3. Gamma Function

29 November 2023 10:06

GAMMA FUNCTION $\Gamma(n)$

$\Gamma(n)$ is defined as:

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx}$$

where 'n' is either +ve real no.
OR
-ve non-integer

Evaluating $\Gamma(1)$ by definition

$$\Gamma(1) = \int_0^\infty e^{-x} x^0\, dx$$

$$= \int_0^\infty e^{-x}\, dx$$

$$= \left[-e^{-x}\right]_0^\infty$$

$$= -(0-1)$$

$$\boxed{\Gamma(1) = 1}$$

RECURSIVE FORMULA FOR Γ

To prove: $\Gamma(n) = (n-1)\Gamma(n-1)$

$$\Gamma(n) = \int_0^\infty \underbrace{e^{-x}}_{v}\ \underbrace{x^{n-1}}_{u}\, dx$$

$$= \left[x^{n-1}(-e^{-x})\right]_0^\infty - \int_0^\infty (-e^{-x})(n-1)x^{n-2}\, dx$$

$$= 0 + (n-1)\int_0^\infty e^{-x} \cdot x^{(n-1)-1}\, dx$$

$$\boxed{\Gamma(n) = (n-1)\Gamma(n-1)}$$

Case ①: When n is positive

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$$\Gamma(n) = (n-1)(n-2)\dots(1)\Gamma(1)$$

$$\boxed{\Gamma(n) = (n-1)!}$$

Case ②: When n is +ve fraction

$$\boxed{\Gamma(n) = (n-1)(n-2)\dots(x)\Gamma(x)} \quad \text{where} \quad 0 < x < 1$$

$$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

Case ③: When n = 0

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(n-1) = \frac{\Gamma(n)}{n-1}$$

Put $n = 1$

$$\Gamma(0) = \frac{1}{0} = \infty$$

$$\boxed{\Gamma(0) = \text{not defined}}$$

Case ④: When n is negative

Using above formula,

$$\Gamma(-1) = \frac{\Gamma(0)}{-1} = -\infty$$

$$\boxed{\Gamma(n) = \text{not defined when } n = 0, -1, -2 \dots}$$

Case ⑤: When n is negative fraction

$$\Gamma(n-1) = (n-1)\Gamma(n-1)$$

Putting n as $n+1$

$$\Gamma(n+1) = (n)\Gamma(n)$$

$\Gamma(n+1) = (n)\Gamma(n)$

$$\boxed{\Gamma(n) = \frac{\Gamma(n+1)}{n}}$$

$$\Gamma\left(\frac{-11}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)}{\frac{-11}{4} \times \frac{-7}{4} \times \underbrace{\frac{-3}{4}}}$$

Final value b/w 0 and 1

RESULTS

Prove that

$$\int_0^\infty x^n e^{-ax^m} dx = \frac{1}{m} \frac{\Gamma\left(\frac{n+1}{m}\right)}{a^{\frac{n+1}{m}}}$$

where m is +ve.

$$I = \int_0^\infty e^{-ax^m} x^n \, dx$$

$t = ax^m \Rightarrow x^m = \frac{t}{a}$

$x = \left(\frac{t}{a}\right)^{\frac{1}{m}}$

$dx = \frac{1}{m}\left(\frac{t}{a}\right)^{\frac{1}{m}-1} \cdot \frac{dt}{a}$

When $x = 0,\ t = 0$

$x = \infty,\ t = \infty$

$$I = \int_0^\infty e^{-t}\left(\left(\frac{t}{a}\right)^{\frac{n}{m}}\right) \cdot \frac{1}{m}\left(\frac{t}{a}\right)^{\frac{1}{m}-1} \frac{dt}{a} = \frac{1}{m} \cdot \frac{1}{a^{\frac{n+1}{m}}} \int_0^\infty e^{-t}\, t^{\frac{n+1}{m}-1}\, dt$$

$$\boxed{\int_0^\infty e^{-ax^m} x^n \, dx = \frac{1}{m} \cdot \frac{\Gamma\left(\frac{n+1}{m}\right)}{a^{\frac{n+1}{m}}}}$$

Proved

Put $m = 1$

$$\int_0^\infty e^{-ax} x^n \, dx = \frac{\Gamma(n+1)}{a^{n+1}}$$

Putting $n+1$ as n,

a

Putting $n+1$ as n,

$$\boxed{\Gamma(n) = a^n \int_0^\infty e^{-ax} x^{n-1}\, dx}$$ → Second definition of $\Gamma(n)$

Put $m = 2$

$$\int_0^\infty e^{-ax^2} x^n \, dx = \frac{1}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}}$$

Putting $n+1$ as n,

$$\boxed{\Gamma(n) = 2a^n \int_0^\infty e^{-ax^2} x^{2n-1}\, dx}$$ → Third definition of $\Gamma(n)$

Put $a=1$

$$\boxed{\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx}$$ → Fourth definition of $\Gamma(n)$

PROOF: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

w.k.t.

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx$$

Put $n = \frac{1}{2}$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-x^2} x^{\cancel{2}\left(\frac{1}{\cancel{2}}\right)-1}\, dx$$

$$= 2\int_0^\infty e^{-x^2} dx \quad -(2)$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-y^2} dy \quad -(3)$$ → Change of variable

(2) × (3)

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int e^{-x^2} dx \int^\infty e^{-y^2} dy$$

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$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$x^2+y^2 = r^2$

$x = r\cos\theta$ $\qquad r \longrightarrow 0$ to ∞

$y = r\sin\theta$ $\qquad \theta \longrightarrow 0$ to $\frac{\pi}{2}$

$dx\,dy = \underbrace{J}\,dr\,d\theta$ $\longrightarrow$ Jacobian $= J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$dx\,dy = r\,dr\,d\theta$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = 4\int_{\theta=0}^{\frac{\pi}{2}}\left(\int_{r=0}^{\infty} e^{-r^2}r\,dr\right)d\theta$$

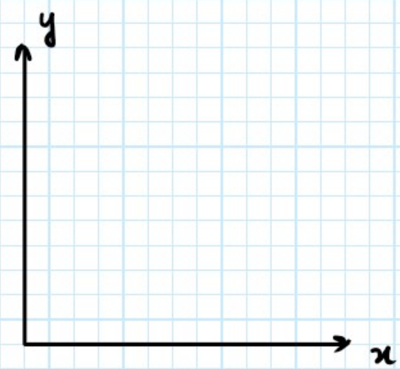
2×2

$$= 2\int_{\theta=0}^{\frac{\pi}{2}} d\theta\left(\int_{t=0}^{\infty} e^{-t}\,dt\right)$$

$$= 2\left[\theta\right]_0^{\frac{\pi}{2}}\left[-e^{-t}\right]_0^{\infty}$$

$$= \cancel{2}\left(\frac{\pi}{\cancel{2}}\right)\left[-(0-1)\right]$$

$$= \pi$$



$$\left(\Gamma\tfrac{1}{2}\right)^2 = \pi$$

$$\boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

PROOF: $\int_0^1 x^m(\log x)^n\,dx$

$\log x = -t$

$x = e^{-t}$ $\qquad dx = -e^{-t}$

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$x = e^{-t} \qquad dx = -e^{-t}$

$x = 0 \longrightarrow t = \infty$

$x = 1 \longrightarrow t = 0$

$$\int_0^1 x^m (\log x)^n \, dx$$

$$= \int_\infty^0 (e^{-t})^m (-t)^n (-e^{-t}) \, dt$$

$$= \int_0^\infty e^{-(m+1)t} (-1)^n (t)^n \, dt$$

$$= (-1)^n \int_0^\infty e^{-(m+1)t} (t)^n \, dt$$

let $(m+1)t = y \qquad\qquad t = 0 \longrightarrow y = 0$

$dt = \frac{dy}{m+1} \qquad\qquad t = \infty \longrightarrow y = \infty$

$$= (-1)^n \int_0^\infty e^{-y} \left(\frac{y}{m+1}\right)^n \left(\frac{dy}{m+1}\right)$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-y} y^n \, dy$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \cdot \Gamma(n+1)$$

PROOF: $\int_0^1 x^p (1 - x^q)^r \, dx$

$x^q = t \Rightarrow x = (t)^{\frac{1}{q}} \qquad\qquad x = 0 \longrightarrow t = 0$

$dx = \frac{1}{q} (t)^{\frac{1}{q} - 1} dt \qquad\qquad x = 1 \longrightarrow t = 1$

$$\Rightarrow \int_0^1 (t^{1/q})^p (1-t) \, \frac{1}{q} \cdot t^{\frac{1}{q} - 1} \, dt$$

$$= \frac{1}{q} \int_0^1 t^{\frac{p+1}{q} - 1} (1-t)^r \, dt$$

$$= \frac{1}{q} \int_0^1 t^{\frac{p+1}{q} - 1} (1-t)^{r} \, dt$$

$$= \frac{1}{q}$$