PH101

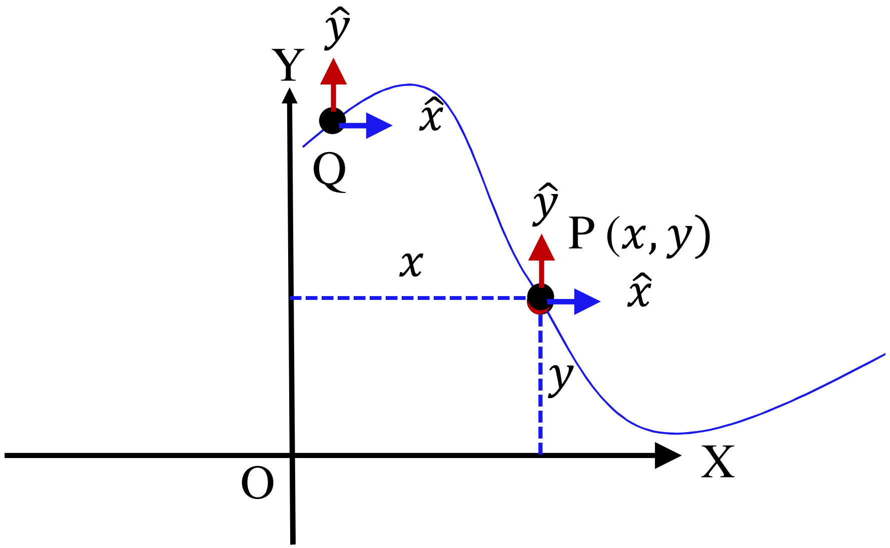
Lecture 2

Coordinate systems

Cartesian coordinate System in plane

In Cartesian coordinate position P is represented by (x, y).

$$\overrightarrow{OP} = \vec{r} = x\,\hat{x} + y\,\hat{y}$$



Cartesian Coordinate System

Note:

- $\hat{x}$ and $\hat{y}$ are unit vectors **pointing the increasing direction** of x and y.

- $\hat{x}$ and $\hat{y}$ are orthogonal and **points in the same direction everywhere** or for any location (x, y).

Another way of looking unit vector Cartesian coordinate in plane

$\hat{x}$ is the unit vector perpendicular to $x = constant$ line (surface)

$\hat{y}$ is the unit vector perpendicular to $y = constant$ line (surface)

Notations

We may interchangeably use the notations:

$$\hat{x} = \hat{\imath}$$
$$\hat{y} = \hat{\jmath}$$
$$\hat{z} = \hat{k}$$

Standard Notations:

$$\frac{dx}{dt} = \dot{x} \quad Or \quad \frac{dr}{dt} = \dot{r}$$

$$\frac{d^2\theta}{dt^2} = \ddot{\theta}$$

For **time** derivatives (only)!

Velocity and acceleration in Cartesian

Velocity $\vec{v} = \frac{d\vec{r}}{dt}$

Velocity in Cartesian: $\boldsymbol{\vec{v}} = \frac{\boldsymbol{d\vec{r}}}{\boldsymbol{dt}} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\ (\boldsymbol{x\,\hat{x} + y\,\hat{y}})$

$$= \dot{x}\hat{x} + x\,\frac{d\hat{x}}{dt} + \dot{y}\hat{y} + y\,\frac{d\hat{y}}{dt}$$

$$\boxed{\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y}}$$

Since, $\frac{d\hat{x}}{dt} = \frac{d\hat{y}}{dt} = 0$

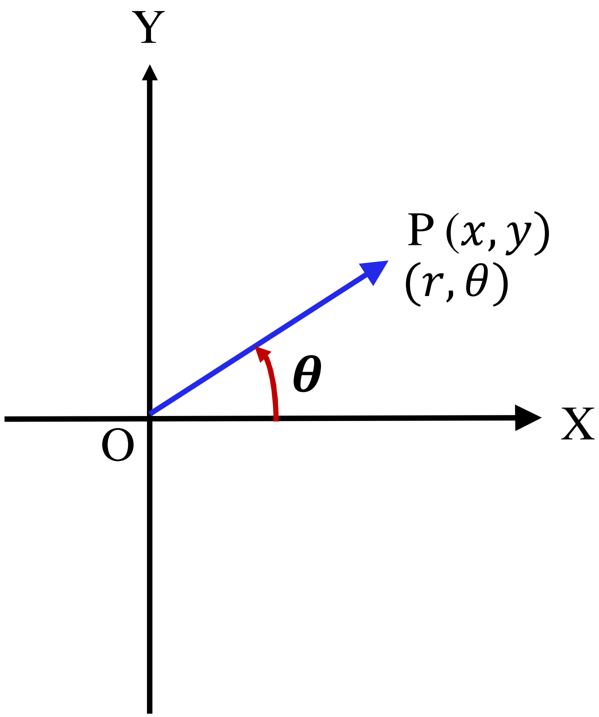
Acceleration,

$$\boxed{\boldsymbol{\vec{a}} = \frac{\boldsymbol{d\vec{v}}}{\boldsymbol{dt}} = \boldsymbol{\ddot{x}\hat{x} + \ddot{y}\hat{y}}}$$

Newton's second law in vector form,

$$\boxed{\boldsymbol{\vec{F}} = \boldsymbol{F_x\hat{x} + F_y\hat{y}} = \boldsymbol{m\frac{d\vec{v}}{dt}} = \boldsymbol{m(\ddot{x}\hat{x} + \ddot{y}\hat{y})}}$$

I. Plane polar coordinate



Each point P (x, y) on the plane can also be represented by its distance (r) from the origin O and the angle (θ) OP makes with X-axis.

Relationship with Cartesian coordinates

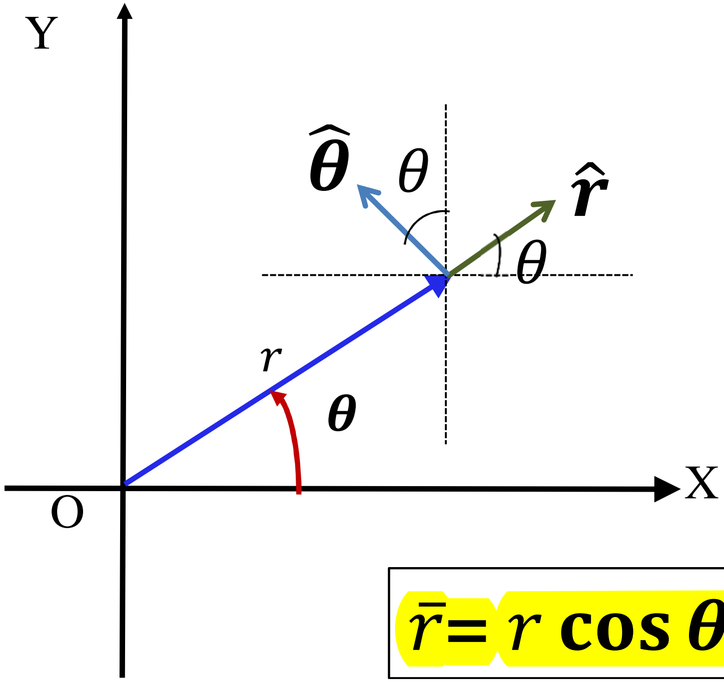
$x = r\ \cos\theta$ & $y = r\ \sin\theta$

Thus ,

$$r = (x^2 + y^2)^{1/2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

Unit vector in plane polar coordinate



- For plane polar unit vectors:

$$\hat{\boldsymbol{r}} \quad \text{and} \quad \hat{\boldsymbol{\theta}}$$

associated to **each point** in the plane.

- $\hat{r}$ and $\hat{\theta}$ are unit vector along increasing direction of coordinate r and θ.

$$\boxed{\bar{r} = r\cos\theta\ \hat{x} + r\sin\theta\ \hat{y}}$$

$$\hat{\boldsymbol{r}} = \cos\theta\ \hat{x} + \sin\theta\ \hat{y}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\ \hat{x} + \cos\theta\ \hat{y}$$

$\hat{r}$ and $\hat{\theta}$ are **orthogonal:** $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\theta}}=0$
but their directions depend on location.

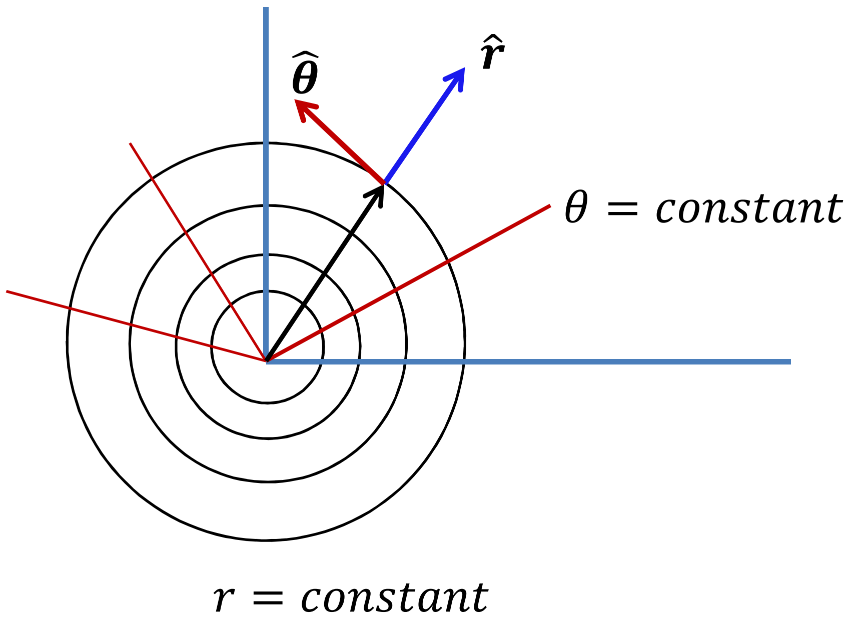
$$\hat{r} = \frac{\partial \bar{r}}{\partial r}$$

$$\hat{\theta} = \frac{\partial \hat{r}}{\partial \theta}$$

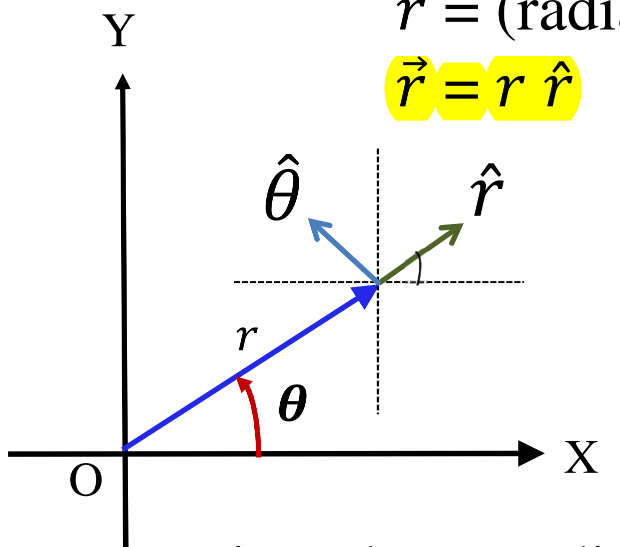
Unit vector in plane polar coordinate

The unit vectors in the polar coordinate can also be viewed in another way. $\hat{r}$ is the unit vector perpendicular to $r = constant$ surface and points in the increasing direction of r .

Similarly, $\hat{\theta}$ is the unit vector perpendicular to $\theta = constant$ surface (i,e tangential to $r = constant$) and points in the increasing direction of θ.



Unit vector in plane polar coordinate



$\vec{r}$ = (radial distance) (unit vector along the vector)

$\vec{r} = r\,\hat{r}$

$$\boxed{\begin{aligned} \hat{x} &= \hat{\imath} \\ \hat{y} &= \hat{\jmath} \\ \hat{z} &= \hat{k} \end{aligned}}$$

Unit vectors in polar coordinate are function of θ only.

$$\frac{\partial \hat{\boldsymbol{r}}}{\partial \boldsymbol{\theta}} = \frac{\partial}{\partial \theta}(\hat{x}\cos\theta + \hat{y}\sin\theta) = -\hat{x}\ \sin\theta + \hat{y}\cos\theta = \hat{\boldsymbol{\theta}}$$

$$\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}} = \frac{\partial}{\partial \theta}(-\hat{x}\sin\theta + \hat{y}\cos\theta) = -(\hat{x}\cos\theta + \hat{y}\sin\theta) = -\hat{\boldsymbol{r}}$$

Velocity in plane polar coordinate

Velocity $\vec{v} = \frac{d\vec{r}}{dt}$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\,\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\hat{r} + r\frac{\partial\hat{r}}{\partial\theta}\frac{d\theta}{dt}$$

Since,

$$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}$$

$$\boxed{\vec{\boldsymbol{v}} = \dot{\boldsymbol{r}}\hat{\boldsymbol{r}} + \boldsymbol{r}\dot{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}}$$

Radial component $\dot{\boldsymbol{r}}$ and

Tangential/transverse component $\boldsymbol{r}\dot{\boldsymbol{\theta}}$

Acceleration in plane polar coordinate

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Note: $\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$ & $\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$

$$= \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\frac{\partial \hat{r}}{\partial \theta}\frac{d\theta}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{\partial \hat{\theta}}{\partial \theta}\frac{d\theta}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}}$$

Radial component of acceleration: $\ddot{r} - r\dot{\theta}^2$

(Note: $-r\dot{\theta}^2$ is the familiar ***Centripetal*** *contribution*!)

Tangential component: $2\dot{r}\dot{\theta} + r\ddot{\theta}$

(Note: $2\dot{r}\dot{\theta}$ is called the ***Coriolis*** contribution!)

Newton's law in plane polar coordinate

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

$$\vec{F} = F_r\hat{r} + F_\theta\hat{\theta} = m[(\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\widehat{\boldsymbol{\theta}}]$$

Newton's law for **radial** direction: $\boldsymbol{F_r} = \mathbf{m}(\ddot{\boldsymbol{r}} - \boldsymbol{r}\dot{\boldsymbol{\theta}}^2)$

Newton's law for **tangential** direction: $\boldsymbol{F_\theta} = \boldsymbol{m}(2\dot{\boldsymbol{r}}\dot{\boldsymbol{\theta}} + \boldsymbol{r}\ddot{\boldsymbol{\theta}})$

Note: Newton's law in polar coordinates **do not** follow its **Cartesian form** as,

$\boldsymbol{F_r} \neq \boldsymbol{m\ddot{r}}$ or $\boldsymbol{F}_\theta \neq \boldsymbol{m}\ddot{\theta}$

Highlights

- Transformation relation between *Cartesian* and *polar coordinate* is given by,

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Reverse transformation

$$r = (x^2 + y^2)^{1/2}$$
$$\theta = \tan^{-1}\frac{y}{x}$$

- Directions of unit vectors $(\hat{x}, \hat{y})$ in **Cartesian** system remain ***fixed*** **irrespective** of the location (x, y).
- Directions of unit vectors $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}})$ in **plane polar** coordinates **depend on the location**.
- **Caution:** Form of Newton's law is different in different coordinate systems.

Questions please