Lecture 06

PH101: Physics 1

Module 2: Special Theory of Relativity - Basics

Girish Setlur & Poulose Poulose

gsetlur@iitg.ac.in

Department of Physics, IIT Guwahati

poulose@iitg.ac.in

RECAP

Momentum and Energy

Consider an object of mass m moving with velocity $\vec{u} = u\hat{x}$ $\quad \gamma = \frac{c}{\sqrt{c^2 - u^2}}$

Relativistic Momentum $\vec{P} = \gamma m \vec{u}$

Total energy of an object of mass m moving with speed u, $E = \gamma mc^2$

Kinetic energy of the object

$$K.E. = E - mc^2$$
$$= (\gamma - 1)\, mc^2$$

For small u

$$\left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} = 1 + \frac{u^2}{2c^2} + \cdots$$

$$\Rightarrow \quad K.E. = \left(1 + \frac{u^2}{2c^2} - 1\right) mc^2 = \frac{1}{2} mu^2$$

Introducing 4-vectors

We had noted earlier that the combination of time and spatial coordinates Collectively can be considered as components of a 4-dimentional vector.

$$X \equiv \left(x_0 = ct, x, y, z\right)$$

Under Lorentz tranformation, this vector transforms as $X' = L \cdot X$

$$\begin{pmatrix} x'_0 \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} x_0 \\ x \\ y \\ z \end{pmatrix}$$

In a similar way, we can consider energy and momentum of an object together as components of a 4-vector

$$p \equiv \left(p_0 = \frac{E}{c}, p_x, p_y, p_z \right)$$

where

$$E = \gamma\, mc^2, \qquad p_i = \gamma\, mu_i, \qquad \gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

p then transforms like $p' = L \cdot p$

In S’ frame, moving with $\vec{v} = v\,\hat{x}$

$$p'_x = \gamma\, p_x - \gamma\beta\, \frac{E}{c}$$

$$p'_y = p_y,$$

$$p'_z = p_z,$$

$$E' = \gamma\, E - \gamma\beta\, c\, p_x$$

Let us verify

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad u'_y = \frac{u_y}{\gamma_v\left(1 - \frac{u_x v}{c^2}\right)} \qquad u'_z = \frac{u_z}{\gamma_v\left(1 - \frac{u_x v}{c^2}\right)}$$

$$E = \gamma_u mc^2, \qquad E' = \gamma_{u'} mc^2$$

$$\vec{p} = \gamma_u m\vec{u}, \qquad \vec{p}\,' = \gamma_{u'} m\vec{u}\,' \qquad \gamma_u = \frac{c}{\sqrt{c^2 - u^2}}, \qquad \gamma_{u'} = \frac{c}{\sqrt{c^2 - u'^2}}$$

$$E' = \gamma_v E - \gamma_v \beta c\, p_x \quad = \gamma_v \gamma_u mc^2 - \gamma_v v \gamma_u m u_x \quad = \gamma_v \gamma_u mc^2\left(1 - \frac{vu_x}{c^2}\right) = \gamma_{u'} mc^2$$

Exercise: $\gamma_{u'} = \dfrac{c}{\sqrt{c^2 - u'^2}} = \gamma_u \gamma_v \left(1 - \dfrac{u_x v}{c^2}\right)$

$$p'_x = \gamma_v p_x - \gamma_v \frac{\beta}{c} E = \gamma_v \gamma_u m u_x - \gamma_v \frac{v}{c^2} \gamma_u mc^2 = \gamma_v \gamma_u m(u_x - v) = \gamma_v \gamma_u m \frac{(u_x - v)}{\left(1 - \frac{u_x v}{c^2}\right)}\left(1 - \frac{u_x v}{c^2}\right) = \gamma_{u'} m u'_x$$

$$p'_y = \gamma_{u'} m u'_y = \gamma_u \gamma_v \left(1 - \frac{u_x v}{c^2}\right) m \left(\frac{u_y}{\gamma_v \left(1 - \frac{u_x v}{c^2}\right)} \right) = \gamma_u m u_y$$

$$p'_z = \gamma_{u'} m u'_z = \gamma_u \gamma_v \left(1 - \frac{u_x v}{c^2}\right) m \left(\frac{u_z}{\gamma_v \left(1 - \frac{u_x v}{c^2}\right)} \right) = \gamma_u m u_z$$

Similar to the invariant space-time interval, we have

$$p^2 = \frac{E^2}{c^2} - \vec{p}\cdot\vec{p} = \left(\vec{p}^2 + m^2c^2\right) - \vec{p}^2 = m^2c^2$$

$$p'^2 = \frac{E'^2}{c^2} - \vec{p}'\cdot\vec{p}' = \left(\vec{p}'^2 + m^2c^2\right) - \vec{p}'^2 = m^2c^2$$

Proper Length and Proper Time

- **Proper Length (interval)** between two events is the physical distance between the spatial locations of the two events as seen from a reference frame in which the two events appear to be simultaneous.
- **Proper Time (interval)** between two events is the time elapsed between the two events as seen from a reference frame in which the two events appear to take place at the same spatial location.

Note that in Special Relativity, the following quantity is a Lorentz invariant (unchanged under Lorentz transformations):

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

This means the following:

A) Imagine there is a reference frame in which the two events $(c\,t, x, y, z)$ and $(c\,t + c\,dt, x + dx, y + dy, z + dz)$ occur at the same spatial location. This means in that special reference frame the two events are going to look like $(c\,\tau, x', y', z')$ and $(c\,\tau + c\,d\,\tau, x', y', z')$.

Since ds^2 is Lorentz invariant this means

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2\, \mathrm{d}\tau^2$$

Hence proper time is defined for events that have time-like separation (otherwise there will be no reference frame in which those two events will appear to take place at the same spatial location).

Hence if $ds^2 > 0$ then proper time $\mathrm{d}\tau$ is also a Lorentz invariant defined as,

$$\mathrm{d}\tau = \frac{ds}{c}$$

Proper time is the time between two ticks of your wrist watch (rather than the time between two ticks of a watch that is running away from you close to the speed of light!)

Conversely,

B) Imagine there is a reference frame in which the two events $(c\,t, x, y, z)$ and $(c\,t + c\,dt, x + dx\,, y + dy, z + dz)$ occur at the same time.
This means in that special reference frame the two events are going to look like $(c\,t', x', y', z')$ and $(c\,t', x' + dx', y' + dy', z' + dz')$.

Since ds^2 is Lorentz invariant this means

$$\mathrm{ds}^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = -dx'^2 - dy'^2 - dz'^2$$

Hence (proper) length is defined for events that have space-like separation (otherwise there will be no reference
frame in which those two events will appear to take place at the same time).

Hence if $ds^2 < 0$ then proper length $\mathrm{d}l$ is also a Lorentz invariant defined as,

$$\mathrm{d}l = \sqrt{|ds^2|}$$

Example 1

A free electron is moving with velocity $\vec{u} = \frac{1}{2\sqrt{2}} c(\hat{x} + \hat{y})$ as seen by an observer on earth (frame: S). What is its momentum, total energy and kinetic energy? Mass of electron may be taken to be $m_e = 10^{-30}\ kg$

(a) As seen by the observer in S.
(b) As seen by an observer in S', which is moving with velocity $\vec{v} = 0.2c\ \hat{x}$

Solution: $\vec{u}=\frac{1}{2\sqrt{2}}c(\hat{x}+\hat{y})$ $|\vec{u}|=0.5c$ $\gamma_u=\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}=\frac{1}{\sqrt{1-0.5^2}}=1.1547$

$$\vec{p}=\gamma_u m\vec{u}=1.1547\times10^{-30}\times\frac{1}{2\sqrt{2}}c(\hat{x}+\hat{y})=1.2247\times10^{-22}(\hat{x}+\hat{y})\ kg\,m/s$$

$$E=\gamma_u mc^2=1.1547\times10^{-30}\times9\times10^{16}=1.0392\times10^{-13}\ J,\qquad K.E.=(\gamma_u-1)mc^2=0.1392\times10^{-13}\ J$$

As seen by S': $\gamma_v=\left(1-0.2^2\right)^{-\frac{1}{2}}=1.0206$

$$E'=\gamma_v(E-\beta c p_x)=1.0206\left(1.0392\times10^{-13}-0.2\times3\times10^8\times1.2247\times10^{-22}\right)J=0.98566\times10^{-13}\ J$$

$$p_x'=\gamma_v\left(p_x-\beta\frac{E}{c}\right)=1.0206\left(1.2247\times10^{-22}-0.2\times\frac{1.0392\times10^{-13}}{3\times10^8}\right)=0.5429\times10^{-22}\quad kgm/s$$

$$p_y'=p_y=1.2247\times10^{-22}\quad kgm/s,\qquad p'_z=p_z=0$$

$$K.E.=E'-mc^2=0.98566\times10^{-13}-10^{-30}\times9\times10^{-16}=0.08566\times10^{-13}\ J$$

Alternatively (in frame S');

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{(0.3536 - 0.2)c}{1 - 0.3536 \times 0.2} = 0.1652c, \quad u'_y = \frac{u_y}{\gamma_v\left(1 - \frac{u_x v}{c^2}\right)} = \frac{0.3536c}{1.0206 \times 0.9293} = 0.3728c, \quad u'_z = 0$$

$$|\vec{u}\,'| = \sqrt{0.1652^2 + 0.3728^2}\, c = 0.40775c, \qquad \gamma_{u'} = \frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.40775^2}} = 1.095178$$

$$E' = \gamma_{u'} mc^2 = 1.095178 \times 10^{-30} \times 9 \times 10^{16} = 0.98566 \times 10^{-13}\ J,$$

$$p'_x = \gamma_{u'} mu'_x = 1.095178 \times 10^{-30} \times 0.1652 \times 3 \times 10^8 = 0.5429 \times 10^{-22} \quad kgm/s,$$

$$p'_y = \gamma_{u'} mu'_y = 1.095178 \times 10^{-30} \times 0.3728 \times 3 \times 10^8 = 1.2247 \times 10^{-22} \quad kgm/s,$$

Example 2

An object moving with velocity $\vec{u} = 0.2c\ \hat{x}$ has momentum $|\vec{p}| = 3\times 10^{-20}\ kg\ m/s$

What is the total energy of the object?

What is the mass of the object?

Solution:

$$|\vec{p}| = p_x = \gamma_u m u_x = \gamma_u m\ 0.2c = 3\times10^{-20} \quad kg\, m/s,$$

$$E = \gamma_u mc^2 = \frac{\vec{p}\cdot\vec{u}}{|\vec{u}|^2}c^2 = \frac{3\times10^{-20}}{0.2}c = 0.45\times10^{-10} \quad J,$$

$$\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-0.2^2}} = 1.0206$$

$$E = \gamma_u mc^2 \quad \Rightarrow \quad m = \frac{E}{\gamma_u c^2} = \frac{0.45\times10^{-10}}{1.0206\times9\times10^{16}} = 0.4899\times10^{-27}\ kg$$

Example 3

In an experiment, an electron and a positron are collided to produce a massive object, Z, of mass $M_Z = 2 \times 10^{-25}\ kg$

The mass of the electron and positron are the same, and can be taken to be

$$m_{e-} = m_{e+} = 10^{-30}\ kg$$

The momenta or the electron and positron are the same in magnitude, and opposite in direction, so that Z is produced at rest (zero momentum).

$$\vec{p}_{e+} = -\vec{p}_{e-}$$

What is the (magnitude of the) momentum of the electron ?

Take the value of speed of light to be $c = 3 \times 10^{8}\ m/s$

Solution:

Let $\vec{p}_{e+} = -\vec{p}_{e-} = p_x\hat{x}$ $\quad m_{e^+} = m_{e^-} \Rightarrow \vec{u}_{e^+} = -\vec{u}_{e^-}$ $\quad E = \frac{\vec{p}\cdot\vec{u}}{|\vec{u}|^2}c^2 \Rightarrow E_{e^+} = E_{e^-}$

In the rest frame of Z: $\vec{p}_Z = 0, \quad E_Z = M_Z c^2$

Energy conservation $\Rightarrow E_{e^+} + E_{e^-} = E_Z \Rightarrow E_{e^+} = E_{e^-} = \frac{M_Z c^2}{2} = 9\times10^{-9}\ J$

$$|\vec{p}_{e-}| = |\vec{p}_{e+}| = \frac{\sqrt{E_{e^+}^2 - m_{e^+}^2 c^4}}{c} = \left(\sqrt{\frac{M_Z^2}{4} - m_{e^+}^2}\right)c = \left(\sqrt{10^{-50} - 10^{-60}}\right)c \approx \frac{E_{e^+}}{c}$$

The speed of electron: $E = \gamma mc^2 \Rightarrow \gamma = \frac{E}{mc^2} = \frac{9\times10^{-9}}{10^{-30}\times9\times10^{16}} = 10^5$

$$\Rightarrow u = \left(\sqrt{1-\gamma^{-2}}\right)c = \left(\sqrt{1-10^{-10}}\right)c = 0.99999999995\,c$$

At these speeds, the energy due to the mass of electron is negligible compared to its kinetic energy. For all practical purposes, we can neglect the mass of the electron at such speeds.

We end the discussion on the Special Theory of Relativity here.