Partial Differentiation, Geometric Interpretation

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PARTIAL DIFFERENTIATION

Say we have $u = F(x,y)$,

Partial differentiation $\longrightarrow$ ordinary derivative w.r.t. to one variable keeping all others constant

- For u w.r.t. x, keeping y constant,
  it is denoted as $\left(\frac{\partial u}{\partial x}\right)_y$ [or] u_x
- Illly w.r.t. y,
  $\left(\frac{\partial u}{\partial y}\right)_x$ [or] u_y
- $$\boxed{\begin{array}{l} \dfrac{\partial u}{\partial x} = \lim\limits_{h\to 0} \dfrac{F(x+h,y) - F(x,y)}{h} \\ \hline \dfrac{\partial u}{\partial y} = \lim\limits_{k\to 0} \dfrac{F(x,y+k) - F(x,y)}{h} \end{array}}$$

PROBLEMS

① $u = \tan^{-1}\left(\frac{x^2+y^2}{xy}\right)$. Find $\frac{\partial u}{\partial x}$

Soln: Say $u = \tan^{-1} z$.

$$z = \frac{x^2+y^2}{xy} = \frac{x}{y} + \frac{y}{x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{1+z^2}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{1}{1+\left(\frac{x^2+y^2}{xy}\right)}\left(\frac{1}{y} + y\left(\frac{-1}{x^2}\right)\right)$$

$$= \left(\frac{xy}{xy+x^2+y^2}\right)\left(\frac{x^2-y}{x^2y}\right)$$

$$= \frac{x^2-y}{x^3+x^2y+xy^2}$$

② $u = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial u}{\partial x} = \left(\frac{1}{x^2+y^2}\right)(2x) + \frac{1}{1+\left(\frac{y}{x}\right)^2}(y)\left(\frac{-1}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} + \frac{y}{\frac{x^2+y^2}{x^2}}\left(\frac{-1}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} - \frac{y\,(\cancel{x^2})}{x^2+y^2\,(\cancel{x^2})} = \frac{2x-y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(2) - (2x-y)(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}(2y) + \frac{x^{\cancel{2}}}{(x^2+y^2)}\left(\frac{1}{\cancel{x}}\right)(1)$$

$$= \frac{2y+x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(2) - (2y+x)(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2x^2+2y^2-(4x^2-2xy)+2x^2+2y^2-(4y^2+2xy)}{(x^2+y^2)^2}$$

$$= \frac{\cancel{2x^2}+\cancel{2y^2}-\cancel{4x^2}+\cancel{2xy}+\cancel{2x^2}+\cancel{2y^2}-\cancel{4y^2}-\cancel{2xy}}{(x^2+y^2)^2}$$

$$= \frac{0}{(x^2+y^2)^2} = 0$$

Hence proved

③ $u = x^y$. Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$

Soln. $\frac{\partial u}{\partial x} = y x^{y-1}$

$\frac{\partial u}{\partial y} = x^y \cdot \log_e(x)$

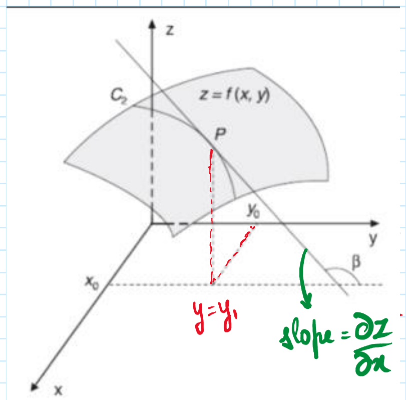
$$\frac{\partial u}{\partial y} = x^y \cdot \log_e(x)$$

GEOMETRIC INTERPRETATION

$\underbrace{z = F(x, y)}$ → represents a surface in 3D space

EXAMPLE: $x^2 + y^2 = z^2$

- Represents a right circular cone



- Any point → $(x, y, F(x, y))$

Let $y = y_1$ (constant)

This represents a plane parallel to x-z plane.

The intersection of the surface $z = F(x, y)$ with the plane $y = y_1$ gives a plane curve.

- Any point on the curve → $(x, y_1, F(x, y_1))$ where y_1 → constant

> Thus, $\frac{\partial z}{\partial x}$ (where y = constant) represents the slope of the tangent to the curve of intersection of the surface $z = F(x, y)$ and the plane $y = y_1$.

IIIly,

$\frac{\partial z}{\partial y}$ $(x = x_1)$ represents slope of the tangent to the curve of intersection of $z = F(x, y)$ and the plane $x = x_1$

RULES OF DIFFERENTIATION

Let u, v, w be functions of x, y, z.

① Partial differentiation of sum/difference

Let $t = u \pm v \pm w$

$$\boxed{\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}(u \pm v \pm w) = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x} \pm \frac{\partial w}{\partial x}}$$

② Partial differentiation of product / Product rule

Let $t = u \times v \times w$

$$\partial t = u \times v \times \partial w + u \times \partial v \times w + \partial u \times v \times w$$

Let $t = u \times v \times w$

$$\boxed{\frac{\partial t}{\partial x} = u \times v \times \frac{\partial w}{\partial x} + u \times \frac{\partial v}{\partial x} \times w + \frac{\partial u}{\partial x} \times v \times w}$$

③ Quotient rule

Let $t = \frac{u}{v}$

$$\boxed{\frac{\partial t}{\partial x} = \frac{v\left(\frac{\partial u}{\partial x}\right) - u\left(\frac{\partial v}{\partial x}\right)}{v^2}}$$

④ Chain rule

$t = f(u)$ where $u = F(x, y, z)$

$$\boxed{\frac{\partial t}{\partial x} = \left(\frac{dt}{du}\right)\left(\frac{\partial u}{\partial x}\right)}$$

This is an ordinary derivative as t is a function of only one variable

MIXED PARTIAL DERIVATIVE

$u = \sin(xy)$

Diff. w.r.t. x

$$\frac{\partial u}{\partial x} = \cos(xy) \times y = y\cos(xy)$$

Diff. w.r.t. y

$$\frac{\partial u}{\partial y} = \cos(xy) \times x = x\cos(xy)$$

Finding higher order derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = y[-\sin(xy)]y = -y^2\sin(xy)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x} = \cos(xy)(1) + y(-\sin(xy)(x))$$

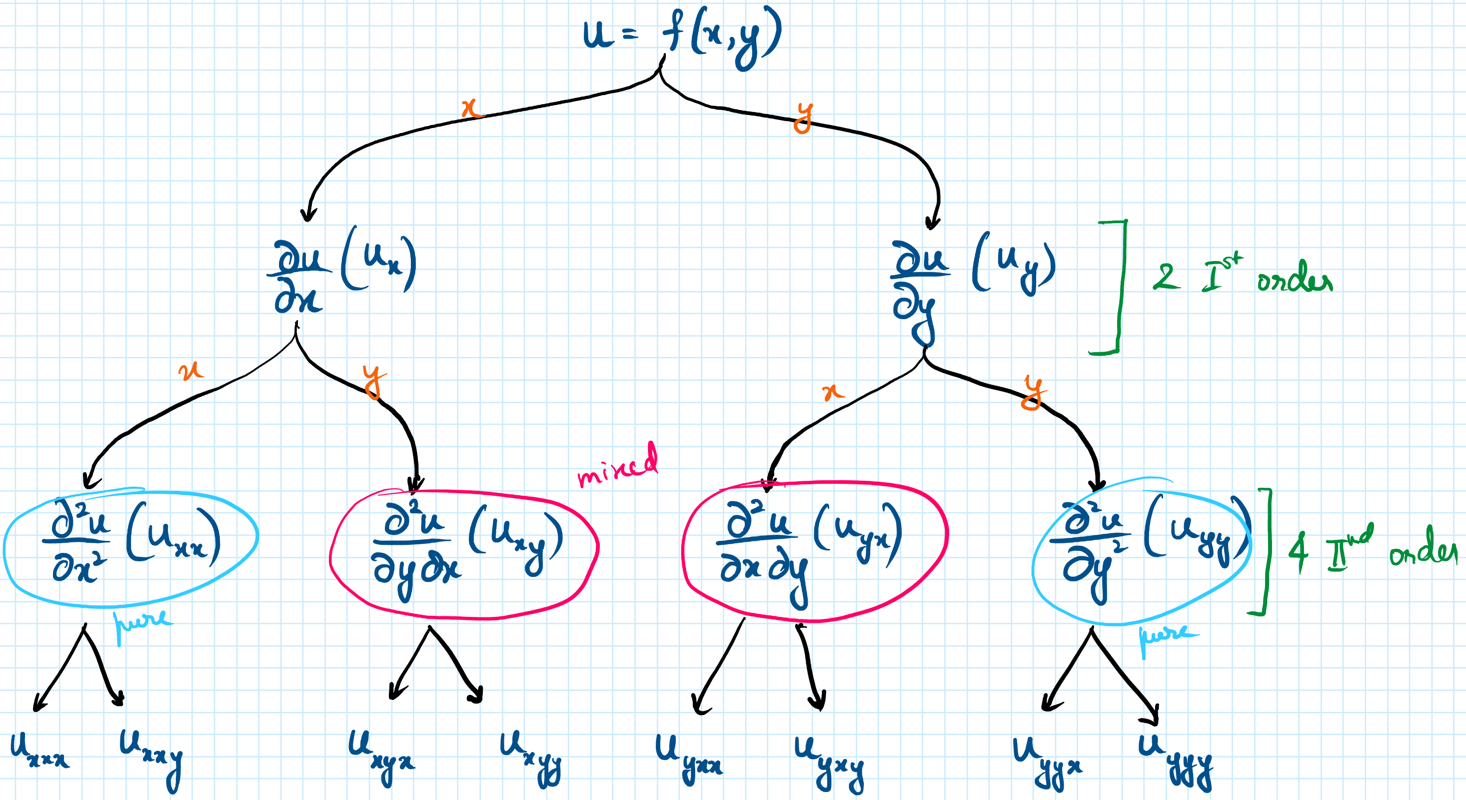
$$= \cos(xy) - xy\cdot\sin(xy)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\,\partial y} = \cos(xy)(1) + x[-\sin(xy)(y)]$$

$$= \cos(xy) - xy\cdot\sin(xy)$$

NOTE:

- $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x} \Longrightarrow$ Derivatives are continuous
- For higher order derivatives, the order of operators in the denominator from right to left show the order of variables w.r.t. which the function was differentiated.



In general:

A function with m independent variables has m^n n^{th} order derivatives.

CLASSWORK PROBLEMS

① Find first order partial derivatives

(i) $f(x,y) = x^4 - x^2y^2 + y^4$ at $(-1,1)$

$\dfrac{\partial}{\partial x} = 4x^3 - (y^2)(2x)$ $\qquad\qquad$ $\dfrac{\partial}{\partial y} = -x^2(2y) + 4y^3$

At $(-1,1)$,

$= -4 - (-2)$ $\qquad\qquad$ $= -1(2) + 4$

$= -2$ $\qquad\qquad$ $= 2$

(ii) $f(x,y) = x^2 e^{-y/x}$ at $(4,2)$

$$\frac{\partial f}{\partial x} = \cancel{x^2}\left[e^{-y/x}(y)\left(\frac{1}{\cancel{x^2}}\right)\right] + e^{-y/x}(2x)$$

$$= (2x+y)e^{-y/x}$$

At $(4,2)$,

$$= 10e^{-\frac{1}{2}}$$

$$\frac{\partial f}{\partial y} = x^2\left[e^{-y/x}\left(\frac{1}{x}\right)(-1)\right]$$

$$= -x\, e^{-y/x}$$

$$= -4\, e^{-\frac{1}{2}}$$

② $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for all $(x,y) \neq 0,0$ when $f(x,y) = x^y$. Prove.

$$\frac{\partial f}{\partial x} = y x^{y-1}$$

$$\frac{\partial^2 f}{\partial y \partial x} = (y)(x^{y-1})(\log_e x) + x^{y-1}(1)$$

$$\frac{\partial f}{\partial y} = x^y . \log x$$

$$\frac{\partial^2 f}{\partial x \partial y} = (\log(x))(y x^{y-1}) + (x^y)\left(\frac{1}{x}\right)$$

③ Find all second order partial derivatives of $f(x,y) = \log\left(\frac{1}{x} - \frac{1}{y}\right)$ at $(1,2)$

$$\frac{\partial f}{\partial x} = \frac{1}{\left(\frac{1}{x} - \frac{1}{y}\right)}\left(\frac{-1}{x^2}\right)$$

$$\frac{y-x}{xy}$$

$$= \frac{xy}{y-x}\left(\frac{-1}{x^2}\right)$$

$$= \frac{xy}{y-x}\left(\frac{-1}{x^2}\right)$$

$$= \frac{-y}{xy-x^2} = -y(xy-x^2)^{-1}$$

$$\frac{\partial^2 f}{\partial x^2} = (-y)(-1)(xy-x^2)^{-2}(y-2x)$$

At $(1,2)$

$$= (-2)(-1)(2-1)^{-2}(2-2)$$

$$= 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{1}{xy-x^2}(-1) + (-y)(-1)(xy-x^2)^{-2}(x)$$

At $(1,2)$

$$= \frac{1}{2-1}(-1) + (2)(2-1)^{-2}(1)$$

$$= -1+2 = 1$$

$$\frac{\partial f}{\partial y} = \left(\frac{xy}{y-x}\right)\left(\frac{1}{y^2}\right) = \frac{x}{y^2-xy} = x(y^2-xy)^{-1}$$

$$\frac{\partial^2 f}{\partial y^2} = (x)(-1)(y^2-xy)^{-2}(2y-x)$$

At $(1,2)$,

$$\frac{\partial^2 f}{\partial y^2} = (1)(-1)(4-2)^{-2}(4-1)$$

$$= \frac{-3}{4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = (y^2-xy)^{-1}(1) + (x)(-1)(-y)$$

$$= \frac{1}{y^2-xy} + xy$$

$$= \frac{xy(y^2-xy)+1}{y^2-xy} = \frac{xy^3-x^2y^2+1}{y^2-xy}$$

Extra.

If $\theta = t^n \cdot e^{\frac{-r^2}{4t}}$, find the value of n for which $\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial \theta}{\partial r}\right)\right] = \frac{\partial \theta}{\partial t}$ is true.

Soln. $\frac{\partial \theta}{\partial t} = \left(e^{\frac{-r^2}{4t}}\right)(n)(t^{n-1}) + (t^n)\left(e^{\frac{-r^2}{4t}}\right)\left(\frac{-r^2}{4}\right)(1)$