Graph Theory :-

Simple Graph: $G = (V, E)$

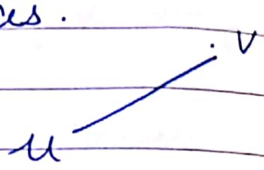
\# vertex

→ V - non empty set finite set - vertices/nodes

→ E - distinct unordered pairs

- edges/links of distinct elements of V

Edge has to be between two vertices.



{ $\{u_1, v_1\}$ u_1 and v_1 distinct

$\{u_2, v_2\}$ u_2 v_2 ——

$\{u_3, v_3\}$ u_3 v_3 ——

}

(u_1, v_2) and (v_2, v_3) are different

(v_2, v_3) and (v_3, v_1) ——

case of a simple Graph ⊥.

General graph :-

→ Multiple Edges

→ Edges from vertex to itself

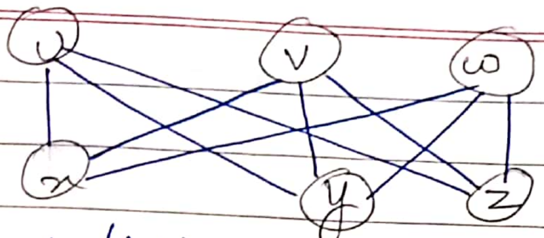
Isomorphism :- $G_1 = (V_1, E_1)$

$G_2 = (V_2, E_2)$

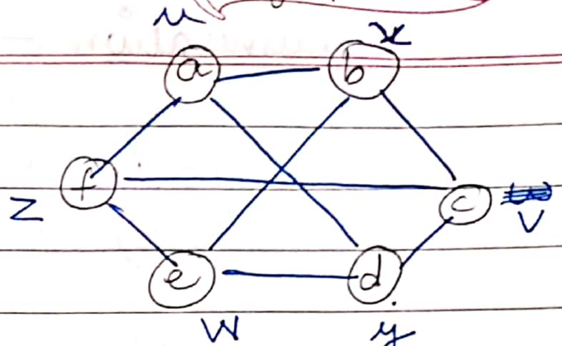
are isomorphic if there exists a one to one mapping f from V_1 onto V_2 s.t edes joining u and v in G_1 = no. of edges joining f(u) and f(v) in G_2

\# change the names of the points but the adjencies are same —

Graph 1.



Graph 2



Edge set / Adjancies are same.

$f(u) = a$
$f(w) = e$
$f(z) = f$
$f(y) = d$
$f(x) = b$.
$f(v) = c$

3 vertin labelled Graph.

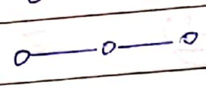
→ one graph → 0 edges
→ 3 graphs → 1 edges $^3C_1$. $(1,2)$ $(2,3)$ $(1,3)$.
(Isomorphic to each other)
→ 3 graphs → 2 edges $^3C_2$. $\{(1,2)(2,3)\}$ $\{(1,2)(1,3)\}$ $\{(1,3)(2,3)\}$
→ 1 graph → 3 edges.

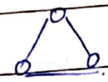
→ Deleating labels results in the graphs becoming identical. Just 4 graphs possible.

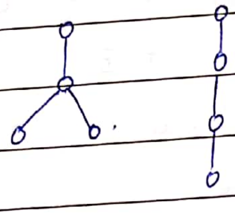
Enumeration :-

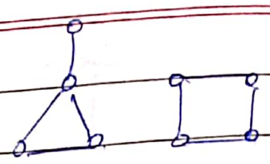
(1,0) 

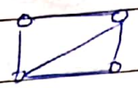
(2,1) 

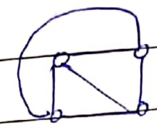
(3,2) 

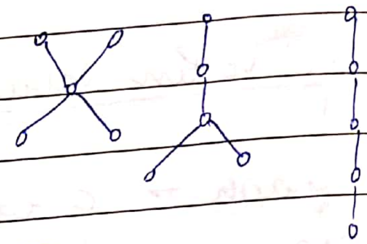
(3,3) 

(4,3) 

(4,4) 

(4,5) 

(4,6) 

(5,4) 

(Distinct Unlabelled Graphs).
↳ No labelling of vertices

(5,5) → 5
(5,6) → 5
(5,7) → 3
(5,8) → 2
(5,9) → 1
(5,10) → 1

(Max edges wid 5 vertices = 10 $^5C_2$)

Remarks :-

→ u and v are adjcent if uv ∈ E
{u, v}

E = unordered pair -

→ uv is adjacent with u and v

→ u is adjacent to uv.

→ e and f are adjacent if they have a common vertex.
$\underbrace{e \text{ and } f}_{\text{edges}}$.

→ Degree of a vertex is the no. of edges incident to it.

→ Degree = zero (Isolated Vertex)
→ Degree ⇒ one (End Vertex)

Scanned by CamScanner

Handshaking lemma :-

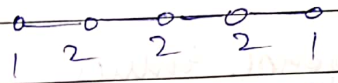
↳ sum of degrees of all vertices = 2(Edges) [always even].

No. of handshakes

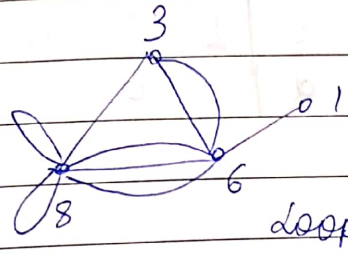
[Sum of degrees ⇒ even.
vertices with even degrees ✓ no. of vertices with odd degrees also should be even - odd + odd = even.]

Degree sequence :-

Degrees in non decreasing monotonic order.



1 1 2 2 2



1, 3, 6, 8.

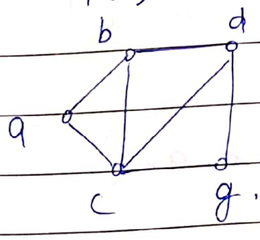
loop → count twice (vertex to itself.)

Ex Find 2 non isomorphic graphs with same deg. seq.

$G' = (V', E')$ is a subgraph of $G = (V, E)$

if $E' \subseteq E \Rightarrow V' \subseteq V$

$G(V)$ is induced subgraph.



$G \cdot [\{a, c, g\}]$

$(\{a, c, g\}, \{\{a, c\}\})$

Every subgraph is not induced but every induced is a subgraph

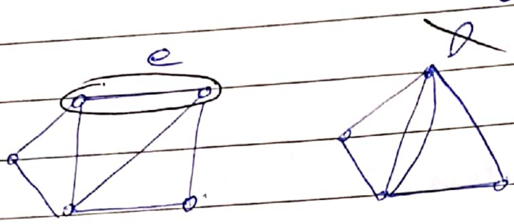
$G - \{b, d\}$
↳ delete the edge bd.

$G - e$. → edge deleted.

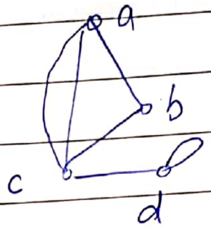
$G - v$ → delete all incident edges.

Edge ~~contraction~~ :- Contraction :-

$G \setminus e$:- Contract that particular edge, such that the 2 vertices collapse on one another.



Matrices : → Adjacency Matrix.



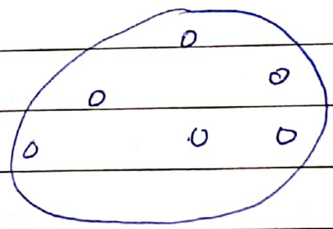
	a	b	c	d
a	0	1	2	0
b	1	0	1	0
c	2	1	0	1
d	0	0	1	2

\# Diagonal Entries
↳ Always even entries (self loop stuff).

→ Incidence Matrix.

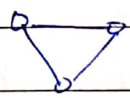
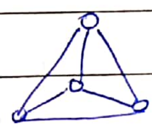
$M_{ij} = 1$ if vertex i is incident to edge j

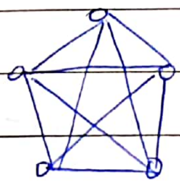
→ Null Graph. # ~~no~~ n vertices –
N_n
(No edges)



N_6

→ complete graph (K_n clique)

$K_1 = $ o K_2  K_3  K_4 

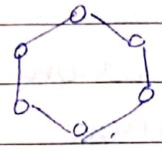
K_5 

(cannot draw planarly without edges crossing)

5 → (Minimum non planarity.)

→ A regular graph :-
All vertices have same deg

r regular :- # 2 regular ⇒ cycle.
(All have 2 degree).



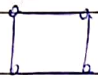
C_6. [Cycle].

3 regular ⇒ Cubic.

Hypercube :-

Q_0 o K_1

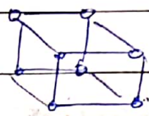
Q_1 o—o K_2

Q_2  C_4

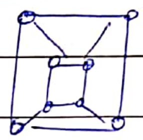
(2 regular graph).

$Q_{k+1} = Q_k + Q_k$
(join there corresponding vertices).

Q_3 → Two copies of Q_2 and join them.



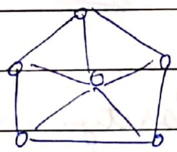
(3 regular graph)



In general, a Q_k graph is a k regular graph.

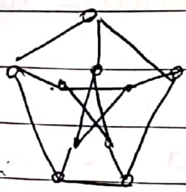
→ Path Graph $C_n - e = P_n$. (A path with n vertices)
↳ straighten out and we get a path.

→ Wheel Graph



W_6.

Note



Peterson Graph
(Deg = 3)

Scanned by CamScanner

Any K_n graph will be $(n-1)$ regular.

Tetrahedron.

(4-clique Graph).

Platonic Graphs :-

Tetrahedron

Octahedron.

Cube.

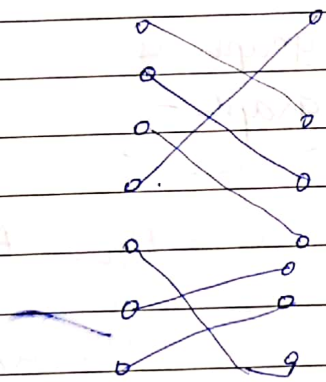
Icosahedron.

Dodecahedron.

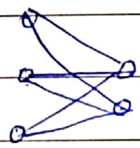
Bipartite Graph :-

There are 2 sets of vertices.

Every vertex is either a right vertex or a left vertex.



$K_{n,m}$ is a complete bipartite graph



$K_{3,2}$

Q_k is a bipartite graph

Number the vertices :-

Parity is no. of 1's in its binary representation

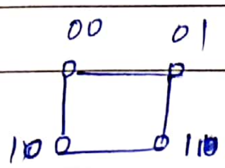
Q_0 0

Q_1

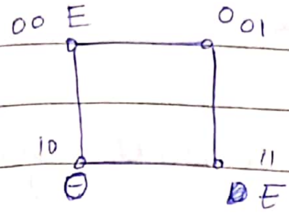
00 → even parity.

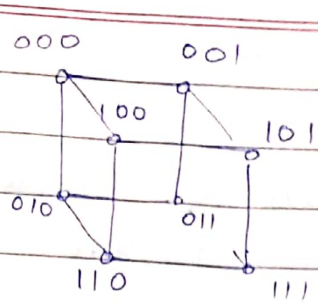
01 → odd parity.

Q_2



Consider Q_2



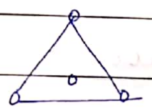


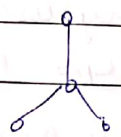
Every connection is between an odd parity and an even parity.

Every Hypercube is a bipartite graph.
Join 2 Q_2's for Q_3. latter face → 0
Fore face → 1. (parity flips)

K^{th} dimension edges between vertices differing in K^{th} dimension

Compliment of a Graph: $\overline{G}$ is $K_{|V|} - G$ (complete graph)





second is comp. of 1st

Walking in the graph ?

$V_0 \to V_1 \to V_2 \to \cdots V_m$

V_i and V_{i+1} are adjacent

Trail: Walk with distinct edges.

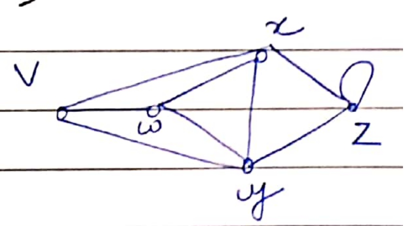
Path: Walk with distinct values except possibly $V_0 - V_m$

Closed trail: $V_0 - V_m$ (circuit)

Closed path: $V_0 - V_m$ (cycle).

v w x y z z y w (walk but not path)

v w x y z z x trail.



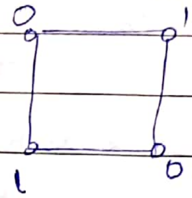
v w x y z Path.

v w x y v closed trail.

v w x y v cycle.

A graph is connected iff there is a path from any vertex to any vertex.

G is bipartite iff every cycle has even length-



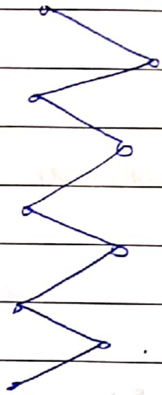
(Two colourable)
↳ Two types basically

↳ length - even (bipartite)

~~[illegible]~~ # given a graph decompose into cycles.

Steps to follow :-

→ Start at any vertex. → Either you come back to starting point / dead end or you find a cycle. (a vertex repeats) (Basically traverse a path)



Case 1 you encounter a cycle → Delete this edges belong cycle and 2 color the remaining graph. Put cycle again and when we traverse the cycle now, Bipartite as colors alternate.

case : 2 you find a path.

If is a graph with n vertices G has k components.

$$|E| = m \quad \text{then} \quad n-k \le m \le (n-k)(n-k+1)/2$$

$m=0$ Basis $N_n \cdot k = n$ $m = 0 \geq x - n = 0.$

G has as few edges as possible, removal of one edge ↑ components.

G − e has n vertices and k+1 components

$m_e - 1$ edges −

$$m_0 - 1 \geq n - (k+1) = (n-k-1)$$

$$m_0 \geq n-k.$$

Assume that n is made up of k components with $n_1, n_2 - n_k$ vertices −

We need to maximize edges −

$$n_1 + n_2 + n_3 \cdots + n_k \qquad n_1 \geq n_2 \cdots n_k.$$

G_1: $\boxed{K_{n_1} \cup K_{n_2}}$ $\quad$ G_2: $\boxed{K_{n_1+1} \cup K_{n_2-1}}$

$n_1 \geq n_2 > 1$

G_1 and G_2 have same no. of vertices :−

Jyada no. of edges ?

$$\frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} \Rightarrow G_1$$

$$\frac{(n_1+1)n_1}{2} + \frac{(n_2-1)(n_2-2)}{2} \Rightarrow G_2$$

$$n_1 - n_2 + 1 > 0.$$

$n_1 = n-k+1$

Isolated vertices $\begin{cases} n_2 = 1 \\ n_3 = 1 \\ \vdots \\ n_k = 1 \end{cases}$ Largest no. of edges −

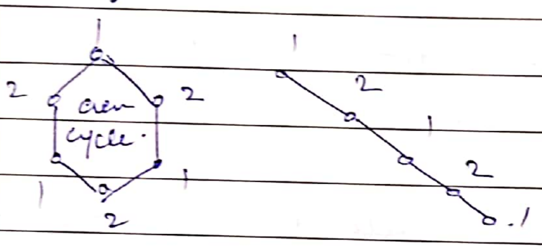
edges − all of em in one component

$$\frac{(n-k+1)(n-k)}{2}$$

Euler Trail: A graph has an Euler trial ~~there~~ iff every vertex has even ~~dg~~ degree.

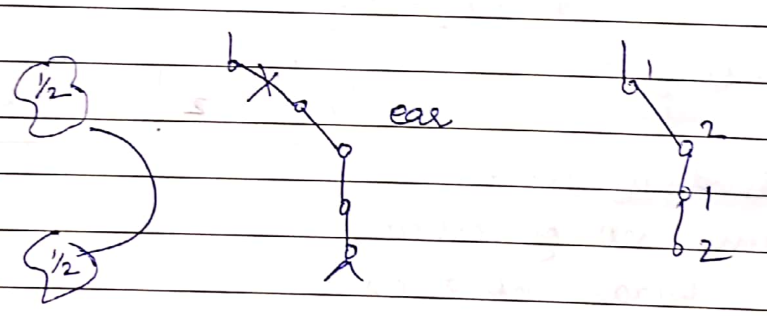
If G has no odd cycle, then G is bipartite.

Basis: o—o. $n=2$ $m=1$
Hypo: All smaller graphs have the stmt true.
Step: G, n, m.
1. If every vertex has a deg ≤ 2.
2. If there is an isolated cycle/chain ✓



3. Start from any vertex of degree > 2

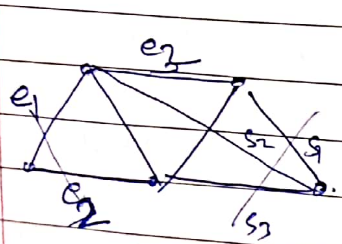
G



\# If g has no odd cycle, g is bipartite

Dis connecting Set :– A set of edges whose removal dis connects G.

A minimal DS is a cutset



(e_1, e_2) is a cutset.

(S_1, S_2, S_3) is a cutset

~~No subset of a cutset should be~~

able to act as a DS.
There can be multiple minimal cutsets.

Size of the smallest cutset is called edge connectivity of G. $\lambda(G)$.

i.e min. of cutset size.

<u>Seperating Set</u>: A set of vertices whose removal disconnects ~~edg~~ the graph

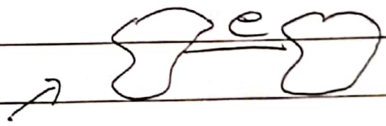
The size of smallest SS. is vertex connectivity of G

$$K(G) = 2$$

When we remove a vertex, we remove an edge.

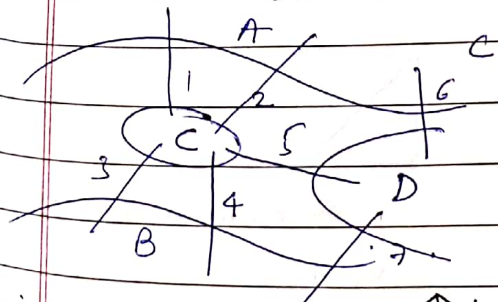
$\therefore$ $K(G) > \lambda(G)$.

<u>Remarks</u> :-



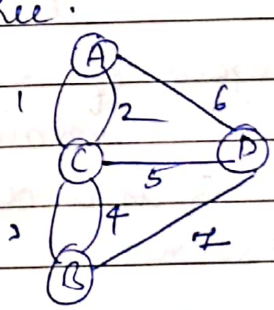
if {e} is a cutset, then e is a bridge
if {v} is a seperating set, then v is a vertex.

<u>Eulerian Graph</u> :-



Konigsberg Problem

In any connected graph there is Eulerian trail iff every vertex has an even degree.

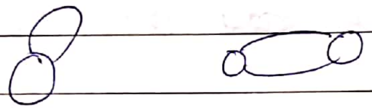


Solution : No we can't.

Proof :-

If G is a graph with deg (v) ≥ 2 ∀v ; ~~[illegible]~~
then G has a cycle.

consider even deg vertex

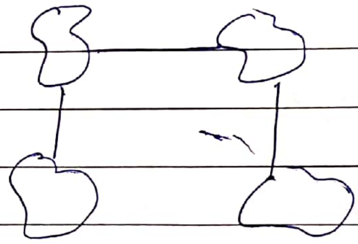


. G is simple.

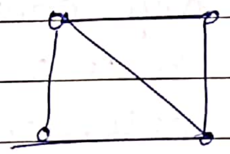
$v \to v_1 \to v_2 \to$ ——

Every vertex has a deg ≥ 2

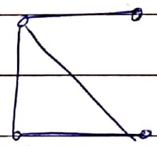
C is a cycle G−C



Hamiltonian Graph :- Goto every vertex and come back to original starting point



Hamiltonian cycle



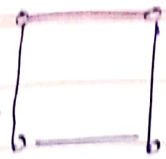
semi hamiltonian

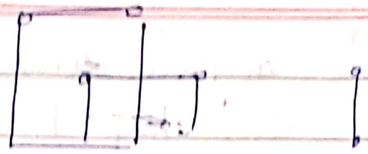
analogous of semi hamiltonian, we have semi Eulerian graph -

G is a semi Eulerian iff it has exactly 2 vertices of odd degree.

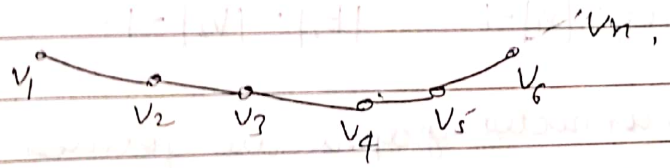
Date
Page

G_2



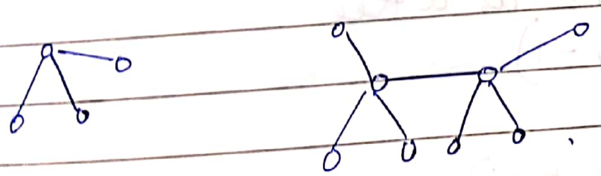


If G is a simple graph $n \geq 3$ and $deg(v) + deg(w) \geq n$ for each v & w non adjacent.
then G is a hamiltonian.



v_i and v_{i-1} (pair of vertices)
so v_1 is adjacent to v_i
and v_{i-1} is adjacent to v_i

Tree :- A forest is a graph without cycles. Acyclic graph. Forest : Each component is a tree. Tree : connected acyclic graph.



1) T is a tree
2) T has no cycles and has $(n-1)$ cycles
3) T is connected and has $(n-1)$ cycles
4) T is connected and each edge is a bridge.
5) Any 2 vertices of T are connected by exactly one path.
6) T has no cycles but addition of a new edge ~~but~~ forms exactly one cycle

T is a simple graph

Scanned by CamScanner

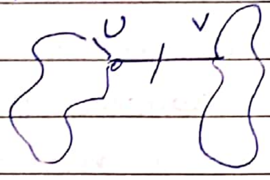
T is a tree

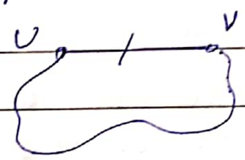
\# ① ⇒ ② Acyclic and connected.

<u>Basis</u> o $|V| = 1$ $|E| = 0$

<u>Induction Step Hypothesis</u> :- Remove any edge. It should split the graph into 2 trees-

$|V_1| + |V_2| - 1$





$|E_1| = |V_1| - 1$ $|E_2| = |V_2| - 1$.

\# Two disconnected graphs are formed with

\# ② ⇒ ③ T is acyclic

(has $k > 1$ components and is connected)

with n_1 n_2 n_3 --- n_k vertices respectively.

Each component is a tree.

No. of Edges = (No. of vertices − 1)

Total no. of edges $= (n_1 - 1) + (n_2 - 1) + (n_3 - 1) + (n_4 - 1)$ --- $n_k - 1$.

$= (n_1 + n_2 \text{ --- } n_k) - k. = n - k < n - 1$

\# ③ ⇒ ④. Remove an edge from T

$m \geq n - k$ in any graph with k components.

$\left\{ \begin{array}{l} n \text{ vertices} \\ (n-2) \text{ edges} \end{array} \right.$ [n−1 pehle. 1 removed]

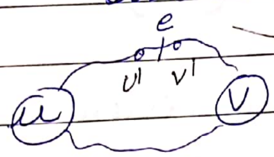
Disconnected.

Use the lower bound on no. of edges for any graph -

Every edge is a cutset on its own [i.e] is a bridge.

④ ⇒ ⑤

T Connected, there is at least one path between any pair of vertices.



(~~this edge is not a bridge~~).

Because removal of e will not stop me from going ~~to~~ from v' to v'

Exactly one path between any pair of vertices.

⑤ ⇒ ⑥ T has a cycle. ✗

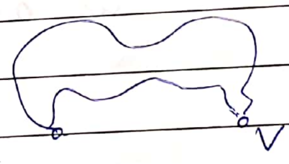
∴ T is acyclic ✓

Now e with u & v forms a cycle (on adding an edge)

T is a forest

But how do you know there is just one cycle?

If there was another cycle, removal of e also results in a cycle.

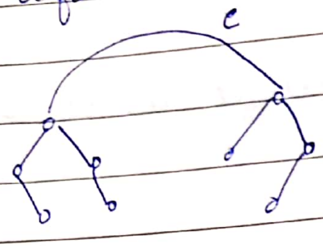


~~Bu~~ But G - e = original graph ⇒ which should be acyclic ✓

⑥ ⇒ ①.

Scanned by CamScanner

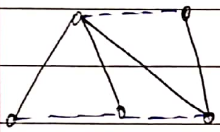
T is a forest   T is nor a tree



addition of e doesn't result in a cycle if T is a tree.

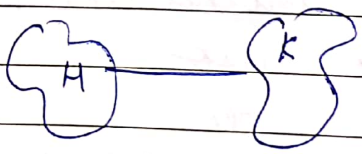
Remark: G is a forest with k ~~edges~~ components.
no of edges $= (n-k)$

Spanning Tree: Every vertex is connected to every vertex.
Removal of specific edges will remove cycles but we can still traverse from one vertex to next.



T is a spanning forest for G.

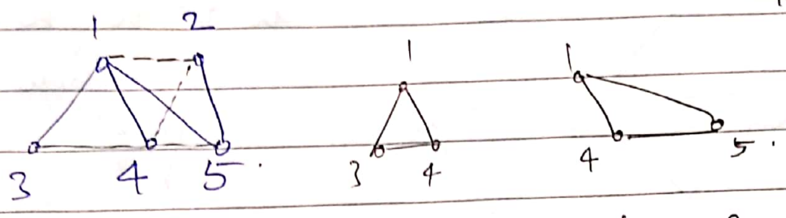
i) Each cutset of G has an edge in common with T
ii) C^* is a cutset of G
Removal of C^* splits G into H and K.
T is a spanning forest.



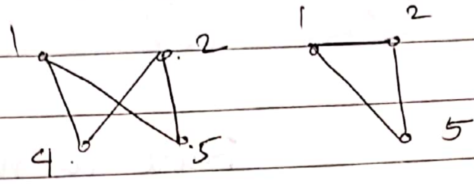
T is a spanning forest in G. Then each cycle in G has an edge common with G−T

T is a spanning forest of G $\boxed{e \le T}$

T+ e has a unique cycle. Fundamental cycle of G wrt T



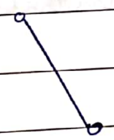
addition of exactly 1 edge ⇒ fundamental cycle.



fundamental cycles-

consider any ~~te~~ tree edge. For removal of each tree edge splits the graph into 2 parts. If we consider these edges in the original graph, then they form a cutset.

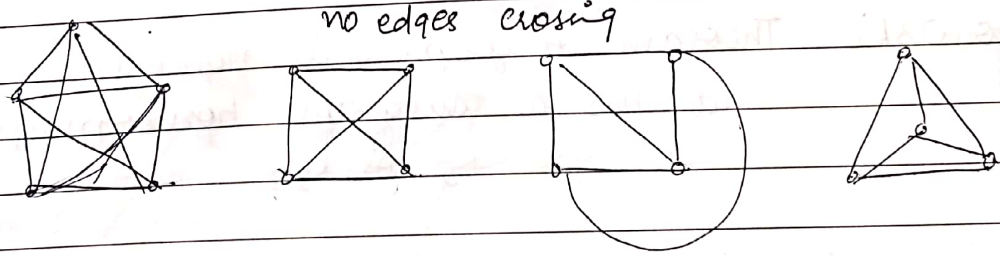
Ex



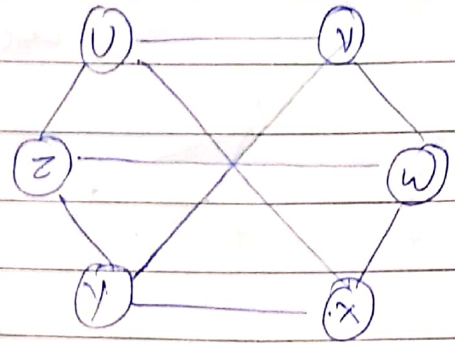
Planar Graph: can be drawn on a plane with non crossing edges-

K_5

whatever can be drawn on a plane with no edges crossing

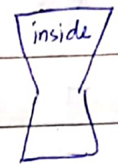


K_5 is nonplanar. $K_{3,3}$ is non-planar

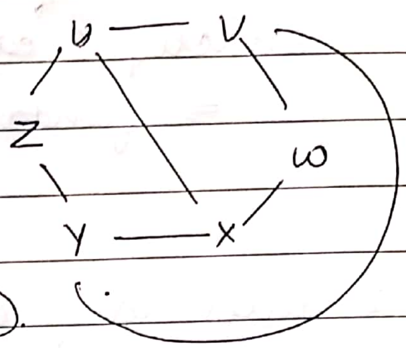


u v w x y z u

outside

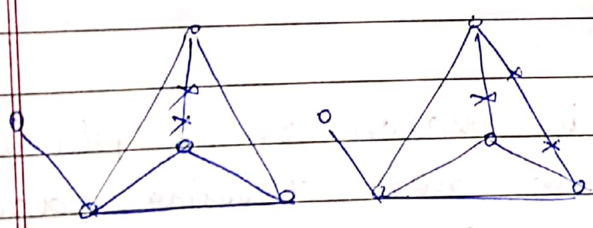


There is no way to draw w-x edge without crossing (v-y).



G_1 and G_2 are homomorphic if both can be obtained from some G by introducing vertices on existing edges-

G_1 is obtained from G by introducing vertices on existing edges-

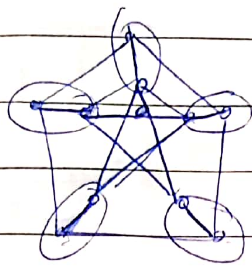


G_1 G_2

Kuratowski Theoram: A graph G is nonplanar iff it has a subgraph homeomorphic to K_5 or $K_{3,3}$

If G can be obtained from H by successively contracting edges. G is contractible into H.

Peterson's graph contracts into K_5

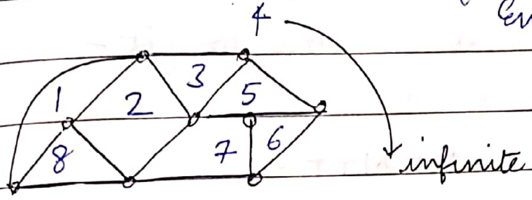


Theorem: A graph is planar iff it contains no subgraph contractible to K_5 or $K_{3,3}$.

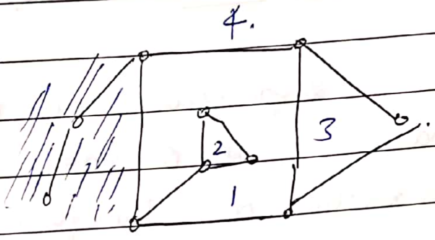
Euler's Formula :-

(Planar Embedding of a Planar graph)-
8 regions so 10 lay-

$n=9$
$m=15$
$f=8$



faces-
Points on edges don't belong to face-
Hanging edges don't but the region belongs to face 4

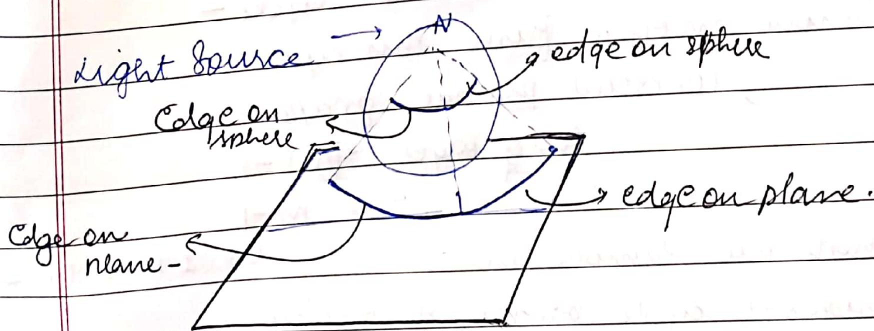


$n = 11$
$m = 13$
$f = 4$.

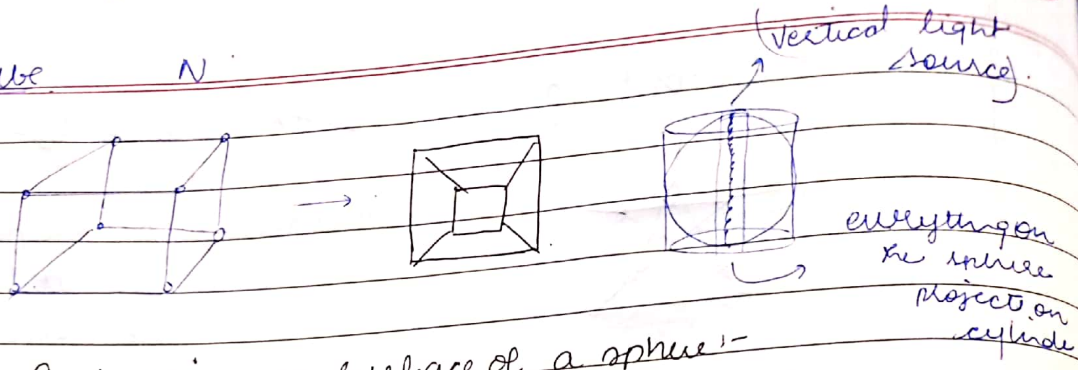
Relation → $n - m + f = 2$. # For every planar graph-

A graph is planar embeddable if it's embeddable on a sphere.

Stereographic Projection :-



Cube N



Planar Embedding on surface of a sphere:-

Infinite face → finite face.

The faces are all elastic :-

One enlarges for another to fit -

Euler's Formula: (1750) $n - M + f = 2$

Induction on m -

Basis $m = 0$ $n = 1$ $f = 1$

Induction ·H· The formula holds for $(m-1)$ edges -

Induction Step = consider an m edge graph - (connected)

Say : Graph is a tree -

No. of edges $= n - 1$ (n vertices)

No cycle. No. of faces $= 1$. $m = n - 1$

$n - m + 1 \Rightarrow n - (n-1) + 1$

$1 + 1 = 2$.

Say : Not a tree -

should have a cycle -

Remove an edge from this cycle.

↳ connected planar graph -

No. of edges $= (m) - 1$

$= m - 1$

↳ I·H holds -

Now when we remove an edges → cycle breaks. n vertices

↳ $(m-1)$ edges

faces → $f - 1$ two faces combine into one.

$n - (m-1) + f - 1 = 2$ (by I·H)

$\Rightarrow n - m + f = 2$

Euler's Polyhedron Formula:-

Both north and south pole should lie inside a face.

G has n vertices, m edges, f faces, k components

$$\frac{n_1 - m_1 \quad f_1}{2} \qquad \frac{n_2 - m_2 \quad f_2}{2} \qquad \frac{n_k - m_k \quad f_k}{2}$$

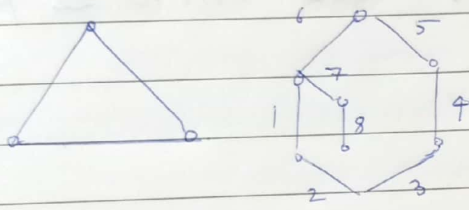
↳ $2k$ ✗

$2k - (k-1) = k+1$

$\boxed{n - m + f = k+1}$ ↳ General formula. ↳ For Euler's formula $k = 1$.

Infinite face is getting added $k-1$ times extra.

Simple Planar Graph:-



$2m \rightarrow$ Every edge gets counted twice when you go into a face and count the no of bounding edges

$\frac{2}{3} m \geq f$ $\quad$ $2m \geq 3f$.

Exact count

↳ For every face that we go to the no. of edges we count is at least 3.

$n - m + f = 2$

$f = 2 - n + m \leq \frac{2}{3} m$

$\boxed{\frac{m}{3} \leq n-2}$ $\Rightarrow$ $\boxed{m \leq 3n-6}$

Any simple planar graph is without Δ's:- Any planar graph.

smallest cycle $\rightarrow 4$

$2m \geq 4f \qquad m \geq 2f$

$f = 2 - n + m \leq \frac{m}{2} \Rightarrow \boxed{m \leq 2n-4}$

Triangle free PG → $m \leq 2n-4$
General PG → $m \leq 3n-6$.

Consider K_5 $m = {}^5C_2 = 10$
↳ $m \leq 3n-6$
$m \leq 15-6 \Rightarrow 9$

Atmost 9 edges should be there for graph to be planar. But $K_5 \to 10$ ∴ Non planar.

$K_{3,3} \Rightarrow$ Triangle free PG → 6 vertices.

$m \leq 2n-4$ $n \Rightarrow K_{3,3} \Rightarrow 6$
$m \leq 8$
But $m = 9$ for $K_{3,3}$ to be planar.