Department of
Electronics & Electrical Engineering

Lecture 6

Sinusoidal Steady State Response

By

Dr. Praveen Kumar
Professor
Department of Electronics & Electrical Engineering

Sinusoidal Functions

- In Fig.1 there are two sinusoids, one denoting the voltage and the other current. The voltage sinusoid is represented as

$$v = V_m \sin(\alpha) = V_m \sin(\omega t) \;\; \text{V} \qquad (1)$$

where

$$\alpha = \omega t \text{ radians} \qquad (2)$$

The terminology is

α is known as the argument

V_m is known as the amplitude

- The eq.1 is commonly referred to as the ***instantaneous voltage***.

- A characteristics of sinusoid is its ***frequency*** and it is defined as the number of cycles of the function which is traversed in ***1s***.
- The frequency is measured in ***cycles per second*** or ***Hertz*** (***f***). The other unit of frequency is ***radians per second***. The relation between ***Hertz*** and ***Radians Per Second*** (ω) is

$$\omega = 2\pi f \qquad (3)$$

Sinusoidal Functions

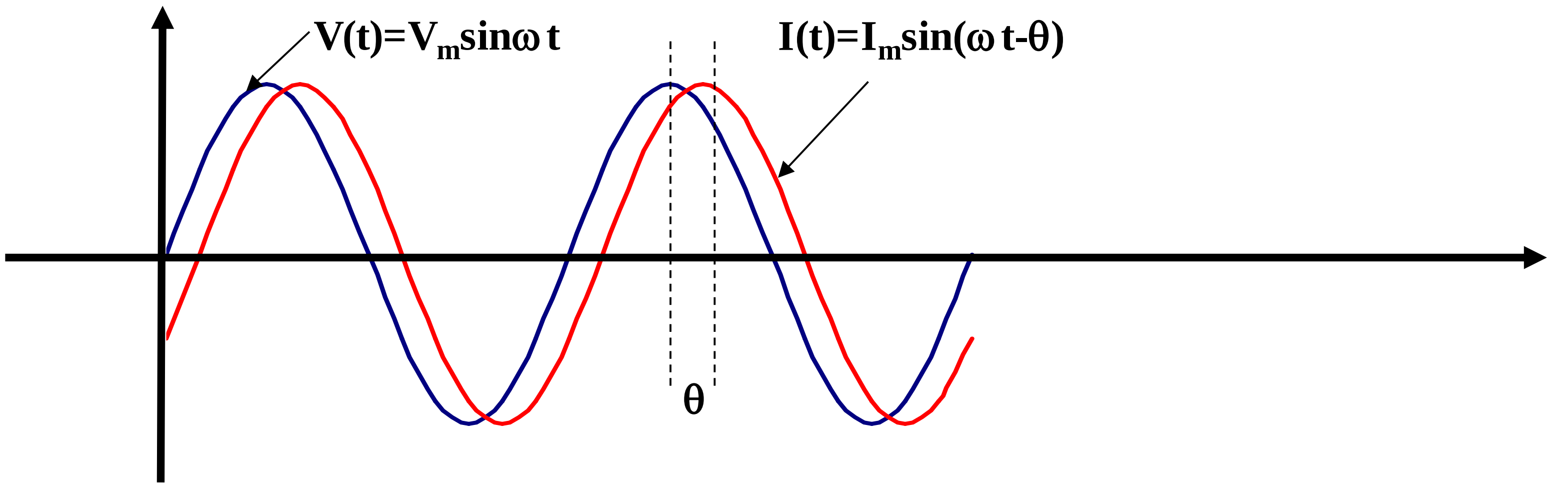

Fig.1: Plot of the sinusoidal voltages and currents

Sinusoidal Functions

- In Fig.1 the sinusoid for the current is displaced by and angle $\boldsymbol{\theta}$ with respect to the voltage sinusoid.

- From Fig.1 it is also seen that the current sinusoid ***lags*** the voltage sinusoid. Hence, the current wave is said to be ***lagging*** behind the voltage wave by the relative phase angle of $\boldsymbol{\theta}$.

- The current sinusoid can be described as having a ***phase lag*** of θ degrees relative to the voltage or the voltage wave has a ***phase lead*** of $\boldsymbol{\theta}$ degrees relative to the current wave.

- The equation describing the current wave including the phase shift is

$$i = I_m \sin(\omega t - \theta) \tag{4}$$

- The negative sign used before $\boldsymbol{\theta}$ indicates that the current lags the voltage by an angle $\boldsymbol{\theta}$.

Sinusoidal Functions

- The form of eq.4 demands that the unit of the argument (ωt-θ) be ***radians***. However, in engineering usage of this equation it is common to express ωt in radians and θ in degrees.

- There exist alternate forms with which to express sinusoids. This is the **Euler's** identity

$$e^{j\omega t} = \cos \omega t + j \sin \omega t \tag{5}$$

- The sine and cosine terms can be expressed as

$$\cos \omega t = \mathrm{Re}\left[e^{j\omega t} \right] \tag{6}$$

$$\sin \omega t = \mathrm{Im}\left[e^{j\omega t} \right] \tag{7}$$

- The sinusoids may also be expressed by the equations:

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \tag{8}$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \tag{9}$$

Average Value of Periodic Functions

- The sinusoidal current is an ***alternating*** current. A general definition of the average value of any function ***f(t)*** over the specified interval between $\mathbf{t_1}$ and $\mathbf{t_2}$ is expressed as

$$F_{av} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)dt \tag{10}$$

- If the function ***f(t)*** is expressed in radians , then the average value of the function is

$$F_{av} = \frac{1}{2\pi}\int_0^{2\pi} f(\omega t)d(\omega t) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)d\alpha \tag{11}$$

- Hence the average value of current shown in Fig.1 is

$$I_{av} = \frac{1}{2\pi}\int_0^{2\pi} I_m \sin(\omega t - \theta)d(\omega t) = \frac{1}{2\pi}\int_0^{2\pi} I_m \sin(\alpha - \theta)d\alpha = 0 \tag{12}$$

$$where$$

$$\omega t = \alpha$$

- The average value of a sinusoid over ***one complete cycle*** is equal to zero.

Effective Value of Periodic Functions

- A finite average value can be found for the sinusoid for the ***positive*** or ***negative*** half cycle. The half cycle average value for the waveform shown in Fig.1 is given by

$$I_{av-1/2cycle} = \frac{1}{\pi}\int_{\theta}^{\theta+\pi} I_m \sin(\alpha-\theta)d\alpha = \frac{2}{\pi} I_m = 0.636 I_m \quad (13)$$

- The average value of either the positive or negative half of a sine function can be found by multiplying the amplitude of the wave by ***0.636***.

- Although the criterion of the ***average value*** of current works well in describing the energy transferring capacity for direct sources, it is a meaningless criterion for symmetrical periodic functions.

- A more suitable definition of the ***average value*** for a symmetric periodic functions is ***effective current***. It is expressed as

$$I_{eff} = I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} \quad (14)$$

Effective Value of Sine Function

- If the function ***I(t)*** in eq.14 is sinusoidal then the ***effective value*** is obtained as

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \sin^2(\omega t)dt} \tag{15}$$

- Using the trigonometric identity

$$\sin^2 \omega t = \frac{1}{2}(1-\cos(2\omega t)) \tag{16}$$

- The ***effective value*** is obtained as

$$I_{eff} = \sqrt{\frac{I_m^2}{2T}\int_0^T (1-\cos(2\omega t))dt} = \frac{I_m}{\sqrt{2}} \tag{17}$$

Instantaneous Power

- Let ***v(t)*** and ***I(t)*** be the instantaneous voltage and instantaneous current (Fig.2) across a network given by

$$v(t) = V_m \sin(\omega t) \tag{18}$$

$$i(t) = I_m \sin(\omega t - \theta) \tag{19}$$

- The expression for instantaneous power is given by

$$p(t) = v(t)i(t) = V_m I_m \sin(\omega t)\sin(\omega t - \theta) \tag{20}$$

using

$$\sin(\omega t - \theta) = \sin(\omega t)\cos(\theta) - \cos(\omega t)\sin(\theta)$$

$$p(t) = V_m I_m \left(\sin^2(\omega t)\cos(\theta) - \sin(\omega t)\cos(\omega t)\sin(\theta)\right)$$

using the identities

$$\sin^2 \omega t = \frac{1}{2}(1 - \cos(2\omega t)) \text{ and } \sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(\omega t)$$

$$p(t) = \frac{V_m I_m}{2}\cos(\theta) - \frac{V_m I_m}{2}\cos(2\omega t - \theta) \tag{21}$$

Instantaneous Power

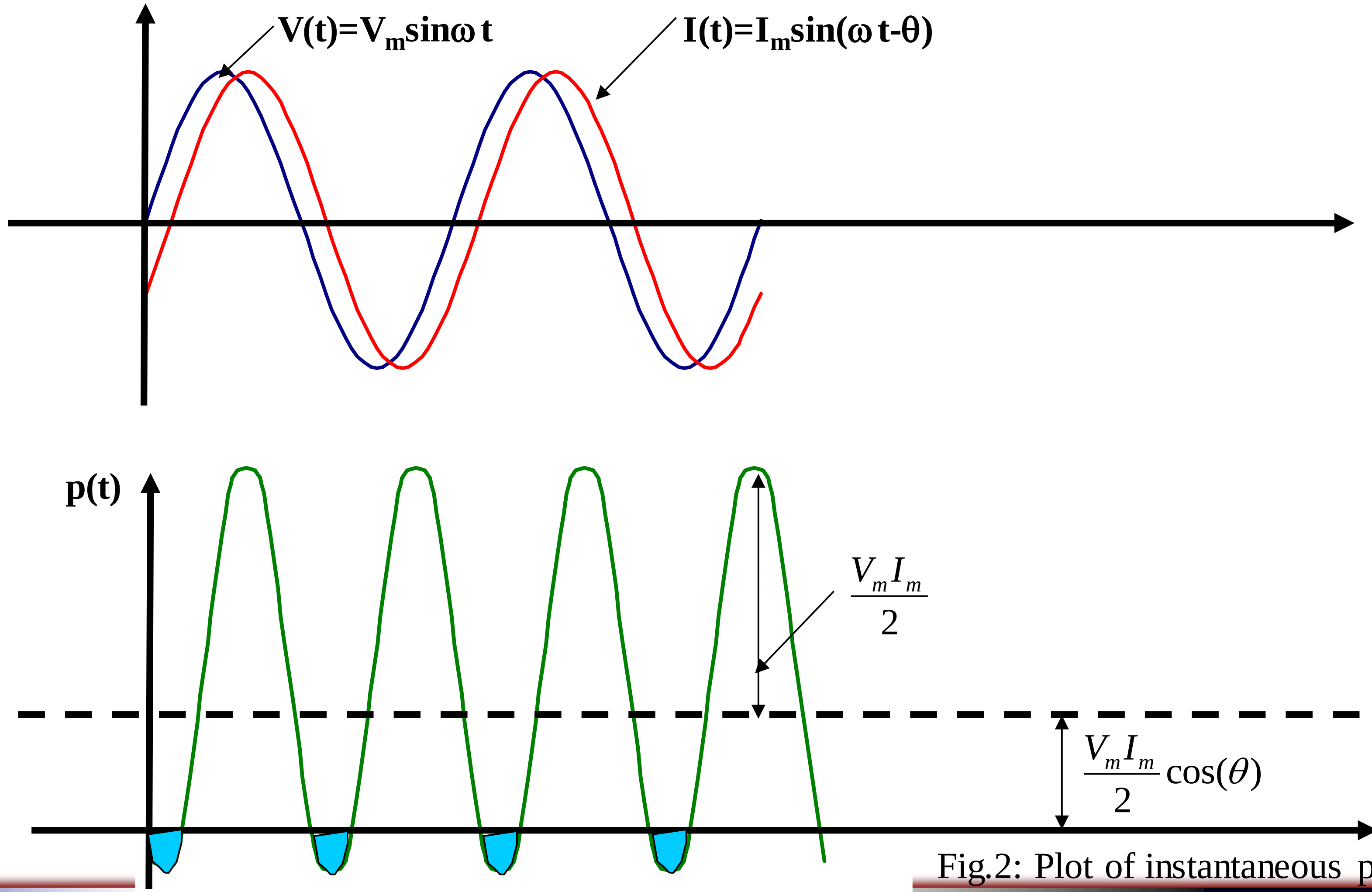

Fig.2: Plot of instantaneous power

Instantaneous Power

- From Fig.2 it can be seen that for a fixed value of angle θ the instantaneous power consists of two components; a constant part and a time varying part.

- The time varying part has a frequency which is ***twice*** that of the voltage and current sinusoids.

- The shaded part of the power in Fig.2 refers to those time intervals when the power is negative.

- The negative power in effect means that the circuit is returning power to the source during these intervals.

- Form Fig.2 it is clear that the positive area of ***p(t)*** curve exceeds the negative area hence, the average power is positive and is equal to the first term in eq.21

Instantaneous Power

- As the angle θ is made smaller and smaller, i.e. as the current ***I*** is brought nearly in phase to the voltage ***v*,** the negative area gets smaller and smaller.

- As θ =0, the current and voltage are in phase, there is no negative area associated with ***p(t)*** curve and all the power in consumed between the circuit branch terminals. This circuit is purely resistive.

- When θ is increased, the negative area increases and less power is consumed by the circuit and more returned to the source.

- At the extreme value of θ, i.e. $\theta=\pi/2$, the ***p(t)*** curve is such that the negative area is equal to the positive area. In this case no power is consumed between the circuit terminals.

Average Power

- The useful quantity in terms of the capability of the circuit to do work is the average power.

- The average power is given by

$$P_{av} = \frac{1}{T}\left[\int_0^T \frac{V_m I_m}{2}\cos\theta dt - \int_0^T \frac{V_m I_m}{2}\cos(2\omega t - \theta)dt\right] \qquad (22)$$

- The second term in eq.22 involves the integration of a **sine** function over a time interval of two period,hence its value is equal to zero.

- The first term is independent of time **t**, the average power is obtained as

$$P_{av} = \frac{V_m I_m}{2}\cos\theta \qquad (23)$$

Average Power

- The eq.23 can also be written as

$$P_{av} = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos\theta = V_{eff} I_{eff} \cos\theta \qquad (24)$$

- Another important term in electric circuit theory can be obtained from eq.24

$$\cos\theta = \frac{P_{av}}{V_{eff} I_{eff}} \qquad (25)$$

- The quantity $\mathbf{P_{av}}$ in eq.25 is the average power in Watts. The denominator involves a quantity whose units are represented by the product of **volts** by **amperes**.

- In case of **direct current** (dc) sources, the units of denominator of eq.25 is watts because it is the **real power** that can be entirely converted to work.

Average Power

- When sinusoidal voltages and current are involved, the denominator of eq.25 does not represent the useful work rather it represents the ***apparent power***.

- The ***apparent power*** is not always realizable in the circuit for doing work. The useful part depends upon the value of ***cosθ*** and because of this ***cosθ*** is called the ***power factor (pf) of the circuit.*** It is expreesed as

$$pf = \cos\theta = \frac{average\ power}{apparent\ power} = \frac{P_{av}}{V_{eff} I_{eff}} \qquad (26)$$

Example 1

- Find the average value of the periodic function shown in Fig.3

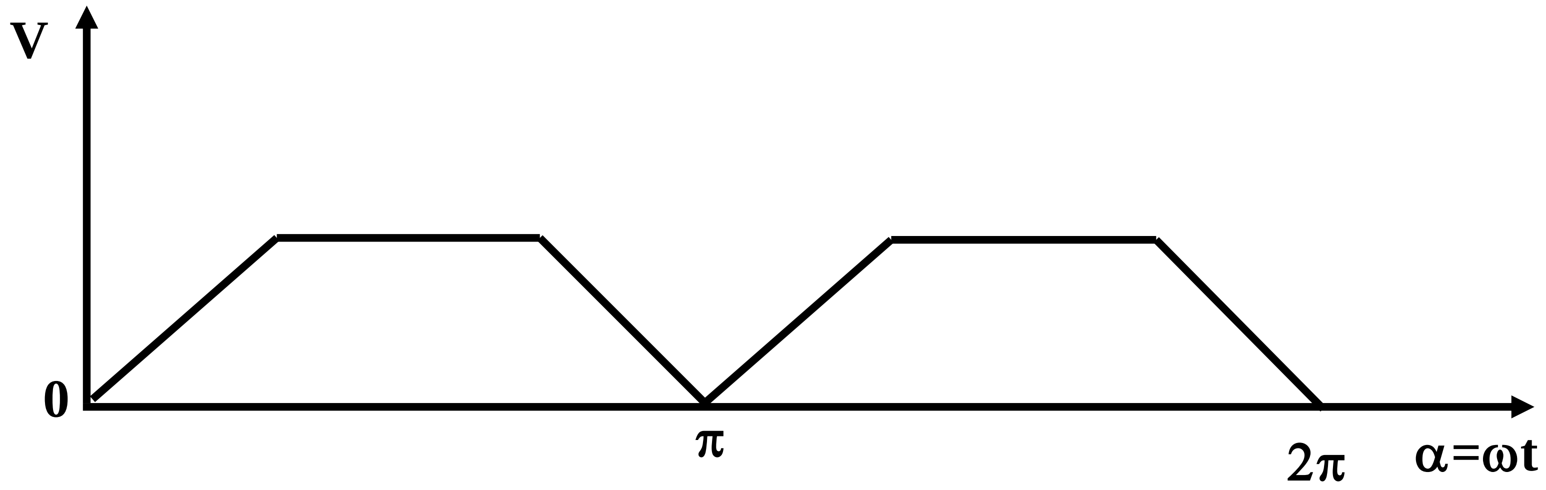

Fig.3: Plot of a periodic function

Example 1

- The entire information about the waveform is contained in the period 0 to π/2.

$$v(t) = \begin{cases} \dfrac{V_m}{\pi/3}(\omega t) = \dfrac{V_m}{\pi/3}\alpha \text{ for } 0 \le \alpha \le \dfrac{\pi}{3} \\ V_m \quad \text{for } \dfrac{\pi}{3} \le \alpha \le \dfrac{\pi}{2} \end{cases}$$

- Hence the average value is obtained as

$$V_{av} = \frac{1}{\pi/2}\left\{\int_0^{\pi/3} \frac{V_m}{\pi/3}\alpha\, d\alpha + \int_{\pi/3}^{\pi/2} V_m\, d\alpha\right\}$$

$$= \frac{1}{\pi/2}\left\{\frac{V_m}{\pi/3}\left[\frac{\alpha^2}{2}\right]_0^{\pi/3} + V_m\left[\alpha\right]_{\pi/3}^{\pi/2}\right\}$$

$$= \frac{V_m}{\pi/2}\left\{\frac{\pi}{6} + \frac{\pi}{6}\right\} = \frac{2}{3}V_m$$

Example 2

- A voltage of $v(t)=170\sin(377t+10^{\circ})$ is applied to a circuit. It causes a steady state current to flow which is given by $I(t)=14.14\sin(377t-20^{\circ})$. Determine the power factor and the average power delivered to the circuit.

Example 2

- A comparison of the expressions for ***v(t)*** and *i(t)* reveals that the relative phase angle is 30º. Hernce,

$$pf = \cos\theta = \cos 30^o = 0.866$$

$$V = \frac{V_m}{\sqrt{2}} = \frac{170}{\sqrt{2}} = 120V$$

$$I = \frac{I_m}{\sqrt{2}} = \frac{14.14}{\sqrt{2}} = 10A$$

$$P_{av} = VI\cos\theta = 120 \times 10 \times 0.866 = 1040W$$