# EQUATIONS REDUCIBLE TO EXACT FORM

## Integrating Factor

non exact DE × IF = exact DE

## Ways to find integrating factor

### Case ①, ②

Given $Mdx + Ndy = 0$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

If:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \phi(x), \quad IF = e^{\int \phi(x).dx}$$

If:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \phi(y) \quad IF = e^{-\int \phi(y).dy}$$

### Case ③

If M and N are homogeneous functions of same degree:

$$IF = \frac{1}{Mx + Ny}$$

### Case ④

Given,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If it can be written as

$$\underbrace{y.\, f_1(xy)}_{M}\, dx + \underbrace{x.\, f_2(xy)}_{N}\, dy = 0$$

Then

$$IF = \frac{1}{Mx - Ny}$$

# APPLICATIONS OF FIRST ORDER DERIVATIVES

## ORTHOGONAL TRAJECTORIES

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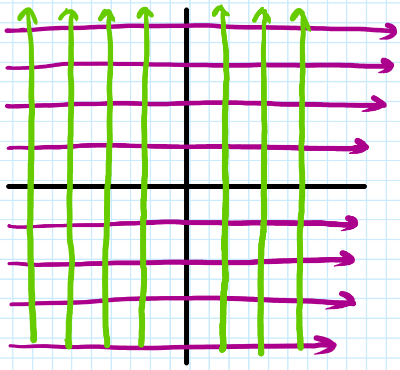
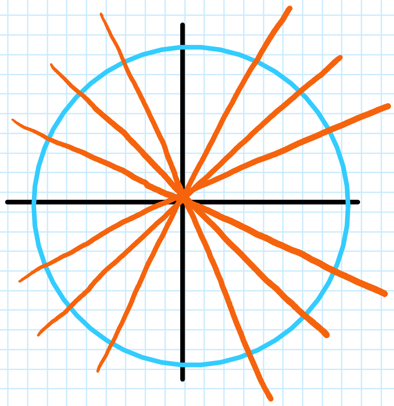
Two families of plane curves $F_1$ and $F_2$ are said to be orthogonal trajectories of each other if every member of one family intersects every member of the other orthogonally (at $90^\circ$).



$x = c_1$

$y = c_2$



$x^2 + y^2 = c^2$

$y = mx$

**Steps to find orthogonal trajectory of some $F_1$ in Cartesian Form**

Let $f(x,y) = c_1$ be the Cartesian equation of $F_1$

① Form differential equation of $F_1$

$$\frac{dy}{dx} = g(x,y)$$

② Let $F_2$ be the OT of $F_1$; $m_1 \longrightarrow$ slope of tangent drawn to some member of $F_1$

$m_2 \rightarrow$ slope of tangent drawn to some member of $F_2$

Since they cut at $90^\circ$,

$$m_1 \times m_2 = -1$$

$$m_2 = \frac{-1}{m_1} = \frac{-1}{g(x,y)}$$

$$\boxed{\frac{dy_2}{dx_2} = \frac{-1}{g(x,y)}} \longrightarrow \text{D.E. of } F_2$$

③ Solve DE of $F_2$.

GS of this eqn represents $F_2$.

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[OR]

① Form DE of $F_1$
② Replace $\frac{dy}{dx}$ by $\left(\frac{-dx_2}{dy_2}\right)$

$y_1$ by $\frac{-1}{y_1}$

or vice versa, to get DE for $F_2$
③ Solve to get eqn for $F_2$

→ Use this for problems

## Self orthogonal curves

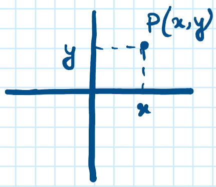
A family of curves is said to be self orthogonal if the members of the family intersect each other at 90°, if at all they intersect.

Eg: $y^2 = 4c(c+x)$

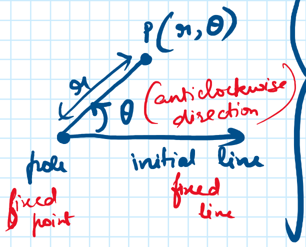
## Algorithm to prove $F_1$ is self orthogonal

① Form DE
② Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ to get DE of orthogonal trajectory ($DE_{orth}$)
③ If $DE_{orth} = DE$,
given family is self orthogonal

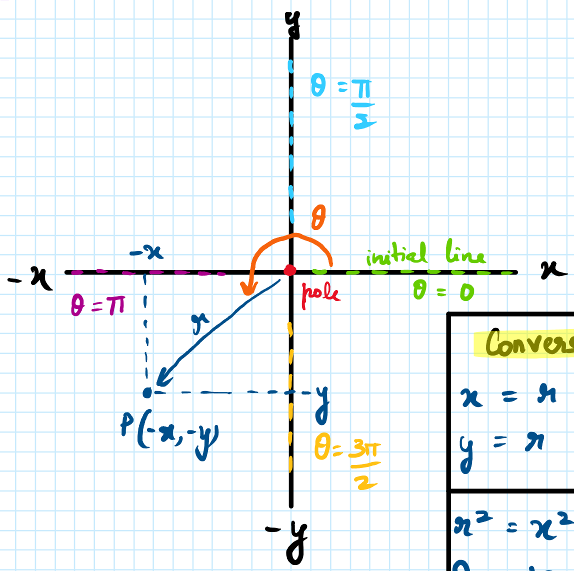
## Orthogonal trajectories in polar form




Cartesian




Polar

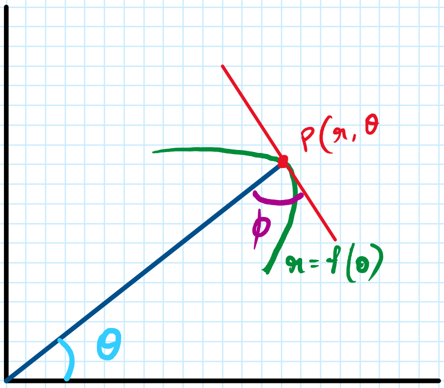



**Conversion:**

$x = r \cos\theta$
$y = r \sin\theta$

$r^2 = x^2 + y^2$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

## Angle b/w radius vector and tangent




$$\phi_2 - \phi_1 = \frac{\pi}{2}$$

$$\phi_2 = \frac{\pi}{2} + \phi_1$$

$$\tan\phi_2 = -\cot\phi_1$$

$$\boxed{\tan\phi_1 \times \tan\phi_2 = -1}$$

## To find orthogonal trajectories in polar form:

Let $F_1$ be a family of polar curves whose eqn. is $f(r,\theta) = C$

① Form DE, $\underbrace{\frac{dr}{d\theta} = g(r,\theta)}_{\text{DE of } F_1}$

$$\tan\phi_1 = \frac{r}{dr/d\theta} = \frac{r}{g(r,\theta)}$$

Let $F_2$ be its OT

$$\tan\phi_2 \times \tan\phi_1 = -1$$

$$\tan\phi_2 = \frac{-1}{\tan\phi_1} = \frac{-g(r,\theta)}{r}$$

w.k.t.

$$\tan\phi_2 = \frac{r}{dr/d\theta}$$

$$\Rightarrow \frac{r}{dr/d\theta} = \frac{-g(r,\theta)}{r}$$

$$\boxed{g(r,\theta) = -r^2\left(\frac{d\theta}{dr}\right)} \text{ } \} \text{ DE of } F_2$$

② Replace:

$$\boxed{\begin{array}{ccc} \frac{dr}{d\theta} & \rightleftharpoons & -r^2\frac{d\theta}{dr} \\ \frac{1}{r}\left(\frac{dr}{d\theta}\right) & \rightleftharpoons & -r\frac{d\theta}{dr} \end{array}}$$

③ Solve