Set Theory

→ A set is a collection of objects called its members

→ A set is an object.

$a \in A$ $\quad a \notin A$.

<u>Principle of Extensionality</u> :- For any 2 sets A & B if for every object x, $x \in A$ iff $x \in B$

then $A = B$.

<u>Empty set</u> :- A set with no members exists

— empty set $\emptyset$

$\boxed{\emptyset \neq \{\emptyset\}}$

★ If every member of A is also a member of B then

$A \subseteq B$ subset.

and $B \supseteq A$ superset.

if $A \neq B$ and $A \subseteq B$ then $A \subset B$, $A \subsetneq B$

(A is a proper subset of B).

$A \cup B$: the set of all of objects in A or in B

$A \cap B$: ____________________ and in B

$U - A$: ____________________ not in A, but in U

★ A stands for a predicate $\alpha(x)$

$x \in A$

↳ First order predicate with one variable.

$A = \{x \mid \alpha(x)\}$

$A \cup B$: $\{x \mid \alpha_A(x) \vee \alpha_B(x)\}$

\# $\cup : \vee$ (union corresponds to ~~and~~ OR)

$\cap : \wedge$ (intersection corresponds to AND)

$U^- : \neg$ (complement corresponds to NOT)

$\cup, \cap, U^-$] Define Boolean algebra

Properties:

① $A \cap U = A$ $\quad A \cup \phi = A$ (Identity)

$x \wedge 1 = x$ $\quad x \vee 0 = x$

② $A \cup U = U$ $\quad A \cap \phi = \phi$ (Domination)

③ $A \cup A = A$ $\quad A \cap A = A$ (Idempotent)

④ $U - (U - A) = A$ (Double negation law)

⑤ $A \cup B = B \cup A$ (commutative)

$A \cap B = B \cap A$

⑥ $A \cup (B \cup C) = (A \cup B) \cup C$ (associative)

$A \cap (B \cap C) = (A \cap B) \cap C$

⑦ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

⑧ $U - (A \cap B) = \cancel{A\phi}(U - A) \cup (U - B)$ (Demorgan's law)

$U - (A \cup B) = (U - A) \cap (U - B)$

⑨ $A \cup (A \cap B) = A$ (Absorption)

$A \cap (A \cup B) = A$

⑩ $A \cup (U - A) = U$ (Negation)

$A \cap (U - A) = \phi$

Power set:

The set of all subset of A

$2^A \qquad \mathcal{P}(A)$

is the power set of A.

$|A| = n$

then $|2^A| = 2^n$

$\bigcup A = \{x \mid \exists b (b \in A \wedge x \in b)\}$

↳ union of all members of members of A

e.g. – $\bigcup \{\underbrace{\{2, 3\}}_{\text{set}}, \underbrace{4}_{\text{Not a set}}, \{5\}\} = \{2, 3, 5\}$

$\bigcap A = \{x \mid \forall b \in A (x \in b)\}$

↳ every member of A has to be set. x should belong to every set of A.

Embedding of Natural no.s in Set Theory

→ successor operator

for set a $\quad a^+ = a \cup \{a\}$

$a = \{2,3,4\}$

$a^+ = \{\{2,3,4\}, 2,3,4\}$

→ Inductive:

A set A is inductive
if $\phi \in A$
and $\forall a \, (a \in A \to a^+ \in A)$

A is closed under the successor operator

$\omega = \bigcap \{A \mid A \text{ is inductive}\}$

$x \in \omega$ iff x belongs to every inductive set.

∴ ω is a subset of every inductive set.

---

* ω is inductive

$\phi \in \omega$

$a \in \omega \to a$ belongs to all ISs

$a^+ \in \omega \leftarrow a^+$ belongs to all ISs

* $x \subseteq \omega \wedge x$ is Inductive $\longrightarrow x = \omega$

OR
No proper subset of ω is inductive.

* ω is the smallest inductive set.

How ω looks like?

$0 = \phi \in \omega$ $\qquad \phi^+ = \phi \cup \{\phi\} = \{\phi\}$

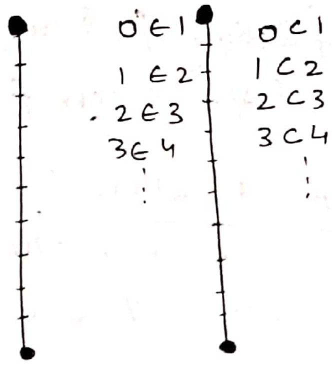
$1 = 0^+ = \phi^+ \in \omega$ $\qquad \phi^{++} = \phi^+ \cup \{\phi^+\} = \{\phi\} \cup \{\{\phi\}\}$

$2 = 1^+ = 0^{++} = \phi^{++} \in \omega$ $\qquad = \{\phi, \{\phi\}\}$

⋮

we defined these sets as natural no.

$0 = \phi$
$1 = \{\phi\} = \{0\}$
$2 = \{\phi, \{\phi\}\} = \{\phi, 1\}$
$3 = \{0, 1, \{0,1\}\} = \{0,1,2\}$
$\vdots$
$n = \{0, 1, 2, \ldots, n-1\}$



For any two distinct objects x, y
$\{x,y\}$ — unordered pair

<u>Ordered pair.</u>
(x,y) : $\{x, \{x,y\}\}$

A Relation is a set of OPs
dom $R = \{x \mid \exists y \; (x,y) \in R\}$
ran $R = \{y \mid \exists x \; (x,y) \in R\}$

<u>field of R</u>
fld $R = (\text{dom } R) \cup (\text{ran } R)$

$A \times B$: The set of all OPs s.t.
the first is from A, the 2nd is from B.

★ R from A to B is a subset of $A \times B$
★ A binary relation R on A is a subset of $A \times A$.

Extending to n elements:
$(a,b,c) = ((a,b),c)$
$(a_1, \ldots, a_n) = ((a_1, \ldots a_{n-1}), a_n)$

n-ary relation:
$(A_1 \times \ldots \times A_n) = (A_1 \times \ldots A_{n-1}) \times A_n$

If $A_1, A_2 \ldots A_n = A$ then the n-ary relations can be written as:- A^2, A^3, ...

~~A function f is a relation s.t.~~
~~$(\forall x \in \mathrm{dom} f)(\exists! y\ (x,y) \in f)$~~

A Function F

is a relation

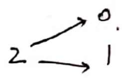
s.t. $\forall x \in \mathrm{dom} F\ (\exists! y\ xFy)$

$F(x)$: the unique y st. xFy.

F maps A into B, if $\mathrm{dom} F = A$ and $\mathrm{ran} F \subseteq B$

F maps A onto B, if $\mathrm{dom} F = A$ and $\mathrm{ran} F = B$

★ A set R is single Rooted
if for each $y \in \mathrm{ran}\ R$
$\exists! x$ s.t. $(x,y) \in R$.



$\{(2,0), (2,1), \epsilon\}$ $\cancel{(3,0)}$
is single Rooted.

★ A function that is 1-rooted is 1-1

Bijection = one-one, onto

Injection = one-one, into

Surjection = onto

bijection iff injection and surjection

$F^{-1} = \{(u,v) \mid (v,u) \in F\}$

$F \circ G = \{(u,v) \mid (u,t) \in G$ and $(t,v) \in F$ for some $t\}$

the Restriction of set F to set A

$F \restriction A = \{(u,v) \mid (u,v) \in F \wedge u \in A\}$

$F[\![A]\!]$ image of A under F

$= \mathrm{ran}\ F \restriction A = \{v \mid \exists u \in A\ (u,v) \in F\}$

$F \circ G(u)$ G: u→ F: t→

1) For a set F;

$$\text{dom } F^{-1} = \text{ran } F$$
$$\text{ran } F^{-1} = \text{dom } F$$

2) For a relation F,

$$(F^{-1})^{-1} = F$$

3) For a set F, F^{-1} is a function iff F is 1-rooted.

4) if F is a 1-1 function,

if $x \in \text{dom } F$, then $F^{-1}(F(x)) = x$

if $y \in \text{ran } F$, then $F(F^{-1}(y)) = y$

5) if F and G are functions, then $F \circ G$ is a function

[There $\circ$ - composition is an operator on set of functions. (it is closed)]

its domain :-

$$\{x \in \text{dom } G \mid G(x) \in \text{dom } F\}$$

for $x \in \text{dom } F \circ G$

$$(F \circ G)(x) = F(G(x))$$

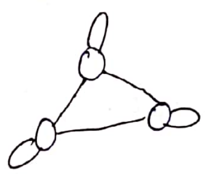
6) For any two sets F and G

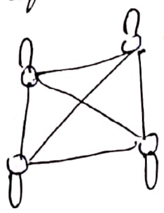
$$(F \circ G)^{-1} = G^{-1} \circ F^{-1}$$

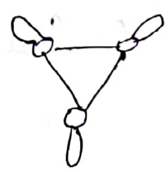
- A binary Rel^n ~~on~~ R on A is <u>reflexive</u> if $\forall x \in A\,(xRx)$
- A binary Rel^n R on A is <u>symmetric</u> if $\forall x, y \in A\,(xRy \rightarrow yRx)$
- A binary Rel^n R on A is <u>transitive</u> if $\forall x, y, z \in A\,(xRy \wedge yRz \rightarrow xRz)$

R is an equivalence relation if R satisfies all the above three properties.

Graphical interpretation of equivalence rel^n :-







equivalence class

clique are being formed.

If there is a path b/w x and y then there is a direct edge b/w them also.

equivalence classes are being formed.

e.g. $xRy \equiv x$ is $y \pmod 5$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19

$\{0, 1, 2, 3, 4\}$

$[0]$
$\equiv \bmod 5$

$x \equiv x \pmod 5$ ~~(symmetric)~~ (reflexive)

if $x \equiv y$ then $y \equiv x \pmod 5$ (symmetric)

if $x \equiv y$, $y \equiv z \rightarrow x \equiv z \pmod 5$ (transitive)

Trichotomy is a binary Relation R

s.t. xRy, $x = y$, yRx

exactly one holds $\forall\, x, y \in A$

e.g. $<$ ✓ $\quad \leq$ ✗

If a Relation is transitive and trichotomy : strict linear order

Antisymmetry property :

$\forall xy\ (xRy \wedge yRx \rightarrow x = y) \rightarrow$ e.g. $\leq$

reflexive, antisymmetric, transitive : linear order
total order

Successor

$a^+ = a \cup \{a\}$

inductive set : contains ϕ, closed under $^+$

$w = \bigcap \{\text{all inductive sets}\}$.

- w is inductive
- w is a subset of every inductive set.

ω is the "smallest" inductive set.

$\omega = \mathbb{N}$

ϕ 0 $\qquad n = \{0, 1, 2, \cdots, n-1\}$

$\Rightarrow \quad 0 \in 1 \in 2 \in 3 \ldots$

ϕ^{+} 1 $\qquad 0 \subset 1 \subset 2 \subset 3 \ldots$

$1^{+} = \phi^{++}$: 2

successor : $+1$

$m + n = m^{+(n)}$

$m \times n = $ + operator applied on m itself $(n-1)$ times.

$m \times 1 = m \qquad m \times 2 = m^{+(m)}$

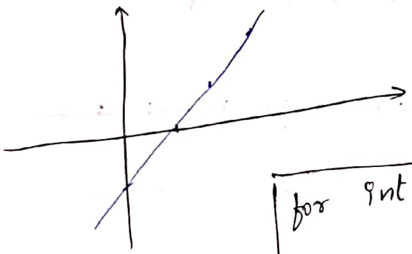
Integers in Set Theory

$\sim$ on $\mathbb{N} \times \mathbb{N}$

$(m, n) \sim (p, q)$ iff $m \dotplus q = n \dotplus p$ equivalence rel^n

$(4,3)\ (3,2)\ (2,1)$: 1

$(1,2)$: -1



$x - y = 1$

$y = x - 1$: represent integer 1

$\boxed{\text{for int } i \text{ :- } \quad y = x - i}$

Integer addition :-

$+_{\mathbb{Z}}$: $[(m,n)]_\sim +_{\mathbb{Z}} [(p,q)]_\sim \longrightarrow [(m+p, n+q)]_\sim$

$\qquad m - n \qquad\qquad p - q \qquad\qquad m + p - n - q$

$\times_{\mathbb{Z}}$: $[(m,n)]_\sim \times_{\mathbb{Z}} [(p,q)]_\sim \longrightarrow [(mp +_{\mathbb{N}} nq, pn +_{\mathbb{N}} mq)]_\sim$

$\star\ \mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim$

$+_{\mathbb{Z}}$

$[(m,n)]_\sim +_{\mathbb{Z}} [(p,q)]_\sim = [(m+p, n+q)]_\sim$

$[(m,n)]_\sim \times_{\mathbb{Z}} [(p,q)]_\sim = [(mp +_{\mathbb{N}} nq, pn +_{\mathbb{N}} mq)]_\sim$

Natural Number multiplication

Scanned by CamScanner

$\times_{\mathbb{Z}}$ is commutative, associative

$\times_{\mathbb{Z}}$ distributive ove $+_{\mathbb{Z}}$

$[(2,1)]_{\sim} = 1_{\mathbb{Z}}$ is id of $\times_{\mathbb{Z}}$

$[(1,1)]_{\sim} = 0_{\mathbb{Z}} \neq 1_{\mathbb{Z}}$

$(\mathbb{Z}, +_{\mathbb{Z}}, \times_{\mathbb{Z}}, 0_{\mathbb{Z}}, 1_{\mathbb{Z}})$ Form an integral domain.

★ $[(m,n)]_{\sim} <_{\mathbb{Z}} [(p,q)]_{\sim} \longrightarrow$ Trichotomy and Transitive $\Rightarrow$ strict linear order.

when $m +_{N} q \in n +_{N} q'$

$m - n < p - q$

$m +_{N} q <_{N} p +_{N} n$

$\Rightarrow m +_{N} q \in p +_{N} n$ (w.r.t natural numbers)

★ $\mathbb{Z}$ is countable

(S is countable iff $\exists$ a 1-1 mapping from S into $\mathbb{N}$)

One-One and onto mapping

Rational Number

$q = \frac{p}{q}$ where $p, r \in \mathbb{Z}$ $r \neq 0$

$\mathbb{Z}' = \mathbb{Z} - \{0_{\mathbb{Z}}\}$

$(a,b) \bowtie (c,d)$ $a, c \in \mathbb{Z}$

bow tie $b, d \in \mathbb{Z}'$.

stands for stands for

$\frac{a}{b} = \frac{c}{d}$

iff $a \times_{\mathbb{Z}} d = b \times_{\mathbb{Z}} c$

$(a,b) \bowtie (a,b)$

$(a,b) \bowtie (c,d) \rightarrow (c,d) \bowtie (a,b)$

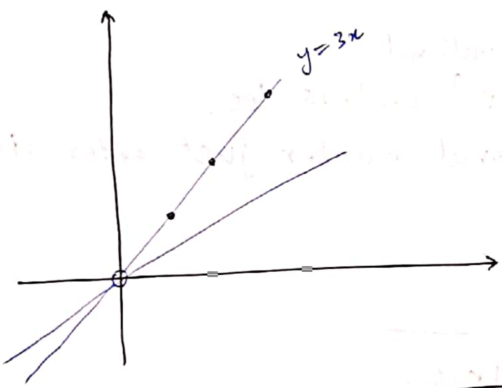
If $(a,b) \bowtie (c,d)$ and $(c,d) \bowtie (e,f)$ then $(a,b) \bowtie (e,f)$

$$(\mathbb{Z} \times \mathbb{Z}')/\!\bowtie \; = \mathbb{Q}$$

$$\frac{1}{3} \equiv \{(1,3),(2,6),(3,9),\ldots,\ (-1,-3),(-2,-6),\ldots\}$$

$$3 \equiv \{(3,1)(6,2)(9,3),\ldots,\ (-3,-1),(-6,-2),(-9,-3),\ldots\}$$

In general $p/q \equiv \{(p,q),(2p,2q),(3p,3q),\ldots,\ (-p,-q),(-2p,-2q)\ldots\}$



$\frac{1}{3} \equiv \{(x,y)$ that satisfies $y=3x\}$

$+_{\mathbb{Q}}$: $[(a,b)]_{\bowtie} +_{\mathbb{Q}} [(c,d)]_{\bowtie} = [(ad+bc,\ bd)]_{\bowtie}$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

Integer addition and multiplication

$\times_{\mathbb{Q}}$: $[(a,b)]_{\bowtie} \times_{\mathbb{Q}} [(c,d)]_{\bowtie} = [(ac,bd)]_{\bowtie}$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

$(\mathbb{Q}, +_{\mathbb{Q}}, \times_{\mathbb{Q}}, 0_{\mathbb{Q}}, 1_{\mathbb{Q}})$ field

---

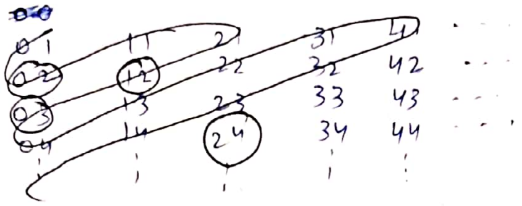
$[(a,b)]_{\bowtie} <_{\mathbb{Q}} [(c,d)]_{\bowtie}$ iff $ad <_{\mathbb{Z}} cb$

$\left(\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc\right)$

Trichotomy and Transitive
$\rightarrow$ Strict linear Order

$\mathbb{Q}$ is countable

Consider $\quad N \times N^+$



(when we count in this manner, we are going through every rational number in 1st quadrant.

but we need a 1-1 mapping, i.e., a number should be counted only once.

→ First assign numbers in the pattern as above. then remove the numbers/mapping for equal numbers except when encountered first.

Now we need to count -ve rational numbers too;
introduce the -ve rational number just after its positive equivalent.

Hence $\mathbb{Q}$ is countable.

There are irrational numbers.

$\sqrt{2}$ is irrational

$\sqrt{2} = \frac{p}{q} \qquad \gcd(p, q) = 1$

$p = \sqrt{2}q$

$p^2 = 2q^2 \quad \Rightarrow \quad p^2$ is an even square $\Rightarrow p$ is also even

$\Rightarrow \quad p^2 = (2k)^2 = 4k^2$

thus $\quad 4k^2 = 2q^2$

$2k^2 = q^2 \quad \Rightarrow \quad q^2$ is an even number

$\Downarrow$

(q is also even)

thus p and q both are even

$\Rightarrow$ they have 2 as a common factor

contradiction to the fact that $\gcd(p, q) = 1$

Real Numbers Done using Dedekind cuts

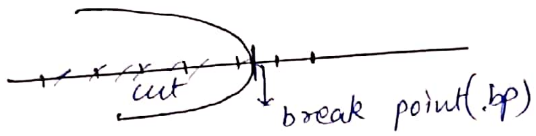
Dedekind Cut :- is a subset of $\mathbb{Q}$ such that

(1) $\phi \neq x \neq \mathbb{Q}$

(2) x is closed downwards

$\forall\, q, r \in \mathbb{Q}\ ((q \in x) \wedge (r < q) \rightarrow (r \in x))$.

(3) x has no largest member belonging to $\mathbb{Q}$



$3 \equiv$ the set of all rational number less than 3

Every dedekind cut is a Real number.

Set of all dedekind cut is equal to set of all ~~rationa~~ Real Numbers

R is uncountable.

$(0,1)$

1	$0.a_1\ a_2\ a_3\ a_4 \ldots$
3	$0.b_1\ b_2\ b_3\ b_4 \ldots$
4	$0.c_1\ c_2\ c_3\ c_4 \ldots$

construct a number as:-

$0.\underbrace{(a_1+1) \bmod 10}_{\text{1st digit is not equal to the 1st rational}}\ \overset{2^{nd}}{0}\ \underbrace{(b_3+1) \bmod 10}_{\text{3rd digit is not equal to the 3rd number}}$

In this way, we can find that the number that we constructed has never been counted.

thus $(0,1)$ is uncountable

$\Rightarrow \mathbb{R}$ is uncountable

Dedekind cut (continued)

if the break point (bp) is rational ;
then the complement of the cut includes the b.p.

★ The set of all Dedekind cuts $\equiv R$

\# $r <_R s$ iff. $r \subset s$

$\Rightarrow$ strict linear order

\# $x +_R y = \{ q+r \mid q \in x$ and $r \in y \}$.

\# $x \times_R y = \{ qr \mid 0 \leq q \in x \wedge 0 \leq r \in y \} \cup 0_R$

– if x and y are non negative real

\# If both x and y are negative real, then

$x \times_R y = |x| \times_R |y|$

\# $x \times_R y = -|x| \times_R |y|$

★ $(R, +_R, \times_R, 0_R, 1_R)$ is a field.

Naive set Theory

– By Georg Cantor

$\{x \mid a(x)\}$ → this is the base for defining sets.

→ this has many problems and paradoxes:–

One of them is — (Berry's Paradox)

$A = \{ x \mid x \in N$ and x can not be defined in at most 100 characters $\}$

$n =$ the smallest member/natural no. not defined ≤ 100 char

Scanned by CamScanner

Russell's Paradox

$S = \{ x \mid x \notin x \}$

Abnormal: self membership. (Paradox)

$S \in S$ iff $S \notin S$

* Russels Type theory

* Zermelo Fraenkel Axioms of set Theory * :-

① Extensionality Axiom

$\forall x \forall y \, (\forall z \, (z \in x \leftrightarrow z \in y) \rightarrow x = y \}$

② empty set Axiom

$\exists x \, \forall y (y \notin x)$

③ Pairing Axiom

$\forall x \forall y \, \exists z \, \forall u \, (u \in z \leftrightarrow u = x \vee u = y)$

given any two elements, we can pair them into a set (an unordered pair)

④ Power set Axiom

$\forall x \, \exists y \, \forall z \, (z \in y \longleftrightarrow z \subseteq x)$

⑤ Subset Axiom Schema

for any formula α with free variable $t_1 \ldots t_k$ u and x the following is an axiom

$\forall t_1 \ldots \forall t_k \; \forall u \; \exists x \; \forall y \left(y \in x \longleftrightarrow y \in u \wedge \alpha(t_1, \ldots t_k, u, x) \right)$

= for every set u, there is a subset x s.t. $y \in x$ iff $y \in u \wedge \alpha(t_1, \ldots t_k, u, x)$

* e.g. $\{ x \mid x \notin x \}$ ⇒ the above axiom says that you cannot talk about self membership in the complement in unrestricted manner.

$\forall t \, \forall u \, \exists x \, \exists y \, (y \in x \leftrightarrow y \in t \wedge y \in u)$

there exist intersection t and u

(6) <u>Union Axiom</u>:- there exist y that is made of members of member of x

(7) <u>Infinity Axiom</u> :- there exist an inductive set

(8) <u>Regularity Axiom</u> :- Every non-empty set x has a member y s.t. $x \cap y = \phi$

For any formula $\mathcal{V}(x,y)$ in which z is not free

if every member of x has a nominee then there is a set made up of these nominees.

(9) <u>Axiom of choice</u> : For any relation R, there is a function $F \subseteq R$ s.t. dom F = dom R

<u>Relation R on A</u> :-

is reflexive if $\forall x \in A (x R x)$

antisymmetric if $\forall x, y \in A \; (x R y \wedge y R x \rightarrow x = y)$

transitive if $\forall x, y, z \in A (x R y \wedge y R z \rightarrow x R z)$

then R is partial order on A

A is a partially ordered set (poset) w.r.t. R

e.g. $(N, \leq)$

For any $x, y \in A$

$x R y \vee y R x$

Not necessarily

* $a \precsim b$ a precedes b

or b succeeds a.

$b \succsim a$ $\succsim = \precsim^{-1}$ } dual

<u>Trichotomy</u>:   <u>aRb</u> xor <u>bRa</u> xor <u>$a=b$</u>

* irreflexive, transitive :- quasi order
* quasi order with trichotomy :- strict partial order

$a \prec b \equiv a \precsim b$ and $a \neq b$

* <u>e.g.</u> $\subseteq$ on 2^A $\longrightarrow$ partial order

$A = \{a, b\}$

$2^A = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$\{a\} \not\subseteq \{b\}$, $\{a\} \not\supseteq \{b\}$

<u>e.g.</u> Divides on N^+ $\longrightarrow$ partial order

$x \mid x$

$x \mid y \wedge y \mid x \rightarrow x = y$

$x \mid y \wedge y \mid z \rightarrow x \mid z$

<u>Hasse diagram</u>

S is a poset

$a, b \in S$.

a is an immediate predecessor of b $(a \ll b)$

if $a < b$ and $\neg\exists c\, (a < c < b)$

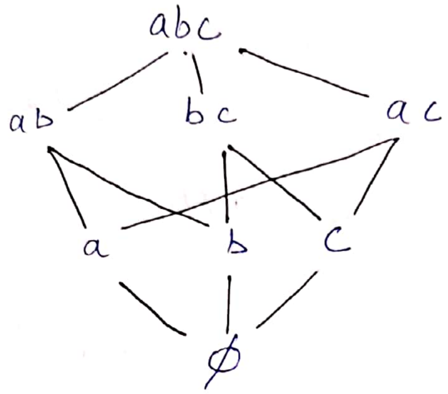
b is an immediate successor of a $(b \gg a)$

Directed Graph

Take a vertex for each $a \in S$

if $a << b$ then construct $(a, b) \in E$

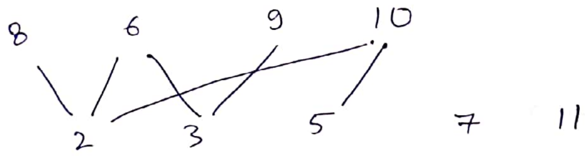
$(a \to b)$



we can draw all order pairs using transitivity

we do not draw a direct edge from a to abc ; but there is an intermediate path via ab or ac.

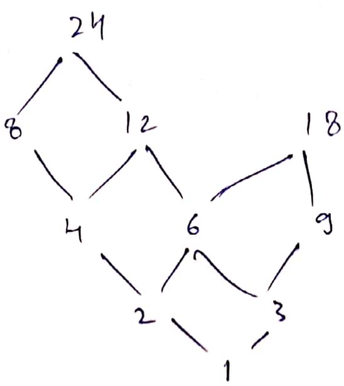
e.g. Divide Relation :-



since $\mathbb{N}$ ~~as~~ has infinite ~~diagram~~ elements, Hasse diagram seems useless.

But when there are finite elements, Hasse diagram seems good to implement.

e.g. Restriction of $|$ onto $\{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$



a and b are comparable in R if

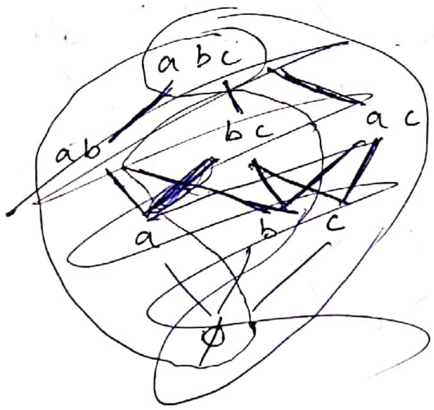
aRb or bRa. (and can have a path)

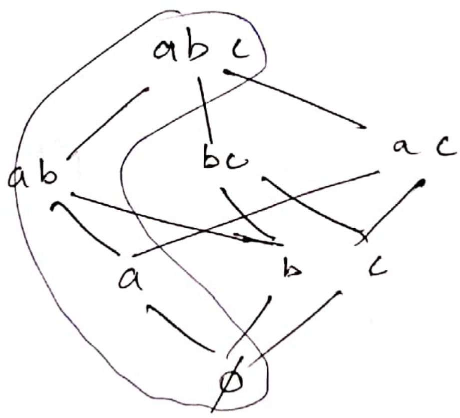
12 and 18 are incomparable $\underline{12 \parallel_R 18}$

★ Total order / Linear order has every pair comparable while this need not be a case in proper partial order.

★ every subset of linear order is a linear order.

But —
A linear order taken from a partial order through restriction is a chain of the POSET.





<u>Anti- Chain:</u> of a POSET is a subset — no two elements of which are comparable

Ordering of ordered tuples

1. Product order

S, T are both linearly ordered sets.

$S \times T$ $\quad (a, b) \leq (a', b')$ iff

$a \leq a'$

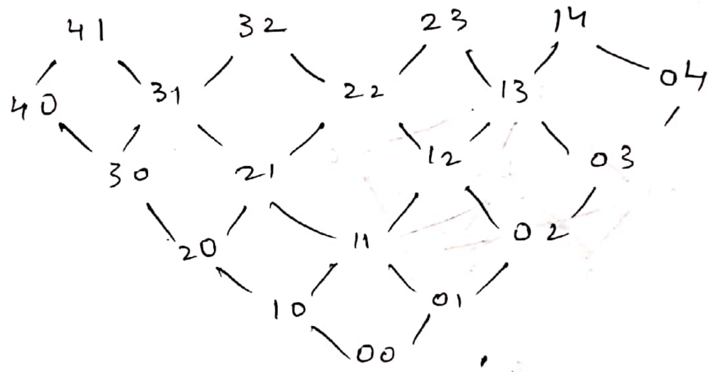
and $b \leq b'$

41 $\quad$ 32 $\quad$ 23 $\quad$ 14 $\longrightarrow$ Not comparable

$4 \geq 3$

but $1 \leq 2$ $\quad \times$

But



2. Lexicographic order

$(a, b) < (a', b')$

If $\quad a < a' \quad$ or $\quad a = a'$ and $b < b'$

Linear order

20
⋮
1n
⋮
12
11
10
⋮
0n
⋮
03
02
01
00

Strings :- A finite sequence of symbols

alphabet $A = \{a, b, c\}$

$$A^* = \left\{ \begin{array}{lll} \epsilon & \rightarrow \text{string of zero length} & \\ a, b, c & \rightarrow \text{unit length.} & (3 \times 1) \\ aa, ab, ac & & \\ ba, bb, bc & \rightarrow 2 \text{ unit length} & (3 \times 3) \\ ca, cb, cc & & \\ & - 3 \text{ unit length} & (3 \times 3 \times 3) \end{array} \right\}$$

A^* → kleene closure of A

Lexicographic order :

\# $u = a u'$ $a, b \in A$

$v = b v'$

$u, v, u', v' \in A^*$

$u < v$ iff $a < b$ or $a = b$ and $u' < v'$

$\epsilon < w$ for any non empty w

e.g. cat < catch

at < atch

t < tch

ϵ < ch

Short lex. order

$u < v$ if $|u| < v$

or $|u| = |v|$

and u precedes v lexicographically

Finite poset S

Draw the Hasse diagram

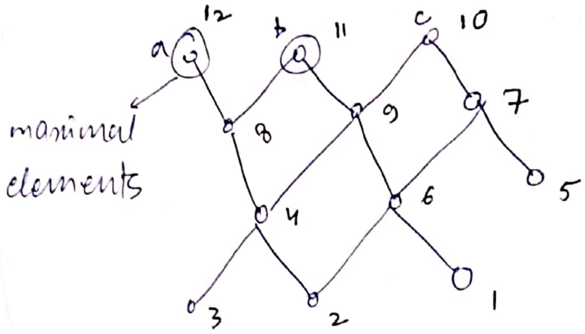
it is a Directed Acyclic Graph

Topographically Sorted

$f : S \rightarrow N$ st $a < b$ then $f(a) < f(b)$

Maximal Element

$a \in S$ is maximal if ~~...~~ $\neg\exists x\,(x \geq a)$



a b c
↓
all 3 are maximal, we can pick any of these three

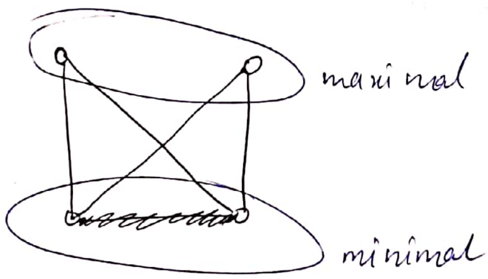
Pick a maximal → name is n
then rest can be numbered.

Basis :- $n = 1$

→ Consistent enumeration of partially ordered set.

Minimal element :-

if $\neg\exists x \in A\ (x \leq A)$



Chain :- A subset of the poset that is in Linear Order

Antichain :- A subset of the poset s.t. no 2 members of it are comparable.

if a poset has a chain of length n
then it has n disjoint anti-chain

→ Remove all the maximal length(then they will form an antichain)
Now the path with n nodes will loose 1 node

⇒ By induction they will have n-1 antichains

⇒ we have a total of $\underline{n-1+1}$ antichain

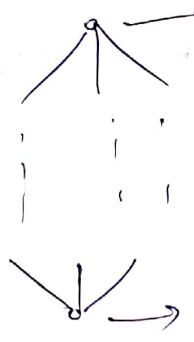
If a finite poset has $nm+1$ elements atleast then either it has a chain of length $>n$ or it has an antichain of ~~length~~ size $>m$.

Proof $\Rightarrow$ Assume that it is wrong $\rightarrow$
suppose that it has the maximum
chain length $= n$
then we can have a total of $n-1$ antichains
If each antichain has elements $\le m$
then total elements $\le m(n)$
$\le nm$

But total number of elements $= nm+1$
thus we have a contradiction.

The First element of a poset is an element a s.t. $\forall x \in S \ (a \le s)$

If the number of minimal elements $= 1$, then that elements is the first element,
similarly of Last element.



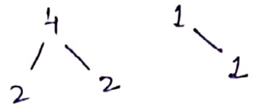
e.g. integer $\rightarrow m \in N^+$
suppose $m = m_1 + m_2 + \dots + m_k \quad m_1, \dots, m_k \in N^+$
assume $m_1 \ge m_2 \ge \dots \ge m_k$
then we say the sequence $m_1, m_2, \dots m_k$ is a partition of m

$P_1 \lesssim P_2$ → If P_1 is a refinement of P_2

(Take P_2, split some of ~~integers~~ integers)

e.g. $m = 5$

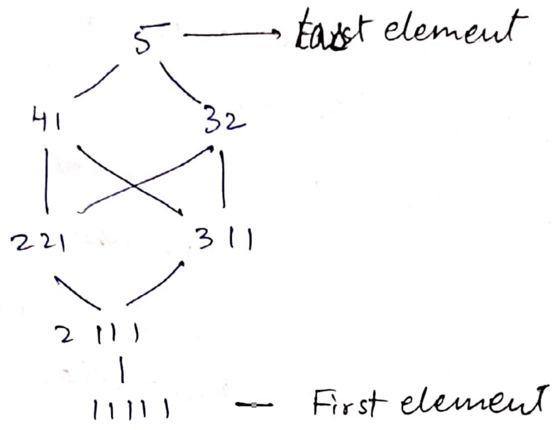
⇒ $5 = 4 + 1$ 41



$5 = 2 + 2 + 1$ 221

Then say :- $221 \lesssim 41$ ★

Partial order:



M is an upper bound of $A \subseteq S$,

if $\forall x \in A \; (M \gtrsim x)$

$\{41, 32\} \longrightarrow 5$

↓ upper bound

m is a lower bound of $A \subseteq S$

if $\forall x \in A \; (m \lesssim x)$

$\underbrace{\{221, 311, 2111, 11111\}}_{LB} \longrightarrow \{41, 32\} \xrightarrow{UB} 5$

$\{221, 311\} \xrightarrow{UB} 41, 32, 5$

$\xrightarrow{LB} 2111, 11111$

Supremum (A):-

LUB / Least Upper Bound (A)

is an UB of A that precedes every UB of A

$\{41, 32\} \xrightarrow{LUB} 5$

$\{221, 311\} \xrightarrow{LUB} 41, 32, 5$

$\xrightarrow{LUB}$ Doesn't ~~t~~ exist (41 and 32 are not comparable.)

Infimum (A) :-

GLB / Greatest Lower Bound (A)

is a LB of A that succeeds every LB of A

$\{221, 311\} \xrightarrow{LB} 2111, 11111$

$\xrightarrow{GLB} 2111$

$\{41, 32\} \xrightarrow{LB} 221, 311, 2111, 11111$

$\xrightarrow{GLB}$ Doesn't exist (221 and 311 are not comparable)

e.g. $(\mathbb{N}^+, |)$

the set of UB (a, b) = the set of all common multiples

LUB (a, b) = LCM (a, b)

GLB (a, b) = ~~#~~ GCD (a, b)

the set of LB (a, b) = the set of all common divisors of a and b.

Isomorphic Posets

$f : X \to Y$ 1 − 1

$a \leq a' \to f(a) \leq f(a')$

if $a \| a'$ then $f(a) \| f(a')$

An ordered set is well ordered if every subset has a first element.

\# A well ordered set is a LO

Every subset of a WOS is a WOS.

if X is WO they $Y \simeq X$ (isomorphic)

then Y is also WO.

* All WOS, of size n are isomorphic to $\{1, 2, \dots n\}$.

* If S is WO, then every $a \in S$ other than the last of S has an imm. successor

Proof :- take any element; say a
then take the set of all elements which succeeds a;
then this will be a subset of S.
and since S is WO, this is also WO
Hence it will have a minimal
that will be the immediate successor of a.

Lattices

A lattice is poset in which every pair a, b has a GLB and a LUB.

→ Set of Natural numbers under division.

L is a non-empty set, closed under meet ($\wedge$) and join ($\vee$)

L is a ~~la~~ lattice if

(L1) $a \wedge b = b \wedge a$
$a \vee b = b \vee a$ } commutative

(L2) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
$a \vee (b \vee c) = (a \vee b) \vee c$ } associative.

(L3) $a \wedge (a \vee b) = a$
$a \vee (a \wedge b) = a$ } Absorptive

Swapping of $\wedge$ with $\vee$ in any theorem will also give us a theorem.

Dual Theorems.

$a \wedge a = a \wedge (a \vee (a \wedge b))$ L3 (b)

$= a$ L3(a)

$a \vee a = a$ (Idempotent Law)

* $(L, \wedge, \vee)$

$a \leq b$ iff $a \wedge b = a$

Theorem :-

if L is a lattice $a \wedge b = a$ iff $a \vee b = b$

and $\leq$ is a partial order (PO)

(reflexive, antisymmetric and transitive)

Proof :-

$\Rightarrow$ $b = b \vee (b \wedge a)$ L 3b

$= b \vee (a \wedge b)$

$= b \vee a$

$= a \vee b$

$\boxed{b = a \vee b}$ H.P.

$\Leftarrow$ If $a \vee b = b$

to show - $a \wedge b = a$

$a = a \wedge (a \vee b)$ L3a

$\boxed{a = a \wedge b}$ H.P.

thus $a \leq b$ iff $a \wedge b = a$

$a \vee b = b$

To Prove now :- $(L, \leq)$ is a partial order

~~Thalle~~

① $\forall a \in L (a \wedge a = a) \Rightarrow \forall a \in L (a \leq a)$

thus $\leq$ is reflexive on L

② $a \leq b$ and $b \leq a$

$\Rightarrow$ $a \wedge b = a$ and $b \wedge a = b = a \wedge b$

$\Rightarrow$ $a \wedge b = a$ as well as $a \wedge b = b$

$\Rightarrow$ $a = b$

thus $\leq$ is antisymmetric on L

③ $a \leq b$ and $b \leq c$

$\Rightarrow$ $a \wedge b = a$ and $b \wedge c = b$

Now: $a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a$

$\Rightarrow$ $a \leq c$

thus $\leq$ is transitive on L

Hence $(L, \leq)$ is a partial order

<u>Claim</u>: LUB and GLB exist for any $a, b \in L$

Consider :- (a, b)

if $a \leq b$ then $GLB(a,b) = a$

and $LUB(a,b) = b$

if $a \geq b$ then $GLB(a,b) = b$

and $LUB(a,b) = a$

If $a \parallel b$ then let $c = a \wedge b \neq a, b$

$a \wedge c = a \wedge (a \wedge b)$

$= (a \wedge a) \wedge b$

$= a \wedge b$

$= c$ $\Rightarrow$ $c \leq a$

the defn :- $a \leq b$ iff $a \wedge b = a$ provides the required ordering such that meet comes down in Hasse diagram.

By similar argument :-

$b \wedge c = c \Rightarrow c \leq b$

Now c is a common lower bound of both a and b.
Now we want to show that c is infact the greatest
Lower bound.

→ say some y is also a LB of a & b
$y \leq a$ and $y \leq b$
$c \wedge y = (a \wedge b) \wedge y = a \wedge (b \wedge y) = a \wedge y = y$.
$\Rightarrow y \leq c$
thus every other LB should precede c;
Hence C → GLB.

Hence :- If a || b then GLB = $a \wedge b$

Hence we can say that for any (a,b) in L

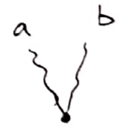
GLB : $a \wedge b$
LUB : $a \vee b$

Let P be a poset with GLB and LUB defined for all $a, b \in P$
$a \wedge b =$ GLB (a, b)
$a \vee b =$ LUB (a, b)

set is closed under $\wedge$ and $\vee$.

Now we need to show that the 3 properties of lattice are defined.

Commutativity :- $a \wedge b = b \wedge a$
GLB(a, b) = GLB(b, a)

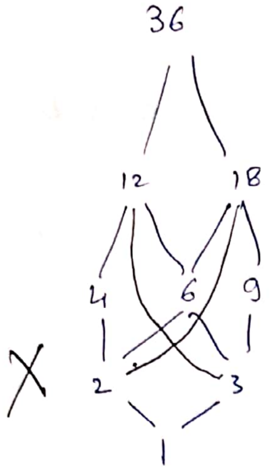


associativity :-

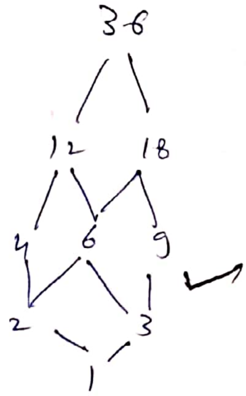
Try This !! absorptive :-

Scanned by CamScanner

D_{36} :- 1, 2, 3, 4, 6, 9, 12, 18, 36



join $\longrightarrow$ only immediate members



★ $(D_{36}, |)$ is a sublattice of $(N^+, |)$

Isomorphic : $f(a \cap b) = f(a) \wedge f(b)$

Bounded lattice:-

L has a LB 0, $\forall x \in L \ (0 \leq x)$

L has an UB 1, I; $\forall x \in L \ (x \leq 1)$

If L has a LB and an UB, it is bounded.

What if L has finite elements?

Can we say that it has a LB and UB.

Consider :-

$$a_i \wedge \underbrace{(a_1 \wedge a_2 \dots \wedge a_n)}_{b} = b$$

$\Rightarrow b \leq a_i$ for every i

thus b is a LB

similar argument for UB.

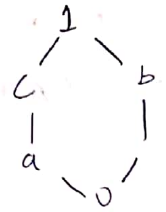
Distributive Lattice

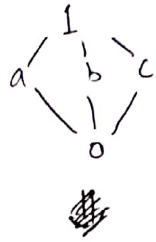
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Scanned by CamScanner

Non distributive lattice :-





If a Lattice has a part isomorphic to any of these two, then the lattice is Non - distributive lattice

$\Downarrow$

$a \vee (b \wedge c) = a \vee 0$
$= a$

$(a \vee b) \wedge (a \vee c)$
$= 1 \wedge c = c$

both are not equal

Similarly for the 2^{nd} :-

$a \vee (b \wedge c) = a \vee 0 = a$

But $(a \vee b) \wedge (a \vee c) = 1 \wedge 1 = 1$

Complements

L is bounded LB − 0
UB − 1

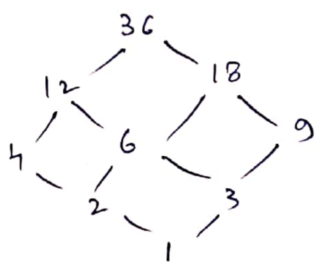
for $a \in L$, the complement $\bar{a}$ of a
$x = \bar{a}$ iff $a \vee x = 1$ and $x \wedge a = 0$

Theorem :- If L is a bounded distributive lattice, then Complements are unique (if they exist)

and L is a complemented lattice, if L is bounded and every element has a complement

* $(L, \wedge, \vee)$ is distributive.
if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



$\wedge$ GCD
$\vee$ LCM.
(Divides)

Q→ $(a, b, c) \rightarrow GCD(a, b, c) = g$

$a = Ag \qquad b = \hat{B} g \qquad c = \hat{C} g$

$= BGg \qquad = CGg$

$GCD(\hat{B}, \hat{C}) = G$

$GCD(B, C) = 1$

$GCD(A, B, C) = 1$

- $a \underline{LCM} (b \underline{GCD} c) = Ag \underline{LCM} (BGg \underline{GCD} CGg)$

$= Ag \underline{LCM} Gg$

$= g \times LCM(A, G) = gAG$

- $a \underline{LCM} b = Ag \underline{LCM} BGg = ABGg$

- $a \underline{LCM} c = ACGg = AGg$

- $a \underline{GCD} (b \underline{LCM} c) = a \underline{GCD} (BGg \underline{LCM} CGg)$

$= Ag \underline{GCD} (BCGg)$

$= g \times GCD(A, BC)$

$a \underline{GCD} b = Ag \; GCD \; BGg$

$= g \times GCD(A, BG)$

$= g \times GCD(A, B)$

$= g \times GCD(A, C)$

Scanned by CamScanner

For any lattice L, 0 is LB, 1 is UB

for $a \in L$ $\quad \underbrace{\bar{a} = x}_{\text{complement of } x}$ iff $a \vee x = 1$ and $a \wedge x = 0$

complements may not exist.

e.g.

without complements.

complements may not be unique,

Theorem :- Let L be a distributive lattice, complements are unique if they exist.

Proof : Let x and y are two complements of a same a

$$a \vee x = a \vee y = 1$$

$$a \wedge x = a \wedge y = 0$$

Now $\quad x = x \vee 0 = x \vee (a \wedge y)$

$$= (x \vee a) \wedge (x \vee y)$$

$$= 1 \wedge (x \vee y)$$

$$= x \vee y$$

similarly $\quad y = y \vee x$; we have $x \vee y = y \vee x$

By commutativity of $\vee$

$$\Rightarrow \boxed{x = y} \qquad \text{H.P.}$$

Lattice L, Lower Bound 0

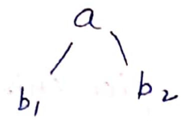
$a \in L$ is join irreducible, if $a = x \vee y \rightarrow a = x \vee a = y$

(if p is a prime
if $p = x \times y \rightarrow p = x \vee p = y$)

0 is Join irreducible

$\Rightarrow 0 = x \vee y \longrightarrow x = 0 \vee y = 0$

if a has 2 immediate predecessors -

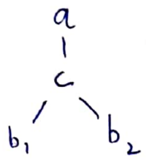


$a = b_1 \vee b_2$

$a \neq b_1 \quad a \neq b_2$

} a is not join irreducible

if a has only 1 immediate predecessor



if $a = x \vee y$

$\rightarrow x = a \vee y = a$

} a is join irreducible

★ a has ≥ 2 immediate predecessors $\Rightarrow$ Not join irreducible

a has < 2 Immediate predecessors $\Rightarrow$ join irreducible

---

★ Immediate successors of 0 are called atoms
(In any divisibility lattice, the atoms are prime numbers)

$\longrightarrow$ ★ All prime powers are also join irreducible.

Finite lattice L

$a \in L$ is not J.I.

$a = b_1 \vee b_2$ where $b_1 \neq a$ and $b_2 \neq a$

Further if both b_1 and b_2 are not J.I.

then $a = c_1 \vee c_2 \vee c_3 \vee c_4$

we can continue like this until we find an exp. of the form $\rightarrow$

$a = d_1 \vee d_2 \vee \dots \vee d_n$

where d_i is J.I.

$\longrightarrow$ every non JI element can be expressed as a join of JI elements.

If $\left.\begin{array}{l} d_i \leq d_j \\ d_i \vee d_j = d_j \end{array}\right\}$ d_i is reduntant.

Let us get rid of all such d_i's

then we will have

$a =$ join of irreduntant J.I. elements

(In fact there is a unique such factorisation)

---

*Let L be a distributive lattice, Then every $a \in L$ can be written uniquely (except for order) as the join of irreduntant JI elements.

Proof :-

$a = b_1 \vee \dots \vee b_r = c_1 \vee \dots \vee c_s$

b's and c's are irredundant and JI

$b_i \leq (b_1 \vee \dots \vee b_r) = a = c_1 \vee \dots \vee c_s$

$b_i = b_i \wedge a = b_i \wedge (c_1 \vee \dots \vee c_s)$

$= \underbrace{(b_i \wedge c_1) \vee (b_i \wedge c_2) \dots (b_i \wedge c_s)}$

$b_i = b_i \wedge c_j$ $\qquad$ b_i is J.I. and it is written as join of different elements.

$\Rightarrow b_i \leq c_j$

Similarly we can argue that $c_j \leq b_k$

Then by transitivity we have $b_i \leq c_j \leq b_k$

$\Rightarrow b_i \leq b_k$

but we removed all reduncencies

$\Rightarrow$ $i = k$

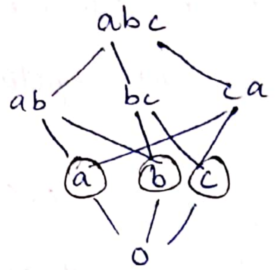
thus $\begin{array}{l} b_i \leq c_j \\ c_j \leq b_i \end{array}$ $\Rightarrow b_i = c_j$

Hence b_i is identical to some c_j

Hence for every b_i there is a unique c_j (which is equal to b_i itself).

Theorem:- Let L be a complemented lattice with unique complements.

Then the J.I. elements of L other than 0 are its atoms.



Proof:- Say a is JI $a \neq 0$ and a is not an atom

Then a has a unique predecessor b $b \neq 0$

then $\bar{b}$ exists. (unique ; property of L)

$b \neq 0 \longrightarrow \bar{b} \neq 1$

① If $a \leq \bar{b}$ then $b \leq a \leq \bar{b}$

$\longrightarrow b \leq \bar{b}$

$\longrightarrow b \vee \bar{b} = \bar{b} = 1$ (contradiction)

thus $a \not\leq \bar{b}$

$\Rightarrow$ Now $a \wedge \bar{b}$ must strictly preced a.

If b is the imm. pred. of $a \longrightarrow a \wedge \bar{b}$ precedes b

$a \wedge \bar{b} \leq \inf(b, \bar{b}) = b \wedge \bar{b} = 0$

$\Rightarrow a \wedge \bar{b} = 0$ # $a \vee b = a$

$a \vee \bar{b} = (a \vee b) \vee \bar{b} = a \vee (b \vee \bar{b}) = a \vee 1 = 1$

$\Rightarrow \bar{a} = \bar{b}$ $\Rightarrow a = b$

(but b is immediate predecessor of a thus contradiction)

Discrete Numeric Functions

$f : N \rightarrow Z$

$\langle a_0, a_1, \ldots \rangle$

e.g. $1, 1, 1, 1, \ldots$

$1, 2, 3, 4, \ldots$

$1, -2, 3, -4, 5, -6, \ldots$

e.g. $a_r = 7r^3 + 1, \quad r \geq 0$

$$b_r = \begin{cases} 2r & 0 \leq r \leq 11 \\ 3^r - 1 & r > 11 \end{cases}$$

$\bar{a} = \langle a_0, a_1, \ldots \rangle$
$\bar{b} = \langle b_0, b_1, \ldots \rangle$ notations.

① $c = a + b = \langle a_0 + b_0, a_1 + b_1, \ldots \rangle$

② $a \times b = \langle a_0 b_0, a_1 b_1, \ldots \rangle$

③ $|a| = \langle |a_0|, |a_1|, \ldots \rangle$

*④ $S^i a = \langle \underbrace{0, 0, \ldots 0}_{i}, a_0, a_1, \ldots \rangle$ ⟼ i shifted (Right)

⑤ $S^{-i} a = \langle a_i, a_{i+1}, \ldots a_{i+2}, \ldots \rangle$ ⟼ (Left shifted)

⑥ forward difference

$\Delta a = \langle a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots \rangle$

backward diff.

$\nabla a = \langle 0, a_1 - a_0, a_2 - a_1, \ldots \rangle$

$$\boxed{\nabla a = S^1(\Delta a)}$$

⑦ $c = a * b$ convolution

$c_r = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \ldots a_{r-2} b_2 + a_{r-1} b_1 + a_r b_0$

$$= \sum_{i=0}^{r} a_i b_{r-i}$$

Scanned by CamScanner

DNF $a = \langle a_0, a_1, a_2, \ldots \rangle$

$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$

$A(z) \longrightarrow$ the generating fn of a

$a = \langle 1, 1, 1, 1, \ldots \rangle$

then $A(z) = 1 + z + z^2 + \ldots z^j \ldots$

$$= \frac{1}{1-z}$$

$a = \langle 1, 3, 3^2, 3^3, \ldots \rangle$ $\quad 1 + 3z + 3^2 z^2 \ldots = \frac{1}{1-3z}$

$a = \langle 7.\cancel{1}, 7.3, 7.3^2, \ldots \rangle$ $\quad \frac{7}{1-3z}$

$a_r = 3^{r+2}, r \geq 0 \quad \Rightarrow \quad \frac{9}{1-3z}$

if $c = a + b$

$C(z) = A(z) + B(z)$

e.g. $a_r = 3^r + 2^r \; ; r \geq 0$

$$A(z) = \frac{1}{1-3z} + \frac{1}{1-2z}$$

$$A(z) = \frac{2 + 3z - 6z^2}{1 - 2z}$$

$$= 3z + \frac{2}{1-2z}$$

$\downarrow$

$2(1 + 2z + 2^2 z^2 + \ldots + 2^r z^r \ldots)$

$+ 3z$

$\Rightarrow \quad 2 + 7z + \cancel{2}^3 z^2 + 2^4 z^3 + \ldots$

$$a_r = \begin{cases} 2 & r = 0 \\ 7 & r = 1 \\ 2^{r+1} & r > 1 \end{cases}$$

e.g. $A(z) = \frac{2}{1-4z^2} = \frac{1}{1-2z} + \frac{1}{1+2z}$

$\downarrow$

$\langle 1, 2, 2^2, 2^3, \ldots \rangle$

$+ \langle 1, -2, 2^2, -2^3, \ldots \rangle$

$= \langle 2, 0, 2^3, 0, 2^5, 0, \ldots \rangle$

① $z^j A(z) \longrightarrow s^j a$

② $z^{-j} \left(A(z) - a_0 - a_1 z - a_2 z^2 + \cdots - a_{j-1} z^{j-1} \right] \longrightarrow s^{-j} a$

③ $A(z) \times B(z) \longrightarrow$ is the GF of the convolution of a & b

$(a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots + b_r z^r + \cdots)$

$= (\ldots\ldots (a_0 b_r + a_1 b_{r-1} + \cdots + a_r b_0) z^r \ldots\ldots)$

④ $b = \langle 1, 1, \ldots 1, \rangle$

a.

$a * b = \langle a_0, a_0 + a, a_0 + a_1 + a_2, \ldots, a_0 + a_1 + a_2 + \cdots a_{r-1} \ldots \rangle$

$\frac{A(z)}{1-z}$

$\rightarrow \frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots$

$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \ldots$

$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots$

$\frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = \frac{(1-z)^2 \cdot 1 + 2z(1-z)}{(1-z)^4}$

$= 1^2 + 2^2 z + 3^2 z^2 + 4^2 z^3 + \ldots$

$z \cdot \frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = \frac{z(1+z)}{(1-z)^3} = 0^2 z^0 + 1^2 z^1 + 2^2 z^2 + \ldots$

$\langle 0^2, 1^2, 2^2, \dots \rangle \longrightarrow \dfrac{z(1+z)}{(1-z)^3}$

$\langle 0^2, 0^2+1^2, 0^2+1^2+2^2, \dots \rangle \longrightarrow \dfrac{z(1+z)}{(1-z)^4}$

coeff z^r in $(1-z)^{-4}$

is $(-4)\dfrac{(-4-1)(-4-2)\dots(-4-r+1)}{r!}(-1)^r$

$\Rightarrow \dfrac{4 \times 5 \times 6 \times \dots (r+3)}{r!} = \dfrac{(r+1)(r+2)(r+3)}{1.2.3}$

thus coeff of z^r in $\dfrac{z(1+z)}{(1-z)^4} \Rightarrow (z^2+z)(1-z)^{-4}$

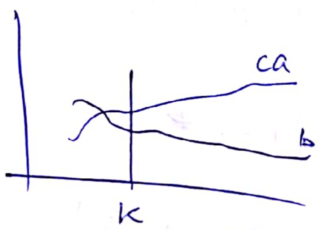
$= \dfrac{(r-1)(r)(r+1)}{6} + \dfrac{r(r+1)(r+2)}{6}$

$= \dfrac{r(r+1)(2r+1)}{6}$

a asymptotically dominates b

if $\exists k, c \in N^+ [\forall r \geq k (|b_r| \leq c.a_r)]$. be d(

Some scaled version of a stays above b from position k onwards



O(a) :- set of all DNFs asymptotically dominated by a.
Usually is literature b is O(a) is also correct.

e.g. $a_r = r+1 \quad r \geq 0$

$b_r = \frac{1}{r} + 7 \quad r \geq 0$

a asymp dominate b.

for $k = 7$ and $c = 1$

e.g. $a_r = \frac{r^2}{100} - 1000 \quad r \geq 0$

$b_r = \begin{cases} 10^6 . & 0 \leq r \leq 10 \\ 10^5 r & r > 10 \end{cases}$

$k = 10^6 + 1 \quad c = 10$

a domin. b

Properties:-

1. a is $O(|a|)$
2. If b is $O(a)$, then for any const α, αb is $O(a)$
3. If b is $O(a)$ then $s^i b$ is $O(a)$, $\forall i \in \mathbb{Z}$
4. If b and c are both $O(a)$ then $\alpha b + \beta c$ is $O(a)$ α, β consts.
5. If c is $O(b)$ and ~~b0~~ b is $O(a)$ then c is $O(a)$.
6. It is possible that a is $O(b)$ and b is $O(a)$.
7. It is possible that a is not $O(b)$ and b is not $O(a)$

e.g. for 6 $\rightarrow$ $a = x^2$
$b = 2x^2$

e.g. for 7 $\rightarrow$ $\sin x$ and $\cos x$

8. It is possible that c is both $O(a)$ and $O(b)$ while a is not $O(b)$ b is not $O(a)$.

A and B are set of DNFs.

$A + B = \{a + b \mid a \in A, b \in B\}$

$\alpha A = \{\alpha a \mid a \in A\}$

$AB = \{ab \mid a \in A, b \in B\}$

Properties:-

1. If b is $O(a)$ then $O(b) \le O(a)$

★ If b is $O(a)$ & a is $O(b)$

$\Rightarrow O(a) = O(b)$

2. ★ $\forall a$ $\quad O(a) + O(a) = O(a)$

★ if $b \in O(a)$

then $O(a) + O(b) = O(a)$

★ for any const. α

$\alpha \cdot O(a) = O(a)$

★ for any a, b

$O(a)\,O(b) = O(ab)$

$\Omega(a)$: the set of all DNFs b s.t.

$\exists\, k; c \in N^{+} \left[\forall\, r \ge k \left(|b_r| \ge c \cdot a_r \right) \right]$

$\Theta(a)$: $O(a) \cap \Omega(a)$

the set of all DNFs b s.t.

$\exists\, c, c', k \ge 0 \left[\forall\, r \ge k \left[c\, a_r \le |b_r| \le c'\, a_r \right] \right]$

$\omega(a)$, $o(a)$:- inequality changes to strict inequality.

e.g. $\quad 10n + 4 = o(n^2)$

$4n^2 + 5n + 3 = \Theta(n^2)$

$\omega(n)$

Scanned by CamScanner

$a_r = 3^r \quad b_r = 2^r \quad r \geq 0$

$A(z) = \frac{1}{1-3z} \qquad B(z) = \frac{1}{1-2z}$

$A(z) \cdot B(z) = \frac{1}{1-3z} \cdot \frac{1}{1-2z} = \frac{3}{1-3z} - \frac{2}{1-2z}$

$\downarrow \qquad \downarrow$

$\langle 3 \cdot 3^r \rangle - \langle 2 \cdot 2^r \rangle$

$\downarrow$

$\langle 3^{r+1} - 2^{r+1} \rangle$

$a_r = \binom{n}{r}$

$A(z) = \binom{n}{0} + \binom{n}{1} z + \binom{n}{2} z^2 + \dots + \binom{n}{n} z^n.$

· r objects from n

↳ divide them into 2 piles.

$k \qquad n-k$

$i \qquad r-i$

$\binom{n}{r} = \sum_{i=0}^{r} \binom{k}{i} \binom{n-k}{r-i}$

if $D(z) = \binom{k}{0} + \binom{k}{1} z + \dots + \binom{k}{k} z^k.$

$E(z) = \binom{n-k}{0} + \binom{n-k}{1} z + \dots \binom{n-k}{n-k} z^{n-k}.$

$A(z) = D(z)\, E(z) \qquad D(z) = (1+z)^k$

$E(z) = (1+z)^{n-k}.$

? $\langle \binom{1}{0} \binom{1}{1}, 0, 0, \dots \rangle$

$(1+z)^n = A(z).$

Put $z = 1$: $\quad 2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$

Put $z = -1$: $\quad 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \dots$

$\Rightarrow \binom{n}{1} + \binom{n}{3} + \binom{n}{5} \dots = \binom{n}{0} + \binom{n}{2} + \dots$

Diff. $(1+z)^n$

$$\binom{n}{1} + \binom{n}{2} \cdot 2z + \binom{n}{3} 3z^2 + \dots + \binom{n}{n} n z^{n-1}$$

$$= n(1+z)^{n-1}$$

Put $z = 1$

$$\Rightarrow \quad n \cdot 2^{n-1} = \binom{n}{1} + \binom{n}{2} 2 + \binom{n}{3} 3 + \dots + \binom{n}{n} n$$

$\binom{n}{i} i$

$\downarrow$

choose i members from n people and we have i candidates for head.

Fibonacci Numbers

$F_0 = 0 \qquad F_1 = 1$

$\langle 0, 1, 1, 2, 3, 5, 8, 13, \dots \rangle$

$F_i = F_{i-1} + F_{i-2}$

$$F(z) = z + z^2 + 2z^3 + 3z^4 + 5z^5 + \dots$$

$$-z F(z) = z^2 + z^3 + 2z^4 + 3z^5 + \dots \qquad \infty$$

$$-z^2 F(z) = z^3 + z^4 + 2z^5 + \dots$$

$$(1 - z - z^2) F(z) = z + 0 + \dots$$

thus $(1 - z - z^2) F(z) = z$

$\dfrac{1}{1-1-1}$ thus $F(z) = \dfrac{z}{1 - z - z^2}$

Now $F(z) = \dfrac{z}{(1 - \phi_1 z)(1 - \phi_2 z)}$

$$\phi_1 = \frac{1+\sqrt{5}}{2} \qquad \phi_2 = \frac{1-\sqrt{5}}{2}$$

$$F(z) = \frac{1/\sqrt{5}}{1-\phi_1 z} - \frac{1/\sqrt{5}}{1-\phi_2 z}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{1}{\sqrt{5}} \langle a_r = \phi_1^r \rangle \qquad \frac{1}{\sqrt{5}} \langle b_r = \phi_2^r \rangle$$

Difference

$$\frac{1}{\sqrt{5}} \langle \phi_1^r - \phi_2^r \rangle$$

thus $\boxed{F_r = \frac{1}{\sqrt{5}} (\phi_1^r - \phi_2^r)}$

Cassini's Theorem

For $n > 0$,

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n$$

$n=1 \Rightarrow F_2 F_0 - F_1^2 = 1 \times 0 - 1^2 = -1 = (-1)^1$ ✓ (Basis)

Induction step:- $F_{n+2} F_n - F_{n+1}^2 = (F_{n+1} + F_n) F_n - (F_{n+1})^2$

$$= (F_{n+1} + F_n) F_n - (F_n + F_{n-1}) F_{n+1}$$

$$= F_n^2 - F_{n-1} F_{n+1}$$

$$= (-1)^n (-1) = (-1)^{n+1}$$

e.g. $a_0 = 1$

$a_r = 3a_{r-1} + 2, \quad r \geq 1$

$1, 5, 17, 53, 160, \ldots$

$$a_r z^r = 3a_{r-1} z^r + 2z^r$$

$$\underbrace{\sum_{r=1}^{\infty} a_r z^r}_{A(z) - a_0} = 3 \sum_{r=1}^{\infty} a_{r-1} z^r + 2 \sum_{r=1}^{\infty} z^r$$

$$A(z) - \cancel{a_0}\ 1 = 3z\, A(z) + 2 \cdot \frac{z}{1-z}$$

$$\Rightarrow (1-3z) A(z) = 1 + \frac{2z}{1-z} = \frac{1+z}{1-z}$$

$$A(z) = \frac{1+z}{(1-z)(1-3z)} = \frac{2}{1-3z} - \frac{1}{1-z} \rightarrow \langle a_r = 1 \rangle$$

$$\downarrow$$

$$\langle a_r = 2.3^r \rangle$$

$$\langle a_r = 2.3^r - 1 \rangle$$

Linear Recurrences With constant coefficients (LRCC)

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \dots + c_k a_{r-k} = f(r)$$

$$c_0, c_1, c_2 \dots, c_k \in \mathbb{R}$$

If $c_0, c_k \neq 1$, then we say that the order is k.

e.g. $3a_r + 2a_{r-1} = r^2$ (1st order)

$7a_r - a_{r-2} = 3$ (2^{nd} order)

e.g. $3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5$ (order = 2)

$a_3 = 0 \quad a_4 = 1$

$r=5$: $3a_5 - 5a_4 + 2a_3 = 5^2 + 5$

$\Rightarrow 3a_5 - 5 + 0 = 30 \quad \therefore \Rightarrow 3a_5 = 35$

$\Rightarrow \boxed{a_5 = \frac{35}{3}}$

$r=6$: $3a_6 - 5a_5 + 2a_4 = 6^2 + 5$

$\Rightarrow 3a_6 - 5\left(\frac{35}{3}\right) + 2(1) = 41$

$\Rightarrow 3a_6 = \frac{39 \times 3 + 175}{3} = \frac{117 + 175}{3} = \frac{292}{3}$

$$\boxed{a_6 = \frac{292}{6}}$$

$r=4$: $3a_4 - 5a_3 + 2a_2 = 4^2 + 5$

$\Rightarrow 3(1) - 5(0) + 2a_2 = 21$

$\Rightarrow 2a_2 = 18 \Rightarrow \boxed{a_2 = 9}$

$r=3$: $3a_3 - 5a_2 + 2a_1 = 9 + 5$

$\Rightarrow \cancel{3f}\ 3(0) - 5a_2 + 2a_1 = 14$

$\Rightarrow \quad 2a_1 = 14 + 5(9) = 14 + 45 = 59$

$$\boxed{a_1 = \frac{59}{2}}$$

* k consecutive elements for LRCC of order k form a boundary conditions. (unique solⁿ for given boundary conditions)

$\langle a_r = p_r \rangle$ as a particular solⁿ.

$c_0 a_r + c_1 a_{r-1} + \dots + c_k a_{r-k} = f(r)$ —— ①

$c_0 a_r + c_1 a_{r-1} + \dots + c_k a_{r-k} = 0$ —— ②

homogeneous LRCC

$\langle a_r = h_r \rangle$ is a solⁿ of homogeneous LRCC

$\langle a_r = p_r + h_r \rangle$ is also a solⁿ for —①

hLRCC of order k. —— ②

$A\alpha^r$

$\cancel{A}(c_0 \alpha^r + c_1 \alpha^{r-1} + \dots + c_k \alpha^{r-k}) = 0$ —— ③

$\boxed{\langle a_r = A\alpha^r \rangle}$

characteristic eqⁿ of the LRCC

$c_0 \alpha^k + c_1 \alpha^{k-1} + \dots + c_{k-1}\alpha + c_k = 0$ —— ④

α_1 is a ch. root of the LRCC

Say, α_1 is a root of this

solⁿ :- $A_1 \alpha_1^r$ is a solⁿ for —②

$(\underbrace{A_1 \alpha_1^r + A_2 \alpha_2^r + \dots A_k \alpha_k^r}_{k \text{ unknowns}}) + p_r$ → Total solⁿ

*

α_1 is a root with multiplicity $m > 1$

α_1 is a root of ④

$(\alpha - \alpha_1)$ is a factor of LHS ④

$(\alpha - \alpha_1)$ is a factor of LHS ③

$(\alpha - \alpha_1)\, g(\alpha)$.

$(\alpha - \alpha_1)\, g'(\alpha) + g(\alpha) = \frac{d}{d\alpha}(\text{LHS ③})$.

$\Rightarrow 2a_1 = 14 + 5(9) = 14 + 45 = 59$

$\boxed{a_1 = \frac{59}{2}}$

* k consecutive elements for LRCC of order k form a boundary conditions. (unique sol^n for given boundary conditions)

$\langle a_r = p_r \rangle$ as a particular sol^n.

$c_0 a_r + c_1 a_{r-1} + \dots + c_k a_{r-k} = f(r)$ —— (1)

$c_0 a_r + c_1 a_{r-1} + \dots + c_k a_{r-k} = 0$ —— (2)

homogeneous LRCC

$\langle a_r = h_r \rangle$ is a sol^n of homogeneous LRCC

$\langle a_r = p_r + h_r \rangle$ is also a sol^n for —(1)

hLRCC of order k. —— (2)

$A\alpha^r$

$\cancel{A}(c_0 \alpha^r + c_1 \alpha^{r-1} + \dots + c_k \alpha^{r-k}) = 0$ —— (3)

$\boxed{\langle a_r = A\alpha^r \rangle}$

$c_0 \alpha^k + c_1 \alpha^{k-1} + \dots + c_{k-1}\alpha + c_k = 0$ —— (4)

characteristic eq^n of the LRCC

α_1 is a root of the ch. LRCC

Say, α_1 is a root of this

sol^n :- $A_1 \alpha_1^r$ is a sol^n for —(2)

$\underbrace{(A_1 \alpha_1^r + A_2 \alpha_2^r + \dots A_k \alpha_k^r)}_{k \text{ unknowns}} + p_r \longrightarrow$ Total sol^n

α_1 is a root with multiplicity $m > 1$

α_1 is a root of (4)

$(\alpha - \alpha_1)$ is a factor of LHS (4)

$(\alpha - \alpha_1)$ is a factor of LHS (3).

$(\alpha - \alpha_1)\, g(\alpha)$.

$(\alpha - \alpha_1)\, g'(\alpha) + g(\alpha) = \frac{d}{d\alpha}(\text{LHS (3)})$.

Derivative of ③ -

$$\frac{d(LHS ③)}{d\alpha}\Bigg|_{\alpha=\alpha_1} C_0 r_1 \alpha_1^{r-1} + C_1 (r-1)\alpha_1^{r-2} + \dots + C_k (r-k)\alpha_1^{r-k-1} = 0$$

$$C_0 A_2 r \alpha_1^{r} + C_1 A_2 (r-1)\alpha_1^{r-1} + \dots + C_k A_{\cancel{2}} (r-k)\alpha_1^{r-k} = 0$$

$$\langle a_i = A_2 i \alpha_1^{i} \rangle$$

$$\langle a_r = (A_1 + A_2 r + A_3 r^2 + \dots + A_m r^{m-1})\alpha_1^{r} \rangle$$

is a solution of HLRCC

$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \dots$

$\underbrace{m_1 \quad m_2 \quad m_3}_{\text{multiplicity}} \quad \dots$

e.g. $4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$

$\frac{1}{2}, \frac{1}{2}, 4$

$$a_r = \langle (A_1 + A_2 r)\left(\tfrac{1}{2}\right)^r + A_3 (4)^r \rangle$$

e.g. $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$

Assume a particular solⁿ.

$P_1 r^2 + P_2 r + P_3$

$$P_1 r^2 + P_2 r + P_3 + 5\left(P_1(r-1)^2 + P_2(r-1) + P_3\right) + 6\left(P_1(r-2)^2 + P_2(r-2) + P_3\right) = 3r^2$$

$$\Rightarrow 12P_1 r^2 - (34P_1 - 12P_2) r + (29P_1 - 17P_2 + 12P_3) = 3r^2$$

$12P_1 = 3 \Rightarrow \boxed{P_1 = 1/4}$

$\boxed{P_2 = \frac{17}{24}}$ $\quad \boxed{P_3 = \frac{115}{288}}$

$$\langle P_r = \frac{1}{4} r^2 + \frac{17}{24} r + \frac{115}{288} \rangle$$

Scanned by CamScanner

$\alpha^2 + 5\alpha + 6 = 0 \longrightarrow$ Homogeneous solⁿ

① If $f(r)$ is a polynomial of degree t, then P.S. is also a polynomial of degree t.

Linear Recurrences with constant coefficients

$$C_0 a_r + C_1 a_{r-1} + \dots + C_k a_{r-k} = f(r)$$

e.g. If $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$

P.S. $= P_1 r^2 + P_2 r + P_3$

$\Rightarrow 12P_1 r^2 - (34P_1 - 12P_2) r + (29P_1 - 17P_2 + 12P_3)$

$= 3r^2 - 2r + 1$

$\boxed{P_1 = 1/4}$

$34P_1 - 12P_2 = 2 \Rightarrow \boxed{P_2 = 13/24}$

$\boxed{P_3 = \frac{71}{288}}$

Hence $\boxed{P.S. = \frac{1}{4}r^2 + \frac{13}{24}r + \frac{71}{288}}$

P.S=

e.g. $a_r - 5a_{r-1} + 6a_{r-2} = 1.$

P.S = P

$\Rightarrow P - 5P + 6P = 2P = 1$ thus $\boxed{P = 1/2}$

If $f(r)$ is $c\beta^r$, where β is not a char. root ②
then PS is of form $P\beta^r$

e.g. $a_r + 5a_{r-1} + 6a_{r-2} = 42 \times 4^r$

$\alpha^2 + 5\alpha + 6 = 0$

$(\alpha+2)(\alpha+3) = 0$

$\alpha = -2 / -3$

$P4^r + 5 \cdot P \cdot 4^{r-1} + 6 \cdot P \cdot 4^{r-2}$

$= 42 \cdot 4^r$

$\Rightarrow 16P + 20P + 6P = 42 \times 16$

$\Rightarrow 42P = 42 \times 16$

$\rightarrow \boxed{P = 16}$

(3) $f(r)$ is of the form

$$(F_1 r^t + \dots + F_{t+1})\beta^r$$

(β is not a char. root)

then the P.S. is also

$$(P_1 r^t + \dots + P_{t+1})\beta^r$$

e.g.

$$a_r + a_{r-1} = 3r \cdot 2^r$$

$\alpha + 1 = 0$ and 2 is not a root of it

then $P.S = (P_1 r + P_2) 2^r$

$$(P_1 r + P_2)2^r + (P_1 r - P_1 + P_2)2^{r-1}$$

$$= \left(P_1 r + \frac{P_1 r}{2}\right)2^r + \left(P_2 - \frac{P_1}{2} + \frac{P_2}{2}\right)2^r$$

$$= (2P_1 r + P_1 r)2^{r-1} + (2P_2 - P_1 + P_2)2^{r-1}$$

$$= (3P_1 r)2^{r-1} + (3P_2 - P_1)2^{r-1}$$

$$= 3r \cdot 2^r$$

thus $\frac{3P_1}{2} = 3$ $\quad$ $3P_2 - P_1 = 0$

$\Rightarrow \boxed{P_1 = 2}$ $\quad$ $\boxed{P_2 = 2/3}$

(4) $f(r)$ is

$$(F_1 r^t + \dots + F_{t+1})\beta^r$$

β is a chr. root of multiplicity m.

then $P.S. = r^m (P_1 r^t + \dots + P_{t+1})\beta^r$

e.g. $a_r - 2a_{r-1} = 3 \cdot 2^r$

(P.S $\Rightarrow$ $t = 0$ and $\beta = 2$

$\alpha - 2 = 0$ (β is a char. root)

with multiplicity $m = 1$

thus $P.S \Rightarrow r^1 (P)(2^r)$

$\Rightarrow P \cdot r \cdot 2^r - P \cdot (r-1)(2^{r-1}) = 3 \cdot 2^r$ $\quad$ $\boxed{P = 3}$

Hence $\boxed{3r2^r \text{ is } P.S.}$

e.g. $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$

char. eq^n $\Rightarrow$ $\alpha^2 - 4\alpha + 4 = 0$

$(\alpha-2)^2 = 0$ and $\beta = 2$ and $t = 1$

thus β is a char. root with multiplicity $= 2 = m$

thus P.S. $= r^2(P_1 r + P_2)2^r$ (After plugging this in, in the eq^n)

$\Rightarrow$ $\boxed{P_1 = 1/6}$ $\boxed{P_2 = 1}$

Hence P.S. $\Rightarrow$ $\langle P_r = r^2\left(\frac{r}{6}+1\right)2^r \rangle$

e.g. $a_r = a_{r-1} + 7$

$\Rightarrow$ $a_r - a_{r-1} = 7 \cdot 1^r$

char. $\Rightarrow$ $\alpha - 1 = 0$ | $\boxed{\alpha = 1}$

and $\beta = 1$ and $m = 1$

thus P.S. $= r^1(P) \cdot 1^r = P \cdot r$

thus $Pr - P(r-1) = 7$

$\Rightarrow$ $\boxed{P = 7}$

Hence $\boxed{P.S. = 7r}$

7 is a const polyno. why can't we assume that the P.S $\Rightarrow$ constant

e.g. $a_r - 2a_{r-1} + a_{r-2} = 7 \cdot 1^r$

$(\alpha^2 - 2\alpha + 1) = (\alpha - 1)^2 = 0$

$\beta = 1$ and $m = 2$

$\Rightarrow$ $r^2(P) \cdot 1^r = Pr^2$

$\Rightarrow$ $P \cdot r^2 - 2P(r-1)^2 + P(r-2)^2 = 7$

$\Rightarrow$ $2P = 7$ $\Rightarrow$ $\boxed{P = 7/2}$

Hence $\boxed{\text{sol}^n = \frac{7}{2}r^2}$

e.g. $a_r + 5a_{r-1} + 6a_{r-2} = 42 \cdot 4^r$

given (Boundary cond.) $a_2 = 278$, $a_3 = 962$

$\langle P_r = 16 \times 4^r \rangle$

$\alpha^2 + 5\alpha + 6 = 0$ | $\alpha = -2$ or -3

$\langle a_r = A_1(-2)^r + A_2(-3)^r + 16 \times 4^r \rangle$ $\rightarrow$ Total sol^n

$278 = 4A_1 + 9A_2 + 256$

$962 = -8A_1 - 27A_2 + 1024$

$\Big\}$ $A_1 = 1$, $A_2 = 2$

Thus the sol^n that satisfies the given Boundary condition →

$$\langle a_r = (-2)^r + 2(-3)^r + 16 \times 4^r \rangle$$

LRCC Using generating fns

$$c_0 a_r + c_1 a_{r-1} + \dots + c_k a_{r-k} = f(r)$$

$r \geq s$

$s \geq k$

Multiply both sides by z^r and apply sum.

$$\sum_{r=s}^{\infty} (c_0 a_r + c_1 a_{r-1} + \dots + c_k a_{r-k}) z^r = \sum_{r=s}^{\infty} f(r) z^r$$

Now 1) $\sum_{r=s}^{\infty} c_0 a_r z^r = c_0 [A(z) - (a_0 + a_1 z + \dots + a_{s-1} z^{s-1})]$

2) $\sum_{r=s}^{\infty} c_1 a_{r-1} z^r = c_1 z [A(z) - (a_0 + a_1 z + \dots + a_{s-2} z^{s-2})]$

⋮

k) $\sum_{r=s}^{\infty} c_k a_{r-k} z^r = c_k z^k [A(z) - (a_0 + a_1 z + \dots + a_{s-k-1} z^{s-k-1})]$

$$RHS = LHS = A(z)[c_0 + c_1 z + \dots c_k z^k] - \left\{ \begin{array}{l} c_0 (a_0 + a_1 z + \dots + a_{s-1} z^{s-1}) \\ + c_1 z (a_0 + a_1 z + \dots + a_{s-2} z^{s-2}) \\ \vdots \\ + c_k z^k (a_0 + \dots + a_{s-k-1} z^{s-k-1}) \end{array} \right.$$

$$A(z) = \frac{\sum_{r=s}^{\infty} f(r) z^r + \{\quad\}}{c_0 + c_1 z + \dots + c_k z^k}$$

In Fibonacci sequence

$$A(z) = \frac{z}{1 - z - z^2}$$