Tutorial - 1: Quantum mechanics

Q.1 According to Boltzmann, a system in contact with surroundings at temperature T and exchanging energy with it, has a probability of having an energy between E and $E + dE$ given by $dp(E)$,

$$dp(E) = \frac{e^{-\frac{E}{k_B T}}}{Z}\ dE$$

where Z is adjusted to make $\int dp(E) = 1$. Planck tried to apply this idea to radiation contained in a cavity, but he added the postulate that energy is discrete since he thought at that time, the walls of the container is capable (for some unknown reason) of absorbing and emitting radiation only in multiples of the energy $h\nu$ where ν is the frequency of radiation and h is a fundamental constant. Hence in this case the energy is $E = nh\nu$ and $E + dE = (n+1)h\nu$. This means that the probability of the system having energy $E_n = nh\nu$ is just $p_n = \frac{e^{-\frac{nh\nu}{k_B T}}}{Z}$ (this makes a lot more sense if $n \gg 1$ in which case there is hardly any difference between n and $n+1$) where Z is adjusted to make $\sum_{n=0}^{\infty} p_n = 1$. Find the average energy $< E >$ for a given frequency ν. The number of states (standing wave modes) per unit volume available for light between frequency ν and $\nu + d\nu$ can be shown to be $d\rho(\nu) = \frac{2\nu^2}{c^2} d\nu$. This means the intensity is $dI(\nu) = c\ d\rho(\nu)\ \ < E >$. Prove Planck's Law from these ideas. What is the corresponding result if you integrated over all possible energies while evaluating $< E >$ (classical result)?

Q.2 Show that both

$$\psi(x,t) = \begin{cases} A\ e^{\frac{i}{\hbar}(p_0\ x - E_0\ t)}, & -L/2 < x < L/2; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

and

$$\psi(x,t) = \frac{h\ sin[\frac{\pi X\ \Delta p)}{h}]}{\pi X\ \sqrt{h\ \Delta p}} \tag{2}$$

where $X = (x - x_0) - \frac{p_0}{m}(t - t_0)$, $E_0 = \frac{p_0^2}{2m}$, $\Delta p > 0$ obey the time-dependent Schrodinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} \tag{3}$$

Which of these two better describes a classical particle moving with speed p_0/m in the x direction and is found at $x = x_0$ at $t = t_0$? Explain by calculating $< p >$ and $< (p- < p >)^2 >$ in both cases.

Q.3 A particle is in a quantum state $\psi(x,t)$. Find a function $\varphi(p,t)$ (in terms of ψ) which has the property that $|\varphi(p,t)|^2\ dp$ is the probability of find the momentum of the particle in state ψ to be between p and $p + dp$.

Q.4 Since $p = -i\hbar\frac{d}{dx}$ prove that the commutator $[x,p] = i\hbar$ where $[A,B] \equiv AB - BA$. We could choose to work in momentum domain where the position becomes an operator $x = i\hbar\frac{d}{dp}$. Show that this choice also obeys $[x,p] = i\hbar$. Now write down the Schrodinger equation for a particle falling in a uniform gravitational field in terms of $\frac{d}{dp}$. Solve for the stationary states.