Network Theorems

Linearity and Superposition

- Basically, a mathematical equation is said to be linear if the following properties hold.
  - homogeneity
  - additivity
- What does this mean? We first look at the property of homogenity.

Linearity: Homogeneity

Homogeneity requires that if the input (excitation) to a system (equation) is multiplied by a constant, then the output should be obtained by multiplying by the same constant to obtain the correct solution.

- Sometimes equations that we think are linear, turn out not be be linear because they fail the homogeneity property.
- We next consider such an example.

Linearity: Homogeneity (Scaling)

- Does homogeneity hold for the following equation?

Given,

$$y = 4x \qquad (1)$$

- If $x = 1$, $y = 4$. If we double x to $x = 2$ and substitute this value into Eq 1 we get $y = 8$.

- Now for homogeneity to hold, scaling should hold for y, that is, y has a value of 4 when $x = 1$.

Linearity: Homogeneity (Scaling)

- Does homogeneity hold for the following equation?

$$y = 4x + 2 \qquad (2)$$

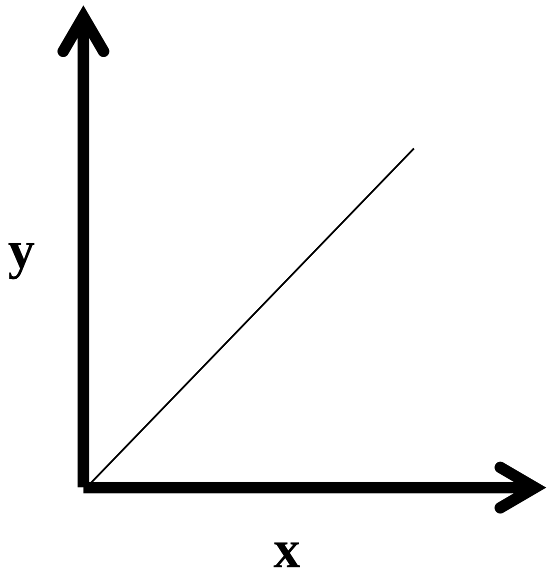
- If x = 1, then y = 6. If we double x to x=2, then y = 10.

- Now, since we doubled x we should be able to double the value that y. Which obviously is not

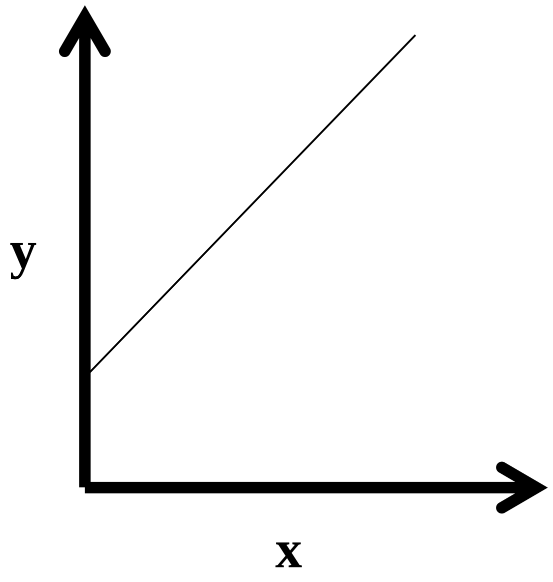
- We conclude that Eq 2 is not a linear equation.

Linearity

- Many of us were brought-up to think that if plotting an equation yields a straight line, then the equation is linear.
- From the following illustrations we have;



Linear



Not linear

Linearity: Additive property

- The additive property is equivalent to the statement that the response of a system to a sum of inputs is the same as the responses of the system when each input is applied separately and the individual responses summed (added together).

- This can be explained by considering the following illustrations.

Given, $y = 4x$.
Let $x = x_1$, then $y_1 = 4x_1$
Let $x = x_2$, then $y_2 = 4x_2$

Then $y = y_1 + y_2 = 4x_1 + 4x_2$ 3

Also, we note,
$y = f(x_1 + x_2) = 4(x_1 + x_2) = 4x_1 + 4x_2$ 4

- Since Equations (3) and (4) are identical, the additive property holds.

Linearity: Additive property

Given, $y = 4x + 2$.

Let $x = x_1$, then $y_1 = 4x_1 + 2$
Let $x = x_2$, then $y_2 = 4x_2 + 2$

Then $y = y_1 + y_2 = 4x_1+2 + 4x_2+2 = 4(x_1+x_2) + 4$ 5

Also, we note,
$y = f(x_1 + x_2) = 4(x_1 + x_2) + 2$ 6

- *Since Equations (5) and (6) are not identical, the additive property does not hold.*

Superposition Theorem

- According to superposition theorem, in a linear network containing two or more sources, the response in any element is equal to the algebraic sum of the responses caused by individual sources acting alone while the other sources are non-operative.

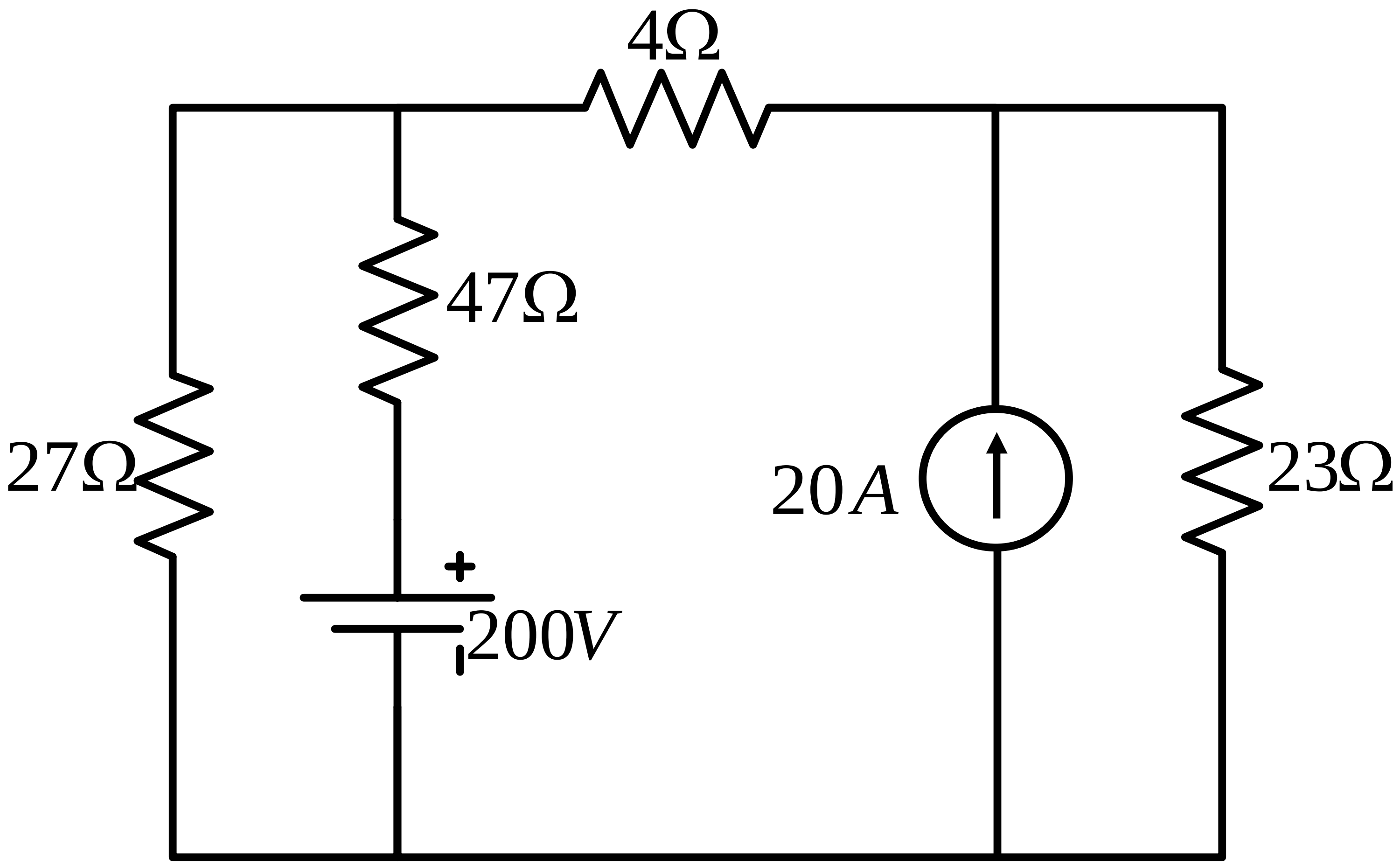
- This theorem is useful for analyzing networks that have large number of independent sources as it makes it possible to consider the effects of each source separately.

- Superposition theorem is applicable to any linear circuit having time varying or time invariant elements.

- The limitations of superposition theorem are
  - *It is not applicable to the networks consisting of non linear elements like transistors, diodes, etc.*
  - *It is not applicable to the networks consisting of any dependent sources*

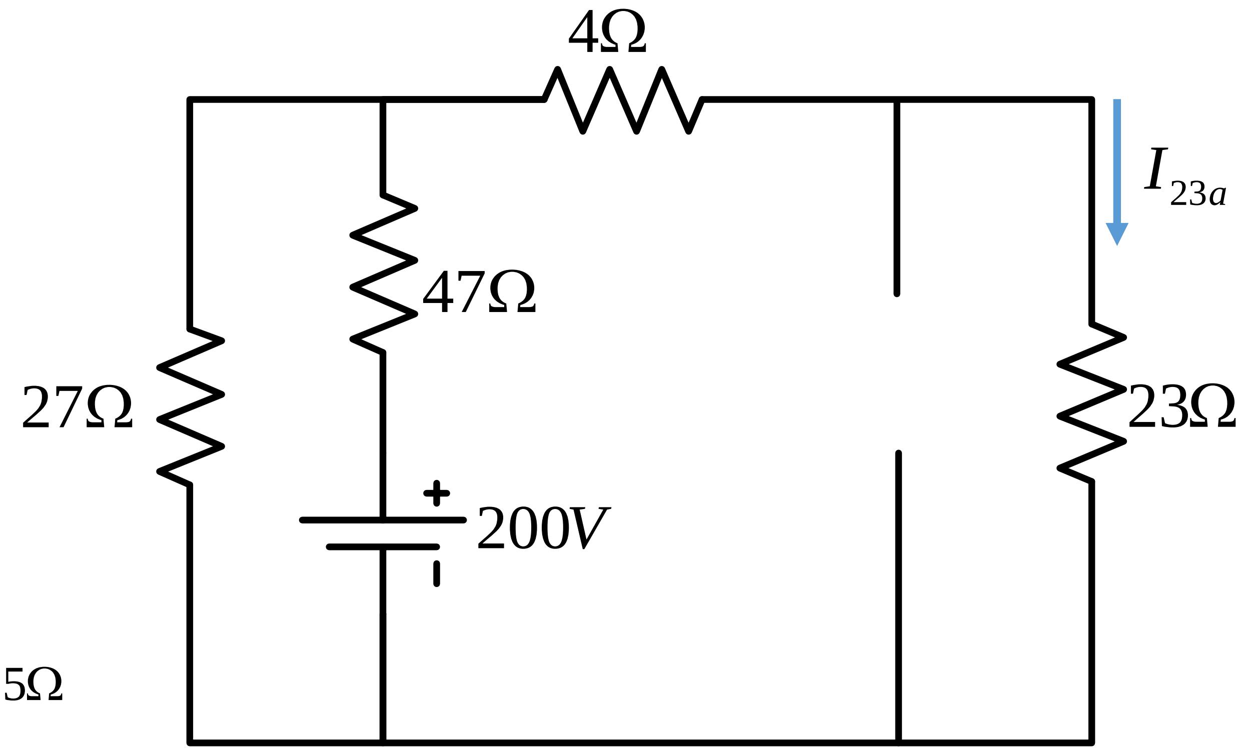
Superposition Theorem: Example

- Compute the current in the 23Ohm resistor using superposition theorem.



Superposition Theorem: Example

- With 200V source acting alone and the 20A current open circuited:



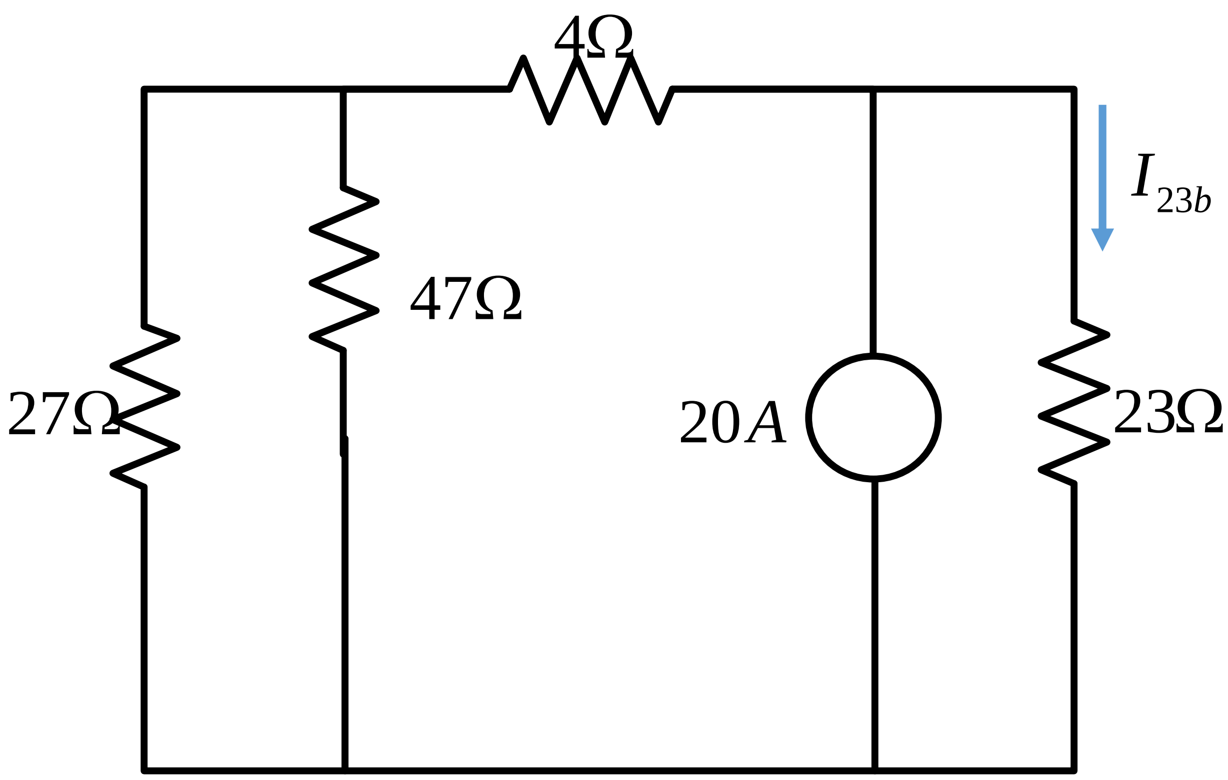
$$R_{eq} = 47 + \frac{27 \times (4+23)}{54} = 60.5\Omega$$

$$I_{total} = \frac{200}{60.5} = 3.31A$$

$$I_{23a} = \left(\frac{27}{54}\right) \times 3.31 = 1.655A$$

Superposition Theorem: Example

- With 20A source acting alone and the 200V source short circuited:



$$R_{eq} = 4 + \frac{27 \times 47}{74} = 21.15\Omega$$

$$I_{23b} = \left(\frac{21.15}{21.15 + 23} \right) \times 20 = 9.58A$$

Curent through the 23Ω is

$$I_{23} = I_{23a} + I_{23b}$$

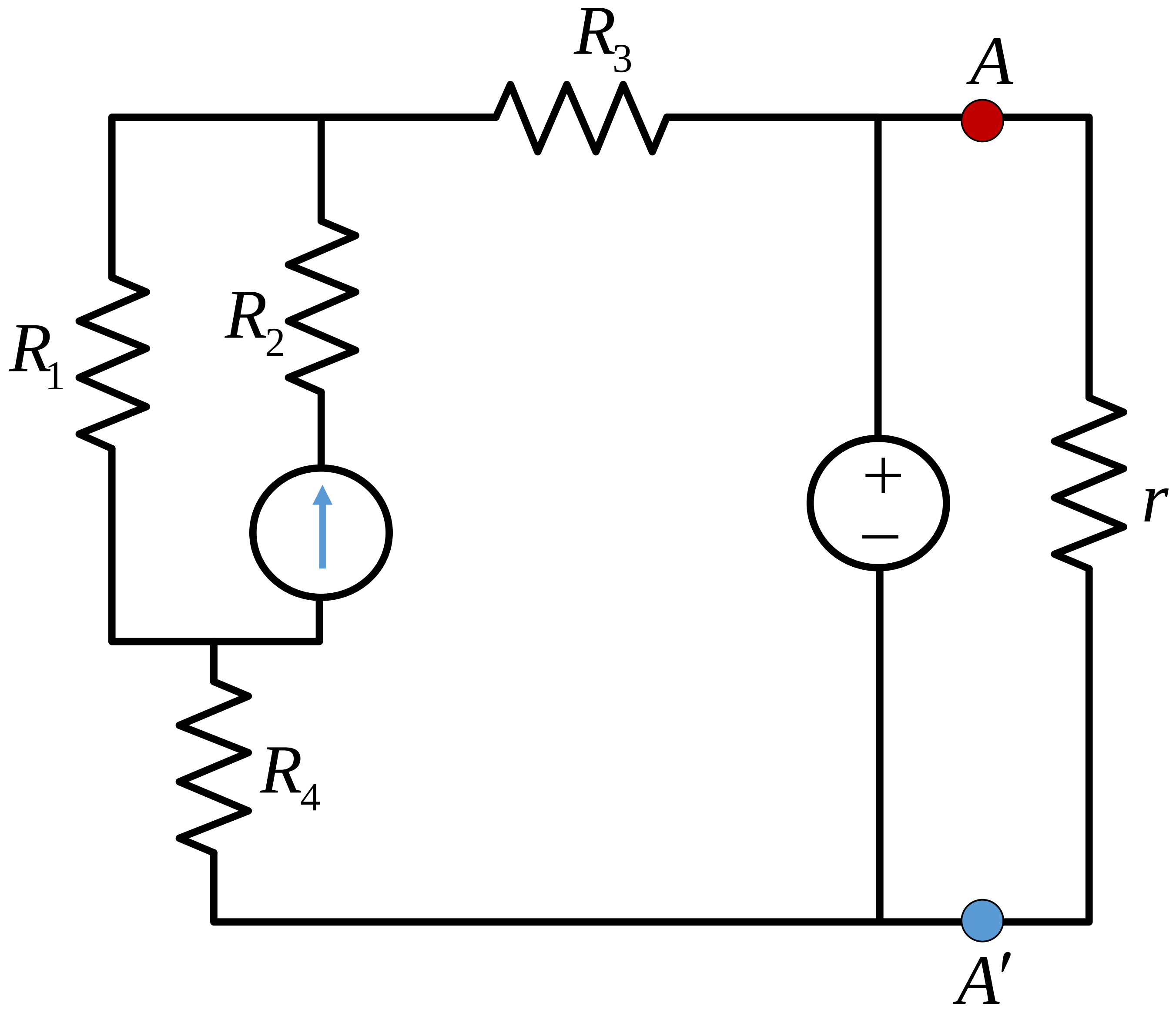
Thevenin: The Person

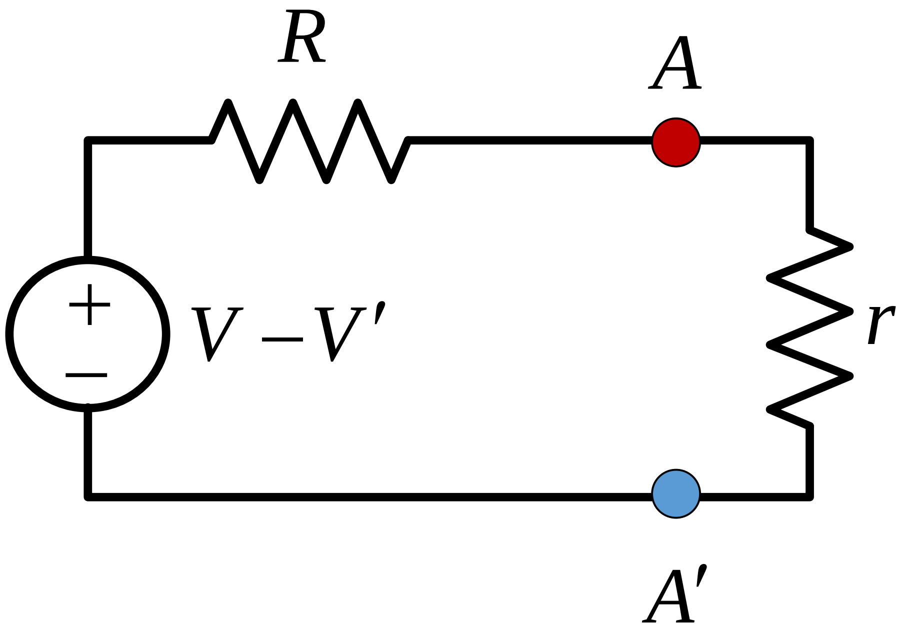
- Léon Charles Thévenin was born in Meaux, France (located some 20 miles from Paris) on March 30, 1857.
- He graduated from the École Polytechnique in 1876 (the year the telephone was developed by Bell)
- In 1878, joined the France's national electrical communication company Postes et Télégraphes, where he spent all of his career..
- He retired in 1914 to his family home in Meaux, and died in Paris on September 21, 1926.
- In 1882, he was appointed to teach courses for training inspectors in the engineering department.
- In developing and teaching his courses, he found novel ways of explaining known results and new techniques as well, the equivalent circuit being one of them.

Thevenin: The Theorem

Assuming any system of linear interconnected[2] conductors, and containing some electromotive forces E1, E2,…En distributed in any way, one considers two points ***A*** and ***A'*** belonging to the system and actually having the potentials ***V*** and ***V'***. If the points ***A*** and ***A'*** are connected by a wire ***ABA'*** having resistance ***r***, not having an electromotive force, the potentials at the points ***A*** and ***A'*** take on different values of ***V*** and ***V'***, but the current ***i*** flowing in the wire is given by the formula ***i=(V-V')/(r+R)***, in which ***R*** represents the *resistance of the primitive system, measured between the points **A** and **A'** considered as electrodes*

Thevenin: The Theorem





Thevenin: The Theorem

- Thevenin's theorem states that any two-terminal linear network having a number of sources and resistances can be replaced by a simple equivalent circuit consisting of a single voltage source in series with a resistance.

- The value of the voltage source is equal to the open circuit voltage across the two terminals of the network.

- The resistance is equal to the equivalent resistance measured between the terminals with all the energy sources replaced by their internal resistances.

Procedure to Obtain Thevenin's Equivalent Circuit

- Temporarily remove the load resistance across which current is required
- Find the open circuit voltage V_{OC} that appears across the two terminals from where the load resistance has been removed. This is known the Thevenin's voltage V_{TH}.
- Calculate the resistance of the whole network as seen from these two terminals, after all voltage sources are replaced by short circuit and all current sources are replaced by open circuit leaving internal resistance (if any). This is called Thevenin's resistance, R_{TH}.
- Replace the entire network by a single Thevenin's voltage source V_{TH} and resistance R_{TH}.
- Connect the resistance (R_L), across which the current value is desired, back to its terminals from where it was previously removed.
- Finally, calculate the current flowing through RL using the following expression:

$$I_L = \frac{V_{TH}}{R_{TH} + R_L} \tag{4}$$

Norton’s Theorem

- Norton’s theorem states that any two-terminal linear network with current sources, voltage sources and resistances can be replaced by an equivalent circuit consisting of a current source in parallel with a resistance

- The value of the current source is equal to the short circuit current between the two terminals of the network and the resistance is equivalent resistance measured between the terminals of the network with all the energy sources replace by their internal resistance.

Procedure to Obtain Norton’s Equivalent Circuit

- Temporarily remove the load resistance across the two terminals and short circuit these terminals.

- Calculate the short circuit current I_N.

- Calculate the resistance of the whole network as seen at these two terminals, after all voltage sources are replaced by short circuit and current sources are replaced by open circuit leaving internal resistances (if any). This is Norton’s resistance R_N.

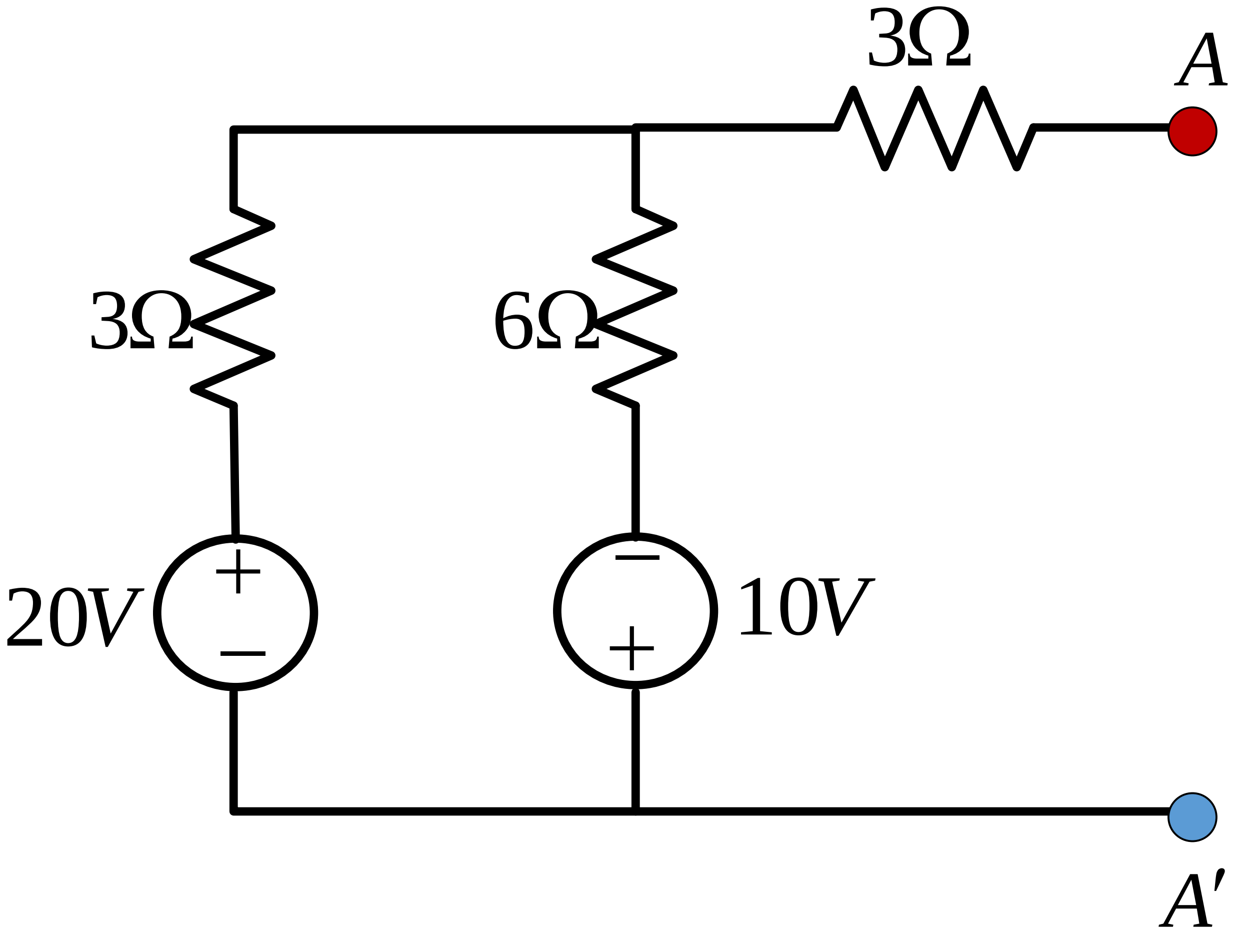
- Replace the entire network by a single Norton’s current source whose short circuit current is I_N and parallel with Norton’s resistance R_N.

- Connect the load resistance (R_L) back to its terminals from where it was previously removed.

- Finally calculate the current flowing through R_L.

Thevenin's and Norton's Equivalent Circuit

- Obtain the Thevenin's and Norton's Equivalent Circuit for the network:



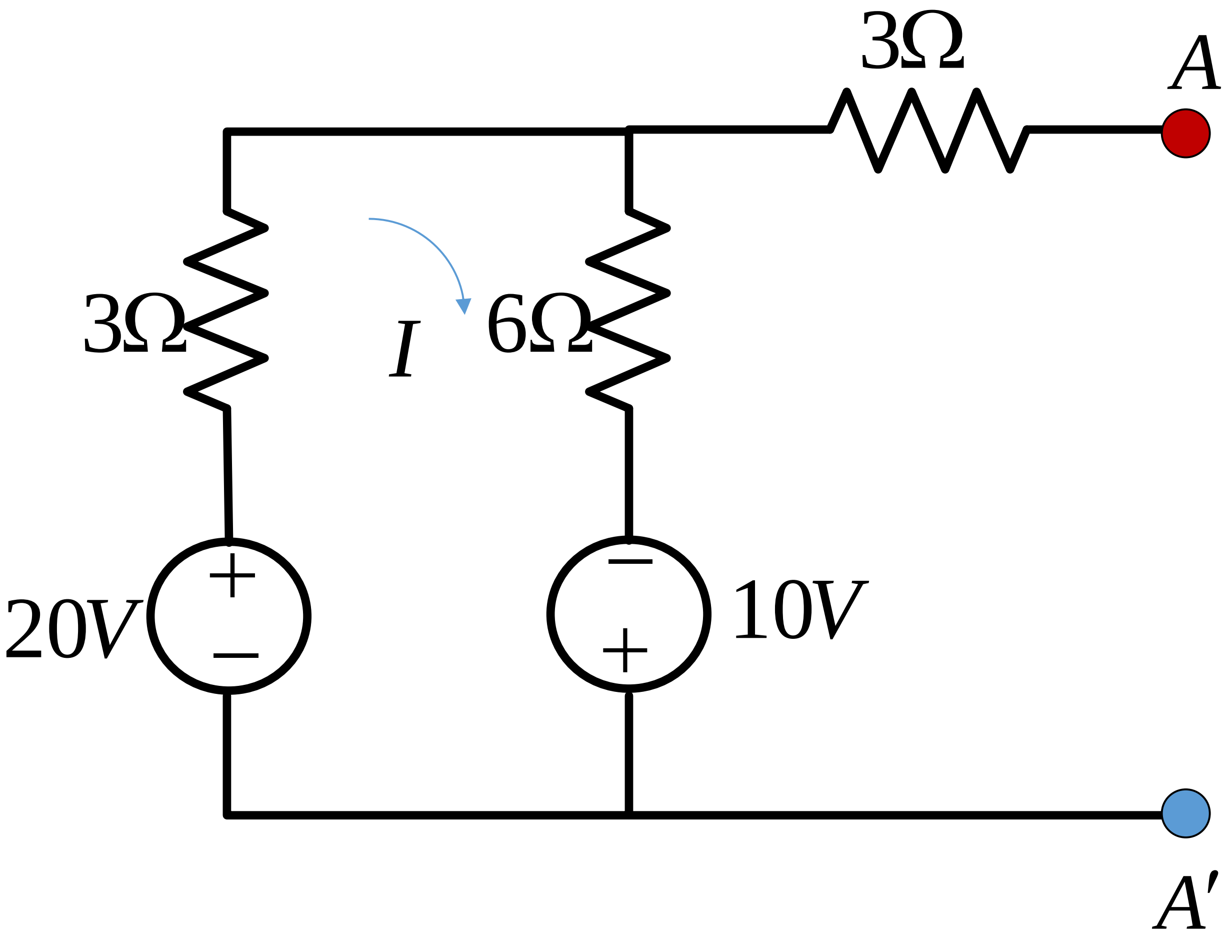
Thevenin's and Norton's Equivalent Circuit

- With the terminals *a* and *b* open, the two voltage sources drive a clockwise current through the 3Ohm and 6Ohm resistors

$$I = \frac{20+10}{3+6} = \frac{30}{9} A$$

- Since no current passes through the upper right 3Ohm resistor, the Thevenin's voltage can be taken from either active branch:

$$V_{TH} = 20 - \left(\frac{30}{9}\right) \times 3 = 10V$$



Thevenin's and Norton's Equivalent Circuit

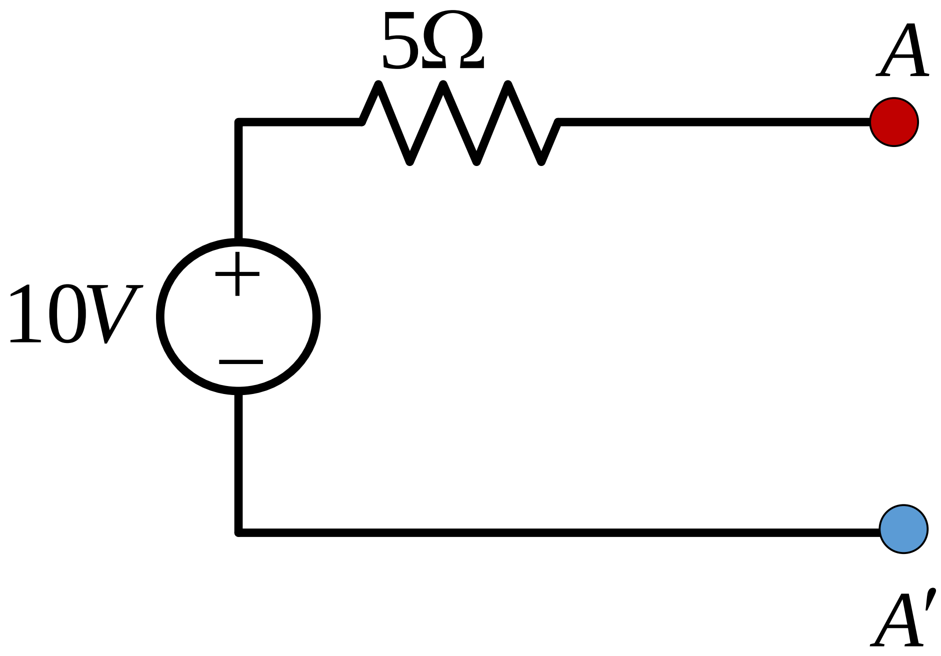
- The equivalent resistance as seen from terminals *a* and *b* is

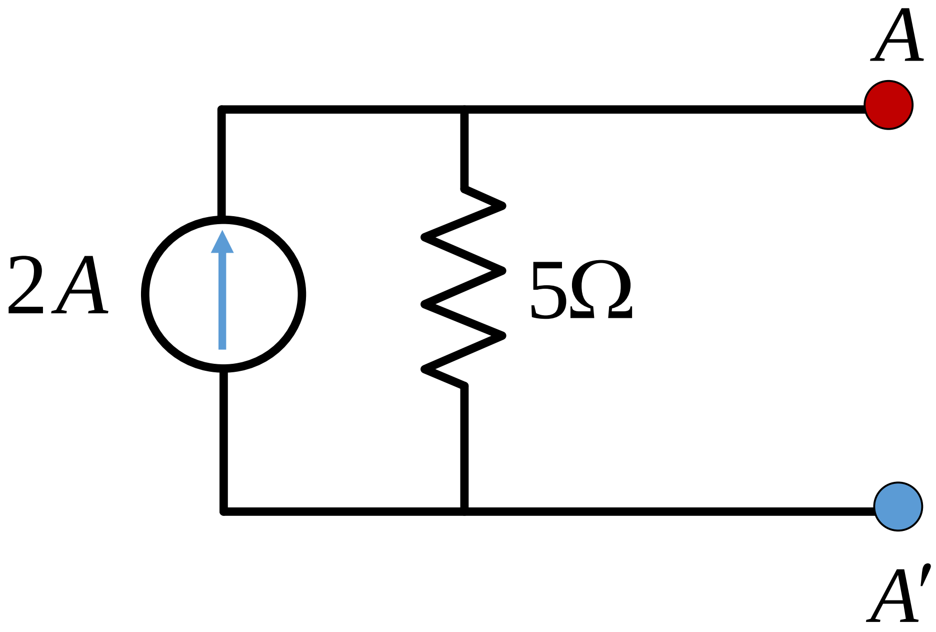
$$R_{eq} = 3 + \frac{3\times 6}{9} = 5\Omega$$

- When a short circuit is applied to the terminals, current I_{sc} results from the two sources. By superposition theorem, this current is given by

$$I_{sc} = \left(\frac{6}{6+3}\right)\left[\frac{20}{3+\frac{3\times 6}{9}}\right] - \left(\frac{3}{3+3}\right)\left[\frac{10}{6+\frac{3\times 3}{6}}\right] = 2A$$

Thevenin's and Norton's Equivalent Circuit



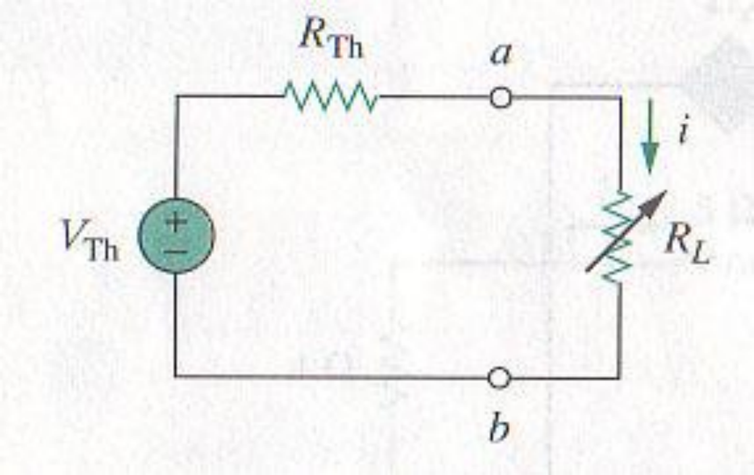


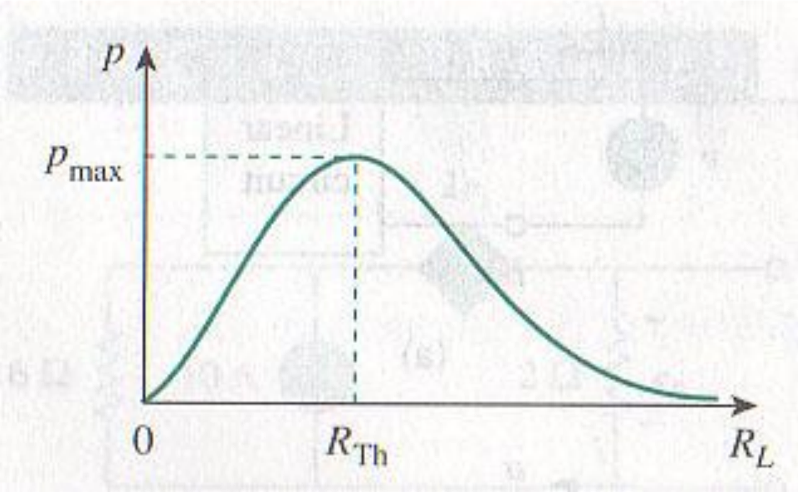
Maximum Power Transfer Theorem

- The maximum power transfer theorem states that:

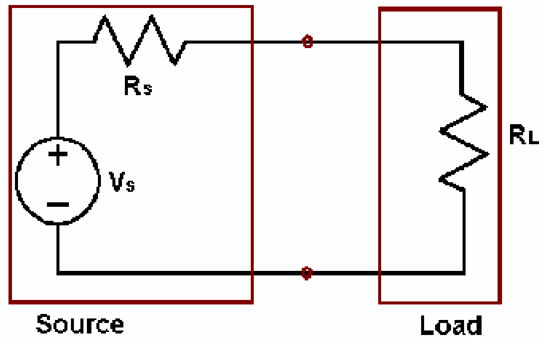
A Load will receive maximum power from a linear network when its total resistance value is exactly equal to the R_{TH} of the network

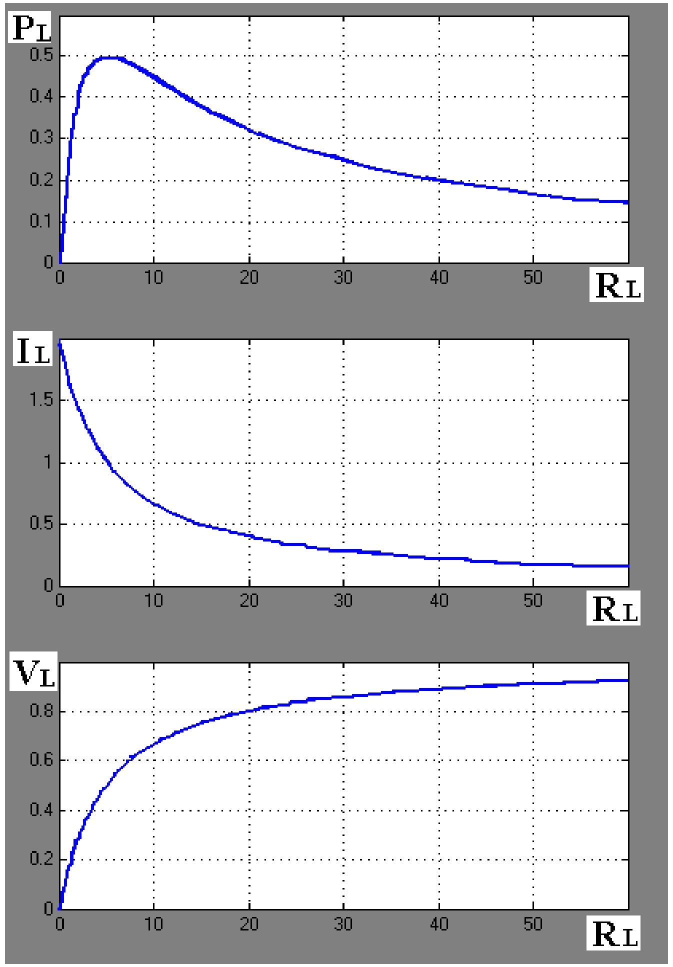
- Maximum power transfer is extremely important for maximum efficiency of a transmission and distribution network of an electric utility such as Western Power.
- The theorem also find application in electronic circuits such as matching input impedance of a speaker system to the output impedance of an amplifier.





Maximum Power Transfer Theorem





Maximum Power Transfer Theorem

$$P_L = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L$$

Taking derivative w.r.t R_L we get,

$$\frac{dP_L}{dR_L} = V_{Th}^2\left\{\frac{(R_{Th} + R_L)^2 - 2R_L(R_{Th} + R_L)}{(R_{Th} + R_L)^4}\right\} = V_{Th}^2\left\{\frac{(R_{Th} + R_L - 2R_L)}{(R_{Th} + R_L)^3}\right\} = 0$$

This imply that $(R_{Th} - R_L) = 0$

$$\therefore\ R_{Th} = R_L$$

$$P_{max} = \frac{V^2_{Th}}{4R_{Th}}$$