PH 102 : Electromagnetism
Tutorial 12
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1. A flat wire of radius a carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$ forms a parallel plate capacitor as shown in figure 1.
   (a) Find the electric and magnetic fields in the gap as functions of the distance s from

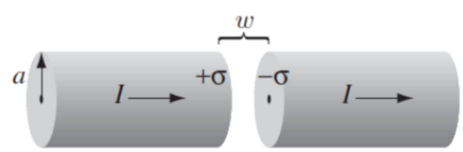

Figure 1: Figure for problem 1.

   the axis and the time t.
   (b) Find the energy density u_{em} and the Poynting vector $\vec{S}$ in the gap. Note specially the direction of $\vec{S}$. Check that $\partial u_{\text{em}}/\partial t = -\vec{\nabla} \cdot \vec{S}$ is satisfied.
   (c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Poynting's Theorem-in this case $W = 0$, because there is no charge in the gap).

Solution: (a) $\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{z}};\ \sigma = \frac{Q}{\pi a^2};\ Q(t) = It\ \Rightarrow\ \mathbf{E}(t) = \frac{It}{\pi\epsilon_0 a^2}\,\hat{\mathbf{z}}.$

$$B\,2\pi s = \mu_0\epsilon_0\frac{\partial E}{\partial t}\pi s^2 = \mu_0\epsilon_0\frac{I\pi s^2}{\pi\epsilon_0 a^2}\ \Rightarrow\ \mathbf{B}(s,t) = \frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}.$$

(b) $u_{\mathrm{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{1}{2}\left[\epsilon_0\left(\frac{It}{\pi\epsilon_0 a^2}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right] = \frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + (s/2)^2\right].$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(\frac{It}{\pi\epsilon_0 a^2}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)(-\hat{\mathbf{s}}) = -\frac{I^2 t}{2\pi^2\epsilon_0 a^4}\,s\,\hat{\mathbf{s}}.$$

$$\frac{\partial u_{\mathrm{em}}}{\partial t} = \frac{\mu_0 I^2}{2\pi^2 a^4}2c^2 t = \frac{I^2 t}{\pi^2\epsilon_0 a^4};\quad -\boldsymbol{\nabla}\cdot\mathbf{S} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\boldsymbol{\nabla}\cdot(s\,\hat{\mathbf{s}}) = \frac{I^2 t}{\pi^2\epsilon_0 a^4} = \frac{\partial u_{\mathrm{em}}}{\partial t}.$$

(c) $U_{\mathrm{em}} = \int u_{\mathrm{em}} w 2\pi s\,ds = 2\pi w\frac{\mu_0 I^2}{2\pi^2 a^4}\int_0^b[(ct)^2 + (s/2)^2]s\,ds = \frac{\mu_0 w I^2}{\pi a^4}\left[(ct)^2\frac{s^2}{2} + \frac{1}{4}\frac{s^4}{4}\right]\Big|_0^b$

$$= \frac{\mu_0 w I^2 b^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{8}\right].$$

Over a surface at radius b :

$$P_{\mathrm{in}} = -\int \mathbf{S}\cdot d\mathbf{a} = \frac{I^2 t}{2\pi^2\epsilon_0 a^4}\left[b\,\hat{\mathbf{s}}\cdot(2\pi b w\,\hat{\mathbf{s}})\right] = \frac{I^2 w t b^2}{\pi\epsilon_0 a^4}.$$

$$\frac{dU_{\mathrm{em}}}{dt} = \frac{\mu_0 w I^2 b^2}{2\pi a^4}2c^2 t = \frac{I^2 w t b^2}{\pi\epsilon_0 a^4} = P_{\mathrm{in}}.$$ (Set b = a for total .)

2. A parallel plate capacitor is made up of two circular discs of diameter d each, spaced a distance h apart. A potential difference of V is applied between the plates and the spacing between the plates is increased at a uniform rate to $2h$ in one second. Find the induced magnetic field and the Poynting vector at the edge of the capacitor plates as the plates are separated. Use the Poynting's theorem to correlate the change of stored energy to the energy loss as the plates are being separated. (Ignore the effects due to edges of the plates)

Solution: If z is the distance between the plates at time t, then $z = h(1+t)$. This can be found from the uniform rate at which the spacing is increased to $2h$ in one second. Thus $dz/dt = h$. Ignoring the edge effects, the electric field can be written as

$$E = \frac{V}{z} \implies D = \frac{\epsilon V}{z}$$

The displacement current density can be found as

$$\frac{dD}{dt} = \epsilon V \frac{d}{dt}\left(\frac{1}{z}\right) = -\frac{\epsilon V}{z^2}\frac{dz}{dt} = -\frac{\epsilon V h}{z^2}$$

Now, by applying Ampere's law to a loop of radius r (coaxial with the centre of the circular discs), the magnetic field B at the radius can be found as

$$2\pi r B = -\frac{\mu\epsilon V h}{z^2}\pi r^2 \implies B = -\frac{\mu\epsilon V h r}{2z^2}$$

At the edge of the plates, $r = d/2$ and hence the magnetic field is

$$B(r = d/2) = -\frac{\mu\epsilon V h d}{4z^2}$$

Poynting vector is $\vec{S} = \frac{1}{\mu}(\vec{E}\times\vec{B})$. In this case, the electric field is in the z direction while the magnetic field is in the circumferential (ϕ) direction. Therefore

$$\vec{S} = \frac{1}{\mu}(\vec{E}\times\vec{B}) = -\frac{V}{z}\frac{\epsilon V h d}{4z^2}(\hat{z}\times\hat{\phi}) = \frac{\epsilon V^2 h d}{4z^3}\hat{r}$$

Thus the energy flowing through the curved surface in between the two circular discs is

$$\oint \frac{1}{\mu}(\vec{E}\times\vec{B})\cdot d\vec{a} = \frac{\epsilon V^2 h d}{4z^3}(z\pi d)$$

where the area of the curved surface is $z\pi d$ and it is in the $\hat{r}$ direction. Now, the stored energy in electromagnetic fields are

$$\int \frac{1}{2}\epsilon E^2 d\tau = \frac{1}{2}\epsilon\left(\frac{V}{z}\right)^2\frac{\pi d^2 z}{4} = \frac{\pi\epsilon V^2 d^2}{8z}$$

$$\int \frac{1}{2\mu} B^2 d\tau = \frac{1}{2} \int \left(-\frac{\mu\epsilon V hr}{2z^2} \right)^2 rdrd\phi z$$

$$\implies \int \frac{1}{2\mu} B^2 d\tau = \frac{\pi\mu\epsilon^2 V^2 h^2 d^4}{256 z^3} = \frac{\pi\epsilon V^2 h^2 d^4}{256 c^2 z^3}$$

where we are not integrating the z variable as we are considering it to be fixed at a particular instant of time t, while integrating the variable r from 0 to $d/2$. Since the speed at which the plates are being separated is very small compared to the speed of light $h \ll c = 1/\sqrt{\mu\epsilon}$, the magnetic energy is negligible. Thus the rate of decrease of stored energy is dominantly from

$$-\frac{d}{dt} \int \frac{1}{2} \epsilon E^2 d\tau = \frac{\pi\epsilon V^2 d^2 h}{8z^2}$$

Now, the force between the capacitor plates for a separation z is

$$F = -\frac{\epsilon E^2}{2} \frac{\pi d^2}{4} = -\frac{\pi\epsilon V^2 d^2}{8z^2}$$

Since the plates move a distance hdt in time dt, the work done in moving them per unit time is

$$\frac{dW}{dt} = -\frac{\pi\epsilon V^2 h d^2}{8z^2}$$

Thus the work done per unit time is equal to the rate of decrease of electromagnetic energy stored in the fields less the energy flowing out through the surface, in agreement with the Poynting's theorem.

3. The linearly polarised wave is denoted by $\vec{\tilde{f}}(z,t) = \tilde{A} e^{i(kz-\omega t)} \hat{n}$. Linear polarisation results from the combination of horizontally and vertically polarised waves of the same phase. If the two components are of equal amplitude, but out of phase by $\pi/2$ (say, $\delta_v = 0, \delta_h = \pi/2$), the result is a circularly polarised wave. In that case:
(a) At a fixed point z, show that the string moves in a circle about the z axis. Does it go clockwise (right circular polarised) or counterclockwise (left circular polarised), as you look down the axis toward the origin? How would you construct a wave circling the other way?
(b) Sketch the string at time $t = 0$.
(c) How would you shake the string in order to produce a circularly polarised wave?

Solution:

(a) The vertical polarisation is $\vec{f_v}(z,t) = A\cos(kz - \omega t)\ \hat{x}$ and the horizontal one is $\vec{f_h}(z,t) = A\cos(kz - \omega t + 90^\circ)\ \hat{y} = -A\sin(kz - \omega t)\ \hat{y}$.

Now, $f_v^2 + f_h^2 = A^2$, so $\vec{f} = \vec{f_v} + \vec{f_h}$ lies on a circle of radius A. At $t = 0$, $\vec{f} = A\cos(kz)\ \hat{x} - A\sin(kz)\ \hat{y}$ and at $t = \frac{\pi}{2\omega}$, $\vec{f} = A\cos(kz - 90^\circ)\ \hat{x} - A\sin(kz - 90^\circ)\ \hat{y} = A\sin(kz)\ \hat{x} + \cos(kz)\ \hat{y}$. It is circling *counterclockwise*, to make it circling the other way, use $\delta_h = -\pi/2$.

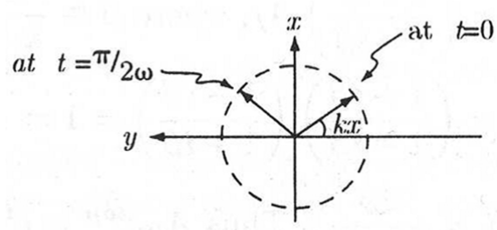

Figure 2: Figure for solution to problem 3.

(b) The sketch of the string at $t = 0$ is shown in figure 3.

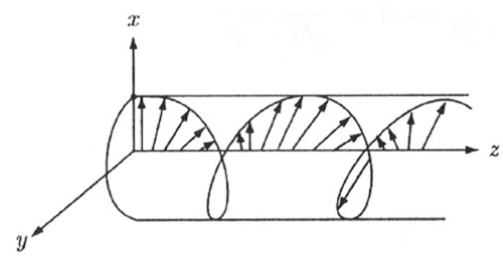

Figure 3: Figure for solution to problem 3.

(c) To produce a circularly polarised wave in a string, one needs to shake it around in a circle instead of up and down.

4. The simplest possible spherical wave can be represented by

$$E(r, \theta, \phi, t) = A\frac{\sin\theta}{r}\left[\cos(kr - \omega t) - \frac{1}{kr}\sin(kr - \omega t)\right]\hat{\phi}, \quad \frac{\omega}{k} = c.$$

(a) Show that $\vec{E}$ obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic field.
(b) Calculate the Poynting vector. Average $\vec{S}$ over a full cycle to get the intensity vector $\vec{I}$. Does it point in the expected direction? Does it fall off like r^{-2}, as it should?
(c) Integrate $\vec{I} \cdot d\vec{a}$ over a spherical surface to determine the total power radiated.

Solution:

$$E(r, \theta, \phi, t) = A\frac{\sin\theta}{r}\left[\cos(kr - \omega t) - \frac{1}{kr}\sin(kr - \omega t)\right]\hat{\phi}, \quad \frac{\omega}{k} = c.$$

For simplicity, we choose $(kr - \omega t) \equiv u$ throughout so that

$$\frac{\partial}{\partial r}\cos u = -k\sin u, \quad \frac{\partial}{\partial r}\sin u = k\cos u.$$

Also, the amplitude A is denoted as E_0 in the calculations below. Recall that in curvilinear coordinates

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{h_1h_2h_2}\left[\frac{\partial}{\partial u_1}(h_2h_3V_1) + \frac{\partial}{\partial u_2}(h_1h_3V_2) + \frac{\partial}{\partial u_3}(h_1h_2V_3)\right]$$

$$\vec{\nabla}\times\vec{V} = \frac{1}{h_1h_2h_2}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_2} \\ h_1V_2 & h_2V_2 & h_3V_3 \end{vmatrix}$$

For spherical polar coordinates, $u_1, u_2, u_3 \equiv r, \theta, \phi, h_1 = 1, h_2 = r, h_3 = r\sin\theta$.
(a)
(i) Gauss's/Coulomb's law:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} = 0$$

(ii) Faraday's law:

$$\begin{aligned}
-\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\vec{E} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\hat{\theta} \\
&= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[E_0\frac{\sin^2\theta}{r}\left(\cos u - \frac{1}{kr}\sin u\right)\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left[E_0\sin\theta\left(\cos u - \frac{1}{kr}\sin u\right)\right]\hat{\theta} \\
&= \frac{1}{r\sin\theta}\frac{E_0}{r}2\sin\theta\cos\theta\left(\cos u - \frac{1}{kr}\sin u\right)\hat{r} - \frac{1}{r}E_0\sin\theta\left(-k\sin u + \frac{1}{kr^2}\sin u - \frac{1}{r}\cos u\right)\hat{\theta}.
\end{aligned}$$

Integrating with respect to t, and noting that $\int\cos u\ dt = -\frac{1}{\omega}\sin u$, $\int\sin u\ dt = \frac{1}{\omega}\cos u$, we obtain

$$\vec{B} = \frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\hat{r} + \frac{E_0\sin\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\hat{\theta}.$$

(iii) Divergence of $\vec{B}$:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{B} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{2E_0\cos\theta}{\omega}\left(\sin u + \frac{1}{kr}\cos u\right)\right] \\
&+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{E_0\sin^2\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\right] \\
&= \frac{1}{r^2}\frac{2E_0\cos\theta}{\omega}\left(k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u\right) \\
&+ \frac{1}{r\sin\theta}\frac{2E_0\sin\theta\cos\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right) \\
&= \frac{2E_0\cos\theta}{\omega r^2}\left(k\cos u - \frac{1}{kr^2}\cos u - \frac{1}{r}\sin u - k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)
\end{aligned}$$

$$\implies \vec{\nabla} \cdot \vec{B} = 0$$

(iii) Ampere's Law (After Maxwell's Correction):

$$\begin{aligned}
\vec{\nabla} \times \vec{B} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{\phi} \\
&= \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[\frac{E_0\sin\theta}{\omega}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)\right]\right. \\
&\left. - \frac{\partial}{\partial\theta}\left[\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\right]\right\}\hat{\phi} \\
&= \frac{E_0\sin\theta}{\omega r}\left(k^2\sin u - \frac{2}{kr^3}\cos u - \frac{1}{r^2}\sin u - \frac{1}{r^2}\sin u + \frac{k}{r}\cos u + \frac{2}{r^2}\sin u + \frac{2}{kr^3}\cos u\right)\hat{\phi} \\
&= \frac{k}{\omega}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} = \frac{1}{c}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} &= \frac{1}{c^2}\frac{E_0\sin\theta}{r}\left(\omega\sin u + \frac{\omega}{kr}\cos u\right)\hat{\phi} = \frac{1}{c^2}\frac{\omega}{k}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} \\
&= \frac{1}{c}\frac{E_0\sin\theta}{r}\left(k\sin u + \frac{1}{r}\cos u\right)\hat{\phi} = \vec{\nabla} \times \vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}.
\end{aligned}$$

(b) Poynting Vector:

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \\
&= \frac{E_0 \sin\theta}{\mu_0 r}\left(\cos u - \frac{1}{kr}\sin u\right)\left[\frac{2E_0\cos\theta}{\omega r^2}\left(\sin u + \frac{1}{kr}\cos u\right)\hat{\theta}\right. \\
&+ \left.\frac{E_0 \sin\theta}{\omega r}\left(-k\cos u + \frac{1}{kr^2}\cos u + \frac{1}{r}\sin u\right)(-\hat{r})\right] \\
&= \frac{E_0^2 \sin\theta}{\mu_0 \omega r^2}\left\{\frac{2\cos\theta}{r}\left[\sin u\cos u + \frac{1}{kr}(\cos^2 u - \sin^2 u) - \frac{1}{k^2r^2}\sin u\cos u\right]\hat{\theta}\right. \\
&- \left.\sin\theta\left(-k\cos^2 u + \frac{1}{kr^2}\cos^2 u + \frac{1}{r}\sin u\cos u + \frac{1}{r}\sin u\cos u - \frac{1}{k^2r^3}\sin u\cos u - \frac{1}{kr^2}\sin^2 u\right)\hat{r}\right\} \\
&= \frac{E_0^2 \sin\theta}{\mu_0 \omega r^2}\left\{\frac{2\cos\theta}{r}\left[\left(1 - \frac{1}{k^2r^2}\right)\sin u\cos u + \frac{1}{kr}(\cos^2 u - \sin^2 u)\right]\hat{\theta}\right. \\
&+ \left.\sin\theta\left[\left(-\frac{2}{r} + \frac{1}{k^2r^3}\right)\sin u\cos u + k\cos^2 u + \frac{1}{kr^2}(\sin^2 u - \cos^2 u)\right]\hat{r}\right\}.
\end{aligned}$$

Averaging over a full cycle, using $\langle \sin u \cos u \rangle = 0, \langle \sin^2 u \rangle = \langle \cos^2 u \rangle = \frac{1}{2}$, it is clear that only one term in the Poynting vector above will contribute to the time averaged Poynting vector. We get the intensity vector as

$$\vec{I} = \langle \vec{S} \rangle = \frac{E_0^2 \sin\theta}{\mu_0 \omega r^2}\left(\frac{k}{2}\sin\theta\right)\hat{r} = \frac{E_0^2 \sin^2\theta}{2\mu_0 c r^2}\hat{r}.$$

It points in the radial direction $\hat{r}$ and falls off as $1/r^2$, as we would expect for a spherical wave.

(c) Total power radiated is

$$P = \int \vec{I} \cdot d\vec{a} = \frac{E_0^2}{2\mu_0 c}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta d\theta d\phi = \frac{E_0^2}{2\mu_0 c}(2\pi)\int_0^\pi \sin^3\theta d\theta = \frac{4\pi}{3}\frac{E_0^2}{\mu_0 c}$$

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