# Linear Algebra

Department of Mathematics
Indian Institute of Technology Guwahati

January – May 2019

MA 102 (RA, RKS, MGPP, KVK)

# Similarity and diagonalization

Topics:

- Similarity transformation
- Diagonalization of matrices and operators
- Triangularization of complex matrices

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Note that similarity of matrices is a transitive relation on $\mathcal{M}_n(\mathbb{F})$.

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Exercise: Let $A \in \mathcal{M}_n(\mathbb{F})$. Then geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

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3. The algebraic multiplicity of each eigenvalue $A$ equals its geometric multiplicity.

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Example: The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is not diagonalizable because 1 is the only eigenvalue with algebraic multiplicity 3 but geometric multiplicity 1. Indeed, the eigenspace $E_1$ is given by

$$E_1 = \text{null}\left( \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span}\left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

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Find the eigenvalues of $T$. Is $T$ diagonalizable?

Solution: Consider $B := [1, x, x^2]$. Then $[T]_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

Hence $1, 2, 3$ are the eigenvalues of $T$. Hence $T$ is diagonalizable.

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Now $\text{Trace}(A) = \text{Trace}(PUP^{-1}) = \text{Trace}(U) = \lambda_1 + \cdots + \lambda_n$. ■

Exercise: Let $A, B \in \mathcal{M}_n(\mathbb{F})$. Show that $\text{Trace}(AB) = \text{Trace}(BA)$.