4. Heisenberg's Uncertainty Principle, Eigenvalue Equation

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HEISENBERG'S UNCERTAINTY PRINCIPLE

$$\underbrace{\Delta x}_{\text{uncertainty in position}} \overbrace{\Delta p_x}^{\text{uncertainty in x-component of momentum}} \geq \frac{h}{4\pi}$$

$$\underbrace{\Delta E}_{\text{uncertainty in energy of a state}} \overbrace{\Delta t}^{\text{uncertainty in lifetime of the state}} \geq \frac{h}{4\pi}$$

$$\underbrace{\Delta L}_{\text{uncertainty in angular momentum}} \overbrace{\Delta \theta}^{\text{uncertainty in angle b/w angular momentum vector}} \geq \frac{h}{4\pi}$$

$$E = \frac{hc}{\lambda}$$

$$\frac{dE}{d\lambda} = \frac{hc}{\lambda^2}$$

$$\boxed{\Delta E = \frac{hc}{\lambda^2} \Delta\lambda}$$

Example ①

$\lambda = 400\text{ nm}$

$\Delta\lambda = 1\text{ nm}$

$\Delta t = ?$

$$\Delta E = \frac{hc}{\lambda^2} \Delta\lambda$$

$$\Delta E = \frac{hc}{(400\times10^{-9})^2}(10^{-9}) = 1.2423\times10^{-21}$$

$$\Delta E\,\Delta t \geq \frac{h}{4\pi}$$

$$\Delta t \geq 4.24\times10^{-14}\text{ s}$$

Show that an electron cannot exist inside a nucleus

Beta decay:

$$X^Z_A \Longrightarrow Y^{Z-1}_A \longrightarrow e^-$$

Assume nucleus of some diameter $d = 2\text{ fm} = 2\times10^{-15}\text{ m}$

Max $\Delta x = 2\times10^{-15}\text{ m}$

$$\Delta p = m\Delta v$$

$$\Delta p \geq \frac{h}{4\pi\Delta x}$$

$$\Rightarrow \Delta p \geq 2.6\times10^{-20}\text{ kg m/s}$$

$$\Delta v = \frac{\Delta p}{m} = 2.89\times10^{10}\text{ m/s}$$

v would be greater than Δv, but here we observe that

$$v > \Delta v > c$$

which is not possible.

$V > \Delta v > c$

which is not possible.

IIIly, $E = \frac{p^2}{2m} \gg \frac{(\Delta p)^2}{2m}$

$E \gg 2300$ MeV

Actual experimental value of E is around 2-20 MeV while here it suggests that E is much greater than 2300 MeV which is absurd.

Thus from the above we can conclude the electron cannot exist inside the nucleus.

Example (2)

An electron's velocity is measured to be 4.5×10^6 m/s with uncertainty of 1m/s. Find the uncertainty in its position.

OBSERVABLES

Dynamical Variables → involved in the dynamics of a particle

Eg: Position, velocity, momentum, KE, PE, TE

Ψ, t are <u>NOT</u> observables.

- All measurable quantities
- Every observable → associated with an operator

EIGEN VALUE EQUATION

$$\boxed{\underbrace{\hat{G}}_{\text{operator}}\underbrace{\Psi_n}_{\text{set of eigen functions } \{\Psi_1, \Psi_2 \cdots \Psi_n\}} = \lambda_n \Psi_n}$$

λ_n → set of eigen values $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ where all $\lambda_i \in \mathbb{R}$ → only in quantum mechanics

Eg: $\frac{d}{dx} e^{ax} = a e^{ax}$

$\frac{d^2}{dx^2}(\cos 4x) = -16 \cos 4x$

wave func Ψ

$\Psi = A e^{-i(\omega t - kx)}$

$\omega = 2\pi\nu$ $\quad$ $E = h\nu$ $\quad$ $k = \frac{2\pi}{\lambda}$ $\quad$ $\lambda = \frac{h}{p}$

$\omega = \frac{2\pi E}{h}$ $\quad$ $\nu = \frac{E}{h}$ $\quad$ $k = \frac{2\pi p}{h}$

$\omega = \frac{2\pi E}{h}$ $\qquad \nu = \frac{E}{h}$ $\qquad \lambda$

$k = \frac{2\pi p}{h}$

$$\boxed{\omega = \frac{E}{\hbar}} \qquad \boxed{k = \frac{p}{\hbar}}$$

$$\psi = A e^{\frac{-i}{\hbar}(Et - px)}$$

$$\frac{\partial \psi}{\partial x} = A e^{\frac{-i}{\hbar}(Et - px)} \left(\frac{ip}{\hbar}\right)$$

$$\psi = Ae^{u}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x}[\psi] = \frac{ip}{\hbar} \psi$$

$$\frac{\hbar}{i} \frac{\partial}{\partial x}[\psi] = p\psi$$

$$-i\hbar \frac{\partial}{\partial x}[\psi] = p\psi$$

$$\boxed{\hat{p}\psi = p\psi}$$

momentum operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

IIIly,

$$\psi = Ae^{\frac{-i}{\hbar}(Et - px)}$$

$$\frac{\partial \psi}{\partial t} = Ae^{\frac{-i}{\hbar}(Et - px)} \left(\frac{-i}{\hbar} E\right)$$

$$\frac{\hbar}{-i} \frac{\partial \psi}{\partial t} = E \; Ae^{\frac{-i}{\hbar}(Et - px)}$$

$$i\hbar \frac{\partial}{\partial t}[\psi] = E\psi$$

$$\boxed{\hat{E}\psi = E\psi}$$

energy operator $\hat{E} = i\hbar \frac{\partial}{\partial t}$

IIIly,

$$\frac{\partial \psi}{\partial x} = \frac{ip}{\hbar} Ae^{\frac{-i}{\hbar}(Et - px)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{ip}{\hbar}\right)^2 Ae^{\frac{-i}{\hbar}(Et - px)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{-p^2}{\hbar^2} \psi$$

$$-\hbar^2 \frac{\partial^2}{\partial x^2}[\psi] = p^2 \psi$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{-p^2}{\hbar^2} \Psi$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\Psi] = \underbrace{\frac{p^2}{2m}}_{\text{kinetic energy}} \Psi$$

$$\boxed{\underbrace{\hat{K}}_{\text{kinetic energy operator}} \Psi = K\Psi}$$

kinetic energy operator $\hat{K} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

Operator for position x

$$\hat{x}\Psi = x\Psi$$

$$\boxed{\hat{x} = x}$$

Operator for potential energy V(x)

$$\boxed{V(x) \text{ itself is the operator}}$$

For SHM

$$V(x) = \frac{1}{2}kx^2$$

For hydrogen atom

$$V(x) = \frac{1}{4\pi\varepsilon_0}\frac{e}{x}$$

$$\hat{E} = \hat{K} + \hat{V}$$

$$\boxed{\underbrace{\hat{H}}_{\text{Hamiltonian operator}} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \hat{V}(x)}$$

Comparing $\hat{E}$, $\hat{H}$:

$$\hat{E}\Psi = E\Psi \qquad i\hbar\frac{\partial}{\partial t}\Psi = E\Psi$$

$$\hat{H}\Psi = E\Psi$$

$$\boxed{\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi = E\Psi}$$

given $V(x)$ is a function of x alone $\Longrightarrow$ <u>Schrodinger's Time Independent Equation</u>

Also,

$$\hat{E}\Psi = \hat{H}\Psi$$

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\psi = i\hbar\frac{\partial}{\partial t}\psi$$

$$\psi(x,t)$$

Schrodingers Time Dependent Equation