Classroom Problems

Thursday, September 7, 2023 10:38 PM

① Find first order partial derivatives

(i) $f(x,y) = x^4 - x^2y^2 + y^4$ at $(-1,1)$

$\frac{\partial}{\partial x} = 4x^3 - (y^2)(2x)$ $\qquad$ $\frac{\partial}{\partial y} = -x^2(2y) + 4y^3$

At $(-1,1)$,

$= -4 - (-2)$ $\qquad$ $= -1(2) + 4$

$= -2$ $\qquad$ $= 2$

(ii) $f(x,y) = x^2 e^{-y/x}$ at $(4,2)$

$\frac{\partial f}{\partial x} = \cancel{x^2}\left[e^{-y/x}(y)\left(\frac{1}{\cancel{x^2}}\right)\right] + e^{-y/x}(2x)$

$= (2x+y)e^{-y/x}$

At $(4,2)$,

$= 10e^{-\frac{1}{2}}$

$\frac{\partial f}{\partial y} = x^2\left[e^{-y/x}\left(\frac{1}{x}\right)(-1)\right]$

$= -x\, e^{-y/x}$

$= -4\, e^{-\frac{1}{2}}$

② $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for all $(x,y) \neq 0,0$ when $f(x,y) = x^y$. Prove.

$\frac{\partial f}{\partial x} = y x^{y-1}$

$\frac{\partial^2 f}{\partial y \partial x} = (y)(x^{y-1})(\log_e x) + x^{y-1}(1)$

$\frac{\partial f}{\partial y} = x^y . \log x$

$\frac{\partial^2 f}{\partial x \partial y} = (\log(x))(y x^{y-1}) + (x^y)\left(\frac{1}{x}\right)$

③ Find all second order partial derivatives of $f(x,y) = \log\left(\frac{1}{x} - \frac{1}{y}\right)$ at $(1,2)$

$\frac{\partial f}{\partial x} = \frac{1}{\left(\ldots\right)}\left(\frac{-1}{x^2}\right)$

(3) find all second order partial derivatives of $f(x,y) = \log\left(\frac{1}{x} - \frac{1}{y}\right)$ at $(1,2)$

$$\frac{\partial f}{\partial x} = \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)}\left(\frac{-1}{x^2}\right) \qquad \frac{y-x}{xy}$$

$$= \frac{xy}{y-x}\left(\frac{-1}{x^2}\right)$$

$$= \frac{-y}{xy - x^2} = -y(xy-x^2)^{-1}$$

$$\frac{\partial^2 f}{\partial x^2} = (-y)(-1)(xy-x^2)^{-2}(y-2x)$$

At $(1,2)$

$$= (-2)(-1)(2-1)^{-2}(2-2)$$

$$= 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{1}{xy-x^2}(-1) + (-y)(-1)(xy-x^2)^{-2}(x)$$

At $(1,2)$

$$= \frac{1}{2-1}(-1) + (2)(2-1)^{-2}(1)$$

$$= -1+2 = 1$$

$$\frac{\partial f}{\partial y} = \left(\frac{xy}{y-x}\right)\left(\frac{1}{y^2}\right) = \frac{x}{y^2-xy} = x(y^2-xy)^{-1}$$

$$\frac{\partial^2 f}{\partial y^2} = (x)(-1)(y^2-xy)^{-2}(2y-x)$$

At $(1,2)$,

$$\frac{\partial^2 f}{\partial y^2} = (1)(-1)(4-2)^{-2}(4-1)$$

$$= \frac{-3}{4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = (y^2-xy)^{-1}(1) + (x)(-1)(-y)$$

$$= \frac{1}{y^2-xy} + xy$$

$$= \frac{xy(y^2-xy)+1}{y^2-xy} = \frac{xy^3 - x^2y^2 + 1}{y^2-xy}$$

Extra.

If $\theta = t^n \cdot e^{\frac{-r^2}{4t}}$, find the value of n for which

$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right)\right] = \frac{\partial \theta}{\partial t}$ is true.

Soln.

$$\frac{\partial \theta}{\partial t} = \left(e^{\frac{-r^2}{4t}}\right)(n)(t^{n-1}) + (t^n)\left(e^{\frac{-r^2}{4t}}\right)\left(\frac{-r^2}{4}\right)\left(\frac{-1}{t^2}\right)$$

$$\frac{\partial \theta}{\partial r} = (\cancel{t^n})\,t^{n-1}\left(e^{\frac{-r^2}{4t}}\right)\left(\frac{-1}{\cancel{4t}}\right)(\cancel{2}r)$$

$$= (t^{n-1})\left(e^{\frac{-r^2}{4t}}\right)\left(\frac{-1}{2}\right)(r)$$

$$r^2 \frac{\partial \theta}{\partial r} = (t^{n-1})\left(e^{\frac{-r^2}{4t}}\right)(r^3)\left(-\frac{1}{\ }\right)$$

$$r^2 \frac{\partial \theta}{\partial r} = (t^{n-1})\left(e^{\frac{-r^2}{4t}}\right)(r^3)\left(\frac{-1}{2}\right)$$

$$\frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial \theta}{\partial r}\right) = \left(\frac{-t^{n-1}}{2}\right)\left[(r^3)\left(e^{\frac{-r^2}{4t}}\right)\left(\frac{-1}{\cancel{4}_{2}t}\right)(\cancel{2}r) - \left(e^{\frac{-r^2}{4t}}\right)(3r^2)\right]$$

$$\frac{1}{r^2}[\dots] = \left(\frac{-t^{n-1}}{2}\right)\left[(r^2)\left(e^{\frac{-r^2}{4t}}\right)\left(\frac{-1}{2t}\right) - \left(e^{\frac{-r^2}{4t}}\right)(3)\right]$$

$$= \frac{(t^{n-2})(r^2)\left(e^{\frac{-r^2}{4t}}\right)}{4} - \frac{(t^{n-1})\left(e^{\frac{-r^2}{4t}}\right)(3)}{2}$$

Equating LHS and RHS,

$$\frac{(t^{n-2})(r^2)\left(e^{\frac{-r^2}{4t}}\right)}{4} - \frac{(3)(t^{n-1})\left(e^{\frac{-r^2}{4t}}\right)}{2} = \frac{(t^n)\left(e^{\frac{-r^2}{4t}}\right)(r^2)}{4t^2} + (n)(t^{n-1})\left(e^{\frac{-r^2}{4t}}\right)$$

Comparing coefficients of like terms,

$$n = \frac{-3}{2}$$

(5) [HW]

$f_{xxx} = e\log\left(\frac{\pi}{2}\right)$ $\quad f_{xxy} = \frac{2e}{\pi} + 1$ $\quad f_{xyy} = \frac{-4e}{\pi^2}$ $\quad f_{yyy} = \frac{16e}{\pi^3}$

(4) [HW]

(7) $v = r^n(3\cos^2\theta - 1)$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) = 0$$

Soln. $\frac{\partial v}{\partial r} = (3\cos^2\theta - 1)(n)(r^{n-1})$

$$r^2\left(\frac{\partial v}{\partial r}\right) = (3\cos^2\theta - 1)\, n\, r^{n+1}$$

$$\frac{\partial}{\partial r}[\dots] = (3\cos^2\theta - 1)(n)(n+1)r^n \longleftarrow (1)$$

$$v = r^n(3\cos^2\theta - 1)$$

$$\frac{\partial v}{\partial \theta} = r^n\left(3(2\cos\theta)(-\sin\theta)\right)$$

$$\sin\theta\left(\frac{\partial v}{\partial \theta}\right) = -6r^n \sin^2\theta\cos\theta$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta\left(\frac{\partial v}{\partial \theta}\right)\right) = -6r^n\left[(\cos\theta)(2)(\sin\theta)(\cos\theta) + (\sin^2\theta)(-\sin\theta)\right]$$

$$= -6r^n\left[2\sin\theta\cos^2\theta - \sin^3\theta\right]$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 6r^n\left[(\cos\theta)(-2\sin\theta)(\cos\theta) + (\sin\theta)(-\sin\theta)\right]$$

$$= -6r^n\left[2\sin\theta\cos^2\theta - \sin^3\theta\right]$$

$$\frac{1}{\sin\theta}\left[\cdots\right] = -6r^n\left[2\cos^2\theta - \sin^2\theta\right] \text{ ———— } 2$$

$1 + 2 = 0$

$$(3\cos^2\theta - 1)(n)(n+1)r^n = 6r^n\left[2\cos^2\theta - \sin^2\theta\right]$$

$$(3\cos^2\theta - 1)(n^2+n)r^n = 6r^n\left(2\cos^2\theta - (1-\cos^2\theta)\right)$$

$$\cancel{(3\cos^2\theta - 1)}(n^2+n)\cancel{r^n} = 6\cancel{r^n}\left[\cancel{3\cos^2\theta - 1}\right]$$

$$n^2 + n = 6$$

$$n^2 + n - 6 = 0$$

$$n^2 - 2n + 3n - 6 = 0$$

$$n(n-2) + 3(n-2) = 0$$

$$(n+3)(n-2) = 0$$

$n = -3$

[OR]

$n = 2$

6) For the point on the surface $x^x \cdot y^y \cdot z^z = c$ where $x = y = z$, show that

$$\frac{\partial^2 z}{\partial x \partial y} = -\left[x \log(ex)\right]^{-1}$$

Soln $x\log x + y\log y + z\log z = \log c$

$$z\log z = \log c - x\log x - y\log y$$

$$\cancel{z}\left(\frac{1}{\cancel{z}}\right)\left(\frac{\partial z}{\partial y}\right) + (\log z)\left(\frac{\partial z}{\partial y}\right) = 0 + 0 + \left[\log y(-1) - \cancel{y}\left(\frac{1}{\cancel{y}}\right)\right]$$

$$(\log z + 1)\frac{\partial z}{\partial y} = -\left[\log y + 1\right]$$

$$\frac{\partial z}{\partial y} = \frac{-[\log y + 1]}{\log z + 1} = -[\log y + 1][\log z + 1]^{-1}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -[\log y + 1](-1)(\log z + 1)^{-2}\left(\frac{1}{z}\right)\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial z}{\partial x} = \frac{-[\log x + 1]}{\log z + 1} \text{ ———— } 2)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -[\log y + 1](-1)[\log z + 1]^{\cancel{-2}\,-3}\left(\frac{1}{z}\right)\frac{-[\log x + 1]}{\cancel{[\log z + 1]}}$$

At the point where $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(-1)[\cancel{\log x + 1}](-1)(-1)[\cancel{\log x + 1}]}{[\log x + 1]^{3}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\left[\log x + \log e\right]^{-1}$$
$$= -\left[\log(ex)\right]^{-1}$$

(8) $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}$

(9) $u = \frac{x^{1/3} - y^{1/3}}{x+y}$

(a) $x\, u_x + y\, u_y$

(b) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy}$

Soln: $u = \frac{x^{\frac{1}{3}}\left[1 - \left(\frac{y}{x}\right)^{\frac{1}{3}}\right]}{x\left[1 + \frac{y}{x}\right]} = x^{-\frac{2}{3}} \cdot \phi\left(\frac{y}{x}\right)$

$\Rightarrow$ u is homogeneous

(a) By Euler's theorem,

$x\, u_x + y\, u_y = nu = \frac{-2}{3}\left[u\right] = \frac{-2}{3}\left(\frac{x^{1/3} - y^{1/3}}{x+y}\right)$

(b) By extension of Euler's theorem,

$x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = n(n-1)u)$

$= \frac{-2}{3}\left(\frac{-5}{3}\right)(u) = \frac{10}{3}u$

(10) $u = \sqrt{y^2 - x^2}\, \sin^{-1}\left(\frac{y}{x}\right)$

Find: $x\, u_x + y\, u_y$

Soln: $u = \sqrt{x^2\left(\left(\frac{y}{x}\right)^2 - 1\right)}\, \sin^{-1}\left(\frac{y}{x}\right) = x\sqrt{\left(\frac{y}{x}\right)^2 - 1}\, \sin^{-1}\left(\frac{y}{x}\right) \Rightarrow$ Degree $= 1$

By Euler's theorem,

$x\, u_x + y\, u_y = nu = \sqrt{y^2 - x^2}\, \sin^{-1}\left(\frac{y}{x}\right)$

(3) $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^3 \tan^{-1}\left(\frac{y}{x}\right)$

Find $x^2 u_{xx} + y^2 u_{yy} + 2xy \cdot u_{xy}$

Soln: $x^3\left[\sin^{-1}\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^3 \tan^{-1}\left(\frac{y}{x}\right)\right]$

$\therefore$ Fn is homogeneous with degree 3.

By extension of Euler's theorem,

$x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = n(n-1)u$

By extension of Euler's theorem,

$$x^2 u_{xx} + y^2 u_{yy} + 2xy \cdot u_{xy} = n(n-1)u$$
$$= 3(2)u$$
$$= 6u$$
$$= 6x^3 \sin^{-1}\left(\frac{y}{x}\right) + 6y^3 \tan^{-1}\left(\frac{y}{x}\right)$$

(4) $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^2\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$

$A = x^3 \sin^{-1}\left(\frac{y}{x}\right)$, $B = y^2 \tan^{-1}\left(\frac{x}{y}\right)$ } u isn't homogeneous on its own, but can be split into homogeneous functions.

$$\underbrace{x\frac{\partial A}{\partial x} + y\frac{\partial A}{\partial y} = 3A}_{(1)}, \quad \underbrace{x\frac{\partial B}{\partial x} + y\frac{\partial B}{\partial y} = 2B}_{(2)}$$

(1) + (2)

$$x\frac{\partial A}{\partial x} + x\frac{\partial B}{\partial x} + y\frac{\partial A}{\partial y} + y + \frac{\partial B}{\partial y} = 3A + 2B$$

$$x\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial x}\right) + y\left(\frac{\partial A}{\partial y} + \frac{\partial B}{\partial y}\right) = 3A + 2B$$

$$x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 3x^3 \sin^{-1}\left(\frac{y}{x}\right) + 2y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

(5) $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + y^{-3} \tan^{-1}\left(\frac{x}{y}\right)$

Evaluate : $x^2 u_{xx} + 2xy \cdot u_{xy} + y^2 u_{yy} + x u_x + y u_y$

Soln: Say $A = x^3 \sin^{-1}\left(\frac{y}{x}\right)$ and $B = y^{-3} \tan^{-1}\left(\frac{x}{y}\right)$

For A,

$A = x^3 \cdot \phi\left(\frac{y}{x}\right) \Rightarrow$ Homogeneous w/ degree 3

IIIly B,

B is homogeneous w/ degree -3

$$x\left(\frac{\partial A}{\partial x}\right) + y\left(\frac{\partial A}{\partial y}\right) = 3A$$

$$x\left(\frac{\partial B}{\partial x}\right) + y\left(\frac{\partial B}{\partial y}\right) = -3B$$

For A,

$$x^2 A_{xx} + 2xy A_{xy} + y^2 A_{yy} = (3)(2)A$$
$$= 6A$$

IIIly for B,

$$x^2 B_{xx} \dots = (-3)(-4)(B) = 12B$$

$$\Rightarrow x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 6A + 12B$$

$$x\left(\frac{\partial B}{\partial x}\right)+y\left(\frac{\partial B}{\partial y}\right)=-3B$$

$$\Rightarrow x\left(\frac{\partial u}{\partial x}\right)+y\left(\frac{\partial u}{\partial x}\right)=3(A-B) \quad \text{--- } (1)$$

$$\Rightarrow x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}=6A+12B \quad \text{--- } (2)$$

From (1) and (2), we have

$$x^2u_{xx}+2xy\cdot u_{xy}+y^2u_{yy}+xu_x+yu_y=(1)+(2)$$

$$=3A-3B+6A+12B$$

$$=9A+9B$$

$$=9x^3\sin^{-1}\left(\frac{y}{x}\right)+9y^{-3}\tan^{-1}\left(\frac{x}{y}\right)$$

Q: $u=\tan^{-1}\left(\frac{x^4+y^4}{xy}\right)$

(a) Prove that $xu_x+yu_y=\sin 2u$

(b) Prove that $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}=\sin 2u\left(1-4\sin^2u\right)$

Q. $u=\tan^{-1}\left(\frac{y^2}{x}\right)$

$$x^2u_{xx}+2xy\cdot u_{xy}+y^2u_{yy}=-\sin^2u\cdot\sin 2u$$

Q. Find the total differential of:

$$u=xy$$

Soln: $du=\frac{du}{dx}\cdot dx+\frac{du}{dy}\cdot dy$

$$=y\cdot dx+x\cdot dy$$

Say it is equal to 0.

$$y\cdot dx+x\cdot dy=0$$

$$d(xy)=0$$

$$\int d(xy)=\int 0$$

$$xy=c$$

Q: Total differential:

$$\frac{y\,dx-x\,dy}{y^2}=0$$

Soln: $d\left(\frac{x}{y}\right)=0$

$$\frac{x}{y} = c$$

Q. Find $\frac{du}{dt}$, $u = x^2 - y^2$, $x = e^t \cos t$, $y = e^t \sin t$

Soln.

$$\begin{tikzcd} & x \arrow[dr] & \\ u \arrow[ur] \arrow[dr] & & t \\ & y \arrow[ur] & \end{tikzcd}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x)\left[(\cos t)(e^t) + (e^t)(-\sin t)\right] + \left[(-2y)\left[(\sin t)(e^t) + (e^t)(\cos t)\right]\right]$$

At $t = 0$:

$$\frac{du}{dt} = (2x)[1]$$

$$= \underline{\underline{2}} \qquad [\because x = 1]$$

Q. Find $\frac{dF}{dt}$ at $t = 0$, where

$$F(x,y,z) = x^3 + xz^2 + y^3 + xyz \,;\quad x = e^t \,;\quad y = \cos t \,;\quad z = t^3$$

Soln: $$\frac{dF}{dt} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt}$$

$$= \left[3x^2 + z^2 + yz\right]\left[e^t\right] + \left[3y^2 + xz\right]\left[-\sin t\right] + \left[2xz + xy\right]\left[3t^2\right]$$

At $t = 0$,

$x = 1$, $y = 1$, $z = 0$

$$= [3][1] + [0] + [1][0]$$

$$= \underline{\underline{3}}$$

Q. Find the rate at which the area of rectangle is increasing at a given instant when the sides of the rectangle are 5 feet and 4 feet, and are increasing at the rate of 1.5 ft/sec and 0.5 ft/sec respectively.

Soln: Target

$\frac{dA}{dt} = ?$, at $x = 5$, $y = 4$

Given:

$A = xy$

$\frac{dx}{dt} = 1.5$, $\frac{dy}{dt} = 0.5$

$$\frac{dx}{dt} = 1.5, \quad \frac{dy}{dt} = 0.5$$

Soln:

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial A}{\partial y}\cdot\frac{dy}{dt}$$

$$= y(1.5) + x(0.5)$$

At $x = 5, \quad y = 4$

$$\frac{dA}{dt} = 4(1.5) + 5(0.5)$$

$$= 6 + 2.5$$

$$= \underline{\underline{8.5 \ ft^2/sec}}$$

Area is increasing by $\underline{\underline{8.5 \ ft^2/sec}}$

Q The altitude of a cone is 15 cm, and is increasing at 0.2 cm/sec. The radius of the base is 10 cm, and is decreasing at the rate 0.3 cm/sec. How fast is the volume changing?

Ans: Target:

$\frac{dV}{dt} = ?$ at $h = 15$ cm, $r = 10$ cm

$$V = \frac{1}{3}\pi r^2 h$$

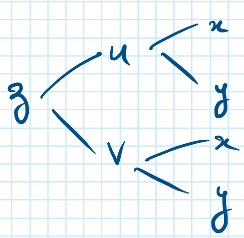
$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

$$= \frac{1}{3}\pi h(2r)(-0.3) + \frac{1}{3}\pi r^2 ($$

Q. If $z = \log(u^2+v)$, where $u = e^{x+y^2}$, $v = x+y^2$,

find the value of $2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$.

Soln:



Using chain rule,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \left(\frac{1}{(u^2+v)}(2u)\right)\left(e^{x+y^2}\right) + \left(\frac{1}{(u^2+v)}(1)\right)(1)$$

$$= \frac{2u\cdot e^{x+y^2} + 1}{}$$

$$= \left(\frac{1}{(u^2+v)}(2u)\right)\left(e^{x+y^2}\right) + \left(\frac{1}{(u^2+v)}\right)(1)$$

$$= \frac{2u \cdot e^{x+y^2} + 1}{u^2+v} \quad \text{—— (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \left(\frac{2u}{u^2+v}\right)\left(e^{x+y^2}\right)(2y) + \left(\frac{1}{u^2+v}\right)(2y)$$

$$= \frac{2y}{u^2+v}\left(2u \cdot e^{x+y^2} + 1\right) \quad \text{—— (2)}$$

$2y \times (1) - (2)$

$$\frac{2y\left(2u \cdot e^{x+y^2}+1\right)}{u^2+v} - \frac{2y}{u^2+v}\left(2u \cdot e^{x+y^2}+1\right)$$

$$= \underline{\underline{0}}$$

Q. Show that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$, where $u = F(x, y)$

Polar eqns.
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Soln:

$$u \begin{cases} x \begin{cases} r \\ \theta \end{cases} \\ y \begin{cases} r \\ \theta \end{cases} \end{cases}$$

Consider RHS.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot [\cos\theta] + \frac{\partial u}{\partial y}[\sin\theta] \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}[-r\sin\theta] + \frac{\partial u}{\partial y}[r\cos\theta] \quad \text{—— (2)}$$

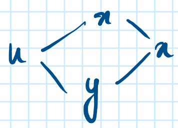
Changing (1) and (2) to match RHS.

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

$$= \left[\frac{\partial u}{\partial x}[\cos\theta] + \frac{\partial u}{\partial y}[\sin\theta]\right]^2 + \frac{1}{r^2}\left[\frac{\partial u}{\partial x}[-r\sin\theta] + \frac{\partial u}{\partial y}[r\cos\theta]\right]^2$$

Q. Find $\frac{du}{dx}$, $u = \tan(x^2+y^2)$, $x^2-y^2=2$

Soln:



$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\left(\frac{dy}{dx}\right)$$

$$= \sec^2(x^2+y^2)(2x) + \sec^2(x^2+y^2)(2y)\left(\frac{dy}{dx}\right) \quad \text{—— (1)}$$

$$F(x,y) = x^2-y^2-2 = 0$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-2x}{-2y} = \frac{x}{y} \quad \text{—— (2)}$$

Putting (2) in (1),

$$\frac{du}{dx} = \sec^2(x^2+y^2)(2x) + \sec^2(x^2+y^2)(2\cancel{y})\left(\frac{x}{\cancel{y}}\right)$$

$$= 2\left[2x\sec^2(x^2+y^2)\right]$$

$$= 4x\sec^2(x^2+y^2)$$

Q. Find $\frac{dy}{dx}$ at the point $(1,1)$ for $e^y - e^x + xy = 1$

Soln: $F(x,y) = e^y - e^x + xy - 1 = 0$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(-e^x+y)}{e^y+x}$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(e^x + y)}{e^y + x}$$

At point $(1,1)$,

$$\frac{dy}{dx} = \frac{e-1}{e+1}$$

Q. Find $\frac{du}{dx}$, $u = x\log(xy)$, $x^3 + y^3 - 3xy - 1 = 0$

Q Compute $\tan^{-1}\left(\frac{0.9}{1.1}\right)$ approximately using series expansion

Soln: Let $F(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

Writing Taylor's series expansion for $F(x,y)$ about $(1,1)$

$$F(1,1) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$F_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot y \cdot \left(\frac{-1}{x^2}\right)$$

$$= \frac{\cancel{x^2}}{x^2+y^2}(y)\left(\frac{-1}{\cancel{x^2}}\right)$$

$$F_x = -\frac{y}{x^2+y^2} = \frac{-1}{1+1} = -\frac{1}{2}$$

$$F_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x^{\cancel{2}}}{x^2+y^2}\left(\frac{1}{\cancel{x}}\right)$$

$$F_y = \frac{x}{x^2+y^2} = \frac{1}{2}$$

$$F_{xx} = -y\left[-\frac{1}{(x^2+y^2)^2}\left[2x\right]\right]$$

$$= -1\left[\frac{-1}{\cancel{4}\,2}\left[\cancel{2}\right]\right]$$

$$F_{xx} = \frac{1}{2}$$

$$F_{xy} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{2(-1) - (-1)(2)}{4} = \frac{-2+2}{4}$$

$$F_{xy} = 0$$

$$F_{yy} = x\left[\frac{-1}{(x^2+y^2)^2}\right]\left[2y\right]$$

$$= \frac{(1)(-1)(\cancel{2})^1}{\cancel{4}\,2} = \frac{-1}{2}$$

$$F(x,y) = F(a,b) + \frac{1}{1!}\left[(x-a)(F_x) + (y-b)(F_y)\right] + \frac{1}{2!}\left[(x-a)^2 F_{xx} + (y-b)^2 F_{yy} + 2(x-a)(y-b)F_{xy}\right]$$

$$F(x,y) = \frac{\pi}{4} + \left[(x-1)\left(\frac{-1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right] + \frac{1}{2}\left[(x-1)^2\left(\frac{1}{2}\right) + (y-1)^2\left(\frac{-1}{2}\right) + 2(x-1)(y-1)(0)\right]$$

$$= \frac{\pi}{4} - (x-1)\left(\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right) + \frac{(x-1)^2}{4} - \frac{(y-1)^2}{4}$$

$$F(1.1, 0.9) = \frac{\pi}{4} - (0.1)\frac{1}{2} + (-0.1)\left(\frac{1}{2}\right) + \frac{(0.1)^2}{4} - \frac{(-0.1)^2}{4}$$

$$= \frac{\pi}{4} - 0.1 + \frac{(0.1)^2}{2}$$

$$= \underline{\underline{0.690}}$$

Q. Find the Taylor's series expansion of $F(x,y) = \sqrt{1+x+y^2}$ in powers of $(x-1)$ and $(y-0)$ upto second degree terms.

Soln: Writing Taylor series expansion about $(1,0)$.

$$F(1,0) = \sqrt{1+1+0^2} = \sqrt{2}$$

$$F_x = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{1+x+y^2}}\right)(1) = \frac{1}{2\sqrt{1+1+0^2}} = \frac{1}{2\sqrt{2}}$$

$$F_y = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{1+x+y^2}}\right)(2y) = \frac{y}{\sqrt{1+x+y^2}} = 0$$

$$F_{xx} = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(1+x+y^2)^{-\frac{3}{2}}(1) = \frac{-(1+x+y^2)^{-3/2}}{4} = -0.088$$

$$F_{yy} = \frac{(1+x+y^2)^{\frac{1}{2}}(1) - y(- - -)}{1+x+y^2}$$

$$F_{yy} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$F_{xy} = \frac{1}{2}\left(\frac{-1}{2}\right)(1+x+y^2)^{-\frac{3}{2}}(2y) = 0$$

$$F(x,y) = \sqrt{2} + \frac{1}{1!}\left[(x-1)\left(\frac{1}{2\sqrt{2}}\right) + (y-0)(0)\right] + \frac{1}{2}\left[(x-1)^2(-0.088) + (y-0)^2\left(\frac{1}{\sqrt{2}}\right) + 2(x-1)(y-0)(0)\right]$$

$$= \sqrt{2} + \frac{x-1}{\ldots} \ldots (x-1)^2(-0.044) \ldots$$

$$+2(x-1)(y-0)(0))$$

$$= \sqrt{2} + \frac{x-1}{2\sqrt{2}} - (x-1)^2(0.044) + \frac{(y-0)^2}{2\sqrt{2}}$$

Q. Expand $F(x,y) = xy^2 + y\cos(x-y)$ in Taylor series upto second order terms about the point $(1,1)$.