1. Laplace Transforms

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LAPLACE TRANSFORMS

$\xrightarrow[\text{to}]{\text{applied}}$ differential equations $\xrightarrow[\text{to}]{\text{reduces}}$ algebraic expressions

Let $f(t)$ be any real valued fn. where $t \geq 0$ and S may be a real/complex parameter. Then the Laplace transform of $f(t)$ is denoted by $L[f(t)]$ or $F(s)$ and is defined as

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

provided integral exists.

Laplace transforms of standard functions

① $L[a]$, where a is a constant

$$L[a] = \int_0^\infty e^{-st} a\, dt = \left[a \frac{e^{-st}}{-s}\right]_0^\infty = \frac{-a}{s}(-1)$$

$$L[a] = \frac{a}{s}$$

② $L[e^{at}]$

$$L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{(a-s)t} dt = \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{-1}{s-a}(0-1)$$

$$L[e^{at}] = \frac{1}{s-a} \quad ; \quad s>a$$

③ $L[e^{-at}] = \frac{1}{s+a}$

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④ $L[\sin(at)] = \frac{a}{s^2+a^2}$

$$L[\sin(at)] = \int_0^\infty e^{-st} \cdot \sin at\, dt = \frac{-e^{-st}}{s} \sin at$$

$u = \sin at \qquad dv = e^{-st} dt$

$v = \frac{-e^{-st}}{s} \qquad du = \cos at$

s

⑤ $L[\cos(at)] = \dfrac{s}{s^2+a^2}$

⑥ $L[\sinh(at)]$

$$L[\sinh(at)] = \int_0^\infty e^{-st}\sinh(at)\,dt$$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t} + e^{-(s+a)t}\,dt$$

$$= \frac{1}{2}\left(\left[\frac{-e^{-(s-a)t}}{-(s-a)}\right]_0^\infty + \left[\frac{-e^{-(s+a)t}}{-(s+a)}\right]_0^\infty\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$$

$$= \frac{1}{2}\left(\frac{(\cancel{s}+a-\cancel{s}+a)}{s^2-a^2}\right)$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}a}{s^2-a^2}\right)$$

$$= \frac{a}{s^2-a^2}$$

⑦ $L[\cosh(at)] = \dfrac{s}{s^2-a^2}$

⑧ $L[t^n]$

$$L[t^n] = \int_0^\infty e^{-st}\,t^n\,dt$$

$st = x$

$s\,dt = dx$

$$= \int_0^\infty e^{-x}\frac{x^n}{s^n}\frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^n\,dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) \quad \Big\} \text{ n is a fraction}$$

$$= \frac{n!}{s^{n+1}} \quad \Big\} \text{ n is an integer}$$

FORMULAE

	$f(t)$	$L[f(t)]$
1	a	$\frac{a}{s}$
2	e^{at}	$\frac{1}{s-a}$; $s > a$
3	e^{-at}	$\frac{1}{s+a}$
4	$\sin(at)$	$\frac{a}{s^2-a^2}$
5	$\cos(at)$	$\frac{s}{s^2-a^2}$
6	$\sinh(at)$	$\frac{a}{s^2+a^2}$
7	$\cosh(at)$	$\frac{s}{s^2+a^2}$
8	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$ $\frac{n!}{s^{n+1}}$

LINEAR PROPERTY

If $f(t)$ & $g(t)$ $\longrightarrow$ any 2 functions, a & b $\longrightarrow$ any 2 constants, then:

$$\boxed{L[a f(t) \pm b g(t)] = a \cdot L[f(t)] \pm b \cdot L[g(t)]}$$

FIRST SHIFTING PROPERTY

If $\mathcal{L}[f(t)] = F(s)$, then

If $\mathcal{L}[f(t)] = F(s)$, then

$$\boxed{\mathcal{L}[e^{at} f(t)] = F(s-a)}$$

CHANGE OF SCALE PROPERTY

If $\mathcal{L}[f(t)] = F(s)$, then

$$\boxed{\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

$$\boxed{\mathcal{L}[t^n f(t)] = (-1)^n \left[\frac{d^n}{ds^n} F(s)\right]}$$

Example

$\mathcal{L}[t e^{4t} \cos 2t]$

$$\mathcal{L}[\cos 2t] = \frac{s}{s^2-4}$$

$$\mathcal{L}[t \cos 2t] = \frac{(s^2-4) - s(2s)}{(s^2-4)^2}$$

$$\mathcal{L}[t e^{4t} \cos 2t] = \frac{(s-4)^2 - 4 - 2(s-4)^2}{[(s-4)^2-4]^2}$$

III

$$\boxed{\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds}$$

Example

$$\mathcal{L}\left[\frac{e^{-at} - e^{-bt}}{t}\right]$$

$$\mathcal{L}[e^{-at} - e^{-bt}] = \frac{1}{s+a} - \frac{1}{s+b}$$

$$\mathcal{L}\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) dt$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^{\infty}$$

$$= \left[\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)\right]_s^{\infty}$$

$$= \cancelto{0}{\log 1} - \log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

IV

$$\boxed{\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)}$$

Example

$$\mathcal{L}\left[\int_0^t \frac{1-e^{at}}{t}\,dt\right]$$

$$\mathcal{L}\left[\frac{1-e^{at}}{t}\right] = \int_s^{\infty} \frac{1}{s} -$$

$$= \frac{1}{s}\log\left(\frac{s-a}{s}\right)$$