Lecture 30 Continuous time Markov process 1

CTMP

- So far we discussed Markov chains where the state space is discrete.

DTMC , CTMC

- In many practical situations, the state space may be continuous within a possible range. The motion of dust particles in air is an example. In such situations the *continuous time continuous state Markov process* or simply *continuous time Markov process* (CTMP) model may help.

Consider ia Markov Process $\{X(t), t \in \Gamma\}$ with the state space $V = \mathbb{R}$. Here Γ is continuous and the state transition can take at any instant of time t. Similarly, V is continuous means that transition can occur to any real value in V. The state transitions are now characterized by the *transition probability density function*.

Suppose the process is at state x_0 at time $t = t_0$. The state transition probability density function at t is given by $f_{X(t)/X(t_0)}(x / x_0)$. For notational simplicity, we denote this pdf by $f(x,t / x_0,t_0)$.

$f_{X(t)|X(t_0)}(x|x_0)$

Further assume that the process is homogeneous.

Chapman-Kolmogorov equation

The state transition PDF $f(x_1, t_1 / x_0, t_0)$ can be obtained as

$$f(x_1, t_1 / x_0, t_0) = \int_{-\infty}^{\infty} f(x, t / x_0, t_0)\, f(x_1, t_1 / x, t)\, dx$$

x_0 t_0 x t x_1 t_1

Proof:

Consider the random variables $X(t)$ at a time t and $X(t_1)$ at a time $t_1 > t$. Given $X(t_0) = x_0$, the conditional joint PDF of $X(t_1)$ and $X(t)$ is $f(x_1, t_1; x, t / x_0, t_0)$.

Then the marginal density $f(x_1, t_1 / x_0, t_0)$ can be obtained from as:

$$f(x_1, t_1 / x_0, t_0) = \int_{-\infty}^{\infty} f(x, t; x_1, t_1 / x_0, t_0)\, dx$$

Using the chain rule and the Markov property, we get

$$\begin{aligned} f(x_1, t_1 / x_0, t_0) &= \int_{-\infty}^{\infty} f(x, t / x_0, t_0)\, f(x_1, t_1 / x, t, x_0, t_0)\, dx \\ &= \int_{-\infty}^{\infty} f(x, t / x_0, t_0)\, f(x_1, t_1 / x, t)\, dx \end{aligned}$$

$f(x_1, t \mid x_0, t_0)$

$= \int_{-\infty}^{\infty} f(x, t \mid x_0, t_0)\, f(x_1, t_1 \mid x, t)\, dx$

Probabilistic Evolution

- We have to know how the process evolves. Similar to the Kolmogorov forward and backward equations for the evolution of the CTMC, we can get those equations for a continuous time Markov process. Note that both time and state change continuously.
- The probabilistic evolution will be a partial differential equation (PDE). Particularly, the the corresponding forward Kolmogorv equation is known as the Fokker Planck (FP) equations.

$$\frac{\partial f(x,t/x_0,t_0)}{\partial t} = -\mu(x,t)\frac{\partial f(x,t/x_0,t_0)}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f(x,t/x_0,t_0)}{\partial x^2}$$

where $\mu(x,t) = \lim_{\Delta t \to 0} \frac{E((X(t+\Delta t) - X(t))/X(t)=x)}{\Delta t}$ and

$$\sigma^2(x,t) = \lim_{\Delta t \to 0} \frac{E((X(t+\Delta t) - X(t))^2/X(t)=x)}{\Delta t}$$

- We omit the derivation of the FP equation here.

FP Equations

The FP equation is given by:

$$\frac{\partial f(x,t/x_0,t_0)}{\partial t} = -\mu(x,t)\frac{\partial f(x,t/x_0,t_0)}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f(x,t/x_0,t_0)}{\partial x^2}$$

where $\mu(x,t) = \lim_{\Delta t \to 0} \frac{E((X(t+\Delta t) - X(t))/X(t) = x)}{\Delta t}$ and

$$\sigma^2(x,t) = \lim_{\Delta t \to 0} \frac{E((X(t+\Delta t) - X(t))^2 / X(t) = x)}{\Delta t}$$

- Note that the FP equations are linear PDE with the time and space varying coefficients. The solution is generally difficult.
- The FP equation has diverse applications as in the dispersion of suspended particles, the dynamics of electrons in a semiconductor, aeronautics, image processing and stochastic finance

Diffusion Equation

When $\mu(x,t)$ and $\sigma^2(x,t)$ are constants, the FP equation simplifies to the *diffusion equation* is given by:

$$\frac{\partial f(x,t/x_0,t_0)}{\partial t} = -\mu \frac{\partial f(x,t/x_0,t_0)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f(x,t/x_0,t_0)}{\partial x^2}$$

with μ and σ^2 respectively known as the drift and the diffusion coefficients. For the Wiener process, the transition pdf follows the above PDE.

Theorem: Considering $t_0 = 0$ and $x_0 = 0$ the solution to the diffusion equation

$$\frac{\partial f(x,t/x_0,t_0)}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 f(x,t/x_0,t_0)}{\partial x^2}$$

is given as

$$f(x,t/x_0=0,t_0=0) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2 t}\right)}$$

$X(0) = 0$ $\quad t_0 = 0 \quad x_0 = 0$

Gaussian process with the variance $\sigma^2 t$.

Thus the transition PDF is Gaussian with time-varying variance. With partial differentiations of $f(x,t/x_0=0,t_0=0)$ with respect to t and x it is easy to show that the above Gaussian PDF satisfies the diffusion equation.

Proof:

Consider the diffusion equation

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 f(x,t)}{\partial x^2}$$

with initial condition $X(0) = 0$ with probability 1.

$f_{X(0)}(x) = \delta(x)$

$$\therefore f(x,0) = \delta(x,0)$$

We can solve the above PDE with the given initial condition using the Fourier transform

Proof.

Let $Y\left(\omega,t\right) = FT(f(x,t)) = \int\limits_{-\infty}^{\infty} f(x,t)e^{-j\omega x}dx$

$$\text{Then, } FT\left[\frac{\partial f\left(x,t\right)}{\partial t}\right] = \int\limits_{-\infty}^{\infty} \frac{\partial f\left(x,t\right)}{\partial t} e^{-j\omega x}dx$$

$$= \frac{\partial}{\partial t}\int\limits_{-\infty}^{\infty} f(x,t)e^{-j\omega x}dx$$

$$\therefore FT\left[\frac{\partial f\left(x,t\right)}{\partial t}\right] = \frac{\partial}{\partial t} Y\left(\omega,t\right)$$

$$\text{Similarly, } FT\left(\frac{\partial^2}{\partial x^2} f(x,t)\right) = \int\limits_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} f(x,t)e^{-j\omega x}dx$$

$$= \frac{\partial}{\partial x} f(x,t)e^{-j\omega x}\Big]_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty} \frac{\partial f(x,t)}{\partial x}\left(-j\omega\right)e^{-j\omega x}dx$$

Proof.

Note that $\lim_{x\to\infty} F(x,t) = 1$ and $\lim_{x\to-\infty} F(x,t) = 0$ (constants).

$$\therefore \frac{\partial f}{\partial x} = 0 \text{ as } x \to \infty \text{ and } x \to -\infty.$$

$$\therefore FT\left(\frac{\partial^2}{\partial x^2} f(x,t)\right) = \int_{-\infty}^{\infty} (j\omega) \frac{\partial f(x,t)}{\partial x} e^{-j\omega x} dx$$

$$= (j\omega) f(x,t)\Big|_{-\infty}^{\infty} - \omega^2 \int_{-\infty}^{\infty} f(x,t) e^{-j\omega x} dx$$

$$= -\omega^2 \int_{-\infty}^{\infty} f(x,t) e^{-j\omega x} dx$$

$$= -\omega^2 Y(\omega,t)$$

$$\therefore \frac{\partial Y(\omega,t)}{\partial t} = -\frac{1}{2}\sigma^2\omega^2 Y(\omega,t)$$

Taking the Fourier transform of both sides of the initial condition $f(x,0)=\delta(x)$, we get

FT

$$Y(\omega,0)=1.$$

The differential equation in the Fourier transform domain is given by

$$\frac{\partial Y(\omega,t)}{\partial t}=-\frac{1}{2}\sigma^2\omega^2 Y(\omega,t)$$

with the initial condition $Y(\omega,0)=1$.

The above equation can be solved for t as

$$Y(\omega,t)=e^{-\frac{1}{2}\sigma^2\omega^2 t}$$

Taking the inverse Fourier transform, we get

$$f(x,t)=\frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2 t}\right)}$$

Note that $X(t)$ is symmetric about horizontal axis and the variance increases linearly with time.

Brownian motion process

The CTMP $X(t)$ with

$$f(x,t)=\frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2 t}\right)}$$

Is called the standard Brownian motion process.

If $\mu(x,t)=\mu\neq 0$, then

$$f(x,t)=\frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{1}{2}\left(\frac{(x-\mu t)^2}{\sigma^2 t}\right)}$$

Wiener process or Brownian motion process

Definition: The random process $\{X(t), t \ge 0\}$ is called a ***Wiener process or the Brownian motion process*** if it satisfies the following conditions:

(1) $X(0) = 0$ with probability 1.

(2) $X(t)$ is an independent increment process.

(3) For each $t_0 \ge 0, t \ge 0$ $\quad X(t+t_0) - X(t_0)$ has the normal distribution with mean 0 and variance $\sigma^2 t$.

$$f_{X(t+t_0)-X(t_0)}(x) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2 t}}$$

- Wiener process was used to model the Brownian motion – microscopic particles suspended in a fluid are subject to continuous molecular impacts resulting in the zigzag motion of the particle named Brownian motion after the British botanist Robert Brown. (1773-1858)
- The Wiener process is characterized by the parameter σ. When $\sigma = 1$, the process is called *the standard Wiener process.*

A realization of the Wiener process is shown in the figure below