IIT GUWAHATI
Department of Electronics and Electrical Engineering
End-semester Examination EE 693 Advanced Topics in Random Processes Maximum Marks:100 Time: 14:00-17:00 hours
Date 25.11.2021

Q.1 (a) Let $\{Z_n, n \geq 0\}$ be a sequence of independent random variables with a constant mean $\mu > 0$. Define $X_n = \sum_{i=0}^{n} Z_i, \ \ n \geq 0$. Examine if $\{Z_n\}$ is a martingale and find whether $\{X_n\}$ is a martingale, a sub-martingale or a super-martingale (4)

(b) Let $\{N(t)\}$ be Poisson process with the rate parameter λ. Examine if (i) $\{N(t)\}$ is a and (ii) $Y(t) = N(t) - \lambda t$ is a martingale with respect to $N(t)$. (6)

(c) Consider a martingale $\{X_n\}_{n=0}^{\infty}$ with $X_0 = 0$ and $EX_n^2 = n$.

Find (i) EX_n, (ii) $E\left(X_{n+5} / X_0, X_1, ..., X_n\right)$ (iii) $EX_n X_{n+2}$ (6)

Q.2. Suppose $\{X(t)\}$ is a zero-mean Gaussian random process with the autocorrelation function $EX(t)X(t+s) = e^{-s}$. (4)+(4)+(2)

(a) Find the joint PDF $f_{X(t),X(t+5)}(x_1, x_2)$

(b) Find $EX(t) \mid X\left(t\right) = x$ and $EX(t+5) \mid X\left(t\right) = x$

(c) Examine if $\{X(t)\}$ is strict-sense stationary.

Q.3. Let $\{X(t)\}$ be a standard Weiner process with the probability density function

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\frac{x^2}{t}}$$ (2)+(4)+(6)+(4)

(a) Write down the corresponding diffusion equation

(b) Find the probability $P(|X(1)| > 3)$ in terms of the Q function.

$$\left(\text{Note } Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} du \right)$$

(c) If $Y = X(2) + X(5) - X(3)$, find EY, $\text{var}(Y)$ and the PDF of Y.

(d) If $Y(t) = X^2(t) - t$ and $t > s$, find $EY(t) \mid X(u), u \leq s$.

Q.4. Consider a Poisson process $\{N(t), t \geq 0\}$ with the rate parameter $\lambda = 2$ and the corresponding interarrival times $T_1, T_2, \ldots$. (4)+(3)+(3)

(a) Find the transition probability matrix of the embedded Markov chain and the generator matrix for the process.

(b) Find the probability that the first arrival occurs after $t = 0.5$.

(c) Given that the third arrival occurs at $t = 2$, find the probability that the 4th arrival occurs after $t = 4$

Q.5. Customers arrive at a service station on the average 10 customers per hour. Assuming the Poisson model, find the probability that (4)+(4)

(a) 2 customers arrive between 9:35 hr and 9:55 hr.
(b) 3 customers arrive between 9:35 hr and 9:55 hr and 7 customers arrive between 10:15 hr and 10:55 hr.

Q.6. (a) Consider a birth-death process $\{N(t), t \geq 0\}$ with the birth-rate parameter $\lambda_n = n\lambda$ and the death rate parameter $\mu_n = 0$.

(i) Write down the forward and backward Kolmogorov equations for the process.
(ii) Assuming $N(0) = 1$, obtain the expression for $P(N(t) = 1)$ (4)+(4)

(b) Consider an M/M/1 queueing system with the constant arrival rate λ and departure rate μ.

(i) Draw the probability rate diagram and hence write down the global balance equations.
(ii) If N is the number of jobs in the system in the steady state, obtain the expression for the steady-state probability $P(N = j)$ using the probability sum constraint for the system
(iii) Find $E(N)$ and $Var(N)$ (4)+(4)+(4)

Q.7. Consider the discrete Markov chain (DTMC) represented by the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0.3 & 0.7 & 0 & 0 \\ 0.4 & 0.5 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

4)+(2)+(2)

(a) Draw the transition probability diagram for the chain
(b) Partition the state-space into communicating classes.
(c) Find the closed communicating class of the chain.

Q.8. Suppose $\{X_n, n \geq 0\}$ is a DTMC with $V = \{0,1,2\}$ and the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(3)+(3)+(4)+(4)

(a) Examine if the chain is aperiodic
(b) Examine if the chain is irreducible
(c) Find the stationary probability distribution of the states.
(d) Find the average first return times to the states.