Assignment 1

1. Suppose $\{A_n\}_{n=1}^{\infty}$ is a sequence of events in a probability space. Show that

   (a) If $A_n = \begin{cases} \text{A}, & n \text{ odd} \\ \text{B}, & n \text{ even} \end{cases}$ then $\lim_{n\to\infty} \sup A_n = A \bigcup B$ and $\lim_{n\to\infty} \inf A_n = A \bigcap B$

   (b) If $A \supset A_2 .... \supset A_n ......$,then

   $$\lim_{n\to\infty} \sup A_n = \bigcap_{n=1}^{\infty} A_n = \lim_{n\to\infty} \inf A_n$$

   (c) If $A \subset A_2 .... \subset A_n ......$,then

   $$\lim_{n\to\infty} \sup A_n = \bigcup_{n=1}^{\infty} A_n$$

   (d.) If $A_i \cap A_j = \phi, i \neq j$, then $\lim_{n\to\infty} A_n = \phi$

2. Find the $\lim_{n\to\infty} \sup A_n$ and $\lim_{n\to\infty} \inf A_n$ of the sequence:

   $$A_n = \begin{cases} \left(1,5-\frac{1}{n}\right) \text{ when n is odd} \\ \left(2,5+\frac{1}{n}\right) \text{ when n is even} \end{cases}$$

3. Suppose $\{X_n\}$ is a sequence of independent random variables with:

   $P(X_n = 0) = 1 - \frac{1}{n^2}$ and $P(X_n = n^2) = \frac{1}{n^2}$

   Examine individually if $\{X_n\}$ converges *a.s.*, in probability, in *m.s.* and in distribution to $\{X = 0\}$.

   With this example show that $\{X_n\} \xrightarrow{a.s.} X \not\Rightarrow \{X_n\} \xrightarrow{m.s.} X$.

4. Suppose $\{X_n\} \xrightarrow{m.s.} X$.

   Show that $EX_n^2 \to EX^2$.

5. Suppose $\{X_n\}_{n=1}^{\infty}$ is a sequence of iid random variables and $X_n \sim U[0,1]$. Define $Z_n = n \min(X_1, X_2, ... X_n)$ and $Z \sim \exp(-1)$, show that $\{Z_n\} \xrightarrow{d.} Z$

6. Suppose $\{X_n\}$ is a sequence of non-negative random variables with

   $$F_{\alpha_n}(x) = 1 - \frac{1}{1+nx}$$

   Examine if $\{X_n\}$ converges in *d,p* and *m.s.* to $\{X = 0\}$.

7. Suppose $\{S_n\}$ represents the number of tails obtained in *n* independent tossing of a fair coin. Find the minimum value of *n* such that the probability that the number of tails deviates from

the expected value by 1% is less than 0.05. Note that for a Gaussian random variable $X \sim N(\mu, \sigma^2)$, $P(\mu - 1.6\sigma < X < \mu + 1.6\sigma) = 0.95$.

8. Let $\{X_n\}$ be a sequence of iid *N(0,1)* random variables. Find the approximate value of $P\left(X_1^2 + X_2^2 + ... + X_n^2 \geq 2n\right)$ when *n* is large. Use large deviation theory.

9. $\{X_n\}$ is a sequence of independent and identically distributed random variables with variance σ^2 and

   $$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \hat{\mu}\right)^2$$

   where $\hat{\mu} = \sum_{i=1}^{n}\frac{X_i}{n}$

   Show that $\hat{\sigma}^2 \xrightarrow{P} \sigma^2$