3. Gamma Function

29 November 2023 10:06

GAMMA FUNCTION $\Gamma(n)$

$\Gamma(n)$ is defined as:

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx}$$

where 'n' is either +ve real no.
OR
-ve non-integer

Evaluating $\Gamma(1)$ by definition

$$\Gamma(1) = \int_0^\infty e^{-x} x^0\, dx$$

$$= \int_0^\infty e^{-x}\, dx$$

$$= \left[-e^{-x}\right]_0^\infty$$

$$= -(0-1)$$

$$\boxed{\Gamma(1) = 1}$$

RECURSIVE FORMULA FOR Γ

To prove: $\Gamma(n) = (n-1)\Gamma(n-1)$

$$\Gamma(n) = \int_0^\infty \underbrace{e^{-x}}_{v}\ \underbrace{x^{n-1}}_{u}\, dx$$

$$= \left[x^{n-1}\left(-e^{-x}\right)\right]_0^\infty - \int_0^\infty (-e^{-x})(n-1)\, x^{n-2}\, dx$$

$$= 0 + (n-1)\int_0^\infty e^{-x} \cdot x^{(n-1)-1}\, dx$$

$$\boxed{\Gamma(n) = (n-1)\Gamma(n-1)}$$

Case (1): When n is positive

Case ①: When n is positive

$\Gamma(n) = (n-1)(n-2)\ldots(1)\Gamma(1)$

$\boxed{\Gamma(n) = (n-1)!}$

Case ②: When n is +ve fraction

$\boxed{\Gamma(n) = (n-1)(n-2)\ldots(x)\Gamma(x)}$ where $0 < x < 1$

$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

Case ③: When n = 0

$\Gamma(n) = (n-1)\Gamma(n-1)$

$$\Gamma(n-1) = \frac{\Gamma(n)}{n-1}$$

Put $n = 1$

$$\Gamma(0) = \frac{1}{0} = \infty$$

$\boxed{\Gamma(0) = \text{not defined}}$

Case ④: When n is negative

Using above formula,

$$\Gamma(-1) = \frac{\Gamma(0)}{-1} = -\infty$$

$\boxed{\Gamma(n) = \text{not defined when } n = 0, -1, -2 \ldots}$

Case ⑤: When n is negative fraction

$\Gamma(n-1) = (n-1)\Gamma(n-1)$

Putting n as n+1

$\Gamma(n+1) = (n)\Gamma(n)$

$\Gamma(n+1) = (n)\Gamma(n)$

$$\boxed{\Gamma(n) = \frac{\Gamma(n+1)}{n}}$$

$$\Gamma\left(\frac{-11}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)}{\frac{-11}{4} \times \frac{-7}{4} \times \underbrace{\frac{-3}{4}}}$$

Final value b/w 0 and 1

RESULTS

Prove that

$$\int_0^\infty x^n e^{-ax^m} dx = \frac{1}{m} \frac{\Gamma\left(\frac{n+1}{m}\right)}{a^{\frac{n+1}{m}}}$$

where m is +ve.

$$I = \int_0^\infty e^{-ax^m} x^n \, dx$$

$t = ax^m \Rightarrow x^m = \frac{t}{a}$

$x = \left(\frac{t}{a}\right)^{\frac{1}{m}}$

$dx = \frac{1}{m}\left(\frac{t}{a}\right)^{\frac{1}{m}-1} \cdot \frac{dt}{a}$

When $x = 0$, $t = 0$
$x = \infty$, $t = \infty$

$$I = \int_0^\infty e^{-t}\left(\left(\frac{t}{a}\right)^{\frac{1}{m}}\right) \cdot \frac{1}{m}\left(\frac{t}{a}\right)^{\frac{1}{m}-1} \frac{dt}{a} = \frac{1}{m} \cdot \frac{1}{a^{\frac{n+1}{m}}} \int_0^\infty e^{-t} t^{\frac{n+1}{m}-1} dt$$

$$\boxed{\int_0^\infty e^{-ax^m} x^n \, dx = \frac{1}{m} \cdot \frac{\Gamma\left(\frac{n+1}{m}\right)}{a^{\frac{n+1}{m}}}}$$ Proved

Put $m = 1$

$$\int_0^\infty e^{-ax} x^n \, dx = \frac{\Gamma(n+1)}{a^{n+1}}$$

Putting $n+1$ as n,

a

Putting $n+1$ as n,

$$\boxed{\Gamma(n) = a^n \int_0^\infty e^{-ax} x^{n-1}\, dx}$$

→ Second definition of $\Gamma(n)$

Put $m = 2$

$$\int_0^\infty e^{-ax^2} x^n \, dx = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}}$$

Putting $n+1$ as n,

$$\boxed{\Gamma(n) = 2a^n \int_0^\infty e^{-ax^2} x^{2n-1}\, dx}$$

→ Third definition of $\Gamma(n)$

Put $a=1$

$$\boxed{\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx}$$

→ Fourth definition of $\Gamma(n)$