Core:

Given a game $v \in G^N$, the core of v is the set of all imputations x in $I(c)$ such that $x(S) \geq v(s)$ for all non-empty coalitions $S \subset N$. The core of a game $v \in G^N$ is denoted by $C(v)$.

Example 1

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 0$
$\{2\}$	$v(\{2\}) = 0$
$\{3\}$	$v(\{3\}) = 0$
$\{1,2\}$	$v(\{1,2\}) = 1$
$\{1,3\}$	$v(\{1,3\}) = 1$
$\{2,3\}$	$v(\{2,3\}) = 1$
$\{1,2,3\}$	$v(\{1,2,3\}) = 3$

We want to find the core allocation of the above coalition game. We have $v(\{1\}) = 0 \leq x_{A_1},\ v(\{2\}) = 0 \leq x_{A_2},\ v(\{3\}) = 0 \leq x_{A_3}$. $v(\{1,2\}) = 1 \leq x_{A_1} + x_{A_2},\ v(\{1,3\}) = 1 \leq x_{A_1} + x_{A_3},\ v(\{2,3\}) = 1 \leq x_{A_2} + x_{A_3}$.
$v(\{1,2,3\}) = 3 \leq x_{A_1} + x_{A_2} + x_{A_3}$.

Substituting $v(\{2,3\}) = 1 \leq x_{A_2} + x_{A_3}$ in $v(\{1,2,3\}) = 3 = x_{A_1} + x_{A_2} + x_{A_3}$. We have $2 \geq x_{A_1}$, similarly we get $2 \geq x_{A_2}$ and $2 \geq x_{A_3}$.
Therefore, core allocations are $2 \geq x_{A_1} \geq 0$, $2 \geq x_{A_2} \geq 0$, $2 \geq x_{A_3} \geq 0$ and $3 = x_{A_1} + x_{A_2} + x_{A_3}$.
It is shown in figure 1 and 2.

Example 2

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 1$
$\{2\}$	$v(\{2\}) = 1$
$\{3\}$	$v(\{3\}) = 1$
$\{1, 2\}$	$v(\{1, 2\}) = 2$
$\{1, 3\}$	$v(\{1, 3\}) = 2$
$\{2, 3\}$	$v(\{2, 3\}) = 2$
$\{1, 2, 3\}$	$v(\{1, 2, 3\}) = 3$

We want to find the core allocation of the above coalition game.
We have $v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.
$v(\{1,2\}) = 2 \leq x_{A_1} + x_{A_2},\ v(\{1,3\}) = 2 \leq x_{A_1} + x_{A_3},\ v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$.
$v(\{1,2,3\}) = 3 \leq x_{A_1} + x_{A_2} + x_{A_3}$.

Substituting $v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$ in $v(\{1,2,3\}) = 3 = x_{A_1} + x_{A_2} + x_{A_3}$. We have $1 \geq x_{A_1}$, similarly we get $1 \geq x_{A_2}$ and $1 \geq x_{A_3}$.
And we have
$v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.

Therefore, core allocation is $(x_{A_1}, x_{A_2}, x_{A_3}) = (1,1,1)$.
See figure 3 and 4.

Example 3

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 1$
$\{2\}$	$v(\{2\}) = 1$
$\{3\}$	$v(\{3\}) = 1$
$\{1,2\}$	$v(\{1,2\}) = 2$
$\{1,3\}$	$v(\{1,3\}) = 2$
$\{2,3\}$	$v(\{2,3\}) = 2$
$\{1,2,3\}$	$v(\{1,2,3\}) = 4$

We want to find the core allocation of the above coalition game.
We have $v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.
$v(\{1,2\}) = 2 \leq x_{A_1} + x_{A_2},\ v(\{1,3\}) = 2 \leq x_{A_1} + x_{A_3},\ v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$.
$v(\{1,2,3\}) = 4 \leq x_{A_1} + x_{A_2} + x_{A_3}$.

Substituting $v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$ in
$v(\{1,2,3\}) = 4 = x_{A_1} + x_{A_2} + x_{A_3}$. We have $2 \geq x_{A_1}$, similarly we get $2 \geq x_{A_2}$ and $2 \geq x_{A_3}$.
And we have
$v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.
Therefore, core allocation are
$2 \geq x_{A_1} \geq 1$, $2 \geq x_{A_2} \geq 1$, $2 \geq x_{A_3} \geq 1$ and $4 = x_{A_1} + x_{A_2} + x_{A_3}$
It is shown in figure 5 and 6.

Example 4

Suppose one person A owns an old car which values nothing to him. There are two potential buyers, Buyer B values it at 1000 and buyer C values it at 1050. The trade between these people can be analysed based on coalition formation.

Coalitions	Value or worth of coalitions
$\{A\}$	0
$\{B\}$	0
$\{C\}$	0
$\{A, B\}$	1000
$\{A, C\}$	1050
$\{B, C\}$	0
$\{A, B, C\}$	1050
$\{\emptyset\}$	0

What is the core allocation?

We have $x_A \geq 0$, $x_B \geq 0$, $x_C \geq 0$,
$x_A + x_B \geq 1000, \quad x_A + x_C \geq 1050, \quad x_B + x_C \geq 0$
$x_A + x_B + x_C \geq 1050$.

Substituting $x_A + x_C \geq 1050$ in $x_A + x_B + x_C = 1050$, we get $x_B \leq 0$. We have $x_B \geq 0$, thus $x_B = 0$.
Substituting $x_A + x_B \geq 1000$ in $x_A + x_B + x_C = 1050$, we get $x_C \leq 50$. We have $x_C \geq 0$, thus $50 \geq x_C \geq 0$.
From this we get the core allocation as
$C(v) = \{(x_A, x_B, x_c) = (1050 - d, 0, d) | 0 \leq d \leq 50\}$.

Example 5

In a three player game v let player A_1 be a firm that uses an input in its production and each of the players A_2 and A_3 is supplier of its input. Some output having value 1 is created if ownership of input is transferred from either or both of A_2 and A_3 to A_1. No value is created if this interaction does not take place. What is the core allocation?

$v(\{A_1\}) = v(\{A_2\}) = v(\{A_3\}) = 0.$
$v(\{A_1, A_2\}) = v(\{A_1, A_3\}) = v(\{A_1, A_2, A_3\}) = 1$ and
$v(\{A_2, A_3\}) = 0.$

Substituting $x_{A_1} + x_{A_2} \geq 1$ in $x_{A_1} + x_{A_2} + X_{A_3} = 1$, we get $x_{A_3} \leq 0$.
We have $x_{A_3} \geq 0$, thus $x_{A_3} = 0$.
Substituting $x_{A_1} + x_{A_3} \geq 1$ in $x_{A_1} + x_{A_2} + X_{A_3} = 1$, we get $x_{A_2} \leq 0$.
We have $x_{A_2} \geq 0$, thus $x_{A_2} = 0$.
From above we get $X_1 = 1$. Thus core allocation is
$C(v) = ((x_{A_1}, x_{A_2}, x_{A_3}) = (1, 0, 0)\}$.

Consider a game $v \in G^N$; the payoff vectors x and y and an arbitrary coalition $S \subseteq N$. x dominates y via coalition S , if $x_{A_i} > y_{A_i}$ for all $A_i \in S$ and $x(S) \leq v(S)$.

In figure 1, we see the allocations which are dominated. The allocations which are not dominated. In the bargaining game where $v(A_1) = 0, v(A_2) = 0$, and $v(A_1, A_2) = 1$. The allocations given by the line $x_{A_1} + x_{A_2} = 1$, $x_{A_1} \geq 0$, and $x_{A_2} \geq 0$ are not dominated by any other allocation.

Any two allocations which are imputation one cannot dominate the other.
Result:
Consider a game $v \in G^N$ and suppose $x \in I(v)$ is an arbitrary imputation. Then the following statement are equivalent.
$i)$ $x \in C(v)$.
$ii)$ There is no payoff vector that dominates x.

Proof

First we prove $i) \to ii)$. Suppose there exists an allocation y which dominates x. This implies $y_{A_i} > x_{A_i}$ for all $A_i \in S$. This implies $x(S) = \sum_{A_i \in S} x_{A_i} < \sum_{A_i \in S} y_{A_i} = y(S) \leq v(S)$.
We have $x \in C(v)$. It implies that $\sum_{A_i \in S} x_{A_i} \geq v(S)$. A contradiction. Therefore, there exists no allocation y that dominates x.
Now we prove $ii) \to i)$.
Suppose x is not a core allocation. This implies that $x(S) < v(S)$ for some coalition S. This implies that there can be an allocation y of the following nature,

$$y_{A_i} = \begin{cases} x_{A_i} + \frac{v(S) - x(S)}{|S|}, & \text{if} A_i \in S \\ 0, A_i & \notin S \end{cases}$$

We have $y_{A_i} > x_{A_i}$ for all $A_i \in S$ and
$y(S) = x(S) + |S|\frac{v(S)-x(S)}{|S|} = v(S)$. We get that y dominates x.
This implies that if x is not in core then there exists an allocation which dominates x. This proves $ii) \to i)$.
This is another way to find core allocations.

Is the set of core allocations always non-empty?

Consider the following coalition game.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 1$
$\{2\}$	$v(\{2\}) = 1$
$\{3\}$	$v(\{3\}) = 1$
$\{1,2\}$	$v(\{1,2\}) = 3$
$\{1,3\}$	$v(\{1,3\}) = 3$
$\{2,3\}$	$v(\{2,3\}) = 3$
$\{1,2,3\}$	$v(\{1,2,3\}) = 4$

We have $v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.
$v(\{1,2\}) = 3 \leq x_{A_1} + x_{A_2},\ v(\{1,3\}) = 3 \leq x_{A_1} + x_{A_3},\ v(\{2,3\}) = 3 \leq x_{A_2} + x_{A_3}$.
$v(\{1,2,3\}) = 4 \leq x_{A_1} + x_{A_2} + x_{A_3}$.

Substituting $v(\{2,3\}) = 3 \leq x_{A_2} + x_{A_3}$ in $v(\{1,2,3\}) = 4 = x_{A_1} + x_{A_2} + x_{A_3}$. We have $1 \geq x_{A_1}$, similarly we get $1 \geq x_{A_2}$ and $1 \geq x_{A_3}$.
We $v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$. This implies $x_{A_1} = 1,\ x_{A_2} = 1,\ x_{A_3} = 1$.
$x_{A_1} + x_{A_2} + x_{A_3} = 3 < v(\{1,2,3\}) = 4$. Therefore, $x_{A_1} = 1,\ x_{A_2} = 1,\ x_{A_3} = 1$ is not a core allocation.
It is shown in figure 2 and 3.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 1$
$\{2\}$	$v(\{2\}) = 1$
$\{3\}$	$v(\{3\}) = 1$
$\{1,2\}$	$v(\{1,2\}) = 2$
$\{1,3\}$	$v(\{1,3\}) = 2$
$\{2,3\}$	$v(\{2,3\}) = 2$
$\{1,2,3\}$	$v(\{1,2,3\}) = 3$

This game is inessential though a constant sum game. We have a core allocation.
We have $v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.
$v(\{1,2\}) = 2 \leq x_{A_1} + x_{A_2},\ v(\{1,3\}) = 2 \leq x_{A_1} + x_{A_3},\ v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$.
$v(\{1,2,3\}) = 3 \leq x_{A_1} + x_{A_2} + x_{A_3}$.

Substituting $v(\{2,3\}) = 2 \leq x_{A_2} + x_{A_3}$ in $v(\{1,2,3\}) = 3 = x_{A_1} + x_{A_2} + x_{A_3}$. We have $1 \geq x_{A_1}$, similarly we get $1 \geq x_{A_2}$ and $1 \geq x_{A_3}$.
And we have
$v(\{1\}) = 1 \leq x_{A_1},\ v(\{2\}) = 1 \leq x_{A_2},\ v(\{3\}) = 1 \leq x_{A_3}$.

Therefore, core allocation is $(x_{A_1}, x_{A_2}, x_{A_3}) = (1,1,1)$.
See figure 4 and 5.

Result

Let the game $v \in G^N$ be constant-sum and essential. Then $C(v)$ (set of core allocation) is empty.

Proof

Suppose there exists a constant - sum essential game $v \in G^N$ such that $C(v)$ is non-empty. Suppose $x \in C(v)$. It implies that $x_{A_i} \geq v(\{A_i\})$ and $\sum_{A_j \in N \setminus A_i} x_{A_j} \geq v(N \setminus A_i)$. We have $x(N) = x_{A_i} + \sum_{A_j \in N \setminus \{A_i\}} x_{A_j} = v(N)$.
We have $-\sum_{A_j \in N \setminus A_i} x_{A_j} \leq -v(N \setminus A_i)$. This implies $x_{A_i} \leq v(N) - v(N \setminus A_i)$. Since v is constant sum game, we have $v(\{A_i\}) + v(N \setminus \{A_i\}) = v(N)$. This together with the above we get $v(N) - x_{A_i} \geq v(N) - V(\{A_i\})$. This implies that $x_{A_i} \leq v(\{A_i\})$. Thus, we have $x_{A_i} = v(\{A_i\})$. Since x is in core, so we have $\sum_{A_i \in N} x_{A_i} = v(N)$. Since v is essential game, so $\sum_{A_i \in N} v(\{A_i\}) < v(N)$. Thus we have $v(N) < v(N)$. A contradiction. Therefore, $x \notin C(v)$. Core is empty.

The set of core allocation is not always non-empty.
There is a theorem which characterizes the non-emptiness of core for based on weights assigned to coalitions. We will not do that.

Problem:
Suppose a building worth 2000 per month to its owner. A cloth merchant is ready to pay a monthly rent of 2500, whereas a bank offers to pay 3000 per month. Find the core allocation of this game.

Solution:
First we have to formulate the characteristic function.
Suppose players are A, B and C, where A is owner, C is cloth merchant and B is the bank. $v_A = 2000$, $v_B = 0$, $v_C = 0$,
$v_{AC} = 2500$, $v_{AB} = 3000$, $v_{BC} = 0$ and $v_{ABC} = 3000$.
$x_A \geq 2000$, $x_B \geq 0$, $x_C \geq 0$.
$x_A + x_C \geq 2500$, $x_A + x_B \geq 3000$, $x_B + x_C \geq 0$

$x_A + x_C + x_B \geq 3000.$

Substituting $x_B + x_C \geq 0$ in $x_A + x_C + x_B = 3000$, since core allocation is an imputation we have, $x_A \leq 3000$ And we have $x_A \geq 2000$. So we get that
$2000 \leq x_A \leq 3000.$

Again we have $x_B \leq 500$, by substituting $x_A + x_C \geq 2500$ in $x_A + x_C + x_B = 3000$. This implies that $0 \leq x_B \leq 500$.

We have $x_C \leq 0$, by substituting $x_A + x_B \geq 3000$ in $x_A + x_C + x_B = 3000$. This implies that $0 = x_C$.
Thus, core allocations are
$2500 \leq x_A \leq 3000$, $0 \leq x_B \leq 500$, and $x_C = 0$.

Problem:
Suppose there are four players $\{A_1, A_2, A_3, A_4\}$, suppose there are two disjoint set L and R of these players. $R = \{A_1, A_2, \}$ and $L = \{A_3, A_4, \}$. Players of set R has right shoe and players in set L has left shoe. A pair of shoe contains two shoes - left and right. A pair of shoe worth 1 and if there is only left or only right, it has no value. Consider set $S = \{A_1, A_3, A_4\}$, so one right shoe. $S \cap R = \{A_1\}$ and $L \cap S = \{A_3, A_4\}$, two left shoes. If there is cooperation between the players in S, then it will have one pair of shoe. The minimum number of left or right shoes determine the number of pairs. So the worth of coalition is $\min\{|S \cap R|, |S \cap L|\}$, when $|S| \geq 2$.
We get the following characteristic function.

$$v(s) = \begin{cases} 0, \ \ \textit{if} \ \ |S| \in \{0,1\}, \\ \min\{|S \cap R|, |S \cap L|\}, \ \ \textit{if} \ \ |S| \geq 2. \end{cases}$$

$v(N) = \min\{|R|, |L|\}$.

We need to find core allocation.

coalitions	v()
$\emptyset$	0
$\{1\}$	$v(\{1\}) = 0$
$\{2\}$	$v(\{2\}) = 0$
$\{3\}$	$v(\{3\}) = 0$
$\{4\}$	$v(\{4\}) = 0$
$\{1,2\}$	$v(\{1,2\}) = 0$
$\{1,3\}$	$v(\{1,3\}) = 1$
$\{1,4\}$	$v(\{2,3\}) = 1$
$\{2,3\}$	$v(\{2,3\}) = 1$
$\{2,4\}$	$v(\{2,4\}) = 1$
$\{4,3\}$	$v(\{4,3\}) = 0$
$\{1,2,3\}$	$v(\{1,2,3\}) = 1$
$\{1,2,4\}$	$v(\{1,2,4\}) = 1$

coalitions	v()
$\{1,3,4\}$	$v(\{1,4,3\}) = 1$
$\{2,3,4\}$	$v(\{4,2,3\}) = 1$
$\{1,2,3,4\}$	$v(\{1,4,2,3\}) = 2$

$x_i \geq 0,\ i = 1,2,3,4.$
$x_1 + x_3 \geq 1,\ x_1 + x_4 \geq 1,\ x_2 + x_3 \geq 1,\ x_2 + x_4 \geq 1,$
$x_1 + x_2 \geq 0,\ x_4 + x_3 \geq 0.$
$x_1 + x_2 + x_3 \geq 1,\ x_1 + x_2 + x_4 \geq 1,\ x_1 + x_4 + x_3 \geq 1,$
$x_4 + x_2 + x_3 \geq 1.$
$x_1 + x_2 + x_3 + x_4 \geq 2$

We have $x_1 + x_3 \leq 1$, by substituting $x_2 + x_4 \geq 1$ in $x_1 + x_2 + x_3 + x_4 = 2$. And we have $x_1 + x_3 \geq 1$, so $x_1 + x_3 = 1$ Similarly we have $x_2 + x_4 \leq 1$ and we have $x_2 + x_4 \geq 1$, so $x_2 + x_4 = 1$.

We also have $x_1 + x_4 = 1$ and $x_2 + x_3 = 1$.
We also get $x_1 \leq 1$ by substituting $x_4 + x_2 + x_3 \geq 1$ in $x_1 + x_2 + x_3 + x_4 = 2$. Similarly we have $x_2 \leq 1$, $x_3 \leq 1$, $x_4 \leq 1$.

From $x_1 + x_3 = 1$, $x_2 + x_4 = 1$, $x_1 + x_4 = 1$ $x_2 + x_3 = 1$ and $x_1 + x_2 + x_3 + x_4 = 2$. We have $x_1 = x_2$ and $x_3 = x_4$.

The core allocations are
$0 \leq x_i \leq 1, i = 1, 2, 3, 4$, $x_1 = x_2$, $x_3 = x_4$, and
$x_1 + x_2 + x_3 + x_4 = 2$.

Sum of games:

A cooperative game is a pair $(N; v)$ where N is the set of players and v is the characteristic function.

Different characteristic functions give different games.

$(N; u)$ game is

$N = \{1, 2, 3\}$

$u(1) = u(2) = u(3) = 0,\ u(\emptyset) = 0.$

$u(1, 2) = 10,\ \ u(1, 3) = 20,\ \ u(2, 3) = 30$

$u(1, 2, 3) = 40.$

Shown in fig 1.

Another game $(N; w)$
$N = \{1, 2, 3\}$
$w(1) = 5,\ w(2) = 10,\ w(3) = 15, w(\emptyset) = 0$
$w(1, 2) = 20,\ w(1, 3) = 25,\ w(2, 3) = 30$
$w(1, 2, 3) = 35.$
Shown in figure 2.

Another game $(N; v)$ is
$N = \{1, 2, 3\}$
$w(1) = 5,\ w(2) = 10,\ w(3) = 15, w(\emptyset) = 0$
$w(1, 2) = 30,\ w(1, 3) = 45,\ w(2, 3) = 60$
$w(1, 2, 3) = 75.$
Shown in figure 3.

Note we can write
$v(S) = u(S) + w(S)$ for all $S \subset N$ in the above example.

The game $(N; v)$ is called the sum of two games $(N; u)$ and $(N; w)$ if for every colaition S from the set of players $N(S \subseteq N)$
$v(S) = u(S) + w(S)$

Example:
$N\{1,2,3\}$ and the characteristic function is
$v(1) = 10,\ v(2) = 5,\ v(3) = 15,\ v(\emptyset) = 0$
$v(1,2) = 15,\ v(1,3) = 30,\ v(2,3) = 25$
$v(1,2,3) = 40.$
We can split the above game in the following way
$(N; u)$
$u(1) = 5,\ u(2) = 5,\ u(3) = 5,\ u(\emptyset) = 0$
$u(1,2) = 10,\ u(1,3) = 15,\ u(1,3) = 15$
$u(1,2,3) = 20$

Another game is
$(N; w)$
$w(1) = 5,\ w(2) = 0,\ w(3) = 10$
$w(1, 2) = 5,\ w(2, 3) = 10,\ w(1, 3) = 15$
$w(1, 2, 3) = 20$

We have
$v(S) = u(S) + w(S)$ for every $S \subset N$ in this example.
It is shown in figure 4.

Shapley Value:
Consider a game (N, v).
A value function ϕ assigns to each possible characteristics function of an n- person game v, an n- tuple $\phi(v) = (\phi_1, \phi_2, \phi_3, ...\phi_N)$ of real numbers. Each ϕ_i represents the worth or value of player i in the game with characteristic function v.
For example
In game
$(N; w)$
$w(1) = 5,\ w(2) = 0,\ w(3) = 10$
$w(1,2) = 5,\ w(2,3) = 10,\ w(1,3) = 15$
$w(1,2,3) = 20$
$w(1,2,3)$ can be distributed among the players $(5 + \frac{5}{3}, \frac{5}{3}, 10 + \frac{5}{3})$.

Here, $\phi(v) = (5 + \frac{5}{3}, \frac{5}{3}, 10 + \frac{5}{3})$.

How to get this division?

Axiom 1: The total amount $v(N)$ is divided among all the players.
Efficiency: $\sum_{i \in N} \phi_i(v) = v(N)$.

Axiom 2: Symmetric players get equal payoffs.
If i and j are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition S not containing i and j then $\phi_i(v) = \phi_j(v)$.

It gives fair division, if players are equal in terms of its contribution to a coalition, they are treated equally.

Axiom 3: The payoff to a null player is zero.
If i is such that $v(S) = v(S \cup \{i\})$ for every colaition S not containing i, then $\phi_i(v) = 0$.

If a player contributes nothing to the coalition, it gets nothing.

Axiom 4: If we split the original game into a sum of individual game games, the division of payoffs among the players in the original game should be the sum of divisions obtained in the individual games.

If u and v are characteristic functions, then $\phi(u + v) = \phi(u) + \phi(v)$.

Using these four axioms, we derive the Shapley value of a game.
Consider the following game

$N = \{1, 2, 3\}$

$v(1) = 6,\ v(2) = 12,\ v(3) = 18$

$v(1, 2) = 30,\ v(1, 3) = 60,\ v(2, 3) = 90$

$v(1, 2, 3) = 120,\ v(\emptyset) = 0.$

In figure 5, we derive the Shapley value.

We split the game into two games in such a way that , one of the game has a special property, it contains a coalition S such that $v(T) = v(S)$ whenever T contains S and $V(T) = 0$ for every other coalition. Such a game is called a carrier game and coalition S is called its carrier.

A carrier game $(N; v)$ is a game in which there is a coalition S called the carrier of the game, such that
$v(T) = v(S)$, whenever $S \subseteq T$
$v(T) = 0$, otherwise.

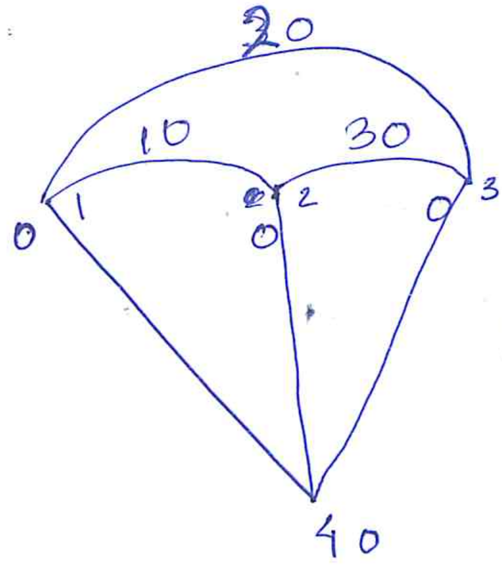


Fig 1

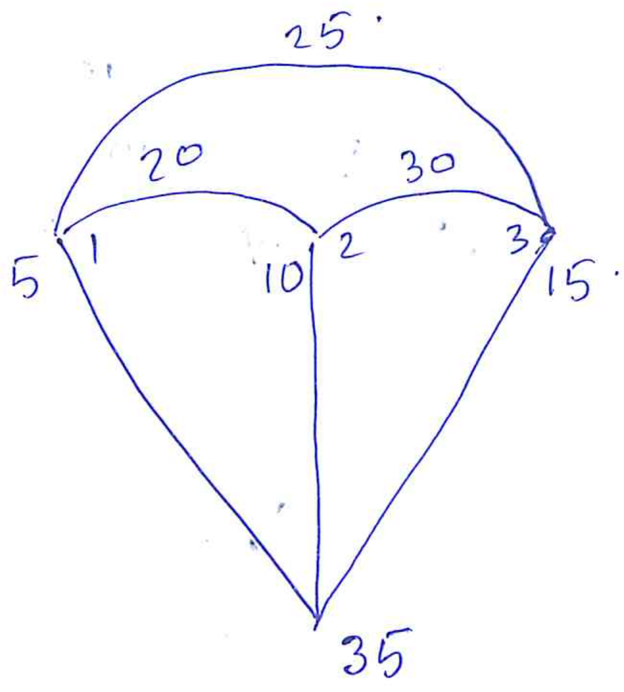


Fig 2.

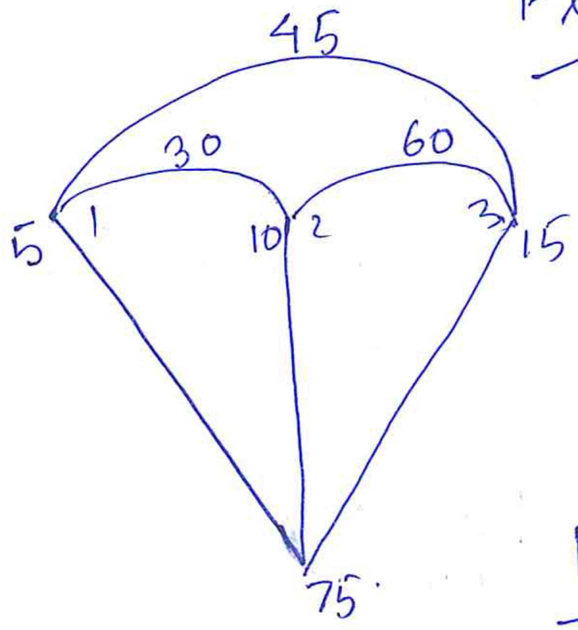
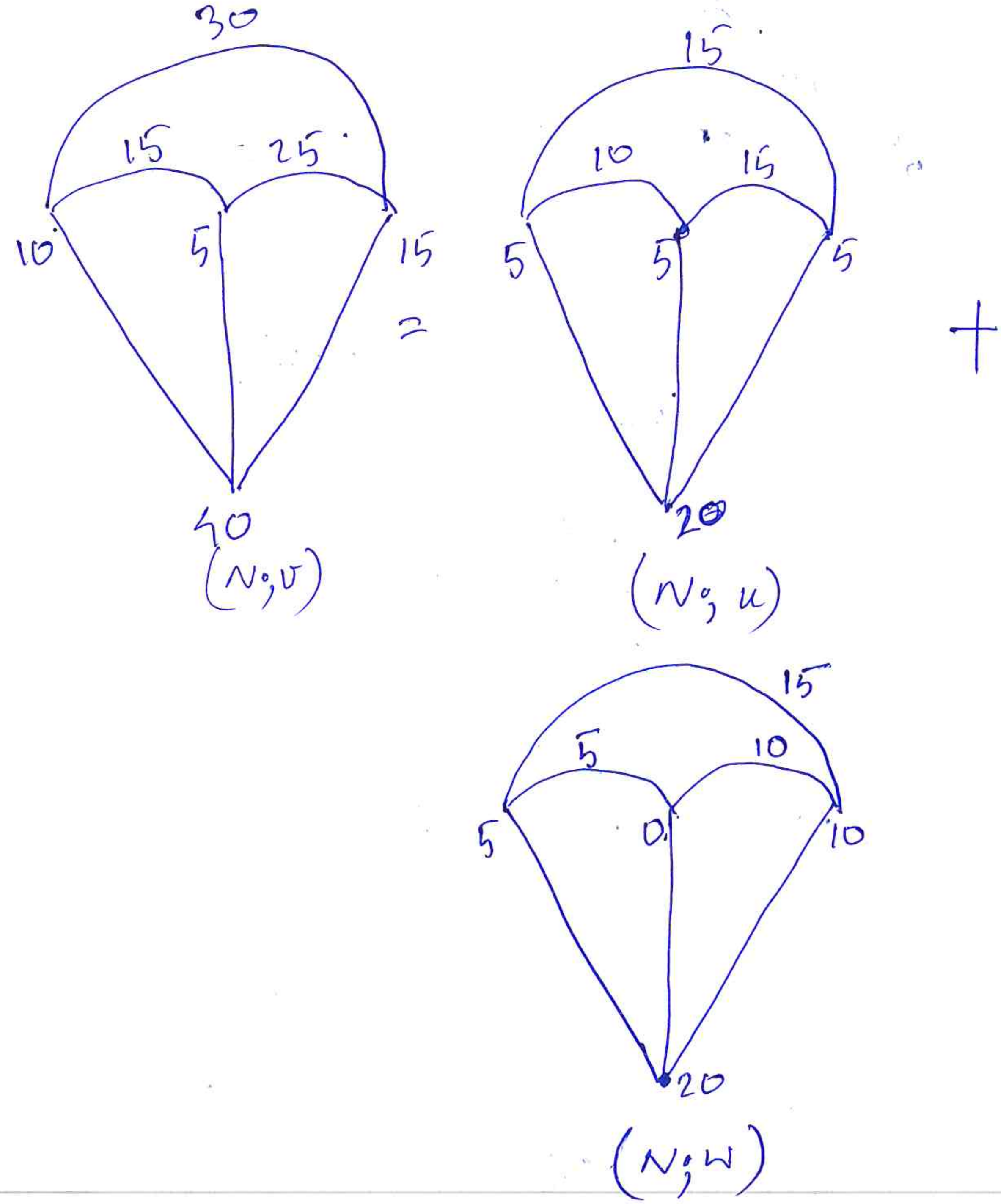


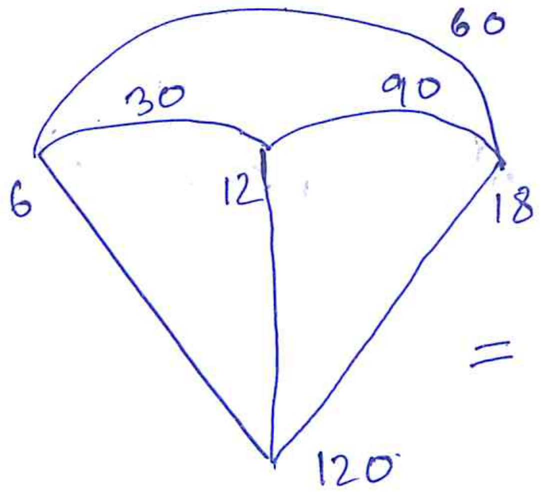
Fig 3.



- Split (N; v) into (N; u) and (N; w).

Fig 4.

Consider the game



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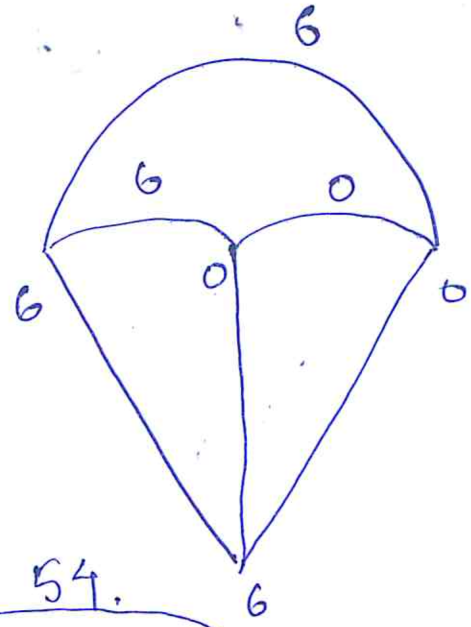

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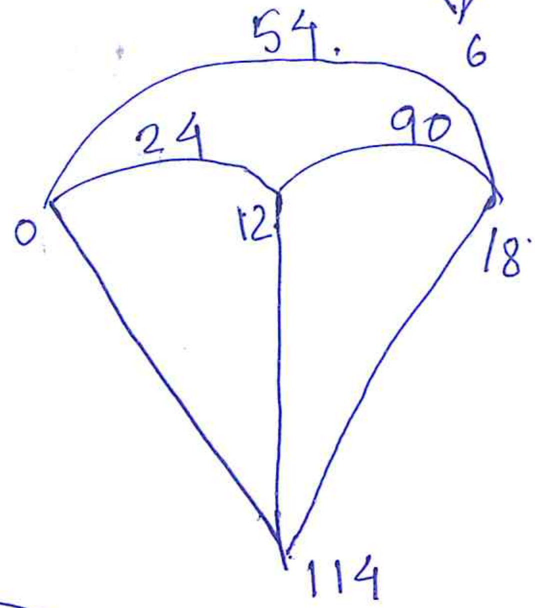

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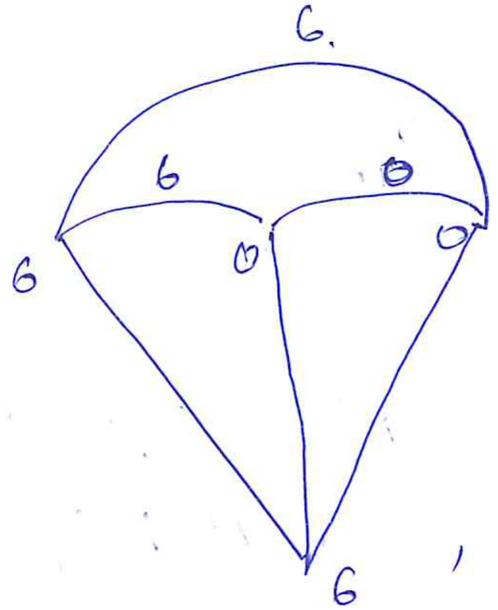
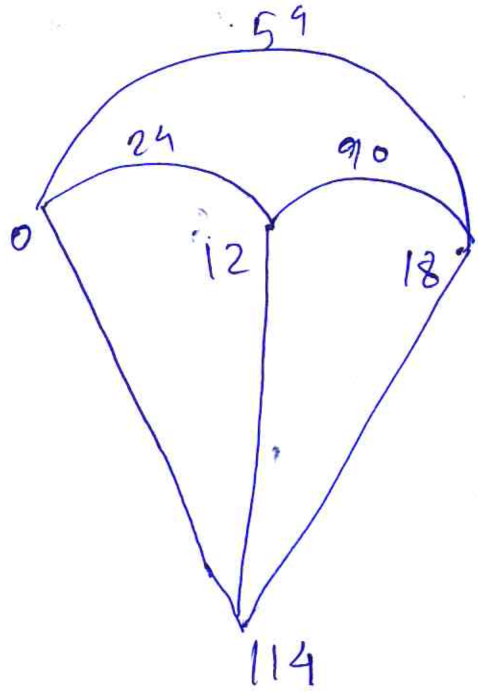


player 2, 3 are null players.

Division is $(6,0,0)$



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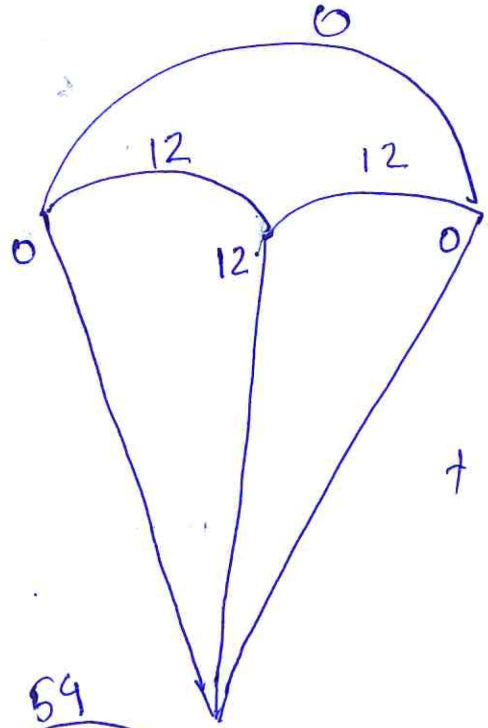

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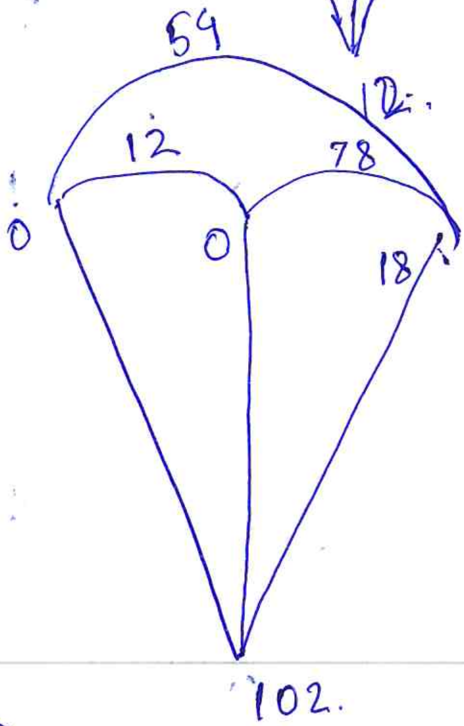
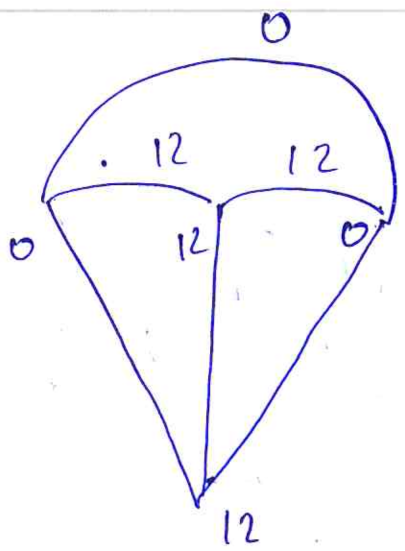
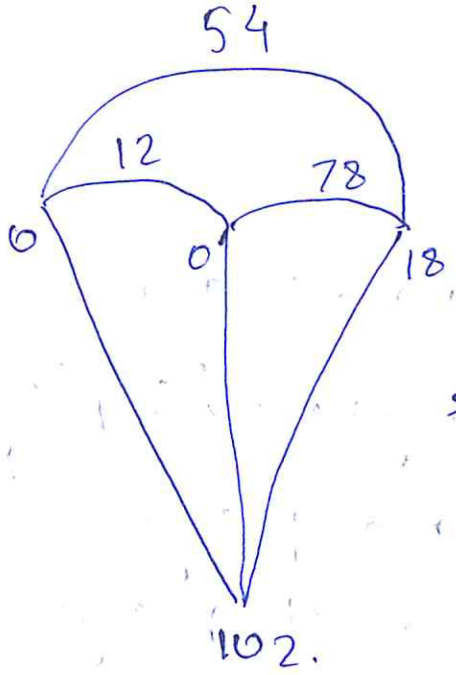




,

Player 1, 3 are
null player.

Decision is
(0, 12, 0).



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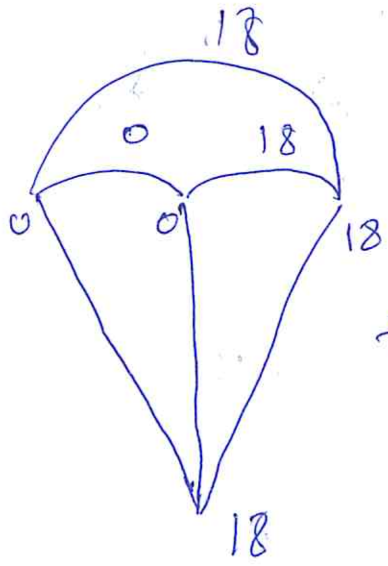

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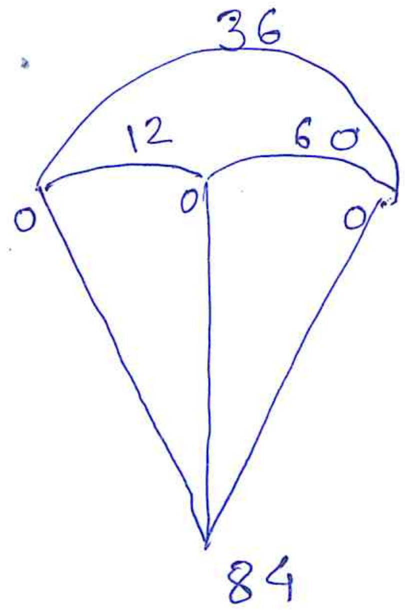
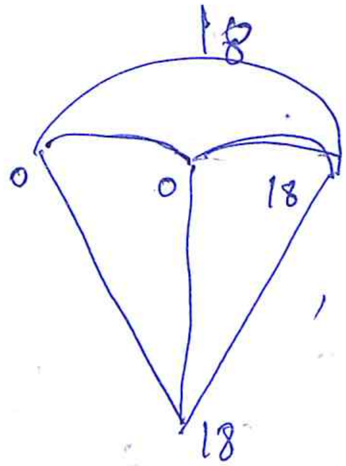
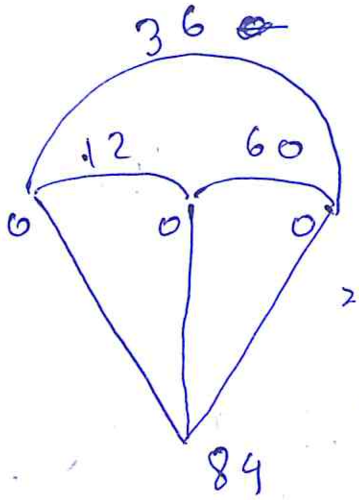




, Player 1, 2 are null players.

Decision is = (0, 0, 18).



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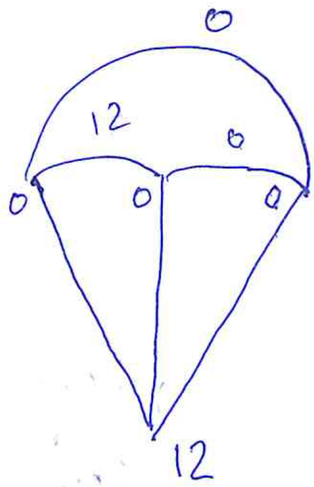

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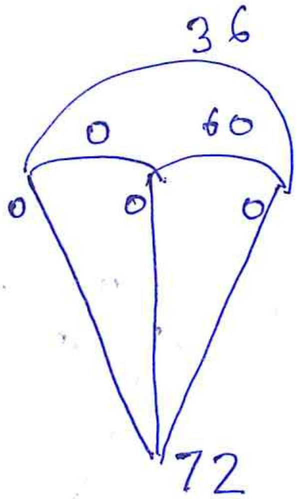
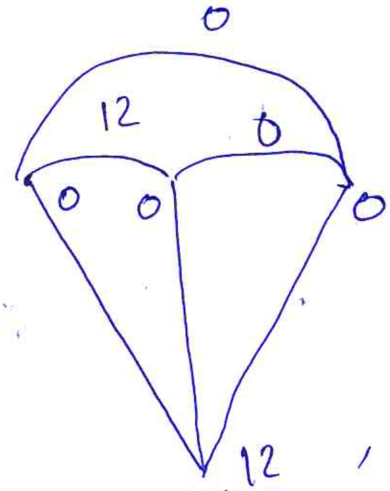


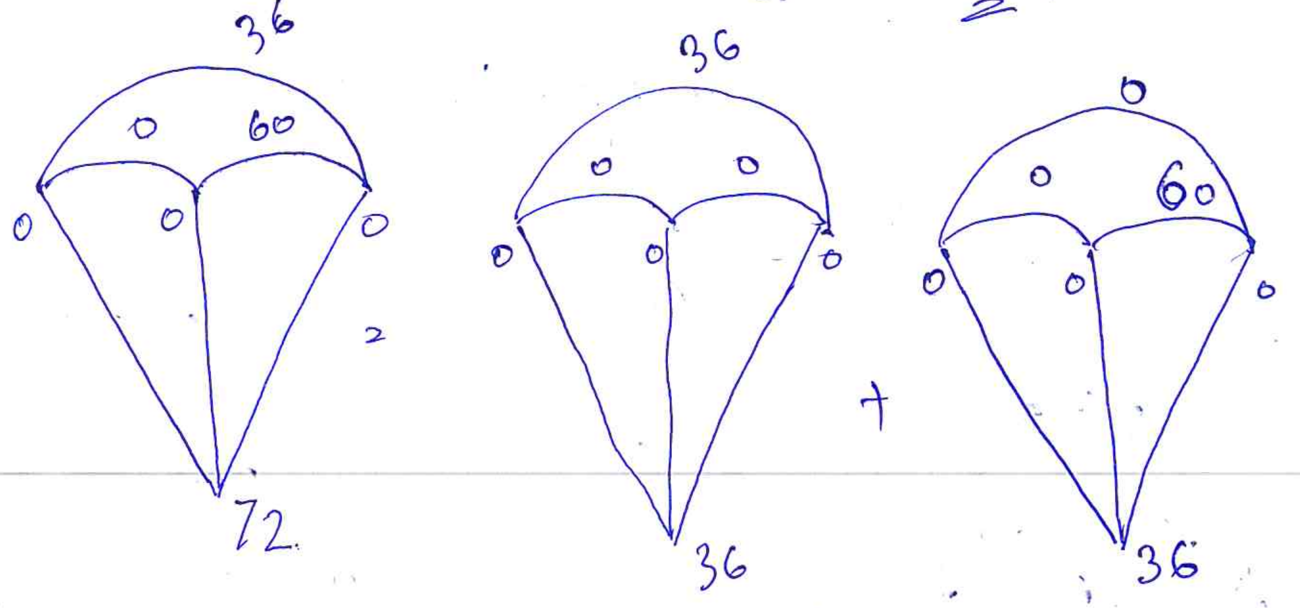
2n



1 and 2 are symmetric player and player 3 is null player.

so Division is

(6, 6, 0).



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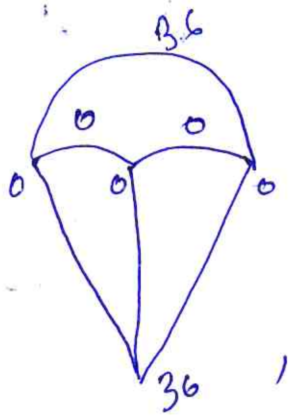
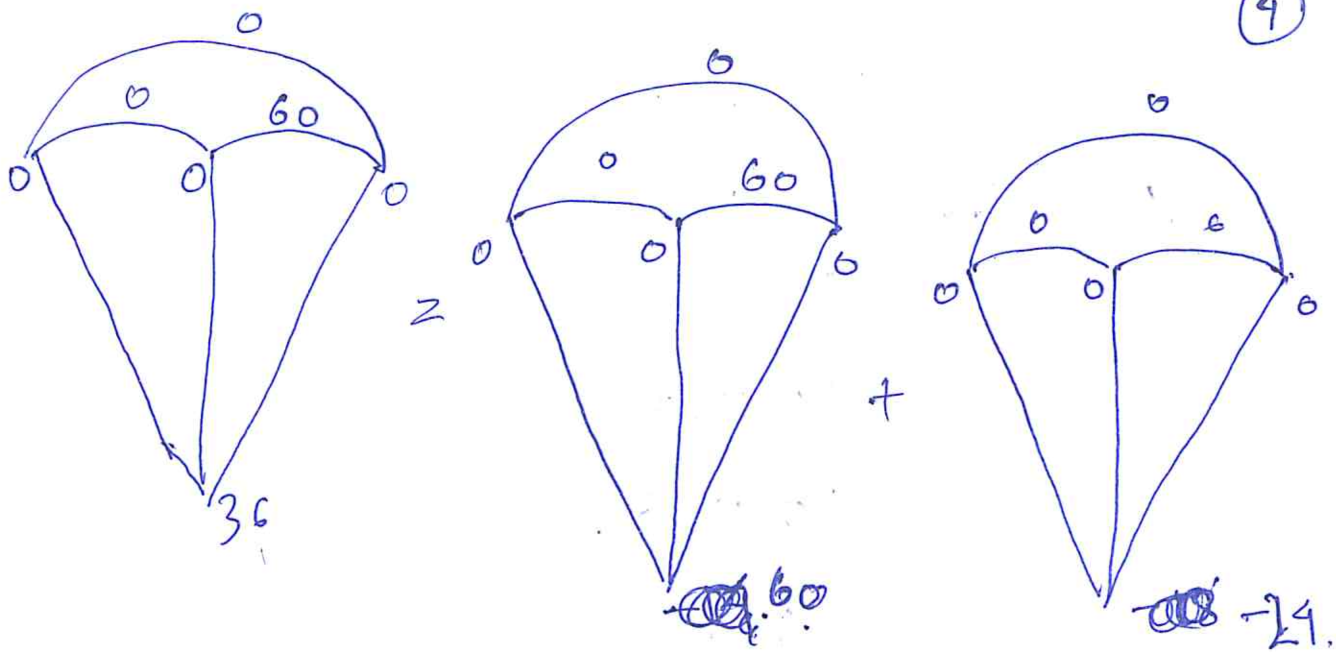


player 1, 3 are symmetric player. player 2 is a null player..

So. (18, 0, 18)



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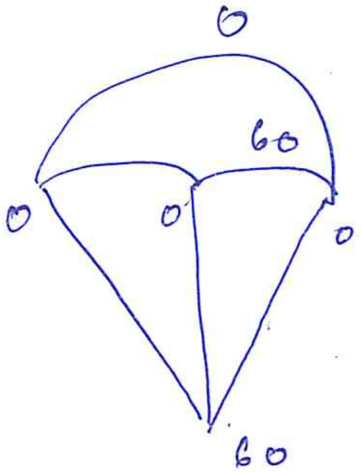
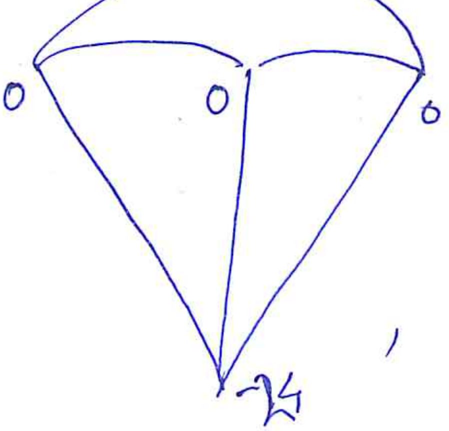


, player 2 and 3 are symmetric player
player 1 is null player

So. Division is
$(0, 30, 30)$.

In



, players 1, 2, and 3 are symmetric.
So the decision is
$(-24/3, -24/3, -24/3)$.

Divisions are:

$(6, 0, 0)$

$(0, 12, 0)$

$(0, 0, 18)$

$(6, 6, 0)$

$(18, 0, 18)$

$(0, 30, 30)$

$\left(\frac{-24}{3}, \frac{-24}{3}, \frac{-24}{3}\right)$.

$$= \left(6+0+0+6+18+0-\tfrac{24}{3},\ 0+12+0+6+0+30-\tfrac{24}{3},\ 0+0+18+0+18+30-\tfrac{24}{3}\right).$$

$= (22, 40, 58)$: Shapley Value

Another way to compute Shapley value
Consider the following game
$N = \{1, 2, 3\}$
$v(1) = 6, \; v(2) = 12, \; v(3) = 18$
$v(1, 2) = 30, \; v(1, 3) = 60, \; v(2, 3) = 90$
$v(1, 2, 3) = 120, \; v(\emptyset) = 0.$

In the last class using carrier coalitions, we compute the Shapley value as $(22, 40, 58)$.

Allocation based on marginal contribution of each player to the coalition.
$v(1,2) - v(2) =$ marginal contribution of player 1.
$v(1,2) - v(1) =$ marginal contribution of player 2.
$v(1,2,3) - v(1,2) =$ marginal contribution of player 3.
We can take the following sequence
$v(1) = 6$, marginal contribution of player 1
$v(1,2) - v(1) = 30 - 6 = 24$ marginal contribution of player 2
$v(1,2,3) - v(1,2) = 120 - 30 = 90$ marginal contribution of player 3.
The sequence is player 1, player 2, player 3.

Suppose the sequence is player 2, player 3, player 1.
$v(2) = 12$, marginal contribution of player 2.
$v(2,3) - v(2) = 90 - 12 = 78$ marginal contribution of player 3.
$v(1,2,3) - v(2,3) = 120 - 90 = 30$ marginal contribution of player 1.
As we change the position, the marginal contribution changes.

Shapley value is the average of marginal contribution of each player taking into account all the possible order.

In case of the above game, the computation is shown below.

Order of entry	Player 1	Player 2	Player 3
1 2 3	6	30-6=24	120-30=90
1 3 2	6	120-60=60	60-6=54
2 3 1	120-90=30	12	90-12=78
2 1 3	30-12=18	12	120-30 90
3 1 2	60-18=42	120-60= 60	18
3 2 1	120-60=30	90-18=72	18
average	22	40	58

$$\phi_1 = \frac{6+6+30+18+42+30}{6},$$
$$\phi_2 = \frac{24+60+12+12+60+72}{6}$$
$$\phi_3 = \frac{90+54+78+90+18+18}{6}$$

For any coalition game $(N; v)$, the Shapley value is

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(N-|S|)!}{N!} \Big[v(S) - v(S - \{i\}) \Big],$$
$i = 1, 2, 3...N$ and $S \subset N$

In a game with N players, total number of order of players is $N!$. Suppose player i enter the coalition. It will join a coalition which will have $S - \{i\}$ players. The contribution of player i is $v(S) - v(S - \{i\})$. $S - \{i\}$ player can come first in $(|S| - 1)!$ ways. The remaining $N - |S|$ players can be ordered in $(N - |S|)!$ ways. So, in $(|S| - 1)!(N - |S|)!$ ways out of $N!$ ways, player i is going to join the coalition after $S - \{i\}$ players have joined.

In the above example, Probability of Player 1 in first position; $\dfrac{2!0!}{3!}$

Probability of Player 1 in second position; $\dfrac{1!1!}{3!}$ and $\dfrac{1!1!}{3!}$

Probability of player 1 in third position ; is $\dfrac{2!1!}{3!}$

Similarly we can find for player 2 and player 3.

Compare core and Shapley value

Theorem: There exists a unique function ϕ satisfying the four axioms of Shapley. It is given by

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(N-|S|)!}{N!} \Big[v(S) - v(S - \{i\}) \Big],$$

$i = 1, 2, 3 ... N.$

Proof:

For a given coalition $S \subset N$, suppose w_S is a characteristic function such that

$$w_S(T) = \begin{cases} 1 \text{ if} S \subset T \\ 0 \quad otherwise \end{cases}$$

So, $(N; w_S)$ is a carrier game.

From above, it is clear that $w_S(S \cup \{i\}) = w_S(S) = 1$ when $i \notin S$.

This implies that $\phi_1(W_S) = 0$. From the null player axiom.

Now if $i, j \in S$, then $\phi_i(w_S) = \phi_j(w_s)$ using the axiom that symmetric players must get same payoffs. Take a coalition $S^{'} \subset S$ and $i, j \notin S^{'}$. $w_S(S^{'}) = 0$, then $w_S(S^{'} \cup \{i\}) = w_S(S^{'} \cup \{j\})$.

Now for a given S we get $\sum_{i \in N} \phi_i(w_S) = w_S(N) = 1$ from efficiency axiom. This implies $\phi_i(w_S) = \frac{1}{|S|}$ for all $i \in S$.

Suppose we take the characteristic function cW_S where c is real number, using the similar steps we get

$$\phi_i(cw_S) = \begin{cases} \frac{c}{|S|} & \text{for} i \in S \\ 0 & \text{for} i \in S \end{cases}$$

Now we show that any characteristic function v defining a coalition game, can be represented as a weighted sum of these characteristic function giving us a carrier game. It is $v = \sum_{S \subset N} c_S w_S$ where c_S is chosen appropriately.

Using the axiom that a characteristic function can be sum of characteristic function, we get that

$$\phi_i(v) = \sum_{S \subset N, i \in N} \frac{c_S}{|S|}, \ i \in N$$

where summation is taken over all coalitions in which i belongs. Here c_S can be negative also.

We have to show that any v can be represented as $v = \sum_{S \subset N} c_S w_S$.

To do this we need to find the c_S. Assume that $c_\emptyset = 0$. Note that c_S is indexed on coalitions.

For all $T \subset N$, we define

$c_T = v(T) - \sum_{S \subset T,\ S \neq T} c_S$.

Each c_T is defined in terms of c_S where S has less number of members than T. We are using induction, $c_i = v(i)$ for all $i \in N$.

We have $v(T) = c_T + \sum_{S \subset T,\ S \neq T} c_S$

This implies $c_T + \sum_{S \subset T,\ S \neq T} c_S = \sum_{S \subset T} c_S$.

This can be written as

$\sum_{S \subset T} c_S = \sum_{S \subset N} c_S w_S$. Thus, $v(T) = \sum_{S \subset N} c_S w_S$.

Now, we have to show that $\phi_i(v) = \sum_{S \subset N, i \in N} \frac{c_S}{|S|}$ satisfies all the four axioms. This part is obvious.

Weighted majority game

The game is defined in the following way

$[q, w_1, w_2, w_3..., w_N]$, where N players, weight of each player is w_i and q is the quota.

If $\sum_{i \in S} w_i \geq q$ then S is a winning coalition. The characteristic function is

$$v(S) = \begin{cases} 1 \text{ ‘if } S \textit{ is a winning coalition} \\ 0 \textit{ if } S \textit{ is a losing coalition} \end{cases}$$

In these types of game, we measure the power or strength of a party.

Consider a weighted majority game;
$[10; 5, 8, 2, 3]$

$$v(S) = \begin{cases} 1 \text{ if } S \text{ is a winning coalition} \\ 0 \text{ if } S \text{ is not a winning coalition} \end{cases}$$

Take the order of players $1, 2, 2, 4$
The marginal contribution of player 1 is zero, marginal contribution of player 3 and 4 also zero.
The marginal contribution of player 2 is 1.
Here player 2 is a pivotal player.

Take another order of players $3, 4, 1, 2$
The marginal contribution of player 13 and 4 is zero, marginal contribution of player 1 also 1.
The marginal contribution of player 2 is 0.
Here player 1 is a pivotal player.

Result: In a weighted majority game, there is exactly one pivotal player in every order.

We add player one by one in a coalition, there is always a player i who turns a losing coalition into a winning coalition. Suppose, we need to add more then one player to turn a winning coalition into a winning coalition. This set of players can be partitioned where each partition contains only one player. Suppose there are k such partitions. Now, if we add one by one in the coalitions, after adding $k-1$ players, if the coalition is still not a winning coalition then we need to add the kth player. That makes kth player pivotal player. Thus, there is only one player who is pivotal. If $k-1$ players can make the coalition a winning coalition, in this case kth player is no more pivotal. If we drop the $k-1$ th player and the coalition is no more winning coalition then $k-1$ th player is pivotal. We can continu in this way, till it has only one player, and that one player is going to be pivotal. If that player is not pivotal, then the initial coalition is not a losing coalition. We have shown that only one player can be pivotal.

$[10; 5, 8, 2, 3]$, in this game, number of ways player 1, can be pivotal
If player 1 is in position 1, it is not possible.
If player 1 is in second position, $2134, 2143$; 2 ways
If player 1 is in third position, $3412, 4312$: 2 ways
If player 1 is in fourth position, the possible ways
$2341, 2431, 3421, 3241, 4231, 4321,$; not pivotal in any one of them.
In 4 ways player 1 can be pivotal out of 24 ways they can be positioned. So $\frac{4}{24}$ is the strength of player 1. Also Shapley value.

Player 2, number of ways it can be pivotal

If player 2 is position 1; zero number of ways

If player 2 is in second position:

$1234, 1243, 3214, 3241, 4213, 4231$; 6 ways

If player 2 is in third position : $3421, 4321, 1423, 4123, 1324, 3124$; 6 ways

If player 2 is in fourth position: not possible.

The strength/power of player 2 is $\frac{12}{24}$, its Shapley value.

Player 3, number of ways it can be pivotal
If player 3 is first position: not possible
If player 3 is in second position: $2314, 2341$: two ways
If player 3 is in third position: $1432, 4132$ two ways
If player 3 is in fourth position: not possible
Strength of player 3 is $\frac{4}{24}$, its shapley value.

Player 4, number of ways it can be pivotal
If player 4 is first position: not possible
If player 4 is in second position: $2413, 2431$: two ways
If player 4 is in third position: $1342, 3142$ two ways
If player 4 is in fourth position: not possible
Strength of player 4 is $\frac{4}{24}$, its Shapley value.

Result:

In a weighted majority game of N players, a players Shapley value is: $\frac{\text{number of times player is pivotal}}{\text{number of possible orders } (N!)}$

Shapley Shubik power index of players is $\frac{\text{number of times player is pivotal}}{\text{number of possible orders } (N!)}$

Cost Sharing

Cost sharing game
$(N; c)$ where c is the characteristic function denoting cost going to be shared between the members of the coalition.

The cost of setting up distribution network; cable TV, water supply, electricity etc

The cost allocation can be done using Shapley value

Order	1	2	3
123	24	12	24
132	24	12	24
213	18	18	24
231	0	18	42
312	0	12	48
321	0	12	48
Average	11	14	35

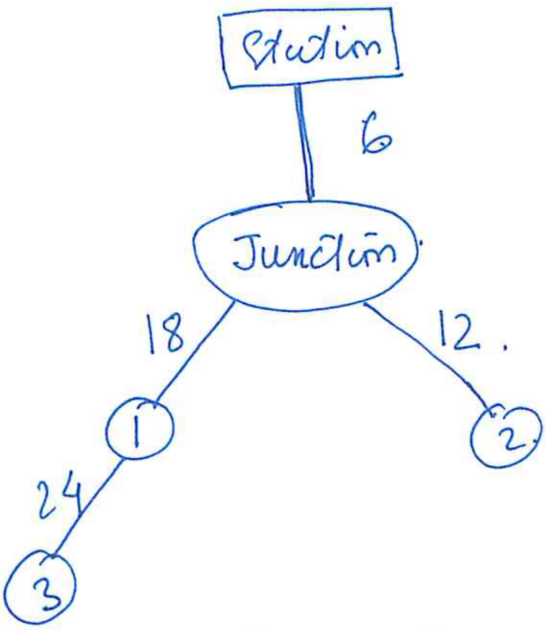
Shapley value is $(11, 14, 35)$

Another way to get this outcome
Every one uses the first segment (edge) , suppose segment 1. The segment (edge) connecting junction and player 2 is is only used by player 2, suppose segment 2. The segment (edge) connecting junction and 1 is shared between player 1 and 3, suppose 3. The segment connected player 1 and player 3 is only used by player 3, suppose segment 4.

Each segment is equally shared among the users.

segment	1	2	3
segment 1	2	2	2
Segment 2	0	12	0
Segment 3	9	0	9
Segment 4	0	0	24
aggregate	11	14	35

Result:
Equal sharing of cost of each segment among the players who use the segment gives Shapley value of the cost sharing game.



$N = \{1, 2, 3\}$.

$C(\emptyset) = 0$

$C(1) = 6 + 18 = 24$

$C(2) = 6 + 12 = 18$

$C(3) = 6 + 18 + 24 = 48$

$C(1,2) = 6 + 12 + 18 = 36$

$C(1,3) = 6 + 18 + \cancel{48}\, 24 = 48$

$C(2,3) = 6 + 18 + 24 + 12 = 60$

$C(1,2,3) = 6 + 12 + 18 + 24 = 60$

Fair Allocation

Bankruptcy problem:

- Suppose a person borrows money from two creditors, c_1 and c_2.
- The borrower cannot repay the amount $c-1$ and c_2 as he has gone bankrupt.
- The worth of the asset created from these borrow in M, and $M \leq c_1 + c_2$.
- How do we allocate M between creditor 1 and 2?
- We have to do some form of rationing.

Surplus sharing problem:

- Suppose two person forms a joint venture.
- Person 1 invests c_1 amount and person 2 invest c_2 amount.
- Suppose the revenue generated from this joint venture is M and $M > c_1 + c_2$. It means the joint venture has made profit.
- How do we divide this M between the partners 1 and 2?
- If the $M < c_1 + c_2$, the joint venture has made loss. How to divide this M between the partners 1 and 2?

Rationing of medical supplies:

- Suppose there are two patients, each needs c_i amount of injections.
- The hospital authority has only M units and suppose $M < c_1 + c_2$.
- How do we divide this M between these two patients?

It is based on

- Allocation/division rules:
  - Proportional rule
  - Constrains equal award rule (equal sharing)
  - Constraint equal loss or constraint equal gain rule
- Allocation procedure:
  - Contested garment method
  - Rif method
  - O'Neill's division method.

Proportional method:

Given $((M, c)$ where c is claim vector and M is the endowment. $x = (x_1, x_2)$ is the final allocation to person 1 and 2 or claimant 1 and 2.

$$x_1 = \frac{c_1 M}{c_1 + c_2}, \quad x_2 = \frac{c_2 M}{c_1 + c_2}.$$

Each claimant gets proportional to its claim.

Constraint equal awards rule:

Given $((M, c)$ where c is claim vector and M is the endowment.
$x_1 = \min\{c_1, \lambda\}, \quad x_2 = \min\{c_2, \lambda\}$, where λ is such that
$\sum_{i=1}^{2} x_i = M$.

Each claims gets which ever is less its claim or an amount which is same for all the claimants.

First divide M equally $\frac{M}{N}$ among the players. If $\frac{M}{N} > c_i$ for some i players . Allocate c_i to those players and rest $M - \sum\limits_{i \in S} c_i$ where $S \subset N$, is allocated equally among the remaining players. In this remaining players, if there are some j players whose $\frac{M - \sum\limits_{i \in S} c_i}{|N - S|} < c_j$, allocate c_j to those players and the remaining $M - \sum\limits_{i \in S} c_i - \sum j \in S_1 c_j$ is equally divided among the rest of the claimants.

Constraint equal losses rule:

Given $((M, c)$ where c is claim vector and M is the endowment.
$x_1 = \max\{c_1 - \theta, 0\}, \quad x_2 = \max\{c_2 - \theta, 0\}$ where θ is such that $x_1 + x_2 = M$.
$\theta = \dfrac{c_1 + c_2 - M}{2}$, if $\dfrac{c_1 + c_2 - M}{2} < c_i, i = 1, 2.$
$\theta = M - c_1 - c_2$ if any c_i is less than $\dfrac{c_1 + c_2 - M}{2}$. Both cannot be less because in that case $C_1 + c_2 > M$ is not satisfied.

In the Bankruptcy problem:

$(C; M)$ where C is vector of claims $C = (c_1, c_2, c_3 ... c_N)$ and M is the endowment or the size of the cake that needs to be divided.

$\sum_{i=1}^{N} c_i > M$.

From the figure 1, it is clear that the claimants with higher claims prefer constraint equal loss rule and the claimants with lower claims prefer constraint equal awards rule.

The proportional rule is preferred to constraint equal award rule when the claims are higher.

The constraint equal loss rule is preferred to proportional rule for the claimants with higher claims.

The claimants with lower claims prefer constraint equal award rule.

In case of surplus sharing problem:
The proportional rule remains same.
The constraint equal awards rule is simply sharing equally the total value of the endowment (egalitarian rule).
The constraint equal loss rule is equal surplus sharing. The surplus amount $M - c_1 - c_2$ is equally shared.

Surplus sharing problem:
$(C; M)$ where C is vector of investments $C = (c_1, c_2, c_3...c_N)$ and M is the total revenue or the size of the cake that needs to be divided.
$\sum_{i=1}^{N} c_i < M$.
From the figure 2, it is clear that the claimants with higher claims prefer proportional rule rule and the claimants with lower claims prefer constraint equal awards rule (egalitarian rule).
The proportional rule is preferred to equal surplus sharing rule when the claims are higher.
The equal surplus sharing rule is preferred to constraint equal awards rule (egalitarian rule) for the claimants with higher claims.
The claimants with lower claims prefer constraint equal award rule (egalitarian rule).

Properties of rule

- Feasibility: Given (C, M), $\sum_{i=1}^{N} = M$, sum of the allocations must be equal to the endowment or size of the cake.
- Bounds on each allocation, $0 \leq x_i \leq c_i$ for all $i \in N$.
- Equal treatment of equals: Given any claims problem (C, M), and for any i, j claimants, if $c_i = c_j$, then $x_i(c, M) = x_j(C, M)$. If the claims of any two claimants are same, they must be allocated same amount.

- Claims monotonicity: Given a claims problem (C, M), for any claimant i, if $c_i' > c_i$, then $x_i(C, M) \leq x_i(c_i', c_{-i}, M)$.
  If the claim of any claimant i increases from c_i to c_i' and teh claim of teh rest of the claimants remains same, the amount allocated to claimant i should be greater than equal to the amount allocated when his claim was c_i.
  Here $C = (c_1, c_2, c_3, c_3....c_i, c_{i+1}...c_N)$, the claim vector $c_{-i} = (c_1, c_2, c_3, c_3....c_{i-1}, c_{i+1}...c_N$, the claims of all the claimants except claimant i is an element of this vector.
  In case of three claimants, $C = (c_1, c_2, c_3)$,
  $c_{-2} = (c_1, c_3), \;\; (c_2, c_{-2}) = (c_1, c_2, c_3)$.

- Order preservation in awards: Given any claims problem (C, M), For any two claimants i, j, if $c_i \geq c_j$, then $x_i(C, M) \geq x_j(C, M)$.
  If the claim of a claimant is greater than some other claimant, then the claimant with greater claims should be allocated greater amount.

  Another version of this property is
  Given any claims problem (C, M), For any two claimants i, j, if $c_i \geq c_j$, then $x_i(C, M) \geq x_j(C, M)$ and $c_i - x_i(C, M) \geq c_j - x_j(C, M)$.
  If the claim of a claimant is greater than some other claimant, then the claimant with greater claims should be allocated greater amount and also, the loss made by the claimants with higher claims is greater than the loss made by the claimant with lower claims.

Bankruptcy.

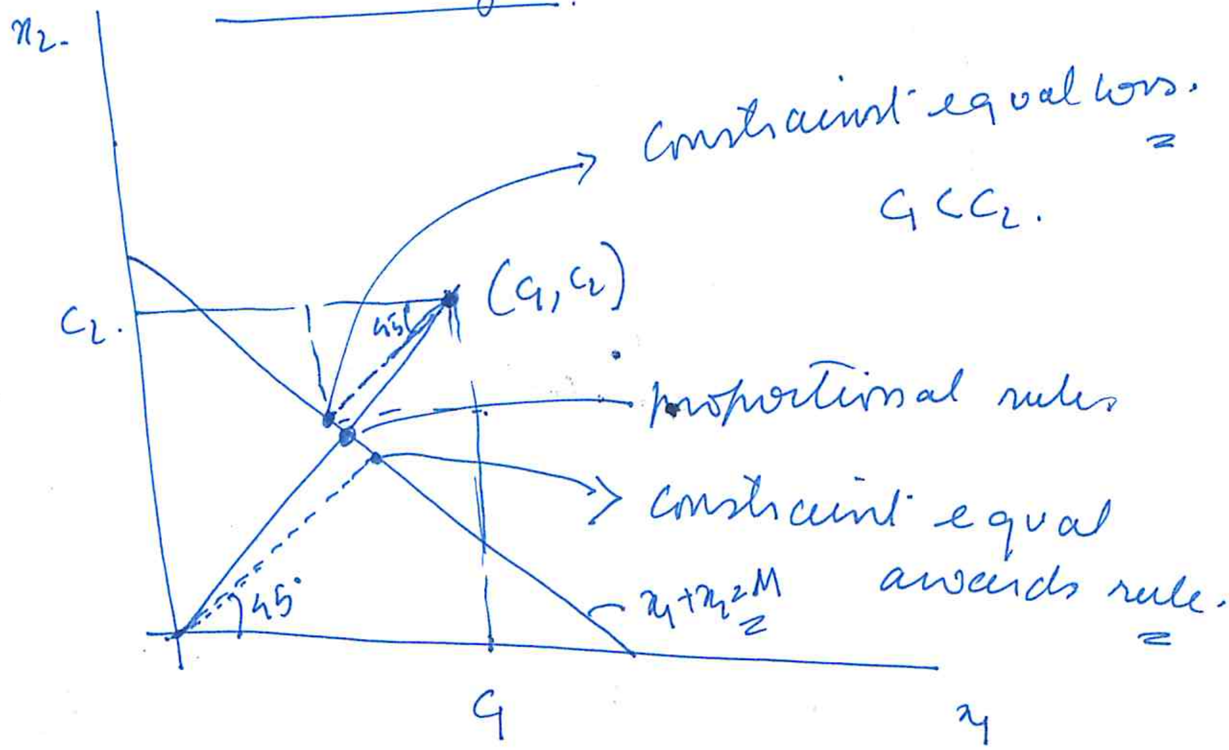


Fig 1

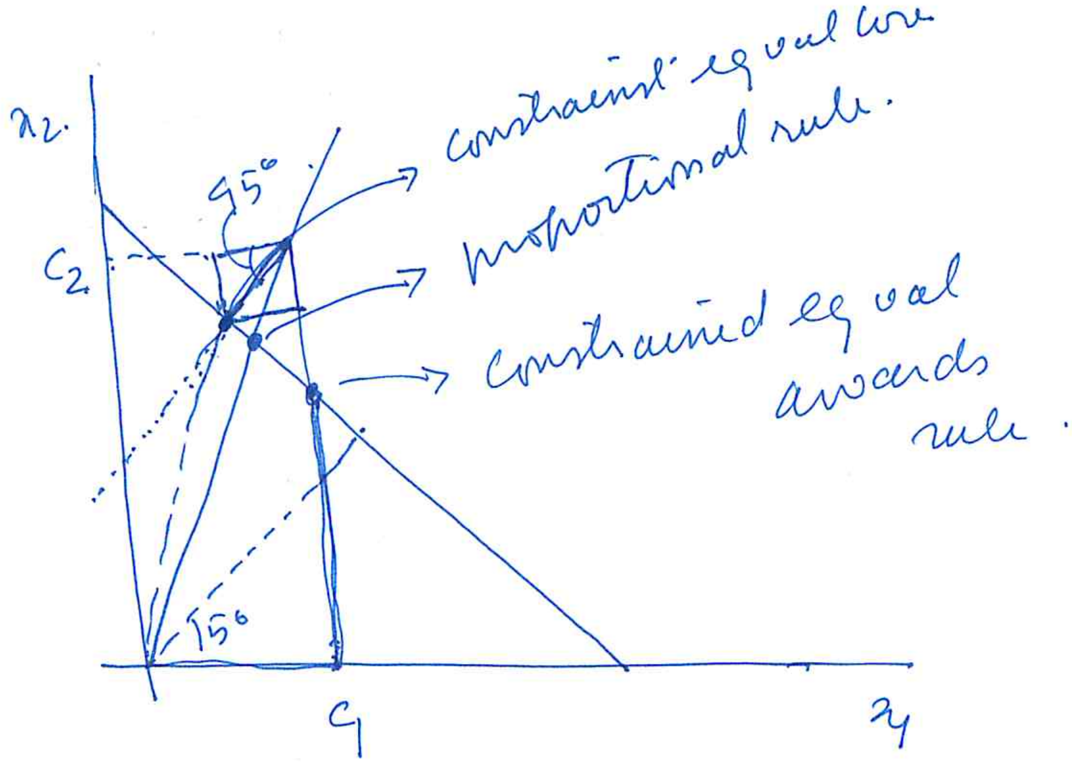
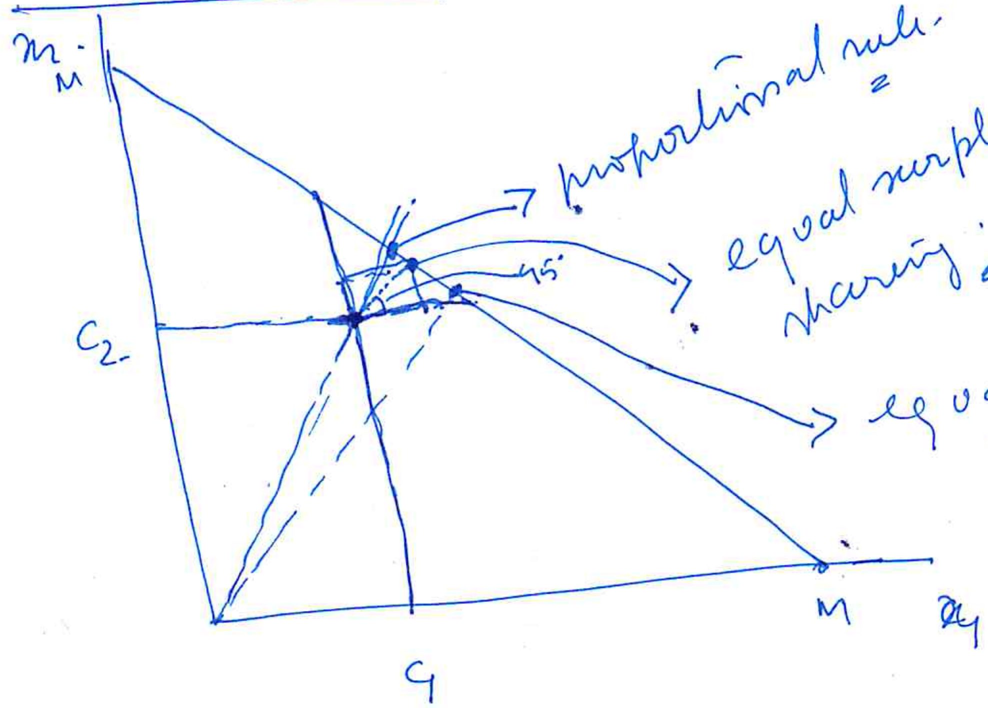


Fig 1'

Surplus sharing.



$c_1 < c_2$

Equal surplus:

$$x_1 = c_1 + \left(\frac{M - c_1 - c_2}{2}\right),$$

$$x_2 = c_2 + \left[\frac{M - c_1 - c_2}{2}\right]$$

Equal sharing

$$x_1 = \frac{M}{2}$$

$$x_2 = M/2.$$

Figure 2

Properties

- Resource monotonicity: Given a claims problem (C, M), for each $M' \in R_+$, if $\sum_{i=1}^{N} c_i \geq M" > M$, then $x_i(C, M') \geq x_i(C, M)$ for all $i \in N$.
  If the size of the cake or endowment increases, then all the claimants should get atleast what they were getting previously.
- Invariance under claims truncation: Given a claims problem (C, M), $x(C, M) = x((\min\{c_i, M\}_{i \in N}, M)$.
  Suppose the claims of some claimants are greater than M the size of cake, their claim is truncated and made M. This rule says that the allocation is such that each claimant should get same amount when their claim is truncated and when it is not truncated.
  Suppose claims are; $(10, 20, 100)$ and $M = 60$, if we follow constraint equal awards rule $x = (10, 20, 30)$.
  Instead, if we truncate the claim of claimant 3, the claim vector is $(10, 20, 60)$, constraint equal awards rule allocate,

If you use proportional rule, the allocation is $(\frac{60}{130}, \frac{120}{130}, \frac{600}{130})$. After truncation, the allocation is $x = (\frac{20}{3}, \frac{40}{3}, 40)$. They are not same. Constrainst equal award rule satisfies this rule.

- Minimal rights first: Given a claims problem (C, M),
$$x(C, M) = m(C, M) + x(c - m(C, M), M - \sum_{i=1}^{N} m_i(C, M)),$$
where $m_i(c, M) = \max\{M - \sum_{j \in N - \{i\}} c_j, 0\}$ for each claimant i.
Each claimant is given its minimal right. The minimal right of a player is the amount remaining after giving all the other claimants its claim. The sum of the minimal amount is less than the size of the cake. After giving the minimal rights amount, the remaining is allocated using a division rule, here the claim of each claimant is their original claim minus the minimal amount.

Example: Claim, $C = (10, 20, 60)$ and $M = 40$
The minimal amounts are:
$m_1 = \max\{40 - (20 + 60, 0\} = 0$, $m_2 = \max\{40 - (10 + 60), 0\} = 0$, $m_3 = \max\{40 - (10 + 20), 0\} = 10$.
Now we have to allocated $40 - 10 = 30$ among the three claimants. Their claims are $(10 - 0, 20 - 0, 60 - 10)$. We can use an allocation rule for this division.
A division rule satisfying minimal rights first must allocate in the same way, if the allocation is done based original claims problem and when first minimal rights amount is given to each claimants and then the remaining cake is allocated using the same rule taking the clams after subtracting the minimal amount of each player.
In the above example, if we use constraint equal loss rule, the allocation is $x = (0, 0, 40)$.
The minimal rights are; $m = (0, 0, 10)$. The claims problem is $(10, 20, 50)$ and $40 - 10 = 30$ size of the cake.

Apply constraint equal loss rule to $C - m = (10, 20, 50)$ and $M - \sum_{i=1}^{3} m_i = 30$, the allocation is $(0, 0, 30)$. So, the final allocation is $(0, 0, 40)$. It is same as constraint equal loss rule. Constraint equal awards rule does not satisfy minimal rights rule for all claims problem.

- Composition down: Given a claims problem (C, M), for each $M' < M$, we have $x(C, M') = x(x(c, M), M')$.
  Suppose allocation is promised for the problem with claims C and M as size of the cake using a particular division rule. If the size of the cake depreciates and its M', then the allocation is done based on taking promised allocations as the new claims using the same division rule. The division rule satisfying this property must give same allocation when the allocation is done using C as claims and M' as the size of the cake and when $x(C, M)$ as the claims and M' as the size of the cake.

Example: $C = (10, 30, 40)$ and $M = 40$. If we use proportional rule, the allocation is $x(C, M) = (5, 15, 20)$.
Now suppose the size of the cake $M' = 30$, we take the claims to be $x(C, M) = (5, 15, 20)$. If we use proportional use, the allocation is $x(x(C, M), M') = (\frac{15}{4}, \frac{45}{4}, 15)$.
If we apply proportional rule to the problem $(10, 30, 40)$ and 30, the allocation is $x(C, M') = (\frac{30}{8}, \frac{90}{8}, 15)$. It is same as the above. Proportional rule satisfies composition down.

- Composition up: Given a claims problem (C, M) and each M', if $\sum_{i=1}^{N} c_i \geq M' > M$, then
  $x(C, M') = x(C, M) + x(c - x(C, M), M' - M)$.
  Suppose given a claims problem, there is a promised allocation to the claimants using a particular division rule. Now, suppose the size of the cake appreciates and it is $M' > M$. The allocation is done first as promised earlier. Next, the claims of each claimant is reduced by the amount received and the remaining size of the cake is divided using the same rule.

Example:
$C = (10, 30, 40)$ and $M = 30$. Suppose we use proportional rule, the allocation is $x = (\frac{30}{8}, \frac{90}{8}, 15)$. Now suppose the size of the cake increases to 40.
We first allocation based on the claims C and cake 30, so $x = (\frac{30}{8}, \frac{90}{8}, 15)$. The remaining cake is $40 - 30 = 10$, the reduced claims are $c'_1 = 10 - \frac{30}{8}, \quad c'_2 = 30 - \frac{90}{8}, \quad c'_3 = 40 - 15 = 25$.
Apply proportional rule to this problem, the allocation is $x(c', M' - M) = (\frac{5}{4}, \frac{15}{4}, 5)$. The final allocation is $(5, 15, 20)$.
When proportional rule is applied to the problem $(10, 30, 40)$ and 40, the allocation is $(5, 15, 20)$. Proportional rule satisfies it.

- No advantageous transfer: Given a claims problem (C, M), for each $S \subset N$ and each $(c_i^{'})_{i \in S} \in R_+^{|S|}$, if $\sum_{i \in S} c_i^{'} = \sum_{i \in S} c_i$ then $\sum_{i \in S} x_i(C, M) = \sum_{i \in S} x_i((c_i^{'})_{i \in S}, c_{N-S}, M)$.
  Suppose a subset of claimants form a group S, the claims in S transfer their claims among themselves but the sum of the claims remain same, these claimants should not gain any additional amount by this transfer of claims.
  Example:
  $C = (10, 30, 40)$ and $M = 30$. Suppose there is group of claimant 1 and 2. The sum of their claims is $10 + 30 = 40$. If their is transfer between them say $c_1^{'} = 10 + a, c_2^{'} = 30 - a$, the sum remains same. If we apply proportional rule , the allocation is $(\frac{(10+a)30}{80}, \frac{(30-a)30}{80}, 15)$.

The sum of allocation of claimant 1 and 2 is $\frac{((10+a)+(20-a))30}{80}$. It remains same, whatever may be the transfer. So claimant 1 and 2 have no advantage of forming a group (coalition).

- No advantageous transfer: Given a claims problem (C, M), for each $S \subset N$ and each $(c_i^{'})_{i \in S} \in R_{+}^{|S|}$, if $\sum_{i \in S} c_i^{'} = \sum_{i \in S} c_i$ then $\sum_{i \in S} x_i(C, M) = \sum_{i \in S} x_i((c_i^{'})_{i \in S}, c_{N-S}, M)$.
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The sum of allocation of claimant 1 and 2 is $\frac{((10+a)+(20-a))30}{80}$. It remains same, whatever may be the transfer. So claimant 1 and 2 have no advantage of forming a group (coalition).

Take the same problem and use constraint equal loss rule.
The allocation is $(0, 10, 20)$. Suppose claimant 1 and 2 form a group and claimant 1 transfer some of its claim to claimant 2. The new claims of 1 and 2 is 5 and 35 respectively.
The claims are $(5, 35, 40)$ and size of the cake is same.
Constraint equal loss rule allocates $(0, 12.5, 17.5)$. Now the sum of the allocation of claimant 1 and 2 is greater than the previous situation. Thus, there is tendency among the claimants to form group (sub-coalition) when constraint equal loss rule is used.

- Self-duality: Given a claims problem (C, M), we have $x(C, M) = c - x(C, \sum_{i \in N} c_i - M)$.
  The allocation proposed by a rule must be equal to the allocation where each claim is reduced is an amount. The amount is the allocation we get by using the same division rule for the bankruptcy problem where claims are same a before and the size of the cake is the deficit amount.
  It means that what is available and what is missing, must be treated symmetrically.

Example:

Suppose claims are $(10, 30, 40)$ and 40.

If you use constrained equal award rule is used, $(10, 15, 15)$.

The deficit amount is $80 - 40 = 40$. So, claims problem with this size of the cake is $(10, 30, 40$ and 40.

The allocation is $(10, 15, 15)$.

$c - x(C, \sum_{i \in N} c_i - M) = (0, 15, 25)$. Thus, constrained equal awards rule does not satisfy this property.

Take proportional rule, same claims problem, the allocation is $(5, 15, 20)$. The allocation in the new claims problem is also $(5, 15, 20)$.

So $c - x(C, \sum_{i \in N} c_i - M) = (5, 15, 20)$. We get $x(C, M) = c - x(C, \sum_{i \in N} c_i - M)$.

Proportional satisfy this property.

Results:
The constrained equal awards rule is the only rule satisfying equal treatment of equals, invariance under claim truncation, and composition up, composition down and claims monotonicity.

Results:

Constrained equal losses is the only rule satisfying equal treatment of equals, minimal rights first, composition down, composition up, claim monotonicity.

Results:
Proportional rule is the only rule satisfying no advantageous transfer, self-duality, composition up, composition down equal treatment of equals.

Contested garment method:

Suppose the worth of a garment is 100. There are two claimants, one claims 60 and another claims 80. How should it be divided?

Claimant 1 claims 60, so claimant 2 can get offered 40.

Claimant 2 claims 80, so claimant 1 can be offered 20.

The remaining size of the cake is $100 - 60 = 40$.

This 40 is equally divided between the two.

So claimant 1 gets $20 + 20 = 40$

Claimant 2 gets $40 + 20 = 60$.

How to implement it for general claims problem?
Consider the table given in figure 1.
Equally divide the cake till the claimants with lowest claims get the half of its claims. After that allocate half of the claims to the claimants with lowest claims and divided the rest of the cake equally among the other claimants. Continue till every one gets the half of their claims.
After that divided the cake is such way that the deficit is equally distributed among the claimants but the final allocation must be greater than the half of their claims.

Given a claims problem (C, M), if $\sum_{i=1}^{N} \frac{c_i}{2} \geq M$, then $x_i(C, M) = \min\{\frac{c_i}{2}, \lambda\}$, where λ is such that $\sum_{i \in N} \min\{\frac{c_i}{2}, \lambda\} = M$.

If $\sum_{i=1}^{N} \frac{c_i}{2} \leq M$, then $x_i(C, M) = c_i - \min\{\frac{c_i}{2}, \lambda\}$, where λ is such that $\sum_{i \in N}[c_i - \min\{\frac{c_i}{2}, \lambda\}] = M$.

Contested Garment Method:

Given a claims problem (C, M), if $\sum_{i=1}^{N} \frac{c_i}{2} \geq M$, then $x_i(C, M) = \min\{\frac{c_i}{2}, \lambda\}$, where λ is such that $\sum_{i \in N} \min\{\frac{c_i}{2}, \lambda\} = M$.

If $\sum_{i=1}^{N} \frac{c_i}{2} \leq M$, then $x_i(C, M) = c_i - \min\{\frac{c_i}{2}, \lambda\}$, where λ is such that $\sum_{i \in N}[c_i - \min\{\frac{c_i}{2}, \lambda\}] = M$.

It satisfies consistency principle. Suppose the allocation is $x(C, M)$ based on the above rule. Take any two claimants, the sum of the amount allocated to them is size of the cake. Use contest garment method to allocate between these two claimants taking their original claims. The allocation done by the contest garment method is going to match with the allocation done by the above rule for these two claimants.
It is shown in figure 1.

Rif's method:

Allocate equally among all the claimants till the claimant with the lowest claims get his full amount.

Give the claim amount to the claimant with smallest claims. The additional amount is equally distributed among the remaining claimants.

If the size of the cake increases, continue in the same way.

Give the claim amount to the claimant with lowest claims and then the claim amount to the claimant with the second lowest claims.

Remaining amount is divided equally among the rest.

It is same as constrained equal awards rule.

O’ Neill’s method of division:
First to arrive gets its claim or the whole cake.
Second to arrive gets its claim or the remaining portion.
Third to arrive gets its claim or the remaining portion.
It continue in this way. If the remaining portion is zero, the claimants get zero.
In this way, the allocation is computed for all possible order in which claimants can arrive at the arbitrator. After that average of these allocation is done. The final allocation is based on average of the all the allocation done based on the all possible order of arrival.

Example:
$C = \{100, 200, 300\}, \quad M = 250.$

Order of arrival	1	2	3
123	100	150	0
132	100	0	150
213	50	200	0
231	0	200	50
312	0	0	250
321	0	0	250
average	$\frac{250}{6}$	$\frac{550}{6}$	$\frac{700}{6}$

O'Neill's method is same as the Shapley value.

$v(S) = [\text{size of the cake} - \text{sum of the claimns who are not in S}]_+$

It means the above expression takes a positive or zero value.

In case of the same example, Shapley value is

$N = \{1, 2, 3\}$

$v(1) = [250 - (200 + 300)]_+ = 0$

$v(2) = [250 - (100 + 300)]_+ = 0$

$v(3) = [250 - (100 + 200)]_+ = 0$

$v(1, 2) = [250 - 300]_+ = 0$

$v(1, 3) = [250 - 200]_+ = 50$

$v(2, 3) = [250 - 100]_+ = 150$

$v(1, 2, 3) = [250 - 0]_+ = 250$

$v(\emptyset) = [250 - (100 + 200 + 300)]_+ = 0$

The computation of Shapley value

Order of arrival	1	2	3
123	0	0	250
132	0	200	50
213	0	0	250
231	100	0	150
312	50	200	0
321	100	150	0
average	$\frac{250}{6}$	$\frac{550}{6}$	$\frac{700}{6}$

Nash Bargaining

How to divide a dollar between two individuals?

Suppose philanthropist gives a dollar to two individuals, if they can agree upon a division of this dollar. If they cannot agree on any division of this dollar, the philanthropist dies give the dollar.

Let (m_1, m_2) be the division of one dollar received by individual 1 and 2 respectively.
The set of possible divisions, $M = \{m = (m_1, m_2) : m_1 + m_2 \leq 1\}$.
M is also the feasible set. The feasible set is shown in figure 1.
The two individuals have to agree on any point in the set M.
The bargaining problem is to choose a particular point from this feasible set.

Another example can be wage bargaining between an owner of a firm and the worker.

The bargaining problem is defined in terms of utility of individuals. Each individual receives utility or attain satisfaction from the possession of a share of the dollar they are dividing. There is function which map the amount of currency (share of dollar)received an individual into utility level of that individual. $u_i(m_i)$ is a function which gives the amount of utility attained by individual i when it gets m_i amount of currency.

We assume that $u_i(m_i)$ is strictly increasing in m_i and differentiable in m_i.

We assume that individuals are risk averse. The assumption of risk averse gives a particular type of strictly increasing differentiable utility function. The utility function is concave in nature. It is shown in figure 2.

What do we mean by concave utility function in this context?
Suppose a risk averse individual has m^0 amount of currency. With currency this guy can buy a lottery. The lottery is of following nature:
It gives m^1 with probability p
It gives m^2 with probability $(1-p)$.
Suppose $m^0 = m^1 p + m^2(1-p)$, this is called the expected return of the lottery.
$0 < m^1 < m^0 < m^2$.
These are shown in figure 2.

A risk averse person is such that
$u(m^0 = m^1 p + m^2(1-p)) > u(m^1) \times p + u(m^2) \times (1-p)$.
This risk averse person is going to buy this lottery when its price is m'' as shown in figure 3.

Lec 32

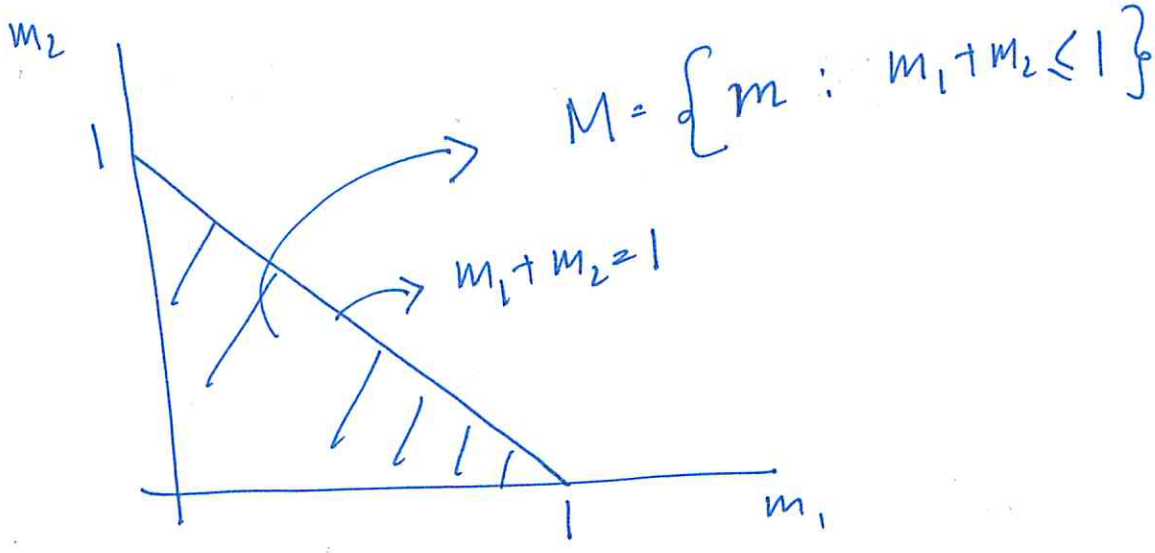


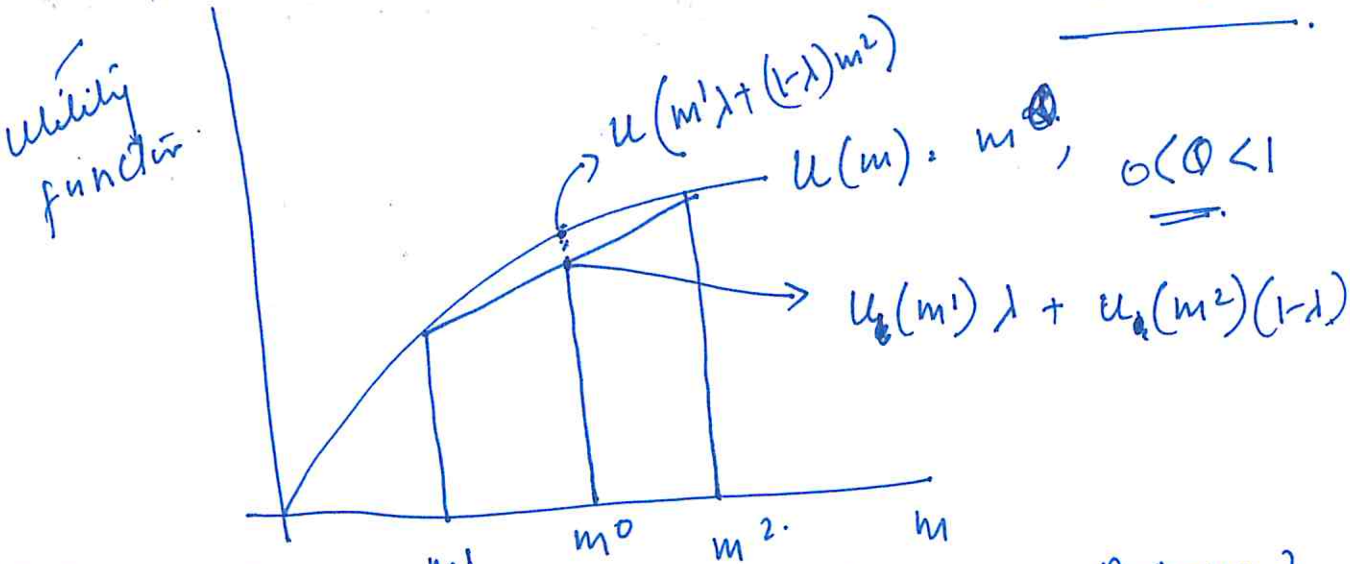
Figure 1

$$X = \{ u = (u_1, u_2) : m_i \in M \}.$$

$u_1 = u_1(m_1)$ – utility f^n of person 1.

$u_2 = u_2(m_2)$ – utility f^n of person 2

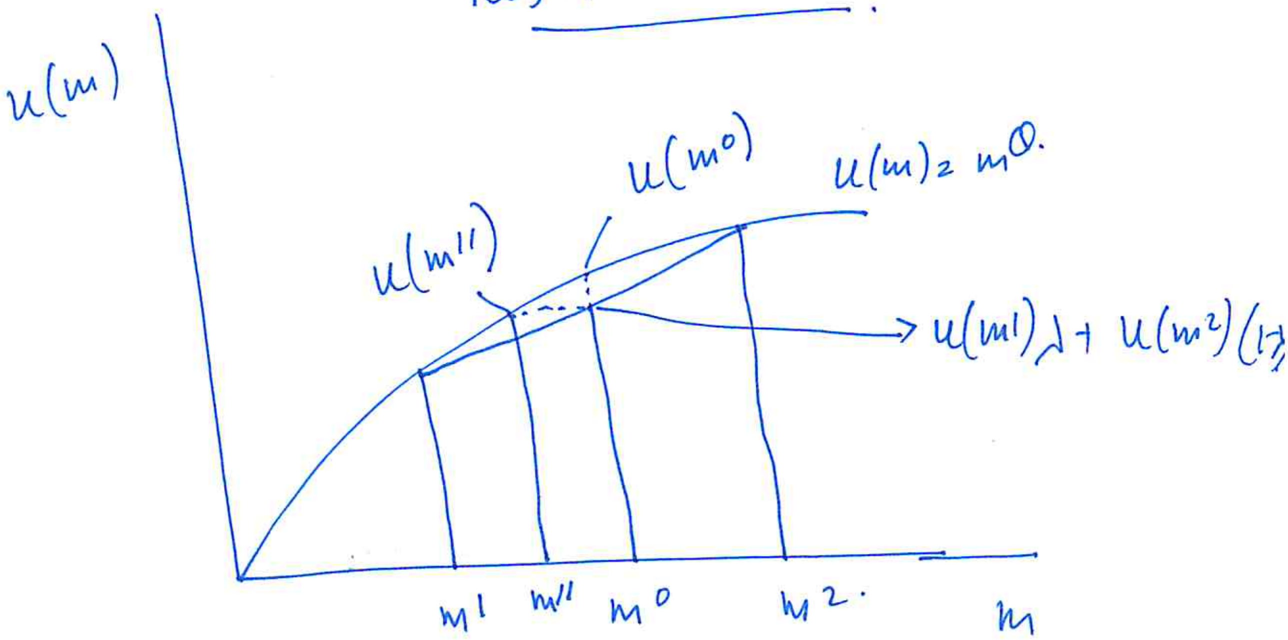
Risk averse:



$m^0 = \lambda m^1 + (1-\lambda) m^2$. Figure 2.

$$u(m^1\lambda + m^2(1-\lambda)) > u(m^1)\lambda + u(m^2)(1-\lambda).$$

Risk averse



$$m^0 = m^1\lambda + m^2(1-\lambda)$$

$$u(m'') = u(m^0 = m^1\lambda + m^2(1-\lambda))$$

: Figure 3

A risk neutral person has following type of utility function. It is a strictly increasing differentiable linear function. It is shown in figure 1.
Take the same lottery as before.
Suppose a risk averse individual has m^0 amount of currency. With currency this guy can buy a lottery. The lottery is of following nature:
It gives m^1 with probability p
It gives m^2 with probability $(1-p)$.
Suppose $m^0 = m^1 p + m^2(1-p)$, $0 < m^1 < m^0 < m^2$.
A risk neutral person is such that
$u(m^0 = m^1 p + m^2(1-p)) = u(m^1) \times p + u(m^2) \times (1-p)$. It is shown in figure 1.
The risk neutral person is indifferent between buying this lottery and holding the money m^0.
An example is $u(m) = am$ where $a > 0$.

A person is risk lover if the utility function is strictly increasing in nature. It is shown in figure 2.
Take the same lottery as before.
Suppose a risk averse individual has m^0 amount of currency. With currency this guy can buy a lottery. The lottery is of following nature:
It gives m^1 with probability p
It gives m^2 with probability $(1-p)$.
Suppose $m^0 = m^1 p + m^2(1-p)$, $0 < m^1 < m^0 < m^2$.
A risk lover person is such that
$u(m^0 = m^1 p + m^2(1-p)) < u(m^1) \times p + u(m^2) \times (1-p)$. It is shown in figure 2.
The risk lover person prefers to buy this lottery rather than holding the money m^0.
An example is $u(m) = m^\alpha$ where $\alpha > 1$.

From the feasible set in terms of money or currency $M = \{m = (m_1, m_2) : m_1 + m_2 \leq 1\}$. This set was shown in last class. It is feasible set.

We define the feasible set in terms of utility of the two individuals. It is
$X = \{u = (u_1(m_1), u_2(m_2)) : m \in M\}$.
We want the feasible set to be a convex set.
A set S is convex set, if for all $x, y \in S$ then $\lambda \times x + (1 - \lambda) \times y \in S$ where $0 < \lambda < 1$. It is shown in figure 3.
It is obvious that M is a convex set.

We have to show that X is also a convex set when the individuals are risk averse.

We want the following properties to be satisfied by the feasible set X in the bargaining problem.
1) X is a convex set.
2) X is a closed and bounded.
3) Free disposal is allowed.

X is closed and bounded. It means that if there a sequence belonging to the set X then the limit point should also belong to that set.
There are numbers $\bar{u}_1, \bar{u}_2 > 0$ such that for all $u \in X$, we have $u = (u_1, u_2) < \bar{u} = (\bar{u}_1, \bar{u}_2)$. This means that the set X is bounded. It is obvious that X contains only non-negative number.

Free disposal means that if individual 1 has m_1 amount of the cake and want to consume only $m_1 - \epsilon$ amount. No additional cost is incurred to dispose off ϵ amount. It also means that If $u(m) \in X$ then $u(m - \epsilon) \in X$.

We show that X is a convex set.
Take any m and n belonging to M and $u(m) \in X$ and also $u(n) \in X$. We have to show that $\lambda \times u(m) + (1 - \lambda) \times u(n) \in X$, where $0 < \lambda < 1$.
We know that $u_i(m_i)$, $i = 1, 2$ is strictly concave in nature because individuals are risk averse in nature. So we have
$u_i(m_i\lambda + n_i(1 - \lambda)) > u_i(m_i)\lambda + u_i(n_i)(1 - \lambda),\ i = 1, 2$
We know that $(u_1(m_1\lambda + n_1(1 - \lambda)), u_2(m_i\lambda + n_i(1 - \lambda))) \in X$ since $m_1\lambda + n_1(1 - \lambda) \in M$. Because M is a convex set.
$(u_1(m_1)\lambda + u_1(n_1)(1 - \lambda), u_2(m_2)\lambda + u_2(n_2)(1 - \lambda)) \in X$ from the free disposal condition. Thus, X is a convex set.

A Nash bargaining problem is given by a pair (X, d) where X is the feasible set and d is the disagreement point. The disagreement point $d = (d_1, d_2)$ is such that that $d \in X$ and is the payoff received by each individual when the two players (person) cannot come to agreement on a particular division.
It is also called the status-qua point, the payoff received when there is disagreement. It is shown in figure 4.

Suppose bargaining solution or the payoff on which both the person (players) agree upon is $u^* = (u_1^*, u_2^*)$. It should be such that $u_1^* \geq d_1$ and $u_2^* \geq d_2$ and $u^* \in X$.

Given the bargaining problem (X, d) objective is to find a feasible solution u^* which is acceptable to both persons (players).

The solution of the Nash bargaining problem is given by
$\max \quad (u_1(m_1) - d_1)(u_2(m_2) - d_2)$
subject to $(u_1, u_2) \in X$.
This is also called as Nash product. The optimal (m_1, m_2) is obtained by maximising the Nash product by taking $m_1 = m$ and $m_2 = 1 - m$.

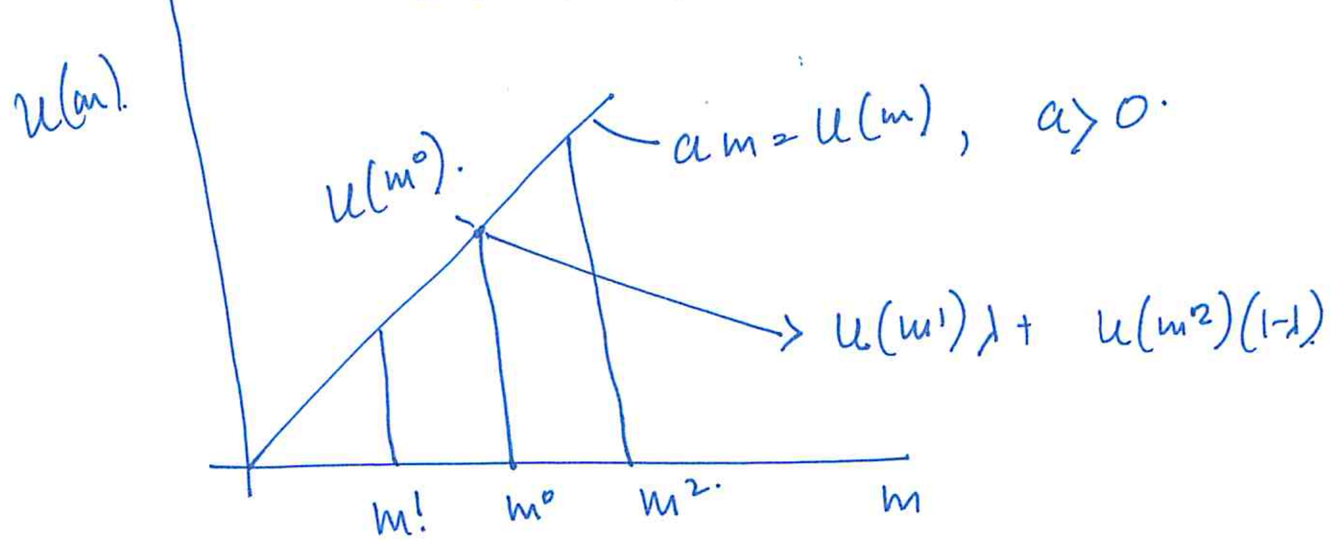
The optimal solution is
differentiate $(u_1(m) - d_1)(u_2(1 - m) - d_2)$ with respect to m. This gives
$u_1'(m)(u_2(1 - m) - d_2) - (u_1(m) - d_1)u_2'(1 - m)$
The first order condition gives
$u_1'(m)(u_2(1 - m) - d_2) = (u_1(m) - d_1)u_2'(1 - m)$
We get a unique m satisfying the above equation (first order condition).

The second order condition is
$u_1''(m)(u_2(1-m)-d_2) - u_1'(m)u_2'(1-m) + (u_1(m)-d_1)u_2''(1-m) - u_2'(1-m)u_1'(m)$.
It is negative for all $m > 0$ since $u_i''() < 0, \ i = 1, 2$. It is due to concavity of the utility function.
Thus, the optimal m^*, we get from the first order condition is maximizing the Nash product. It is shown in figure 5.

The Nash solution to the bargaining problem (X, d) is given by the maximization of the Nash product. This optimization is characterized based on four axioms.

Lec 33.

Risk Neutral.



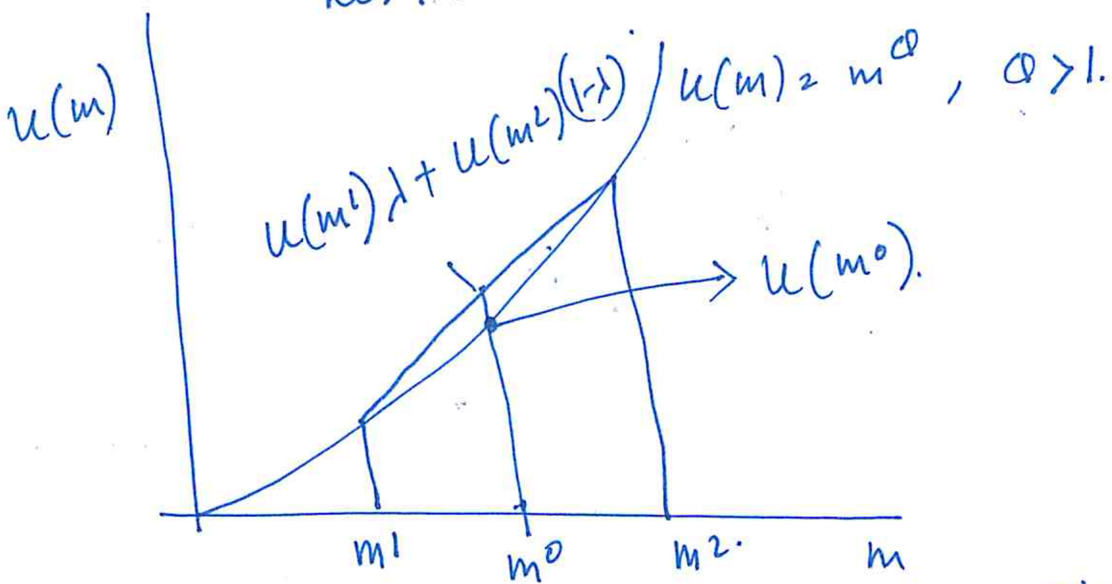
$m^0 = m^1\lambda + (1-\lambda)m^2.$

we have

$$u(m^0) = u(m^1)\lambda + u(m^2)(1-\lambda)$$

Figure 1.

Risk lover

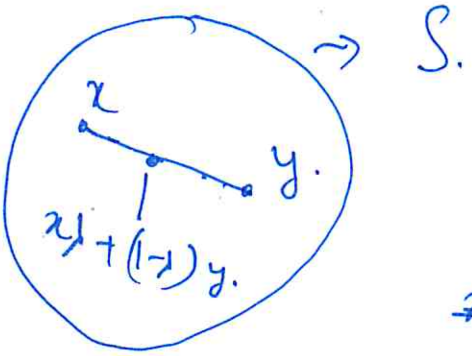


$m^0 = m^1\lambda + (1-\lambda)m^2$ Figure 2

We have

$$u(m^0) < u(m^1)\lambda + u(m^2)(1-\lambda)$$

Convex set.

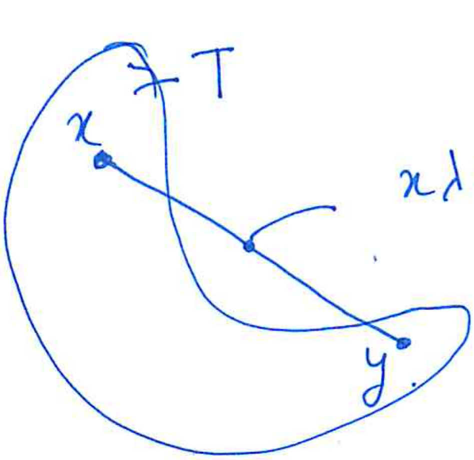


Any $x, y \in S$.

~~$xa + by$~~

$x\lambda + (1-\lambda)y \in S$, $0<\lambda<1$

so S is a convex set.

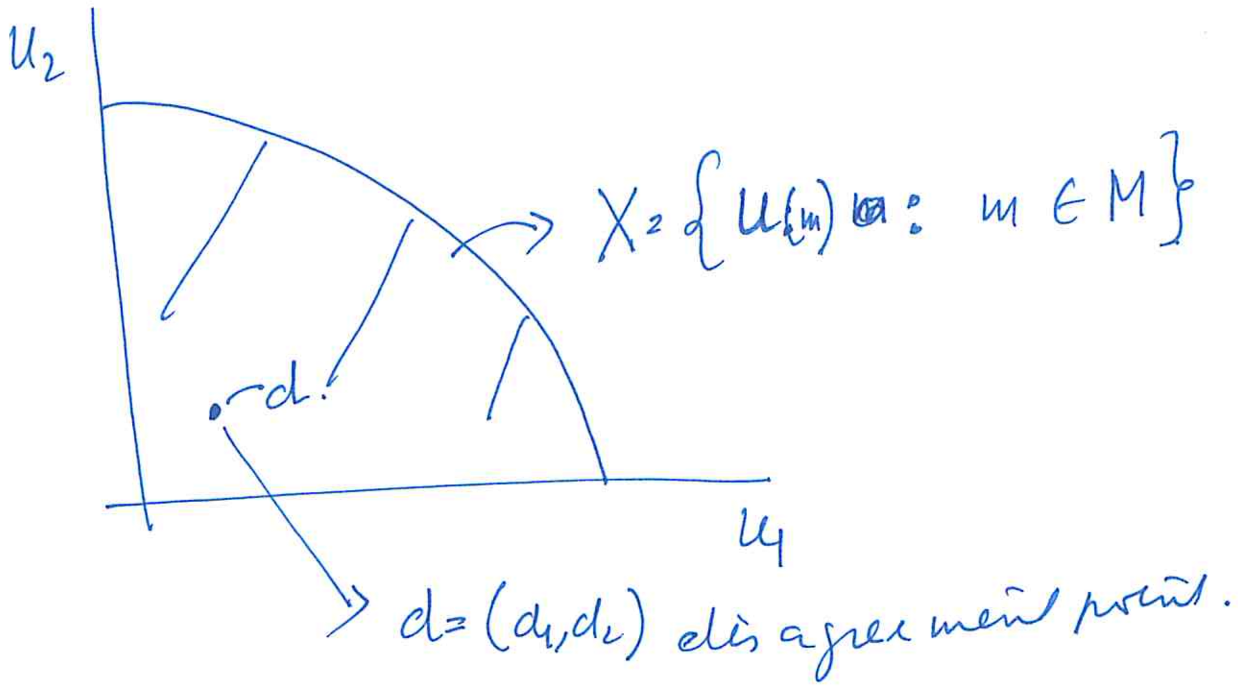


$x\lambda+(1-\lambda)y$, $0<\lambda<1$

$x\lambda+(1-\lambda)y \notin T$.

so T is not a convex set.

Figure 3



Representation of (X, d).

Figure 4

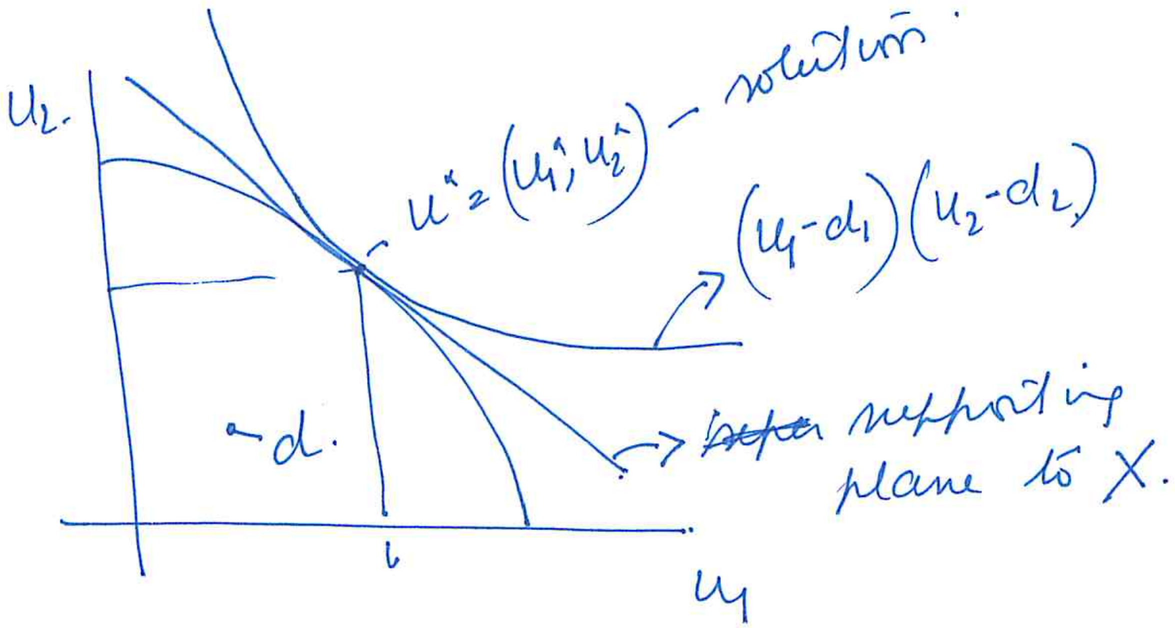


Figure 5

Given a bargaining problem (X, d), the solution is defined by a function
$F : \mathcal{B} \longrightarrow X$, where $\mathcal{B}$ is a set of all pairs (X, d).
We define the axioms.

Axiom 1: Pareto efficiency
The solution or agreed payoff pair should always be in the bargaining set (feasible set) X.
$F(X, d) = (u_1^*, u_2^*) \geq d = (d_1, d_2)$
If $y = (u_1, u_2) > F(X, d) = (u_1^*, u_2^*)$ then $y \notin X$.
The solution pair must be greater than equal to the disagreement payoff. If a payoff pair is greater than the solution pair, then that payoff is not in the feasible set. The solution will always be in the boundary of X.

Axiom 2:Invariance to the transformation of the utility functions
The solution pair should be independent of utility scale and origin.
Suppose $t_1(u_1(m_1)) = a_1 u_1 + b_1$ and $t_2(u_2(m_2)) = a_2 u_2 + b_2$ where $a_i > 0$ and $b_i > 0$, $i = 1, 2$.
If $F(X, d) = (u_1^*, u_2^*)$ is a solution to the bargaining problem (X, d), then $F(t(X), t(d)) = t(F(X, d))$.
If the utility functions are transformed such that multiplied by a positive real number and origin is changed, the solution to this new bargaining problem is attained based on simple transformation of the solution we got before the transformation.

Axiom 3: Independence of irrelevant alternative

Suppose $Y \subset X$ as shown in figure 1. If $d \in Y$, then if $F(X,d) \in Y$ then $F(X,d) = F(Y,d)$.

Consider a subset of X, suppose the disagreement point belongs to this subset Y, and also the solution when X is the feasible set also belongs to Y. Then the solutions of the bargaining problems are same in case of Y and X being the feasible sets .

Axiom 4: Symmetry
If $d_1 = d_2$ and
If $(u_1, u_2) \in X$ then $(u_2, u_1) \in X$
then $u_1^* = u_2^*$ where (u_1^*, u_2^*) is a solution to the bargaining model (X, d).
If the disagreement point is same and also the feasible set is such that whatever is feasible to person 1 is also feasible to person 2 then the solution of the bargaining problem is such that both the person get similar payoff.

Result:
If $F : \mathcal{B} \longrightarrow X$ satisfies Axiom 1 to 4, then F gives the unique solution to the bargaining problem (X, d) and F is the solution of maximization of $(u_1 - d_1)(u_2 - d_2)$ subject to $U \in X$.
The converse is also true.
The sketch of the proof is given based on diagram.

An example

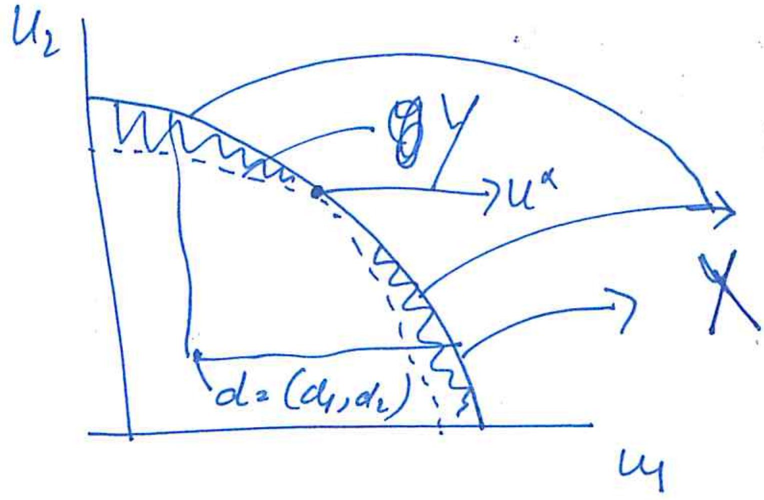
Suppose $u_1 = (m)^{0.5}$ and $u_2 = (m)^{0.6}$. The size of the cake is 1. So $m_1 + m_2 = 1$. The disagreement point is $(0.2, 0.2)$. We need to find the solution to this bargaining problem.

We maximize $((m)^{0.5} - 0.2)((1-m)^{0.6} - 0.2)$ with respect to m.
We differentiate with respect to m and get
$0.5m^{-0.5}((1-m)^{0.6} - 0.2) - (m^{0.5} - 0.2)(1-m)^{-0.4}0.6$.
The first order condition implies
$0.5m^{-0.5}((1-m)^{0.6} - 0.2) = (m^{0.5} - 0.2)(1-m)^{-0.4}0.6$.
We solve the above equation to get the optimal m. This gives the solution to the bargaining problem.

There can be another important point in the bargaining problem. It is break down point. This point gives the payoff pair to the persons when the bargaining breaks down. Like in case of wage bargaining between the firm owner and the workers, if the bargaining breaks down. It means the workers are going to search for work in other place and also the owner is going to look for other way to invest its money. This is also called outside option. It is given by the point $b = (b_1, b_2)$. It is such that $b \in X$ and $u^* \geq b$ where u^* is bargaining solution.

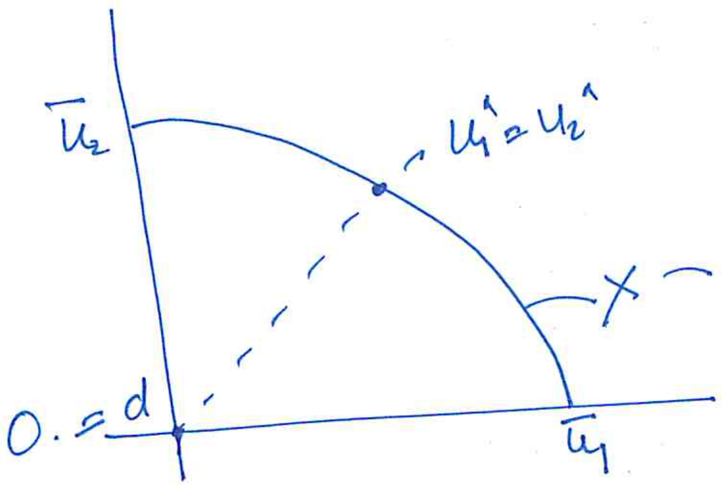
The generalized Nash bargaining solution is given by
Maximization of $(u_1(m_1) - d_1)^\alpha (u_2(m_2) - d_2)^\gamma$
subject to $u_1 \geq b_1$ and $u_2 \geq b_2$.

We have $0 < \alpha < 1$ and $0 < \gamma < 1$. These two parameters indicate some form of bargaining power due to eagerness of the person to agree on a solution because of the fear of break down. When these two parameters takes higher values, the share in the bargaining goes up.



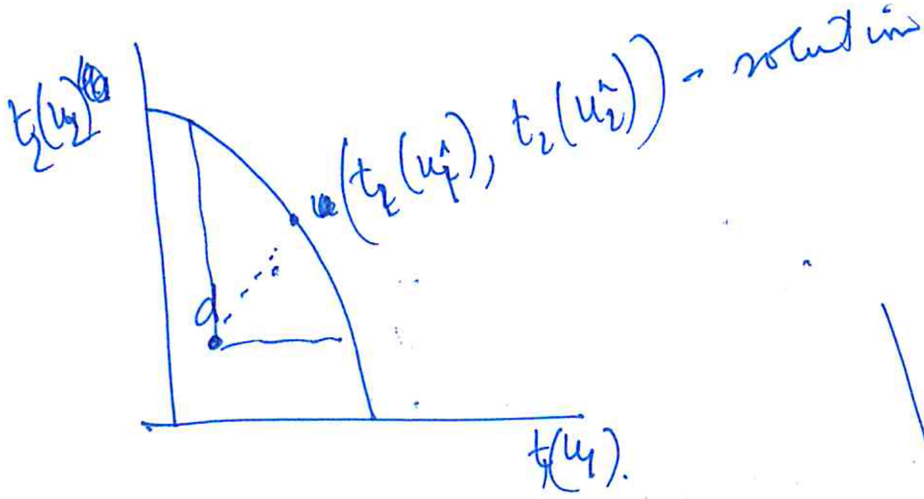
$u^* \in Y$, $d \in Y$.

Figure 1

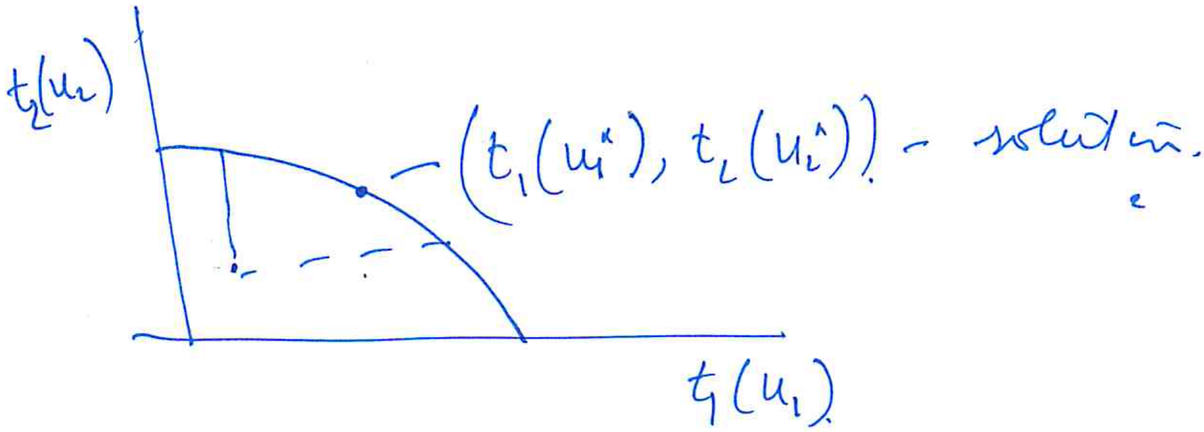


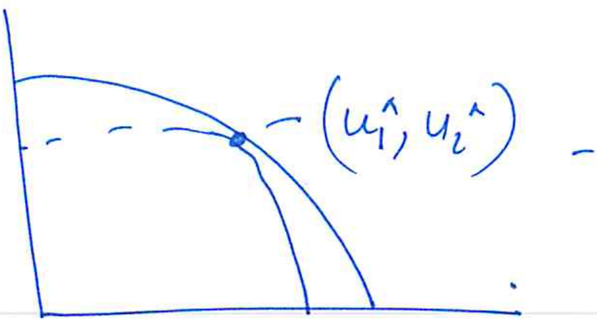
Then using axiom 4 $u_1 = u_2$.

Now using axiom 2, we can find solution for any transformation.



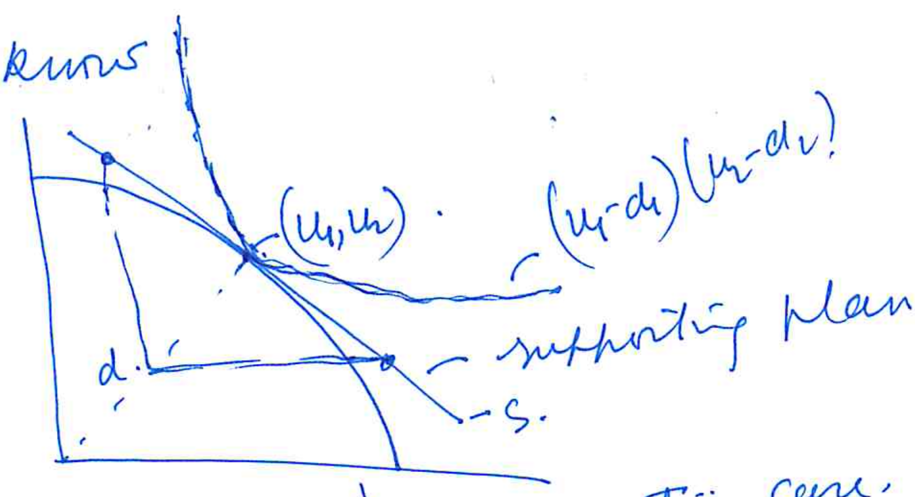
→ From axiom 2





From axiom 3

- we know



→ symmetric case.

Matching

Origin of the problem: Matching students with universities at the time of admission.
Each student apply to u universities and there are n students.
Each university has v vacancies and their are there m universities.
How to assign (match) students to universities?
Another area:
Matching of men and women in a community when there are n men and n women.
We want a good matching so that the matching could be sustained. To design a matching first we need information on the preference of the individuals, that is preferences of men and women.
We need the preferences of each woman over the set of men. The preference of each man over the set of women.

Suppose there are four women, Ann, Beth, Cher, and Dot.
There are four men; Al, Bob, Cal, and Dan.
The preference of women are given in figure 1.
The preference of mean are given in figure 2. The number in each cell represents the rank or position of each woman. In figure 1, the first column gives the ranking of Ann over the set of men. In figure 2, the first row gives the ranking of Al over the set of women.

Consider a possible matching proposal given below

AL	*Bob*	*Cal*	*Dan*
Dot	*Ann*	*Beth*	*Cher*
2×2	2×2	2×3	2×4

In this proposal, all the men are getting their second choice. Ann and Dot are getting their second choice. Beth has got her third choice and Cher has got fourth choice. Is this proposal going to be sustained? If Cher proposes to Al, Al will accept the proposal. Because Al prefers Cher over Dot. Cher will get her third choice. It is better than fourth choice. So, Al and Cher as a pair can be better off than the given proposal. Here, Al and Cher prefer each other to their actual mate provided by the proposal. So, this proposal cannot be sustained.

Another proposal

AL	*Bob*	*Cal*	*Dan*
Ann	*Cher*	*Dot*	*Beth*
3×1	4×1	4×1	4×4

Ann, Cher, and Dot get their first choice. Beth gets her fourth choice. If Beth proposes to Bob who has got his fourth choice. Bob will accept the proposal of Beth because she is his third choice better than fourth choice. Thus, this proposal also cannot be sustained. We have Bob and Beth who prefer each other than to their actual mates.

Consider another proposal

AL	*Bob*	*Cal*	*Dan*
Cher	*Dot*	*Ann*	*Beth*
1×3	1×3	1×3	4×4

In this proposal three men are getting their first choice. Dan is getting his fourth choice. Cher, Dot and Ann are getting their third choice and Beth is getting her fourth choice. Beth cannot propose to any one else. It will not be accepted because all of the three are getting their first choice. Dan is in the fourth position for all the women, so none among Cher, Dot and Ann is going to propose to Dan. Thus, this proposal is going to be sustained. We do not have any pair of man and women who prefer each other than to their actual mate.

A matching system (proposal) is called stable, if under it their cannot be found a man and a woman who are not paired off with each other but prefer each other to their actual mates. When unmatched pair will find it beneficial to deviate from the matching and form their own matching.

Another example showing stable matching

The set of women are A, B, C, and D.

The set of men are a, b, c, d.

The preferences are given in figure 3.

Consider the proposal below.

A	B	C	D
b	c	d	a
2×2	1×1	3×1	2×3

We find that there is no pair of man and women who prefer each other than their actual mates. Thus, it is stable proposal.

Try to find another stable proposal for this example.

Gale-Shapley algorithm for stable matching

1st Stage: Every man turns to the woman who is first on his list and proposes. Every woman who receives more than one proposal selects her favourite from among who propose to her and rejects the other proposal. Every man not rejected is put on a waiting list of the woman to whom he proposed.

2nd Stage: Every man who was rejected turns to the woman who is second on his list and proposes her. Every woman who receives more than one proposal, including any proposal from the previous stage (man in the waiting list), selects her favourite and put him on her waiting list. She rejects the others.

3rd Stage: Every man who is rejected turns to the woman who is next on his list and proposes her. Once again every woman selects her favourite from the set of proposals received including the one in the waiting list and rejects others.

The procedure continues until no man is rejected or each woman has one proposal.

Example:
The set of women are A,B, C,D.
The set of men are a, b,c,d.
The preferences of woman and men are given in figure 1.
The procedure to find a proposal is shown in figure 2.
The procedure terminates in 6th stage. In stage 1 all the men proposes to their first choice. In stage 1 b is rejected by A.
In stage 2, b proposes to D and d is rejected by D.
In stage 3, d proposes to B and B rejects c.
In stage 4, c proposes to A and A rejects a.
In stage 5, a proposes to B and B rejects a.
In stage 6, a proposes to C. In this stage each woman has one proposal and also no man is rejected. Thus, the procedure terminates.

The procedure terminates in the following proposal.

A	B	C	D
c	d	a	b
2×2	2×2	2×3	2×2

a has got third choice, and all others have second choice. a has been rejected by his first and second choice. So, is in third choice. Note, all the first choice of men has been rejected. So, there is no pair of man and women who prefer each other than their actual mates. Thus, the proposal is stable.

Result:
The Gale-Shapley algorithm terminates after finite number of steps (stages).
Proof:
The number of men is equal to number of women. If there is a woman with more than one proposal, then atleast one women is without any proposal.
Once a woman has a proposal, she will always have one, because some one will always be on her waiting list.
When evey woman has a proposal, every woman will have exactly one proposal, because the number of men equals number of women.

It is possible to get to the stage at which every woman has a proposal, because at every stage the men propose to the woman who are next on their list, therefore they cannot back track and propose again to the women who rejected them. Because there are finite number of men and woman in the community, no going back, a stage must be reached at which every woman has a proposal. The procedure terminates at this stage.

Result:
The Gale -Shapley algorithm terminates in a stable matching system.
Proof:
Suppose there are n men and women. There Gale-Shaley algorithm terminates in the following proposal.
$A...\quad ...R...\quad S..X..$
$a...\quad ...r...\quad s..x..$
Consider one man Mr s. Suppose Mr s prefers R to S and Mr s is match with S. So, Mr s must have proposed Mrs R before Mrs S. Mrs S must have rejected Mr s. It may not be because of Mr r. Mrs R may have rejected Mr s for some Mr x. After series of rejections Mr r proposes. She does not reject Mr r. So Mrs S is not going to accept the proposal of Mr s. She prefers Mr r over s. Thus, the proposal at which Gale-Shapley algorithm terminates is going to be a stable outcome.

another example:
It is given in figure 3.
In stage 1, all the men proposes to their best choice. Mr B is rejected by Mrs a.
In stage 2, b proposes to B and B rejects c.
In stage 3, c proposes to C and C rejects d.
In stage 4, d proposes to A and A rejects a.
In stage 5, a proposes B and B rejects b.
In stage 6, b proposes to C and C rejects c.
In stage 7, c proposes to A and A rejects d.
In a stage 8, d proposes to B and B rejects a.
In stage 9, a proposes to C and C rejects b.
In stage 10, b proposes to D.
All the women have one proposal, so the procedure terminates. It is stable. This Gale-Shapley algorithm took 10 steps (stages) to reach a stable outcome.

Lec 35

Preferences of Women

	Ann	Beth	Cher	Dot
AL	1	1	3	2
Bob	2	2	1	3
Cal	3	3	2	1
Dan	4	4	4	4

Fig 1

Preferences of Men

	Ann	Beth	Cher	Dot
AL	3	4	1	2
Bob	2	3	4	1
Cal	1	2	3	4
Dan	3	4	2	1

Fig 2

Col - preferences.

	A	B	C	D
a	1	2	4	2
b	2	4	2	1
c	3	1	1	3
d	4	3	3	4.

Row - preference.

	A	B	C	D
a	4	2	1	3
b.	2	1	3	4
c	3	1	4	2
d	2	4	1	3.

Fig 3.

Lec 36

~~Wom~~ Preferences of women

	A	B	C	D
a	3	3	2	3
b	4	1	3	2
c	2	4	4	1
d	1	2	1	4

Preferences of men

	A	B	C	D
a	1	2	3	4
b	1	4	3	2
c	2	1	3	4
d	4	2	3	1

Fig 1

	A	B	C	D
Stage 1	a	c		d
	b*			

	A	B	C	D
Stage 2	a	c		d*
				b

	A	B	C	D
Stage 3	a	c*		b
		d		

	A	B	C	D
Stage 4	a*	d		b
	c			

	A	B	C	D
Stage 5	c	d		b
		a¹		

	A	B	C	D
Stage 6	c	d	a	b

A	B	C	D	
c	d	a	b.	→ stable
2x2	2x2	2x3	2x2.	

Fig 2

Example 2

Preference of Woman

	A	B	C	D
a	3	2	1	3
b	4	3	2	4
c	1	4	3	2
d	2	1	4	1

	A	B	C	D
a	1	2	3	4
b	1	2	3	4
c	3	1	2	4
d	2	3	1	4

Figure 3

Gale - Shapley algorithm

Stage1	A	B	C	D
	a, b^{*}	c	d	
2	a	c^{*}, b	d	
3	a	b	d^{*}, c	
4	a^{*}, d	b	c	
5	d	b^{*}, a	c	
6	d	a	c^{*}, b	
7	d^{*}, c	a	b	
8	c	a^{*}, d	b	
9	c	d	b^{*}, a	
10	c	d	a	b

A	B	C	D
c	d	a	b
1x3	1x2	1x2	4x4.

It took 10 steps to termination.