7. Differential Equations, Methods of Solving DEs (Linear, Bernoulli, Exact)

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Ordinary Differential Equations of First Order

An equation containing dependent variable, independent variables and the derivatives of dependent variable w.r.t. independent variables is called differential eqn.

ODEs
Contains ordinary derivative

PDEs
Contains partial derivative

Solution

General
Family of plane curves

Particular
Single specific curve; some condition specified

$$\boxed{\text{Note: No. of arbitrary constants in soln = Order of DE}}$$

Forming a Differential Equation

To form a differential eqn., eliminate arbitrary constant in given eqn. by differentiating.

Methods of Solving Differential Equation

① Variable separable

$$y\,dx = x\,dy$$

$$\frac{y}{dy} = \frac{x}{dx}$$

Integrate:

$$\log y = \log x + \log c$$

$$y = cx$$

② Homogeneous diff. eqns.

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$$, where P and Q are homogeneous functions of same degree

Put $y = vx$

$\Rightarrow y = v,\ x = 1$

③ Linear Differential Equation

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

③ LINEAR DIFFERENTIAL EQUATION

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

In general, an eqn is linear if:

① It contains dependent variable and its derivatives all with degree 1

② Dependent variable cannot occur in product form with its derivative

Leibnitz form: Coefficient of derivative is 1

Eg: $\left[2x^2\left(\frac{d^2y}{dx^2}\right) - 5x\left(\frac{dy}{dx}\right) - 9(y) = \sin x\right] \rightarrow$ LDE w/ variable coefficients

$\left[2\left(\frac{d^2y}{dx^2}\right) - 5\left(\frac{dy}{dx}\right) - 9(y) = \sin x\right] \rightarrow$ LDE w/ constant coefficients

$2\left(\frac{d^2y}{dx^2}\right) - 5\left(\frac{dy}{dx}\right) - 9(y) = 0 \rightarrow$ Homogeneous LDE $(=0)$

$\rightarrow$ Non-homogeneous LDE $(\neq 0)$

Integrating Factor

$$IF = e^{\int P.dx}$$

Multiplying a non-exact DE by IF converts it into exact.

General Solution

Multiply DE by IF on both sides

$$e^{\int P.dx}\left(\frac{dy}{dx} + P(x)\cdot y\right) = (Q(x))\, e^{\int P.dx}$$

$$e^{\int P.dx}\cdot dy + y\cdot P(x)\cdot e^{\int P.dx} = e^{\int P.dx}\cdot Q(x)\cdot dx$$

$$d\left[y\cdot e^{\int P.dx}\right] = d\left[e^{\int P.dx}\cdot Q(x)\,dx\right]$$

$$\boxed{y\cdot e^{\int P.dx} = \int e^{\int P.dx}\cdot Q(x)\cdot dx}$$

→ DOUBT

④ BERNOULLI'S FORM

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n$$

- not linear but can be reduced to Leibnitz form

Dividing by y^n

$$y^{-n}\left(\frac{dy}{dx}\right) + P(x)\cdot y^{1-n} = Q(x) \quad \text{—— ①}$$

Dividing by y^n

$$y^{-n}\left(\frac{dy}{dx}\right) + P(x)\, y^{1-n} = Q(x) \quad \text{——} \quad ①$$

Put $t = y^{1-n}$

Differentiating w.r.t. x:

$$\frac{dt}{dx} = (1-n)\, y^{1-n-1} \cdot \frac{dy}{dx}$$

$$\frac{1}{1-n}\left(\frac{dt}{dx}\right) = y^{-n}\left(\frac{dy}{dx}\right) \quad \text{——} \quad ②$$

Using ② in ①:

$$\frac{1}{1-n}\left(\frac{dt}{dx}\right) + P(x)\cdot t = Q(x) \rightarrow \text{linear}$$

Multiplying by $(1-n)$:

$$\boxed{\begin{aligned} &\frac{dt}{dx} + (1-n)(P(x))\,t = (1-n)\,Q(x) \\ &IF = e^{\int (1-n) P(x)\, dx} \end{aligned}} \rightarrow \text{Leibnitz form}$$

General soln.

$$\boxed{t \times IF = \int IF \times Q(x) \times (1-n)\, dx + C}$$

⑤ EXACT DIFFERENTIAL EQUATION

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = d\big(U(x,y)\big) = 0}$$

A differential eqn. of the type

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be exact if there exists a potential function $U(x,y)$ such that

$$M(x,y)\,dx + N(x,y)\,dy = d\big(U(x,y)\big)$$

Another definition

For an exact differential eqn,

$$\boxed{M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}}$$

TEST FOR EXACT DIFF. EQN.

A differential eqn of the type $M\,dx + N\,dy = 0$ is said to be exact

$$\Downarrow$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(1)

Given $M\,dx + N\,dy = 0$ is exact,

<u>To prove</u> : $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

For exact,

$$M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial u}{\partial x\,\partial y}, \quad \frac{\partial N}{\partial y} = \frac{\partial u}{\partial y \cdot dx}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

<u>Hence proved</u>

(2)

Given $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

<u>To prove</u>: $M\,dx + N\,dy = 0$ is exact

Let $u(x,y) = \int_{y\ \text{const.}} M \cdot dx$. Differentiating partially w.r.t x,

$$\frac{\partial u}{\partial x} = M \quad \text{— (1)}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} = \underbrace{\frac{\partial N}{\partial x}}_{\text{given}}$$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$; integrating partially w.r.t. x, keeping y constant

$$N = \frac{\partial u}{\partial x} + \phi(y) \quad \text{— (2)}$$

From (1) and (2),

$$\frac{\partial u}{\partial x}\,dx + \left(\frac{\partial u}{\partial y} + \phi(y)\right) dy = 0$$

$$du + \phi(y)\,dy = 0$$

$$d\left(\int M \cdot dx\right) + d\left(\int \phi(y) \cdot dy\right) = d(c)$$

$$d\left(\int M \cdot dx + \int \left(\begin{matrix}\text{terms of } N \\ \text{that do not} \\ \text{contain } x\end{matrix}\right) dy\right) = d(c)$$

Thus given eqn is exact.

General solution

$$\boxed{\int_{'y'\ \text{constant}} M \cdot dx + \int \left(\begin{matrix}\text{terms of } N \text{ not} \\ \text{containing } x\end{matrix}\right) dy = C}$$

$$\int_{y\ \text{const.}} M \cdot dx + \int N(y)\,dy = C$$

$$\boxed{U(x,y) = c}$$