

Classroom Problems

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① Find first order partial derivatives

(i) $f(x, y) = x^4 - x^2y^2 + y^4$ at $(-1, 1)$

$$\frac{\partial f}{\partial x} = 4x^3 - (y^2)(2x) \quad \frac{\partial f}{\partial y} = -x^2(2y) + 4y^3$$

At $(-1, 1)$,

$$\begin{aligned} &= -4 - (-2) &&= -1(2) + 4 \\ &= \cancel{-2} &&= \cancel{2} \end{aligned}$$

(ii) $f(x, y) = x^2 e^{-\frac{y}{x}}$ at $(4, 2)$

$$\begin{aligned} \frac{\partial f}{\partial x} &= x^2 \left[e^{-\frac{y}{x}} (y) \left(\frac{1}{x^2} \right) \right] + e^{-\frac{y}{x}} (2x) \\ &= (2x+y)e^{-\frac{y}{x}} \end{aligned}$$

At $(4, 2)$,

$$= \cancel{10e^{\frac{1}{2}}}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= x^2 \left[e^{-\frac{y}{x}} \left(\frac{1}{x} \right) (-1) \right] \\ &= -x e^{-\frac{y}{x}} \\ &= -4 e^{\frac{1}{2}} \end{aligned}$$

② $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for all $(x, y) \neq 0, 0$ when $f(x, y) = x^y$. Prove.

$$\frac{\partial f}{\partial x} = y x^{y-1}$$

$$\frac{\partial^2 f}{\partial y \partial x} = (y)(x^{y-1})(\log_e x) + x^{y-1}(1)$$

$$\frac{\partial f}{\partial y} = x^y \cdot \log x$$

$$\frac{\partial^2 f}{\partial x \partial y} = (\log(x))(y x^{y-1}) + (x^y)\left(\frac{1}{x}\right)$$

③ Find all second order partial derivatives of $f(x, y) = \log\left(\frac{1}{x} - \frac{1}{y}\right)$ at $(1, 2)$

$$\frac{\partial f}{\partial x} = \cancel{\frac{1}{x-1}} \left(\frac{-1}{x} \right)$$

(3) find all second order partial derivatives of $f(x, y) = \log\left(\frac{1}{x} - \frac{1}{y}\right)$ at $(1, 2)$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)} \left(\frac{-1}{x^2} \right) \\ &= \frac{xy}{y-x} \left(\frac{-1}{x^2} \right) \\ &= \frac{-y}{xy-x^2} = -y(xy-x^2)^{-1}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= (-y)(-1)(xy-x^2)^{-2}(y-2x) \quad \left| \begin{array}{l} \frac{\partial^2 f}{\partial y \partial x} = \frac{1}{xy-x^2}(-1) + (-y)(-1)(xy-x^2)^{-2} \\ \text{At } (1, 2) \end{array} \right. \\ &\quad \begin{array}{l} = (-2)(-1)(2-1)^{-2}(2-2) \\ = 0 \end{array} \\ &\quad \begin{array}{l} = \frac{1}{2-1}(-1) + (2)(2-1)^{-2}(1) \\ = -1+2 = 1 \end{array}\end{aligned}$$

$$\frac{\partial f}{\partial y} = \left(\frac{xy}{y-x} \right) \left(\frac{1}{y^2} \right) = \frac{x}{y^2-xy} = x(y^2-xy)^{-1}$$

$$\frac{\partial^2 f}{\partial y^2} = (x)(-1)(y^2-xy)^{-2}(2y-x)$$

At $(1, 2)$,

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= (1)(-1)(4-2)^{-2}(4-1) \\ &= \frac{-3}{4}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= (y^2-xy)^{-1}(1) + (x)(-1)(-y) \\ &= \frac{1}{y^2-xy} + xy \\ &= \frac{xy(y^2-xy)+1}{y^2-xy} = \frac{xy^3-x^2y^2+1}{y^2-xy}\end{aligned}$$

extra.

If $\theta = t^n \cdot e^{-\frac{x^2}{4t}}$, find the value of n for which

$$\frac{1}{9t^2} \left[\frac{\partial}{\partial n} \left(n \cdot \frac{\partial \theta}{\partial n} \right) \right] = \frac{\partial \theta}{\partial t} \text{ is true.}$$

$$\begin{aligned}\frac{\partial \theta}{\partial t} &= \left(e^{-\frac{x^2}{4t}} \right) (n)(t^{n-1}) + (t^n) \left(e^{-\frac{x^2}{4t}} \right) \left(\frac{-x^2}{4t} \right) \left(\frac{-1}{t^2} \right)\end{aligned}$$

$$\begin{aligned}\frac{\partial \theta}{\partial n} &= \left(t^{n-1} \right) \left(e^{-\frac{x^2}{4t}} \right) \left(\frac{-1}{4t} \right) \left(\frac{1}{n} \right)\end{aligned}$$

$$= \left(t^{n-1} \right) \left(e^{-\frac{x^2}{4t}} \right) \left(\frac{-1}{2} \right) (x)$$

$$\therefore \frac{\partial^2 \theta}{\partial n^2} = \left(t^{n-1} \right) \left(e^{-\frac{x^2}{4t}} \right) \left(x^3 \right) / (-1)$$

$$r^2 \frac{\partial \theta}{\partial r} = (t^{n-1})(e^{-\frac{r^2}{4t}})(x^3) \left(\frac{-1}{2} \right)$$

$$\frac{\partial}{\partial r} \left(r^2 \cdot \frac{\partial \theta}{\partial r} \right) = \left(-\frac{t^{n-1}}{2} \right) \left[(x^3) \left(e^{-\frac{r^2}{4t}} \right) \left(\frac{-1}{4t} \right) (2r) - (e^{-\frac{r^2}{4t}})(3r^2) \right]$$

$$\begin{aligned} \frac{1}{r^2} [\dots] &= \left(-\frac{t^{n-1}}{2} \right) \left[(x^3) \left(e^{-\frac{r^2}{4t}} \right) \left(\frac{-1}{2t} \right) - (e^{-\frac{r^2}{4t}})(3) \right] \\ &= \underbrace{\left(t^{n-2} \right) \left(x^2 \right) \left(e^{-\frac{r^2}{4t}} \right)}_4 - \underbrace{\left(t^{n-1} \right) \left(e^{-\frac{r^2}{4t}} \right) (3)}_2 \end{aligned}$$

Equating LHS and RHS,

$$\frac{(t^{n-2})(x^2)(e^{-\frac{r^2}{4t}}) - (3)(t^{n-1})(e^{-\frac{r^2}{4t}})}{4} = \frac{(t^n)(e^{-\frac{r^2}{4t}})(x^2)}{4t^2} + (n)(t^{n-1})(e^{-\frac{r^2}{4t}})$$

Comparing coefficients of like terms,

$$\underline{n = -\frac{3}{2}}$$

⑤ [HW]

$$f_{xxx} = e \log \left(\frac{\pi}{2} \right) \quad f_{xxy} = \frac{2e}{\pi} + 1 \quad f_{xyy} = \frac{-4e}{\pi^2} \quad f_{yyy} = \frac{16e}{\pi^3}$$

④ [HW]

$$⑦ v = r^n (3 \cos^2 \theta - 1)$$

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v}{\partial \theta} \right) = 0$$

$$\text{Soln: } \frac{\partial v}{\partial r} = (3 \cos^2 \theta - 1)(n)(r^{n-1})$$

$$r^2 \left(\frac{\partial v}{\partial r} \right) = (3 \cos^2 \theta - 1)n r^{n+1}$$

$$\frac{\partial}{\partial r} [\dots] = (3 \cos^2 \theta - 1)(n)(n+1)r^n \quad \text{--- ①}$$

$$v = r^n (3 \cos^2 \theta - 1)$$

$$\frac{\partial v}{\partial \theta} = r^n (3(2 \cos \theta)(-\sin \theta))$$

$$\sin \theta \left(\frac{\partial v}{\partial \theta} \right) = -6r^n \sin^2 \theta \cos \theta$$

$$\begin{aligned} \frac{\partial}{\partial \theta} \left(\sin \theta \left(\frac{\partial v}{\partial \theta} \right) \right) &= -6r^n [(\cos \theta)(2)(\sin \theta)(\cos \theta) + (\sin^2 \theta)(-\sin \theta)] \\ &= -6r^n [2 \sin \theta \cos^2 \theta - \sin^3 \theta] \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= -6x^n [2 \sin \theta \cos^2 \theta - 8 \sin^3 \theta] \\ \frac{1}{\sin \theta} [-] &= -6x^n [2 \cos^2 \theta - \sin^2 \theta] \quad \text{--- } ② \end{aligned}$$

$$① + ② = 0$$

$$\begin{aligned} (3 \cos^2 \theta - 1)(n)(n+1)x^n &= 6x^n [2 \cos^2 \theta - \sin^2 \theta] \\ (3 \cos^2 \theta - 1)(n^2+n)x^n &= 6x^n (2 \cos^2 \theta - (1 - \cos^2 \theta)) \\ (3 \cos^2 \theta - 1)(n^2+n)x^n &= 6x^n [3 \cos^2 \theta - 1] \end{aligned}$$

$$n^2 + n = 6$$

$$n^2 + n - 6 = 0$$

$$n^2 - 2n + 3n - 6 = 0$$

$$n(n-2) + 3(n-2) = 0$$

$$(n+3)(n-2) = 0$$

$$n = -3$$

[OR]

$$n = 2$$

⑥ For the point on the surface $x^x \cdot y^y \cdot z^z = c$ where $x=y=z$, show that

$$\frac{\partial^2 z}{\partial x \partial y} = -[x \log(x)]^{-1}$$

Take $x \log x + y \log y + z \log z = \log c$

$$z \log z = \log c - x \log x - y \log y$$

$$\cancel{\frac{\partial}{\partial x}} \left(\frac{\partial z}{\partial y} \right) + (\log z) \left(\frac{\partial z}{\partial y} \right) = 0 + 0 + [\log y (-1) - \cancel{\frac{\partial}{\partial x}} \left(\frac{1}{\partial y} \right)]$$

$$(\log y + 1) \frac{\partial z}{\partial y} = -[\log y + 1]$$

$$\frac{\partial z}{\partial y} = \frac{-[\log y + 1]}{\log z + 1} = -[\log y + 1][\log z + 1]^{-1}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -[\log y + 1](-1)(\log z + 1)^{-2} \left(\frac{1}{z} \right) \left(\frac{\partial z}{\partial x} \right)$$

$$\frac{\partial z}{\partial x} = -\frac{[\log x + 1]}{\log z + 1} \quad \text{--- } ②$$

$$\frac{\partial^2 z}{\partial x \partial y} = -[\log y + 1](-1)[\log z + 1]^{-2} \left(\frac{1}{z} \right) - \frac{[\log x + 1]}{[\log z + 1]^3}$$

At the point where $x=y=z$,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(-1)[\log x + 1](-1)(-1)[\log x + 1]}{[\log x + 1]^3}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -[\log x + \log y]^{-1}$$

$$= -[\log(xy)]^{-1}$$

(8) $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$

(9) $u = \frac{x^{1/3} - y^{1/3}}{x+y}$

(a) $x u_x + y u_y$

(b) $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy}$

Soln: $u = \frac{x^{1/3} \left[1 - \left(\frac{y}{x}\right)^{1/3} \right]}{x \left[1 + \frac{y}{x} \right]} = x^{-2/3} \cdot \phi\left(\frac{y}{x}\right)$

$\Rightarrow u$ is homogeneous

(a) By Euler's theorem,

$$x u_x + y u_y = n u = -\frac{2}{3} [u] = -\frac{2}{3} \left(\frac{x^{1/3} - y^{1/3}}{x+y} \right)$$

(b) By extension of Euler's theorem,

$$x^2 u_{xx} + y^2 u_{yy} + 2xy u_{xy} = n(n-1) u$$

$$= -\frac{2}{3} \left(-\frac{5}{3} \right) (u) = \frac{10}{3} u$$

(10) $u = \sqrt{y^2 - x^2} \sin^{-1}\left(\frac{y}{x}\right)$

Find: $x u_x + y u_y$

Soln: $u = \sqrt{x^2 \left(\frac{y}{x}\right)^2 - 1} \sin^{-1}\left(\frac{y}{x}\right) = x \sqrt{\left(\frac{y}{x}\right)^2 - 1} \sin^{-1}\left(\frac{y}{x}\right) \Rightarrow \text{Degree} = 1$

By Euler's theorem,

$$x u_x + y u_y = n u = \sqrt{y^2 - x^2} \sin^{-1}\left(\frac{y}{x}\right)$$

(3) $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^3 \tan^{-1}\left(\frac{y}{x}\right)$

Find $x^2 u_{xx} + y^2 u_{yy} + 2xy u_{xy}$

Soln: $x^3 \left[\sin^{-1}\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^3 \tan^{-1}\left(\frac{y}{x}\right) \right]$

\therefore u is homogeneous with degree 3.

By extension of Euler's theorem,

$$x^2 u_x + y^2 u_y + 2xy u_{xy} = n(n-1) u$$

By extension of Euler's theorem,

$$\begin{aligned}
 x^2 u_{xx} + y^2 u_{yy} + 2xy \cdot u_{xy} &= n(n-1) u \\
 &= 3(2) u \\
 &= 6u \\
 &= \underline{\underline{6x^3 \sin^{-1}\left(\frac{y}{x}\right) + 6y^3 \tan^{-1}\left(\frac{x}{y}\right)}}
 \end{aligned}$$

④ $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^2 \left(\tan^{-1}\left(\frac{x}{y}\right)\right)$

$A = x^3 \sin^{-1}\left(\frac{y}{x}\right)$, $B = y^2 \tan^{-1}\left(\frac{x}{y}\right)$ } u isn't homogeneous on its own, but can be split into homogeneous functions.

$$\underbrace{x \frac{\partial A}{\partial x} + y \frac{\partial A}{\partial y}}_{\textcircled{1}} = 3A, \quad \underbrace{x \frac{\partial B}{\partial x} + y \frac{\partial B}{\partial y}}_{\textcircled{2}} = 2B$$

① + ②

$$x \frac{\partial A}{\partial x} + x \frac{\partial B}{\partial x} + y \frac{\partial A}{\partial y} + y \frac{\partial B}{\partial y} = 3A + 2B$$

$$x \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial x} \right) + y \left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial y} \right) = 3A + 2B$$

$$x \left(\frac{\partial u}{\partial x} \right) + y \left(\frac{\partial u}{\partial y} \right) = \underline{\underline{3x^3 \sin^{-1}\left(\frac{y}{x}\right) + 2y^2 \tan^{-1}\left(\frac{x}{y}\right)}}$$

⑤ $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^{-3} \tan^{-1}\left(\frac{x}{y}\right)$

Evaluate: $x^2 u_{xx} + 2xy \cdot u_{xy} + y^2 u_{yy} + xu_x + yu_y$

Given: say $A = x^3 \sin^{-1}\left(\frac{y}{x}\right)$ and $B = y^{-3} \tan^{-1}\left(\frac{x}{y}\right)$

For A,

$$A = x^3 \cdot \phi\left(\frac{y}{x}\right) \Rightarrow \text{Homogeneous w/ degree 3}$$

Willy B,

B is homogeneous w/ degree -3

$$x \left(\frac{\partial A}{\partial x} \right) + y \left(\frac{\partial A}{\partial y} \right) = 3A$$

$$x \left(\frac{\partial B}{\partial x} \right) + y \left(\frac{\partial B}{\partial y} \right) = -3B$$

For A,

$$\begin{aligned}
 x^2 A_{xx} + 2xy A_{xy} + y^2 A_{yy} &= (3)(2)A \\
 &= 6A
 \end{aligned}$$

Willy for B,

$$x^2 B_{xx} \dots = (-3)(-4)(B) = 12B$$

$$\Rightarrow x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = \underline{\underline{6A + 12B}}$$

$$x \left(\frac{\partial u}{\partial x} \right) + y \left(\frac{\partial u}{\partial y} \right) = -3B$$

$$\Rightarrow x \left(\frac{\partial u}{\partial x} \right) + y \left(\frac{\partial u}{\partial y} \right) = 3(A-B) \quad \text{--- (1)}$$

From (1) and (2), we have

$$x^2 u_{xx} + 2xy \cdot u_{xy} + y^2 u_{yy} + xu_x + yu_y = (1) + (2)$$

$$= 3A - 3B + 6A + 12B$$

$$= 9A + 9B$$

$$= 9x^3 \sin^{-1}\left(\frac{y}{x}\right) + 9y^3 \tan^{-1}\left(\frac{x}{y}\right)$$

Q. $u = \tan^{-1}\left(\frac{x^2 + y^2}{xy}\right)$

(a) Prove that $xu_x + yu_y = \sin 2u$

(b) Prove that $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = \sin 2u (1 - 4 \sin^2 u)$

Q. $u = \tan^{-1}\left(\frac{y^2}{x}\right)$

$$x^2 u_{xx} + 2xy \cdot u_{xy} + y^2 u_{yy} = -\sin^2 u \cdot \sin 2u$$

Q. Find the total differential of:

$$u = xy$$

$$\begin{aligned} \text{Soln: } du &= \frac{du}{dx} \cdot dx + \frac{du}{dy} \cdot dy \\ &= y \cdot dx + x \cdot \cancel{dy} \end{aligned}$$

Say it is equal to 0.

$$y \cdot dx + x \cdot \cancel{dy} = 0$$

$$d(xy) = 0$$

$$\int d(xy) = \int 0$$

$$\underline{xy = c}$$

Q. Total differential:

$$\underline{y \frac{dx}{y^2} - x \frac{dy}{y^2}} = 0$$

$$\underline{d\left(\frac{x}{y}\right)} = 0$$

$$\frac{x}{y} = c$$

Q. Find $\frac{du}{dt}$, $u = x^2 - y^2$, $x = e^t \cos t$, $y = e^t \sin t$

Ans:

$$u \begin{cases} x \\ y \end{cases} \rightarrow t$$

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \\ &= (2x)[(\cos t)(e^t) + (e^t)(-\sin t)] + [-2y][(\sin t)(e^t) + (e^t)(\cos t)]\end{aligned}$$

At $t = 0$:

$$\begin{aligned}\frac{du}{dt} &= (2x)[-1] \\ &= \underline{\underline{2}} \quad [\because x=1]\end{aligned}$$

Q. Find $\frac{dF}{dt}$ at $t = 0$, where

$$F(x, y, z) = x^3 + xy^2 + y^3 + xyz; \quad x = e^t; \quad y = \cos t; \quad z = t^3$$

$$\begin{aligned}\text{Ans: } \frac{dF}{dt} &= \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt} \\ &= [3x^2 + yz][e^t] + [2y^2 + xz][- \sin t] + [yz + xy][3t^2]\end{aligned}$$

At $t = 0$,

$$\begin{aligned}x &= 1, \quad y = 1, \quad z = 0 \\ &= [3][1] + [0] + [1][0] \\ &= \underline{\underline{3}}\end{aligned}$$

Q. Find the rate at which the area of rectangle is increasing at a given instant when the sides of the rectangle are 5 feet and 4 feet, and are increasing at the rate of 1.5 ft/sec and 0.5 ft/sec respectively.

Ans: Target

$$\frac{dA}{dt} = ?, \quad \text{at } x = 5, y = 4$$

Given:

$$A = xy$$

$$\frac{dx}{dt} = 1.5, \quad \frac{dy}{dt} = 0.5$$

$$\frac{dx}{dt} = 1.5, \quad \frac{dy}{dt} = 0.5$$

Soln:

$$\begin{aligned}\frac{dA}{dt} &= \frac{\partial A}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial A}{\partial y} \cdot \frac{dy}{dt} \\ &= y(1.5) + x(0.5)\end{aligned}$$

$$\text{At } x = 5, \quad y = 4$$

$$\begin{aligned}\frac{dA}{dt} &= 4(1.5) + 5(0.5) \\ &= 6 + 2.5 \\ &= 8.5 \text{ ft}^2/\text{sec}\end{aligned}$$

Area is increasing by $8.5 \text{ ft}^2/\text{sec}$

- Q. The altitude of a cone is 15 cm, and is increasing at 0.2 cm/sec. The radius of the base is 10 cm, and is decreasing at the rate 0.3 cm/sec. How fast is the volume changing?

Ans: Forget:

$$\frac{dV}{dt} = ? \quad \text{at } h = 15 \text{ cm} \\ r = 10 \text{ cm}$$

$$V = \frac{1}{3}\pi r^2 h$$

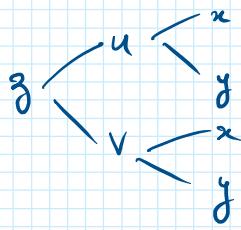


$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt} \\ &= \frac{1}{3}\pi h(2r)(-0.3) + \frac{1}{3}\pi r^2(0.2)\end{aligned}$$

- Q. If $g = \log(u^2 + v)$, where $u = e^{x+y^2}$, $v = x + y^2$,

find the value of $2y \frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}$.

Soln:



Using chain rule,

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= \left(\frac{1}{u^2+v}(2u)\right)(e^{x+y^2}) + \left(\frac{1}{u^2+v}(1)\right)(1)$$

$$= \frac{2ue^{x+y^2}}{u^2+v} + 1$$

$$= \frac{(u^2+v^2)(e^{x+y^2}) + (u^2+v^2)'(1)}{u^2+v^2} \quad \text{--- } \textcircled{1}$$

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial y} \\ &= \left(\frac{2u}{u^2+v^2}\right)(e^{x+y^2})(2y) + \left(\frac{1}{u^2+v^2}\right)(2y) \\ &= \frac{2y}{u^2+v^2} \left(2u \cdot e^{x+y^2} + 1\right) \quad \text{--- } \textcircled{2}\end{aligned}$$

$$2y \times \textcircled{1} - \textcircled{2}$$

$$\begin{aligned}\cancel{2y} \left(\frac{2u \cdot e^{x+y^2} + 1}{u^2+v^2} \right) - \cancel{2y} \left(\frac{2u \cdot e^{x+y^2} + 1}{u^2+v^2} \right) \\ = \underline{\underline{0}}\end{aligned}$$

Q. Show that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta}\right)^2$, where $u = F(x, y)$

Polar eqns.

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

Soln: $u < \begin{matrix} x \\ y \end{matrix} < \begin{matrix} \theta \\ \theta \end{matrix}$

Consider RHS.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} [\cos \theta] + \frac{\partial u}{\partial y} [\sin \theta] \quad \text{--- } \textcircled{1}$$

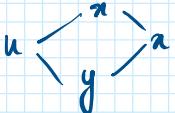
$$\begin{aligned}\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} \\ &= \frac{\partial u}{\partial x} [-r \sin \theta] + \frac{\partial u}{\partial y} [r \cos \theta] \quad \text{--- } \textcircled{2}\end{aligned}$$

Changing $\textcircled{1}$ and $\textcircled{2}$ to match RHS.

$$\begin{aligned}\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta}\right)^2 \\ = \left[\frac{\partial u}{\partial x} [\cos \theta] + \frac{\partial u}{\partial y} [\sin \theta] \right]^2 + \frac{1}{r^2} \left[\frac{\partial u}{\partial x} [-r \sin \theta] + \frac{\partial u}{\partial y} [r \cos \theta] \right]^2\end{aligned}$$

Q. Find $\frac{du}{dx}$, $u = \tan(x^2 + y^2)$, $x^2 - y^2 = 2$

Soln:



$$\begin{aligned}\frac{du}{dx} &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \left(\frac{dy}{dx} \right) \\ &= \sec^2(x^2 + y^2)(2x) + \sec^2(x^2 + y^2)(2y) \left(\frac{dy}{dx} \right) \quad \text{--- (1)}\end{aligned}$$

$$F(x, y) = x^2 - y^2 - 2 = 0$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-2x}{-2y} = \frac{x}{y} \quad \text{--- (2)}$$

Putting (2) in (1),

$$\begin{aligned}\frac{du}{dx} &= \sec^2(x^2 + y^2)(2x) + \sec^2(x^2 + y^2)(2y) \left(\frac{x}{y} \right) \\ &= 2 \left[2x \sec^2(x^2 + y^2) \right] \\ &= \underline{\underline{4x \sec^2(x^2 + y^2)}}\end{aligned}$$

Q. Find $\frac{dy}{dx}$ at the point $(1, 1)$ for $e^y - e^x + xy = 1$

Soln: $F(x, y) = e^y - e^x + xy - 1 = 0$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(e^x + y)}{e^y + x}$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(e^x+y)}{e^y+x}$$

At point $(1, 1)$,

$$\frac{dy}{dx} = \frac{e-1}{e+1}$$

Q. Find $\frac{du}{dx}$, $u = x \log(xy)$, $x^3 + y^3 - 3xy - 1 = 0$