PH101: Physics 1

Module 2: Special Theory of Relativity - Basics

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RECAP

Lorentz Transformations

$$\gamma_v = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$x^{'} = \gamma(x - vt);\;\; y' = y;\;\; z' = z;\;\; t^{'} = \gamma_v(t - \frac{v}{c^2}x)$$

Law of addition of velocities

$$u_x^{'} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad u_y^{'} = \frac{u_y}{\gamma_v(1 - \frac{u_x v}{c^2})} \qquad u_z^{'} = \frac{u_z}{\gamma_v(1 - \frac{u_x v}{c^2})}$$

Distance between events (a Lorentz invariant)

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$ds^{'2} = ds^2$$

Definition of relativistic momentum

Should we continue to define the momentum as $\mathbf{p} = m\,\mathbf{u}$, where m is a velocity independent constant called the rest mass?

It can be shown that this choice is not suitable as total momentum conserved in one reference frame can be shown to be **not** conserved in a moving reference frame related to the earlier one by a Lorentz transformation.

Given that the momentum has to be parallel to the velocity of the particle we write

$$\mathbf{p} = \Gamma(u)\; m\; \mathbf{u}$$

where $\Gamma(u)$ is a scalar function of $u = |\mathbf{u}|$.

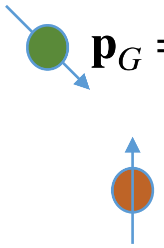
The goal now is to find $\Gamma(u)$ in such a way that if momentum conserved in one reference frame it is also seen to be conserved in a Lorentz transformed reference frame.

Two-body collision

Reference frame S

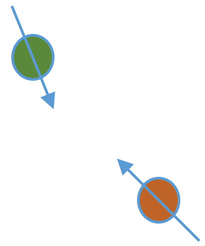
Reference frame S'

Before



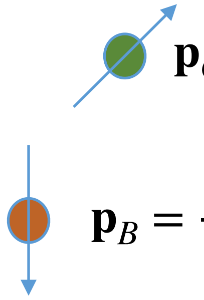
$\mathbf{p}_G = p\ (\hat{x} - \hat{y})$

$\mathbf{p}_B = p\ \hat{y}$



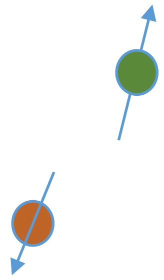
G = Green, B = Brown

After



$\mathbf{p}_G = p\ (\hat{x} + \hat{y})$

$\mathbf{p}_B = -\,p\ \hat{y}$



BEFORE COLLISION AS SEEN FROM **S**

$$\mathbf{p}_G = p\ (\hat{x} - \hat{y}) = \Gamma(u_G)\ m\ (u\ \hat{x} - u\ \hat{y})$$

$$\mathbf{p}_B = p\ \hat{y} = \Gamma(u_B)\ m\ u_B\ \hat{y}$$

AFTER COLLISION AS SEEN FROM **S**

$$\mathbf{p}_G = p\ (\hat{x} + \hat{y}) = \Gamma(u_G)\ m\ (u\ \hat{x} + u\ \hat{y})$$

$$\mathbf{p}_B = -\,p\ \hat{y} = -\,\Gamma(u_B)\ m\ u_B\ \hat{y}$$

Here $u_G = \sqrt{u^2 + u^2} = \sqrt{2}u$

BEFORE COLLISION AS SEEN FROM $\mathbf{S}'$

$$\mathbf{p}'_G = \Gamma(u'_{Gb})\ m\ (u'_{Gb,x}\ \hat{x} + u'_{Gb,y}\ \hat{y});$$

$$u'_{Gb,x} = \frac{u - v}{1 - \frac{uv}{c^2}}; \qquad u'_{Gb,y} = \frac{-u}{\gamma_v(1 - \frac{uv}{c^2})}$$

$$\mathbf{p}'_B = \Gamma(u'_{Bb})\ m\ (u'_{Bb,x}\ \hat{x} + u'_{Bb,y}\ \hat{y});$$

$$u'_{Bb,x} = -v; \qquad u'_{Bb,y} = \frac{u_B}{\gamma_v}$$

AFTER COLLISION AS SEEN FROM $\mathbf{S}'$

$$\mathbf{p}'_G = \Gamma(u'_{Ga})\ m\ (u'_{Ga,x}\ \hat{x} + u'_{Ga,y}\ \hat{y});$$

$$u'_{Ga,x} = \frac{u - v}{1 - \frac{uv}{c^2}}; \qquad u'_{Ga,y} = \frac{u}{\gamma_v(1 - \frac{uv}{c^2})}$$

$$\mathbf{p}'_B = \Gamma(u'_{Ba})\ m\ (u'_{Ba,x}\ \hat{x} + u'_{Ba,y}\ \hat{y});$$

$$u'_{Ba,x} = -v; \qquad u'_{Ba,y} = \frac{-u_B}{\gamma_v}$$

Here Gb means Green ball before collision etc.

$$u_{Gb} = \sqrt{u_{Gb,x}^2 + u_{Gb,y}^2} \qquad u_{Ga} = \sqrt{u_{Ga,x}^2 + u_{Ga,y}^2}$$

AS SEEN FROM $\mathbf{S}$

$$p = \Gamma(\sqrt{2}u)\ m\ u$$

$$p = \Gamma(u_B)\ m\ u_B$$

Total momentum conservation

AS SEEN FROM $\mathbf{S}'$

$$\Gamma(u'_{Gb})\ m\ (u'_{Gb,x}\ \hat{x} + u'_{Gb,y}\ \hat{y}) + \Gamma(u'_{Bb})\ m\ (u'_{Bb,x}\ \hat{x} + u'_{Bb,y}\ \hat{y}) =$$

$$\Gamma(u'_{Ga})\ m\ (u'_{Ga,x}\ \hat{x} + u'_{Ga,y}\ \hat{y}) + \Gamma(u'_{Ba})\ m\ (u'_{Ba,x}\ \hat{x} + u'_{Ba,y}\ \hat{y})$$

$$u'_{G,a} = u'_{G,b} = |\,u'_G\,| \qquad\qquad u'_{B,a} = u'_{B,b} = |\,u'_B\,|$$

$$\Gamma(|u'_G|)\ m\ u'_{Gb,y} + \Gamma(|u'_B|)\ m\ u_{Bb,y} = \Gamma(|u'_G|)\ m\ u'_{Ga,y} + \Gamma(|u'_B|)\ m\ u'_{Ba,y}$$

or

$$\Gamma(|u'_G|)\ m\ (u'_{Gb,y} - u'_{Ga,y}) + \Gamma(|u'_B|)\ m\ (u_{Bb,y} - u_{Ba,y}) = 0$$

or

$$\boxed{\begin{array}{c} \Gamma(|u'_G|)\ m\ \left(\dfrac{-2u}{\gamma_v(1-\frac{uv}{c^2})}\right) + \Gamma(|u'_B|)\ m\ \left(\dfrac{2u_B}{\gamma_v}\right) = 0 \\ \\ |u'_G| = \sqrt{\left(\dfrac{u-v}{1-\frac{uv}{c^2}}\right)^2 + \left(\dfrac{-u}{\gamma_v(1-\frac{uv}{c^2})}\right)^2} \qquad |u'_B| = \sqrt{v^2 + \left(\dfrac{u_B}{\gamma_v}\right)^2} \\ \\ p = \Gamma(\sqrt{2}u)\ m\ u \qquad p = \Gamma(u_B)\ m\ u_B \end{array}}$$

The goal is to find the form of the unknown function $\Gamma(u)$ by solving the boxed equations of the previous page. This looks formidable but there is an easy way of doing this. Expand the boxed equation to first order in u.

To first order in u it is easy to see that, $u_B \approx u$ and $|u_B^{'}| \approx |v|$

$$|u_G^{'}| \approx v + u\left(\frac{v^2}{c^2} - 1\right)$$

This means the first boxed equation becomes

$$-\frac{\Gamma\left(u\left(\frac{v^2}{c^2}-1\right)+v\right)}{\left(1-\frac{uv}{c^2}\right)} + \Gamma(v) \approx 0$$

Since u is small, we have to remember to retain only upto first order in u by performing Taylor expansion in u to get,

$$(c-v)(c+v)\Gamma'(v) - v\Gamma(v) = 0$$

This simple differential equation is easily solved to get (we have used the relation $\Gamma(0) = 1$)

$$\Gamma(v) = \frac{c}{\sqrt{c^2 - v^2}} \equiv \gamma_v$$

This means the correct definition of momentum in Special Relativity that ensures that momentum conserved in one reference frame is also conserved in a Lorentz transformed frame is,

$$\mathbf{p} = \frac{m\,\mathbf{u}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Note that the above definition is very different from the usual definition of momentum in Galilean Relativity viz. $\mathbf{p} = m\,\mathbf{u}$. The relativistic definition becomes the usual definition valid only in Galilean Relativity when $|\mathbf{u}| \ll c$.

Kinetic Energy in Special Relativity

In the absence of forces acting on a particle, the work done is equal to the change in kinetic energy. We may now use this idea to find a relativistic expression for the kinetic energy of a particle. The work done going from an initial state "i" to a final state "f" is

$$W_{i\to f} = \int_i^f \mathbf{F} \cdot d\mathbf{r}$$

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$W_{i\to f} = \int_i^f \frac{d\mathbf{p}}{dt} \cdot d\mathbf{r} = \int_i^f d\mathbf{p} \cdot \frac{d\mathbf{r}}{dt} = \int_i^f d\mathbf{p} \cdot \mathbf{u}$$

The relation between momentum and velocity may be inverted to give,

$$\mathbf{u} = \frac{c\ \mathbf{p}}{\sqrt{p^2 + (mc)^2}}$$

Exercise

This means the work done is

$$W_{i\to f} = \int_i^f d\mathbf{p} \cdot \mathbf{u} = \int_i^f d\mathbf{p} \cdot \mathbf{u} = \int_i^f d\mathbf{p} \cdot \mathbf{u} = \int_i^f d\mathbf{p} \cdot \frac{c\ \mathbf{p}}{\sqrt{p^2 + (mc)^2}}$$

This work done is equal to change in kinetic energy between the initial and final states if no forces are acting.

$$E_f - E_i = \int_i^f dp \frac{c}{\sqrt{p^2 + (mc)^2}}$$

$$E_a = \sqrt{c^2 p_a^2 + (mc^2)^2}\ ; \quad a = i, f$$

Classwork: Show that the kinetic energy can also be written as

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Exercise

From this it follows that if the particle is at rest it still has an energy equal to mass times the square of the speed of light:

$$E = mc^2$$

This is arguably the most famous equation in all of physics. It is the operating principle behind both a nuclear reactor and also an atomic bomb.
It is also the basic principle behind the Large Hadron Collider and all its predecessors.