4. Separation of Variables, Higher Order PDE

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SEPARATION OF VARIABLES

This method is used to solve first order PDE if the soln is of the form:

$$u = X(x)\, Y(y)$$

<u>Example</u>: $x^2 \dfrac{\partial u}{\partial x} + y^2 \dfrac{\partial u}{\partial y} = 0$ — (1)

Let the solution be $u = X(x)\, Y(y)$

Diff wrt y, x partially

$\dfrac{\partial u}{\partial x} = X'Y \qquad \dfrac{\partial u}{\partial y} = XY'$ — (2)

$$x^2(X'Y) + y^2(Y'X) = 0$$

$x^2 \cdot X'Y + y^2\, Y'X = 0 \rightarrow$ Dividing by XY

$$x^2 \cdot \frac{X'}{X} + y^2 \frac{Y'}{Y} = 0$$

$$x^2 \cdot \frac{X'}{X} = -y^2\left(\frac{Y'}{Y}\right) = k$$

$$x^2 \frac{X'}{X} = k$$

$$\frac{X'}{X} = kx^{-2}$$

$$\log X = \frac{-k}{x} + \log c_1$$

$$\log \frac{X}{c_1} = \frac{-k}{x}$$

$$\boxed{X = c_1 e^{-\frac{k}{x}}}$$

$$-y^2 \frac{Y'}{Y} = k$$

$$\frac{Y'}{Y} = -y^{-2} k$$

$$\log Y = \frac{k}{y} + \log c_2$$

$$\boxed{Y = c_2 e^{\frac{k}{y}}}$$

HOMOGENEOUS, LINEAR HIGHER ORDER PDE

$$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = \phi(x,y)$$

$\rightarrow$ all derivatives have the same order

∂x^2 $\partial x\partial y$ ∂y^2

constants

If $\phi(x,y) = 0$

$$\underline{\underline{z = CF}}$$

If $\phi(x,y) \neq 0$

$$\underline{\underline{z = CF + PI}}$$

FINDING CF

Consider the general equation

$$a_0\left(\frac{\partial^2 z}{\partial x^2}\right) + a_1\left(\frac{\partial^2 z}{\partial x\partial y}\right) + a_2\left(\frac{\partial^2 z}{\partial y^2}\right) = \phi(x,y)$$

$$\frac{\partial}{\partial x} = D_x \ , \ \frac{\partial}{\partial y} = D_y \quad [\text{or}] \quad \frac{\partial}{\partial x} = D \ , \ \frac{\partial}{\partial y} = D'$$

Writing eqn in operator form:

$$a_0 D_x^2(z) + a_1 D_x D_y(z) + a_2 D_y^2(z) = \phi(x,y)$$

$$\left(a_0 D_x^2 + a_1 D_x D_y + a_2 D_y^2\right) z = \phi(x,y)$$

$$\boxed{F(D_x, D_y)\, z = \phi(x,y)}$$

Taking $D_x = m$, $D_y = 1$

$$\boxed{\text{Auxiliary Equation} : \ a_0 m^2 + a_1 m + a_2 = 0 = F(m,1)}$$

Let the roots be m_1 and m_2

Case (1): If the roots are distinct (real or complex)

$$\boxed{z = F_1(y + m_1 x) + F_2(y + m_2 x)}$$

Case (2): If the roots are equal $(m_1 = m_2 = m)$

$$z = F_1(y+mx) + xF_2(y+mx)$$

Case ②: If the roots are equal $(m_1 = m_2 = m)$

$$\boxed{z = F_1(y+mx) + xF_2(y+mx)}$$

FINDING PI

Consider the following PDE:

$$(a_0 D_x^2 + a_1 D_x D_y + a_2 D_y^2) z = \phi(x,y)$$

$$F(D_x, D_y)\, z = \phi(x,y)$$

$$z = \frac{\phi(x,y)}{F(D_x, D_y)}$$

Type ①: $\phi(x,y) = e^{ax+by}$

Replace D_x by a, D_y by b

$$\frac{e^{ax+by}}{F(D_x+D_y)} = \frac{e^{ax+by}}{F(a,b)} \quad \text{provided } F(a,b) \neq 0$$

$D_x \longrightarrow a$

$D_y \longrightarrow b$

If $F(a,b) = 0$ on replacement,

$$\frac{e^{ax+by}}{F(D_x, D_y)} = \frac{x \cdot e^{ax+by}}{\underbrace{F_{D_x}}_{\text{differentiate w.r.t. } D_x}(D_x, D_y)}$$

then proceed with the replacement as before

Type ②: $\phi(x,y) = \sin(ax+by)$ [or] $\cos(ax+by)$

$$\frac{\sin(ax+by)}{F(D_x^2, D_x D_y, D_y^2)} = \frac{\sin(ax+by)}{F(-a^2, -ab, -b^2)}$$

$D_x^2 \longrightarrow -a^2$

$D_x D_y \longrightarrow -ab$

$D_y^2 \longrightarrow -b^2$

Type ③: $\phi(x,y) =$ Polynomial

Example: $D_x^2 + 3D_xD_y + 2D_y^2 = x+y$

$$PI = \frac{x+y}{D_x^2 + 3D_xD_y + 2D_y^2}$$

$$= \frac{1}{D_x^2} \cdot \frac{x+y}{\left(1 + \frac{3D_y}{D_x} + \frac{2D_y^2}{D_x^2}\right)}$$

$$= \frac{1}{D_x^2}\left(1 + \frac{3D_y}{D_x} + \frac{2D_y^2}{D_x^2}\right)^{-1}(x+y)$$

$$= \frac{1}{D_x^2}\left(1 - \frac{3D_y}{D_x} - \frac{2D_y^2}{D_x^2}\right)(x+y)$$

$$\boxed{(1+x)^{-1} = 1 - x + x^2 - x^3 \dots}$$

$$\frac{D_y}{D_x}(x+y) = D_y\left(\frac{x^2}{2} + xy\right)$$

$$= 0 + x$$

$$\frac{D_y}{D_x}\left(\frac{D_y}{D_x}(x+y)\right) = \frac{D_y}{D_x}\cdot(x) = D_y\left(\frac{x^2}{2}\right) = 0$$

Type ④: $\phi(x,y) \cdot e^{ax+by} \cdot V(x,y)$

$$\frac{e^{ax+by} \cdot V(x,y)}{F(D_x, D_y)} = e^{ax+by} \cdot \frac{V(x,y)}{F(D_x+a, D_y+b)}$$

outside the operator

$D_x \longrightarrow D_x + a$

$D_y \longrightarrow D_y + b$