1. Natural numbers or Counting numbers : $\mathbb{N} = \{1, 2, 3, \ldots\}$.
   (a) Addition is closed, associative and commutative.
   (b) Multiplication is closed, associative and commutative. 1 is the unique multiplicative identity.
   (c) Multiplication distributes over addition.
   (d) There is no additive identity. We cannot talk of additive inverses.
   (e) There are no multiplicative inverses except for 1.
   (f) Principle of mathematical induction is valid, *viz.*, Assume $P(n)$ is a well-defined statement for each natural number n which is either true or false. If (i) $P(1)$ is true and (ii) For each natural k, $P(k)$ true implies $P(k+1)$ true, then: $P(n)$ is true for each natural n.
   (g) There is an order relation $1 < 2 < 3 < \cdots$
   (h) {Well–ordering principle} Every non-empty subset of naturals has a least element.
   (i) ...... many other derived/inferred properties.
   (j) Subtraction?

2. Integers : $\mathbb{Z} = \{1, 2, 3, \ldots, 0, -1, -2, -3, \ldots\}$.
   (a) Addition is closed, associative and commutative. 0 is the unique additive identity.
   (b) Multiplication is closed, associative and commutative. 1 is the unique multiplicative identity.
   (c) Multiplication distributes over addition.
   (d) Every integer has a unique additive inverse.
   (e) There are no multiplicative inverses except for ± 1.
   (f) There is an order relation $\cdots -3 < -2 < -1 < 0 < 1 < 2 < 3 < \cdots$
   (g) Subtraction is a closed operation.
   (h) ...... many other derived/inferred properties
   (i) Principle of mathematical induction?
   (j) Does every non-empty subset of integers have a least element?
   (k) Division?

3. Rationals : $\mathbb{Q} = \{1, 2, 3, \ldots, 0, -1, -2, -3, \ldots \frac{1}{2}, \frac{3}{2}, \ldots, -\frac{1}{2}, -\frac{3}{2}, \ldots, \frac{1}{3}, \frac{2}{3}, \ldots, -\frac{1}{3}, -\frac{2}{3}, \ldots, \}$.
   (a) Addition is closed, associative and commutative. 0 is the unique additive identity.
   (b) Multiplication is closed, associative and commutative. 1 is the unique multiplicative identity.
   (c) Multiplication distributes over addition.
   (d) Every rational has a unique additive inverse.
   (e) Every non–zero integer has a unique multiplicative inverse.
   (f) There is an order relation ...
   (g) Subtraction is a closed operation.
   (h) Division of a rational by any non-zero rational is possible
   (i) ...... many other derived/inferred properties

4. Despite earlier education on these matters, who can prove:
   (a) -1 times -1 equals $+1$
   (b) $\frac{2}{3} \div \frac{5}{7} = \frac{2 \cdot 7}{3 \cdot 5}$

5. Fundamental drawback: Negative numbers and Rationals were introduced through notation!

6. How to rectify? Study Robert Anderson's Set theory and construction of numbers or equivalents or wikipedia

7. A very brief hint:
   (a) Put equivalence relation on $\mathbb{N} \times \mathbb{N}$ where $(a, b) \sim (c, d)$ if $a + d = b + c$ to get equivalence classes as integers. So, the integer -2 is the equivalence class $[(5, 3)]$
   (b) Put equivalence relation on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ where $(p, q) \sim (r, s)$ if $ps = qr$ to get equivalence classes as rationals. So, the rational $\frac{-2}{3}$ is the equivalence class $[(-2, 3)]$

1. Fundamental Question:

   What is the set of numbers which is sufficient to measure physical quantity like 'Length'? [and likewise Mass and Time]

2. More precisely: What magical set M do we need so that there is a one-to-one correspondence between elements of M and points on a idealized physical straight line L?

   Attempted answers: Naturals, Integers, Rationals ... all necessary but insufficient!

3. Recall that square root of 2 is not a rational number. Inspired by this we have an incomplete answer: Add $\sqrt{2}, \sqrt{3}, \ldots - \sqrt{2}, -\sqrt{3}, \ldots \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}}, \ldots, \sqrt[3]{2}, \sqrt[3]{3}, \ldots$???

4. Some tools: Pick a stick and call it of 'standard' length, say 1 foot or 1 metre or 1 unit length or merely 1. By trial and error, take two sticks of equal length [equal as far as you can see with eye, magniying glass, microscope etc.] and line them up and match with the standard. Then each of these has length $\frac{1}{2}$. Likewise other fractional lengths. Including lengths of $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$

5. Now by experience we can say a measurement of the length ℓ of a stick is between 1 and 2 metres, is between 1.4 and 1.5 metres, is between 1.41 and 1.42 metres, etc. ...

6. Mathematically a *measurement* is an interval $I_1 = [s_1, b_1]$ where s_1 and b_1 are rational numbers and we implicitly assume that $s_1 < b_1$ and we want to indicate that the length ℓ is between the smaller number s_1 and the bigger number b_1.

7. Second measurement is an interval $I_2 = [s_2, b_2]$ where s_2 and b_2 are rational numbers. It is an improvement over the first if and only if $I_1 \supset I_2$.

8. A *lab measurement* for length is thus a finite sequence of intervals with rational endpoints such that $I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n$, for some natural number n.

9. What is a perfect measurement for length? Is it an infinte sequence of intervals $[s_1, b_1] \supset [s_2, b_2] \supset [s_3, b_3] \supset \cdots \supset \cdots$, such that:

   (a) $s_1 \le s_2 \le s_3 \le \cdots$ and $b_1 \ge b_2 \ge b_3 \ge \cdots$? –or –

   (b) $s_1 < s_2 < s_3 < \cdots$ and $b_1 > b_2 > b_3 > \cdots$? –or –

   (c) ?

10. A *perfect measurement* for length is an infinte sequence of intervals $[s_1, b_1] \supset [s_2, b_2] \supset [s_3, b_3] \supset \cdots \supset \cdots$, such that:

    (a) For each natural number k, there is a natural number n such that width of $I_n = b_n - s_n < \frac{1}{10^k}$.

    (b) Equivalently, for each positive rational number ϵ, [no matter how small], there there is a natural number n such that width of $I_n = b_n - s_n < \epsilon$.

    (c) Equivalently, "limit" of widths of interval, $\lim_{n\to\infty} \text{width } (I_n) = 0$.

11. Can two perfect measurements represent the same length? If so, under what conditions? Two perfect measurements $I_1 \supset I_2 \supset I_3 \supset \cdots \supset \cdots$ and $J_1 \supset J_2 \supset J_3 \supset \cdots \supset \cdots$ are equivalent if for each natural number n, there is a natural k such that $J_k \subset I_n$ and vice-versa. Equivalently, if $I_n \cap J_n$ is non-empty for each natural n.

12. Real numbers are exactly equivalence classes of perfect measurements. The set of real numbers is denoted by $\mathbb{R}$.

13. Some hints: How to add, subtract, multiply, divide? How to put order relation? How to prove some properties of these operations and of the order relation?

14. Short-cut: Axioms

1. Axioms for the *complete ordered field* $\mathbb{R}$.

   (a) There is a function $f_+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, called the addition function. Assume four axioms about addition, viz., commutativity, associativity, existence of an identity and existence of an additive inverse for each real.
   Inferences:

      i. Prove that there is a unique additive identity. Denote it by 0 and call it *zero*.
      ii. Prove that every real has a unique additive inverse. Denote the additive inverse of a real a by $-a$.

   (b) There is a function $f_\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, called the multiplication function. Assume four axioms about multiplication, viz., commutativity, associativity, existence of an identity and existence of a multiplicative inverse for each real not equal to zero.
   Inferences:

      i. Prove that there is a unique multiplicative identity. Denote it by 1 and call it *one*.
      ii. Prove that every real not equal to zero has a unique multiplicative inverse. Denote the multiplicative inverse of a non–zero real a by $1/a$.

   (c) Multiplication distributes over addition.

   (d) There exists a non–empty subset, $\mathbb{P} \subset \mathbb{R}$, called the set of positive real numbers which is closed under addition and multiplication. Further given any real x, exactly one and no more of the following is true:

      (N) $-x \in \mathbb{P}$
      (Z) $x = 0$
      (P) $x \in \mathbb{P}$.

   Further definitions and inferences:

      i. Given any reals a and b, define $a < b$ if and only if $b + (-a) \in \mathbb{P}$. Say $a \leq b$ if either $a < b$ or $a = b$.
      ii. Given a subset $S \subset \mathbb{R}$, define $u \in \mathbb{R}$ to be an *upper bound* of S if $s \leq u$ for every $s \in S$. We say a set is *bounded above* if it has an upper bound.
      iii. Given a subset $S \subset \mathbb{R}$, define $l \in \mathbb{R}$ to be a *least upper bound* of S if l is an upper bound of S and $l \leq u$ for any upper bound u of S.

   (e) Every non–empty subset of $\mathbb{R}$ which is bounded above has a least upper bound. This is the completeness axiom.

   Following is a list of some of the derived properties of $\mathbb{R}$.

2. $a \cdot 0 = 0$ for every $a \in \mathbb{R}$.

3. $a + x = a + y$ implies $x = y$. Corollary: $a + x = a$ implies $x = 0$.

4. $a \cdot x = a \cdot y,\ a \neq 0$ implies $x = y$. Corollary: $a \cdot x = a,\ a \neq 0$ implies $x = 1$.

5. $a \cdot b = 0$ implies $a = 0$ or $b = 0$.

6. Define subtraction of any two reals $a - b := a + (-b)$. It is neither commutative nor associative. However, 0 works as the identity and every element is its own inverse. Addition and subtraction are opposites of each other, viz., $(a + b) - b = a$ and $(a - b) + b = a$ for any reals a and b.

7. Define division of any two reals $a/b := a \cdot \frac{1}{b}$, for $b \neq 0$. It is neither commutative nor associative. 1 works as the identity and every non–zero element is its own inverse. Multiplication and division are opposites of each other, viz., $(a \cdot b)/b = a$ and $(a/b) \cdot b = a$ for any reals a and $b \neq 0$.

8. Multiplication distributes over subtraction and division distributes over both addition and subtraction.

9. The relation $\leq$ is transitive, compatible with addition and compatible with multiplication.

10. For any non–zero $a \in \mathbb{R}$, $0 < a^2$. Corollary: $0 < n$ for every natural n.

11. Archimedean Property viz., for any real number x, there exists a natural N such that $x < N$.

12. Density of Rationals viz., given any two real numbers $x < y$, there exists a rational $x < r < y$.

1. Give a precise definition of the maximum (and the minimum) of a finite collection of reals.
2. Is it legitimate to use the concept of max/min for infinite sets?
   (a) For example, consider the set of rationals $S := \{\frac{n}{n+1} | n \in \mathbb{N}\}$. What is the maximum of S?
   (b) If there is no element $u \in S$ such that $s \leq u$ for all $s \in S$, can you find such a $u \in \mathbb{R}$? Can you find all such u?
   (c) Among all such u that you found, what is special about 1? What is wrong with saying 1 is the minimum of all such u? Can you give a better definition which will pick out such a maximum (more precisely, extended concept of maximum) in all cases?
3. Complete a similar exercise of finding the 'minimum' for the set $T := \{1 + \frac{1}{n} | n \in \mathbb{N}\}$.
4. Definitions of *upper bound*, *lower bound* and *bound* for a subset of reals. Definition of a *bounded* set. Examples.
5. Definition of *supremum* [sup] or *least upper bound* [lub] and *infimum* [inf] or *greatest lower bound* [glb] for a subset of reals. Examples.
6. Existence of sup as guaranteed by the completeness axiom. Can you prove existence of inf using the completeness axiom?
   (a) Write statements of completeness property for naturals, integers and rationals.
   (b) Show that completeness is valid for naturals and integers, but not for rationals.
7. sup and inf of a set, if they exist, are uniquely determined.
8. (a) {Anything less than the supremum is not an upper bound}
   Proposition: If h is the supremum of $S \subset \mathbb{R}$, then for any real $\epsilon > 0$, $h - \epsilon$ is not an upper bound for S.
   (b) {Anything more than the infimum is not a lower bound}
   Proposition: If m is the infimum of $S \subset \mathbb{R}$, then for any real $\epsilon > 0$, $m + \epsilon$ is not a lower bound for S.
9. (a) {If anything less than an upper bound is not an upper bound, then it is the least upper bound}
   Proposition: Let $S \subset \mathbb{R}$ be non-empty and let u be an upper bound for S. If for every real $\epsilon > 0$, there exists a $t \in S$ such that $u - \epsilon < t \leq u$, then $u = \sup S$.
   (b) {If anything more than a lower bound is not a lower bound, then it is the greatest lower bound}
   Proposition: Let $S \subset \mathbb{R}$ be non-empty and let ℓ be a lower bound for S. If for every real $\epsilon > 0$, there exists a $t \in S$ such that $\ell \leq t < \ell + \epsilon$, then $\ell = \inf S$.
10. Proof of Archimedean Property using completeness.

    If there is no natural number bigger than a given real number r_0, the set of naturals is bounded above by r_0. Using completeness, let s be the supremum of naturals. Then, $s - 1$ is not an upper bound for the naturals and hence there is a natural n such that $s - 1 < n$. This implies $s < n + 1$ with $n + 1$ a natural greater than s, the supremeum of naturals: a contradiction.
11. Proof of density of rationals: Read up
12. Discussion topic:

    Archimedean Property can be interpreted simply as: 'there is no greatest real number'.

    Question: Is there a smallest real number?

    More interesting: Is there a smallest positive real number?

    Isn't ∞ the greatest real number?

1. Finite sequence example to illustrate growth of terms where finite analysis is adequate: The size of economy of USA is \$15 trillion, while that of India is \$1.5 trillion. US economy grows at a rate of 3%pa while India's economy grows at a rate of 8%. Will the Indian economy ever overtake US' economy? If so when?

2. Pocket money: Would you rather have 1 crore rupees for a month or a paisa doubled every-day for a month?

n days in a month	Rupees: $\frac{1}{100}\sum_{k=0}^{n} 2^k$
28	26,84,354.55
29	53,68,709.11
30	1,07,37,418.23
31	2,14,74,836.47

Hint:

3. A corrupt telecom minister has set up a kickback plan where he gets 1 Rupee for the first phone call, 1/2 of a Rupee for the second call, 1/3 of a rupee for the third and so on. Will he ever become a crorepati?

n	$\sum_{k=1}^{n} \frac{1}{k}$
1	1
10	2.928968254
100	5.1873775176
1000	7.4854708606
10000	9.787606036
100000	12.0901461299
1000000	14.3927267229
10000000	16.6953113659
100000000	18.9978964139
1000000000	21.3004815023
10000000000	22.064778

Hint:

4. Zeno's Paradox: World famous runner Achilles who runs at a speed of 1Km/min is pitted against a tortoise which crawls at a speed of 100m/min. They start a race with the tortoise given a head start of 1Km. By the time (1min) Achilles runs 1Km, the tortoise has moved 100m. In the next 1/10 of a minute, Achilles covers the latter 100m, but the tortoise has moved 10m. In the next 1/100 of a minute, Achilles covers this 10m – but the tortoise covers another 1m. And so on for ever. If Achilles is always catching up with the marks left by tortoise at min., 1+1/10 of a minute, 1+1/10+1/100 of a minute and so on, how can Achilles ever overtake the tortoise?

5. Grandi's series: $1-1+1-1+1-1+\cdots$ What is the sum?

Zero? $(1-1)+(1-1)+(1-1)+\cdots = 0+0+0+\cdots = 0$

One? $1-(1-1)-(1-1)-(1-1)-\cdots = 1-0-0-0-\cdots = 1$,

Half? $S = 1-1+1-1+1-\cdots$ and $S = 1-(1-1+1-1+1-1+1-\cdots)$ implies $S = 1-S$ and $S = 1/2$

next Try $S_2 = 1-2+3-4+5-6+\cdots$. Add it to itself (with a shift) to get $S_2+S_2 = S_1$ so that S_2 equals 1/4.

more Take $S_3 = 1+2+3+4+\cdots$ and $S_3 - S_2 = 0+4+0+8+0+12+\cdots = 4S_3$ and hence $S_3 = -1/12$. Really?

6. Before you analyse the latter three questions, first prove that the multi–variable sum function $\Sigma : \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R} \to \mathbb{R}$ can be defined for finitely many variables. Using induction establish that this sum is both independent of ordering of the variables and the bracketing which dictates the partial sums.

7. The three questions raised, viz., Zeno's paradox and about the corrupt minister and the Grandi's series demand a definition for a sum of infinitely many reals. What would be your definition for the infinite sum $a_1+a_2+a_3+\cdots$ where each a_n is real?

8. On can define a sequence of partial sums viz., $\sigma_1 = a_1$, $\sigma_2 = a_1+a_2$, $\sigma_3 = a_1+a_2+a_3$, ... and see whether the sequence of numbers $\sigma_1, \sigma_2, \sigma_3, \ldots$ is getting *closer and closer* or *approaching* a fixed real number. This will be formalized via Cauchy's definition in the next lecture.

9. Discussion topics:

(a) In how many ways can you add 2 numbers? 3 numbers? 4? 5? ...

Answers: 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452,..

1. Definition and notation for a sequence.

2. Given a sequence (a_n), there is no concept of the *eventual value* of (a_n) – simply because we have not defined a_∞. However, we have some idea of whether the given sequence (a_n) is or is not getting closer and closer to a fixed real number a. Denote this by $a_n \to a$. Let us attempt to give a criterion which should satisfy the conditions:

   (a) in every example where we believe $a_n \to a$, the criterion should be true,

   (b) in every example where we believe $a_n \not\to a$ the criterion should be false and

   (c) there should be no need to modify the criterion in the face of new examples.

3. (a) Take a small real number, say, $\epsilon_1 = 1$.
   Criterion 1: We say $a_n \to a$ if there exists a natural number N_1 such that $|a_n - a| < \epsilon_1 = 1$ for all $n \geq N_1$.
   Note that this criterion demands that all but a finite number of terms of the sequence be within a distance of 1 from a.
   This criterion works in proving: (i) every constant sequence $a_n = a$ satisfies $a_n \to a$, (ii) $(\frac{1}{n}) \to 0$ and even (iii) $(-1)^n \not\to -1, 0, 1$. However, consider the sequence $b_n = b + \frac{1}{2} \cdot (-1)^n$ for all n. We do not believe (b_n) is getting closer and closer to b, but Criterion 1 makes $b_n \to b$.

   (b) Perhaps $\epsilon_1 = 1$ is not small enough. Take $\epsilon_2 = \frac{1}{2}$.
   Criterion 2: We say $a_n \to a$ if there exists a natural number N_2 such that $|a_n - a| < \epsilon_2 = \frac{1}{2}$ for all $n \geq N_2$.
   This criterion works in cases (i)–(iii) listed above. This criterion works in the case of (b_n) given above to show $b_n \not\to b$. So Criterion 2 is better than Criterion 1. However, consider the sequence $c_n = c + \frac{1}{4} \cdot (-1)^n$ for all n. We do not believe (c_n) is getting closer and closer to c, but Criterion 2 makes $c_n \to c$.

   (c) Perhaps $\epsilon_2 = \frac{1}{2}$ is not small enough. Take $\epsilon_3 = \frac{1}{3}$.
   Criterion 3: We say $a_n \to a$ if there exists a natural number N_3 such that $|a_n - a| < \epsilon_3 = \frac{1}{3}$ for all $n \geq N_3$.
   This criterion works in all the cases Criteria 1 & 2 work given above. It also works to show $c_n \not\to c$. However, consider the sequence $d_n = d + \frac{1}{6} \cdot (-1)^n$ for all n. We do not believe (d_n) is getting closer and closer to d, but Criterion 3 is true here.

   (d) Perhaps $\epsilon_3 = \frac{1}{3}$ is not small enough. Take $\epsilon_0 > 0$ to be some fixed *small* real number.
   Criterion 0: We say $a_n \to a$ if there exists a natural number N_0 such that $|a_n - a| < \epsilon_0$ for all $n \geq N_0$.
   This criterion works in all cases where Criteria 1–3 work, if $\epsilon_0 < \frac{1}{3}$. Also, one can show $(d_n) \not\to d$ if $\epsilon_0 \leq \frac{1}{6}$. However, consider the sequence $a_n = a + \frac{\epsilon_0}{2} \cdot (-1)^n$ for all n. We do not believe (a_n) is getting closer and closer to a, but Criterion 0 is true here.

4. Observation: Each of the criteria 0–3 has to be necessarily true in the examples we have of sequences *approaching* a real number. Whereas, on the contrary, given any fixed criterion among them, there is an example for a sequence for which the criterion believes that the sequence approaches a real number – while we do not believe this to be so. Moreover, varying the value of ϵ_0, Criterion 0 is actually a collection of infinitely many criteria.

5. Thus we are faced with a situation where infinitely many criteria are necessary for our notion of a sequence getting closer and closer to a real number, whereas, no single one of them is sufficient. Cauchy gathered all the conditions together to capture our notion in the definition below.

6. Cauchy's definition: We say $a_n \to a$ if (and only if) the following is true:
   For any given real $\epsilon > 0$, there exists a natural N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

7. Constant sequence, Tail of a sequence

8. We claim the following limits

   (a) $(\frac{1}{n}) \to 0$

   (b) $(\frac{1}{n^2}) \to 0$

   (c) $(\frac{1}{n^p}) \to 0$ for each fixed natural p

   (d) $(\frac{1}{1+n\alpha}) \to 0$ for each fixed real $\alpha > 0$

   (e) $(b^n) \to 0$ for every fixed real $-1 < b < 1$

   (f) $(c^{1/n}) \to 1$ for every fixed real $c > 0$

   (g) $(n^{1/n}) \to 1$

   Brief hints: Given any real $\epsilon > 0$, we have $\frac{1}{\epsilon}$ as a real number and there exists a natural number N such that $\frac{1}{\epsilon} < N$, which implies that for all $n > N$, we have $\frac{1}{n} < \frac{1}{N} < \epsilon$.

$\frac{1}{1+n\alpha} < \frac{1}{n\alpha}$

Since $0 < b < 1$, we can write $b = 1/(1+a)$, where $a := (1/b) - 1$ so that $a > 0$. By Bernoulli's Inequality, we have $(1+a)^n > 1 + na$. Hence $0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}$

9. Prove $(-1)^n$ does not converge, *i.e.*, diverges.

10. Prove $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} =?$

11. Template for applying Cauchy's definition to prove $a_n \to a$:

Rough Work		Credit worthy work	
START:	Given a real $\epsilon > 0$	START:	Given a real $\epsilon > 0$
DO SOMETHING:	$\boxed{? \qquad\qquad ?}$	SAY:	My N is equal to ...
FIND:	A natural $N =?$	ASSUMING:	$n \geq N$
		DO SOMETHING*	$\boxed{? \qquad\qquad ?}$
		GET FINALLY:	$\|a_n - a\| < \epsilon$

1. Uniqueness of limits. If $a_n \to a, b$, then $a = b$.

   Proof: Use the lemma: if a non-negative number is smaller than every positive number, it has to be zero.

   Can you make $|a - b|$ smaller than every positive number?

2. If two sequences converge to the same real number, are the two sequences 'equal'? If two sequences converge to the same real number, do they have to be on 'different' sides of the limit?

3. A sequence $X = (x_n)$ is *bounded* if the set $\{x_n | n \in \mathbb{N}\}$ is bounded or equivalently there exists a real B such that for every natural n, $|x_n| \leq B$. Picture.

4. Why can't one take $B = \max(x_1, x_2, x_3, \ldots)$?

5. Proposition: Convergent implies bounded.

   Proof:

   Method 1 Except for finitely many terms, all others cluster around the limit.

   Method 2 Can you explain when $a_n \not\to a$?

6. Building new sequences from one given sequence (a_n):

   (a) Constant multiple sequence $(c \cdot a_n)$ for some real c
   (b) Square sequence (a_n^2)
   (c) Cube sequence (a_n^3)
   (d) p–th power sequence (a_n^p) for natural p. The latter can be extended to include $p = 0$
   (e) To extend this to all integral p, need to assume none of the $a_n = 0$
   (f) Similarly get fractional powers under additional assumptions, if necessary
   (g) Further, let $f : \mathbb{R} \to \mathbb{R}$ be any polynomial function, viz., $f(x) = c_k x^k + c_{k-1} x^{k-1} + \cdots + c_1 x + c_0$ for real numbers $c_0, c_1, \cdots, c_k$ and natural k. Then $(f(a_n))$ is a new sequence
   (h) Fundamental question: If $a_n \to a$, does $f(a_n) \to f(a)$?

7. Building new sequences from two given sequences (a_n) and (b_n):

   (a) their sum $(a_n + b_n)$
   (b) their difference $(a_n - b_n)$
   (c) product $(a_n \cdot b_n)$ and
   (d) quotient (a_n/b_n) [assuming none of the b_n is zero]
   (e) Proposition: If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.
   (f) Proposition: If $a_n \to a$ and $b_n \to b$, then $a_n - b_n \to a - b$.
   (g) If $a_n \to a$ and $b_n \to b$, then $a_n \cdot b_n \to a \cdot b$.
   (h) If $a_n \to a$ and $b_n \to b$, and none of the $b_n = 0$ and $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$.
   (i) How about proofs of these propositions?

8. Building new sequences from more than two given sequences.

9. Discussion topic:

   (a) Examples for bounded sequences which are not convergent.
   (b) Sequences which seem to have two "limit–like" points? [NOT a formal phrase: don't use it!]
   (c) three limit–like points?
   (d) four limit–like points?,...
   (e) infinitely many?
   (f) all rationals as limit–like points?
   (g) all irrationals?
   (h) all reals???
   (i) any given subset of reals?
   (j) when is a real number c, a "limit-like" point of a given sequence a_n: definition?

1. Proposition: Suppose $a_n \to a$ and for every natural n, $a_n \geq 0$. Then $a \geq 0$.

   Picture and Proof: If $a < 0$, take $\epsilon = -a > 0$ and get a natural N from the definition such that for all $n \geq N$, we have $a - \epsilon < a_n < a + \epsilon$. In particular for $n = N$, we get $a - (-a) < a_N < a + (-a) = 0$ contradicting the hypothesis that $a_N \geq 0$.
   Question: If $a_n \to a$ and for every natural n, $a_n > 0$, then is $a > 0$? If $a_n \geq c$ for every n, then is $a \geq c$? Similar questions with $\leq$, etc..

2. Proposition: Suppose $a_n \to a$ and $b_n \to b$ with $a_n \leq b_n$ for every n. Then, $a \leq b$.

   Picture and Proof: Apply previous proposition to the difference of given sequences.

3. Proposition: Suppose $a_n \to a$ and $\alpha \leq a_n \leq \beta$. Then $\alpha \leq a \leq \beta$.

   Picture and Proof: Apply previous proposition to the constant sequence $b_n = \beta$, etc..

4. Squeeze/Sandwich/Pinching Theorem: For three sequences, $a_n \leq b_n \leq c_n$ with $a_n \to l$ and $c_n \to l$. Then the sequence b_n converges and the limit is l.

   Picture and Proof: Given a real $\epsilon > 0$, find a natural N such that for every $n \geq N$, both $|a_n - a|, |b_n - b| < \epsilon$. Then, $-\epsilon < a_n - a \leq b_n - a \leq c_n - a < \epsilon$ for all $n \geq N$. This proves the required.

5. Nested interval property: Let for each natural n, I_n be an interval of real numbers, *viz.*, $I_n = [a_n, b_n]$ for some real numbers $a_n \leq b_n$. Of course, each such interval is non-empty and bounded. If $I_1 \supset I_2 \supset I_3 \supset \cdots$ and width of $I_n = b_n - a_n \to 0$, then $\cap_1^\infty I_n$ is a set with exactly one real number.

   Is this property true for rational numbers?

6. *Increasing, decreasing* and *monotone* sequences. Additional qualifier: 'strictly'

7. Every increasing sequence is bounded below. There are examples of increasing sequences which are not bounded above. Analogous statements for decreasing sequences.

8. Whereas boundedness for a general sequence does not imply convergence, it does for the restricted class of monotone sequences.

9. Monotone Convergence Theorem: An increasing sequence which is bounded above converges to the supremum of the set formed by the sequence.

   Picture and Proof: Let s be the supremum. Given any real $\epsilon > 0$, recall $s - \epsilon$ is not an upper-bound of the sequence and hence there is a natural N such that $s - \epsilon < a_N$. What can you say about a_n for $n \geq N$? Where are they??

   Analogous statement for decreasing sequences and a proof.

10. Given any strictly increasing sequence of naturals $n_1 < n_2 < n_3 < \cdots$, and a sequence of real numbers (a_n), the sequence (a_{n_k}) is called a subsequence of the given sequence (a_n).

    Examples

11. Proposition: If a sequence $a_n \to a$, then every subsequence $a_{n_k} \to a$.

    Contrapositive gives a divergence criterion.

12. Bolzano–Weierstrass Theorem: A bounded sequence of real numbers has a convergent subsequence.

    Picture and Proof: Let $I_1 = [\inf S, \sup S]$, where $S = \{a_n | n \in \mathbb{N}\}$, is the set of terms of the sequence.

    Set $L_2 = [\inf S, \frac{1}{2}(\inf S + \sup S)]$ and $R_2 = [\frac{1}{2}(\inf S + \sup S), \sup S]$. Let $A_2 = \{n | a_n \in L_2\}$ and $B_2 = \{n | a_n \in R_2\}$. At least one of A_2 or B_2 is infinite and if A_2 is infinite, set $I_2 = L_2$ and if not, set $I_2 = R_2$.

    Continue ... and apply nested interval property

1. If we want to apply Cauchy's definition of convergence, we need to know the limiting point in advance. However, in many cases we may guess a sequence to be convergent without knowing the limiting point. {Example: A sequence (x_n) defined by $x_1 = 1; \quad x_2 = 2; \quad x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$: See text for full discussion} To handle such cases, is it possible to give a criterion which, if true, would enable us to conclude convergence of the sequence? Note that this criterion should be in terms of the given sequence only.

2. Such a criterion was given by Cauchy. It is based on differences between terms of the sequence.

3. Before we see that condition, let us prove: If a sequence $b_n \to b$ then, the sequences $b_{n+1} - b_n,\ b_{n+2} - b_n,\ b_{n+3} - b_n, \ldots \to 0$. Thus, if $b_n \to b$, then given any natural p the sequence $b_{n+p} - b_n \to 0$. Thus if a sequence is convergent, its p–th difference sequence should converge to 0 for every natural p. Let us call this *poochy* for short. Thus being poochy is necessary for being convergent.

4. Warning: *poochy* is a non-standard term. Please do not use it after the end of this semester!

5. Does poochy imply convergent? The example of $\sum \frac{1}{n}$ illustrates that this is not true. We need something stronger than poochy.

6. Cauchy found such a condition by making the difference sequences converge to zero uniformly in p. See tutorial sheet #2 for more inspiration.

7. A sequence (c_k) is *cauchy* if for every (real) $\epsilon > 0$, there exists a natural number N such that for all $m, n \geq N$, $|c_n - c_m| < \epsilon$.

8. Proposition: Convergent implies cauchy.

   Proof: Let c be the limit of a cauchy sequence (c_k). Given any real $\epsilon > 0$, there exists a natural number K such that for all $k \geq K$, $|c_k - c| < \frac{\epsilon}{2}$. Now, using triangle inequality: For all $m, n \geq K$, we have $|c_m - c_n| = |c_m - c - (c_n - c)| \leq |c_m - c| + |(c_n - c)| < \epsilon$.

9. Proposition: Cauchy implies bounded.

   Proof: Taking $\epsilon = 1$, there exists a natural number K such that for all $m, n \geq K$, $|c_m - c_n| < 1$.

   Thus for all $n \geq K$, $-1 + c_K < c_n < 1 + c_K$. Set $L = \min(c_1, c_2, \ldots, c_{K-1}, -1 + c_K)$ and $U = \max(c_1, c_2, \ldots, c_{K-1}, 1 + c_K)$ and all terms of the sequence are between L and U.

10. Proposition: Cauchy implies convergent.

    Proof: Given a cauchy sequence (c_n), define the sequence of infimums $\alpha_n := \inf\{c_k | k \geq n\}$ Prove the following inequalities
    $$\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \alpha_n \leq \cdots.$$
    By the Monotone Convergence Theorem, $\alpha_n \to \alpha^*$ for some real α^*.

    Given a positive real ϵ, by definition of cauchy sequence, there exists a natural L such that for all natural $m, n \geq L$, we have $|c_m - c_n| < \frac{1}{3} \cdot \epsilon$. . . . (1)
    Since $\alpha_n \to \alpha^*$, there exists a natural M' such that $\alpha^* - \frac{1}{3} \cdot \epsilon < \alpha_{M'} \leq \alpha^*$. However, the sequence of infimums is increasing and we can find a natural $M \geq \max(M', L)$ such that $\alpha^* - \frac{1}{3} \cdot \epsilon < \alpha_M \leq \alpha^*$, i.e., $|\alpha_M - \alpha^*| < \frac{1}{3} \cdot \epsilon$. . . . (2)
    Since $\alpha_M + \frac{1}{3} \cdot \epsilon > \alpha_M$, it is not the infimum of the set $\{c_k | k \geq M\}$. Hence, there exists a natural $N \geq M$ such that $\alpha_M \leq c_N < \alpha_M + \frac{1}{3} \cdot \epsilon$, i.e., $|c_N - \alpha_M| < \frac{1}{3} \cdot \epsilon$. . . . (3)
    Now, combine (1), (2) and (3) in the following.
    Given any natural $m \geq N$, $|c_m - \alpha^*| \leq |c_m - c_N| + |c_N - \alpha_M| + |\alpha_M - \alpha^*| < \epsilon$.

11. For extra-credit:

    Define the sequence of supremums (β_n) by $\beta_n := \sup\{c_k | k \geq n\}$. Prove the following inequalities
    $$\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_3 \leq \beta_2 \leq \beta_1.$$
    By the Monotone Convergence Theorem, $\beta_n \to \beta^*$ for some real β^*. One can prove that $\alpha^* = \beta^*$.

12. Remark: The α^*, β^* in the above proof are respectively termed the $\liminf$ and the $\limsup$ of the given sequence. You can try to prove that a given sequence (c_n) is convergent if and only if these two $\liminf c_n$ and the $\limsup c_n$ exist and are equal. This may be viewed as an alternative to cauchy's formulation of convergence criterion.

1. Infinite series generated by a sequence, terms of the series, partial sums and sequence of partial sums, convergent/divergent series, sum or value of series. Notation.

2. Proof of coonvergence of geometric series $\sum r^n = \frac{1}{1-r}$ for $|r| < 1$.

   Proof of divergence for $r \geq 1$: Partial sums are unbounded.

   Proof of divergence for $r = -1$: Odd and even terms of sequence of partial sums have different limits.

   Proof of divergence for $r < -1$: ?

   Simplify proofs ...

3. Telescoping series $\sum \frac{1}{n(n+1)} = 1$.

   Other examples like $\sum \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$.

4. n–th term test – necessary but not sufficient for convergence of a series. Proof: $a_n = s_n - s_{n-1}$.

5. As a corollary to monotone convergence theorem: A non–negative series converges if and only if the sequence of partial sums is bounded.
   Applications: 2–series converges and in general p–series for $p > 1$ converges by arguing on s_{2^k-1}.
   Harmonic series diverges by arguing $s_{2^k} \geq 1 + \frac{k}{2}$. By comparison, p–series diverges for $0 < p \leq 1$.

6. Cauchy criterion for series as a direct application of cauchy criterion for convergence.

7. Discussion topic:

   (a) How many examples do you know where a series converges *and* you know the sum?

   (b) Can you find "the answer" to $\sum_1^\infty \frac{1}{n^2}$?
   Answer: $\frac{\pi^2}{6}$

   (c) Can you find "the answer" to $\sum_1^\infty \frac{1}{n^3}$?
   Answer: "Unknown!": What is "knowing" a real number anyway?
   Only known to be rational

   (d) $\sum_1^\infty \frac{1}{n^4}$?
   Answer: Known: find out

   (e) $\sum_1^\infty \frac{1}{n^5}$?
   Answer: Not known whether rational or irrational!

   (f) Read about *Basel problem*

1. Given two numbers a and b, they can be added in two ways, viz., $a+b$ and $b+a$. The two answers are equal by commutativity. Given three numbers a, b and c, they can be added in twelve ways, viz., $a+(b+c), a+(c+b), b+(c+a), b+(a+c), c+(a+b), c+(b+a), (a+b)+c, (b+a)+c, (b+c)+a, (c+b)+a, (c+a)+b, (a+c)+b$. These twelve answers are equal by commutativity and associativity.

2. Given any finite collection of real numbers, we expect the different ways of adding them to yield the same answer. Having proved this expectation, the coinciding answer can be defined as the *sum* of these numbers.

3. The axioms for reals provide a 2–input addition function $f_+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The identity function may be taken, artificially, as the 1–input addition function. Inductively, given a natural $k \geq 3$, define a k*–input* addition function from $\mathbb{R}^k$ to $\mathbb{R}$ as a function of the form $f_+(g(* \cdots *), h(* \cdots *))$ for an l–input addition function g and an m–input addition function h such that $l+m=k$ and $l, m \leq k-1$. {Here, the first l components of the k component input are fed as input to g and the remaining m components as input to h.}
   As an exercise, find the number of k–input addition functions. By induction, prove that any two k–input addition functions are equal.

4. Given a k–tuple $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, a *reordering* of x is a k–tuple $x_\varphi = (x_{\varphi(1)}, x_{\varphi(2)}, \ldots, x_{\varphi(k)})$ for a bijection $\varphi : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$.
   As an exercise, find the number of reorderings of a given k–tuple. If f_0 is a k–input addition function, prove that for any two reorderings x and y of each other, the evaluations $f_0(x)$ and $f_0(y)$ are equal.

5. Given k real numbers, a *way* of adding them is an evaluation $f(x)$ of a particular k–input addition function f at x – a particular ordering of the given numbers.
   Find the number of ways of adding k real numbers. Prove that all such ways yield the same answer. This is called the *sum* of the given numbers. The following theorems were necessary in the steps leading up to our definition of sum.

   Finite Regrouping Theorem: Given a k tuple $x \in \mathbb{R}^k$, $f(x) = g(x)$ for any two k–input addition functions f and g.
   Finite Rearrangement Theorem: Given a k–input addition function, f, $f(x) = f(y)$ for any two reorderings x and y of the same k–tuple.

6. Can we extend the above two theorems to infinite series?

7. Devise a definition and find an analogue of the regrouping theorem for infinite series.

8. If $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection, the series $\sum a_{\varphi(n)}$ is called a *rearrangement* of the $\sum a_n$. We explore an analogue of rearrangement theorem for infinite series.

9. We have seen $\sum \frac{1}{n}$ diverges. How about $\sum \frac{1}{2n}$ and $\sum \frac{1}{2n-1}$?

10. Alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$ converges. For a proof, observe that s_{2n} is monotonically increasing and s_{2n+1} is monotonically decreasing. Further, $0 < s_{2n} < s_{2n} + \frac{1}{2n+1} = s_{2n+1} < 1$ which implies that the even and odd partial sums are bounded and hence converge. By squeeze theorem to the same inequality, they converge to the same limit. Finally conclude that the series is convergent.

11. Based on this example, we make the definitions: For a given series $\sum a_n$, if $\sum |a_n|$ converges the series converges *absolutely* and if $\sum a_n$ converges while $\sum |a_n|$ diverges, the series converges conditionally. By Cauchy's criterion, one sees that $\sum |a_n|$ converges implies $\sum a_n$ converges.

12. Rearrangement theorem: If $\sum a_n$ converges absolutely, then every rearrangement of the series converges to the same value. For a proof, let $\sum a_n = a$, s_n and t_n the sequence of partial sums of the given series and its rearrangement. Pick a natural N such that for $n \geq N$, both $|s_n - a| < \frac{1}{2}\epsilon$ and $\sum_N^\infty |a_n| < \frac{1}{2}\epsilon$ are true. Let M be a natural number such that the terms $a_1, a_2, \ldots a_N$ appear in the rearranged partial sums t_n for $n \geq M$. Naturally, $M \geq N$. Then, for $n \geq M \geq N$, we have $|t_n - a| \leq |t_n - s_n| + |s_n - a| \leq \sum_N^\infty |a_n| + \frac{1}{2}\epsilon < \epsilon$. Done.

13. Demonstrate that the alternating harmonic series can be rearranged to converge to any real number.

14. Statement of Riemann's rearrangement theorem.

1. Alternating Series Test: If (z_n) is monotonically decreasing sequence of non–negatives converging to zero, the series $\sum(-1)^{n+1}z_n$ converges. Proof is analogous to the example of alternating harmonic series. One just observes that the even subsequence of partial sums s_{2n} is monotonically increasing and the odd subsequence of partial sums s_{2n+1} is monotonically decreasing. Further $0 \le s_{2n} \le s_{2n} + z_{2n+1} = s_{2n+1} \le z_1$. This makes the two subsequences bounded and hence convergent. Using $z_{2n+1} \to 0$ and squeeze theorem they converge to the same value. Using this establish that the alternating series converges.

2. Comparison test for non–negative series. Similar one for negative series. Statement:
   Let $0 \le x_n \le y_n$ for $n \ge K$, some natural. Then (i) $\sum y_n$ converges implies $\sum x_n$ converges. Also, (ii) $\sum x_n$ diverges implies $\sum y_n$ diverges.
   How about if $\sum x_n$ converges or $\sum y_n$ diverges?

3. Limit comparison test: (x_n) and (y_n) are positive sequences and $r = \lim \frac{x_n}{y_n}$ exists. Then:
   (i) If $r \neq 0$, $\sum x_n$ is convergent if and only if $\sum y_n$ is convergent.
   Proof: Take $\epsilon = \frac{1}{2}r$. There exists a natural K such that $n \ge K$ implies $|\frac{x_n}{y_n} - r| < \frac{1}{2}r$, i.e., $\frac{1}{2}r < \frac{x_n}{y_n} < \frac{3}{2}r$ whence $(\frac{1}{2}r)y_n < x_n < (\frac{3}{2}r)y_n$. Now apply comparison test.
   (ii) If $r = 0$, and $\sum y_n$ is convergent then $\sum x_n$ is convergent.
   Proof: Take $\epsilon = 1$. There exists a natural K such that $n \ge K$ implies $-1 < 0 < \frac{x_n}{y_n} < 1$ whence $0 < x_n < y_n$. Now apply comparison test.

4. Examples: $\sum \frac{1}{n^2+n+1}$ by comparison with $\sum \frac{1}{n^2}$. Limit comparison of $\sum \frac{1}{n^2-n+1}$ with $\sum \frac{1}{n^2}$. Limit comparison of $\sum \frac{1}{\sqrt[3]{n+9}}$ with $\sum \frac{1}{\sqrt[3]{n}}$

5. Value Root Test: Let (a_n) be a sequence of reals.
   Suppose for some real r, $|a_n|^{\frac{1}{n}} \le r$ for all $n \ge K$ for some natural K. If $r < 1$, the series $\sum a_n$ (and $\sum |a_n|$) are convergent. For a proof, compare with geometric series.
   Suppose $|a_n|^{\frac{1}{n}} \ge 1$ for all $n \ge K$ for some natural K. Then the series $\sum a_n$ (and $\sum |a_n|$) are divergent. For a proof, use n–th term test.

6. Limit Root Test: Suppose (a_n) be a sequence of reals such that $r = \lim |a_n|^{\frac{1}{n}}$ exists.
   If $r < 1$, then $\sum a_n$ (and $\sum |a_n|$) are convergent.
   If $r > 1$, then $\sum a_n$ (and $\sum |a_n|$) are divergent.

7. Value Ratio Test: Let (a_n) be a sequence of non–zero reals.
   Suppose that for some real r, $\left|\frac{a_{n+1}}{a_n}\right| \le r$ for all $n \ge K$ for some natural K. If $r < 1$, the series $\sum a_n$ (and $\sum |a_n|$) are convergent. For a proof, compare with geometric series.
   Suppose $\left|\frac{a_{n+1}}{a_n}\right| \ge 1$ for all $n \ge K$ for some natural K. Then, the series $\sum a_n$ (and $\sum |a_n|$) are divergent. For a proof, use n–th term test.

8. Limit Ratio Test: Suppose that (a_n) is a sequence of non–zero reals such that $r = \lim \left|\frac{a_{n+1}}{a_n}\right|$ exists.
   If $r < 1$, then $\sum a_n$ (and $\sum |a_n|$) are convergent.
   If $r > 1$, then $\sum a_n$ (and $\sum |a_n|$) are divergent.

1. An IITian is *smart* if there exists some day N of his (indefinite) stay in IIT such that for any day $n \geq N$, the IITian solves tutorial sheet problems on day n. Who is a non–smart IITian?

2. An IITian is *super smart* if there exists some day N of her stay in IIT such that for any day $n \geq N$, if n is a working day, the IITian solves tutorial sheet problems on day n. Who is a non–super smart IITian?

3. Examples of graphs of functions with and without breaks at a certain point of their domain.

4. (SC) Sequential criterion: for every sequence $x_n \to c$, assuming $x_n \in \text{domain}(f)$, it should be true that $f(x_n) \to f(c)$.
   OPP.(SC): there exists a sequence $x_n \to c$ such that $x_n \in \text{domain}(f)$ but $f(x_n) \not\to f(c)$.

5. (WC) Weierstrass criterion: for any real $\epsilon > 0$, there exists a real $\delta > 0$ such that for every x satisfying $|x - c| < \delta$, if $x \in \text{domain}(f)$, it should be true that $|f(x) - f(c)| < \epsilon$.
   OPP. (WC): there exists a real $\epsilon_0 > 0$ such that for any real $\delta > 0$, there exists an $x_\delta \in \text{domain}(f)$ which satisfies $|x_\delta - c| < \delta$ but it is true that $|f(x_\delta) - f(c)| \geq \epsilon_0$.

6. Proof of OPP.(WC) implies OPP.(SC). This proves (SC) implies (WC).

7. Proof of (WC) implies (SC).

8. Domain: A subset $A \subset \mathbb{R}$ is a *domain* if for every $c \in A$, at least one one of the following three types of sets $[c, c + \alpha_0)$, $(c - \alpha_0, c + \alpha_0)$ or $(c - \alpha_0, c] \subset A$ for some real $\alpha_0 > 0$.

9. Suppose $f : A \to \mathbb{R}$ is a function on a domain A. Say f is *continuous at c* by (SC) $\equiv$ (WC). Say f is *continuous* if it is continuous at every point of its domain.

10. In domain(f), 'domain' associates an object to a function f. In a subset $A \subset \mathbb{R}$ being a domain, 'domain' is an adjective for subsets of reals.

11. Proof that the constant, identity and square functions are continuous by (WC).

12. Exercise: continuity of power, modulus, exponential and root functions.

1. Rules about continuity of resultant function at a point/over domain with respect to operations of addition, subtraction, multiplication, division and composition.

2. Applications: Continuity of polynomial and rational functions.

3. Non-traditional functions like Dirichlet's and Thomae's examples.

4. Thousand Dollar Challenge: Find a function from reals to reals which is discontinuous at irrationals and continuous at rationals.

5. A function is *bounded* if its range is bounded. *Supremum of a function* is the supremum of its range. Likewise infimum.

6. A function *has an absolute maximum* if it attains its supremum. Likewise for *absolute minimum*.

7. Examples for functions which are unbounded and do not attain their sup/inf on $\mathbb{R}$, on half–rays, on bounded intervals of various kinds.

8. Failure to produce such examples on intervals of type $[a, b]$ for real $a \leq b$.

1. Intermediate Value Property for a subset of $\mathbb{R}$. An *interval* is a non–empty subset of $\mathbb{R}$ satisfying the INTERmediate VALue Property. Examples for intervals.

2. Exhibition theorem for intervals: Every interval is of exactly one of the following forms: (1) $\{a\} = [a, a]$ (2) $[a, b]$ (3) $(a, b]$ (4) $[a, b)$ (5) (a, b) (6) $(-\infty, b]$ (7) $[a, \infty)$ (8) $(-\infty, b)$ (9) (a, ∞) (10) $(-\infty, \infty) = \mathbb{R}$, for some reals $a < b$. Proof: Use sup/inf.

3. IVP and Continuity: Recall that a subset $A \subset \mathbb{R}$ is a *domain* if for every $a \in A$, there exists an interval $I \neq [a, a]$ of $\mathbb{R}$ such that $a \in I \subset A$. We expand our study of continuous functions to those defined on domains.
   (a) A continuous function preserves IVP.
   (b) A continuous function takes an interval to an interval.
   (c) (Weak Preservation of Intervals Theorem) If $f : I \to \mathbb{R}$ is a continuous function defined on an interval I, then Range(f)=$f(I)$ is an interval.
   (d) (Strong Preservation of Intervals Theorem) If $f : A \to \mathbb{R}$ is a continuous function on a domain A, then for any interval $I \subset A$, Range(f)=$f(I)$ is an interval.
   (e) (Bolzano's Intermediate Value Theorem) Let $f : A \to \mathbb{R}$ be a continuous function on a domain A such that the interval $[a, b] \subset A$. Case (I) For any real k satisfying $f(a) < v < f(b)$ there exists a real $i \in [a, b]$ such that $f(i) = v$. Case (II) Analogous.
   (f) (Trap Root Theorem) Let $f : A \to \mathbb{R}$ be a continuous function on a domain A such that the interval $[a, b] \subset A$. If $f(a)f(b) < 0$, there exists a $z \in (a, b)$ such that $f(z) = 0$.

4. Can you prove the equivalence of the latter four statements?

5. Trap Root Algorithm: For a function satisfying the above hypotheses, find a zero with an error of less than a real $\epsilon > 0$. The procedure is

   INITIALIZE left$_1$=a, right$_1$=b, mid$_1$=$\frac{1}{2}$(left$_1$+right$_1$), error$_1$= $\frac{1}{2}$(right$_1$−left$_1$).
   DO

   IF f(mid$_n$)=0, THEN a root has been located.
   SET left$_{n+1}$ = $\frac{1}{2}$(left$_n$+mid$_n$), right$_{n+1}$ = $\frac{1}{2}$(mid$_n$+right$_n$)
   SET mid$_{n+1}$ =mid$_n$, error$_{n+1}$= $\frac{1}{2}$(right$_{n+1}$−left$_{n+1}$)
   (On a machine, exit LOOP.)

   IF (f(mid$_n$)>0), THEN
   IF f(left$_n$ > 0), THEN SET left$_{n+1}$=mid$_n$, right$_{n+1}$=right$_n$.
   IF f(left$_n$ < 0), THEN SET left$_{n+1}$=left$_n$, right$_{n+1}$=mid$_n$.
   SET mid$_{n+1}$=$\frac{1}{2}$(left$_{n+1}$+right$_{n+1}$), error$_{n+1}$= $\frac{1}{2}$(right$_{n+1}$−left$_{n+1}$)

   IF (f(mid$_n$)<0), THEN
   IF f(left$_n$ > 0), THEN SET left$_{n+1}$=left$_n$, right$_{n+1}$=mid$_n$.
   IF f(left$_n$ < 0), THEN SET left$_{n+1}$=mid$_n$, right$_{n+1}$=right$_n$.
   SET mid$_{n+1}$=$\frac{1}{2}$(left$_{n+1}$+right$_{n+1}$), error$_{n+1}$= $\frac{1}{2}$(right$_{n+1}$−left$_{n+1}$)

   WHILE error$_{n+1} \geq \epsilon$.

   When the procedure exits on iteration N, take mid$_{N+1}$ as an approximation for the root. Either f(mid$_{N+1}$)=0 in which case we have the root. Or $|$mid$_{N+1} - z| <$error$_{N+1} < \epsilon$, in which case we have the root to desired accuracy. (Note the $N + 1$, not N)

6. Proof of Trap Root Theorem: Use the procedure to define sequences left$_n$, mid$_n$, right$_n$ and error$_n$ inductively. Verify that left$_n$ and right$_n$ are monotonic and bounded and hence convergent to say L and R. Since $|$right$_n$−left$_n| = 2\cdot$error$_n = \frac{2(b-a)}{2^n}$, by squeeze theorem $L = R = z$, say. Further, since left$_n <$ mid$_n <$ right$_n$, by squeeze theorem again mid$_n \to z$. We claim that $f(z) = 0$. For a proof, note that in the special case if f(mid$_k$)=0 for some natural k, then, by definition, mid$_n$=mid$_k$ for all $n \geq k$. Hence mid$_n$ is an eventually constant sequence converging to z with $f(z) = 0$. If f(mid$_n$)$\neq$ 0, for every natural n, then f(left$_n$)f(right$_n$) < 0 for every n and continuity implies $f(z)f(z) \leq 0$ and being real $f(z) = 0$.

7. Now, write a program in a suitable language, compile and run for a few functions.

1. Especial Property of compact intervals: If α_n is a sequence in any compact interval $[a, b]$, then there exists a subsequence $\alpha_{n_k} \to \alpha \in [a, b]$. Proof: Bolzano–Weierstrass and Squeeze.

2. Boundedness Theorem: A continuous function on a compact interval is bounded. If $f : [a, b] \to \mathbb{R}$ is unbounded, then there exists a sequence $\alpha_n \in [a, b]$ such that $|f(\alpha_n)|$ is an unbounded monotonic sequence. By Especial Property, a subsequence $\alpha_{n_k} \to \alpha \in [a, b]$ which implies that $|f(\alpha_{n_k})|$ converges to $f(\alpha)$ by continuity. This contradicts unboundedness of $|f(\alpha_{n_k})|$.

3. Lemma: If $S \subset \mathbb{R}$ is bounded, there exists sequences (y_n) and (z_n) s.t. $y_n,\ z_n \in S$ for every natural n and $y_n \to \sup S$ and $z_n \to \inf S$.

4. Min–Max Theorem/Attainment Theorem: A continuous function $f : [a, b] \to \mathbb{R}$ on a compact interval $[a, b]$ for real $a \leq b$ attains its supremum and infimum.
   By the lemma, there is a sequence (α_n) such that $\alpha_n \in [a, b]$ and $f(\alpha_n) \to \sup f$. By the Especial Property, there exists a subsequence $\alpha_{n_k} \to \alpha \in [a, b]$. Being a subsequence, $f(\alpha_{n_k}) \to \sup f$ and by continuity, $f(\alpha_{n_k}) \to f(\alpha) = \sup f$. Done. Likewise with inf.

5. Compactness Theorem: The range of a continuous function is a compact interval, if the domain is a compact interval. Proof: Now, the range is a subset of $[\inf f, \sup f]$, by definition of inf/sup. By Attainment Theorem, the end points are in the range. But since the range is an interval, the range is exactly $[\inf f, \sup f]$.

6. Converse/contrapositive of above property/theorems?

7. Remark: Especial Property fails for every other type of interval. These three theorems fail for continuous functions on other types of intervals and also for discontinuous functions on compact intervals. Provide explicit examples.

8. Fake proof of boundedness theorem: Start with continuity at $x = a$ and get $\delta_1 > 0$ such that $|f(x) - f(a)| < 1$ for all $|x - a| < \delta_1$. Next continuity at $x = a + 0.99\delta_1$ yields a $\delta_2 > 0$ such that $|f(x) - f(a + 0.99\delta_1)| < 1$ for all $|x - (a + 0.99\delta_1| < \delta_2$. These two together give $|f(x) - f(a)| < 2$ for all $|x - a| < 0.99\delta_1 + 0.99\delta_2$. Continue this argument and reach b. Done. Where is the flaw?

9. Fake proof leads to the definition of uniform continuity as a convenient hypothesis to obtain boundedness. A function $f : A \to \mathbb{R}$ on a domain A is *uniformly continuous* if for every real $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x, y \in A$ satisfying $|x - y| < \delta$, it should be true that $|f(x) - f(y)| < \epsilon$. A uniformly continuous function is continuous. Prove that a uniformly continuous function on a bounded interval is bounded. Further a continuous function on a compact interval is uniformly continuous.

1. Let I and J be intervals. For functions $f : I \to J$ for $A \subset \mathbb{R}$ definitions of *increasing, decreasing, monotonic, strictly increasing, strictly decreasing, strictly monotonic.*

2. Examples for: monotonic $\not\Rightarrow$ injective, strictly monotonic $\not\Rightarrow$ surjective (continuous), continuous $\not\Rightarrow$ monotonic (injective, surjective), injective $\not\Rightarrow$ monotonic (continuous), surjective $\not\Rightarrow$ monotonic (continuous).

3. Failure to provide examples contradicting: strictly monotonic $\Rightarrow$ injective, continuous and injective $\Rightarrow$ strictly monotonic, monotonic and surjective $\Rightarrow$ continuous, monotonic and bijective $\Rightarrow$ inverse is strictly monotonic.

4. If you are giddy, here is the general principle: Suppose $A_1, A_2, A_3, \ldots A_n$ are adjectives which can be applied to a collection of nouns (objects) in N. It is only natural to ask the questions:

   (a) First Order: For any $x \in N$, if x is $A_1 \overset{?}{\Rightarrow} x$ is A_2 and similar $n(n-1)$ questions.

   (b) Second Order: For any $x \in N$, if x is A_1 and x is $A_2 \overset{?}{\Rightarrow} x$ is A_3 and similar $\frac{1}{2}(n)(n-1)(n-2)$ questions.

   (c) Third Order: Formulate and count.

   (d) Fourth–$(n-1)^{\text{th}}$ Order: Formulate and count.

   Show that the total number of such questions is $n(2^{n-1} - 1)$.

5. In our example, our nouns are 'functions from intervals to intervals'. Our six adjectives are 'monotonic', 'strictly monotonic', 'injective','surjective','bijective', 'continuous'. How many questions of 1–5 orders can I ask you? How many can you answer?

6. Injective & continuous implies strictly monotonic.
   Let $f : I \to J$ be continuous and injective from an interval I to an interval J. Then f is strictly monotonic.
   Proof: Assume f is not strictly monotonic. Then f is not strictly increasing or strictly decreasing (and hence there exists $\xi_1 < \xi_2$ and $\xi_3 < \xi_4$ such that $f(\xi_1) \le f(\xi_2)$ *and* $f(\xi_3) \ge f(\xi_4)$). This implies (fill the gap!) we can find three points in the domain $x_1 < x_2 < x_3$ such that either (i) $f(x_1) \le f(x_2) \ge f(x_3)$ or (ii) $f(x_1) \ge f(x_2) \le f(x_3)$. Injectivity of f allows us to infer that either (i) $f(x_1) < f(x_2) > f(x_3)$ or (ii) $f(x_1) > f(x_2) < f(x_3)$ is true. Case (i) has three sub-cases: (i.a) $f(x_1) < f(x_3)$, (i.b) $f(x_1) = f(x_3)$ and (i.c) $f(x_1) > f(x_3)$. Sub-case (i.b) violates injectivity. For other two sub-cases, apply intermediate value property to get contradictions to injectivity. Case (ii) is similar.

7. Invertible & monotonic implies monotonically invertible.
   $f : A \to B$ be an increasing bijective function from a set A to a set B. Then the inverse of f is strictly increasing. Like wise for a decreasing bijective function.
   Proof: Let $g : B \to A$ be the uniquely defined inverse of f. Suppose $b_1 < b_2$ are any two elements in B. Since g is injective, $g(b_1) = g(b_2)$ is not allowed. Suppose $g(b_1) > g(b_2)$. f being increasing and injective, we get $f(g(b_1)) > f(g(b_2))$, i.e., $b_1 > b_2$, a contradiction. Conclude that $g(b_1) < g(b_2)$.

8. Lemma on monotonic and discontinuous

   Let $f : (a, b) \to B$ be monotonically increasing. For each $c \in (a, b)$, define

   $$L_c := \sup\{f(x)|x < c\} \quad \text{and} \quad U_c := \inf\{f(x)|x > c\}.$$

   Then, f is continuous at $c \in (a, b)$ if and only if $L_c = U_c$.

   (continuity at c implies equality) Proof: Clearly $L_c \le f(c) \le U_c$. If $L_c = f(c) < U_c$, find a sequence (x_n) such that $x_n > c$, $x_n \to c$, and $f(x_n) \to U_c \ne f(c)$, contradicting continuity. Similarly, if $L_c < f(c) = U_c$, find a sequence (x_n) such that $x_n < c$, $x_n \to c$, and $f(x_n) \to L_c \ne f(c)$, contradicting continuity.

   (equality of L_c and U_c implies continuity) If $L_c = f(c) = U_c$, by making given ϵ smaller, if necessary, take $\delta = min(\delta_1, \delta_2)$ where $\delta_1 =$

9. Monotonic & surjective implies continuous.
   Let $f : A \to B$ be a monotonic and surjective function from a domain A to a domain B. Then f is continuous.
   Proof: We write the proof for f being increasing, the other case being analogous. Assume f is discontinuous at $c \in A$. Then, there exists a real $\epsilon_0 > 0$ and a sequence $\delta_n \to 0$ such that $|f(c + \delta_n) - f(c)| \ge \epsilon_0$. By

passing to subsequences, we may assume that either $\delta_n < 0$ for each n or that $\delta_n > 0$ for each n. In the former case, $\sup\{f(x)|x < c, x \in A\} = \sup\{f(c+\delta_n)|n \in \mathbb{N}\} \leq -\epsilon_0 + f(c) < f(c)$ while in the latter case $f(c) < f(c) + \epsilon_0 \leq \inf\{f(c+\delta_n)|n \in \mathbb{N}\} = \inf\{f(x)|x > c, x \in A\}$. As a result, in the former case, f, being monotonic does not realize the value $-\frac{1}{2}\epsilon_0 + f(c)$, while in the latter it does not realize $f(c) + \frac{1}{2}\epsilon_0$. This contradicts its surjectivity. (There is a bug in this proof as it assumes that $f(c) + \frac{1}{2}\epsilon_0$ or $-\frac{1}{2}\epsilon_0 + f(c)$ belongs to B. For this to happen, B should be an interval. The error was pointed out by a tutor to me on 19th Nov. So I am modifying my claim.)

10. Invertible & continuous implies continuously invertible.

    Inverse Function Theorem (in the world of continuous functions): Let I and J be intervals and $f : I \to J$ a bijective continuous function. Then its inverse $g : J \to I$ is continuous.
    Proof: Relying on the previous three results, f is injective and continuous implies f is strictly monotonic, f is monotonic and bijective implies g is monotonic, g is monotonic and surjective implies g is continuous. Done.

11. Application: Existence and continuity of n–th root functions.

1. Recall definition of a domain A in $\mathbb{R}$. Quick mention of *limit of g at c*, for a function $g : A \to \mathbb{R}$ at a point $c \in A$. (Deliberately exclude $x = c$, anticipating definition of derivative).

2. Analogous to limit of a sequence and definition of continuity. Comparable results are true: uniqueness of limit if it exists and the rules: power, sum, difference, product, quotient (be delicate), chain/composition. Squeeze theorem. Write down proofs by yourself.

3. Galileo's law of motion states that an object/body/particle without any external force/disturbance continues its state of rest *or* uniform velocity (constant speed in a straight line).
   Challenges faced by Newton:

   (a) To give an *absolutely accurate* definition of *instantaneous* velocity from the crude calculations of *average* velocity.

   (b) Having answered (a), to quantify amount of *instantaneous* change in velocity.

4. Certainly, you have seen the definition of instantaneous velocity for a particle whose position has been co–ordinated by real numbers. It is the 'eventual value' of average velocity quotient $\frac{f(c+h)-f(c)}{h}$ as the time window h 'becomes'/'approaches' zero. The definition is accurate. Indeed, using this and an analogous definition for *instantaneous* acceleration, Newton calculated that moon is falling towards earth at a rate $\frac{1}{60^2}$ that of a stone thrown up. Knowing that the distance between moon and earth is 60 times the radius of earth – he hypothesised the Universal Law of Gravitation.

5. Latter day mathematicians have made these notions rigorous. Stripped of any and all physical interpretation, consider a function $f : A \to \mathbb{R}$ for a domain A in $\mathbb{R}$. For a $c \in A$, we say that
   $$f \text{ is } \textit{differentiable} \text{ at } c \in A \text{ if } \qquad \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ at } c \text{ exists.}$$
   Note that $g(h) = \frac{f(c+h)-f(c)}{h}$ is defined on a subset of $\mathbb{R}$ *not* containing 0 and we are seeking $\lim\limits_{h\to 0} g(h)$.

6. Define *differentiable* to mean differentiable at all points of the domain.

7. Notation for derivative.

8. Proposition: Differentiable at c implies continuous at c.

9. Rules: Power, Sum, Difference, Product, Quotient, Chain. Proof of chain rule is delicate. See BaSh.

10. The inverse function theorem in the world of continuous functions assures a continuous inverse for a bijective continuous function. Bijectivity is necessary if you want to invert a function. And if the inverse of a function is continuous, by the inverse function theorem, the function is continuous. (all this is for functions between intervals). What should one do to get a differentiable inverse? Certainly we need a bijective function. And if the inverse is differentiable, the inverse is continuous and by our previous observation, the function is continuous. But is the function differentiable? The function $x \mapsto x^{\frac{1}{3}}$ from $(-1, 1)$ to $(-1, 1)$ is not differentiable (at zero) but has a differentiable inverse, viz., $x \mapsto x^3$. Likewise, the function $x \mapsto x^3$ is differentiable and has a local differentiable inverse at all points of its domain except zero.
    These two examples illustrate that for bijective functions, differentiability is neither necessary nor sufficient to get a differentiable inverse. This is explained by the observation (in examples, at least) $f'(x_n) \to 0$ as $x_n \to c$ causes $|f^{-1\prime}(y_n)| \to \infty$ as $y_n = f(x_n) \to f(c)$.
    Challenge: Give an example for a function $f : \mathbb{R} \to \mathbb{R}$ such that f is bijective, f is differentiable at all points of a pre-specified subset $S \subset \mathbb{R}$ and not differentiable at points of S^c.
    Another one: Can you find an example of a bijective function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable nowhere but the inverse is differentiable (everywhere).

11. Infinitesimal Inverse Function Theorem (in the world of differentiable functions): Let $f : I \to J$ be a bijective continuous function between intervals I and J, with $g : J \to I$ as the inverse. If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $f(c)$ and $g'(f(c)) = \frac{1}{f'(c)}$.

12. Local Inverse Function Theorem: Let $f : I \to J$ be a bijective continuous function between intervals I and J, with $g : J \to I$ as the inverse. Suppose that for a $c \in I$, there is a real $\epsilon > 0$ such that f is differentiable and f' is

continuous on $(c-\epsilon, c+\epsilon)$ with $f'(c) \neq 0$. Then, there is a real $\delta > 0$ such that at every $x \in (c-\delta, c+\delta) \cap I$, g is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$.

13. Global Inverse Function Theorem: Let $f : I \to J$ be a bijective differentiable function between intervals I and J, with $g : J \to I$ as the inverse. If f is differentiable and $f'(x) \neq 0$ for every $x \in I$, then g is differentiable and $g'(f(x)) = \frac{1}{f'(x)}$ for every $x \in I$.

1. Definitions for *relative minimum, maximum, extremum* for a function defined on an interval or even a domain. Note the use of the words *minimum/maximum* instead of *infimum/supremum*. The word *relative* contrasts these definitions with *absolute* used earlier.

2. Examples of existence and non–existence.

3. Derivative vanishes at interior points of relative extrema.
   On an interval I, let $f : I \to \mathbb{R}$ be a function. Let c be an interior point of I, f have a relative extremum at c and let f be differentiable at c. Then $f'(c) = 0$.
   Proof: Assume $f'(c) = \ell \neq 0$. Then there exists a real $\delta > 0$ such that if $0 < |h| < \delta$, then $\ell - \frac{1}{2}\ell < \frac{f(c+h)-f(c)}{h} < \ell + \frac{1}{2}\ell$. Consider:

   (a) Assume c is a point of relative maximum of f.
      i. If $\ell > 0$, using the above inequality, we have for all $0 < h < \delta$ that $f(c+h) - f(c) > \frac{1}{2}\ell \cdot h > 0$ which contradicts the assumption that c is a point of relative maximum.
      ii. If $\ell < 0$, using the above inequality, we have for all $-\delta < h < 0$ that $f(c+h) - f(c) > \frac{3}{2}\ell \cdot h$ which contradicts the assumption that c is a point of relative maximum.

   (b) The case of c being a point of relative minimum of f is similar.

   Remark: To make sense of $f(c+h)$, δ has to be sufficiently small. Since we are using both $h > 0$ and $h < 0$, c should be an interior point.

4. The proof above tells us that if f is differentiable (even just) at c and $f'(c) > 0$, then there exists a real $\delta > 0$ such that for all $0 < h < \delta$, $f(c+h) - f(c) > 0$ and $f(c-h) - f(c) < 0$. What this does not tell us is that $f(x) - f(y) > 0$ for all $x < y \in (c - \delta, c + \delta)$ – i.e., f is locally increasing. Indeed this is false as the example of $x \mapsto x + x^2 \sin(1/x)$ with $0 \mapsto 0$ illustrates. Upshot: From the infinitesimal data $f'(c) > 0$, we cannot conclude f is locally increasing.

5. Critical points of a function on a domain are defined as the collection which includes end points of the domain, those points where derivative fails to exist or those where the derivative exists and equals zero. If a function has a relative extremum at a point, then that point is a critical point.

6. Rolle's Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b)$. Then there exists a (at least one) point $\xi \in (a, b)$ such that $f'(\xi) = 0$. (Here $a < b$)
   Proof: If $\sup f = \inf f = f(a) = f(b)$, then f is a constant and any element in (a, b) may be used for ξ. Otherwise, an interior point ξ of the interval is a point of absolute extremum and hence relative extremum. By previous theorem, $f'(\xi) = 0$.

7. Mean Value Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a (at least one) point $\xi \in (a, b)$ such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$. (Here $a < b$)
   Proof: Use Rolle's Theorem on $\varphi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$.
   Example of Speeding Fine.

8. First Derivative Test.

1. Lengths: Definition of a unit length as a standard. Lengths of straight line segments which are natural numbers, fractions and real numbers.

2. Areas: Definition of a unit square as standard.

3. Area of a rectangle whose sides are of lengths which are natural numbers/non–negative integers. Extension to rectangles having sides of lengths which are non–negative fractions. How about if sides are non–negative reals?

4. Area of a parallelogram. Area of a triangle. Area of a polygon.

5. Remark that with the definitions we have, the most general curves for which we can *ask* about their lengths are actually only straight line segments. Similarly the most general regions for which we can *ask* about its area are polygons. Of course, minor variants like lengths of piece–wise straight line segments and a collection of polygons any two of which intersect either in an empty set or in a point are allowed. It is *inappropriate* to ask about circumference of a circle or an ellipse or area of a circular or an elliptical region, for instance.

6. Despite the remark, ancients considered the problem of lengths of more general curves and areas of more general planar regions. For calculating lengths of "curved" curves, their method was to approximate the curve by a piece–wise straight line segment and take "finer and finer" approximations. The method for calculating areas of regions with "curved" boundaries is to approximate the region by a collection of triangles or rectangles and take "finer and finer" approximations.

7. What is the circumference of a circle of radius R? What is its area?

8. Proposition: There exists a universal constant denoted by π such that for any circle $\frac{\text{circumference}}{\text{diameter}} = \frac{\text{area}}{(\text{radius})^2} = \pi$.
Proof: (Using Trigonometry) Let $\mathcal{C}$ and $\mathcal{A}$ denote the circumference and area of a circle of radius R. Let p_n, P_n, a_n, A_n denote respectively the perimeters and areas of regular n-gons ($n \geq 3$) circumscribed by and circumscribing the given circle. We then have
$$p_n < \mathcal{C} < P_n \quad \& \quad a_n < \mathcal{A} < A_n \qquad \text{for every natural } n \geq 3.$$
Using trigonometry, one verifies
$$p_n = 2R \cdot 180 \cdot \frac{\sin\left(\frac{180^\circ}{n}\right)}{\frac{180}{n}}, \quad P_n = 2R \cdot 180 \cdot \frac{\tan\left(\frac{180^\circ}{n}\right)}{\frac{180}{n}}, \qquad a_n = R^2 \cdot 180 \cdot \frac{\sin\left(\frac{360^\circ}{n}\right)}{\frac{360^\circ}{n}}, \quad A_n = R^2 \cdot 180 \cdot \frac{\tan\left(\frac{180^\circ}{n}\right)}{\frac{180}{n}}$$
Can you prove that there is a real number α such that $\lim\limits_{\theta^\circ \to 0} \frac{\sin(\theta^\circ)}{\theta^\circ} = \lim\limits_{\theta^\circ \to 0} \frac{\tan(\theta^\circ)}{\theta^\circ} = \alpha$? ...(*). Define π to be that real number which is 180α. Then, one has $\lim p_n = \lim P_n = \mathcal{C} = 2\pi R$ and $\lim a_n = \lim A_n = \mathcal{A} = \pi R^2$. This completes the proof.
Note that your proof of (*) should *not* use $\lim \frac{\sin \xi}{\xi} = 1$, when ξ is measured in radians. This is owing to the fact that radian measure presupposes establishment of π and length of circular arc.

9. Can one find the circumference and area of a circle without using trigonometry?

10. Lemma: If s_n and S_n are the lengths of sides of regular n–gons circumscribed by and circumscribing a circle of radius R, then
$$\frac{s_{2n}}{R} = \sqrt{2 - 2\sqrt{1 - \left(\frac{s_n}{2R}\right)^2}} \quad \& \quad \frac{S_{2n}}{R} = \frac{4}{\frac{S_n}{R}} \left(\sqrt{1 + \left(\frac{S_n}{2R}\right)^2} - 1 \right)$$
Proof: Elementary Euclidean Geometry.

11. Antiphon found $\lim p_{2^k}$, Bryson found $\lim P_{2^k}$ and Archimedes found $\lim A_{2^k}$ using the above lemma as follows. One can verify that $s_4 = \sqrt{2}R$ and $S_4 = 2R$. Using the lemma, the sequences (s_{2^k}) and (S_{2^k}) are well defined. As an exercise prove that $(2^k s_{2^k}), (2^k S_{2^k})$ and $(2^k \frac{1}{2} R S_{2^k})$ converge to $2\pi R, 2\pi R$ and πR^2 for some real number π.

12. Corollary: Define $\alpha_2 = \sqrt{2}$ and $\alpha_{k+1} = \sqrt{2 - 2\sqrt{1 - \left(\frac{\alpha_k}{2}\right)^2}}$ for each $k \geq 2$.
Similarly, $\beta_2 = 2$ and $\beta_{k+1} = \frac{4}{\beta_k} \left(\sqrt{1 + \left(\frac{\beta_k}{2}\right)^2} - 1 \right)$ for each $k \geq 2$.
$2^{k-1}\alpha_k$ and $2^{k-1}\beta_k$ are monotonic sequences converging to π.

1. Ancient Problem of Areas: Given a region in a plane, find its area. In this form, the problem is ill–posed. Despite this, like the example of circle, mathematicians of antiquity found area of elliptical regions, area of parabolic regions etc.

2. Issues with their methods: What is a definition for *area* of a region? If a sequence of polygons approximate a region, over what kinds of such polygonal approximations should one take the limit to get the area of the region? Does every such limit yield the same answer?

3. The biggest challenge was to get the *exact* answer. Exact answers could be found in very few examples. There was nothing anyone could do for almost two thousand years.

4. We shall consider the problem of *area under a function* $f : [a, b] \to \mathbb{R}$. By this we mean the area of the region whose boundary is determined by the curves $x = a$, $x = b$ and $y = f(x)$. Question: Is this restricted problem sufficient to solve the Ancient Problem of Areas?

5. Given an interval $[a, b]$, define a *partition* for that interval, *subinterval*, *norm* of a partition, *tag* for a subinterval and *tagged* partition.

6. The Riemann sum of a function defined on a compact interval w.r.t. a tagged partition is an approximation to the yet-to-be-defined area under that function. One should find limits of Riemann sums over a sequence of tagged partitions whose norm goes to zero. If all such limits yield the same answer, we say that *area under the function* exists/well–defined and take that common answer to be the value of the area.

7. We say that $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* or simply *integrable* if any of the following equivalent conditions hold:
   Sequential Criterion There exists a real number L such that if $\dot{\mathcal{P}}_n$ is a sequence of tagged partitions of $[a, b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$, then $\lim S(f, \dot{\mathcal{P}}_n) = L$.
   Riemann's Criterion There exists a real number L such that for any real $\epsilon > 0$, there exists a real $\delta > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a, b]$ with $||\dot{\mathcal{P}}|| < \delta$, we have $|S(f, \dot{\mathcal{P}}) - L| < \epsilon$.
   Cauchy's Criterion For every real $\epsilon > 0$, there exists a real $\eta > 0$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are tagged partitions of $[a, b]$ with $||\dot{\mathcal{P}}||, ||\dot{\mathcal{Q}}|| < \eta$, we should have $|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \epsilon$.

8. If any of the above criteria hold, we say that f is integrable (over $[a, b]$) and its (Riemann) integral is L. We write $\int_a^b f(x)dx = L$. This notation is unfortunate as it anticipates fundamental theorem of calculus.

9. Uniqueness of limit.

10. Proof: RC implies SC, CC are easy. For SC implies RC, do rather OPP(RC) implies OPP(SC). For CC implies RC, see BaSh.

11. Examples of integrability of constant, identity and square functions on compact intervals.

12. Rules: Sum, Difference and Constant Multiple Rules of Riemann Integrability. Dominance/Comparison Rule. All proofs are easy with Sequential Criterion.

13. Integrability of restrictions, Summation over restrictions.

14. Theorems: integrable implies bounded, continuous implies integrable and monotonic implies integrable. How about their converses?

15. Definition of integral for $a = b$ and $b < a$. Interpretation for negative areas.