2.06. Prove $\oint d\vec{r} \times \vec{B} = \int\int_S (\hat{n} \times \vec{\nabla}) \times \vec{B}\, dS$.

Consider a vector $\vec{A} = \vec{B} \times \vec{C}$, where $\vec{C}$ is a constant vector.

Then by Stoke's theorem applied to the vector field $\vec{A}$:

$$\iint_S \underbrace{\left[\vec{\nabla} \times (\vec{B} \times \vec{C})\right]}_{\vec{\nabla} \times \vec{A}} \cdot d\vec{S} = \oint_C \underbrace{(\vec{B} \times \vec{C})}_{\vec{A}} \cdot d\vec{r}$$

Use identity: $\vec{\nabla} \times (\vec{B} \times \vec{C}) = (\vec{C} \cdot \vec{\nabla})\vec{B} - \vec{C}(\vec{\nabla} \cdot \vec{B}) - (\vec{B} \cdot \vec{\nabla})\vec{C} + \vec{B}(\vec{\nabla} \cdot \vec{C})$ (Prove Take home)

Since $\vec{C}$ is a constant vector, 3rd and 4th terms of the above vanish.

$$\Rightarrow \iint_S \left[\vec{\nabla} \times (\vec{B} \times \vec{C})\right] \cdot d\vec{S}$$

$$= \iint_S \left[(\vec{C} \cdot \vec{\nabla})\vec{B} - \vec{C}(\vec{\nabla} \cdot \vec{B})\right] \cdot \underbrace{d\vec{S}}_{\hat{n}\, dS}$$

$$= \iint_S \left[(\vec{C} \cdot \vec{\nabla})\vec{B} - \vec{C}(\vec{\nabla} \cdot \vec{B})\right] \cdot \hat{n}\, dS$$

Now, consider $\left[(\vec{C} \cdot \vec{\nabla})\vec{B}\right] \cdot \vec{D}$

$$= D_j C_i \partial_i B_j = C_i \partial_i (D_j B_j) - B_j C_i \partial_i D_j$$

$$= \vec{C}\cdot\left[\vec{\nabla}(\vec{D}\cdot\vec{B})\right] - \vec{B}\cdot\left[(\vec{C}\cdot\vec{\nabla})\,\vec{D}\right]$$

Choose $\vec{D} = \hat{n}$

$$\Rightarrow \left[(\vec{C}\cdot\vec{\nabla})\vec{B} - \vec{C}\,(\vec{\nabla}\cdot\vec{B})\right]\cdot\hat{n}$$

$$= \vec{C}\cdot\left[\vec{\nabla}(\hat{n}\cdot\vec{B})\right] - \vec{B}\cdot\left[(\vec{C}\cdot\vec{\nabla})\,\hat{n}\right] - \vec{C}\cdot\left[\hat{n}(\vec{\nabla}\cdot\vec{B})\right]$$

(By using above identity)

$$= \vec{C}\cdot\left[\vec{\nabla}(\vec{B}\cdot\hat{n}) - \hat{n}(\vec{\nabla}\cdot\vec{B})\right] - \vec{B}\cdot\left[(\vec{C}\cdot\vec{\nabla})\,\hat{n}\right]$$

Next, $\left[(\vec{D}\times\vec{\nabla})\times\vec{B}\right]_i$

$$= \epsilon_{ijk}(\vec{D}\times\vec{\nabla})_j\, B_k = \epsilon_{ijk}\,\epsilon_{jlm}\, D_l\,\partial_m B_k$$

$$= \underbrace{\epsilon_{jki}\,\epsilon_{jlm}}_{\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}} D_l\,\partial_m B_k = D_k\,\partial_i B_k - D_i\,\partial_k B_k$$

$$= \partial_i(D_k B_k) - B_k\,\partial_i D_k - D_i(\vec{\nabla}\cdot\vec{B})$$

$$= \partial_i(\vec{D}\cdot\vec{B}) - D_i(\vec{\nabla}\cdot\vec{B}) - \vec{B}\cdot(\partial_i\vec{D})$$

$$= \left[\vec{\nabla}(\vec{D}\cdot\vec{B})\right]_i - \left[\vec{D}(\vec{\nabla}\cdot\vec{B})\right]_i - \vec{B}\cdot(\partial_i\vec{D})$$

$$\therefore\ \vec{C}\cdot\left[(\vec{D}\times\vec{\nabla})\times\vec{B}\right]$$

$$= C_i\left[(\vec{D}\times\vec{\nabla})\times\vec{B}\right]_i$$

$$= \vec{C}\cdot\left[\vec{\nabla}(\vec{D}\cdot\vec{B}) - \vec{D}(\vec{\nabla}\cdot\vec{B})\right] - \vec{B}\cdot\left(C_i\,\partial_i\,\vec{D}\right)$$

$$= \vec{C}\cdot\left[\vec{\nabla}(\vec{D}\cdot\vec{B}) - \vec{D}(\vec{\nabla}\cdot\vec{B})\right] - \vec{B}\cdot\left[(\vec{C}\cdot\vec{\nabla})\,\vec{D}\right]$$

So, for $\vec{D} = \hat{n}$, the above yields

$$\vec{C}\cdot\left[(\hat{n}\times\vec{\nabla})\times\vec{B}\right] = \left[(\vec{C}\cdot\vec{\nabla})\vec{B} - \vec{C}(\vec{\nabla}\cdot\vec{B})\right]\cdot\hat{n}$$

(Give this as take home assignment)

Therefore we have,

$$\iint_S \left[\vec{\nabla}\times(\vec{B}\times\vec{C})\right]\cdot d\vec{s} = \iint_S \vec{C}\cdot\left[(\hat{n}\times\vec{\nabla})\times\vec{B}\right] ds$$

$$= \vec{C}\cdot\iint_S \left[(\hat{n}\times\vec{\nabla})\times\vec{B}\right] ds$$

(As $\vec{C}$ is a constant vector)

On the other hand

$$\oint_C (\vec{B}\times\vec{C})\cdot d\vec{r} = \oint_C \vec{C}\cdot(d\vec{r}\times\vec{B})$$

$$= \vec{C}\cdot\oint_C d\vec{r}\times\vec{B}.$$

Now since $\vec{c}$ is completely arbitrary, equating the above two one obtains,

$$\iint_S \left[(\hat{n} \times \vec{\nabla}) \times \vec{B} \right] dS = \oint_C d\vec{r} \times \vec{B}.$$