Unit 1 Formulae, Imp Points

Monday, October 23, 2023 9:40 AM

PARTIAL DIFFERENTIATION

- $$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{F(x+h, y) - F(x, y)}{h}$$

  $$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{F(x, y+k) - F(x, y)}{h}$$

GEOMETRIC INTERPRETATION

$$\underbrace{z = F(x, y)}$$

→ represents a surface in 3D space

The intersection of the surface $z = F(x,y)$ with the plane $y = y_0$ gives a plane curve.

- Any point on the curve → $(x, y_0, F(x, y_0))$ where y_0 → constant

Thus, $\frac{\partial z}{\partial x}$ (where y = constant) represents the slope of the tangent to the curve of intersection of the surface $z = F(x,y)$ and the plane $y = y_0$.

RULES OF DIFFERENTIATION

Let u, v, w be functions of x, y, z.

① Partial differentiation of sum/difference

Let $t = u \pm v \pm w$

$$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}(u \pm v \mp w) = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x} \pm \frac{\partial w}{\partial x}$$

② Partial differentiation of product / Product rule

Let $t = u \times v \times w$

$$\frac{\partial t}{\partial x} = u \times v \times \frac{\partial w}{\partial x} + u \times \frac{\partial v}{\partial x} \times w + \frac{\partial u}{\partial x} \times v \times w$$

③ Quotient rule

Let $t = \frac{u}{v}$

$$\frac{\partial t}{\partial x} = \frac{v\left(\frac{\partial u}{\partial x}\right) - u\left(\frac{\partial v}{\partial x}\right)}{v^2}$$

④ Chain rule

$t = f(u)$ where $u = F(x, y, z)$

$$\frac{\partial t}{\partial x} = \underbrace{\left(\frac{dt}{du}\right)}\left(\frac{\partial u}{\partial x}\right)$$

→ This is an ordinary derivative as t is a function of only one variable

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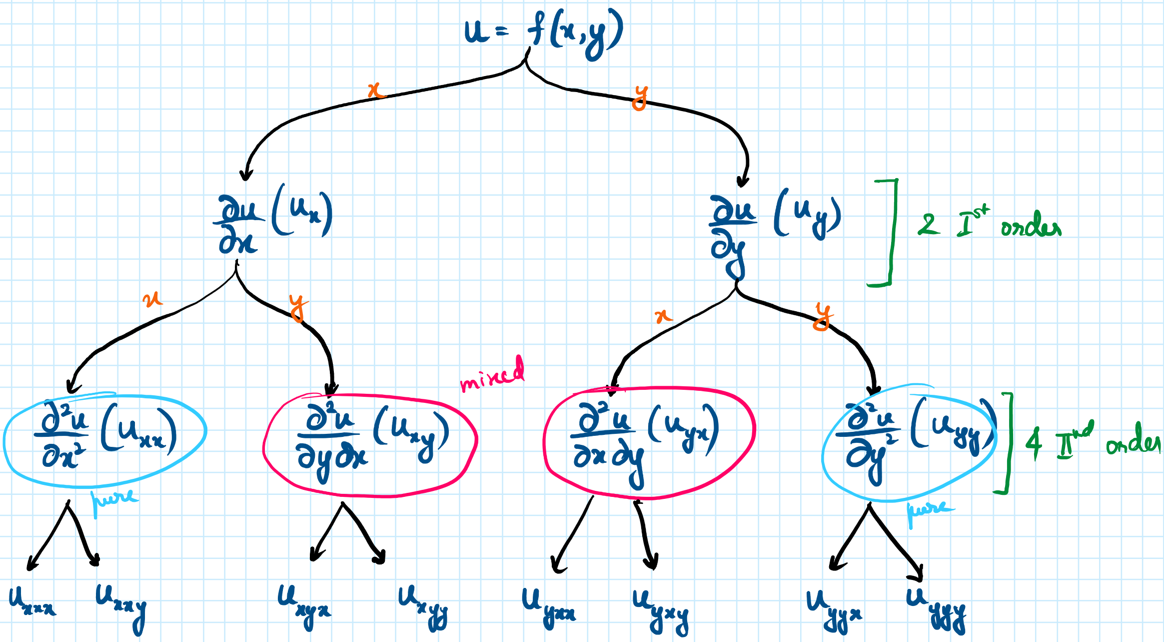
Mixed Partial Derivative

Note:

- $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x} \Longrightarrow$ Derivatives are continuous
- For higher order derivatives, the order of operators in the denominator from right to left show the order of variables w.r.t. which the function was differentiated.



In general:
A function with m independent variables has m^n n^{th} order derivatives.

Euler's Theorem (for homogeneous functions)

If $u(x,y)$ is a homogeneous function of degree n,

$$\boxed{x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu}$$

and so on for any no. of independent variables

Extension of Euler's Theorem for 2nd Order Derivatives

If $u(x,y)$ is a homogeneous fn. of degree n, then

$$\boxed{x^2 \cdot \frac{\partial^2 u}{\partial x^2} + 2xy \cdot \frac{\partial^2 u}{\partial y\,\partial x} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = n(n-1)\,u}$$

Total Differential

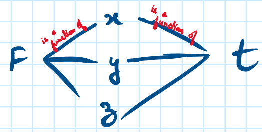
The total or exact differential of a function $F(x, y, z)$ is given by

$$\boxed{dF = \frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy + \frac{\partial F}{\partial z} \cdot dz}$$

Composite Functions

Type ①

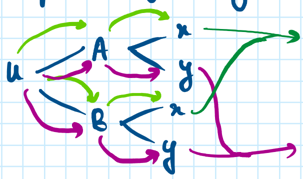
$$\boxed{\frac{dF}{dt} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt}}$$



This can be extended to any no. of variables.

Type ②

If u is a function of A, B and A, B are functions of x, y, then u is a composite fn. of x, y through A and B.

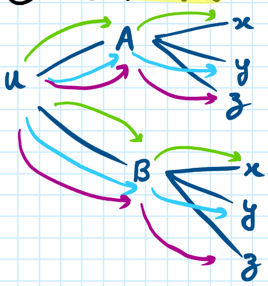


$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial x} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial y} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial y}$$

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}}$$

Type ③ : 3 variables



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial x} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial y} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial y}$$

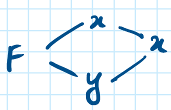
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial z} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial z}$$

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}}$$

where u is a function of t.

Derivative of implicit function

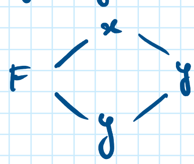
Consider an implicit function $F(x, y) = C$



Not independent variables

$$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\partial F}$$

Illly, for y being the independent variable,



$$\frac{dx}{dy} = \frac{-\frac{\partial F}{\partial y}}{\partial F}$$

$$\boxed{\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}}$$

$$\boxed{\frac{dx}{dy} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}}$$

Taylor Series Expansion of $f(x,y)$ about the point (a,b)

- Expansion in terms of $(x-a)$, $(y-b)$

$$F(x,y) = F(a,b) + \frac{1}{1!}\Big[(x-a)[F_x(a,b)] + (y-b)[F_y(a,b)]\Big]$$
$$+ \frac{1}{2!}\Big[(x-a)^2[F_{xx}(a,b)] + 2(x-a)(y-b)[F_{xy}(a,b)] + (y-b)^2[F_{yy}(a,b)]\Big]$$
$$+ \frac{1}{3!}\Big[(x-a)^3[F_{xxx}(a,b)] + 3(x-a)^2(y-b)[F_{xxy}(a,b)] + 3(x-a)(y-b)^2[F_{xyy}(a,b)] + (y-b)^3[F_{yyy}(a,b)]\Big]$$
$$+ \dots$$

McLaurin's Series of $F(x,y)$

Put $(a,b) = 0$

$$F(0,0) = F(0,0) + \frac{1}{1!}\Big[x[F_x(0,0)] + y[F_y(0,0)]\Big]$$
$$+ \frac{1}{2!}\Big[x^2[F_{xx}(0,0)] + 2xy[F_{xy}(0,0)] + y^2[F_{yy}(0,0)]\Big]$$
$$+ \frac{1}{3!}\Big[x^3[F_{xxx}(0,0)] + 3x^2y[F_{xxy}(0,0)] + 3xy^2[F_{xyy}(0,0)] + y^3[F_{yyy}(0,0)]\Big]$$
$$+ \dots$$

MAXIMA AND MINIMA

For some $F(x,y)$,

Condition for (a,b) to be maxima/minima:

$$\left(\frac{\partial F}{\partial x}\right)_{(a,b)} = 0 \quad , \quad \left(\frac{\partial F}{\partial y}\right)_{(a,b)} = 0$$

Let $F_{xx} = r$, $F_{xy} = s$, $F_{yy} = t$

$rt - s^2 > 0$ and $r < 0$ $\longrightarrow$ (a,b) is a maximum point

$rt - s^2 > 0$ and $r > 0$ ⟶ (a,b) is a minimum point

$rt - s^2 < 0$ ⟶ (a,b) is a saddle point

$rt - s^2 = 0$ ⟶ Insufficient information to draw any conclusion; further investigation required.

Lagrange's Method of Undetermined Multipliers

- For any number of variables that are not independent; variables are constrained by some equation called constraint equation.

Let $F(x,y,z)$ be a function whose extreme value is to be determined, subjected to a constraint $\phi(x,y,z) = c$.

$$\frac{\partial F}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial F}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

Lagrange's Equations: when solved, gives critical points

Note: You only get critical points using this; you cannot find the nature of the point

Methods of Solving Differential Equation

Refer notes