★ Gale Shapley's Algorithm
(Valentine's Day Special)

→ Stable Matching

$M = \{m_1, m_2, \ldots, m_n\}$

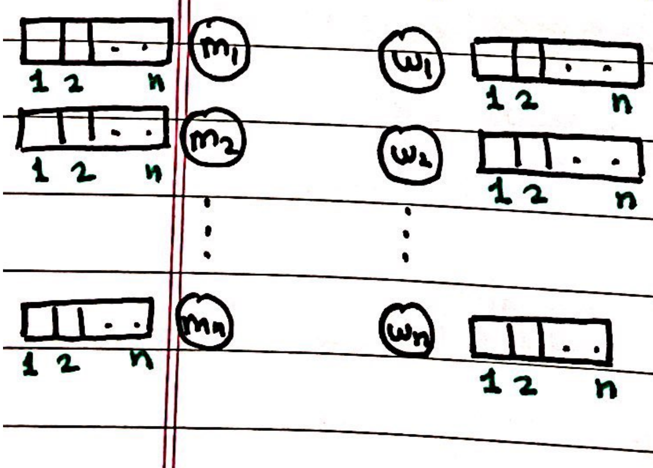
$W = \{w_1, w_2, \ldots, w_n\}$

$M \times W : \{(m, w) \mid m \in M \text{ and } w \in W\}$

★ Matching S is a set of ordered pairs each from $M \times W$ ∋ each member of M & each member of W appears ~~exactly~~ atmost in one pair in S.

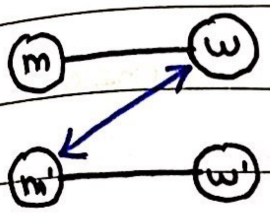
★ Had it been exactly then we have a Perfect matching

★ each man ranks all the women according to his preference. |||'ly each woman ranks all men according to her preference.



★ Prefence list for each man & woman

★ Unstable Matching:



ⓐ → ⓑ : a looking at b

$m' : w > w'$
$w : m' > m$
} Preferences

★ Matching is Stable

⇔ ① It is perfect

② No instability

→ G-S Algo

Initially all $m \in M$ and $w \in W$ are free

While ∃ man m who is free and hasn't proposed to every woman

Choose such a man m

Let w be the ↑ prefered woman by m

whom m hasn't proposed yet   m: | x | x | x | w | |

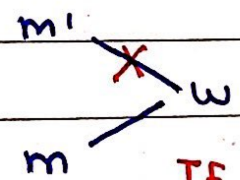
If w is free then

(m,w) are engaged

else (w is currently engaged to m')

If w prefers $m' > m$

m remains free



else (m,w) engaged   If $m > m'$

m' becomes free

End While

★ engaged & matched is same here

★ Once a woman matched she remains matched

★ The algo always outputs a matching.

Proof By contradiction

→ G-S algorithm terminates after $O(n^2)$ iterations.

Proof: Let $P(t)$ denote the set of pairs (m, w) $\ni$ m proposed to w by end of iteration t

But $P(t+1) > P(t)$ always.

But there are only n^2 possible pairs

$\therefore \exists$ $O(n^2)$ iterations.

★ If m is free at some point $\exists$ a woman to whom he hasn't proposed.

Proof: Suppose m has proposed all the n women the each woman is engaged at this point. Since engaged pairs form a matching $\Rightarrow$ there must be n engaged men at this point. Hence #

★ The set S returned at termination is a perfect matching.

Proof:

Let ∃ a free man m

∴ ∃ a women who is not matched (w)

algorithm terminates only after m proposed all the women.

m: $\boxed{\;}\boxed{\;}\boxed{\;}\boxed{w}$ ∴ if m proposed w at any point of time w must be engaged to someo

to m being left.

★ G-S algo gives a Stable matching.

Let ∃ a instability

m —— w   w: m' > m

m' —— w'   m': w > w'

∴ m' must ofo have proposed w before w'

pref: w > w'

If w matched with m (# to given matching)

else

∃ m'' ∈ m'' > m' in w's pref

But w is matched to m finally

∴ Pref: m > m'' > m'

(# to given : m' > m)

★ $\boxed{w \mid w'}$ m w $\boxed{m' \mid m}$

$\boxed{w' \mid w}$ m' w' $\boxed{m \mid m'}$

∃ more than one stable matching for the above example.

→ m — w m ╳ w

m' — w' m' ╳ w'

→ Hence uniqueness fails

★ We will say woman w is a valid partner of man m if ∃ a stable matching with pair (m, w) in it.

→ w is the best valid partner of m if w is a valid partner of m & no woman whom m ranks ↑ than w is a valid partner.

w → best valid partner

X	X	X	X	✓	X	✓	✓	X

★ III'y w: X X X ✓ X ✓ ✓ X X

m → worst valid partner

★ In G-S algo each man gets BEST valid partner and each woman gets WORST valid partner.

Proof:

Let m be the man who is rejected by his best valid partner (m: first such man)

m: × × ✓ × × × ✓✓ × ×

ⓦ : Hence m is rejected by w

m — ~~ⓧ~~ — w → may or may not happen
~~but still the following holds:~~

m'

∴ w: m' > m

But ∃ a stable matching:

m — w

m' — w'

w: m' > m

→ Here we have 2 cases

① m': . . . w . . . w' ⇒ Instability in the above stable matching.

② m': . . . w' . . . w . . .

In G-S

m — w

m' — w'

→ Hence m' would have been rejected by w' before m was rejected by w

to given

→ for women:

m —— w : x x x m x x m' x x

↓ worst valid partner

∃ stable matching

m' —— w   If w doesnt get m'

m —— w'   she must be matched with someone prior ranked man m

m: w > w' (# previous result)

Hence above match unstable (#)

→ No of iterations : $O(n^2)$

Free List: [m] → [ ] → [ ]   (Linked List)

Take the head of the list

(If some man is inserted he is inserted in the front of linked list.)

m [| | w | | |]

First ↑ ~~best~~ unproposed woman

w [| m | | m' | |]

If free match w & m

else compare ranks of m & m': $O(1)$

★ Pre Processing : $O(n^2)$ for all woman

w: [m_3 | m_7 | m_5 | . . . .]   Pref list

[| | 1 | | 2 | | 3 |] : Rank list

m_3 m_7 m_5

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