CONTINUITY AND DIFFERENTIATION OF RANDOM PROCESSES

- We know that the dynamic behaviour of a system is described by differential equations involving the input and the output of the system. For example, the behavior of an RC network is described by a first order linear differential equation with the source voltage as the input and the capacitor voltage as the output. *What happens if the input voltage to the network is a random process?*

- Each realization of the random process is a deterministic function and the concepts of differentiation and integration can be applied to it. Our aim is to extend these concepts to the ensemble of realizations and apply calculus to random process.

We discussed the convergence and the limit of a sequence of random variables. The continuity of the random process can be defined with the help of the convergence and limits of a sequence of random variables arising from a random process. We can define *almost-sure continuity*, *mean-square continuity*, and *continuity in probability* etc. according to the mode of convergence considered. Here, we shall discuss the concept of mean-square continuity and introduce elementary concepts of corresponding *mean-square calculus*.

Mean-square continuity of a random process

Recall that a sequence of random variables $\{X_n\}$ converges to a random variable X if

$$\lim_{n\to\infty} E\left[X_n - X\right]^2 = 0$$

and we write

$$\underset{n\to\infty}{\text{l.i.m.}} X_n = X$$

We also note the following properties:

(a) $\underset{n\to\infty}{\text{l.i.m.}} c = c$ where c is a constant.

(b) $\underset{n\to\infty}{\text{l.i.m.}} (c_1 X_n + c_2 Y_n) = c_1 \underset{n\to\infty}{\text{l.i.m.}} X_n + c_2 \underset{n\to\infty}{\text{l.i.m.}} Y_n$

(c) $\lim_{n\to\infty} EX_n = E \underset{n\to\infty}{\text{l.i.m.}} X_n$

Proof (b)

Suppose $\underset{n\to\infty}{\text{l.i.m}} X_n = X$ and $\underset{n\to\infty}{\text{l.i.m.}} Y_n = Y$.

Then

$$E(c_1X_n + c_2Y_n - c_1X - c_2Y)^2 = E(c_1(X_n - X) + c_2(Y_n - Y))^2$$
$$= c_1^{\,2}E(X_n - X)^2 + c_2^{\,2}E(Y_n - Y)^2 + 2c_1c_2E(X_n - X)(Y_n - Y)$$
$$\therefore \lim_{n\to\infty} E(c_1X_n + c_2Y_n - c_1X - c_2Y)^2 = \lim_{n\to\infty} c_1^{\,2}E(X_n - X)^2 + \lim_{n\to\infty} c_2^{\,2}E(Y_n - Y)^2$$
$$+ 2c_1c_2 \lim_{n\to\infty} E(X_n - X)(Y_n - Y)$$
$$\leq 0 + 0 + 2|c_1c_2| \lim_{n\to\infty} |E(X_n - X)(Y_nY)|$$
$$\leq 0 + 0 + 2|c_1c_2| \lim_{n\to\infty} \sqrt{|E(X_n - X)^2 E(Y_n - Y)^2|}$$
$$= 0$$

The following **Cauchy criterion** gives the condition for m.s. convergence of a random sequence without actually finding the limit. The sequence $\{X_n\}$ converges in m.s. if and only if $E(x_m - x_n)^2 \to 0$ as $m,n \to \infty$

The concept of the limit of a deterministic function can be extended to a random process to define limit in the sense of mean-square, probability etc. The limit of a continuous-time random process $\{X(t)\}$ at point $t = t_0$ can be defined using a sequence of random variables at the neighbourhood of t_0. The limit in the m.s. sense is defined as follows:

Definition: Suppose $\{X(t), t \in \Gamma\}$ is a random process and Y be a random variable defined on the same probability space $(S, \mathbb{F}, P)$. Y is called the m.s. limit of $\{X(t)\}$ at point $t = t_0$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $E\left[X(t) - Y\right]^2 < \varepsilon$ whenever $|t - t_0| < \delta$. We write $\underset{t\to t_0}{\text{l.i.m.}}\, X(t) = Y$

We also note that

$$\lim_{t\to t_0} EX(t) = E\, \underset{t\to t_0}{\text{l.i.m.}}\, X(t)$$

With the above definition for the m.s. limit, we can define the mean-square continuity of a random process.

Mean-square continuity

Definition: A random process $\{X(t), t \in \Gamma\}$ defined on a probability space $(S, \mathbb{F}, P)$ is said to be continuous at a point $t = t_0$ in the mean-square sense if $\underset{t\to t_0}{\text{l.i.m.}}\, X(t) = X(t_0)$ or equivalently

$$\lim_{t\to t_0} E\left[X(t) - X(t_0)\right]^2 = 0$$

The random process $\{X(t), t \in \Gamma\}$ is called m.s. continuous if it is m.s. continuous at each $t \in \Gamma$.

Mean-square continuity and autocorrelation function

The m.s. continuity of a random process $\{X(t), t \in \Gamma\}$ at point $t = t_0$ depends on the two-dimensional continuity of the autocorrelation function $R_X(t_1, t_2)$ as stated in the following theorem

Theorem 1: A random process $\{X(t), t \in \Gamma\}$ is MS continuous at t_0 if and only if its autocorrelation function $R_X(t_1, t_2)$ is continuous at (t_0, t_0).

Proof: We shall first prove the necessary part first. We have,

$$\begin{aligned}
\left|R_X(t_1, t_2) - R_X(t_0, t_0)\right| &= \left|EX(t_1)X(t_2) - EX(t_0)X(t_0)\right| \\
&= \left|EX(t_1)(X(t_2) - X(t_0)) + EX(t_0)(X(t_1) - X(t_0))\right| \\
&\le \left|EX(t_1)(X(t_2) - X(t_0))\right| + \left|EX(t_0)(X(t_1) - X(t_0))\right| \\
&\le \sqrt{EX^2(t_1)E(X(t_2) - X(t_0))^2} + \sqrt{EX^2(t_0)E(X(t_1) - X(t_0))^2}
\end{aligned}$$

(Using the Cauchy Schwartz inequality)

Now $\lim_{t_2 \to t_0} E(X(t_2) - X(t_0))^2 = 0$ and $\lim_{t_1 \to t_0} E(X(t_1) - X(t_0))^2 = 0.$

$$\therefore \lim_{t_1 \to t_0, t_2 \to t_0} R_X(t_1, t_2) - R_X(t_0, t_0) = 0$$

Thus $R_X(t_1, t_2)$ is continuous at (t_0, t_0).

To prove the sufficient part, suppose $R_X(t_1, t_2)$ is continuous at (t_0, t_0). Then,

$$\begin{aligned}
E\left[X(t) - X(t_0)\right]^2 &= E\left(X^2(t) - 2X(t)X(t_0) + X^2(t_0)\right) \\
&= R_X(t, t) - 2R_X(t, t_0) + R_X(t_0, t_0)
\end{aligned}$$

$$\begin{aligned}
\therefore \lim_{t \to t_0} E\left[X(t) - X(t_0)\right]^2 &= \lim_{t \to t_0} R_X(t, t) - 2R_X(t, t_0) + R_X(t_0, t_0) \\
&= R_X(t_0, t_0) - 2R_X(t_0, t_0) + R_X(t_0, t_0) \\
&= 0
\end{aligned}$$

Thus $\{X(t), t \in \Gamma\}$ is MS continuous at t_0

Theorem 2: If $\{X(t)\}$ is MS continuous at t_0 its mean is continuous at t_0.

This follows from the fact that

$$\left(EX(t)-EX(t_0)\right)^2=\left(E(X(t)-X(t_0)\right)^2$$
$$\leq E\left(X(t)-X(t_0)\right)^2$$
$$\therefore \lim_{t\to t_0}\left[E\left[X(t)-X(t_0)\right]\right]^2$$
$$\leq \lim_{t\to t_0}E\left[X(t)-X(t_0)\right]^2=0$$

$\therefore EX(t)$ is continuous at t_0.

The converse of this theorem is not true. For example, a white noise process $\{X(t)\}$ has the mean $EX(t)=0$ which is continuous at every *t*. But its autocorrelation function $R_X(t_1,t_2)=c\delta(t_1-t_2)$ is discontinuous at point (t,t).

Remark

(1) If the autocorrelation function $R_X(t_1,t_2)$ is continuous at all points of the form $(t,t)\in\Gamma\times\Gamma$, then $\{X(t),t\in\Gamma\}$ m.s. continuous over all points of Γ.

(2) If $\{X(t)\}$ is a WSS random process, then $EX(t_1)X(t_2)=R_X(\tau)$ where $\tau=t_1-t_2$. Then Theorem 1 is simplified as follows:

A WSS random process $\{X(t),t\in\Gamma\}$ is MS continuous over all points of Γ if and only if its autocorrelation function $R_X(\tau)$ is continuous at $\tau=0$.

This is because,

$$E\left[X(t_0+\tau)-X(t_0)\right]^2=R_X(0)-2R_X(\tau)+R_X(0)$$

If $\lim_{\tau\to 0}R_X(\tau)=R_X(0)$, then from above $\lim_{\tau\to 0}E\left[X(t_0+\tau)-X(t_0)\right]^2=0$. This proves the sufficient condition. The necessary condition can similarly be proved.

<u>Example 1</u>

Consider the random binary wave $\{X(t)\}$. A typical realization of the process is shown in Figure 1. The realization is a discontinuous function.

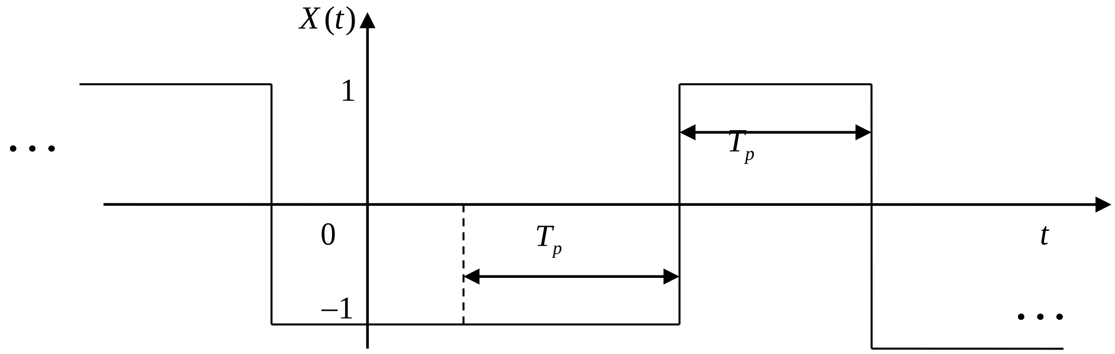

Figure 1

The process has the autocorrelation function given by

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T_p} & |\tau| \le T_p \\ 0 & \text{otherwise} \end{cases}$$

We observe that $R_X(\tau)$ is continuous at $\tau = 0$. Therefore, $R_X(\tau)$ is continuous at all τ.

<u>Example 2</u>

For a Wiener process $\{X(t)\}$,

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2)$$

where α is a constant.

$$\therefore R_X(t, t) = \alpha \min(t, t) = \alpha t$$

Thus the autocorrelation function of a Wiener process is continuous everywhere implying that a Wiener process is m.s. continuous everywhere. We can similarly show that the Poisson process is m.s. continuous everywhere.

Mean-square differentiability

Definition: The random process $\{X(t)\}$ on a probability space $(S, \mathbb{F}, P)$ is said to have the mean-square derivative $X'(t)$ at a point $t \in \Gamma$, provided $\dfrac{X(t+\Delta t) - X(t)}{\Delta t}$ approaches $X'(t)$ in the mean square sense as $\Delta t \to 0$. In other words, the random process $\{X(t)\}$ has a m-s derivative $X'(t)$ if $\lim\limits_{\Delta t \to 0} E\left[\dfrac{X(t+\Delta t) - X(t)}{\Delta t} - X'(t)\right]^2 = 0$

Remark

- If all the sample functions of a random process $X(t)$ are differentiable, then the above condition is satisfied and the m-s derivative exists.

Example 3

Consider the random-phase sinusoid $\{X(t)\}$ given by

$X(t) = A\cos(\omega_0 t + \phi)$ where A and w_0 are constants and $\phi \sim U[0,\ 2\pi]$. Then for each ϕ, $X(t)$ is differentiable.

Therefore, the M.S. derivative is $X'(t) = -A\omega_0 \sin(\omega_0 t + \phi)$

M.S. Derivative and Autocorrelation functions

Theorem: The random process $\{X(t)\}$ is m.s. differentiable at $t \in \Gamma$ if and only if $\frac{\partial^2 R_X(t_1, t_2)}{\partial t_1 \partial t_2}$ exists at $(t,t) \in \Gamma \times \Gamma$.

To prove the theorem, we shall take the help of the Cauchy criterion for m.s. convergence:

The sequence $\{X_n\}$ converges in m.s. if and only if

$$\lim_{n\to\infty, m\to\infty} E|x_n - x_m|^2 = 0.$$

Applying the Cauchy criterion, the condition for existence of m.s. derivative is

$$\lim_{\Delta t_1, \Delta t_2 \to 0} E\left[\frac{X(t+\Delta t_1) - X(t)}{\Delta t_1} - \frac{X(t+\Delta t_2) - X(t)}{\Delta t_2}\right]^2 = 0$$

Expanding the square and taking expectation results,

$$\lim_{\Delta t_1,\Delta t_2\to 0} E\left[\frac{X(t+\Delta t_1)-X(t)}{\Delta t_1}-\frac{X(t+\Delta t_2)-X(t)}{\Delta t_2}\right]^2$$

$$= \lim_{\Delta t_1,\Delta t_2\to 0}\left[\frac{R_X(t+\Delta t_1,t+\Delta t_1)+R_X(t,t)-2R_X(t+\Delta t_1,t)}{\Delta t_1^{\,2}}\right]+\left[\frac{R_X(t+\Delta t_2,t+\Delta t_2)+R_X(t,t)-2R_X(t+\Delta t_2,t)}{\Delta t_2^{\,2}}\right]$$

$$-2\left[\frac{R_X(t+\Delta t_1,t+\Delta t_2)-R_X(t+\Delta t_1,t)-R_X(t,t+\Delta t_2)+R_X(t,t)}{\Delta t_1\Delta t_2}\right]$$

Each of the above terms within square bracket converges to $\left.\dfrac{\partial^2 R_X(t_1,t_2)}{\partial t_1\partial t_2}\right]_{t_1=t,t_2=t}$ if this second partial derivative exists.

$$\therefore \lim_{\Delta t_1,\Delta t_2\to 0} E\left[\frac{X(t+\Delta t_1)-X(t)}{\Delta t_1}-\frac{X(t+\Delta t_2)-X(t)}{\Delta t_2}\right]^2=\left.\frac{\partial^2 R_X(t_1,t_2)}{\partial t_1\partial t_2}+\frac{\partial^2 R_X(t_1,t_2)}{\partial t_1\partial t_2}-2\frac{\partial^2 R_X(t_1,t_2)}{\partial t_1\partial t_2}\right]_{t=t_1,t=t_2}$$

$$=0$$

Thus, $\{X(t)\}$ is m-s differentiable at $t\in\Gamma$ if $\dfrac{\partial^2 R_X(t_1,t_2)}{\partial t_1\partial t_2}$ exists at $(t,t)\in\Gamma\times\Gamma$.

Particularly, if $X(t)$ is WSS,

$R_X(t_1,t_2)=R_X(t_1-t_2)$

Substituting $t_1-t_2=\tau$, we get

$$
\begin{aligned}
\frac{\partial^2 R_X\left(t_1,t_2\right)}{\partial t_1 \partial t_2} &= \frac{\partial^2 R_X\left(t_1-t_2\right)}{\partial t_1 \partial t_2} \\
&= \frac{\partial}{\partial t_1}\left(\frac{dR_X\left(\tau\right)}{d\tau}.\frac{\partial\left(t_1-t_2\right)}{\partial t_2}\right) \\
&= -\frac{d^2 R_X\left(\tau\right)}{d\tau^2}\frac{\partial \tau}{\partial t_1} \\
&= -\frac{d^2 R_X\left(\tau\right)}{d\tau^2}
\end{aligned}
$$

Therefore, a WSS process $X(t)$ is m-s differentiable if $R_X(\tau)$ has second derivative at $\tau = 0$.

Example 4

Consider a WSS process $\{X(t)\}$ with autocorrelation function

$R_X(\tau) = \exp(-a|\tau|)$

$R_X(\tau)$ does not have the first and second derivative at $\tau = 0.$ $\{X(t)\}$ is not mean-square differentiable.

Example 5

The random binary wave $\{X(t)\}$ has the autocorrelation function

$$
R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T_p} & |\tau| \leq T_p \\ 0 & \text{otherwise} \end{cases}
$$

$R_X(\tau)$ does not have the first and second derivative at $\tau = 0.$ Therefore, $\{X(t)\}$ is not mean-square differentiable.

Example 6

For a Wiener process $\{X(t)\}$,

$$R_X(t_1,t_2) = \alpha \min(t_1,t_2)$$

where α is a constant.

$$\therefore R_X(0,t_2) = \begin{cases} \alpha t_2 & \text{if } t_2 < 0 \\ 0 & \text{other wise} \end{cases}$$

$$\therefore \frac{\partial R_X(0,t_2)}{\partial t_1} = \begin{cases} \alpha & \text{if } t_2 < 0 \\ 0 & \text{if } t_2 > 0 \\ \text{does not exist if if } & t_2 = 0 \end{cases}$$

$$\therefore \frac{\partial^2 R_X(t_1,t_2)}{\partial t_1 \partial t_2} \text{ does not exist at } (t_1 = 0, t_2 = 0)$$

Thus a Wiener process is m.s. differentiable nowhere.

Mean and Autocorrelation of the Derivative process

We have,

$$\begin{aligned} EX'(t) &= E \underset{\Delta t \to 0}{\text{l.i.m.}} \frac{X(t+\Delta t) - X(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{EX(t+\Delta t) - EX(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\mu_X(t+\Delta t) - \mu_X(t)}{\Delta t} \\ &= \mu_X'(t) \end{aligned}$$

For a WSS process $EX'(t) = \mu_X'(t) = 0$ as $\mu_X(t) =$ constant.

$$R_{XX'}(t_1,t_2) = EX(t_1)X'(t_2)$$
$$= EX(t_1)\underset{\Delta t_2 \to 0}{\text{l.i.m.}} \frac{X(t_2+\Delta t_2) - X(t_2)}{\Delta t_2}$$
$$= \lim_{\Delta t_2 \to 0} \frac{E\left[X(t_1)X(t_2+\Delta t_2) - X(t_1)X(t_2)\right]}{\Delta t_2}$$
$$= \lim_{\Delta t_2 \to 0} \frac{R_X(t_1,t_2+\Delta t_2) - R_X(t_1,t_2)}{\Delta t_2}$$
$$= \frac{\partial R_X(t_1,t_2)}{\partial t_2}$$
$$\therefore R_{XX'}(t_1,t_2) = \frac{\partial R_X(t_1,t_2)}{\partial t_2}$$

Similarly we can show that
Using the above result,

$$R_{X'}(t_1,t_2) = EX'(t_1)X'(t_2)$$
$$= \frac{\partial R_{X,X'}(t_1,t_2)}{\partial t_1}$$
$$= \frac{\partial^2 R_{X,X}(t_1,t_2)}{\partial t_1 \partial t_2}$$

For a WSS process

$$\mu_X^{'}(t) = E\mu_X = 0,$$
$$EX(t_1)X'(t_2) = \frac{\partial}{\partial t_2} R_X(t_1 - t_2) = -\frac{dR_X(\tau)}{d\tau}$$

and

$$EX'(t_1)X'(t_2) = \frac{\partial^2 R_X(t_1 - t_2)}{\partial t_1 \partial t_2}$$
$$= -\frac{d^2 R_X(\tau)}{d\tau^2}$$
$$\therefore \operatorname{var}\left(X'(t)\right) = -\left.\frac{d^2 R_X(\tau)}{d\tau^2}\right|_{\tau=0}$$

Mean Square Integral

Recall that the Riemannian integral of a function $x(t)$ over the interval $[t_0,\ t]$ is defined as the limiting sum given by

$$\int_{t_0}^{t} x(\tau)d\tau = \lim_{n\to\infty \Delta k\to 0} \sum_{k=0}^{n-1} x(\tau_k)\Delta_k$$

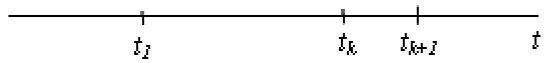

Figure 2

where $t_0 < t_1 < ... < t_{n-1} < t_n = t$ is a partition on the interval $[t_0,\ t]$ and $\Delta_k = t_{k+1} - t_k$ and $\tau_k \in [t_{k-1},\ t_k]$. For a random process $\{X(t)\}$, the m-s integral can be similarly defined.

Definition: Suppose $t_0 < t_1 < .... < t_{n-1} < t_n = t$ is a partition on the interval $[t_0,\ t]$ and $\Delta_k = t_{k+1} - t_k$ and $\tau_k \in [t_{k-1},\ t_k]$. The random process $\{X(t)\}$ on a probability space $(S, \mathbb{F}, P)$ is said to be *m.s. integrable* over $[t_0,\ t]$ if $\underset{n\to\infty \Delta k\to 0}{\text{l.i.m.}} \sum_{k=0}^{n-1} X(\tau_k)\Delta_k$ exists. This limit is called the mean-square integral of $\{X(t)\}$ and denoted by

$$Y(t) = \int_{t_0}^{t} X(\tau)d\tau .$$

Thus

$$Y(t) = \int_{t_0}^{t} X(\tau)d\tau = \underset{n\to\infty \Delta k\to 0}{\text{l.i.m.}} \sum_{k=0}^{n-1} X(\tau_k)\Delta_k$$

Note that the mean-square integral is a random process as the integral is over time.

Existence of M.S. Integral

- It can be shown that the sufficient condition for the m.s. integral $\int_{t_0}^{t} X(\tau)d\tau$ to exist is that the double integral $\int_{t_0}^{t}\int_{t_0}^{t} R_X(\tau_1,\tau_2)d\tau_1 d\tau_2$ exists. In other words, the random process $\{X(t)\}$ is m.s. integrable over $[t_0,\ t]$ if and only if $R_X(t_1,t_2)$ is Riemann integrable over $[t_0,\ t]\times[t_0,\ t]$.
- If $\{X(t)\}$ is m.s. continuous, then the above condition is satisfied and the process is m.s. integrable.

Mean and Autocorrelation of the Integral of a WSS process

We have

$$\begin{aligned} EY(t) &= E\int_{t_0}^{t} X(\tau)d\tau \\ &= \int_{t_0}^{t} EX(\tau)d\tau \\ &= \int_{t_0}^{t} \mu_X d\tau \\ &= \mu_X (t - t_0) \end{aligned}$$

Therefore, if $\mu_X \neq 0$, $\{Y(t)\}$ is necessarily non-stationary.

$$\begin{aligned} R_Y(t_1, t_2) &= EY(t_1)Y(t_2) \\ &= E\int_{t_0}^{t_1}\int_{t_0}^{t_2} X(\tau_1)X(\tau_2)d\tau_1 d\tau_2 \\ &= \int_{t_0}^{t_1}\int_{t_0}^{t_2} EX(\tau_1)X(\tau_2)d\tau_1 d\tau_2 \\ &= \int_{t_0}^{t_1}\int_{t_0}^{t_2} R_X(\tau_1 - \tau_2)d\tau_1 d\tau_2 \end{aligned}$$

which is a function of t_1 and t_2.

Thus the integral of a WSS process is always non-stationary.

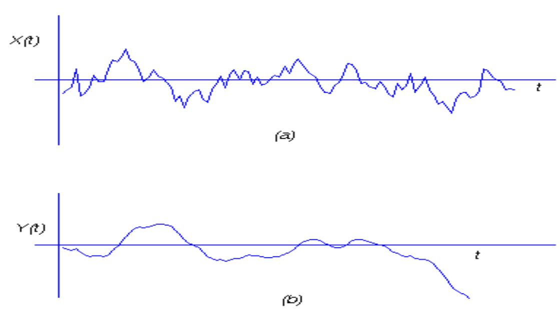

Figure 2. (a) Realization of a WSS process $X(t)$
(b) corresponding integral $Y(t)$

Remark

- The nonstationary of the M.S. integral of a random process has physical importance – the output of an integrator due to stationary noise rises unboundedly.

<u>Example 7</u>

The random binary wave $\{X(t)\}$ has the autocorrelation function

$$R_X(\tau) = \begin{cases} 1 - \frac{|\tau|}{T_p} & |\tau| \le T_p \\ 0 & \text{otherwise} \end{cases}$$

$R_X(\tau)$ is continuous at $\tau = 0$ implying that $\{X(t)\}$ is M.S. continuous. Therefore, $\{X(t)\}$ is mean-square integrable.