# Existence and Uniqueness of Solutions to First-Order IVPs

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In this lecture, we shall find answers to the following questions.

- When a solution to an IVP exists?
- If a solution to the IVP exists, Is it unique?
- Under what conditions, a solution to the IVP is unique?
- As initial conditions/ functions vary, how the solutions to the IVP vary? Will it vary continuously?

Example-1.

$$\text{ODE:} \qquad x\,y'(x) = 4y \qquad \text{for } x \in I = [-1,\ 1]\ ,$$
$$\text{IC:} \qquad y(0) = 1\ .$$

Any function $y(x)$ satisfying the ODE $xy'(x) = 4y$ in a neighbourhood of 0 will take value $y(0) = 0$ and hence it will not satisfy $y(0) = 1$. Therefore, this IVP has no solution.

Example-2.

$$\text{ODE:} \qquad y'(x) = 4y \qquad \text{for } x \in \mathbb{R}\ ,$$
$$\text{IC:} \qquad y(0) = 1\ .$$

There is a unique solution to this IVP and is given by $y(x) = e^{4x}$ for $x \in \mathbb{R}$.

Example-3.

$$\text{ODE:} \qquad y'(x) = \sqrt{|y|} \qquad \text{for } x \in \mathbb{R}\,,$$
$$\text{IC:} \qquad y(0) = 0\,.$$

The solutions to this IVP are: $y_1(x) = 0$ for all $x \in \mathbb{R}$ and $y_2(x) = x^2/4$ for $x \geq 0$ and $y_2(x) = 0$ for $x < 0$. Note that, this IVP has two solutions $y_1(x)$ and $y_2(x)$.

Example-4.

$$\text{ODE:} \qquad y' = 3\,y^{2/3} \qquad \text{for } x \in \mathbb{R}\,,$$
$$\text{IC:} \qquad y(0) = 0\,.$$

The solutions to this IVP are:

$$y_c(x) = \begin{cases} 0 & \text{if} \quad x \leq c \\ (x-c)^3 & \text{if} \quad x \geq c \end{cases} \qquad \text{where } c \geq 0\,.$$

For each real number $c \geq 0$, we have a solution $y_c(x)$ to the IVP. Therefore, this IVP has infinitely many solutions.

Observation: Thus, an IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0$$

may have none, precisely one, or more than one solution.

Well-posed IVP: An IVP is said to be well-posed if

- it has a solution,
- the solution is unique and,
- the solution is continuously depends on the initial data $y_0$ and $f$.

# (Cauchy)-Peano Theorem

Let $D$ be bounded domain containing the point $(x_0, y_0)$. Consider the IVP:

$$\text{ODE:} \qquad y' = f(x,\ y) \quad \text{on } D\ , \tag{1}$$

$$\text{IC:} \qquad y(x_0) = x_0\ . \tag{2}$$

Let

$R = \{(x,\ y) \in \mathbb{R}^2 : |x - x_0| \le a,\ |y - y_0| \le b,\ a > 0,\ b > 0\}$ be a closed rectangle in $D$.

Theorem ((Cauchy)-Peano Theorem)

*If $f(x, y) \in C(R)$, then there exists a solution $y(x)$ to the IVP on the interval $|x - x_0| \le h$, where $h = \min(a,\ b/M)$ and $|f(x, y)| \le M$ for all $(x, y) \in R$.*

Note: The Cauchy-Peano's theorem ensures the existence of solutions to the IVP (1)-(2) locally and does not say about the uniqueness of solutions.

Definition (Lipschitz Condition)

Let $f(x, y)$ be a real valued function defined on a bounded domain $D$. We say that $f$ satisfies Lipschitz condition in the (second) variable $y$ with a Lipschitz constant $K$ if

$$|f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2|$$

for any $(x, y_1)$ and $(x, y_2)$ in $D$.

Example: The function $f(x, y) = x|y|$ satisfies Lipschitz condition with respect to the variable $y$ on the bounded domain $D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 < x < 2 \text{ and } -4 < y < 4\}$. For any $(x, y_1)$ and $(x, y_2)$ in $D$, we have

$$|f(x, y_1) - f(x, y_2)| = x||y_1| - |y_2|| \leq 2|y_1 - y_2|.$$

Let $R = \{(x,\ y) \in \mathbb{R}^2\ :\ |x - x_0| \leq a,\ |y - y_0| \leq b\}$, where $a, b > 0$.

Theorem (Picard Theorem)

*If $f(x, y)$ is continuous and satisfies Lipschitz condition in the variable $y$ with Lipschitz constant $K$ on the closed rectangle $R$, then there exists a unique solution $y(t)$ to the IVP*

$$\begin{aligned} \textit{ODE:} \qquad & y' = f(x,\ y) \quad \textit{on } R\ , \\ \textit{IC:} \qquad & y(x_0) = y_0\ . \end{aligned}$$

*on the interval $|x - x_0| \leq h$, where $h = \min(a,\ b/M)$ and $|f(x, y)| \leq M$ for all $(x, y) \in R$.*

By applying the Picard theorem, we can conclude that the solution $y(x) = e^{4x}$ to the IVP of Example-2 ($y' = 4y$ with $y(0) = 1$) is unique in the interval $|x| \leq h = \min(a, (1/4))$.

Note: In Examples 3 and 4, $f(x, y)$ fails to satisfy Lipschitz condition in every neighborhood of the origin.

Corollary to Picard Theorem: If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ is continuous on the closed rectangle $R$, then there exists a unique solution $y(x)$ to the IVP (1)-(2) on the interval $|x - x_0| \leq h$, where $h = \min(a,\ b/M)$ and $|f(x, y)| \leq M$ for all $(x, y) \in R$.

Justification: If $\frac{\partial f}{\partial y} \in C(R)$, then by applying the MVT for the variable $y$, it follows that

$$|f(x, y_1) - f(x, y_2)| \leq \max_{(x,y)\in R} \left|\frac{\partial f}{\partial y}\right| |y_1 - y_2| \leq K\, |y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in R$. Thus, $f$ satisfies Lipschitz condition in the variable $y$ with Lipschitz constant $K$ on $R$. Now, Picard theorem ensures the existence of unique solution to IVP (1)-(2).

Note: There exists a function $f(x, y)$ for which $\frac{\partial f}{\partial y}$ is not continuuous on a closed rectangle $R$, but $f$ satisfies Lipschitz condition in the variable $y$ with Lipschitz constant $K$ on $R$.

Example: Take $f(x, y) = |y|$ for $(x, y) \in R$, where $R = \{(x, y) \in \mathbb{R}^2 \ : \ |x| \leq 1, \ |y| \leq 1\}$.

Note that $\frac{\partial f}{\partial y}$ does not exist at $(x, 0)$. But, $f$ satisfies Lipschitz condition in the variable $y$ with Lipschitz constant $K = 1$.

Example: Let $R \ : \ |x| \leq 5, \ |y| \leq 3$ be the rectangle. Consider the IVP

$$y' = 1 + y^2, \quad y(0) = 0$$

over $R$.

Here, $a = 5, \ b = 3$. Then

$$\max_{(x,y)\in R} |f(x,y)| = \max_{(x,y)\in R} |1 + y^2| \leq 10(= M),$$

$$\max_{(x,y)\in R} \left|\frac{\partial f}{\partial y}\right| = \max_{(x,y)\in R} 2|y| \leq 6(= K).$$

$$\alpha = \min\{a, \frac{b}{M}\} = \min\{5, \ \frac{3}{10}\} = 0.3 < 5.$$

Note that the solution of the IVP is $y = \tan x$. This solution is valid in the interval $|x| \leq 0.3$ in stead of the entire interval $|x| \leq 5$.

# Method to find the solution to the IVP provided by Picard Theorem (Method of Successive Approximations)

Step 1: Set $y_0(x) = y_0$ for all $x \in \mathbb{R}$.
Step 2: Compute

$$y_n(x) = y_0(x) + \int_{x_0}^{x} f(s, y_{n-1}(s))\, ds \qquad \text{for } n = 1, 2, \cdots$$

This $y_n(x)$ is called the Picard Successive Approximation or Picard Iterate.
Step 2: Limit of Iterates

$$y(x) := \lim_{n\to\infty} y_n(x) \qquad \text{for } x \in I = [x_0 - h, x_0 + h]\ .$$

Under the hypothesis of Picard's theorem, $\{y_n(x)\}$ converges uniformly on the interval $I$ and the limit function $y(x)$ is the unique solution of the given IVP (1)-(2) in $I$.

Example. Solve $y' = 2y$ with $y(0) = 1$ by the method of successive approximations.
Here $f(x, y) = 2y$ for $(x, y) \in \mathbb{R}^2$.

Step 1: Initial Approximation
Set $y_0(x) = y_0 = 1$ for all $x \in \mathbb{R}$.

Step 2: Computing Successive Approximations

$$\begin{aligned}
y_1(x) = y_0(x) + \int_{x_0}^{x} f(s,\ y_0(s))\ ds &= 1 + \int_0^x 2\ ds = 1 + 2x\ . \\
y_2(x) = y_0(x) + \int_{x_0}^{x} f(s,\ y_1(s))\ ds &= 1 + \int_0^x 2(1 + 2s)\ ds \\
&= 1 + 2x + 2x^2. \\
y_3(x) = y_0(x) + \int_{x_0}^{x} f(s,\ y_2(s))\ ds &= 1 + \int_0^x 2(1 + 2s + 2s^2)\ ds \\
&= 1 + 2x + 2x^2 + \frac{4x^3}{3}\ .
\end{aligned}$$

## Step 3: Limit of Successive Approximations (if possible compute)

$$y_n(x) = \sum_{k=0}^{n} \frac{(2x)^k}{k!} \to e^{2x} =: y(x) \text{ for } x \in \mathbb{R} \qquad \text{as } n \to \infty \ .$$

Exercise. Solve $y' = 2x - y$ with $y(0) = 1$ by the method of successive approximations.

Answer:

$$y_0(x) = 1; \quad y_1(x) = 1 - x + x^2; \quad y_2(x) = 1 - x + \frac{3x^2}{2} - \frac{x^3}{3}$$

$$y_3(x) = 1 - x + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{x^4}{12}$$

$$y_n(x) = (2x - 2) + \left(3\sum_{k=0}^{n} \frac{(-x)^k}{k!}\right) + (-1)^{n+1}\frac{2x^{n+1}}{(n+1)!}$$

Observe that $y_n(x) \to y(x) := (2x - 2) + 3e^{-x}$ for $x \in \mathbb{R}$.
$y(x) = 2x - 2 + 3e^{-x}$ is the unique solution to the given IVP.

# Continuous Dependence of Solutions on Initial Conditions

Let $f(x, y)$ be continuous and satisfy Lipschitz condition in the variable $y$ with Lipschitz constant $K$ on the closed rectangle $R$. Let $\phi(x, y_0)$ be the unique solution to the IVP:

$$y' = f(x, y) \quad y(x_0) = y_0 \text{ in } |x - x_0| \le h.$$

Let $N_r(y_0)$ be the open neighborhood of $y_0$ with some radius $r > 0$.
Assume that for each $y^* \in N_r(y_0)$, there exists a unique solution $\phi(x, y^*)$ to new IVP:

$$y' = f(x, y) \quad \text{with } y(x_0) = y^* \text{ in } |x - x_0| \le h.$$

Then the function $g : y^* \mapsto \phi(x, y^*)$ is a continuous function at $y_0$. That is, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|y^* - y_0| < \delta \implies |\phi(x, y^*) - \phi(x, y_0)| < \epsilon.$$

That is, the solutions to the IVP depend continuosly on the initial conditions.

## Continuous Dependence of Solutions on functions $f$

Let $f(x, y)$ be continuous and satisfy Lipschitz condition in the variable $y$ with Lipschitz constant $K$ on the closed rectangle $R$.

Let $\phi(x)$ be the unique solution to the IVP: $y' = f(x, y)$ with $y(x_0) = y_0$ in $|x - x_0| \leq h$.

Let $g(x, y)$ be a continuous function such that

$$|g(x, y) - f(x, y)| \leq \epsilon \text{ for all } (x, y) \in R.$$

Let $\psi(x)$ be the unique solution to the IVP: $y' = g(x, y)$ with $y(x_0) = y_0$ in $|x - x_0| \leq h$.
Then the solutions satisfy

$$|\phi(x) - \psi(x)| < \frac{\epsilon\,(e^{Kh} - 1)}{K} \qquad \text{on } |x - x_0| \leq h\ .$$

That is, the solutions to the IVP depend continousl y on the functions $f$.

# Application of Previous Slide

Consider the IVP $y'(x) = f(x, y)$ with $y(0) = 0$, where $f(x, y) = x^2 + y^2 + y + 1$.

Let $\psi(x)$ denote its soluion on $|x - 0| \leq h$.

We can obtain information about the solution $\psi(x)$ in a sufficiently small neighborhood of $(0, 0)$ from the solution $\phi(x)$ to the IVP $y'(x) = y + 1$ with $y(0) = 0$.

Reason: If $x$ and $y$ sufficiently small, then $|(x^2 + y^2 + y + 1) - (y + 1)| = |x^2 + y^2|$ can be made less than any given $\epsilon > 0$ and, hence we can apply the result mentioned in the previous slide to get

$$|\phi(x) - \psi(x)| < \frac{\epsilon\,(e^{Kh} - 1)}{K} \qquad \text{on } |x - 0| \leq h \, .$$

*** End ***