PH101: Physics 1

Module 2: Special Theory of Relativity - Basics

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RECAP

Lorentz transformation:

Relating space and time coordinates in frames S and S', where S' is moving with constant velocity $\vec{v}$ along the x-axis.

$$x' = \gamma(x - \beta x_0); \qquad y' = y; \quad z' = z \qquad x_0 = ct \qquad \beta = \frac{v}{c} \qquad \gamma = \sqrt{\frac{1}{1-\beta^2}}$$

$$x'_0 = \gamma(x_0 - \beta x)$$

The invariant interval is

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2$$
$$= c^2\,dt'^2 - dx'^2 - dy'^2 - dz'^2 = ds'^2$$

Light-like interval $(ds^2 = 0)$ remains like light-like in all inertial frames

Time-like interval $(ds^2 > 0)$ remains like time-like in all inertial frames

Space-like interval $(ds^2 < 0)$ remains like space-like in all inertial frames

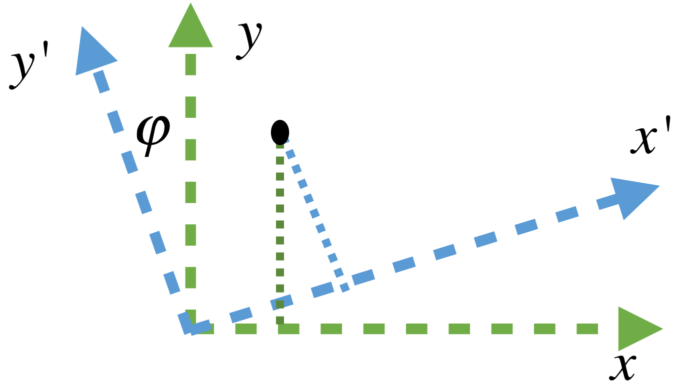
Transformation Matrix

Representing the Lorentz transformations in matrix form

First, consider Rotation in 2D

$$x' = x\cos\varphi + y\sin\varphi$$

$$y' = -x\sin\varphi + y\cos\varphi$$



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In Short notation $X' = R \cdot X$

Can we write the Lorentz transformation in a similar way?

$$x' = \gamma(x - \beta x_0); \qquad y' = y; \quad z' = z$$

$$x'_0 = \gamma(x_0 - \beta x)$$

$$\begin{pmatrix} x'_0 \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x \\ y \\ z \end{pmatrix}$$

$X' = L \cdot X$

Noting that $\gamma^2 = \dfrac{1}{1-\beta^2}$

we have $\gamma^2 - (\gamma\beta)^2 = 1$

one may parametrise: $\cosh\vartheta = \gamma, \quad \sinh\vartheta = \gamma\beta$

$$\begin{pmatrix} x'_0 \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\vartheta & -\sinh\vartheta & 0 & 0 \\ -\sinh\vartheta & \cosh\vartheta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x \\ y \\ z \end{pmatrix}$$

S’ moving in arbitrary direction (with constant velocity) (General Lorentz transformation)

$$t' = \gamma\left(t - \frac{\vec{r}\cdot\vec{v}}{c^2}\right), \qquad \vec{r}\,' = r + \left(\frac{\gamma - 1}{v^2}\vec{r}\cdot\vec{v} - \gamma t\right)\vec{v}$$

$$\beta_i = \frac{v_i}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{\vec{v}\cdot\vec{v}}{c^2}}}$$

$$\begin{pmatrix} x'_0 \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} x_0 \\ x \\ y \\ z \end{pmatrix}$$

Simultaneity

Consider two different frames S and S',
with S' moving along along the x direction with speed v with respect to S.

Consider two events with coordinates $\left(t_1, x_1, y_1, z_1\right)$ and $\left(t_2, x_2, y_2, z_2\right)$ in S

$\left(t'_1, x'_1, y'_1, z'_1\right)$ and $\left(t'_2, x'_2, y'_2, z'_2\right)$ in S'

Intervals of the two events be denote by: $dx = x_2 - x_1, \quad dt = t_2 - t_1$ etc.

Space and time intervals in the two frames are related by

$$dx' = \gamma\left(dx - \beta c\, dt\right); \qquad dy' = dy; \quad dz' = dz$$

$$c\, dt' = \gamma\left(c\, dt - \beta dx\right)$$

where

$$\beta = \frac{v}{c}, \qquad \gamma = \sqrt{\frac{c^2}{c^2 - v^2}}$$

Consider the events as simultaneous in S: $dt = t_2 - t_1 = 0$

The time interval of the same events seen from S': $dt' = \gamma\left(dt - \frac{\beta}{c} dx\right) = -\gamma \frac{\beta}{c} dx$

Two events which are simultaneous in one frame are not necessarily simultaneous in another inertial frame.

Length Contraction

Consider a stick at rest in S' kept along the x axis.

coordinates of the two ends: (t, x_1, y, z) and (t, x_2, y, z) in S

(noted at the same time)

$$(t_1', x_1', y' = y, z' = z) \text{ and } (t_2', x_2', y' = y, z' = z) \text{ in S'}$$

Length as measure in the rest frame of the stick (S'): dx'

$$dx' = \gamma(dx - \beta c\, dt) \quad \Rightarrow \quad dx = \frac{dx'}{\gamma} < dx'$$

Length as measured in the rest frame of the observer is less than the actual length of the rod as measured in its own rest frame

Note that if the rod is at rest relative to you you can measure the location of the ends at different times and still get the correct length, but if it is moving relative to you it is important that you locate the ends at the same time.

This is why $t_1 = t_2 = t$ but t_1' need not be equal to t_2'

Length contraction: Imagine a rod is placed on a table inside a spaceship S' moving relative to earth S with speed v.

The astronaut at rest relative to the rod wishes to measure the length of the rod. He could do one of two things.

i) Note down the location of the left end of the rod and right end of the rod at the same time according to the astronaut. This is guaranteed to give the correct length since the rod is not moving relative to the astronaut.

ii) Since the rod is not moving relative to the astronaut, there is another way to measure the length of the rod. He marks the location of the left end of the rod. Then he leaves the room goes to bed wakes up the next day, comes back to the table where the rod is still there and then locates the right end of the rod. The distance between the two marks is STILL THE CORRECT length of the rod since the rod was not moving when the astronaut was asleep.

iii) However for someone looking at the rod through a telescope on earth it is imperative that the location of the left end and right end of the rod be located AT THE SAME TIME according to the person watching the rod through the telescope. The person on earth does not have the luxury of marking the location of the left end of the rod, then going to bed, waking up and then marking the right end of the rod. This is because when the person on earth was asleep, the spaceship would have travelled a great distance and the right end of the rod will be really far away compared to the location of the left end of the rod noted by this observer the previous day.

Hence for length contraction:

$X'_R - X'_L$ is the correct length of the rod EVEN IF t'_R is not equal to t'_L.
(since the rod is at rest in the spaceship according to S')

But $X_R - X_L$ is NOT the correct length of the rod as seen by S UNLESS $t_R = t_L$ (since the rod is in a moving spaceship according to S). Note that this length is less that the “proper” length measured by S’.

Time Dilation

Consider clocks clicking (kept at the same spatial point)

$$dt = \gamma(dt' + v\,dx'/c^2) = \gamma dt'$$

$$\gamma = \sqrt{\frac{c^2}{c^2 - v^2}} > 1$$

$$dt > dt'$$

Clocks run slower in a moving frame, compared to the frame at rest.

Time (properly measured) always dilates, length (properly measured) always contracts.

To measure time, you have to freeze space (hold the clock at rest relative to you) and to measure length you have to freeze time (note down location of the two ends at the same time).

Example1: A microwave oven in a spaceship moving at speed 0.95c relative to the earth is set to heat a frozen pizza for 3 minutes. To an observer on the earth, for how long does the microwave oven heat the pizza?

Let S' be the reference frame on the spaceship and S be the reference frame on the earth.

Event **E1**: Start of the microwave oven

Event **E2:** Microwave oven switches off

The two events happen at the same spatial location in S' (let us set this to be the origin).

$$t'_1 = 0 = x'_1 = y'_1 = z'_1$$

$$t'_2 = 3m, \; x'_2 = y'_2 = z'_2 = 0$$

We also synchronise the coordinates of the two frames so that E1 happens as time $t_1 = 0$ and at the origin of the reference frames in both the cases.

$$t_1 = 0 = x_1 = y_1 = z_1$$

How does E2 look, when seen from the two frames?

	As seen from S':	As seen from S:
E_1 :	$(0,0,0,0)$	$(0,0,0,0)$
E_2 :	$(t'_2 = 3m,0,0,0)$	$(t_2,x_2,0,0)$

$$t = \gamma\left(t' + \beta x'\right) \qquad x = \gamma\left(x' + \beta t'\right)$$

$$\beta = \frac{v}{c} = 0.95, \qquad \gamma = \sqrt{\frac{c^2}{c^2 - v^2}} = 3.20256$$

$$t_2 = \gamma\left(t'_2 + \beta x'_2\right) = \gamma t'_2 = 3.20256 \times 3\, m$$

$$\Rightarrow t_2 = 9.60768\, m$$

The pizza heats up for more than 9 minutes !!

Example2: A Physics class on Earth lasts for 1 hour. An astronaut in a spaceship moving at speed 0.95c relative to the earth is keenly watching the class. For how long does the class last for the astronaut?

Let S' be the reference frame on the spaceship and S be the reference frame on the earth.

Event **E1**: Start of the class

Event **E2:** End of the class

The two events happen at the same spatial location in S (let us set this to be the origin).

$$t_1 = 0 = x_1 = y_1 = z_1$$

$$t_2 = 60m, \; x_2 = y_2 = z_2 = 0$$

We also synchronise the coordinates of the two frames so that E1 happens as time $t'_1 = 0$ and at the origin of the reference frames in both the cases.

$$t'_1 = 0 = x'_1 = y'_1 = z'_1$$

How does E2 look, when seen from the two frames?

	As seen from S':	As seen from S:
$E_1:$	$(0,0,0,0)$	$(0,0,0,0)$
$E_2:$	$(t'_2, x'_2, 0, 0)$	$(t_2 = 60m, 0, 0, 0)$

$$t' = \gamma\left(t - \frac{\beta}{c}x\right) \qquad x' = \gamma\left(x - \beta ct\right)$$

$$\beta = \frac{v}{c} = 0.95, \qquad \gamma = \sqrt{\frac{c^2}{c^2 - v^2}} = 3.20256$$

$$t'_2 = \gamma\left(t_2 - \frac{\beta}{c}x_2\right) = \gamma t_2 = 3.20256 \times 60\ m$$

$$\Rightarrow t_2 = 3.20256\ hrs.$$

The class goes on for more than 3 hours !!

Example3: Imagine an astronaut on the same spaceship (that we considered earlier today, moving with a speed 0.95c) is also an athlete and is practicing long jump. He jumps a distance of 6m in the direction of motion of the spaceship as measured by his friend, who is also on the spaceship. To an observer on the earth, how far does this athlete appear to jump?

A. After the jump the observer on earth measures the starting location E_1 and the finish location E_2 at the same time (according to him). It is important to measure the starting location and ending location at the same time when these locations are moving along with the spaceship to get the distance between them. But on the spaceship it is not important when you note down the starting and ending location you will still get the correct distance.

$$x_1 = y_1 = z_1 = t_1 = 0\,; \quad x_2 = ?\,, y_2 = z_2 = 0\,; t_2 = 0$$

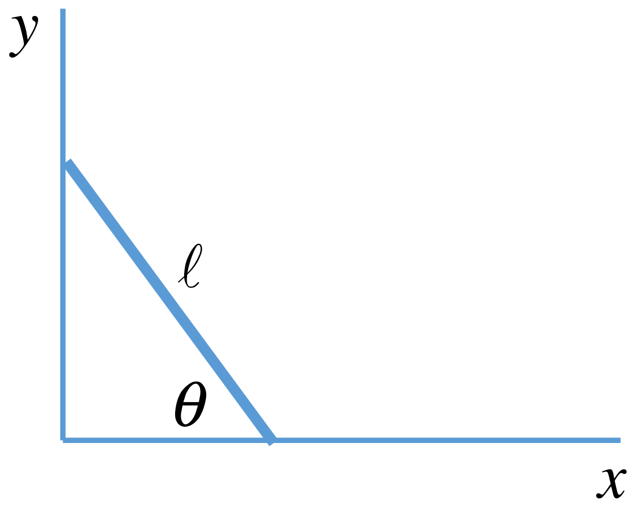
$$x_1' = y_1' = z_1' = 0; \quad x_2' = 6m,\, y_2' = z_2' = 0$$

$$x'_1 = \gamma\,(x_1 - \mathrm{v}\,t_1)\,; \qquad x'_2 = \gamma\,(x_2 - \mathrm{v}\,t_2)$$

$$6\,m = \gamma\,(x_2 - \mathrm{v} \times 0)\ ; \quad x_2 = \frac{6\,m}{3.20256} = 1.8735\,m$$

The observer on earth is not at all impressed by this jump whereas the athlete on the spaceship is celebrating his longest ever jump!

Example4: A long thin rod is kept on slanted on a wall so as to make an angle θ with the horizontal, as measured in the rest frame of the rod (on earth). The length of the rode as measured in this frame is ℓ. What is the angle, as measured from a frame moving with speed 0.7c along the horizontal x-axis (see figure)?



The two ends of the rod are to be measured simultaneously in the moving frame, S'.

Left end $(t'_L, x'_L, y'_L, 0)$

Right end $(t'_R, x'_R, y'_R, 0)$ with $t'_L = t'_R = t'$

As measured in the rest frame of the rod, (S):

Left end $(t_L, x_L, y_L, 0)$ $\quad x_L = \gamma\left(x'_L - \beta c t'\right)$ $\quad x_R - x_L = \gamma\left(x'_R - x'_L\right)$

Right end $(t_R, x_R, y_R, 0)$ $\quad x_R = \gamma\left(x'_R - \beta c t'\right)$ $\quad y_R - y_L = y'_R - y'_L$

$$\tan\theta' = \frac{y'_R - y'_L}{x'_R - x'_L} = \frac{\gamma\left(y_R - y_L\right)}{\left(x_R - x_L\right)} = \gamma \tan\theta$$

The angle will increase, as the perpendicular component remain the same, and the horizontal component undergoes length contraction.

We shall continue the discussion in the next class