Time Series

Components of TSD

Trend (T_t)
- Upward upward / downward direction
- Consistent long term

Seasonal (S_t) — e.g. heatwave
Repetitive upward / downward movement within a calendar year, fixed periodicity

Seasonality index (Holt-Winters) represents a repeating pattern in data at fixed intervals

Cyclic (C_t) — e.g. stock market
Fluctuations around trend of more than a year. Non-constant periodicity

Irregular (I_t)
White noise
$\sim N(0, c)$

Models:

① Additive
$$Y_t = T_t + S_t + C_t + I_t$$
→ rare for trend to be indep. of S, C.
→ spikes the same size

② Multiplicative
$$Y_t = T_t \cdot S_t \cdot C_t \cdot I_t$$
simpler form: $Y_t = T_t S_t$
→ increasing spike size

③ Seasonally Adjusted TS (SA_t)
Remove seasonality
Add: $SA_t = Y_t - S_t$
Multi: $SA_t = \frac{Y_t}{S_t}$

④ Pseudo-Additive Time Series
combo wombo
$$\boxed{Y_t = T_t + T_t(S_t - 1) + T_t(I_t - 1)}$$
→ Assumes: S_t, I_t dependent on T_t
but S_t, I_t not dependent on each other
→ used when $\log Y_t$ has a lot of small values, $|S_t|$ and $|I_t|$ depend on trend
→ $T_t \to$ same unit
$S_t, I_t \to$ unitless

Forecasting

- Simple
  - moving avg: $\boxed{F_{t+1} = \frac{1}{N}\sum_{k=t+1-N}^{t} Y_k}$
  - weighted avg moving: $\boxed{F_{t+1} = \frac{1}{N}\sum_{k=t+1-N}^{t} W_k \cdot Y_k}$ $\boxed{\sum_k W_k = 1}$
  - Simple Exponential Smoothing (EWMA)
    $$\boxed{F_{t+1} = \alpha Y_t + (1-\alpha)F_t}$$
    α = smoothing constant
    $\alpha \uparrow$: recent value weightage
    $\alpha \downarrow$: old value "
- Regression
- Auto Regression
- AR with Moving Avg
- AR Integrate MA
- ARIMA with X

- short term use
- dynamic
- no minimum data

- makes use of all past data unlike MA
- progressive decreasing weights to old data
- as n increases, becomes less sensitive to changes in data
- always lags behind trend
- larger n → greater lag
- does not take strong trend/seasonal data into account

Double Exponential Smoothing (Holt's Method)

account for trend by adding another equation

Level: $L_t = \alpha Y_t + (1-\alpha)F_{t-1}$

Trend: $T_t = \beta(L_t - L_{t-1}) + (1-\beta)T_{t-1}$

Forecast: $F_{t+n} = L_t + nT_t$

$0 < \alpha < 1$
$0 < \beta < 1$

Initialisation:
$L_t = Y_t$
$T_t = (Y_t - Y_{t-1})$ OR $(Y_t - Y_1)/(t-1)$

Triple Exponential Smoothing (Holt-Winter)

account for seasonality

Level: $L_t = \alpha\left(\frac{Y_t}{S_{t-c}}\right) + (1-\alpha)\overbrace{\left[L_{t-1} + T_{t-1}\right]}^{F \text{ w/o } S}$
remove S

Trend: $T_t = \beta[L_t - L_{t-1}] + (1-\beta)T_{t-1}$

Seasonality: $S_t = \gamma\left[\frac{Y_t}{L_t}\right] + (1-\gamma)S_{t-c}$

Forecast: $F_{t+1} = [L_t + T_t]S_{t+1-c}$

c = no of seasons
monthly = 12
daily = 7

Additive

Level: $L_t = \alpha[Y_t - S_{t-c}] + (1-\alpha)[L_{t-1} + T_{t-1}]$

Trend: $T_t = \beta[L_t - L_{t-1}] + (1-\beta)T_{t-1}$

Seasonality: $S_t = \gamma[Y_t - L_t] + (1-\gamma)S_{t-c}$

Forecast: $F_{t+m} = L_t + mT_t + S_{t+m-c}$

Initialisation
$L_t = Y_t$ [or] $\frac{1}{c}(Y_1 + Y_2 + \dots + Y_c)$
$$T_t = \frac{1}{c}\left[\frac{Y_t - Y_{t-c}}{c} + \frac{Y_{t-1} - Y_{t-1-c}}{c} + \dots + \frac{Y_{t-c+1} + Y_{t-2c+1}}{c}\right]$$

Seasonality Index

① Find average Y for every season: $(\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_c)$
② Avg of avgs: $\bar{\bar{Y}}$
③ Seasonality index for season k = $\frac{\bar{Y}_k}{\bar{\bar{Y}}}$

Regression Forecasting

instead of past values, use current values of other related vars
$$F_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \dots + \beta_n X_{nt} + \varepsilon_t$$
→ check for auto-correlation amongst errors → Durbin Watson test
→ Eval with R^2 score
auto-correlation → inclusion of non-significant variable in the model

Seasonal variation

① Estimate seasonal index S_t (avg of avgs)
② Deseasonalize data:
multi: $Y_{d,t} = \frac{Y_t}{S_t}$
add: $Y_{d,t} = Y_t - S_t$
③ Set up regression to find $F_{d,t+1}$
④ Add seasonality back:
$F_{t+1} = F_{d,t+1} \times S_{t+1}$

Croston's method for intermittent demand

forecast → time b/w demand, magnitude of demand

① Z_t: non-zero demand values
P_t: period between demand arrivals
② Apply SES to both independently
③ $F_t = \frac{Z_t}{P_t}$

Stationarity

"the way the time-series changes does not change over time"

TS data that looks about the same regardless of when you look at it
→ properties do not depend on t
→ trend, seasonality ✗
→ cyclic, white noise ✓
→ no predictable patterns in the long term

Conditions:
① Mean of Y_t at different t → constant
② Variances of Y_t at " " → " (homoscedasticity)
③ Covariances of Y_t and Y_{t-k} solely depend on k, not t

AR, MA, ARMA → assume stationarity
→ assume errors follow white noise $N(0, c)$

Auto Regression (AR)

Regression but on past values of variable
$$\boxed{Y_{t+1} = \beta Y_t + \varepsilon_{t+1}}$$ } AR(1) process, 1 lag value

Mean centring: $(Y_{t+1} - \mu) = \beta(Y_t - \mu) + \varepsilon_{t+1}$

You can continue recursively expanding
$$Y_{t+1} - \mu = \beta(\beta(Y_{t-1} - \mu) + \varepsilon_t) + \varepsilon_{t+1}$$
$$\vdots$$
$$\boxed{Y_{t+1} - \mu = \beta^t(Y_0 - \mu) + \sum_{k=1}^{t-1}\beta^{t-k}\varepsilon_k + \varepsilon_{t+1}}$$

$|\beta| > 1 \to \beta^t(Y_0 - \mu)$ would explode
$|\beta| = 1 \to$ all previous terms w/ same weight → non-stationarity
$|\beta| < 1 \to$ ideal

Estimating β with OLS
$$Y_t - \mu = \beta(Y_{t-1} - \mu) + \varepsilon_t$$
$$\varepsilon_t = \underbrace{(Y_t - \mu)}_{y_i} - \beta\underbrace{(Y_{t-1} - \mu)}_{x_i}$$
$$\boxed{\hat{\beta} = \frac{\sum_{t=2}^{n}(Y_t - \mu)(Y_{t-1} - \mu)}{\sum_{t=2}^{n}(Y_{t-1} - \mu)^2}}$$ } $\hat{\beta} = \frac{\mathrm{cov}(X,Y)}{\sigma_x^2}$

Short term forecast:
Forecast horizon = 1
$F_{t+1} - \mu = \beta(Y_t - \mu)$

Long term forecast:
" " > 1
$F_{t+1} - \mu = \beta(F_t - \mu)$

Moving Average (MA)

Past residuals used for forecasting
$$\boxed{Y_{t+1} = \mu + \alpha_1\varepsilon_t + \varepsilon_{t+1}}$$ } MA(1)
$$\boxed{Y_{t+1} = \mu + \alpha_1\varepsilon_t + \alpha_2\varepsilon_{t-2} + \dots + \alpha_q\varepsilon_{t-q+1} + \varepsilon_{t+1}}$$ } MA(q)

ARMA

ARMA(p,q) = AR(p) + MA(q)
$$\boxed{\underbrace{Y_{t+1} = \beta_1 Y_t + \beta_2 Y_{t-1} + \dots + \beta_p Y_{t-p+1}}_{\text{capture trend}} + \underbrace{\alpha_1\varepsilon_t + \alpha_2\varepsilon_{t-1} + \dots + \alpha_q\varepsilon_{t-q+1}}_{\text{capture deviations}} + \varepsilon_{t+1}}$$