I. Functions of the form $\mathbf{f} : A \subset \mathbb{R} \to \mathbb{R}^n$

1. Write such a function as $\mathbf{f}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$, then f_1, f_2, ..., f_n are real-valued functions of a real variable. The functions f_1, f_2, ..., f_n are called the component functions of $\mathbf{f}$.

2. Examples:

   (a) $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$ defined by $\mathbf{f}(t) = (\cos t, \sin t)$.

   (b) Let $g : \mathbb{R} \to \mathbb{R}$ be a function and define $\mathbf{f}(t) = (t, f(t))$ for $t \in \mathbb{R}$.

   (c) $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$ defined by $\mathbf{f}(t) = (\cos t, \sin t, t)$.

3. Limits: Let $\mathbf{f} : A \subset \mathbb{R} \to \mathbb{R}^n$ and $a \in A$. The function $\mathbf{f}$ is said to have a limit $\mathbf{L} \in \mathbb{R}^n$ as t tends to a, if given $\epsilon > 0$ there is a $\delta > 0$ such that whenever $|t-a| < \delta$ and $t \in A$ then $\|\mathbf{f}(t) - \mathbf{L}\| < \epsilon$. In this case we write $\lim_{t\to a} \mathbf{f}(t) = \mathbf{L}$

4. *Proposition (L):* For $i \in \mathbb{N}$ with $1 \le i \le n$, let f_i denote the i^{th} component functon of a function $\mathbf{f} : A \subset \mathbb{R} \to \mathbb{R}^n$. $\lim_{t\to a} \mathbf{f}(t)$ exists if and only if all of $\lim_{t\to a} f_1(t)$, $\lim_{t\to a} f_2(t)$, ..., $\lim_{t\to a} f_n(t)$ exist. When these limits exist, $\lim_{t\to a} \mathbf{f}(t) = (\lim_{t\to a} f_1(t), \lim_{t\to a} f_2(t), \ldots, \lim_{t\to a} f_n(t))$.

5. Continuity: Let $\mathbf{f} : A \subset \mathbb{R} \to \mathbb{R}^n$ and $a \in A$. The function $\mathbf{f}$ is said to be continuous at a, if given $\epsilon > 0$ there is a $\delta > 0$ such that whenever $|t - a| < \delta$ and $t \in A$ then $\|\mathbf{f}(t) - \mathbf{f}(a)\| < \epsilon$. A function is said to be continuous if it is continuous at every point in its domain.

6. Note that $\mathbf{f}$ is continuous at $a \in A$ if and only if $\lim_{t\to a} \mathbf{f}(t)$ exists and equals $\mathbf{f}(a)$.

7. By the above proposition (L), the function $\mathbf{f}$ is continuous at a point if and only if the component functions are continuous at that point.

8. Derivatives:

   (a) A function $\mathbf{f} : A \subset \mathbb{R} \to \mathbb{R}^n$ is said to be differentiable at a point $t \in A$ if $\lim_{h\to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}$ exists. The derivative of $\mathbf{f}$ at t is the value of this limit, when it exists, and is denoted by $\mathbf{f}'(t)$.

   (b) This is equivalent to the existence of the derivatives of the component functions of $\mathbf{f}$ at the point t.

   (c) Eg. Consider $\mathbf{f}(t) = (\cos t, \sin t, t)$ for $t \in \mathbb{R}$, $\mathbf{f}'(t) = (-\sin t, \cos t, 1)$.

   (d) Chain Rule: Suppose that $\alpha : B \subset \mathbb{R} \to \mathbb{R}$ is differentiable at $t \in B$ and $\alpha(t) \in A \subset \mathbb{R}$. If $\mathbf{f} : A \to \mathbb{R}^n$ is differentiable at $\alpha(t)$, then the composition $\mathbf{f} \circ \alpha$ is differentiable at t and $(\mathbf{f} \circ \alpha)'(t) = \mathbf{f}'(\alpha(t)) \cdot \alpha'(t)$ (proof?)

9. A continuous function $\mathbf{f} : (c, d) \subset \mathbb{R} \to \mathbb{R}^n$ is called a path in $\mathbf{R}^n$. If $n = 2$, we call it a path in the plane and if $n = 3$, we call it a path in the space. Suppose $[a, b] \subset (c, d)$. Then the trace, in $\mathbb{R}^n$, of the restriction of $\mathbf{f}$ to the interval $[c, d]$ is called a curve in $\mathbb{R}^n$ with endpoints $\mathbf{f}(a)$ and $\mathbf{f}(b)$. The restricted function $\mathbf{f}(t)$ $t \in [a, b]$ is said to parametrize this curve. A parametrized curve is a curve with some given parametrization.

10. Example: Consider $\mathbf{f}(t) = (\cos t, \sin t)$ for $t \in \mathbb{R}$. Then the restriction of $\mathbf{f}$ to $[0, 2\pi]$ and to $[0, 4\pi]$ are two different parametrized curves, although both the curves start at $(1, 0)$ and end at $(1, 0)$. The trace of the first restricted function is once around the unit circle, whereas that of the second restricted function is twice around the unit circle.

11. Suppose that a parametrized curve $\mathbf{f}$ is differentiable at a point t in its domain. Then the derivative $\mathbf{f}'(t)$ in $\mathbb{R}^n$ is called the tangent vector to the curve at the point $\mathbf{f}(t)$. Here the differentiability at the end points is considered using the left-handed and the right-handed limits.

12. The equation of the tangent line to the parametrized curve $\mathbf{f}$ at t_0 is given by $\mathbf{f}(t_0) + \mathbf{f}'(t_0)t$ where $t \in \mathbb{R}$. Eg. The tangent line to the helix $(\cos t, \sin t, t)$ at the point $t = \frac{\pi}{2}$ is $(0, 1, \frac{\pi}{2}) + t(-1, 0, 1)$ for $t \in \mathbb{R}$.

13. A parametrized differentiable curve is a parametrized curve which is differentiable at every point in its domain.

14. A parametrized differentiable curve $\mathbf{f} : [a, b] \to \mathbb{R}^n$ is said to be regular, if $\mathbf{f}'(t) \neq \mathbf{0}$ for all $t \in (a, b)$.

15. If a parametrized curve is regular, then it does not turn back at any point in its trace *i.e.* it moves along its trace only in one direction. It also does not stop at a single point in its trace.

16. Consider the problem of calculation of arc length of a regular parametrized curve in the xy plane. As a first crude approximation, we pick the endpoints of the curve and draw a straight line connecting these points. The length of the straight line is our first guess of the length of the curve. Then to improve the approximation, we take three points on the curve. Draw secants to the curve connecting the first endpoint and the middle point and then the middle point and the terminal endpoint. The sum of the lengths of these segments is our better approximation. Continuing thus, we can form a partition of the parameter interval $[a, b]$ and take the lengths of the segments from $\mathbf{f}(x_i)$ to $\mathbf{f}(x_{i+1})$ where $[x_i, x_{i+1}]$ is the i^{th} segment. When we take the limit of this sum as the mesh of the partition goes to zero, we get the value of the arc length. Owing to this discussion, we define arc length for a parametrized curve in $\mathbb{R}^n$ as follows.

17. Let $\mathbf{f} : [a, b] \to \mathbb{R}^n$ be a parametrized differentiable curve. The arc length of this curve from $\mathbf{f}(a)$ to $\mathbf{f}(t)$, for $t \in [a, b]$ is defined as $s(t) = \int_a^t \|\mathbf{f}'(x)\|dx$.

18. We call $\frac{ds}{dt} = \|\mathbf{f}'(t)\|$ as the arc length segment.

19. The arc length of a curve is independent of the parametrization. The following is the statement.

20. *Theorem:* If $\mathbf{f} : [a, b] \to \mathbb{R}^n$ and $\mathbf{g} : [c, d] \to \mathbb{R}^n$ are two regular parametrized differentiable paths which parametrize the same curve with the same endpoints, then the arc length of the curve with respect to either parametrization is equal *i.e.* $\int_a^b \|\mathbf{f}'(x)\|dx = \int_c^d \|\mathbf{g}'(x)\|dx$.

    proof: Let us assume that $\mathbf{f}$ and $\mathbf{g}$ are one-to-one on their domain except possibly at the end points. (If not, cut the domains so that this property is satisfied on each piece and take the matching pieces.) Now, assume that there is a function $\alpha : [a, b] \to [c, d]$ which is monotonically increasing and differentiable with a differentiable inverse such that $\mathbf{f}(t) = \mathbf{g} \circ \alpha(t)$ for every $t \in [a, b]$. (We will need the general Chain rule for differentiability to show that $\alpha = \mathbf{g}^{-1} \circ \mathbf{f}$ and that it has the said properties). Then, by Chain rule as above, $\mathbf{f}'(t) = \mathbf{g}'(\alpha(t)) \cdot \alpha'(t)$. So, $\int_a^b \|\mathbf{f}'(x)\|dx = \int_a^b \|\mathbf{g}'(\alpha(x)) \cdot \alpha'(x)\|dx = \int_a^b \|\mathbf{g}'(\alpha(x))\| \cdot |\alpha'(x)|dx$. Now substitute $y = \alpha(x)$ for $x \in [a, b]$, then $dy = \alpha'(x)dx = |\alpha'(x)|dx$. The lower and upper limits of y are c and d respectively. So the above integral is $\int_c^d \|\mathbf{g}'(y)\|dy$

II. Functions of the form $f : A \subset \mathbb{R}^2 \to \mathbb{R}$

1. Examples:
   (a) $f(x, y) = x + y$ for $(x, y) \in \mathbb{R}^2$
   (b) $f(x, y) = xy$ for $(x, y) \in \mathbb{R}^2$
   (c) Polynomials in two variables, eg. $f(x, y) = x^4y + \pi xy^2y - 39x + 2y^2$ for $(x, y) \in \mathbb{R}^2$
   (d) $f(x, y) = \tan \frac{y}{x}$, for $(x, y) \in \mathbb{R}^2, x \neq 0$ (composition of trigonometric functions with other functions in two variables).
   (e) $f(x, y) = e^{(x^2+y^2)}$ for $(x, y) \in \mathbb{R}^2$ (composition of exponential function with other functions in two variables).
   (f) $f(x, y) = \begin{cases} y & \text{if} x \leq y \\ x & \text{if} y < x \end{cases}$. Discuss domains and ranges of these functions.

2. Visualize these functions in two ways:
   I Draw their graphs in $\mathbb{R}^3$ with $z = f(x, y)$. z records the output for every pair of independent inputs (x, y).
   (a) $f(x, y) = c$ for $(x, y) \in \mathbb{R}^2$.
   (b) $f(x, y) = ax + by$ for $(x, y) \in \mathbb{R}^2$.
   (c) $f(x, y) = \sqrt{1 - x^2 - y^2}$ for $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 < 1$.
   (d) $f(x, y) = x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$.
   II Level curves in $\mathbb{R}^2$: $\{(x, y) \in \mathbb{R}^2 : f(x, y) = c\}$. Above examples.

3. Limits of functions of two variables:
   (a) *Definition:* A function $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ is said to have a limit L as $(x, y) \in \mathbb{R}^2$ tends to $(a, b) \in A$, if for every given $\epsilon > 0$, there is a corresponding $\delta > 0$ such that $||(x, y) - (a, b)|| < \delta$ and $(x, y) \in A$ implies $||f(x, y) - L|| < \epsilon$. When the limit L exists, one writes $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

(b) *Example* $f(x,y) = xy$ for $(x,y) \in \mathbb{R}^2$. As (x,y) tends to $(2,\pi)$, $\lim_{(x,y)\to(2,\pi)} f(x,y) = 2\pi$. Given small $\epsilon > 0$, choose $\delta = \frac{\epsilon}{3\pi}$, so $||(x,y) - (2,\pi)|| < \delta$ implies $||xy - 2\pi|| = ||(x-2)(y-\pi) + \pi x + 2y - 4\pi|| = ||(x-2)(y-\pi) + \pi(x-2) + 2(y-\pi)|| < \delta(\delta + \pi + 2) < \delta(3\pi) = \epsilon$.

(c) *Sequential criterion:* A function $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ is said to have a limit L as $(x,y) \in \mathbb{R}^2$ tends to $(a,b) \in A$, if for every sequence $((x_n, y_n)) \subset A$ converging to (a,b), the corresponding sequence $(f(x_n, y_n))$ converges to L.

(d) *Limit rules:* When $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{(x,y)\to(a,b)} g(x,y)$ exist,
$\lim_{(x,y)\to(a,b)}(f \pm g)(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y)$
$\lim_{(x,y)\to(a,b)}(fg)(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) \lim_{(x,y)\to(a,b)} g(x,y)$
Furthermore when $\lim_{(x,y)\to(a,b)} g(x,y) \neq 0$, $\lim_{(x,y)\to(a,b)}(f/g)(x,y) = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$

4. Continuity of these functions:

(a) A function $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be continuous at a point (a,b) in its domain, if for every given $\epsilon > 0$, there is a corresponding $\delta > 0$ such that $||(x,y) - (a,b)|| < \delta$ and (x,y) is in the domain of f implies $||f(x,y) - f(a,b)|| < \epsilon$. A function which is continuous at all points in its domain is said to be a continuous function.

(b) The above definition is equivalent to the following: f is continuous at (a,b) in its domain if and only if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and the limit equals $f(a,b)$.

(c) Sequential criterion: A function $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be continuous at $(a,b) \in A$, if for every sequence $((x_n, y_n)) \subset A$ converging to (a,b), the corresponding sequence $(f(x_n, y_n))$ converges to $f(a,b)$

(d) Sequential criterion can often be used to show that a function is not continuous at a point. Eg: $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if}(x,y) \neq (0,0) \\ 0 & \text{if}(x,y) = (0,0) \end{cases}$

(e) Algebra of continuous functions: addition, subtraction, multiplication, division(?).

(f) Composition of continuous functions (one has to exercise care with domains and ranges).

5. We will discuss the notion of differentiability of these functions later.

III. Functions of the form $f : A \subset \mathbb{R}^3 \to \mathbb{R}$

1. Examples:

(a) $f(x,y,z) = 3x - y + z$ for $(x,y,z) \in \mathbb{R}^3$

(b) $f(x,y,z) = x^4y^2z + y^4z^2 + 16y - 3z + 2$ for $(x,y,z) \in \mathbb{R}^3$ and other polynomials in three variables

(c) $f(x,y,z) = \tan x^2 + y^2 + z^2$, for $(x,y,z) \in \mathbb{R}^3$ and other compositions of trigonometric functions with other functions in three variables.

(d) $f(x,y,z) = e^{(x-y^2+z^3)}$ for $(x,y,z) \in \mathbb{R}^3$ and other compositions of exponential functions with other functions in three variables. Discuss domains and ranges of these functions.

2. Visualization using level surfaces in $\mathbb{R}^3$: Level- c -surface of a function f is the set $\{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = c\}$. Examples:

(a) $f(x,y,z) = x^2 + y^2 + z^2$ for $(x,y,z) \in \mathbb{R}^3$.

(b) $f(x,y,z) = \sin(x + y + z)$ for $(x,y,z) \in \mathbb{R}^3$.

3. Limits and continuity : Much like functions of two variables.

IV. Functions of the form $f : A \subset \mathbb{R}^n \to \mathbb{R}$

1. The notion of limit and continuity for functions of this type can be defined in a similar manner to the way they have been defined for the cases $n = 2$ or $n = 3$.

2. The sequential criterion for continuity at a point holds for any n.

3. Recall the extreme value theorem which can be rephrased as "continuous image of a compact set is compact". In this form this theorem applies to functions of the form $f : A \subset \mathbb{R}^n \to \mathbb{R}$ as well. A precise statement is the following.

4. If A is a compact subset of $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ is a continuous function, then the set $\{f(a) : a \in A\}$ is a compact interval in $\mathbb{R}$.

5. This statement implies that the range of a function of this type, defined and continuous on a compact set A is such that the range is a compact interval $[c, d]$. This means that there are $\mathbf{x}$, $\mathbf{y} \in A$ such that $f(\mathbf{x}) = c$ and $f(\mathbf{y}) = d$. So the function assumes its maximum and minimum values in A.

V. Functions of the form $\mathbf{f} : A \subset \mathbb{R}^n \to \mathbb{R}^m$

1. A function $\mathbf{f}$ of this type has m component functions, each of which is a real valued function of n variables. It should be thought of as $\mathbf{f}(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n))$.

2. Example: $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{f}(x, y) = (x^2 - y^2, 2xy)$

3. Example: $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\mathbf{f}(x, y) = (\cos x, \sin x, y)$

4. Functions of these types are continuous if and only if the component functions are continuous.