5. Bessel's Function

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BESSEL'S FUNCTION

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0 \quad \text{—(1)}$$

→ Bessel's differential equation, 2nd order

Series solution of equation (1)

$$\boxed{y = J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}}$$

$J_{-n}(x)$ is also a solution

$$\boxed{y = J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}}$$

If you have two solutions to an equation, the linear combination of the solutions is also a solution. Hence,

$$\boxed{y = A\,J_n(x) + B\,J_{-n}(x)}$$

→ when n is a non-integer (linearly independent)

PROOF: $J_{-n}(x) = (-1)^n J_n(x)$ → n is an integer

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}$$

For $r \in [0, n-1]$, we get Γ-ve numbers in the denominator which does not exist. Hence, only for $r \geq n$, it exists.

$$\therefore\ J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}$$

Put $r = n + k$

$k = r - n$

Say $r = n \longrightarrow k = 0$

$r = \infty \longrightarrow k = \infty$

$$\therefore\ J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{(n+k)!\,\Gamma(\cancel{-n}+\cancel{n}+k+1)} \left(\frac{x}{2}\right)^{-n+2(n+k)}$$

$$(n+k)!\,\Gamma(\cancel{-n}+\cancel{n}+k+1)\,($$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(n+k+1)\,k!} \left(\frac{x}{2}\right)^{n+2k}$$

$$\Longrightarrow \boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

PROOF : $J_{1/2}(x)$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$n = \frac{1}{2}$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+3/2)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2r}$$

$$= \frac{1}{\Gamma(3/2)}\left(\frac{x}{2}\right)^{\frac{1}{2}} + \frac{-1}{\Gamma(\frac{5}{2})}\left(\frac{x}{2}\right)^{\frac{5}{2}} + \frac{1}{2\,\Gamma(\frac{7}{2})}\left(\frac{x}{2}\right)^{\frac{9}{2}} + \dots$$

$$= \sqrt{\frac{x}{2}} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+\frac{3}{2})}\left(\frac{x}{2}\right)^{2r}$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma\frac{3}{2}} - \frac{1}{\Gamma\frac{5}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2\,\Gamma\frac{7}{2}}\left(\frac{x}{2}\right)^4 - \dots + \dots\right]$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\frac{1}{2}\sqrt{\pi}} - \frac{x^2}{\frac{3}{2}(\frac{1}{2})(\sqrt{\pi})(2^2)} + \frac{x^4}{\frac{5}{2}(\frac{3}{2})(\frac{1}{2})(\sqrt{\pi})(2\right.$$

$$\sqrt{\frac{2}{\pi x}}\sin x$$

RECURSION FORMULAE

① $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

w.k.t.

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Differentiating w.r.t. x.

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} (n+2r) \left(\frac{x}{2}\right)^{(n+2r-1)} \left(\frac{1}{2}\right)$$

$$= \sum_{r=0}^{\infty} \frac{n(-1)^r (x/2)^{n+2r-1}\left(\frac{1}{2}\right)}{r!\,\Gamma(n+r+1)} + \sum_{r=0}^{\infty} \frac{2r(-1)^r (x/2)^{n+2r-1}(1/2)}{r!\,\Gamma(n+r+1)}$$

Multiplying by x,

$$x J_n'(x) = n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\,\Gamma(n+r+1)} + \sum_{r=1}^{\infty} \frac{(-1)^r (x/2)^{n+2r-1} \cdot x}{(r-1)!\,\Gamma(n+r+1)}$$

$s = r-1$ $\quad$ $r=1 \longrightarrow s=0$

$r = s+1$ $\quad$ $r=\infty \longrightarrow s=\infty$

$(-1)^1$

$$x J_n'(x) = n J_n(x) + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1} (x/2)^{n+2(s+1)-1}}{s!\,\Gamma(n+s+2)}$$

$$x J_n'(x) = n J_n(x) - x \sum_{s=0}^{\infty} \frac{(-1)^s \cdot (x/2)^{(n+1)+2s}}{s!\,\Gamma((n+1)+s+1)}$$

$$\boxed{x J_n'(x) = n J_n(x) - x J_{n+1}(x)}$$

② $J_n'(x) = x J_{n-1}(x) - n J_n(x)$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r+n-n)(x/2)^{n+2r-1}(1/2)}{r!\,\Gamma(n+r+1)}$$

$$r!\,\Gamma(n+r+1)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-n)(x/2)^{n+2r-1}(1/2)}{r!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (\cancel{2})(n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{1}{\cancel{2}}\right) - n \sum_{r=0}^{\infty} (-1)^r (x/2)^n$$

③

w.k.t.

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

$$x J_n'(x) = x J_{n-1} - n J_n(x)$$

Adding the two:

$$2\cancel{x} J_n'(x) = \cancel{x}\left(J_{n-1} - J_{n+1}\right)$$

$$\boxed{J_n'(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)}$$

④

w.k.t.

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

$$x J_n'(x) = x J_{n-1} - n J_n(x)$$

Subtracting,

$$0 = 2n J_n(x) - x\left(J_{n-1} + J_{n+1}\right)$$

$$\boxed{2n J_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right)}$$

⑤ $\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1} x$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_n'(x) + J_n(x) \cdot n x^{n-1}$$

w.k.t. $J_n' = J_{n-1} - \frac{n}{x} J_n$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n \left(J_{n-1} - \frac{n}{x} J_n\right) + n x^{n-1} J_n(x)$$

$$= x^n J_{n-1}(x) - \cancel{n x^{n-1} J_n(x)} + \cancel{n x^{n-1} J_n(x)}$$

$$\boxed{\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)}$$

⑥ $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}$