PH 102: Physics II

Lecture 17 (Spring 2019)

IIT Guwahati
Debasish Borah

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Lectures	Division
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	I, II (4-4:55 pm)
Lec 15	13-3-2019	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	1	III, IV (9-9:55 am)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	I, II (3-3:55 pm)
Lec 16	14-3-2019	Applications of Ampere's law, Magnetic Vector Potential	5.3, 5.4	1	III, IV (10-10:55 am)
Tut 8	19-3-2019	Lec 15, 16			
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic Materials, Magnetization	5.4, 6.1	1	I, II (4-4:55 pm)
Lec 17	20-3-2019	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic Materials, Magnetization	5.4, 6.1	1	III, IV (9-9:55 am)
Lec 18	22-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	I, II (3-3:55 pm)
Lec 18	22-3-2019	Field of a magnetized object, Boundary Conditions	6.2, 6.3, 6.4	1	III, IV (10-10:55 am)
Tut 9	26-3-2019	Lec 17, 18			
Lec 19	27-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	I, II (4-4:55 pm)
Lec 19	27-3-2019	Ohm's law, motional emf, electromotive force	7.1	1	III, IV (9-9:55 am)
Lec 20	28-3-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	I, II (3-3:55 pm)
Lec 20	28-3-2019	Faraday's law, Lenz's law, Self & Mutual Inductance, Energy Stored in Magnetic Field	7.2	1	III, IV (10-10:55 am)
Tut 10	2-4-2019	Lec 19, 20			
Lec 21	3-4-2019	Maxwell's equations	7.3	1	I, II (4-4:55 pm)
Lec 21	3-4-2019	Maxwell's equations	7.3	1	III, IV (9-9:55 am)
Lec 22	4-4-2019	Discussions, problem solving	7.3	1	I, II (3-3:55 pm)
Lec 22	4-4-2019	Discussions, problem solving	7.3	1	III, IV (10-10:55 am)
Tut 11	9-4-2019	Quiz II			

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	I, II (4-4:55 pm)
Lec 23	10-4-2019	Continuity equation, Poynting Theorem	8.1	1	III, IV (9-9:55 am)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	I, II (3-3:55 pm)
Lec 24	11-4-2019	Wave solution of Maxwell's equation, polarisation	9.1, 9.2	1	III, IV (10-10:55 am)
Tut 12	23-4-2019	Lec 23, 24			
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	I, II (4-4:55 pm)
Lec 25	24-4-2019	Electromagnetic waves in matter, Reflection and transmission: Normal incidence	9.3	1	III, IV (9-9:55 am)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	I, II (3-3:55 pm)
Lec 26	25-4-2019	Reflection and transmission: Oblique Incidence	9.3, 9.4	1	III, IV (10-10:55 am)
Tut 13	30-4-2019	Lec 25, 26			
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	I, II (4-4:55 pm)
Lec 27	1-5-2019	Relativity and electromagnetism: Galilean and special relativity	12.1, 12.2, 12.3	1	III, IV (9-9:55 am)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	I, II (3-3:55 pm)
Lec 28	2-5-2019	Discussions, problem solving	12.1, 12.2, 12.3	1	III, IV (10-10:55 am)

RECAP (Lecture 15,16)

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$ Lorentz Force Law

$$\vec{F} = \int I(d\vec{l} \times \vec{B})$$ Force on current carrying wire

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$ Continuity Equation

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l'} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}$$ Biot-Savart Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$ Ampere's Law

$$\vec{\nabla} \cdot \vec{B} = 0$$ Absence of monopole

Magnetic Vector Potential $$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \cdot \vec{A} = 0 \qquad \nabla^2 \vec{A} = -\mu_0 \vec{J} \qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'})}{\mathfrak{r}} d\tau'$$

Magnetostatic Boundary Conditions

- Just like electric field suffers a discontinuity at a surface charge, so the magnetic field is discontinuous at a surface current.

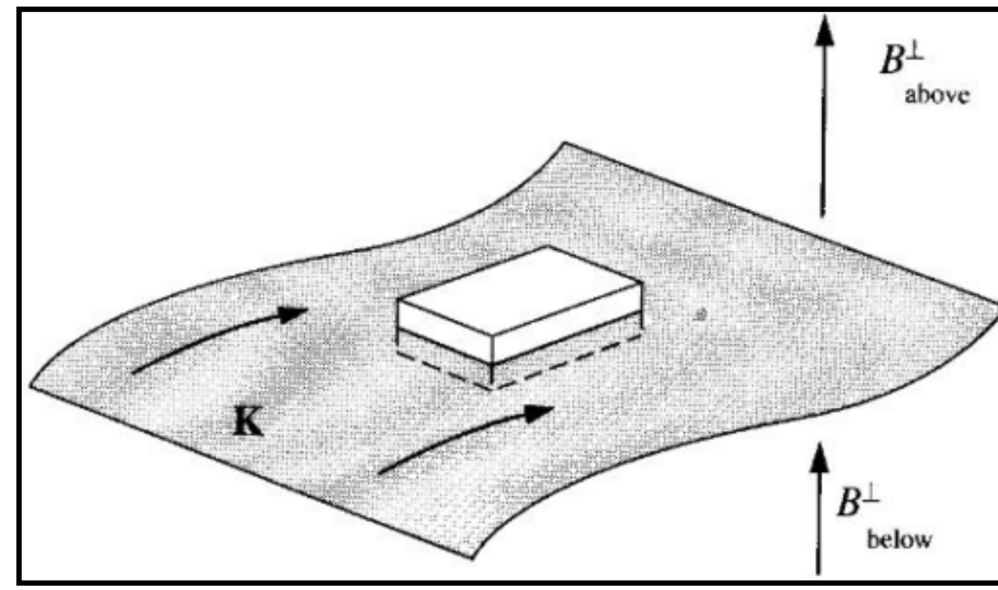


Figure 5.49, Introduction to Electrodynamics, D. J. Griffiths

- Using the integral form of $\vec{\nabla} \cdot \vec{B} = 0$ that is,

$$\oint \vec{B} \cdot d\vec{a} = 0$$

to a thin pillbox straddling the surface, we get $B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$

What about the contributions from other sides of the pillbox?

Notice the contrast with electrostatic analogue: $E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \dfrac{\sigma}{\epsilon_0}$

Magnetostatic Boundary Conditions

- The boundary conditions for tangential components can be found by taking an Amperian loop running perpendicular to the current which gives

$$\oint \vec{B} \cdot d\vec{l} = (B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}})l = \mu_0 I_{\text{enc}} = \mu_0 K l$$

$$\implies B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$$

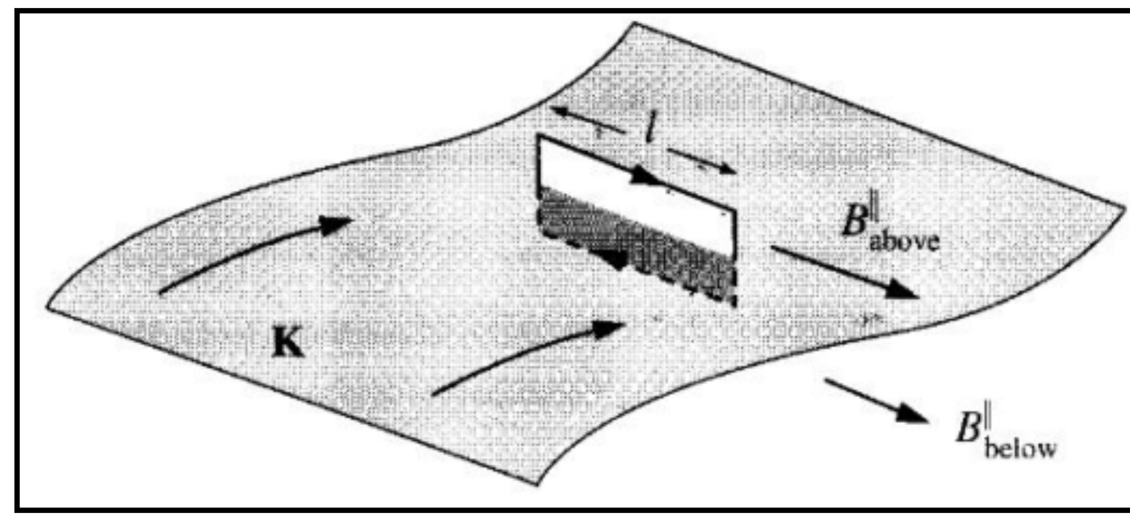


Figure 5.50, Introduction to Electrodynamics, D. J. Griffiths

- In general,

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0(\vec{K} \times \hat{n})$$

where $\hat{n}$ is a unit vector perpendicular to the surface, pointing upward.

Notice the contrast with electrostatic analogue: $E^{\parallel}_{\text{above}} - E^{\parallel}_{\text{below}} = 0$

Magnetostatic Boundary Conditions

- Magnetic vector potential is continuous across any boundary.

- Continuity of normal components is guaranteed by

$$\vec{\nabla} \cdot \vec{A} = 0 \implies \oint \vec{A} \cdot d\vec{a} = 0$$

- For tangential components, we can calculate

$$\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a} = \Phi$$

which is zero for an Amperian loop of vanishing thickness. Thus, tangential components are continuos.

- The derivative of vector potential however, is discontinuous

$$\frac{\partial \vec{A}_{\text{above}}}{\partial n} - \frac{\partial \vec{A}_{\text{below}}}{\partial n} = -\mu_0 \vec{K}$$

Since A is continuous across the boundary we have, at all points on the surface: $\vec{A}_{\text{above}} = \vec{A}_{\text{below}}$

If the boundary is the x-y plane, the above condition means

$\dfrac{\partial A}{\partial x}, \dfrac{\partial A}{\partial y}$ are same above and below.

Only normal derivatives can be discontinuous

Why?

From the boundary condition on magnetic field:

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0(\vec{K} \times \hat{n})$$

The parallel components of B are $\left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right)\hat{x} + \left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right)\hat{y}$

Using the continuity of x, y derivatives, we get:

$$\left(-\frac{\partial A_{y\text{above}}}{\partial z} + \frac{\partial A_{y\text{below}}}{\partial z}\right)\hat{x} + \left(\frac{\partial A_{x\text{above}}}{\partial z} - \frac{\partial A_{x\text{below}}}{\partial z}\right)\hat{y} = \mu_0(\vec{K} \times \hat{n})$$

Considering the surface current to be in x direction, the right hand side of the previous relation is $-\mu_0 K \hat{y}$

Equating x and y components on both sides:

$$\left(-\frac{\partial A_{y\text{above}}}{\partial z} + \frac{\partial A_{y\text{below}}}{\partial z}\right) = 0, \ \left(\frac{\partial A_{x\text{above}}}{\partial z} - \frac{\partial A_{x\text{below}}}{\partial z}\right) = -\mu_0 K$$

Therefore, in general

$$\frac{\partial \vec{A}_{\text{above}}}{\partial n} - \frac{\partial \vec{A}_{\text{below}}}{\partial n} = -\mu_0 \vec{K}$$

Magnetic Dipole

- A current carrying loop with area $\vec{a}$ has a magnetic dipole moment given by $\vec{m} = I\vec{a}$.

- Magnetic dipole moment is independent of the choice of origin. (Prove it. What about electric dipole moment?)

- The dipole term is the leading order term in the multipole expansion of the vector potential.

- The dipole term is identified as the one that is proportional to inverse of distance squared (r^2) in the multipole expansion.

Multipole Expansion of Vector Potential

- Multipole expansion* is used to write the potential in the form of a power series in 1/r.

- The vector potential of a current loop can be written as

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{\mathfrak{r}} d\vec{l'}$$

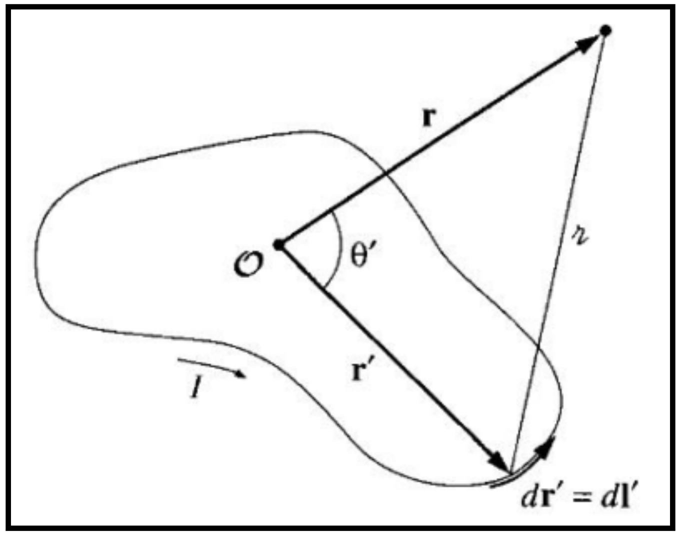


Figure 5.51, Introduction to Electrodynamics, D. J. Griffiths

- Using the standard expansion

$$\frac{1}{\mathfrak{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr' \cos\theta'}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

and using the standard Legendre polynomials*

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \left[\frac{1}{r} \oint d\vec{l'} + \frac{1}{r^2} \oint r' \cos\theta' d\vec{l'} + \frac{1}{r^3} \oint (r')^2 \left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\vec{l'} + \dots \right]$$

*Not required in this course: Don't worry!

Multipole Expansion of Vector Potential

The power series expansion can also be realised as*

$$\mathfrak{r}^2 = r^2 + (r')^2 - 2rr'\cos\theta' = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right]$$

$$\implies \mathfrak{r} = r\sqrt{1+\epsilon}, \epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)$$

$$\frac{1}{\mathfrak{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right)$$

$$\implies \frac{1}{\mathfrak{r}} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2 (3\cos^2\theta' - 1)/2 + \left(\frac{r'}{r}\right)^3 (5\cos^3\theta' - 3\cos\theta')/2 + \ldots\right]$$

The first term in the expansion of vector potential is zero due to the vanishing integral of total vector displacement around a closed loop

$$\oint d\vec{l'} = 0$$

Thus, there is no monopole contribution to vector potential.
Absence of magnetic monopoles: $\vec{\nabla} \cdot \vec{B} = 0$ Lecture 15, 16

*without any reference to special functions!

The same exercise, if done for electrostatic potential gives

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\vec{r'})d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r'})d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r'})d\tau' + ...\right]$$

where the first term is the usual monopole term* while the second term is the dipole contribution

$$V_{\text{dipole}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r'})d\tau' = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

where $\vec{p} \equiv \int \vec{r'}\rho(\vec{r'})d\tau'$ is the electric dipole moment.

*Unlike in case of vector potential, here monopole term contributes!

Magnetic Dipole

- The dipole contribution to the vector potential is

$$\vec{A}_{\text{dipole}}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r' \cos\theta' d\vec{l'} = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r'})d\vec{l'}$$

- Using the definition of area of a loop $\vec{a} = \dfrac{1}{2}\oint \vec{r}\times d\vec{l}$ and the identity $\oint(\vec{c}\cdot\vec{r})d\vec{l} = \vec{a}\times\vec{c}$

Chapter 1, Introduction to Electrodynamics, D J Griffiths

we can write $\oint(\hat{r}\cdot\vec{r'})d\vec{l} = -\hat{r}\times\int d\vec{a'}$

- The dipole contribution can now be written as

$$\vec{A}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}, \quad \vec{m} \equiv I\int d\vec{a} = I\vec{a}$$

Magnetic dipole moment

- In general, $\vec{m} = \dfrac{1}{2}\oint I(\vec{r}\times d\vec{l}) = \dfrac{1}{2}\int(\vec{r}\times\vec{J})d\tau$

Field of a Magnetic Dipole

- Let us consider a magnetic dipole with dipole moment m at the origin, pointing in the z direction.

- The vector potential is given by

$$\vec{A}_{\mathrm{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

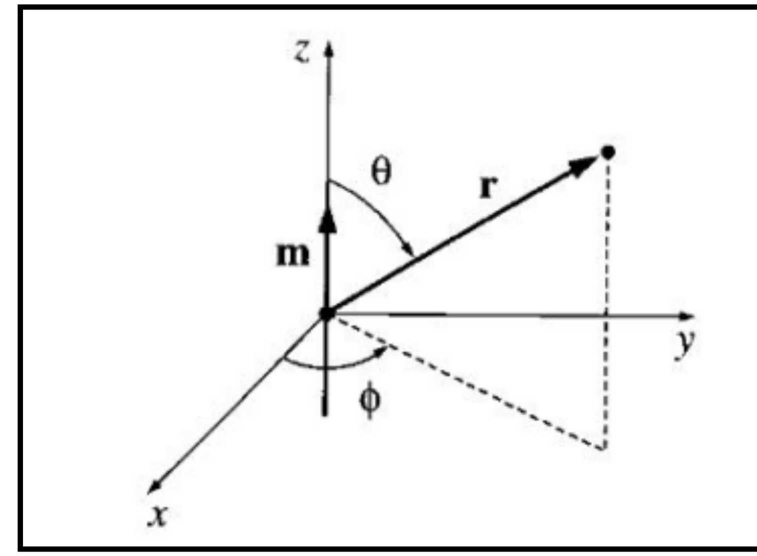


Figure 5.54, Introduction to Electrodynamics, D. J. Griffiths

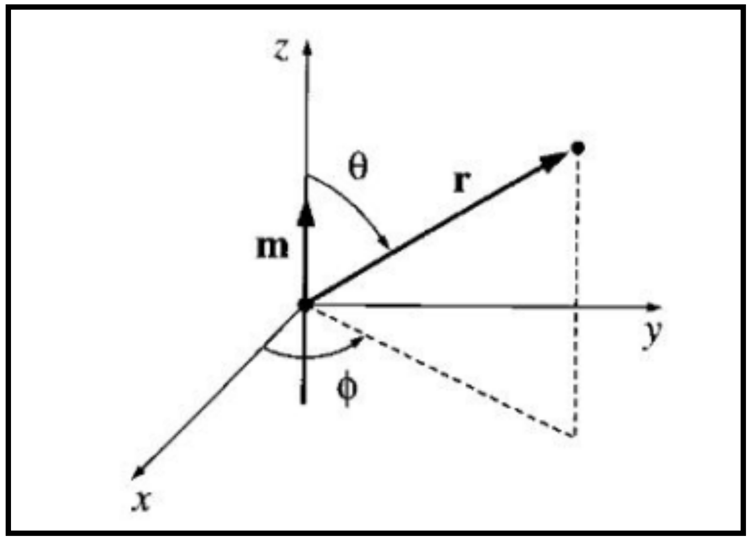
- The magnetic field is given by

$$\vec{B}_{\mathrm{dipole}}(\vec{r}) = \vec{\nabla}\times\vec{A}_{\mathrm{dipole}}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

- The magnetic field of a dipole in coordinate free form can be written as (Problem 5.33, Introduction to Electrodynamics, D. J. Griffiths)

$$\vec{B}_{\mathrm{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}]$$

The dipole moment can be decomposed as*



$$\begin{aligned}\vec{m} &= (\vec{m}\cdot\hat{r})\hat{r} + (\vec{m}\cdot\hat{\theta})\hat{\theta} \\ &= m\cos\theta\hat{r} - m\sin\theta\hat{\theta}\end{aligned}$$

Using this, we get:

$$\begin{aligned}3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m} &= 3m\cos\theta\hat{r} - m\cos\theta\hat{r} + m\sin\theta\hat{\theta} \\ &= 2m\cos\theta\hat{r} + m\sin\theta\hat{\theta}\end{aligned}$$

Therefore,

$$\frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \quad = \vec{B}_{\text{dipole}}(\vec{r})$$

$$\vec{B}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}]$$

*Compare with $\vec{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$ (Tutorial 6)

*No azimuthal component as it is in z direction

Magnetic Field of Earth

Magnetic dipole axis of earth and its geographic (rotation) axis do not coincide. Also, magnetic south pole is in northern hemisphere so that the north pole of the magnetic compass can point towards geographic north pole. Choosing z axis to be the rotation axis and x axis to pass through prime meridian, the dipole moment is given by

$$\vec{m}_E = m_E(\sin\theta_0\cos\phi_0\hat{i} + \sin\theta_0\sin\phi_0\hat{j} + \cos\theta_0\hat{k}) \qquad (\theta_0,\phi_0) = (169^\circ, 109^\circ)$$
$$= m_E(-0.062\hat{i} + 0.18\hat{j} - 0.98\hat{k}) \qquad m_E = 7.79\times 10^{22}\,\mathrm{Am}^2$$

The location of IIT Guwahati is $(\theta_G,\phi_G) = (63.81^\circ, 91.69^\circ) \equiv 26.10^\circ\mathrm{N}, 91.69^\circ\mathrm{E}$

The position vector of IIT Guwahati is

$$\vec{r}_G = r_E(\sin\theta_G\cos\phi_G\hat{i} + \sin\theta_G\sin\phi_G\hat{j} + \cos\theta_G\hat{k})$$
$$= r_E(-0.026\hat{i} + 0.897\hat{j} + 0.44\hat{k})$$

The angle between $-\vec{m}_E, \vec{r}_G$ is

$$\theta_{GE} = \cos^{-1}\left(\frac{-\vec{r}_G\cdot\vec{m}_E}{|\vec{r}_G||\vec{m}_E|}\right)$$
$$= \cos^{-1}(0.268) = 74.4^\circ$$

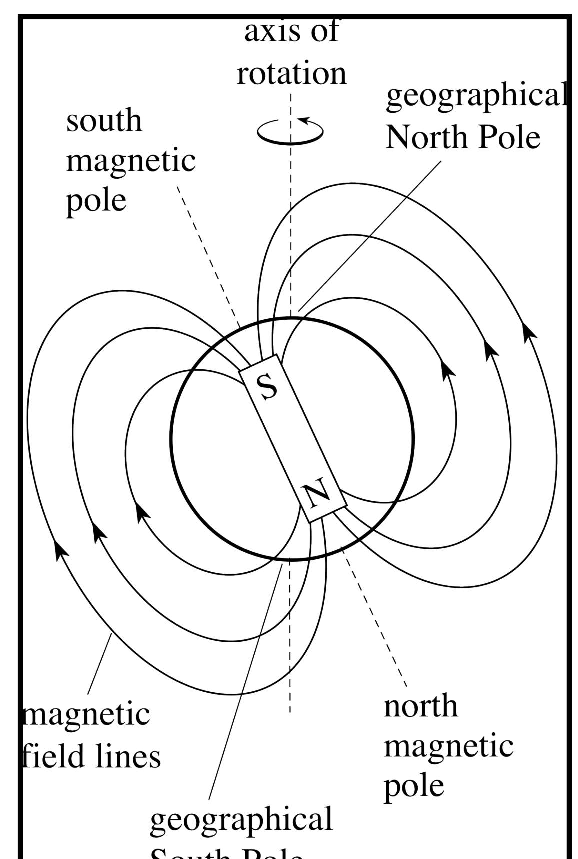


Image credit: http://www.met.reading.ac.uk

Magnetic Field of Earth

Using $\vec{B}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}]$, one can show that

the ratio of the radial and the polar components is

$$\frac{B_r}{B_\theta} = \frac{\frac{\mu_0}{4\pi}\frac{2m}{r^3}\cos\theta}{\frac{\mu_0}{4\pi}\frac{m}{r^3}\sin\theta} = 2\cot\theta$$

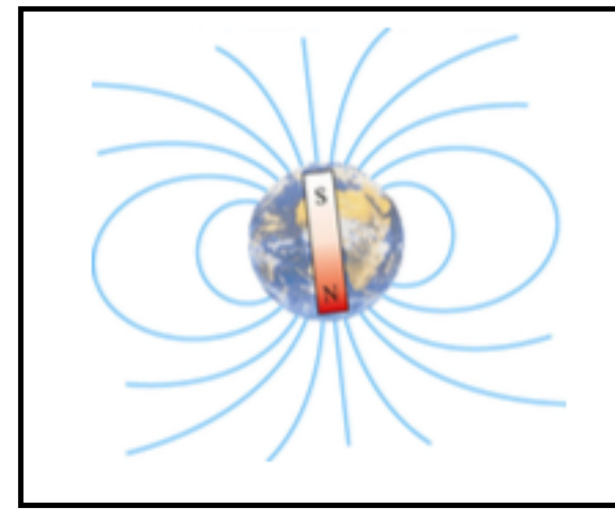


Image credit: MIT

At the location of IIT Guwahati, this value is:

$$\frac{B_r}{B_\theta} = 2\cot\theta_{GE} \approx 0.56$$

Note that here, we are taking θ_{GE} instead of θ keeping in mind the difference between dipole moment axis and the z-axis.

Earth's magnetic field varies in between 0.25-0.65 Gauss on its surface!

Magnetic Field of Earth

- It is generated in the metallic core of the planet.
- Used for navigation (human, birds….)
- Keeps the earth safe from harmful radiation.
- Lead to the formation of Aurora.

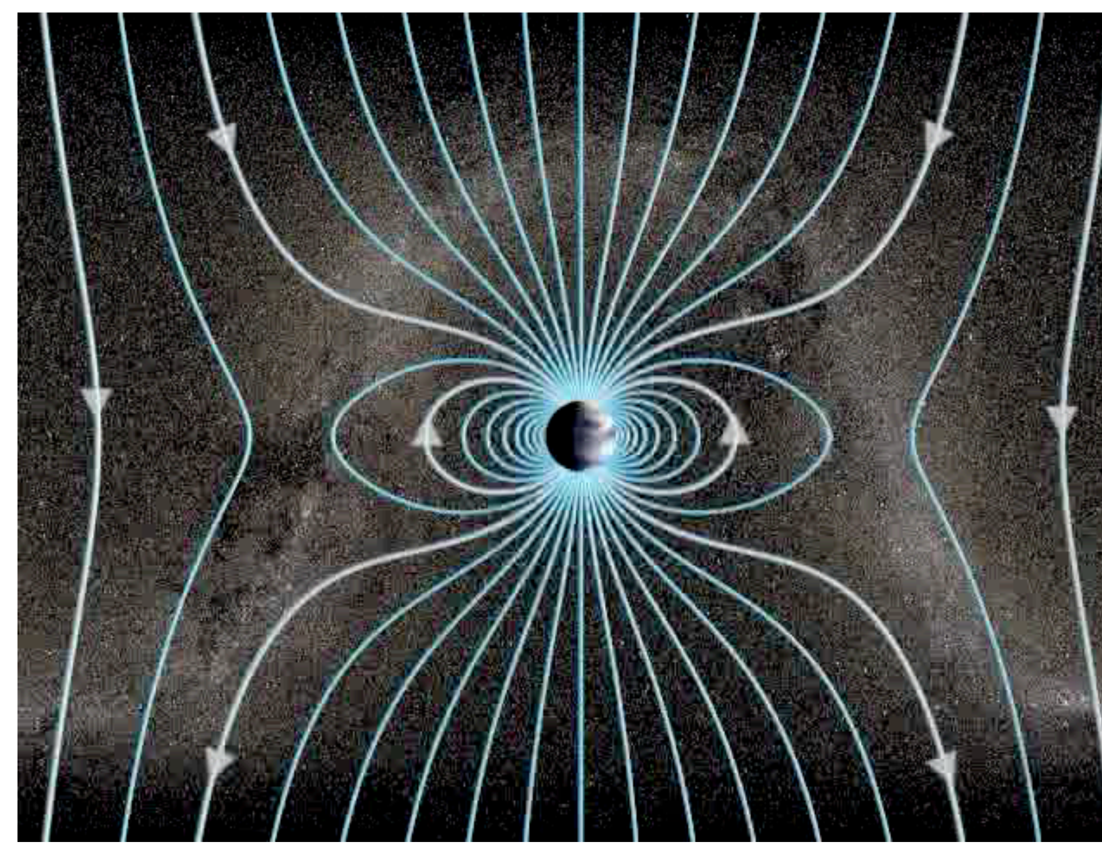
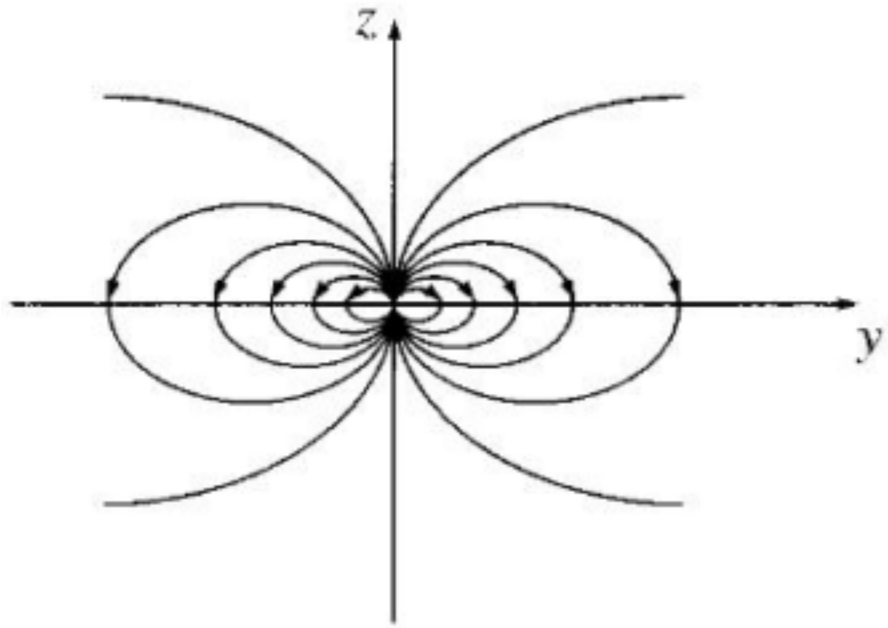
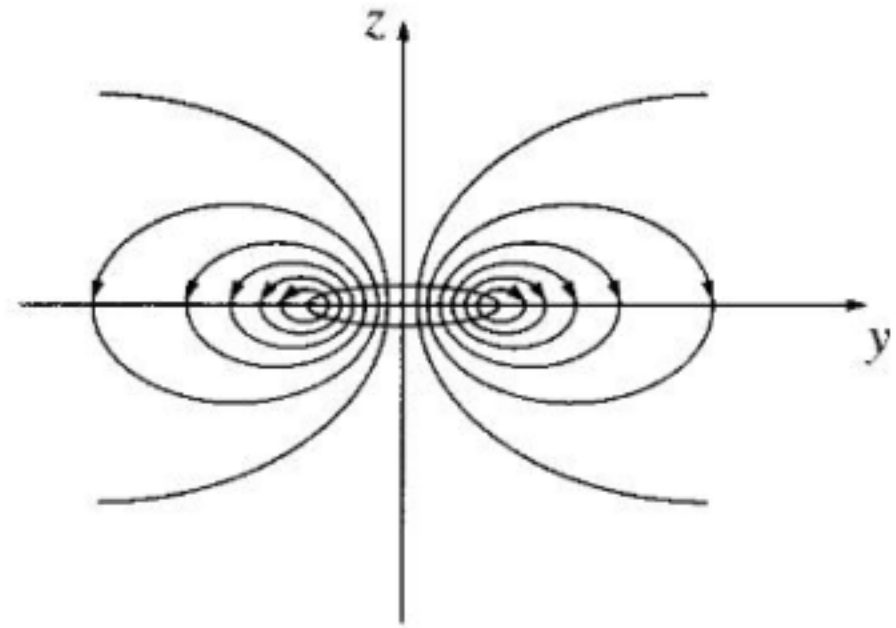


Image credit: NOAA
Visualisation credit: MIT

Field of a Magnetic Dipole



(a) Field of a "pure" dipole



(a) Field of a "physical" dipole

Figure 5.55, Introduction to Electrodynamics, D. J. Griffiths

Torques & Forces on Magnetic Dipoles

- A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field.

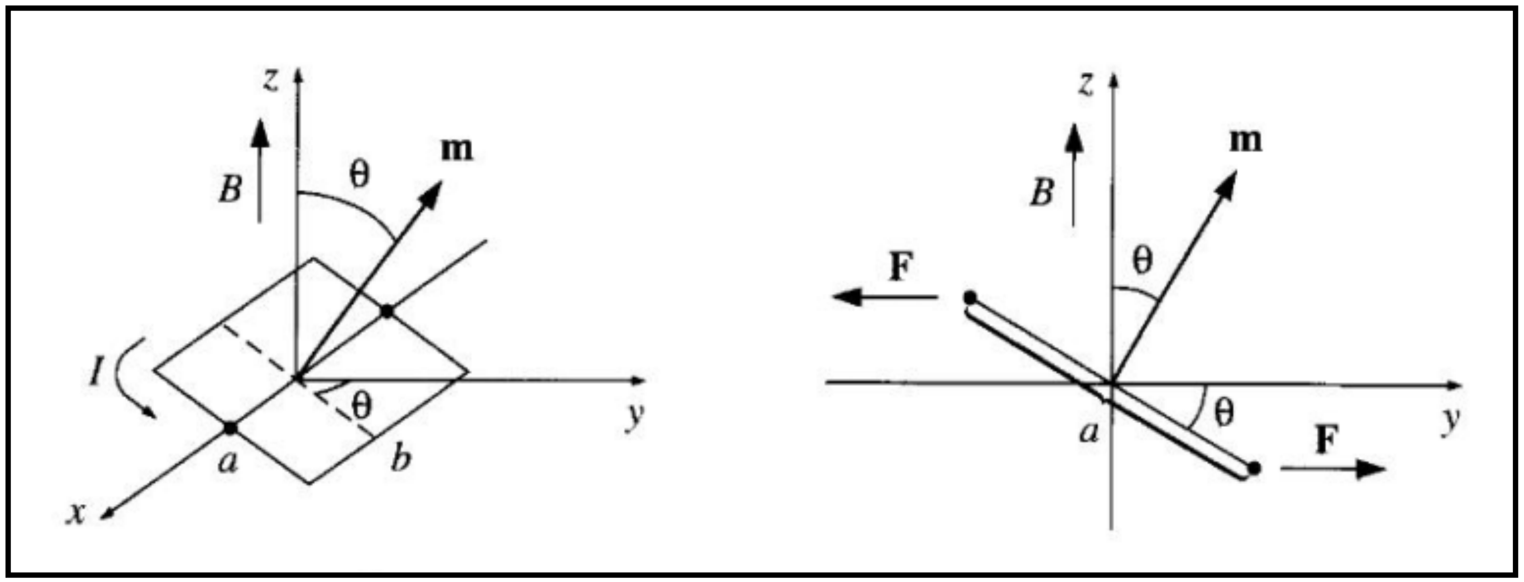


Figure 6.2, Introduction to Electrodynamics, D. J. Griffiths

- Consider a rectangular loop carrying current I, with centre at the origin and magnetic field pointing along the z direction.

- The forces on the two sloping sides (length a) cancel each other.

- The forces on the horizontal sides (length b) cancel each other but they generate a torque.

Torques & Forces on Magnetic Dipoles

- The magnitude of the force on each of the horizontal segments is F=IbB.

- The resulting torque is

$$\vec{N} = aF\sin\theta\hat{x} = IabB\sin\theta\hat{x} = mB\sin\theta\hat{x}$$

$$\implies \vec{N} = \vec{m} \times \vec{B}$$

where m=Iab is the magnetic dipole moment of the loop.

- This gives the exact torque on any localised current distribution, in the presence of a uniform field.

- In a nonuniform field, the above formula gives the exact torque about the centre, for a perfect dipole of infinitesimal size.

Torque on a general current loop

$$\vec{N} = I \oint \vec{r} \times (d\vec{l} \times \vec{B})$$

$$\implies \vec{N} = I \oint (\vec{r} \cdot \vec{B}) d\vec{l} - I \oint (\vec{r} \cdot d\vec{l}) \vec{B}$$

$$(\text{Using } \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C})$$

$$\text{Now } I\vec{B} \oint (\vec{r} \cdot d\vec{l}) = I\vec{B}\frac{1}{2} \oint (\vec{\nabla} r^2) \cdot d\vec{l} = 0 \text{ (Stoke's Theorem)}$$

$$\implies \vec{N} = I \oint (\vec{r} \cdot \vec{B}) d\vec{l}$$

$$\implies \vec{N} = -I \int \vec{\nabla}(\vec{r} \cdot \vec{B}) \times d\vec{a}$$

$$(\text{Using identity } \oint f d\vec{l} = -\int (\vec{\nabla} f) \times d\vec{a})$$

Torque on a general current loop

$$\vec{N} = -I \int \vec{\nabla}(\vec{r} \cdot \vec{B}) \times d\vec{a}$$

$$\implies \vec{N} = -I \int \vec{B} \times d\vec{a} \quad (\text{Using } \vec{\nabla}(\vec{r} \cdot \vec{B}) = \vec{B} \text{ for uniform } \vec{B})$$

$$\implies \vec{N} = -\vec{B} \times (I \int d\vec{a}) = \vec{m} \times \vec{B}$$

The identity used in previous page can be proved as:

$$\oint f\vec{C} \cdot d\vec{l} = \int \vec{\nabla} \times (f\vec{C}) \cdot d\vec{a} \quad (\text{Stoke's Theorem})$$

$$\implies \oint f\vec{C} \cdot d\vec{l} = \int (\vec{\nabla} f \times \vec{C}) \cdot d\vec{a} \quad (\text{For constant } \vec{C})$$

$$\implies \vec{C} \cdot \oint f d\vec{l} = -\vec{C} \cdot \int \vec{\nabla} f \times d\vec{a}$$

$$\implies \oint f d\vec{l} = -\int \vec{\nabla} f \times d\vec{a}$$

Torque on a general current loop (Alternate proof)

Force and torque on an elemental current element of the loop is: $d\vec{F} = I(d\vec{l} \times \vec{B}), d\vec{N} = \vec{r} \times d\vec{F} = I\vec{r} \times (d\vec{l} \times \vec{B})$

Using the identity: $\left[\vec{A} \times (\vec{B} \times \vec{C})\right] + \left[\vec{B} \times (\vec{C} \times \vec{A})\right] + \left[\vec{C} \times (\vec{A} \times \vec{B})\right] = 0$

We have $$\left[\vec{r} \times (d\vec{l} \times \vec{B})\right] = -\left[d\vec{l} \times (\vec{B} \times \vec{r})\right] - \left[\vec{B} \times (\vec{r} \times d\vec{l})\right] \qquad (1)$$

Also, $$d\left[\vec{r} \times (\vec{l} \times \vec{B})\right] = d\vec{r} \times (\vec{l} \times \vec{B}) + \vec{r} \times (d\vec{l} \times \vec{B})$$ (Since B is uniform)

$$\implies \vec{r} \times (d\vec{l} \times \vec{B}) = d\left[\vec{r} \times (\vec{l} \times \vec{B})\right] - d\vec{r} \times (\vec{l} \times \vec{B})$$

Using $d\vec{r} \to d\vec{l}, \vec{l} \to \vec{r}$

$$\implies \vec{r} \times (d\vec{l} \times \vec{B}) = d\left[\vec{r} \times (\vec{r} \times \vec{B})\right] - d\vec{l} \times (\vec{r} \times \vec{B}) \qquad (2)$$

$$(1) + (2) \implies \vec{r} \times (d\vec{l} \times \vec{B}) = \frac{1}{2} d\left[\vec{r} \times (\vec{r} \times \vec{B})\right] - \frac{1}{2}\vec{B} \times (\vec{r} \times d\vec{l})$$

Torque on a general current loop

The net torque on the loop is therefore,

$$N = I\oint \vec{r}\times(d\vec{l}\times\vec{B}) = \frac{1}{2}I\oint d\Big[\vec{r}\times(\vec{r}\times\vec{B})\Big] - \frac{1}{2}I\vec{B}\oint \times(\vec{r}\times d\vec{l})$$

=0

$$-I\vec{B}\times\frac{1}{2}\oint(\vec{r}\times d\vec{l})$$

$$= -\vec{B}\times I\vec{a} = \vec{m}\times\vec{B}$$

Therefore, the torque on a current carrying loop or arbitrary shape is

$$\vec{N} = \vec{m}\times\vec{B}$$

(Compare with $\vec{N} = \vec{p}\times\vec{E}$ for electric dipole)

Work done in rotating the dipole:

$$W = \int_{\theta_0}^{\theta} N d\theta' = mB(\cos\theta_0 - \cos\theta)$$

$$= U - U_0 = \triangle U$$

Taking reference position as $\theta_0 = \pi/2$, the potential energy of the dipole is:

$$U = -mB\cos\theta = -\vec{m}\cdot\vec{B}$$

similar to $U = -\vec{p}\cdot\vec{E}$ of an electric dipole in an electric field.

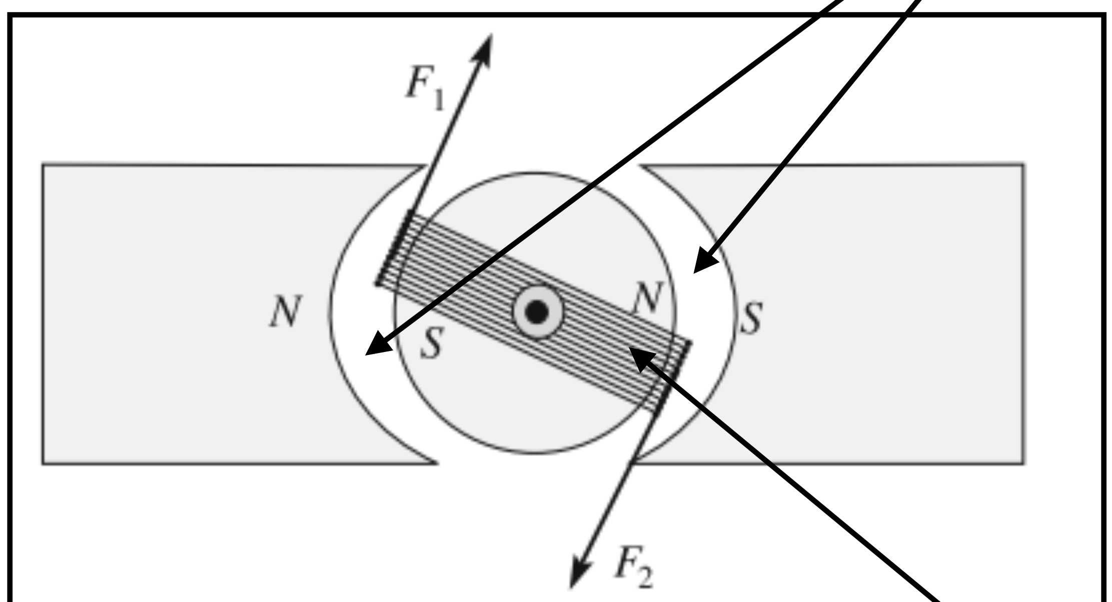
Application: Galvanometer

Radial magnetic field

When current I flows in the coil, there arises a force and a torque.

$$F_1 = F_2 = IaB$$

$$\tau = IBab$$



Equilibrium is reached when torsion moment of the suspension wire balances the torque on the coil.

Coil with N number of turns
and sides a, b

Same principle behind DC electric motor

Image credit: Springer

Force on Magnetic Dipole

- In a uniform field, the net force on a current loop is zero:

$$\vec{F} = I \oint (d\vec{l} \times \vec{B}) = I \left(\oint d\vec{l} \right) \times \vec{B} = 0$$

as the constant field B can be taken outside the integral.

- In a nonuniform field, the net force is not zero. Consider a circular loop of radius R, current I, suspended above a short solenoid in the fringing region. Here the field has a radial component and a net downward force acts on the loop $F = 2\pi IRB\cos\theta$

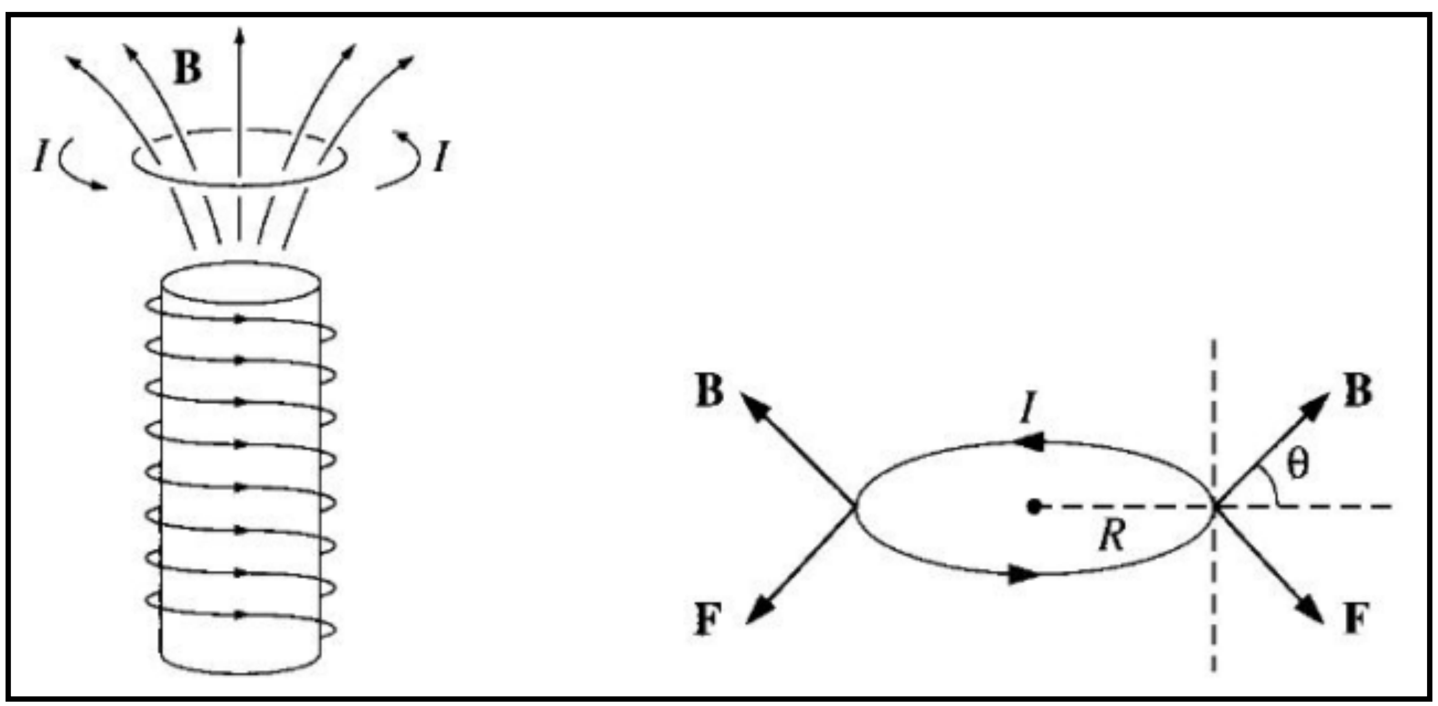


Figure 6.3, Introduction to Electrodynamics, D. J. Griffiths

Force on Magnetic Dipole

For an infinitesimal loop, with dipole moment $\vec{m}$, in a magnetic field $\vec{B}$, the force is $$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

Proof: Assume the dipole to be an infinitesimal square of side e. Choosing the axes as shown in figure, calculate the magnetic force on each of the four sides

$$\vec{F} = I \int (d\vec{l} \times \vec{B})$$

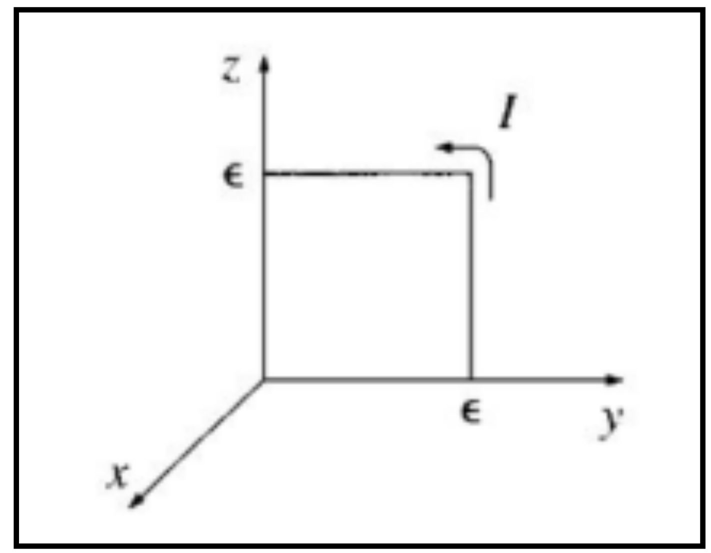


Figure 6.8, Introduction to Electrodynamics, D. J. Griffiths

Force on the elemental square loop is

$$d\vec{F} = I\Big[(dy\hat{y}) \times \vec{B}(0,y,0) + (dz\hat{z}) \times \vec{B}(0,\epsilon,z) - (dy\hat{y}) \times \vec{B}(0,y,\epsilon) - (dz\hat{z}) \times \vec{B}(0,0,z)\Big]$$

$$= I\Big[-(dy\hat{y}) \times \{\vec{B}(0,y,\epsilon) - \vec{B}(0,y,0)\} + (dz\hat{z}) \times \{\vec{B}(0,\epsilon,z) - \vec{B}(0,0,z)\}\Big]$$

Using: $$\vec{B}(0,\epsilon,z) \approx \vec{B}(0,0,z) + \epsilon \frac{\partial \vec{B}}{\partial y}\bigg|_{(0,0,z)}, \vec{B}(0,y,\epsilon) \approx \vec{B}(0,y,0) + \epsilon \frac{\partial \vec{B}}{\partial z}\bigg|_{(0,y,0)}$$

and $\int dy \frac{\partial \vec{B}}{\partial z}\Big|_{(0,y,0)} \approx \epsilon \frac{\partial \vec{B}}{\partial z}\Big|_{(0,0,0)}, \int dz \frac{\partial \vec{B}}{\partial y}\Big|_{(0,0,z)} \approx \epsilon \frac{\partial \vec{B}}{\partial y}\Big|_{(0,0,0)}$ Upto leading order!

we can write the total force as

$$\vec{F} = \int d\vec{F} = I\epsilon^2 \left[\hat{z} \times \frac{\partial \vec{B}}{\partial y} - \hat{y} \times \frac{\partial \vec{B}}{\partial z}\right]$$
$$= I\epsilon^2 \left[\hat{y}\frac{\partial B_x}{\partial y} - \hat{x}\frac{\partial B_y}{\partial y} - \hat{x}\frac{\partial B_z}{\partial z} + \hat{z}\frac{\partial B_x}{\partial z}\right]$$

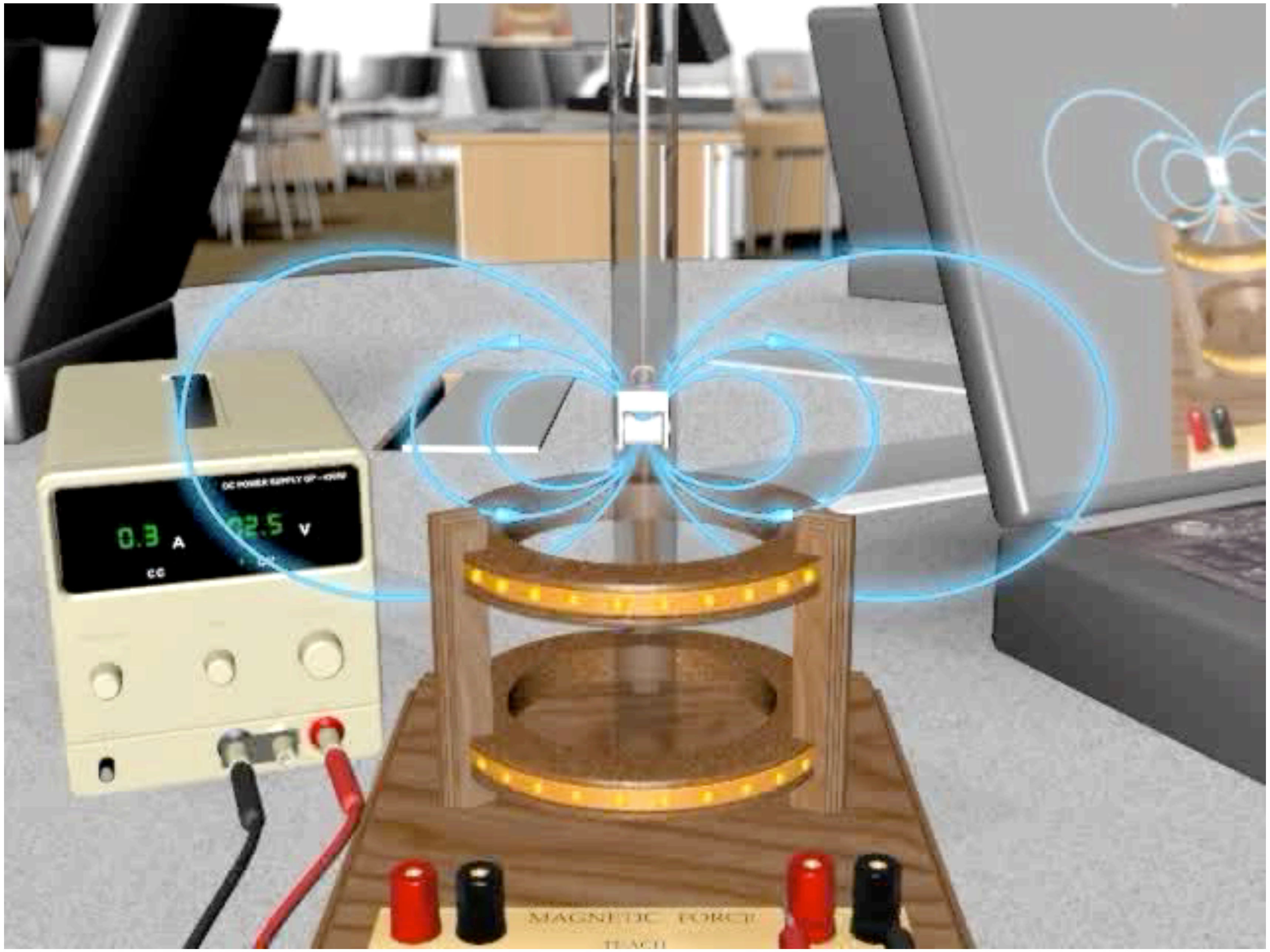
Using $\vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \implies \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = -\frac{\partial B_x}{\partial x}$

$$\vec{F} = m\left[\hat{y}\frac{\partial B_x}{\partial y} + \hat{x}\frac{\partial B_x}{\partial x} + \hat{z}\frac{\partial B_x}{\partial z}\right] \qquad \vec{m} = m\hat{x} = I\epsilon^2\hat{x}$$

$$\implies \vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

Compare with the electric dipole analogue $\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E}$ Tutorial 6

Force on Magnetic Dipole



Visualisation credit: MIT