Department of Electronics and Electrical Engineering

Mid-semester Examination EE 693 Advanced Topics in Random **Processes**

Maximum Marks: 60 Time: 14:00-16:00 hours Date 2**5.9.2021**

Answer all questions. Write answers to parts of a question in the same page or consecutive pages only.

1.(a) Suppose $A_1, A_2, \ldots \in \mathbb{F}$ is a sequence of events. State the two conditions under which the continuity theorem $\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n)$ holds. (4)

(b) Consider the sequence of events given by $A_1 = \{0,4\}, A_2 = \{1,2\}$ and $A_n = \{(-1)^n, 2, 3\}, n > 2$. Find $\lim_{n\to\infty} sup\, A_n$ and $\lim_{n\to\infty} \inf A_n$. Does $\lim_{n\to\infty} A_n$ exist? (6)

2(a). Suppose $\{X_n\}$ is a sequence of independent random variables taking two values $X_n = \sqrt{n}$ with probability $\frac{1}{n}$ and $X_n = 0$ with probability $1\text{-}\frac{1}{n}$. Examine if $\{X_n\}$ converges to $\{X = 0\}$ in probability, in distribution and in the mean-square sense. (9)

(b) Suppose $S = \{s_1, s_2\}$ and $\{X_n\}$ be a sequence of random variables with

$X_n(s_1) = 1 + \frac{1}{n}$ and $X_n(s_2) = (-1)^n$. Find $P(\{s \mid \lim_{n\to\infty} X_n(s) = 1\})$.

Does $\{X_n\} \xrightarrow{a.s.} \{X = 1\}$? (3)

3.(a) Suppose $\{X_n\}$ is a sequence of independent and identically distributed random variables with $P(\{X_n = 2\}) = P(\{X_n = 1\}) = \frac{1}{4}$ and $P(\{X_n = 0\}) = \frac{1}{2}$. Find the limiting value $\mu_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ as $n \to \infty$, by applying (i) the weak law of large number and (ii) the strong law of large numbers. (4)

(b) Suppose S represents the total number of tails obtained in *100* independent tossing of a fair coin. Find $P(48 \le S \le 52)$ using the central limit theorem in terms of the Q function. (Note that the Q funcition is given by $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-\frac{u^2}{2}} du$.) (4)

(c) Suppose $\{X_n\}$ is a sequence of independent random variables with identical moment generating function $M_X(s) = e^{\frac{s^2}{2}}$ and. $S_n = \sum_{i=1}^{n} X_i$. Apply Cramer's theorem to find the the approximate value for $P\left(\frac{S_n}{n} \ge 3\right)$. (4)

4. (a) A WSS random process $\{X(t)\}$ has the autocorrelation function given by

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T}, & \text{if } |\tau| \le T \\ 0 & \text{otherwise.} \end{cases}$$

Examine if $\{X(t)\}$ is m.s. continuous and m.s. differentiable. (4)

(b) Consider a wide-sense stationary random process $\{X(t)\}$ with the mean $\mu_X = 0$ and the autocorrelation function $R_X(\tau) = \dfrac{A^2}{2}\cos\omega_0\tau$. If $Y(t) = X'(t)$ is the mean-square derivative process, find μ_Y and $R_Y(\tau)$ (4)

(c) Suppose $\{X(t)\}$ is a random process defined by $X(t) = Y$ where Y is a ranom variable with mean 0 and a finite variance. Examine if $\{X(t)\}$ is a mean-ergodic process. Is $\{X(t)\}$ wide-sense stationary? (6)

5.(a) Suppose $\{Z_n, n \ge 0\}$ is a sequence of independent and identically distributed random variables with the probability mass function

$p_{Z_n}(0) = 1 - p$ and $p_{Z_n}(1) = p$

It can be shown that the sum $X_n = \sum_{i=0}^{n} Z_i$ can be modelled as a discrete-time Markov chain. Identify (i) the state space V and (ii) the probability transition matrix $\mathbf{P}$ for this Markov chain. (4)

(b) Consider a 2-state homogeneous MC $\{X_n, n \ge 0\}$ with the state space $V = \{0,1\}$. The transition probability matrix and the initial probabilities are given by $\mathbf{P} = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$ and $\mathbf{p}^{(0)} = [0.3\ 0.7]$ respectively.

Find (i) $P(X_1 = 1)$ (ii) $P(X_2 = 1, X_1 = 0, X_0 = 1)$ and (iii) the steady-state probability vector $[\pi_0 \ \ \pi_1] = \lim_{n\to\infty} \mathbf{p}^{(n)}$ (8)