Mean Value Theorem for integrals: Let $f : [a, b] \to \mathbb{R}$ be a **continuous** function. Then there exists $\xi \in [a, b]$ such that

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x)dx.$$

First Fundamental Theorem of Calculus: Let $f : [a, b] \to \mathbb{R}$ be a **continuous** function. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(x)dx.$$

Then, F is uniformly continuous on $[a, b]$, differentiable on (a, b), and

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Let $f : [a, b] \to \mathbb{R}$ be a function. A function F is called an antiderivative or primitive of f if $F'(x) = f(x)$ for all $x \in [a, b]$.

Second Fundamental Theorem of Calculus: Let $f : [a, b] \to \mathbb{R}$ be a **continuous** function and let F be an antiderivative of f. Then,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Second Fundamental Theorem of Calculus: Let $f : [a, b] \to \mathbb{R}$ be a **continuous** function and let F be an antiderivative of f. Then,

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Remark: The theorem holds even if f is not assumed to be continuous.

For $x \in [a,b]$, consider the integral $\int\limits_a^x f(t)dt$. Then by the Fundamental Theorem of Calculus, (applied to f' on the interval $[a,x]$), we get

$$f(x) = f(a) + \int\limits_a^x f'(t_1)dt_1.$$

In other words, in a neighbourhood of a, $f(x)$ and $f(a)$ differ by the *indefinite integral*.

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In other words, in a neighbourhood of a, $f(x)$ and $f(a)$ differ by the *indefinite integral*.

If f is twice differentiable, then we get

$$f(x) = f(a) + \int_a^x f'(a)dt_1 + \int_a^x \int_a^{t_1} f''(t_2)dt_2dt_1$$

Thus,

$$
\begin{aligned}
f(x) &= f(a) + \int\limits_a^x f'(t_1)dt_1 \\
&= f(a) + \int\limits_a^x f'(a)dt_1 + \int\limits_a^x \int\limits_a^{t_1} f''(t_2)dt_2 dt_1 \\
&= f(a) + \int\limits_a^x f'(a)dt_1 + \int\limits_a^x \int\limits_a^{t_1} f''(a)dt_2 dt_1 + \int\limits_a^x \int\limits_a^{t_1} \int\limits_a^{t_2} f'''(t_3)dt_3 dt_2 dt_1
\end{aligned}
$$

Note that

$$\int_a^x f'(a)dt_1 = f'(a)\int_a^x dt_1 = f'(a)(x-a)$$

$$\int_a^x \int_a^{t_1} f''(a)dt_2 dt_1 = f''(a)\int_a^x (t_1-a)dt_1 = f''(a)\frac{(x-a)^2}{2}$$

$$\int_a^x \int_a^{t_1} \int_a^{t_2} f'''(a)dt_3 dt_2 dt_1 = f'''(a)\frac{(x-a)^3}{3\cdot 2}$$

In general,

$$\int_a^x \int_a^{t_1} \cdots \int_a^{t_n} f^{(n)}(a)dt_n \cdots dt_2 dt_1 = f^{(n)}(a)\frac{(x-a)^n}{n!}$$

Questions: Let $f : \mathbb{R} \to \mathbb{R}$, or even $f : [a, b] \to \mathbb{R}$. What can you say about f, if you know that

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In general, when can you say that the given function is a polynomial. If f is not a polynomial, then how far it is from being a polynomial?

One conjectures, then, and proves by induction, that

$$f(x) = P_n(x) + R_n(x),$$

where $P_n(x)$ is the <u>n-th Taylor's polynomial</u>

$$Pn(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \ldots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

and the <u>n-th remainder term</u> $R_n(x)$ is represented as

$$R_n(x) = \int_a^x \int_a^{t_1} \cdots \int_a^{t_n} f^{(n+1)}(t_{n+1})dt_{n+1}\cdots dt_2 dt_1.$$

Taylor's Theorem

Let $f : [a, b] \to \mathbb{R}$ be such that $f, f', f'', \ldots, f^{(n)}$ are continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b). Let $x_0 \in [a, b]$.
Then for any $x \in [a, b]$ there exists $c \in (x, x_0)$ such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

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Remark: If $x_0 < x$, then the interval should be taken as (x_0, x).

Power Series

Let (a_n) be a sequence. Then for $x \in \mathbb{R}$ the series $\sum_{n=0}^{\infty} a_n x^n$ is called a power series.

In general the series for $a \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} a_n (x-a)^n$ is called power series around a.

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We will assume that $a = 0$.

Theorem: Suppose the series $\sum_{n=0}^{\infty} a_n x^n$ converges at some $x = x_0$ and diverges at $x = x_1$. Then

1. $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $|x| < |x_0|$.
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Thus either the series $\sum_{n=0}^{\infty} a_n x^n$ converges only at $x = 0$ or there exists unique $r > 0$, such that the series converges absolutely for all $|x| < r$ and diverges for all $|x| > r$.

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Thus either the series $\sum_{n=0}^{\infty} a_n x^n$ converges only at $x = 0$ or there exists unique $r > 0$, such that the series converges absolutely for all $|x| < r$ and diverges for all $|x| > r$.
This r is called the radius of convergence.

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Formula for radius of convergence:

$$r = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

Taylor's series

The power series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

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If $a = 0$, then the power series is called Maclaurin series.
Remark: If f is infinite times differentiable at a then the corresponding Taylor series is defined. Moreover, $P_n(x)$ is the n-th partial sum of the Taylor series.

Examples

Let $f : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1-x}$. Then the Taylor's series of f around 0 (i.e. Maclaurin's series) is the geometric series

$$\sum_{n=0}^{\infty} x^n.$$

This converges for all $x \in (-1, 1)$ and diverges for $|x| > 1$. Thus the radius of convergence is 1.

Examples

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad x > 0.$$

What is the radius of convergence?