Linear Algebra

Department of Mathematics
Indian Institute of Technology Guwahati

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MA 102 (RA, RKS, MGPP, KVK)

Inner product spaces and spectral theorem

Topics:

- Inner product spaces
- Gram-Schmidt Process

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- Inner product spaces
- Gram-Schmidt Process
- Spectral theorem for Hermitian matrices
- Orthogonal complement and projection theorem

Inner product spaces (IPS)

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Recall that $\mathbb{R}^n$ is an IPS with an inner product

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$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n.$$

Examples

- Consider the ordered basis $B := [1, x, \ldots, x^n]$ of $\mathbb{R}_n[x]$. Then $\mathbb{R}_n[x]$ is an IPS with respect to the inner product

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- The vector space $\mathcal{M}_{m\times n}(\mathbb{F})$ is an IPS with respect to the inner product $\langle A,\, B\rangle := \mathrm{Trace}(B^*A)$, where B^* is the conjugate transpose of B.

Length and distance

Definition: Let $\mathbb{V}$ be an IPS. The norm (or length) of a vector $\mathbf{v}$ in $\mathbb{V}$ is a nonnegative number $\|\mathbf{v}\|$ defined by $\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

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Cauchy-Schwarz Inequality: Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{V}$. Then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

Proof: Suppose that $\mathbf{u} \neq 0$. Set $\alpha := \langle \mathbf{v}, \mathbf{u} \rangle / \|\mathbf{u}\|^2$ and define $\mathbf{w} := \mathbf{v} - \alpha \mathbf{u}$. Then

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3. Triangle inequality: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

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Distance: The distance $\mathrm{d}(\mathbf{u}, \mathbf{v})$ between two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{V}$ is defined by $\mathrm{d}(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\|$.

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Example: For $f(x) := x$ and $g(x) := 3x - 2$ in $C[0, 1]$, we have

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Unit vector: A vector $\mathbf{v}$ in $\mathbb{V}$ is called a unit vector if $\|\mathbf{v}\| = 1$. If $\mathbf{u}$ is a nonzero vector then $\mathbf{v} := \frac{1}{\|\mathbf{u}\|}\mathbf{u}$ is a unit vector in the direction of $\mathbf{u}$. Here $\mathbf{v}$ is referred to as normalization of $\mathbf{u}$.

Example: The polynomials $1, \sqrt{3}\,x, \ldots, \sqrt{2n+1}\,x^n$ are unit vectors in $\mathbb{R}_n[x]$ with respect to the inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)dx$.

Angle and orthogonality

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbb{V}$. By Cauchy-Schwarz inequality $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$. Hence there exists a unique $\theta \in [0, \pi/2]$ such that

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Pythagoras' Theorem: Let **u** and **v** be vectors in $\mathbb{V}$. Then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \Leftrightarrow \langle \mathbf{u}, \mathbf{v}\rangle = 0.$$

Orthonormal Basis

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Definition: Let $\mathbb{V}$ be an IPS. A subset $S \subseteq \mathbb{V}$ is said to be an orthogonal basis (OB) of a subspace $\mathbb{W}$ of $\mathbb{V}$ if S is an orthogonal set and is a basis of $\mathbb{W}$.

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Example

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  The polynomials in S are called Legendre polynomials. By normalizing the Legendre polynomials, we obtain an orthonormal basis of $\mathbb{R}_2[x]$ on the interval $[-1, 1]$.

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Theorem: Let $B := [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be an ordered ONB of $\mathbb{V}$. Let $\mathbf{v} \in \mathbb{V}$. Then $\mathbf{v} = \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \cdots + \langle \mathbf{v}, \mathbf{v}_n \rangle \mathbf{v}_n$ and

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Question: Does every finite dimensional IPS have an orthonormal basis? Yes. In fact, any basis can be transformed to an ONB. The process is called Gram-Schmidt Process.

Gram-Schmidt Process via an example

Suppose $\mathbb{W} = \operatorname{span}\left(\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 11 \\ 4 \end{bmatrix}\right) \preccurlyeq \mathbb{R}^4$.

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Fact: Let $\mathbb{V}$ be an IPS and $B := [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be an ordered ONB. Then $[T]_B = [\langle T\mathbf{v}_j, \mathbf{v}_i\rangle]_{n\times n}$ and $[T^*]_B = ([T]_B)^*$.

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Proof: $[T\mathbf{v}_j]_B = [\langle T\mathbf{v}_j, \mathbf{v}_1\rangle, \ldots, \langle T\mathbf{v}_j, \mathbf{v}_n\rangle]^\top$. ■

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$\mu\langle \mathbf{u}, \mathbf{v}\rangle = \langle A\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, A\mathbf{v}\rangle = \lambda\langle \mathbf{u}, \mathbf{v}\rangle \Rightarrow \langle \mathbf{u}, \mathbf{v}\rangle = 0$. ■

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Spectral Theorem for Hermitian matrices

Theorem: Let $A \in \mathcal{M}_n(\mathbb{C})$. Then A is Hermitian $\Leftrightarrow$ there exist a unitary matrix U and a real diagonal matrix D such that

$$U^*AU = D.$$

Thus a Hermitian matrix is unitarily diagonalizable and has n orthonormal eigenvectors.

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- If $\mathbb{W} = \text{span}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k)$, then $\mathbf{v} \in \mathbb{W}^\perp$ if and only if $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, 2, \ldots, k$.

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Exercises

- Use projection theorem to show:
  (1) $\dim \mathbb{W} + \dim \mathbb{W}^\perp = n$,

  (2) $(\mathbb{W}^\perp)^\perp = \mathbb{W}$.

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- Find a basis for $W^\perp$, where
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