1. Consider the propagation of an electromagnetic wave in a medium characterised by complex dielectric constant $\epsilon = \epsilon_r - i\epsilon_i$. An x-polarised electromagnetic wave propagating in the z-direction is given by $\vec{E} = E_0 e^{i(\omega t - kz)}\hat{x} = E_0 e^{-\gamma z} e^{i(\omega t - \beta z)}\hat{x}$. Calculate the magnetic field and the time averaged Poynting vector.

   Solution: For an x-polarised electromagnetic wave propagating in the z-direction, the electric field is

   $$\vec{E} = E_0 e^{i(\omega t - kz)}\hat{x} = E_0 e^{-\gamma z} e^{i(\omega t - \beta z)}\hat{x}$$

   The corresponding magnetic field can be obtained from Faraday's law

   $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}$$

   Also $\vec{\nabla} \times \vec{E} = \hat{y}\frac{\partial \vec{E}}{\partial z} = \hat{y}(-ik)E_0 e^{i(\omega t - kz)}$. Therefore, the magnetic field is

   $$\vec{B} = \hat{y}\frac{k}{\omega}E_0 e^{i(\omega t - kz)} = \hat{y}\frac{1}{\eta}E_0 e^{-\gamma z} e^{i(\omega t - \beta z)}$$

   where $\eta = \omega/k = \omega/(\beta - i\gamma) = |\eta| e^{i\theta}$ so that $|\eta| = \omega/\sqrt{\beta^2 + \gamma^2}$. Using $k^2 = \omega^2 \epsilon\mu = \omega^2\mu(\epsilon_r - i\epsilon_i)$ we can solve for β, γ by using the corresponding relations that follows from this. They are given by

   $$\beta^2 - \gamma^2 = \omega^2\mu\epsilon_r,\ 2\beta\gamma = \omega^2\mu\epsilon_i$$

   Using these, we can find the equation in terms of β as

   $$\beta^4 - \omega^2\epsilon_r\mu\beta^2 - \frac{1}{4}\omega^4\mu^2\epsilon_i^2 = 0$$

   which can be solved for β^2 to get

   $$\beta^2 = \frac{\omega^2\epsilon_r\mu}{2}\left[1 \pm \sqrt{1 + \left(\frac{\epsilon_i}{\epsilon_r}\right)^2}\right]$$

   where only the + will be realistic as β is real, by definition. Defining $g = \epsilon_i/\epsilon_r$, we can write it in compact form as

   $$\beta = \omega\sqrt{\frac{\mu\epsilon_r}{2}}\left[\sqrt{1 + g^2} + 1\right]^{1/2}$$

Similarly γ can be found as

$$\gamma = \omega\sqrt{\frac{\mu\epsilon_r}{2}}\left[\sqrt{1+g^2}-1\right]^{1/2}$$

Using these we can write

$$\beta^2+\gamma^2 = \frac{\omega^2\epsilon_r\mu}{2}(2\sqrt{1+g^2}) \implies \sqrt{\beta^2+\gamma^2} = \omega\sqrt{\mu\epsilon_r}(1+g^2)^{1/4}$$

Therefore, $|\eta| = (1+g^2)^{-1/4}/\sqrt{\mu\epsilon_r}$. And the phase is given by

$$\tan\theta = \frac{\gamma}{\beta} = \sqrt{\frac{\alpha-1}{\alpha+1}}, \alpha = \sqrt{1+g^2}$$

Since $1 < \alpha < \infty$, we get $0 < \tan\theta < 1 \implies 0 < \theta < \pi/4$.

In order to find the Poynting vector, we write the actual electromagnetic fields, which are given by

$$\vec{E} = E_0 e^{-\gamma z}\cos{(\omega t - \beta z)}\hat{x},\ \vec{B} = \frac{E_0}{\eta}e^{-\gamma z}\cos{(\omega t - \beta z - \theta)}\hat{z}$$

Time averaged Poynting vector is

$$\langle\vec{S}\rangle = \frac{1}{\mu}\langle\vec{E}\times\vec{B}\rangle = \frac{E_0^2}{\mu|\eta|}e^{-2\gamma z}\langle\cos{(\omega t-\beta z)}\cos{(\omega t - \beta z - \theta)}\rangle\hat{z}$$

Using $2\cos A\cos B = \cos A - B + \cos A + B$ and the fact that $\langle\cos\omega t\rangle = 0$, we find the time averaged Poynting vector to be simply

$$\langle\vec{S}\rangle = \frac{1}{\mu}\langle\vec{E}\times\vec{B}\rangle = \frac{E_0^2}{\mu|\eta|}e^{-2\gamma z}\cos\theta\hat{z}$$

Where we used $\langle\cos{(2\omega t - 2\beta z - \theta)}\rangle = 0$. Also, the phase angle θ can be found in terms of given quantities as

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \sqrt{\alpha^2-1} = g = \frac{\epsilon_i}{\epsilon_r}$$

2. Light of angular frequency ω passes from medium 1. through a slab (thickness d) of medium 2, and into medium 3 (for instance, from water through glass into air, as shown in figure 1). Show that the transmission coefficient for normal incidence is given by

$$T^{-1} = \frac{1}{4n_1n_3}\left[(n_1+n_3)^2 + \frac{1}{n_2^2}(n_1^2-n_2^2)(n_3^2-n_2^2)\sin^2\left(\frac{n_2\omega d}{c}\right)\right].$$

Hint: To the left, there is an incident wave and a reflected wave; to the right, there is

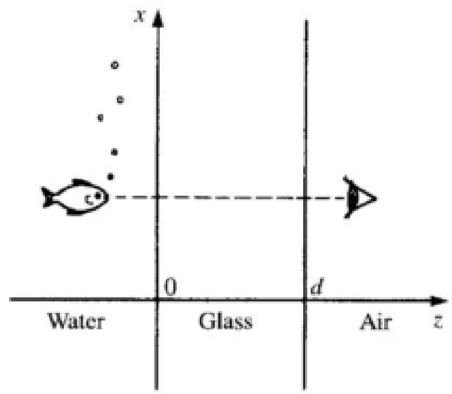

Figure 1: Figure for problem 2, 4.

a transmitted wave; inside the slab there is a wave going to the right and a wave going to the left. Express each of these in terms of its complex amplitude, and relate the amplitudes by imposing suitable boundary conditions at the two interfaces. All three media are linear and homogeneous; assume $\mu_1 = \mu_2 = \mu_3 = \mu_0$.

Solution: The electromagnetic fields in medium 1 ($z < 0$), medium 2 ($0 < z < d$) and

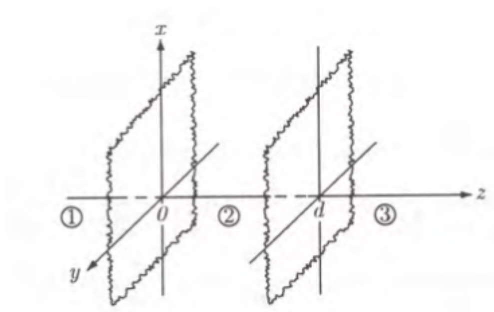

Figure 2: Figure for solution to problem 2.

medium 3 ($z > d$), are given by

$$z < 0 : \quad \begin{cases} \vec{\tilde{E}}_I(z,t) = \tilde{E}_I e^{i(k_1 z - \omega t)}\hat{x}, \ \vec{\tilde{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_I e^{i(k_1 z - \omega t)}\hat{y} \\ \vec{\tilde{E}}_R(z,t) = \tilde{E}_R e^{i(-k_1 z - \omega t)}\hat{x}, \ \vec{\tilde{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_R e^{i(-k_1 z - \omega t)}\hat{y}. \end{cases}$$

$$0 < z < d : \quad \begin{cases} \vec{\tilde{E}}_r(z,t) = \tilde{E}_r e^{i(k_2 z - \omega t)}\hat{x}, \ \vec{\tilde{B}}_r(z,t) = \frac{1}{v_2}\tilde{E}_r e^{i(k_2 z - \omega t)}\hat{y} \\ \vec{\tilde{E}}_l(z,t) = \tilde{E}_l e^{i(-k_2 z - \omega t)}\hat{x}, \ \vec{\tilde{B}}_l(z,t) = -\frac{1}{v_2}\tilde{E}_l e^{i(-k_2 z - \omega t)}\hat{y}. \end{cases}$$

$$z > d : \quad \left\{ \vec{\tilde{E}}_T(z,t) = \tilde{E}_T e^{i(k_3 z - \omega t)}\hat{x}, \ \vec{\tilde{B}}_T(z,t) = \frac{1}{v_3}\tilde{E}_T e^{i(k_3 z - \omega t)}\hat{y}. \right.$$

Assuming $\mu_1 = \mu_2 = \mu_3 = \mu_0$, the boundary conditions for the parallel components of the electromagnetic fields are $E_1^{\parallel} = E_2^{\parallel}, B_1^{\parallel} = B_2^{\parallel}$. For the two interfaces at $z = 0, z = d$, these boundary conditions are

$$z = 0 : \begin{cases} \tilde{E}_I + \tilde{E}_R = \tilde{E}_r + \tilde{E}_l \\ \frac{1}{v_1}\tilde{E}_I - \frac{1}{v_1}\tilde{E}_R = \frac{1}{v_2}\tilde{E}_r - \frac{1}{v_2}\tilde{E}_l \implies \tilde{E}_I - \tilde{E}_R = \beta(\tilde{E}_r - \tilde{E}_l),\ \beta = v_1/v_2. \end{cases}$$

$$z = d : \begin{cases} \tilde{E}_r e^{ik_2d} + \tilde{E}_l e^{-ik_2d} = \tilde{E}_T e^{ik_3d} \\ \frac{1}{v_2}\tilde{E}_r e^{ik_2d} - \frac{1}{v_2}\tilde{E}_l e^{-ik_2d} = \frac{1}{v_3}\tilde{E}_T e^{ik_3d} \implies \tilde{E}_r e^{ik_2d} - \tilde{E}_l e^{-ik_2d} = \alpha\tilde{E}_T e^{ik_3d},\ \alpha = v_2/v_3. \end{cases}$$

Adding the first two of the above boundary conditions, we can eliminate $\tilde{E}_R$:

$$2\tilde{E}_I = (1+\beta)\tilde{E}_r + (1-\beta)\tilde{E}_l.$$

Similarly, adding the last two of the above boundary conditions, we can eliminate $\tilde{E}_R$:

$$2\tilde{E}_r e^{ik_2d} = (1+\alpha)\tilde{E}_T e^{ik_3d}.$$

Subtracting the last two lead to the elimination of $\tilde{E}_r$:

$$2\tilde{E}_l e^{-ik_2d} = (1-\alpha)\tilde{E}_T e^{ik_3d}.$$

Putting the last two of the above three equations in the first, we get

$$2\tilde{E}_I = (1+\beta)\frac{1}{2}(1+\alpha)e^{-ik_2d}e^{ik_3d}\tilde{E}_T + (1-\beta)\frac{1}{2}(1-\alpha)e^{ik_2d}e^{ik_3d}\tilde{E}_T$$

$$\begin{aligned} 4\tilde{E}_I &= [(1+\alpha)(1+\beta)e^{-ik_2d} + (1-\alpha)(1-\beta)e^{ik_2d}]\tilde{E}_T e^{ik_3d} \\ &= [(1+\alpha\beta)(e^{-ik_2d} + e^{ik_2d}) + (\alpha+\beta)(e^{-ik_2d} - e^{ik_2d})]\tilde{E}_T e^{ik_3d} \\ &= 2[(1+\alpha\beta)\cos(k_2d) - i(\alpha+\beta)\sin(k_2d)]\tilde{E}_T e^{ik_3d} \end{aligned}$$

The transmission coefficient is defined as the ratio of transmitted intensity to the incident intensity:

$$T = \frac{I_T}{I_I} = \frac{\frac{1}{2}\epsilon_3 v_3|\tilde{E}_T|^2}{\frac{1}{2}\epsilon_1 v_1|\tilde{E}_I|^2} = \frac{v_3}{v_1}\frac{\mu_3\epsilon_3}{\mu_1\epsilon_1}\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2} \quad (\text{Using } \mu_1 = \mu_2 = \mu_3,\ v = \frac{1}{\sqrt{\mu\epsilon}})$$

$$T = \frac{v_1}{v_3}\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2} = \alpha\beta\frac{|\tilde{E}_T|^2}{|\tilde{E}_I|^2}$$

Using the relation between $\tilde{E}_I$ and $\tilde{E}_T$ derived above, we can now write down

$$\begin{aligned} T^{-1} &= \frac{1}{\alpha\beta}\frac{|\tilde{E}_I|^2}{|\tilde{E}_T|^2} = \frac{1}{\alpha\beta}\left|\frac{1}{2}[(1+\alpha\beta)\cos(k_2d) - i(\alpha+\beta)\sin(k_2d)]e^{ik_3d}\right|^2 \\ &= \frac{1}{4\alpha\beta}[(1+\alpha\beta)^2\cos^2(k_2d) + (\alpha+\beta)^2\sin^2(k_2d)] \\ &= \frac{1}{4\alpha\beta}[(1+\alpha\beta)^2 + (\alpha^2+2\alpha\beta+\beta^2-1-2\alpha\beta-\alpha^2\beta^2)\sin^2(k_2d)] \\ &= \frac{1}{4\alpha\beta}[(1+\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2d)]. \end{aligned}$$

Using the definition of refractive index: $n_1 = c/v_1, n_2 = c/v_2, n_3 = c/v_3$ and hence $\alpha = v_2/v_3 = n_3/n_2, \beta = v_1/v_2 = n_2/n_1$, we can write

$$\begin{aligned} T^{-1} &= \frac{n_1}{4n_3}\left[\left(1+\frac{n_3}{n_1}\right)^2 - \left(1-\frac{n_3^2}{n_2^2}\right)\left(1-\frac{n_2^2}{n_1^2}\right)\sin^2(k_2d)\right] \\ T^{-1} &= \frac{1}{4n_1n_3}\left[(n_1+n_3)^2 + \frac{1}{n_2^2}(n_1^2-n_2^2)(n_3^2-n_2^2)\sin^2(k_2d)\right]. \end{aligned}$$

Using $k_2 = \omega/v_2 = \omega n_2/c$, we get

$$T^{-1} = \frac{1}{4n_1n_3}\left[(n_1+n_3)^2 + \frac{1}{n_2^2}(n_1^2-n_2^2)(n_3^2-n_2^2)\sin^2\left(\frac{n_2\omega d}{c}\right)\right].$$

3. A microwave antenna is radiating at 10 GHz is to be protected from the environment by a plastic shield of dielectric constant 2.5. What is the minimum thickness of the shielding that will allow perfect transmission (assuming normal incidence)? Hint: Use the result for transmission coefficient in problem 2.

Solution: For perfect transmission $T = 1$. Using the expression from the solution to problem 2

$$T^{-1} = \frac{1}{4n_1n_3}\left[(n_1+n_3)^2 + \frac{1}{n_2^2}(n_1^2-n_2^2)(n_3^2-n_2^2)\sin^2(k_2d)\right]$$

it is obvious that T can be maximum if $\sin(k_2d) = 0 \implies kd = n\pi, n \in I$. The minimum non-trivial thickness is $d = \pi/k_2$. Also, $k_2 = \omega/v_2 = 2\pi\nu n_2/c$. The refractive index, for $\mu_1 = \mu_2 = \mu_3 = \mu_0$, is $n_2 = c/v_2 = \sqrt{\epsilon_2\mu_2/(\epsilon_0\mu_0)} = \sqrt{\epsilon_2/\epsilon_0} = \sqrt{\epsilon_r}$. Therefore,

$$d = \frac{\pi c}{2\pi\nu\sqrt{\epsilon_r}} = \frac{c}{2\nu\sqrt{\epsilon_r}} = \frac{3\times10^8}{2\times(10\times10^9)\sqrt{2.5}}\text{ m} = 9.49\times10^{-3}\text{ m} \approx 9.5\text{ mm}.$$

4. Light from an aquarium (figure 1) goes from water ($n = 4/3$) through a plane of glass

($n = 3/2$) into air ($n = 1$). Assuming it is a monochromatic plane wave and that it strikes the glass at normal incidence, find the minimum and maximum transmission coefficients. You can see the fish clearly, how well can it see you? Hint: Use the result for transmission coefficient in problem 2.

Solution: Using the expression from the solution to problem 2

$$T^{-1} = \frac{1}{4n_1n_3}\left[(n_1+n_3)^2 + \frac{1}{n_2^2}(n_1^2-n_2^2)(n_3^2-n_2^2)\sin^2\left(\frac{n_2\omega d}{c}\right)\right].$$

Now we have $n_1 = 4/3, n_2 = 3/2, n_3 = 1$. Using these in the above expression, we get

$$\begin{aligned}
T^{-1} &= \frac{1}{4(4/3)(1)}\left\{[(4/3)+1]^2 + \frac{1}{(9/4)}[(16/9)-(9/4)][1-(9/4)]\sin^2\left(\frac{3\omega d}{2c}\right)\right\} \\
&= \frac{3}{16}\left[\frac{49}{9} + \frac{4}{9}\left(-\frac{17}{36}\right)\left(-\frac{5}{4}\right)\sin^2\left(\frac{3\omega d}{2c}\right)\right] \\
&= \frac{49}{48} + \frac{85}{(48)(36)}\sin^2\left(\frac{3\omega d}{2c}\right).
\end{aligned}$$

Therefore,

$$T = \frac{48}{49 + (85/36)\sin^2\left(\frac{3\omega d}{2c}\right)}.$$

The maximum and minimum values of T will depend upon the value of $\sin^2\left(\frac{3\omega d}{2c}\right)$ which varies between 0 and 1. Thus,

$$T_{\min} = \frac{48}{49 + (85/36)} = 0.935, \quad T_{\max} = \frac{48}{49} = 0.980.$$

Thus, there is not much variation with frequencies and the overall transmission is good ($> 90\%$) for all frequencies. Since the expression for transmission coefficient derived in problem 6 remains same for $n_1 \leftrightarrow n_3$, the fish can see you as well as you see it.

5. Consider the incidence of a plane electromagnetic wave (with its electric field perpendicular to the plane of incidence x-z) at the interface of two dielectrics with $n_2 < n_1$. Assume the angle of incidence θ_1 to be greater than the critical angle θ_c. Calculate time averaged Poynting vector $\langle S_{2x}\rangle, \langle S_{2z}\rangle$ (where $\vec{S}_2$ is the Poynting vector associated with the transmitted wave) and interpret the results physically.

Solution: Let the electric field associated with the incident wave be

$$\vec{E}_1 = \hat{y}E_{10}e^{i(k_{1x}x + k_{1z}z - \omega t)}$$

The corresponding magnetic field can be found using Faraday's law

$$\vec{\nabla} \times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t} = i\omega\vec{B}_1$$

$$\implies \vec{B_1} = (-\hat{x}k_{1z} + \hat{z}k_{1x})\frac{E_{10}}{\omega}e^{i(k_{1x}x+k_{1z}z-\omega t)} = (-\hat{x}\sin\theta_1 + \hat{z}\cos\theta_1)\frac{k_1 E_{10}}{\omega}e^{i(k_{1x}x+k_{1z}z-\omega t)}$$

where we have used $k_{1x} = k_1\cos\theta_1, k_{1z} = k_1\sin\theta_1$. x-direction is considered to be the normal to the interface while x-z being the plane of incidence.

To find the Poynting vector, we use the real parts of the fields only. Therefore

$$\vec{E_1} = \hat{y}E_{10}\cos(k_{1x}x + k_{1z}z - \omega t), \vec{B_1} = (-\hat{x}k_{1z} + \hat{z}k_{1x})\frac{E_{10}}{\omega}\cos(k_{1x}x + k_{1z}z - \omega t)$$

Here $k_{1x} = \frac{\omega}{c}n_1\cos\theta_1, k_{1z} = \frac{\omega}{c}n_1\sin\theta_1$. Thus,

$$\langle S_{1x}\rangle = \frac{k_1 E_{10}^2}{\omega\mu_1}\cos\theta_1\langle\cos^2 k_{1x}x + k_{1z}z - \omega t\rangle = \frac{k_1 E_{10}^2}{2\omega\mu_1}\cos\theta_1$$

Similarly,

$$\langle S_{1z}\rangle = \frac{k_1 E_{10}^2}{2\omega\mu_1}\sin\theta_1.$$

As derived in lecture 26, the ratio of transmitted and incident amplitudes is (assuming $\mu_1 = \mu_2$)

$$\frac{E_{0T}}{E_{0I}} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

where $\theta_{1,2}$ are angle of incidence and transmission respectively. Now, $\sin\theta_2/\sin\theta_1 = n_1/n_2$ and for total internal reflection the critical angle is $\sin\theta_c = n_2/n_1$. Using these

$$\frac{n_2}{n_1}\cos\theta_2 = \sqrt{\sin^2\theta_c - \frac{n_2^2}{n_1^2}\sin^2\theta_2} = i\sqrt{\sin^2\theta_1 - \sin^2\theta_c}$$

$$\frac{E_{0T}}{E_{0I}} = \frac{2\cos\theta_1}{\cos\theta_1 + i\sqrt{\sin^2\theta_1 - \sin^2\theta_c}} = |\frac{E_{0T}}{E_{0I}}|e^{i\phi}$$

where

$$|\frac{E_{0T}}{E_{0I}}| = |t_\perp| = \frac{2\cos\theta_1}{\cos\theta_c}, \cos\phi = \frac{\cos\theta_1}{\cos\theta_c}, \sin\phi = -\frac{\sqrt{\sin^2\theta_1 - \sin^2\theta_c}}{\cos\theta_c}$$

The transmitted wave can now be written as

$$\vec{E_2} = \hat{y}E_{10}|t_\perp|e^{i(k_{2x}x+k_{2z}z-\omega t+\phi)}$$

where

$$k_{2x} = \frac{\omega}{c}n_2\cos\theta_2 = i\frac{\omega}{c}n_1\sqrt{\sin^2\theta_1 - \sin^2\theta_c} = i\kappa$$

$$k_{2z} = \frac{\omega}{c}n_2\sin\theta_2 = \frac{\omega}{c}n_1\sin\theta_1$$

Using these, the transmitted wave is written as

$$\vec{E_2} = \hat{y}E_{10}|t_\perp|e^{-\kappa x}e^{i(k_{2z}z-\omega t+\phi)}$$

whose real part is the actual field

$$\vec{E_2} = \hat{y}E_{10}|t_\perp|e^{-\kappa x}\cos\left(k_{2z}z - \omega t + \phi\right)$$

Using Faraday's law, we get

$$\begin{aligned}
-\frac{\partial \vec{B_2}}{\partial t} &= \vec{\nabla} \times \vec{E_2} = -\hat{x}\frac{\partial E_{2y}}{\partial z} + \hat{z}\frac{\partial E_{2y}}{\partial x} \\
&= \hat{x}k_{2z}E_{10}|t_\perp|e^{-\kappa x}\sin\left(k_{2z}z - \omega t + \phi\right) - \hat{z}\kappa E_{10}|t_\perp|e^{-\kappa x}\cos\left(k_{2z}z - \omega t + \phi\right) \\
\implies \vec{B_2} &= -\hat{x}\frac{k_{2z}}{\omega}E_{10}|t_\perp|e^{-\kappa x}\cos\left(k_{2z}z - \omega t + \phi\right) - \hat{z}\frac{\kappa}{\omega}E_{10}|t_\perp|e^{-\kappa x}\sin\left(k_{2z}z - \omega t + \phi\right)
\end{aligned}$$

Thus, B_{2z} is out of phase by $\pi/2$ with E_{2y}. Therefore, the time averaged Poynting vector $\langle S_{2x}\rangle = \langle E_{2y}B_{2z}\rangle/\mu_2 = 0$ which indicates that there is no power flow along the x-direction and therefore, the transmission coefficient is zero. This implies that the reflection is complete. The time averaged Poynting vector in the z direction is

$$\langle S_{2z}\rangle = \langle E_{2y}B_{2x}\rangle/\mu_2 = \frac{k_{2z}}{2\omega\mu_2}E_{10}^2|t_\perp|^2e^{-2\kappa x}$$

which indicates that there is a net power flow along the z axis. Indeed, when a spatially bounded beam is incident at an interface making an angle greater that the critical angle, then the beam undergoes a lateral shift (z-direction, in the given problem) which can be interpreted as the beam entering the rarer medium and reemerging (from the rarer medium) after reflection. This is known as the Goos-Hanchen shift, shown in figure 3.

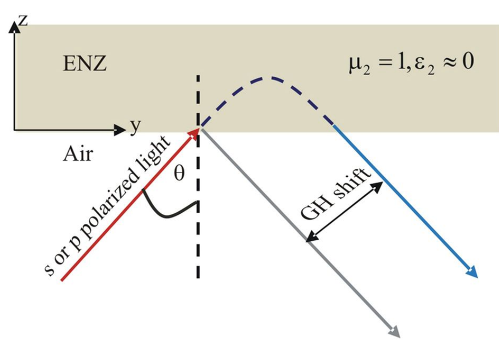

Figure 3: Figure for solution to problem 5. Credit: Nature. For the given problem, z-axis to be replaced by x while y-axis to be replaced by z-axis.