Continuous-time Markov Chain BD process 2

SIDNEY CHAPMAN

ANDREY KOLMOGOROV

Birth-death processes

- State holding time T_i at a state $i \neq 0$ is given by $T_i = \exp(\lambda_i + \mu_i)$.
- Transition probabilities of the embedded MC.

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

At $i = 0$, $\nu_0 = \lambda_0$ and $P_{01} = 1$

The probability rate function is given by

$$q_{i,i+1} = \nu_i P_{i,i+1} = \lambda_i, \quad q_{i,i-1} = \nu_i P_{i,i-1} = \mu_i$$

$$\therefore q_{i,i} = -(\lambda_i + \mu_i)$$

At $i = 0$, $\nu_0 = \lambda_0$ and $q_{01} = \lambda_0$ $q_{00} = -\lambda_0$

The forward Kolmogorov equation is given by

$$p_{ij}'(t) = -v_j p_{ij}(t) + \sum_{k \neq j} p_{ik}(t) q_{kj}$$

$$\therefore \frac{dp_{i,j}(t)}{dt} = -(\lambda_j + \mu_j) p_{i,j}(t) + \lambda_{j-1} p_{i,j-1}(t) + \mu_{j+1} p_{i,j+1}(t)$$

The backward Kolmogorov equation is given by

$$\frac{dp_{i,j}(t)}{dt} = -(\lambda_i + \mu_i) p_{i,i}(t) + \lambda_i p_{i+1,j}(t) + \mu_i p_{i-1,j}(t)$$

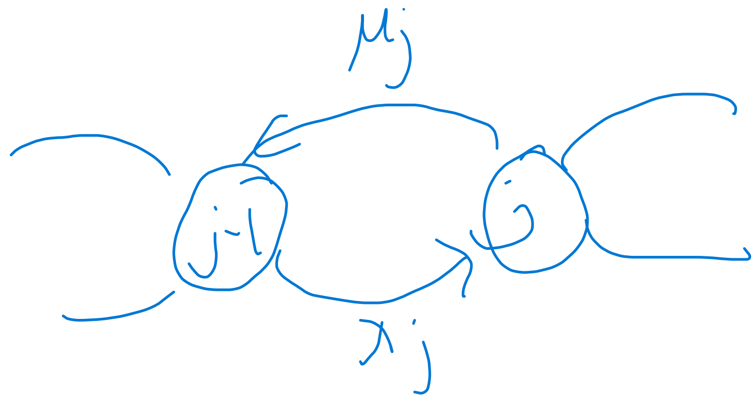
Because of the state varying parameters λ_i and μ_i, the solution of Kolmogorv equations is difficult.

Global Balance quations

We consider the special case when the steady state solution exists. Then as $t \to \infty$, $\lim_{t \to \infty} \frac{dp_{i,j}(t)}{dt} = 0$, $\lim_{t \to \infty} p_{i,j}(t) = \pi_j$ independent of *i*. Putting the above results in the forward Kolmogorv equation, we get

$$\pi_{j-1}\lambda_{j-1} + \pi_{j+1}\mu_{j+1} - (\lambda_j + \mu_j)\pi_j = 0$$

Or $\pi_{j-1}\lambda_{j-1} + \pi_{j+1}\mu_{j+1} = (\lambda_j + \mu_j)\pi_j$



Solution of GB equations

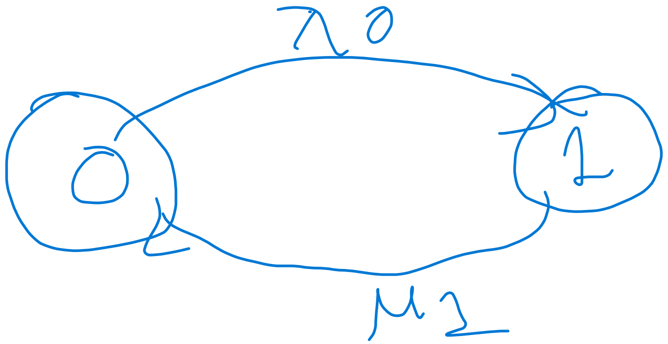
At a state $j \neq 0,$

$$\pi_{j-1}\lambda_{j-1} + \pi_{j+1}\mu_{j+1} = (\lambda_j + \mu_j)\pi_j$$

~~At a state~~

At j=0, there cannot be further death. Therefor GB equation becomes

$$\lambda_0\pi_0 = \mu_1\pi_1$$

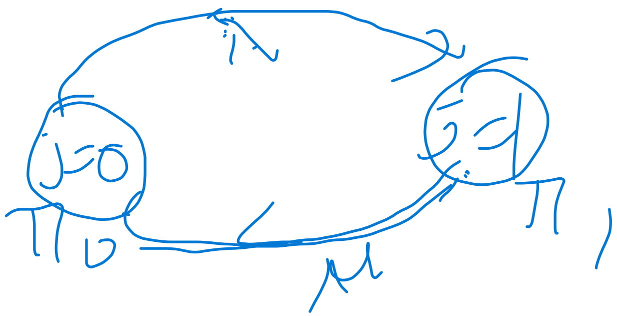


These systems of linear equations are to be solved with the constraint

$$\sum_{j=0}^{\infty}\pi_j = 1$$

Simple case: two-state MC

A certain system has two states – under operation state 1 and under repair state 0. The duration of repair and operation are exponential RVs with rate parameters λ and μ respectively.



At state $j = 0$,

$$\pi_0 \lambda = \mu \pi_1$$

We have to solve the above equation with the probability constraint,

$$\pi_0 + \pi_1 = 1$$

Solving, we get

$$\pi_0 = \frac{\mu}{\lambda + \mu} \text{ and } \pi_1 = \frac{\lambda}{\lambda + \mu}$$

General case:

For a BD process with arrival rate λ_j and departure rate μ_j, the limiting state probabilities, if they exist, is given by

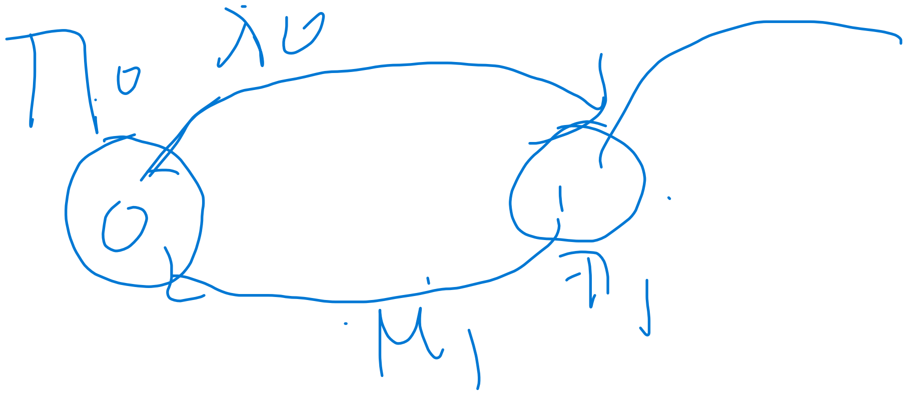
$$\pi_j = \frac{\prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}}{1+\sum_{j=1}^{\infty}\prod_{i=1}^{j}\frac{\lambda_{i-1}}{\mu_i}}$$

$j = 0, 1, \ldots$

Proof We have at $j \neq 0$,

$$\pi_{j-1}\lambda_{j-1} + \pi_{j+1}\mu_{j+1} = (\lambda_j + \mu_j)\pi_j$$

At j=0, $\lambda_0\pi_0 = \mu_1\pi_1$



These systems of linear equations are to be solved with the constraint

$$\sum_{j=0}^{\infty}\pi_j = 1$$

From

$$\lambda_0\pi_0 = \mu_1\pi_1$$

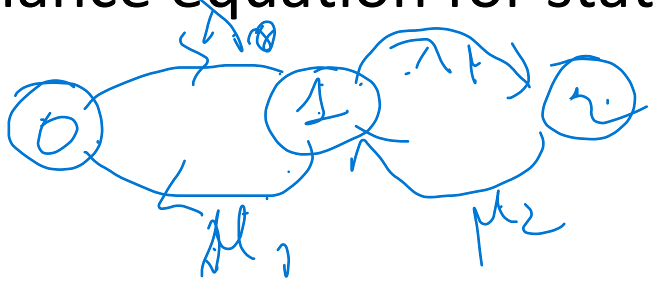
we get

$$\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$$

$\mu_1\pi_1$

Substituting the value of π_1 in the global balance equation for state 1, we get

$$(\lambda_1 + \mu_1)\pi_1 = \lambda_0\pi_0 + \mu_2\pi_2$$



$$\Rightarrow \pi_2 = \frac{\lambda_1}{\mu_2}\pi_1 = \left(\frac{\lambda_1}{\mu_2}\right)\left(\frac{\lambda_0}{\mu_1}\right)\pi_0$$

In the same manner, $\pi_j = \dfrac{\lambda_{j-1}}{\mu_j}\pi_{j-1} = \prod_{i=1}^{j}\dfrac{\lambda_{i-1}}{\mu_i}\pi_0$

$$\because \sum_{j=0}^{\infty} \pi_j = 1$$

$$\Rightarrow \pi_0 + \sum_{j=1} \pi_j = 1$$

$$\therefore \pi_0 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i} \pi_0 = 1$$

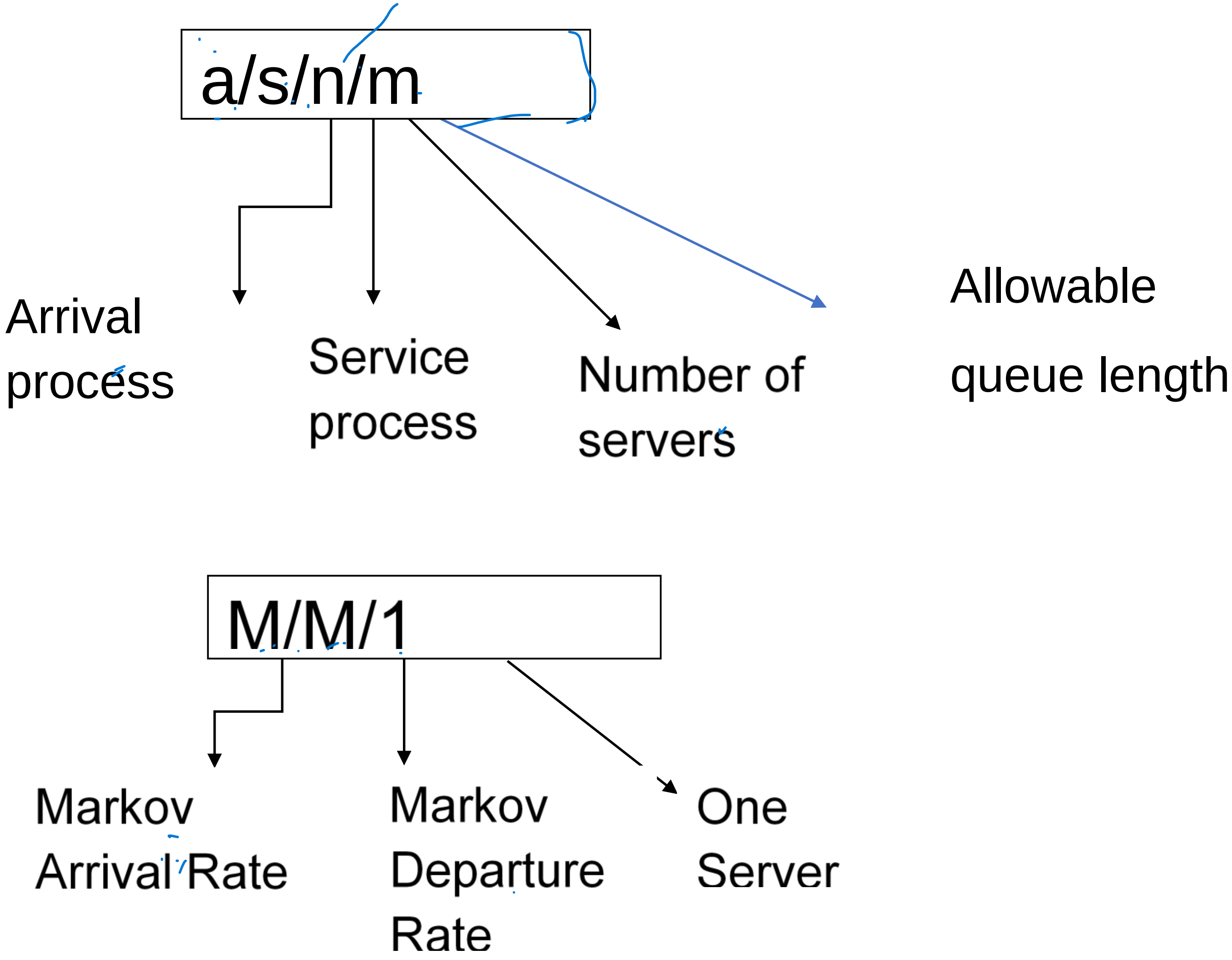
$$\Rightarrow \pi_0 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i} \pi_0 = 1$$

$$\Rightarrow \pi_0 = \frac{1}{1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}}$$

$$\therefore \pi_j = \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i} \pi_0 = \frac{\prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}}{1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}}$$

π_j exist if $\dfrac{\prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}}{1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}}$ converges implying that whether $\sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}$ converges.

Application Queueing Models



M/M/1 queue is a BD process. Here the arrival rate $\lambda_i = \lambda$ for $i = 0,1,....$ and $\mu_i = \mu$ for $i = 1,2,....$

$X(t)$ is the continuous-time Markov Chain representing the number of jobs in the queueing system at time t.

When $\lambda > \mu$, the queue will grow unboundedly. Each state in this case will be transient.

When $\lambda = \mu$, then the process will behave as symmetrical random walk process and each state of $X(t)$ will the null-recurrent.

When $\lambda < \mu$, π_j is positive recurrent.

We have $\lambda_i = \lambda$ for $i = 0,1,....$and $\mu_i = \mu$ for $i = 1,2,...$ and $\mu_0 = 0$. We can get the steady-state probabilities as follows:

For a M/M/1 queue with arrival rate λ_j and departure rate μ_j, the limiting state probabilities, if they exist, is given by

$$\pi_j = (1-\rho)\rho^j.\ j = 0,1,\ldots$$

$= \lambda$ $= \mu$

Geometric pmf

$\rho = \frac{\lambda}{\mu} < 1$

$=$ utility ratio.

$(1-\rho)\rho^0 = 1-\rho$

$$\boxed{\pi_j = \frac{\prod_{i=1}^{j} \frac{\lambda_{i-1}}{\mu_i}}{1 + \sum_{j=1} \prod_{i=1} \frac{\lambda_{i-1}}{\mu_i}}}$$

$$\sum_{j=1}^{\infty}\prod_{i=1}^{j}\left(\frac{\lambda_{i-1}}{\mu_i}\right)=\sum_{j=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^j$$

$$=\frac{\dfrac{\lambda}{\mu}}{1-\dfrac{\lambda}{\mu}}$$

$$\therefore \pi_j=\frac{\left(\dfrac{\lambda}{\mu}\right)^j}{1+\dfrac{\dfrac{\lambda}{\mu}}{1-\dfrac{\lambda}{\mu}}},\quad j=0,1,2,\ldots \Rightarrow \frac{1}{1-\lambda/\mu}$$

$$=\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j=(1-\rho)\rho^j,\ \ \rho=\text{utilization factor}$$

Thus the number of jobs in the queue in the steady state is a geometric random variable.

Suppose $\lim_{t \to \infty} X(t) = X$ number of jobs in the queue. Note that this limit is in the probabilistic sense.

$X \sim Geo(\rho)$

Thus the average number of jobs $EX = \sum_{j=0}^{\infty} j\pi_j = \frac{\rho}{1-\rho}$ and $\mathrm{var}(X) = \frac{\rho^2}{1-\rho}$

Thus the average number of jobs in the queue $\boxed{EX = \frac{\rho}{1-\rho}}$

Example A single server system with $\lambda = 2.7$ jobs per minute and service rate $\mu = 3$ jobs per minute.

$$\rho = \frac{\lambda}{\mu} = \frac{2.7}{3.0} = 0.9$$

Then

Average number of job in the system at steady state

$$= \frac{\rho}{1-\rho} = \frac{0.9}{1-0.9} = 9 \text{ jobs}$$

The probability that there is no job

$$(1-\rho)\rho^{j} = (1-\rho)\rho^{0}$$

$$= 0.1$$

To Summarise

- Birth-death process $\{N(t)\}$ is a well-known CTMC with the forward Kolmogorov equation

$$\frac{dp_{i,j}(t)}{dt} = -(\lambda_j + \mu_j)\,p_{i,j}(t) + \lambda_{j-1}p_{i,j-1}(t) + \mu_{j+1}p_{i,j+1}(t)$$

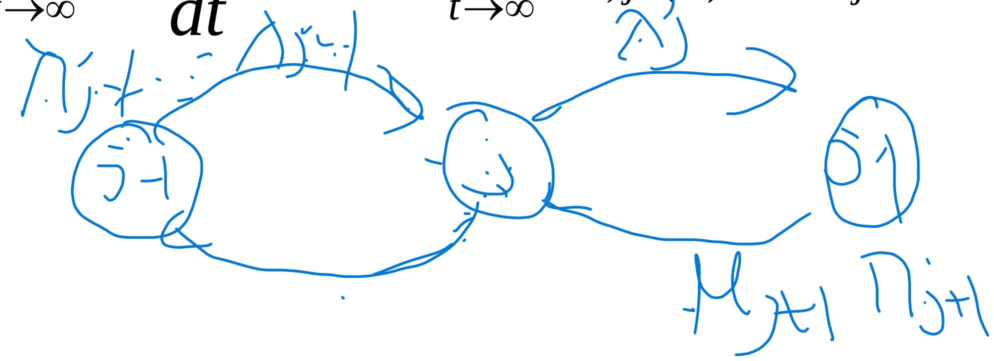
- The backward Kolmogorov equation is given by

$$\frac{dp_{i,j}(t)}{dt} = -(\lambda_i + \mu_i)\,p_{i,j}(t) + \lambda_i p_{i+1,j}(t) + \mu_i p_{i-1,j}(t)$$

If the steady state probabilities exist, then $\lim\limits_{t\to\infty}\dfrac{dp_{i,j}(t)}{dt} = 0, \lim\limits_{t\to\infty} p_{i,j}(t) = \pi_j$.

We get the Global balance equation

$$\pi_{j-1}\lambda_{j-1} + \pi_{j+1}\mu_{j+1} = (\lambda_j + \mu_j)\,\pi_j$$



To Summarise…

- Global Balance Equations are solved with following information

1) $\sum_{j=0}^{\infty} \pi_j = 1$

(2) At j=0, there cannot be further death so that $\lambda_0 \pi_0 = \mu_1 \pi_1$

The steady-state probabilities are given by

$$\pi_j = \frac{\prod_{i=1}^{j} \left(\frac{\lambda_{i-1}}{\mu_i} \right)}{1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \left(\frac{\lambda_{i-1}}{\mu_i} \right)}$$

M/M/1

$\pi_j = (1-\rho)\rho^j$

$\rho = \frac{\lambda}{\mu} < 1$

THANK YOU