]

(c) Take a point on the flat section of the conductor. The coordinates of this pt. are $(x, y, 0)$; $|x|, |y| \geq R$

Now, induced charge density is given by:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n}\Big|_{z=0}$$

$$= -\epsilon_0 \hat{n} \cdot \vec{\nabla} V \Big|_{z=0}$$

Here we have $\hat{n} = \hat{z}$.

$$\therefore \sigma = -\epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}$$

$$= \frac{q}{4\pi}\left[-\frac{2d}{(x^2+y^2+d^2)^{3/2}} + \frac{2R^3}{d^2(x^2+y^2+R^4/d^2)^{3/2}}\right] \rightarrow \left[\begin{array}{c}\text{Marks} \\ 1+\end{array}\right]$$

Total charge on the flat surface:

On the xy-plane, polar coordinates $(s, \phi)$ can be used. Then $x^2+y^2 = s^2$

and $da$ = surface element

$= s\,ds\,d\phi$

$$\therefore \quad Q_{total} = \int \sigma \, da$$

$$= \frac{q}{4\pi} \int\!\!\int \left[\frac{-2d}{(s^2+d^2)^{3/2}} + \frac{2R^3}{d^2(s^2+R^4/d^2)^{3/2}}\right] s\,ds\,d\phi$$

$$= \frac{q}{4\pi} \cdot 2\pi \int \left[ \frac{-2d}{(s^2+d^2)^{3/2}} + \frac{2R^3}{d^2(s^2+R^4/d^2)^{3/2}} \right] s\,ds$$

$$= \frac{q}{2} \int_R^{\infty} \left[ \frac{-2d}{(s^2+d^2)^{3/2}} + \frac{2R^3}{d^2(s^2+R^4/d^2)^{3/2}} \right] s\,ds$$

Substituting, $s^2+d^2 = X^2$ and $s^2 + \frac{R^4}{d^2} = Y^2$ one obtains:

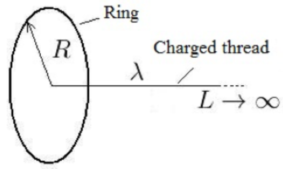
$$Q_{total} = -qd \int_{\sqrt{R^2+d^2}}^{\infty} \frac{dX}{X^2} + \frac{qR^3}{d^2} \int_{\sqrt{R^2+\frac{R^4}{d^2}}}^{\infty} \frac{dY}{Y^2}$$

$$= -qd \left[ -\frac{1}{X} \right]_{\sqrt{R^2+d^2}}^{\infty} + \frac{qR^3}{d^2} \left[ -\frac{1}{Y} \right]_{\sqrt{R^2+\frac{R^4}{d^2}}}^{\infty}$$

$$= \frac{q}{2} \frac{2}{d} \left[ \frac{R^2}{d\,(1+R^2/d^2)^{1/2}} - \frac{d^2}{(d^2+R^2)^{1/2}} \right]$$

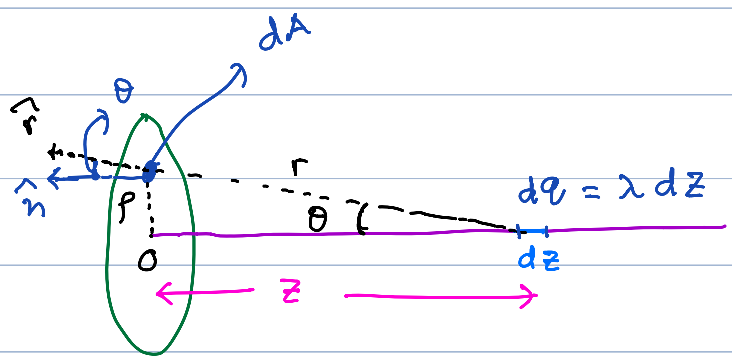
$$= \frac{q}{d} \frac{R^2-d^2}{\sqrt{R^2+d^2}} \,.\; \longrightarrow \left[ \begin{matrix} \text{Mark} \\ 2+ \end{matrix} \right]$$

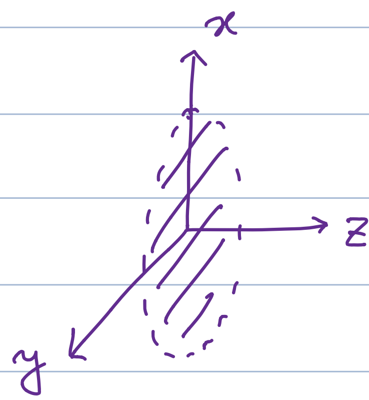
**Question 4:** A very long uniformly charged thread of length $L$ oriented along the axis of a ring of radius $R$ rests on its center with one of the ends. The charge of the thread per unit length is $\lambda$.




a) Find the flux $\Phi_L$ of the electric field vector $\vec{E}$ across the ring.
b) Determine the flux $\Phi_\infty$ through the ring in the infinite length limit (i.e., $L \to \infty$).

[5+1=6]




The electric field on the elementary area $d\vec{A} = dA\,\hat{n}$ due to elementary charge $dq = \lambda dz$ is

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 r^2}\,\hat{r}.$$

$\therefore$ The flux through $d\vec{A}$ area for the electric field $d\vec{E}$ is :

$$d\Phi = d\vec{E}\cdot d\vec{A}$$

$$= \frac{dq}{4\pi\epsilon_0 r^2} (\hat{r} \cdot \hat{n})\, dA$$

$$= \frac{dq}{4\pi\epsilon_0} \frac{dA \cos\theta}{r^2}.$$

$\left[\frac{\text{Mark}}{1 +}\right]$

$\therefore$ The flux through the ring due to charge $dq = \lambda\, dz$ is

$$\Delta\Phi = \int_{\left(\substack{\text{area of}\\ \text{the ring}}\right)} \frac{dq}{4\pi\epsilon_0} \frac{dA \cos\theta}{r^2}$$

Here $dA = \rho\, d\rho\, d\phi$ with $\rho = 0 \longrightarrow R$
$\phi = 0 \longrightarrow 2\pi$;

and $\cos\theta = \frac{z}{r}$ with $r = \sqrt{\rho^2 + z^2}$.

$$\therefore \Delta\Phi = \frac{dq}{4\pi\epsilon_0} \int \rho\, d\rho\, d\phi \cdot \frac{z}{r^3}$$

$$= \frac{z\, dq}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^R \frac{\rho\, d\rho}{(\rho^2 + z^2)^{3/2}}$$

$$= \frac{2\pi z\, dq}{4\pi\epsilon_0} \int_z^{\sqrt{z^2+R^2}} \frac{m\, dm}{m^3}$$

Put $\rho^2 + z^2 = m^2$
$\therefore \rho\, d\rho = m\, dm$

$$= \frac{z\, dq}{2\epsilon_0} \left[-\frac{1}{m}\right]_z^{\sqrt{z^2+R^2}}$$

$$= \frac{z\,dq}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right]$$

$$= \frac{dq}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] = \frac{\lambda\, dz}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]$$

Thus the total flux through the ring due to length L of the charged thread is

$$\Phi_L = \frac{\lambda}{2\epsilon_0}\int_0^L \left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) dz \longrightarrow \left[\frac{\text{Mark}}{2+}\right]$$

Substituting, $z = R\tan\theta \Rightarrow dz = R\sec^2\theta\, d\theta$

$$\Phi_L = \frac{\lambda}{2\epsilon_0}\int_0^{\tan^{-1}(L/R)} \left(1 - \frac{R\tan\theta}{R\sec\theta}\right) R\sec^2\theta\, d\theta$$

$$= \frac{\lambda R}{2\epsilon_0}\int_0^{\tan^{-1}(L/R)} \left(\sec^2\theta - \tan\theta\sec\theta\right) d\theta$$

$$= \frac{\lambda R}{2\epsilon_0}\Big[\tan\theta - \sec\theta\Big]_0^{\tan^{-1}(L/R)}$$

$$= \frac{\lambda R}{2\epsilon_0}\left[\frac{L}{R} - \sec\left(\tan^{-1}(L/R)\right) + 1\right]$$

$$\therefore \boxed{\Phi_L = \frac{\lambda R}{2\epsilon_0}\left[ L/R - \frac{L}{R}\sqrt{1+R^2/L^2} + 1 \right]} \rightarrow \left[\frac{\text{Mark}}{2+}\right]$$

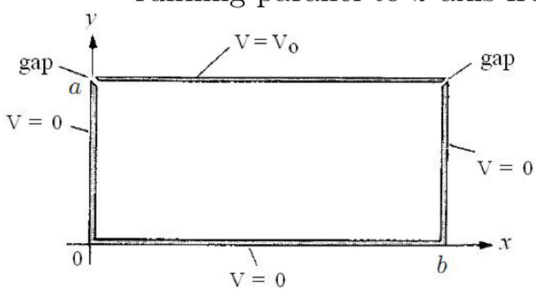
(b) For $L \to \infty$ limit: (keep upto leading order in $(R/L)$):

$$\Phi_L \simeq \frac{\lambda R}{2\epsilon_0}\left[ L/R - \frac{L}{R}\left(1+\frac{R^2}{2L^2}\right) + 1 \right]$$

$$= \frac{\lambda R}{2\epsilon_0} + \mathcal{O}\left(R/L\right) \xrightarrow{L\to\infty} \frac{\lambda R}{2\epsilon_0}$$

$$\therefore \boxed{\phi_\infty = \frac{\lambda R}{2\epsilon_0}} \longrightarrow \left[\frac{\text{Mark}}{1+}\right]$$

**Question 5:** Determine the electrostatic potential function for the region inside the rectangular pipe running parallel to $z$-axis from $[-\infty, +\infty]$ whose cross-section is shown in the figure.




[6]

Since the given problem has a z-symmetry, the solution must be independent of z-coordinate;

i.e. $V(x, y, z) \equiv V(x, y)$.

$$\therefore \quad \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \longrightarrow \left[\text{Mark } \tfrac{1}{2}+\right]$$

is the 2D Laplace's equation satisfying the boundary conditions:

(i) $V(x=0, \; 0 \leq y \leq a) = 0$

(ii) $V(x=b, \; 0 \leq y \leq a) = 0$

(iii) $V(0 \leq x \leq b, \; y=0) = 0$

(iv) $V(0 \leq x \leq b, \; y=a) = V_0$

$\longrightarrow$ [Mark 1+]

We now seek a variable separable solution given by the multiplicative ansatz:

$$V(x, y) = X(x)\, Y(y).$$

$$\therefore \quad \nabla^2 v = 0 \Rightarrow \frac{1}{X} X'' + \frac{1}{Y} Y'' = 0$$

For the given problem, by inspection we choose $\lambda^2 > 0$ such that

$$-\frac{1}{X} X'' = \frac{1}{Y} Y'' = \lambda^2 > 0 .$$

Hence, we obtain two 2nd order ordinary differential equations:

$$X'' + \lambda^2 X = 0$$

and $$Y'' - \lambda^2 Y = 0$$

The solutions are:

$$\left.\begin{aligned} X(x) &= A \sin(\lambda x) + B \cos(\lambda x) \\ Y(y) &= C \sinh(\lambda y) + D \cosh(\lambda y) \end{aligned}\right\} \rightarrow \boxed{\text{Mark } 1+} \quad ***$$

See at the end which is a discussion if one takes the exponential sol$^n$ for $Y(y)$

From BC. (i) : $v(0,y) = 0 \Rightarrow B = 0$

$$\therefore \quad v(x,y) = \sin(\lambda x) \left( C \sinh(\lambda y) + D \cosh(\lambda y) \right)$$

BC (ii) : $v(b,y) = 0 \Rightarrow \sin \lambda b = 0$

$$\Rightarrow \lambda b = n\pi$$

$$\Rightarrow \lambda \equiv \lambda_n = \frac{n\pi}{b} \quad \text{with } n = 1, 2, 3, \ldots$$

BC.(iii) : $v(x,0) = 0 \Rightarrow D = 0 .$

$$\therefore \quad v_n(x,y) = C_n \sin(\lambda_n x) \sinh(\lambda_n y) \longrightarrow (\times)$$

$$\therefore \boxed{V(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{b}\right) \sinh\left(\frac{n\pi y}{b}\right)}$$

[Mark

$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=1\frac{1}{2}$]

is the most general solution to the Laplace's equation.

Imposing the final B.C. (iv) one finds:

$$V_0 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{b}\right) \sinh\left(\frac{n\pi a}{b}\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi x}{b}\right)$ and integrating $0 \leq x \leq b$:

$$\int_0^b dx \; V_0 \sin\left(\frac{m\pi x}{b}\right) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi a}{b}\right) \underbrace{\int_0^b dx \sin\left(\frac{n\pi x}{b}\right) \sin\left(\frac{m\pi x}{b}\right)}_{\frac{b}{2}\delta_{mn}}$$

$$= \frac{b}{2} C_m \sinh\left(\frac{m\pi a}{b}\right)$$

$$\Rightarrow \quad V_0 \int_0^b dx \sin\left(\frac{m\pi x}{b}\right) = \frac{b}{2} C_m \sinh\left(\frac{m\pi a}{b}\right)$$

or, $$\frac{b V_0}{m\pi}\left[1 - \cos(m\pi)\right] = \frac{b}{2} C_m \sinh\left(\frac{m\pi a}{b}\right)$$

or, $$C_m = \frac{2V_0}{m\pi \sinh\left(\frac{m\pi a}{b}\right)} \underbrace{\left[1 - \cos(m\pi)\right]}_{\begin{cases} 2 & \text{for } m \text{ is odd} \\ 0 & \text{for } m \text{ is even.} \end{cases}}$$

$$\Rightarrow \quad C_n = \begin{cases} \dfrac{4V_0}{n\pi \sinh\left(\frac{n\pi a}{b}\right)}, & \text{if } n = \text{odd} \\ 0, & \text{if } n = \text{even} \end{cases}$$

$\rightarrow$ [Mark $1\frac{1}{2}$ +]

Thus the complete solution for the electrostatic potential is

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots} \left[ \frac{\sin\left(\frac{n\pi x}{b}\right) \sinh\left(\frac{n\pi y}{b}\right)}{n \sinh\left(\frac{n\pi a}{b}\right)} \right]$$

[Mark $\frac{1}{2}$ +]

Another form of $Y(y)$ solution:

$$Y(y) = c' e^{\lambda y} + D' e^{-\lambda y}$$

Then BC (i) and (ii) implies:

$$V_n(x,y) = \sin \lambda_n x \left(C' e^{\lambda_n y} + D' e^{-\lambda_n y}\right)$$

with $\lambda_n = \frac{n\pi}{b}$ ; $n = 1, 2, 3, ---$

B.C. (iii) : $V(x,0) = 0$

$\Rightarrow C' = -D'$

$$\therefore \quad V_n(x,y) = C' \sin \lambda_n x \left(e^{\lambda_n y} - e^{-\lambda_n y}\right)$$

$$= 2C' \sin \lambda_n x \sinh \lambda_n y$$

$$\equiv C_n \sin \lambda_n x \sinh \lambda_n y .$$

which is equⁿ (X). Rest of steps are as earlier.