Continuous-time Markov Chain: Poisson Process 3

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Superposition property

The sum of two independent Poisson processes is a Poisson process.
We first proof the following theorem for Poisson RVs.

Theorem: Suppose X_1 and X_2 are independent Poisson random variables with parameters λ_1 and λ_2 respectively. Let $Y = X_1 + X_2$ then $Y \sim Poi(\lambda_1 + \lambda_2)$.

Proof: Given $X_1 \sim Poi(\lambda_1)$ and $X_2 \sim Poi(\lambda_2)$

Then, MGF of X_1

$$M_{X_1}(s) = Ee^{sX_1}$$

$$= \sum_k e^{sk} e^{-\lambda_1} \frac{(\lambda_1)^k}{k!} = e^{-\lambda_1} \sum_k \frac{(e^s \lambda_1)^k}{k!} = e^{-\lambda_1} e^{\lambda_1 e^s}$$

$$=$$

$$= e^{\lambda_1 (e^s - 1)}$$

Superposition property…

Similarly,

$$M_{X_2}(s) = e^{\lambda_2(e^s - 1)}$$

$$M_Y(s) = Ee^{s(X_1+X_2)} = Ee^{sX_1} \cdot e^{sX_2}$$

$$= Ee^{sX_1} Ee^{sX_2} \leftarrow \text{Independence property}$$

$$= e^{\underline{\lambda_1(e^s - 1)}} e^{\underline{\lambda_2(e^s - 1)}}$$

$$\therefore M_Y(s) = e^{(\lambda_1+\lambda_2)(e^s - 1)}$$

which is the MGF of a Poisson random variable with parameter $(\lambda_1 + \lambda_2)$.

$$\therefore Y \sim Poi(\lambda_1 + \lambda_2)$$

Superposition property…

Using the above theorem we can prove the following property of Poisson processes.

Suppose $N_1(t)$ and $N_2(t)$ are independent Poisson processes with rates λ_1 and λ_2 respectively. Then $N(t) = N_1(t) + N_2(t)$ is a Poisson process with a rate $\lambda_1 + \lambda_2$.

Example A petrol pump serves on the average 30 cars and 20 trucks per hour. Assuming the Poisson model, find the probability that during a period of 5 minutes n vehicles come to the station.

Solution: We have $\lambda_1 = \dfrac{1}{2}$ and

$$\lambda_2 = \frac{1}{3}$$

$$\therefore \lambda_1 + \lambda_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$\frac{5}{6} \times \frac{5}{\cancel{60}\,12}$

No of Vehicles $\sim \text{Poi}\left(\frac{5}{72}\right)$

$$P(N(t) = n) = \frac{e^{-\frac{5}{72}}\left(\frac{5}{72}\right)^n}{n!}$$

Nonhomogeneous Poisson process

A *nonhomogeneous Poisson process* allows for the arrival rate $\lambda(t)$ to be a function of time. Such an assumption is valid in many practical cases.

Following are the postulates:

(i) *N*(0)=0 with probability 1.

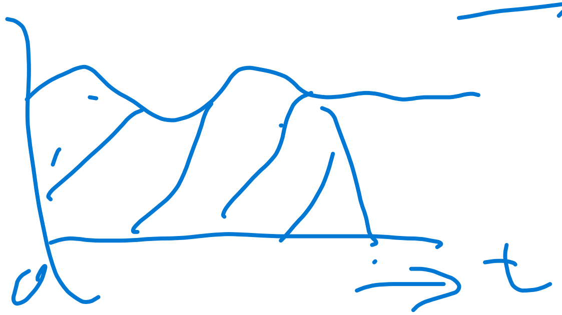
(ii) *N* (*t*) is an *independent increment* process.

(iii) $P\left(N(t+\Delta t)-N(t)=1\right)=\lambda(t)\Delta t+o(\Delta t)$

(iv) $P\left(N(t+\Delta t)-N(t)\geq 2\right)=\cancel{\lambda}o(\Delta t)$

$\lambda(t)$

$$\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$



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Using these postulates, we can derive that

$$P\left(N(t)=n\right)=\frac{\left(m(t)\right)^n e^{-m(t)}}{n!}$$

where $m(t)=\int_0^t \lambda(u)du$

We have,

$m(t) = \int_0^t \lambda(u)du$ is the

When $\lambda(t) = \lambda$, then we get homogeneous Poisson process.

We can show that

$$\boxed{\begin{aligned} EN(t) &= m(t) \\ \mathrm{var}(N(t)) &= m(t) \end{aligned}}$$

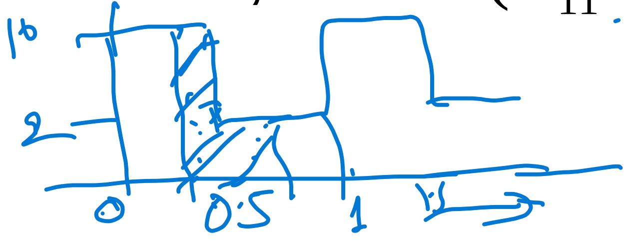
Example: For a nonhomogenous Poisson process the rate is given by

$$\lambda(t) = \begin{cases} 10, \ if \ t \in (0, 0.5],(1\,1.5]... \\ 2, \ if \ \ t \in (0.5, 1],(1.5\ 2].. \end{cases}$$

Find $E(N(1)$. If the 10the event occur at t=0.45, find $P(W_{11} > 0.75)$

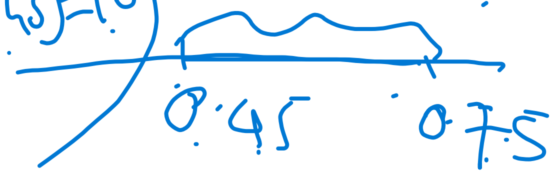
waiting time

$$m(1) = \int_0^1 \lambda(u)\,du = 10\times 0.5 + 2\times 0.5 = 6.$$



$$P(W_{11} > 0.75 \mid W_{10} = 0.45)$$

$$= P((N(0.45\ \ 0.75) = 0) \mid N(0.45) = 10)$$



$N(\ ) = 0 \quad W_{11} > 0.75$

$$= P(N(0.45\ \ 0.75] = 0) =$$

$$= e^{-1}.$$

$$m(t) = 0.05\times 10 + 0.25\times 2 = 1.$$

Compound Poisson Process

Let X_is be iid random variables and $Y(t)=\sum_{i=1}^{N(t)} X_i$, where $N(t)$ is a homogeneous Poisson process which is independent of each of X_is. The process $\{Y(t)\}$is called a *compound Poisson process*.

For example, suppose customers arrive at a departmental store with a Poisson rate λ. Each customer *i* spends some amount of money X_i independently of others and the amount for each customer is assumed to be of identical distribution (say uniform). The ~~the~~ money earned by the stores in the time interval $(0, t]$ is a compound Poisson process.

We can find $EY(t)$ and $Var(Y(t))$ using the properties of conditional expectations.

$EY = EEY|X$

$EY|X$ is random and $EEY|X = EY$

$Var(Y) = E\,Var\,Y|X + Var(EY|X)$.

We have

$$Y(t) = \sum_{i=1}^{N(t)} X_i$$

$$E(Y(t) \mid N(t) = n) = E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \sum_{i=1}^{n} EX_i$$

$$= nEX_1$$

$E(Y(t) | N(t)) = N(t)\,EX_1$

$$\therefore EY(t) = EE(Y(t) / N(t)) = \sum_{n=0}^{\infty} \frac{nEX_1 e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= EX_1.\lambda t$$

$$var(Y) = E\,var(Y|X) + var(EY|X$$

Now

$$Var(Y(t)\,|\,N(t)=n) = \text{var}\left[\sum_{i=1}^{n} X_i\right]$$

$$= \sum_{i=1}^{n} \text{var}(X_i)$$

$$= n\,\text{var}\,X_1$$

$$\therefore\ var(Y(t)|N(t)) = N(t)\,var X_1$$

Thr variance of $\{Y(t)\}$ is given as

$$Var(Y(t)) = E(\text{var}(Y(t)\,|\,N(t) + \text{var}(E(Y(t)\,|\,N(t)))$$

$$= EN(t)\,\text{var}(X_1) + \text{var}(N(t)EX_1)$$

$$= \lambda t\,\text{var}(X_1) + \lambda t(EX_1)^2$$

$$= \lambda t\left(var(X_1) + (EX_1)^2\right)$$

$$= EX_1^2$$

$$= \lambda t E X_1^2$$

Customers arrive at a departmental store in a Poisson manner at rate of 10 customers per hour. Each customer spends uniformly between Rs. 400 -Rs 2000 independently of other customer. What is the average income of the store in a two-hour interval? What is the variance of the total income?