Lecture 33 MARTINGALE 2

MARTINGALE

Recall that a discrete-time random process $\{X_n, n \geq 0\}$ is called martingale process if for all $n \geq 1$,

(i) $E|X_n| < \infty$, and

(ii) $E(X_{n+1} / X_0, X_1, ..., X_n) = X_n$

If the equality sign in(ii) above is replaced by $\leq$, then $\{X_n, n \geq 0\}$ is called a *supermartingale* and if it is replaced by $\geq$, then $\{X_n, n \geq 0\}$ is a *submartingale*.

Doob-type martingale

Definition: Consider two discrete-time random processes $\{X_n, n \geq 0\}$ and $\{Y_n, n \geq 0\}$. Then $\{X_n, n \geq 0\}$ is called a Doob-type martingale process if for all $n \geq 0$

(i) $E|X_n| < \infty$, and

(ii) $E(X_{n+1} / Y_0, Y_1, ..., Y_n) = X_n$

Ex 4: Suppose the random process $\{Y_n, n \geq 0\}$ given by $Y_n = \sum_{i=1}^{n} Z_i$, $n \geq 1$ is a symmetrical random walk process and $Y_0 = 0$ is a martingale. Then the random process $X_n = Y_n^2 - n$ is also a martingale w.r.t. $\{Y_n, n \geq 0\}$.

Proof: We have

$$X_{n+1} = Y_{n+1}^2 - (n+1)$$

$$= (Y_n + Z_{n+1})^2 - (n+1)$$

$$= Y_n^2 + Z_{n+1}^2 + 2Y_n Z_{n+1} - (n+1)$$

$$\therefore EX_{n+1} / Y_0, Y_1, ..., Y_n$$

$$= E(Y_n^2 + Z_{n+1}^2 + 2Y_n Z_{n+1} - (n+1)) / Y_0, Y_1, ..., Y_n$$

$$= Y_n^2 + EZ_{n+1}^2 + 2Y_n EZ_{n+1} - (n+1)$$

$$\text{(Using indepence of } Z_{n+1} \text{with each } Y_i, i = 0,1,...,n)$$

$$= Y_n^2 + 1 + 0 - (n+1)$$

$$= Y_n^2 - n$$

$$= X_n$$

Therefore, $\{X_n\}_{n=0}^{\infty}$ is a Doob-typemartingale.

Example 5: Polya's urn process

An urn contains *R* red balls and *B* black balls. One ball is selected in random and replaced along with one more ball of the same color. Let Y_n be the number of red balls after the n^{th} draw. The fraction of red balls after the n^{th} draw be $X_n = \dfrac{Y_n}{n+R+B}$ Then $\{X_n, n \geq 0\}$ is a martingale w.r.t $\{Y_n\}$

Proof: We have

$$X_{n+1} = \begin{cases} \dfrac{Y_n + 1}{n+1+R+B} & \text{with probability } \dfrac{Y_n}{n+R+B} \\ \dfrac{Y_n}{n+1+R+B} & \text{with probabilty } 1 - \dfrac{Y_n}{n+R+B} \end{cases}$$

$$\therefore E(X_{n+1} / Y_0, Y_1, \ldots, Y_n)$$

$$= \frac{Y_n + 1}{n+1+R+B} \times \frac{Y_n}{n+R+B} + \frac{Y_n}{n+1+R+B} \times (1 - \frac{Y_n}{n+R+B})$$

$$= \frac{Y_n^2 + Y_n + Y_n(n+R+B) - Y_n^2}{(n+1+R+B)(n+R+B)}$$

$$= \frac{Y_n}{n+R+B}$$

$$= X_n$$

Properties of martingales

Martingale has constant mean

Proof: We have

$$E(X_{n+1} / X_0, X_1, ..., X_n) = X_n$$

Taking expectation with respect to the joint random variables $X_0, X_1, ..., X_n$ on both sides, we get

$$EE(X_{n+1} / X_0, X_1, ..., X_n) = EX_n$$
$$\Rightarrow EX_{n+1} = EX_n$$

Continuing in the similar manner, we can show that

$$EX_{n+1} = EX_n = ... = EX_0 = \text{constant}$$

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Martingale Inequalities

Maximal Inequality- If $\{X_n, n \geq 0\}$ is a non-negative martingalle, then $P\left(max_{n\geq 0}\left(X_n\right) \geq a\right) \leq \frac{EX_0}{a}$

Proof Using Markov inequality

$$P\left(X_n \geq a\right) \leq \frac{EX_n}{a}, n \geq 0$$

$$= \frac{EX_0}{a}$$

$$\therefore P\left(max_{n\geq 0}\left(X_n\right) \geq a\right) \leq \frac{EX_0}{a}$$

Kolmogorov Inequality

Suppose $X_0 = 0$. Clearly, $EX_n = 0,\ n = 0,1,\ldots.$

Then by applying Chebyshev inequality

$$P(|X_n| \geq a) \leq \frac{EX_n^2}{a^2},$$ n=0,1,…,n

m-step prediction

We have observed that for a martingale process $\{X_n, n \geq 0\}$,

$$E\left(X_{n+1} / X_0, X_1, ...X_n,\right) = X_n \text{ (one-step prediction)}$$

The following theorem says about the m-step prediction

Theorem: For a Martingale process $\{X_n\}$,

$$E\left(X_{n+m} / X_0, X_1, ..., X_n\right) = X_n$$

Proof Recall the property of the conditional expectation:

$E(EY / X, Z) / X = EY / X$

Therefore,

$$LHS = E\left\{E\left(X_{n+m} / X_0, X_1, ..., X_n, ..., X_{n+m-1}\right)\right\} / X_0, X_1, ...X_n$$

$$= E\left(X_{n+m-1} / X_0, X_1, ...X_n\right)$$

Repeating this we get

$$= E\left(X_{n+1} / X_0, X_1, ..., X_n\right)$$

$$= X_n$$

Corollary 1 For a Martingale process $\{X_n\}$, $E\left(X_nX_{n+m}\right)=EX_n^2,\, m\geq 0$

Proof

$$\begin{aligned}E\left(X_nX_{n+m}\right)&=EE\left(X_nX_{n+m}/X_0,X_1,\ldots,X_n\right)\\&=EX_nE\left(X_{n+m}/X_0,X_1,\ldots,X_n\right)\\&=EX_nX_n\\&=EX_n^2\end{aligned}$$

Corollary 2 A martingale $\{X_n\}$ is an orthogonal increment process, i.e. for $n_1 < n_2 < n_3 < n_4$ $E(X_{n_2} - X_{n_1})(X_{n_4} - X_{n_3}) = 0$

Proof

$$E(X_{n_2} - X_{n_1})(X_{n_4} - X_{n_3})$$
$$= EX_{n_2}X_{n_4} - EX_{n_2}X_{n_3} - EX_{n_1}X_{n_4} + EX_{n_1}X_{n_3}$$
$$= EX_{n_2}^2 - EX_{n_2}^2 - EX_{n_1}^2 + EX_{n_1}^2$$
$$= 0$$

Corollary 3 For a martingale process $\{X_n\}$, EX_n^2 is a monotonically increasing sequence.

Proof We have

$$0 \le E\left(X_{n+1} - X_n\right)^2$$

$$= EX_{n+1}^2 + EX_n^2 - 2EX_n X_{n+1}$$

$$= EX_{n+1}^2 + EX_n^2 - 2EX_n^2$$

$$= EX_{n+1}^2 - EX_n^2$$

$\therefore EX_n^2$ is a monotonically increasing sequence.

To summarise

- A discrete-time random process $\{X_n, n \geq 0\}$ is called martingale process if for all $n \geq 1$,

(i) $E|X_n| < \infty$, and

(ii) $E(X_{n+1} / X_0, X_1, ..., X_n) = X_n$

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- A martingale has constant mean

For a Martingale process $\{X_n\}$,

$$E\left(X_{n+m} / X_0, X_1, ..., X_n\right) = X_n$$

To summarise…

- For a Martingale process $\{X_n\}$, $E\left(X_n X_{n+m}\right) = EX_n^2, m \geq 0$
- A martingale $\{X_n\}$ is an orthogonal increment process, i.e. for

$$n_1 < n_2 < n_3 < n_4$$

$$E(X_{n_2} - X_{n_1})(X_{n_4} - X_{n_3}) = 0$$

- For a martingale process $\{X_n\}$, EX_n^2 is a monotonically increasing sequence.