Power Series Solutions to the Bessel Equation

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The Bessel equation

The equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0, \tag{1}$$

where α is a nonnegative constant, is called the Bessel equation of order α.

This equation occurs in problems concerning vibrations of membranes, heat flow in cylinders, and propagation of electric currents in cylindrical conductors. Some of its solutions are known as Bessel functions.

The point $x_0 = 0$ is a regular singular point. We shall use the method of Frobenius to solve this equation.

Thus, we seek solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad x > 0, \tag{2}$$

with $a_0 \neq 0$.

Differentiation of (2) term by term yields

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}.$$

Similarly, we obtain

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Substituting these into (1), we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r}$$
$$+ \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} \alpha^2 a_n x^{n+r} = 0.$$

This implies

$$x^r \sum_{n=0}^{\infty} [(n+r)^2 - \alpha^2] a_n x^n + x^r \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Now, cancel x^r, and try to determine a_n's so that the coefficient of each power of x will vanish.

For the constant term, we require $(r^2 - \alpha^2)a_0 = 0$. Since $a_0 \neq 0$, it follows that

$$r^2 - \alpha^2 = 0,$$

which is the indicial equation. The roots are $r_1 = \alpha$ and $r_2 = -\alpha$. Let us first determine a solution corresponding to the root $r_1 = \alpha$.

Case I. For $r = \alpha$, the equations for determining the coefficients are:

$$[(1+\alpha)^2 - \alpha^2]a_1 = 0 \text{ and,}$$
$$[(n+\alpha)^2 - \alpha^2]a_n + a_{n-2} = 0, \quad n \geq 2.$$

Since $\alpha \geq 0$, we have $a_1 = 0$. The second equation yields

$$a_n = -\frac{a_{n-2}}{(n+\alpha)^2 - \alpha^2} = -\frac{a_{n-2}}{n(n+2\alpha)}. \tag{3}$$

Since $a_1 = 0$, we immediately obtain

$$a_3 = a_5 = a_7 = \cdots = 0.$$

For the coefficients with even subscripts, we have

$$a_2 = \frac{-a_0}{2(2+2\alpha)} = \frac{-a_0}{2^2(1+\alpha)},$$

$$a_4 = \frac{-a_2}{4(4+2\alpha)} = \frac{(-1)^2 a_0}{2^4 2!(1+\alpha)(2+\alpha)},$$

$$a_6 = \frac{-a_4}{6(6+2\alpha)} = \frac{(-1)^3 a_0}{2^6 3!(1+\alpha)(2+\alpha)(3+\alpha)},$$

and, in general

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(1+\alpha)(2+\alpha)\cdots(n+\alpha)}.$$

Therefore, the choice $r = \alpha$ yields the solution

$$y_\alpha(x) = a_0 x^\alpha \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1+\alpha)(2+\alpha)\cdots(n+\alpha)}\right).$$

Remark: The ratio test shows that the power series formula converges for all $x \in \mathbb{R}$.

For $x < 0$, we proceed as above with x^r replaced by $(-x)^r$. Again, in this case, we find that r satisfies

$$r^2 - \alpha^2 = 0.$$

Taking $r = \alpha$, we obtain the same solution, with x^α is replaced by $(-x)^\alpha$. Therefore, the function $y_\alpha(x)$ is given by

$$y_\alpha(x) = a_0|x|^\alpha \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1+\alpha)(2+\alpha)\cdots(n+\alpha)}\right) \tag{4}$$

is a solution of the Bessel equation valid for all real $x \neq 0$.

Case II. For $r = -\alpha$, determine the coefficients from

$$[(1-\alpha)^2 - \alpha^2]a_1 = 0 \ \text{ and } \ [(n-\alpha)^2 - \alpha^2]a_n + a_{n-2} = 0.$$

These equations become

$$(1-2\alpha)a_1 = 0 \ \text{ and } \ n(n-2\alpha)a_n + a_{n-2} = 0.$$

If 2α is not an integer, these equations give us

$$a_1 = 0 \ \text{ and } \ a_n = -\frac{a_{n-2}}{n(n-2\alpha)}, \quad n \geq 2.$$

Note that this formula is same as (3), with α replaced by $-\alpha$. Thus, the solution is given by

$$y_{-\alpha}(x) = a_0|x|^{-\alpha}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1-\alpha)(2-\alpha)\cdots(n-\alpha)}\right), \tag{5}$$

which is valid for all real $x \neq 0$.

Euler's gamma function and its properties: In order to simplify the form of the solutions, we need some properties of Euler's gamma function. For $s \in \mathbb{R}$ with $s > 0$, define $\Gamma(s)$ by

$$\Gamma(s) = \int_{0+}^{\infty} t^{s-1} e^{-t} dt.$$

The integral converges if $s > 0$ and diverges if $s \leq 0$. Integration by parts yields the functional equation

$$\Gamma(s+1) = s\Gamma(s).$$

In general,

$$\Gamma(s+n) = (s+n-1) \cdots (s+1)s\Gamma(s), \text{ for every } n \in \mathbb{Z}^+.$$

Since $\Gamma(1) = 1$, we find that $\Gamma(n+1) = n!$. Thus, the gamma function is an extension of the factorial function from integers to positive real numbers. Therefore, we write

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \quad s \in \mathbb{R}^+.$$

Using this gamma function, we shall simplify the form of the solutions of the Bessel equation. With $s = 1 + \alpha$, we note that

$$(1+\alpha)(2+\alpha)\cdots(n+\alpha) = \frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)}.$$

Choose $a_0 = \frac{2^{-\alpha}}{\Gamma(1+\alpha)}$ in (4), the solution for $x > 0$ can be written

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1+\alpha)} \left(\frac{x}{2}\right)^{2n}.$$

The function J_α defined above for $x > 0$ and $\alpha \geq 0$ is called the Bessel function of the first kind of order α. Note that it has the form x^α times a power series.

When α is a nonnegative integer, say $\alpha = p$, the Bessel function $J_p(x)$ is given by

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p}, \quad (p = 0, 1, 2, \ldots).$$

This is also a solution of the Bessel equation for $x < 0$.

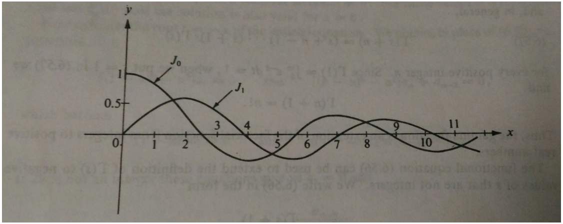

Figure : The Bessel functions J_0 and J_1.

If $\alpha > 0$, $\alpha \notin \mathbb{Z}^+$, define a new function $J_{-\alpha}(x)$ (replacing α by $-\alpha$)

$$J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1-\alpha)} \left(\frac{x}{2}\right)^{2n}, \; x > 0.$$

With $s = 1 - \alpha$, we note that

$$\Gamma(n+1-\alpha) = (1-\alpha)(2-\alpha)\cdots(n-\alpha)\Gamma(1-\alpha).$$

Thus, the series for $J_{-\alpha}(x)$ is the same as that for $y_{-\alpha}(x)$ in (5) with $a_0 = \frac{2^{\alpha}}{\Gamma(1-\alpha)}$, $x > 0$. If α is not positive integer, $J_{-\alpha}$ is a solution of the Bessel equation for $x > 0$.

If α is not an integer, the two solutions $J_{\alpha}(x)$ and $J_{-\alpha}(x)$ are linearly independent on $x > 0$. This yields the following result.

Theorem: If α is not an integer, a general solution of the Bessel equation for $x > 0$ is

$$y(x) = c_1 J_{\alpha}(x) + c_2 J_{-\alpha}(x).$$

For integer α, the Bessel functions $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly dependent, because

$$J_{-\alpha}(x) = (-1)^\alpha J_\alpha(x).$$

If α is a nonnegative integer, say $\alpha = p$, we have found only one solution J_p. We need to look for a second linearly independent solution.
For simplicity, let us take $\alpha = 0$. The Bessel equation takes the form

$$xy'' + y' + xy = 0. \tag{6}$$

The indicial equation has a double root $r = 0$. Thus, the desired solution must have the form

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} A_n x^n.$$

Now, substitute y_2 and its derivatives

$$y_2'(x) = J_0'(x) \ln x + \frac{J_0(x)}{x} + \sum_{n=1}^{\infty} nA_n x^{n-1}$$

$$y_2''(x) = J_0''(x) \ln x + \frac{2J_0'(x)}{x} - \frac{J_0(x)}{x^2} + \sum_{n=1}^{\infty} n(n-1)A_n x^{n-2}.$$

into the differential equation (6) to obtain

$$2J_0' + \sum_{n=1}^{\infty} n(n-1)A_n x^{n-1} + \sum_{n=1}^{\infty} nA_n x^{n-1} + \sum_{n=1}^{\infty} A_n x^{n+1} = 0,$$

where we have used the fact J_0 is a solution of (6). In the above equation, use the series

$$J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n\, x^{2n-1}}{2^{2n}\,(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-1}(n!)(n-1)!}$$

to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-2} n!(n-1)!} + \sum_{n=1}^{\infty} n^2 A_n x^{n-1} + \sum_{n=1}^{\infty} A_n x^{n+1} = 0.$$

Equating the coefficients to zero, we get

$$A_1 = A_3 = A_5 = \cdots = A_{2n+1} = 0,$$

$$A_2 = \frac{1}{4}, \quad \frac{(-1)^{n+1}}{2^{2n}(n+1)!\, n!} + (2n+2)^2 A_{2n+2} + A_{2n} = 0, \ (n = 1, 2, \cdots).$$

and hence,

$$\begin{aligned} A_{2n} &= \frac{(-1)^{n-1}}{2^{2n}(n!)^2} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right], \quad n = 1, 2, \cdots. \\ &= \frac{(-1)^{n-1}}{2^{2n}(n!)^2} h_n, \text{ where } h_n = \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right]. \end{aligned}$$

Thus,

$$\begin{aligned} y_2(x) &= J_0(x)\ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{2n}(n!)^2} h_n x^{2n} \\ &= J_0(x)\ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \cdots \end{aligned}$$

The functions J_0 and y_2 are linearly independent solutions to (**??**). It has been customary to choose a certain special linear combination of J_0 and y_2. Of course, another basis is obtained if we replace y_2 by an independent particular solution of the form $a(y_2 + bJ_0)$, where $a(\neq 0)$ and b are constants. With $a = \frac{2}{\pi}$ and $b = \gamma - \ln 2$, where γ is called Euler's constant and is defined by

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) \approx 0.5772.$$

The standard particular solution thus obtained is known as the Bessel function of the second kind of order zero and is denoed by $Y_0(x)$. Thus

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2}+\gamma\right)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}h_n}{2^{2n}(n!)^2}x^{2n}\right],$$

where h_n is defined as before.
The general solution of (6) for $x > 0$ is given by

$$y(x) = c_1 J_0(x) + c_2 Y_0(x).$$

The Bessel function of the second kind of order α: If α is a nonnegative integer, say $\alpha = p$, we have only one solution J_p. A second linearly independent solution is obtained as follows.
Recall, if y_1 is a nonzero solution of $y'' + p_1 y' + p_2 y = 0$ on an interval I, a second solution y_2 independent of y_1 is given by

$$y_2(x) = y_1(x) \int_c^x \frac{e^{-\int p_1(t)dt}}{(y_1(x))^2} dt.$$

For the Bessel equation, $p_1(x) = 1/x$. Thus, a second solution y_2 is given by the formula

$$y_2(x) = J_p(x) \int_c^x \frac{1}{t(J_p(t))^2} dt, \tag{7}$$

if $c, x \in I$ in which $J_p \neq 0$.

This second solution can be put in other forms. Using

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1+\alpha)} \left(\frac{x}{2}\right)^{2n},$$

we may write

$$\frac{1}{(J_p(t))^2} = \frac{1}{t^{2p}} g_p(t),$$

where $g_p(0) \neq 0$. In the interval I the function g_p has a power-series expansion

$$g_p(t) = \sum_{n=0}^{\infty} A_n t^n$$

which could be determined by equating coefficients in the identity $g_p(t)(J_p(t))^2 = t^{2p}$. Assuming the existence of such an expansion, the integrand in (7) takes the form

$$\frac{1}{t(J_p(t))^2} = \frac{1}{t^{2p+1}} \sum_{n=0}^{\infty} A_n t^n.$$

Integrating term by term from c to x, we find that

$$\begin{aligned}\int_c^x \frac{1}{t(J_p(t))^2}\,dt &= \int_c^x \frac{1}{t^{2p+1}}\sum_{n=0}^{\infty} A_n t^n\,dt,\\ &= A_{2p}\ln x + x^{-2p}\sum_{n=0}^{\infty} B_n x^n.\end{aligned}$$

Therefore.

$$y_2(x) = A_{2p}J_p(x)\ln x + J_p(x)x^{-2p}\sum_{n=0}^{\infty} B_n x^n,$$

where $A_{2p} \neq 0$. Multiplying $y_2(x)$ by $\frac{1}{A_{2p}}$, the resulting solution, denoted by $Y_p(x)$, has the form

$$Y_p(x) = J_p(x)\ln x + x^{-p}\sum_{n=0}^{\infty} C_n x^n.$$

This is the form of the solution promised by Frobenius.

One can verify that a solution of the form actually exists by substituing the right-hand member in the Bessel equation and determine the coefficients C_n so as to satisfy the the equation. The details of the calculation are lengthy. The final result can be expressed as

$$\begin{aligned} Y_p(x) &= J_p(x)\ln x - \frac{1}{2}\left(\frac{x}{2}\right)^{-p}\sum_{n=0}^{p-1}\frac{(p-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n} \\ &\quad -\frac{1}{2}\left(\frac{x}{2}\right)^{p}\sum_{n=0}^{\infty}(-1)^n\frac{h_n+h_{n+p}}{n!(n+p)!}\left(\frac{x}{2}\right)^{2n}, \quad x>0 \end{aligned}$$

where $h_0 = 0$ and $h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ for $n \geq 1$.

This series converges for all real x. The function $Y_p(x)$ is called the Bessel function of the second kind of order p. The general solution for $x > 0$ is

$$y(x) = c_1 J_p(x) + c_2 Y_p(x).$$

Useful recurrence relations for J_α

- $\frac{d}{dx}(x^\alpha J_\alpha(x)) = x^\alpha J_{\alpha-1}(x)$.

$$
\begin{aligned}
\frac{d}{dx}(x^\alpha J_\alpha(x)) &= \frac{d}{dx}\left\{ x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\alpha+n)} \left(\frac{x}{2}\right)^{2n+\alpha} \right\} \\
&= \frac{d}{dx}\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2\alpha}}{n!\,\Gamma(1+\alpha+n) 2^{2n+\alpha}} \right\} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2\alpha) x^{2n+2\alpha-1}}{n!\,\Gamma(1+\alpha+n) 2^{2n+\alpha}}.
\end{aligned}
$$

Since $\Gamma(1+\alpha+n) = (\alpha+n)\Gamma(\alpha+n)$, we have

$$
\begin{aligned}
\frac{d}{dx}(x^\alpha J_\alpha(x)) &= \sum_{n=0}^{\infty} \frac{(-1)^n 2 x^{2n+2\alpha-1}}{n!\,\Gamma(\alpha+n) 2^{2n+\alpha}} \\
&= x^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+(\alpha-1)+n)} \left(\frac{x}{2}\right)^{2n+\alpha-1} \\
&= x^\alpha J_{\alpha-1}(x).
\end{aligned}
$$

Exercise:

- $\frac{d}{dx}(x^{-\alpha}J_\alpha(x)) = -x^{-\alpha}J_{\alpha+1}(x)$.
- $\frac{\alpha}{x}J_\alpha(x) + J'_\alpha(x) = J_{\alpha-1}(x)$.
- $\frac{\alpha}{x}J_\alpha(x) - J'_\alpha(x) = J_{\alpha+1}(x)$.
- $J_{\alpha-1}(x) + J_{\alpha+1}(x) = \frac{2\alpha}{x}J_\alpha(x)$.
- $J_{\alpha-1}(x) - J_{\alpha+1}(x) = 2J'_\alpha(x)$.

*** End ***