

Name of student :	Anupriya Tiwari		Roll No.	102
Course No.		Signature of the student :		

$$A \cap B^c \cap C^c, \quad (A \cap B) \cup (A \cap C) \cup (B \cap C), \quad A \cap B \cap C^c, \\ (A^c \cup B^c) \cap (A^c \cup C^c) \cap (B^c \cup C^c).$$

? (c) No, $P(C)$ is not a probability. Because if it is then we get, $P(A \cup B) = P(A) + P(B)$.

$$1 = P(S) = \sum P(\xi_i) = \sum_{i=1}^{\infty} P(\xi_i = 3) = 10.$$

3. \Rightarrow (Since $U(S \in \mathcal{F}_2) \neq \emptyset$, $\exists S \in \mathcal{F}_2$ such that $S \in \mathcal{F}_2$ and $S \in U(S \in \mathcal{F}_2) \neq \emptyset$.)
~~Since~~ If $A \in \mathcal{F}_2$, then $A \in (\mathcal{F}_2 \setminus \{\emptyset\}) \cup \{A\}$.
 $\Rightarrow A^c \in \mathcal{F}_2 \setminus \{\emptyset\}$ [Since \mathcal{F}_2 is a σ -algebra]
 $\Rightarrow A \in \mathcal{F}_2$.

If $\{A_i\}_{i=1}^{\infty} \in \bigcap F_{\alpha}$, then $\{A_i\}_{i=1}^{\infty} \in F_{\alpha} \neq \emptyset$.

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\alpha \quad \forall \alpha, \exists \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\alpha$$

4. let $S = \mathbb{R}$, let $\mathcal{F}_1 = \{\emptyset, \mathbb{R}, (0, 2), (0, 2)'\}$

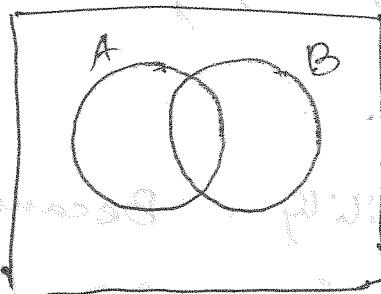
and let $\mathcal{F}_2 = \{\phi, R, (1, 3), (1, 3)^c\}$

Then $\mathcal{F}_1 \cup \mathcal{F}_2 = \{ \emptyset, \mathbb{R}, (0, 2), (1, 3), (0, 2)^c, (1, 3)^c \}$.

$(0, 2) \cap (1, 3) = (1, 2)$ is not a σ -algebra.

Thus $\mathcal{F}_1, \mathcal{F}_2$ is not a σ -algebra.

5. (a) $\{\emptyset, A, A^c, S\}$



(Step 1) Form a partition of S

using A & B

$A \cap B^c, A \cap B, B \cap A^c, A^c \cap B^c$

Step 2 Take all possible unions of the above sets. There will be total 16 sets.

$, A, A \cap B^c, A \cap B, B \cap A^c, A^c \cap B^c, A, (A \cap B^c) \cup (B \cap A^c),$
 $B^c, B, (A \cap B) \cup (A^c \cap B^c), A^c, A \cup B, (A \cap B)^c, B^c \cup A, A^c \cup B,$

Step 3

We prove by induction.

$$\text{Base step: } P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2)$$

Induction Hypothesis: True for $n-1$

Inductive Step: $P(A_1 \cup A_2 \cup \dots \cup A_n) \leq P(A_1 \cup \dots \cup A_{n-1}) + P(A_n)$

$$\leq \sum_{i=1}^n P(A_i)$$

(SAD, 1994, (14) Question 6) Thus (SAD, 1994, 6) Thus (SAD, 1994, 6)

7. Define $B_1 = A_1$, $B_2 = A_2 - A_1$, ..., $B_n = A_n - (A_1 \cup \dots \cup A_{n-1})$.

Signature of the student

Thus (B_i) are disjoint and $B_i \subset A_i$ for i and

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

$$\text{Thus } P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i)$$

$$\leq \sum_{i=1}^{\infty} P(A_i). \quad \begin{array}{l} \text{[Last inequality by} \\ \text{monotonicity property.]} \end{array}$$

8. We prove by induction. $P(A) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

Base step: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

Induction hypothesis: (True for $n-1$) $P(A_1 \cup \dots \cup A_{n-1}) = P(A_1) + \dots + P(A_{n-1})$

$$\text{Inductive Step: } P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1 \cup A_2 \cup \dots \cup A_{n-1})$$

$$+ P(A_n) - P\left(\bigcup_{i=1}^{n-1} A_i \cap A_n\right)$$

$$= P(A_1 \cup \dots \cup A_{n-1}) + P(A_n) - P\left(\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right)$$

$$= \sum_{i=1}^n P(A_i) - \sum_{i=1}^n \sum_{i_2=1}^{n-1} P(A_i \cap A_{i_2}) + \sum_{i_1 < i_2} (-1)^{n+1} P\left(\bigcap_{i=1}^n A_i\right)$$

9. ~~Left~~ \Rightarrow Right hand side inequality already shown in problem 6.

We use induction for Left hand inequality (by (a) \Rightarrow (b))?

Basis Step

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Induction Hypothesis: True for $m-1$, A_1, A_2, \dots, A_m events

Inductive Step: $P(\bigcup_{i=1}^m A_i) = P(\bigcup_{i=1}^{m-1} A_i) + P(A_m) - P(\bigcup_{i=1}^{m-1} A_i \cap A_m)$

$$\geq \sum_{i=1}^{m-1} P(A_i) + P(A_m) - \sum_{i=1}^{m-1} \sum_{j=1}^m A_i \cap A_j$$

$$= (A_1 \cup \dots \cup A_{m-1}) \cap (A_m \cup \dots \cup A_m) = (A_1 \cup \dots \cup A_{m-1}) \cup A_m$$

10. a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C)$

$$= P(B \cap C) + P(A \cap B \cap C) = 0.6 + 0.5 + 0.4 - 0.3 - 0.2$$

~~- 0.2 + 0.1~~ = 0.9

$P(A \cap B \cap C^c) = 1 - P(A \cup B \cup C) = 1 - 0.9 = 0.1$

b) $P((A \cup B) \cap C) = P(A \cup B) + P(C) - P(A \cup B \cup C)$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P(A \cup B \cup C)$$

~~= 0.6 + 0.5 + 0.4 - 0.3 - 0.9~~ = 0.3

$P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap B \cap C)$

$$= 0.6 + 0.2 - 0.1 = 0.7$$

c) $P((A^c \cup B^c) \cap C^c) = P(A^c \cup B^c) + P(C^c) - P(A^c \cup B^c \cup C^c)$
~~= 1 - P(A \cap B) + 1 - P(C) - (1 - P(A \cap B \cap C))~~

$$= 1 - 0.3 + 1 - 0.4 - 1 + 0.1 = 0.4$$

$((A^c \cap B^c) \cup C^c) = P(A^c \cap B^c) + P(C^c) - P(A^c \cap B^c \cap C^c)$

$$= 1 - (P(A) + P(B) - P(A \cap B)) + 1 - P(C) - 0.1 = 1 - 0.6 - 0.5 + 0.3 + 1 - 0.4 = 0.7$$

$$d) 0 \leq P(D \cap B \cap C) \leq P(B \cap D) = 0$$

$$0 \leq P(A \cap C \cap D) \leq P(C \cap D) = 0$$

$$e) P(A \cup B \cup D) = P(A) + P(B) + P(D) - P(A \cap B) - P(A \cap D) - P(B \cap D) + P(A \cap B \cap D) = 0.6 + 0.5 + 0.2 - 0.3 - 0.1 = 0.9$$

$$P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D) - P(A \cap B) - P(A \cap C) - P(A \cap D) - P(B \cap C) + P(A \cap B \cap C) = 0.6 + 0.5 + 0.4 + 0.2 - 0.3 - 0.1 - 0.1 + 0.1 = 1.0$$

$$f) P((A \cap B) \cup (C \cap D)) = P(A \cap B) + P(C \cap D) - P(A \cap B \cap C \cap D) = 0.3$$

$$II) P(A)P(B^c) - P(A \cap B^c)$$

$$= P(A)(1 - P(B)) - P(A \cap B^c) = P(A) - P(A \cap B^c) - P(A)P(B) = P(A \cap B) - P(A)P(B)$$

$$P(A^c)P(B) - P(A^c \cap B) = P(1 - P(A))P(B) - P(A^c \cap B) = P(B) - P(A^c \cap B) - P(A)P(B) = P(A \cap B) - P(A)P(B)$$

$$P((A \cup B)^c) - P(A^c)P(B^c)$$

~~$$= P((A \cup B)^c) - (1 - P(A))(1 - P(B))$$~~

$$= P((A \cup B)^c) - 1 + P(A) + P(B) - P(A)P(B)$$

$$= P(A) + P(B) - P(A \cup B) - P(A)P(B) = P(A \cap B) - P(A)P(B)$$

Q12

Total number of ways in which n persons stand in a row is $n!$

The r persons can be choose in $\binom{n-2}{r}$ ways from $(n-2)$ persons [leaving P_1 & P_2 there are $(n-2)$ persons].

These r persons can be arranged in $r!$ ways between P_1 & P_2 .

Now consider the unit from P_1 to P_2 (including r persons in between) as one ~~one~~ person, then we need to arrange $(n-2-r+1)$ persons in a row, and it can be performed in $(n-r-1)!$ ways.

Again P_1 & P_2 can change their position giving 2 ways to arrange make the row.

Hence the required probability is

$$\frac{2(n-r-1)! \binom{n-2}{r} r!}{n!} = \frac{2(n-r-1)!}{n(n-1)}.$$

□

Q13

Q13

There are 6^3 ways to choose three numbers from $\{1, 2, \dots, 6\}$. Root will be real if $b^2 \geq 4ac$. We have count in how many ways ~~it~~ it can happen.

(a, c)	Possible values of $b \rightarrow b^2 \geq 4ac$	No. of ways
$(1, 1)$	2, 3, 4, 5, 6	5
$(2, 1) \Delta (1, 2)$	3, 4, 5, 6	8
$(3, 1) \Delta (1, 3)$	4, 5, 6	6
$(4, 1) \Delta (1, 4)$	4, 5, 6	6
$(5, 1) \Delta (1, 5)$	5, 6	4
$(6, 1) \Delta (1, 6)$	5, 6	4
$(2, 2)$	4, 5, 6	3
$(3, 2) \Delta (2, 3)$	5, 6	4

(a, c)	Choice of $b \rightarrow b^2 \geq 4ac$	# ways.
$(1, 2) \& (2, 1)$	6	2
$(3, 3)$	6	1
	Total	43.

Hence the required probability is $\frac{43}{6^3}$. \square

Q15

Q14(a) From the set $\{1, 2, \dots, 50\}$, three numbers can be chosen in $\binom{50}{3}$ ways.

Three ~~real~~ real numbers will be in AP if

$a + c = 2b$. (Here $a < b < c$)
It means that if we choose $a & c$ such that the sum of $a + c$ is an even integer, then b can be found automatically (Taking $b = \frac{a+c}{2}$).

Now $a + c$ will be even if both $a & c$ are even or both $a & c$ are odd.

There are 25 evens and 25 odds in the set $\{1, 2, \dots, 50\}$.

Hence total no. of ways $a, b, & c$ can be drawn from the set $\{1, 2, \dots, 50\}$ such that ~~they are in AP~~ they are in AP is $\binom{25}{2} + \binom{25}{2}$.

The required probability is

$$\frac{2 \binom{25}{2}}{\binom{50}{3}} = \frac{600}{\binom{50}{3}}$$

\square

Q15(6)

Q14(b) For $a < b < c$ in GP we have $b = ar$ and $c = ar^2$, where r is the common ratio. We consider $r > 1$.

Here $1 \leq a < ar < ar^2 \leq 50 \Rightarrow 1 < r^2 \leq 50 \Rightarrow 1 < r \leq \sqrt{50}$.

Case-I: r is an integer.

$r \in \{2, 3, \dots, 7\}$. For each r , $1 \leq a \leq \frac{50}{r^2}$

r	Possible values of a	#
2	1, 2, 3, ... 12	12
3	1, 2, 3, 4, 5	5
4	1, 2, 3	3
5	1, 2	2
6	1	1
7	1	1
Total		24

Case-II r is not an integer.

Let $r = \frac{m}{n}$, $m & n$ are co-prime and $m > n > 1$.

$1 \leq a \leq a \frac{m}{n} \Rightarrow a \frac{m^2}{n^2} \leq 50$.

As $a \frac{m^2}{n^2}$ is an integer $\Rightarrow a$ is a multiple of n^2 .

For fixed $r = \frac{m}{n}$, we have $1 \leq a \leq \frac{50m^2}{n^2}$ & a is a multiple of n^2 .

r	Range of a	Possible values of a	#
$\frac{3}{2}$	$[1, 22.22]$	4, 8, 12, 16, 20	5
$\frac{5}{2}$	$[1, 8]$	4, 8	2
$\frac{7}{2}$	$[1, 4.08]$	4	1
$\frac{4}{3}$	$[1, 28.125]$	8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104, 112, 120, 128, 136, 144, 152, 160, 168, 176, 184, 192, 200, 208, 216, 224, 232, 240, 248, 256, 264, 272, 280, 288, 296, 304, 312, 320, 328, 336, 344, 352, 360, 368, 376, 384, 392, 400, 408, 416, 424, 432, 440, 448, 456, 464, 472, 480, 488, 496, 504	3
$\frac{5}{3}$	$[1, 18]$	9, 18	2
$\frac{7}{3}$	$[1, 9.18]$	9	1
$\frac{5}{4}$	$[1, 32]$	16, 32	2

r	Range for a	Possible values of a	#
7/4	[1, 16.33]	16	1
6/5	[1, 34.72]	25	1
7/5	[1, 25.51]	25	1
7/6	[1, 36.73]	36	1
		Total	20.

Hence the required probability is $\frac{24+20}{\binom{50}{3}} = \frac{44}{\binom{50}{3}}$. \square

15 The number of ways four groups (each of size 4) can be made is $\binom{16}{4} \binom{12}{4} \binom{8}{4} \binom{4}{4} / 4!$

The number of ways four groups each having one graduate student can be made is

$$\frac{4! \binom{12}{3} \binom{9}{3} \binom{6}{3} \binom{3}{3}}{4!}$$

Hence the required probability is

$$\frac{4! \cdot \frac{12!}{3! 9!} \times \frac{9!}{3! 6!} \times \frac{6!}{3! 3!} \times \frac{3!}{3! 1!} \times \frac{1!}{4!}}{\frac{16!}{4! 12!} \times \frac{12!}{4! 8!} \times \frac{8!}{4! 4!} \times \frac{4!}{3! 1!} \times \frac{1!}{4!}}$$

$$= \frac{4! \times 12! \times 9! 4! 4! 4! 4!}{16! 3! 3! 3! 3!}$$

$$= \frac{4 \times 3 \times 2 \times 4^4 \times 4^2}{16 \times 15 \times 14 \times 13} = \frac{2 \times 3 \times 4^3}{15 \times 14 \times 13}$$

\square

Q16 Define the event that the i^{th} letter is inserted into the i^{th} envelope by A_i . $i=1, 2, \dots, n$.

We need to find $P\left[\left(\bigcup_{i=1}^n A_i\right)^c\right] = 1 - P\left(\bigcup_{i=1}^n A_i\right)$.

Again $P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i) - \sum_{i < j} P(A_i \cap A_j) + \dots + (-1)^n P\left(\bigcap_{i=1}^n A_i\right)$

Here $P(A_i) = \frac{1}{n}$ $i=1, 2, \dots, n$.

for $i < j$, $P(A_i \cap A_j) = \frac{1}{n(n-1)}$.

for $i_1 < i_2 < \dots < i_m$, $P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_m}) = \frac{1}{n(n-1) \dots (n-m+1)}$

Hence the required probability is

$$1 - \left[n \times \frac{1}{n} - \binom{n}{2} \frac{1}{n(n-1)} + \binom{n}{3} \times \frac{1}{n(n-1)(n-2)} + \dots + (-1)^n \frac{1}{n!} \right]$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!}.$$

□

Three ~~and~~ digits can be drawn in 10^3 ways.

Q17

~~Two digits can be drawn in~~

~~Two distinct digits can be drawn in 10^2 ways.~~

~~Now there are~~

~~10^2 ways.~~

~~permutation can be made~~

~~using these two distinct digits~~

~~$10^2 \times 2 = 20$ ways.~~

Three distinct digits can be drawn in 10^3 ways.

Three same digits can be drawn in 10 ways.

Hence exactly two different digits can be drawn in

$$10^3 - (10 \times 9 \times 8 + 10) \text{ ways}$$

$$= 10^3 - 730$$

$$= 1000 - 730 = 270$$

Hence the required probability is

$$1 - \frac{270}{1000} = 0.73$$

□

Q18 If we arrange r balls in a row and then insert $(n-1)$ bars among the balls, we will get n groups of balls. These groups can be considered as the balls in different cells.

For example in the following picture there are $n=3$ cells and $r=4$ balls

$$\cdot \cdot | \cdot | \cdot \quad r_1=2, r_2=1, r_3=1$$

$$\cdot \cdot \cdot \cdot | | \quad r_1=4, r_2=0, r_3=0.$$

(a) To obtain the distinguishable distⁿ, we have r balls and $(n-1)$ bars. Hence if we have $(n-1+r)$ places and if we choose r places to put the balls and rest $(n-1)$ places to put the bars, we will have distinguishable distributions. Clearly it can be done in $\binom{n+r-1}{r}$ ways. \square

(b) If we arrange the ball in a row, there will be $(r-1)$ places in between. Now if we put $(r-1)$ bars in ~~these~~ places among these $(r-1)$ places then we get no empty cells. Clearly it can be done in $\binom{r-1}{n-1}$. \square

(c) The required probability is $\begin{cases} \frac{\binom{r-1}{n-1}}{\binom{n+r-1}{r}} & \text{for } r \neq n \\ 0 & \text{O.W.} \end{cases}$ \square

Q19 There are $n!$ ways to arrange n keys. The right key will be found in R^{th} trial if the right key is in the R^{th} position. There are $(n-1)!$ ways to arrange n keys such that right key is in the R^{th} position. Hence the required prob is $\frac{1}{n}$. \square

Q20 The no. of ways in which two pairs can be joined
joined is $(2n-1)(2n-3)\dots 1 = \frac{(2n)!}{2^n n!}$
Hence the required probability is $\frac{2^n n!}{(2n)!}$. \square

- Q21
- (a) $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{4}$, $P(A \cap B) = \frac{1}{16}$. Hence A & B are indep. \square
- (b) $P(A) = \frac{1}{4}$, $P(B) = \frac{4}{16} = \frac{1}{4}$, $P(A \cap B) = \frac{1}{16}$. Hence A & B are indep. \square
- (c) $P(A) = \frac{3}{16}$, $P(B) = \frac{4}{16} = \frac{1}{4}$, $P(A \cap B) = \frac{1}{16}$. Hence A & B are not indep. \square

Q22 $A \cap B \cap C = \left(\frac{1}{4}, \frac{1}{2}\right) \Rightarrow P(A \cap B \cap C) = \frac{1}{4} \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{8}$.
 $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$, $P(C) = \frac{2}{3}$. Hence $P(A \cap B \cap C) = P(A)P(B)P(C)$.
 $A \cap B = \left(\frac{1}{4}, \frac{1}{2}\right) \Rightarrow P(A \cap B) = \frac{1}{4} \neq \frac{1}{2} \times \frac{3}{4} = P(A)P(B)$. \square

Q23 Trivial.

Q24 Trivial.

Q25

(a) $P(A \cap B | C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} \times \frac{P(B \cap C)}{P(C)} = P(A | B \cap C)P(B | C)$. \square

* Hence proved.

(b) This statement can be disproved using the problem 23. \square

Q26

$$P\left(\bigcap_{i=1}^n A_i^c\right) = \prod_{i=1}^n P(A_i^c) \quad \text{as } A_1, A_2, \dots, A_n \text{ are independent.}$$

$$= \prod_{i=1}^n (1 - P(A_i))$$

$$\leq \prod_{i=1}^n e^{-P(A_i)} = e^{-\sum_{i=1}^n P(A_i)} \quad \square$$

Q27

(a) Consider ~~the~~ two independent losses of a fair coin.

Let $A = \{HT, HH\}$, $B = \{TT\}$, $C = \{TH, HT\}$.

Then $P(A \cap B) = 0$, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{4}$. Hence A & B are (-ve)ly associated.

$P(B \cap C) = 0$, $P(C) = \frac{1}{2} \Rightarrow$ B & C are (-ve)ly associated.

$P(A \cap C) = \frac{1}{4} = P(A)P(C) \Rightarrow$ A & C are not (-ve)ly associated. \square

(b) Let $A = \{HH, TH, HT\}$

Let $A = \{HT, HH\}$ $B = \{HH, TH, HT\}$, $C = \{TH, HT\}$.

$P(A) = \frac{1}{2} = P(C)$ and $P(B) = \frac{3}{4}$.

$P(A \cap B) = \frac{1}{2}$, $P(B \cap C) = \frac{1}{2}$

$P(A \cap B) > P(A)P(B) \wedge P(B \cap C) > P(B)P(C)$.

Hence A & B are (+ve)ly associated and B & C are (+ve)ly associated.

Now $P(A \cap C) = \frac{1}{4} = P(A)P(C)$.

\Rightarrow A & C are not positively associated. \square

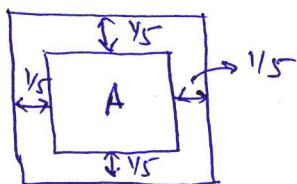
Q28

Given that $P(A \cap B) > P(A)P(B)$.

Given that $P(A \cap B^c) = P(A) - P(A \cap B) < P(A)P(B^c)$. \square .

Now $P(A \cap B^c) = P(A) - P(A \cap B) < P(A)P(B^c)$.

Q29

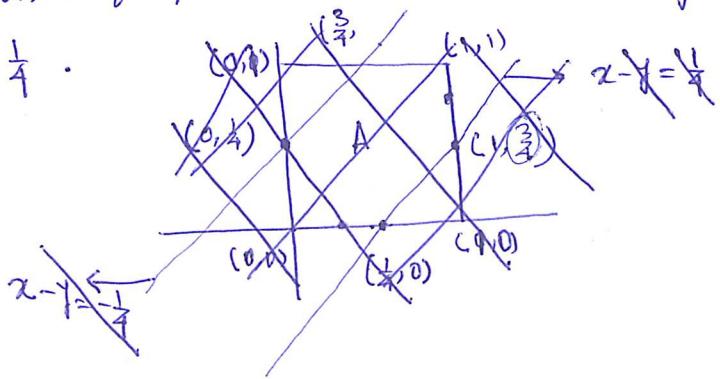
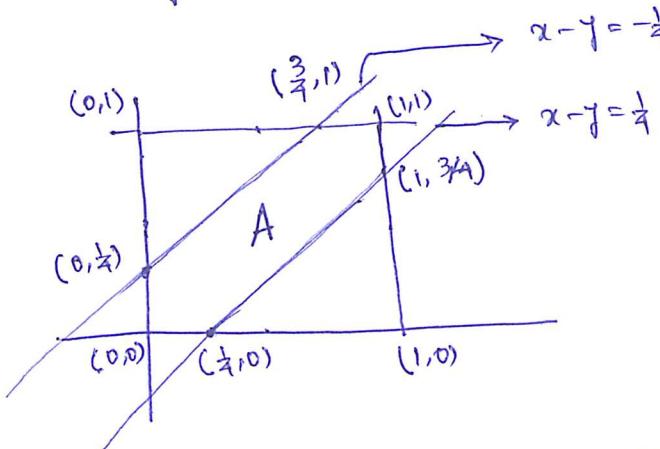


With respect to the diagram, the required probability is

$$\frac{\text{Area of } (S - A)}{\text{Area of } S} = \frac{1 - \left(\frac{3}{5}\right)^2}{1} = \frac{16}{25}. \quad \square$$

[Q30]

Let, x & y be the arrival time of person 1 & 2, respectively, where $0 \leq x \leq 1$ and $0 \leq y \leq 1$. They will meet if $|x-y| \leq \frac{1}{4}$.



$$\frac{\text{Area of } A}{\text{Area of } [0,1]^2}$$

Using the figure above the required prob is

$$= 1 - 2 \times \frac{1}{2} \times \frac{3}{4} \times \frac{3}{4}$$

$$= \frac{7}{16}.$$

□

[Q31]

Either he wins in first game or he loses the 1st and wins the 2nd.
Hence the required prob is $0.5 + (0.5)^2 = \frac{3}{4}$.

[Q32] [Q33]

Let E be the event that sum total is at least 4.
Let A_i be the event that the 1st roll results in ~~{1, 2, 3, 4, 5, 6}~~ and $i = 1, 2, \dots, 4$.

~~{3, 4, 5, 6}~~, respectively.

The required prob is

$$P(E) = P(E|A)P(A) + P(E|B)P(B)$$

$$P(E) = \sum_{i=1}^{14} P(E|A_i)P(A_i)$$

$$= \frac{1}{6} \times \frac{1}{6} + \frac{2}{6} \times \frac{1}{6} + \frac{3}{6} \times \frac{1}{6} + 0 \times \frac{1}{6} + 1 \times \frac{1}{6}$$

$$= \frac{2}{36} \times \frac{1}{4} + \frac{3}{36} \times \frac{1}{4} + 0 \times \frac{1}{4} + 1 \times \frac{1}{4}$$

$$= \frac{9}{16}.$$

□

Q34

Let E_i be the event that the student is upto date after the week i , $i = 1, 2, 3$.
Let E_0 be the event that the student is upto date ~~at~~ at the beginning.
Then $P(E_0) = 1$.

$$\text{Now } P(E_1) = P(E_1|E_0)P(E_0) = 0.8.$$

$$P(E_2) = P(E_2|E_1)P(E_1) + P(E_2|E_1^c)P(E_1^c)$$
$$= 0.8 \times 0.8 + 0.4 \times 0.2$$

$$= 0.24$$

$$P(E_3) = P(E_3|E_2)P(E_2) + P(E_3|E_2^c)P(E_2^c)$$
$$= 0.8 \times 0.24 + 0.4 \times 0.76$$

$$= 0.496.$$

□.

Q32

The required probability is

$$\binom{3}{1} \left(\frac{1}{2}\right)^3 + \binom{3}{2} \left(\frac{1}{2}\right)^3 = \frac{3}{4}.$$

□

3.5: Let C_i , $i=1, 2, 3$, denote the events that the car is behind door i .

Let X_i , $i=1, 2, 3$, denote the events that you choose door i in the beginning.

Let D_i , $i=1, 2, 3$, denote the events that the host opens door i .

$$\text{Then } P(D_3 | C_1, X_1) = \frac{1}{2} \quad P(D_2 | C_1, X_1) = \frac{1}{2}$$

$$P(D_3 | C_2, X_1) = 1 \quad P(D_2 | C_2, X_1) = 0$$

$$P(D_3 | C_3, X_1) = 0 \quad P(D_2 | C_3, X_1) = 1$$

$$P(C_i) = \frac{1}{3}, \quad P(D_3 | X_1) = \frac{1}{2}, \quad P(D_2 | X_1) = \frac{1}{2}$$

$$\begin{aligned} \text{Now } P(C_1 | D_3, X_1) &= \frac{P(C_1 \cap D_3 \cap X_1)}{P(D_3 \cap X_1)} = \frac{P(D_3 | C_1, X_1)}{\underbrace{P(C_1, X_1)}_{P(D_3 \cap X_1)}} \\ &= \frac{\frac{1}{2} \cdot P(C_1) P(X_1)}{P(D_3 | X_1) P(X_1)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}. \end{aligned}$$

$$\begin{aligned} P(C_2 | D_3, X_1) &= \frac{P(C_2 \cap D_3 \cap X_1)}{P(D_3 \cap X_1)} = \frac{P(D_3 | C_2, X_1) P(C_2, X_1)}{P(D_3 \cap X_1)} \\ &= \frac{P(C_2) P(X_1)}{P(D_3 | X_1) P(X_1)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}. \end{aligned}$$

$$\text{Sim. } P(C_1 | D_2, X_1) = \frac{1}{3}, \quad P(C_3 | D_2, X_1) = \frac{2}{3}.$$

Thus winning probability increases with switching.

36. The required event will happen, if in the next three times, 2 black balls are placed and 1 white ball is placed. Thus the required probability is

$$\binom{3}{2} \cdot \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} = 3 \cdot \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} = \frac{4}{9}.$$

37. Let A_{ij} denote the event that j th machine produces code i , where $i=0, 1$, $j=1, 2, 3, 4$.

Using Bayes Theorem, the required probability is

$$P(A_{01} | A_{14}) = \frac{P(A_{14} | A_{01}) P(A_{01})}{P(A_{14} | A_{01}) P(A_{01}) + P(A_{14} | A_{11}) P(A_{11})}$$

Now each code, gets changed maximum 3 times. If M_1 produces code 0 and M_4 produces code 1, then the code has changed odd number of times. Sim., if both M_1 & M_4 produce code 1 then the code has changed even number of times. Thus

$$P(A_{14} | A_{01}) = \binom{3}{1} \left(\frac{3}{4}\right) \cdot \left(\frac{1}{4}\right)^2 + \binom{3}{3} \left(\frac{3}{4}\right)^3$$

$$P(A_{11} | A_{11}) = \binom{3}{2} \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right) + \binom{3}{0} \left(\frac{1}{4}\right)^3.$$

$$\text{Thus } P(A_{01} | A_{14}) = \frac{3}{10}.$$

38- a) $P(B \cap C \cap P \cap M) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} = \frac{1}{120}.$

b) $P(B^c \cap C^c \cap P^c \cap M^c) = \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{5}\right)$
 $= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{24}{120} = \frac{1}{5}.$

$$\begin{aligned}
 \text{c)} \quad & P(B \cap C^c \cap P^c \cap M^c) + P(B^c \cap C \cap P^c \cap M^c) + P(B^c \cap C^c \cap P \cap M) \\
 & + P(B^c \cap C^c \cap P^c \cap M) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \\
 & + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} = \frac{24+12+8+6}{120} \\
 & = \frac{50}{120} = \frac{5}{12}.
 \end{aligned}$$

$$\begin{aligned}
 \text{d)} \quad & P(B \cap C \cap P \cap M^c) + P(B \cap C^c \cap P \cap M^c) + P(B \cap C^c \cap P^c \cap M) + \\
 & P(B^c \cap C \cap P \cap M^c) + P(B^c \cap C \cap P^c \cap M) + P(B^c \cap C^c \cap P \cap M) \\
 = \quad & \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} \\
 \text{e)} \quad & \cancel{P(B \cap C \cap P \cap M^c)} + \cancel{P(B \cap C^c \cap P \cap M^c)} + \cancel{P(B \cap C^c \cap P^c \cap M)} + \cancel{P(B^c \cap C \cap P \cap M^c)} + \cancel{P(B^c \cap C \cap P^c \cap M)} + \cancel{P(B^c \cap C^c \cap P \cap M)} = \frac{12+8+6+4+3+2}{120} = \frac{35}{120}.
 \end{aligned}$$

$$\text{e)} \quad P(B \cup C \cup P \cup M) = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\begin{aligned}
 39. \quad P(H_j | F_j) &= \frac{P(H_j \cap F_j)}{P(F_j)} = \frac{P(F_j | H_j)P(H_j)}{P(F_j | H_j)P(H_j) + P(F_j | H_j^c)P(H_j^c)} \\
 &= \frac{\gamma_j p_j}{\gamma_j p_j + (1-p_j)} \leq P(H_j) = p_j
 \end{aligned}$$

$$\left. \begin{array}{l} \cancel{\gamma_j (1-p_j) < (1-p_j)} \\ \Rightarrow \gamma_j < \gamma_j p_j + (1-p_j) \\ \Rightarrow \frac{\gamma_j}{\gamma_j p_j + (1-p_j)} < 1 \end{array} \right\} \quad \begin{array}{l} P(H_i | F_j) = \frac{p_i}{\gamma_j p_j + (1-p_j)} \\ \cancel{p_i = P(H_i)} \end{array}$$

40. Let A_n be the event that a 6 does not show up after n rolls. Then $P(A^c) \leq P(A_n) = \left(\frac{5}{6}\right)^n$

Thus $P(A^c) = 0$.

$$41. P(\text{disease positive}) = \frac{P(\text{positive} | \text{disease}) P(\text{disease})}{P(\text{positive} | \text{disease}) P(\text{disease}) + P(\text{positive} | \text{no disease}) \times P(\text{no disease})}$$

$$= \frac{\frac{95}{100} \cdot \frac{5}{1000}}{\frac{95}{100} \cdot \frac{5}{1000} + \frac{1}{100} \cdot \frac{995}{1000}} = \frac{475}{1470} = \frac{95}{294}.$$

$$42. P(\text{aircraft} | \text{alarm}) = \frac{P(\text{alarm} | \text{aircraft}) P(\text{aircraft})}{P(\text{alarm} | \text{aircraft}) P(\text{aircraft}) + P(\text{alarm} | \text{no aircraft}) \times P(\text{no aircraft})}$$

$$= \frac{\frac{99}{100} \cdot \frac{5}{100}}{\frac{99}{100} \cdot \frac{5}{100} + \frac{10}{100} \times \frac{95}{100}} = \frac{495}{1445} = \frac{99}{229}$$

$$P(\text{no aircraft} \cap \text{alarm}) = P(\text{alarm} | \text{no aircraft}) P(\text{no aircraft})$$

$$= \frac{10}{100} \times \frac{\frac{19}{2}}{\frac{95}{100}} = \frac{19}{200}.$$

$$P(\text{aircraft} \cap \text{no alarm}) = P(\text{no alarm} | \text{aircraft}) P(\text{aircraft})$$

$$= \frac{1}{100} \cdot \frac{\frac{8}{20}}{\frac{95}{100}} = \frac{1}{2000}.$$

$$\begin{aligned}
 43: P(\text{Disease} \mid \text{positive}) &= \frac{P(\text{positive} \mid \text{disease}) P(\text{disease})}{P(\text{positive})} \\
 &= \frac{\cancel{95}}{100} \cdot \frac{1}{1000} = \frac{\cancel{95}}{1000} \cdot \frac{1}{1000} = \frac{95}{100000} \\
 &= \frac{95}{100} \cdot \frac{1}{1000} + \frac{5}{100} \cdot \frac{999}{1000} = \frac{95}{100000} + \frac{4995}{100000} = \frac{5090}{100000} \\
 &= \frac{19}{1018}
 \end{aligned}$$

$$44. \text{ a) } 1 - \prod_{i=1}^n (1 - p_i(t))^*$$

$$\text{b) } \frac{n}{\prod_{i=1}^n p_i(t)}$$

$$\text{c) } \sum_{i=k}^n \binom{n}{i} (p(t))^i (1 - p(t))^{n-i}$$

$$\begin{aligned}
 45. \text{ Required probability is } & (0.75)(0.95) \\
 & + (0.90)(0.80)(0.90) + (0.90)(0.95)(0.85) - (0.75)(0.95) \\
 & (0.90)(0.80)(0.90) - (0.75)(0.95)(0.90)(0.95)(0.85) \\
 & - (0.90)(0.80)(0.90)(0.95)(0.85) + (0.75)(0.95)(0.90) \\
 & (0.80)(0.90)(0.95)(0.85) = 0.957
 \end{aligned}$$