# Simulation of coupled problems with the Finite Element Method

## Exam project

Within the lecture a thermo-mechanical coupling was introduced. The governing equations assuming small deformations were implemented over the course of the programming exercises. For the exam, a code, based on the Jupyter-Notebooks from the exercises, shall be extended to the fully non-linear case.

The exam consists of submission of a report, a working python code followed by a presentation (20 min per group) and an oral exam ($\approx$ 20 min per group). Within the report – which has to be typeset either in Word or LaTeX– model and tangents derived in the exam-prerequisite including required corrections and the results of the numerical examples (based on the tangents detailed in Appendix A) have to be documented. The presentation should give a short overview over the implementation and the numerical results. To be able to take part in the exam, the exam prerequisite has to be sufficiently solved. The final report (as a pdf-document) and the code (zipped, working python-script) have to be submitted by E-Mail to all three lecturers on **21.09.2025**.

## Tasks - Exam-Prerequisite

The following tasks are considered as an exam-prerequisite and were submitted as a pdf-document on 14.07.2025 on ILIAS.

**(A)** Derive the first Piola-Kirchoff stress tensor $\mathbf{P}$ and the second Piola-Kirchoff stress tensor $\mathbf{S}$ from the given Helmholtz free energy function:

$$\begin{aligned}\rho_0\Psi = {} & \alpha_1\left(J^{-2/3}\mathrm{I}_\mathrm{C} - 3\right) + \alpha_2\left(J^{-4/3}\mathrm{II}_\mathrm{C} - 3\right) + \frac{\alpha_3}{2}(J-1)^2 \\ & - \beta\ln(J)(\theta-\theta_0) - \rho_0 c_\varepsilon\left(\theta\ln\frac{\theta}{\theta_0} - \theta + \theta_0\right)\end{aligned}$$

where $\mathrm{I}_\mathrm{C}$ and $\mathrm{II}_\mathrm{C}$ are the 1st and 2nd principal invariants of right Cauchy-Green deformation tensor $\mathbf{C}$ and $J = \sqrt{\det\mathbf{C}}$ .

**(B)** Derive the strong forms for the given non-linear thermo-mechanical problem in reference configuration. Ignore the body force and heat source.

**(C)** Derive the weak forms and state the residuum in reference configuration.

**(D)** Document the tangents required in order to form the tangent-stiffness-matrix.

# Tasks - Exam

Over the course of the lecture, a nonlinear elastic problem using the Simo and Pister material model was derived. The exam task consists of extending the linear thermo-elastic problem to nonlinearity and its subsequent implementation.

**(E)** Implement the fully non-linear equations for thermo-elasticity into the given code structure. Use the material model and tangents detailed in Appendix A!

In order to validate your implementation, consider the cantilever beam from Appendix B. Analytical solutions for the small and finite strain cases are provided.

**(F)** Implement a cantilever beam under a combination of normal and transverse loads, P and Q, respectively. Distribute the forces over the front surface. Avoid thermal effects by choosing appropriate boundary conditions. Discretise the domain based on $120 \times 8$ rectangles in $x$ and $y$ directions, each divided into 2 triangular elements of second order.

**(G)** Plot the maximum displacement $\delta_{\max}$ over the traverse force $\mathrm{Q} \in [0,\, 100\,\mathrm{kN}]$ for $\mathrm{P} = 120\,\mathrm{kN}$. Use 10 linearly spaced data points.

**(H)** Report the Newton iterations for all cases.

After a successful validation, differences between linear and non-linear model should be studied for the abstracted brake disc.

**(I)** Implement the abstracted brake disc, based on exercise E6_E7 and in Appendix C. Use the plane-strain assumption! Adjust the loading in order to achieve sufficiently large deformations!

**(J)** Analyse the differences in the behaviour of the two models for purely mechanical loading, purely thermal loading and thermo-mechanical loading. Meaningful conclusions and studies on influences and nuances are more important than fancy plots!

**(K)** Make a statement on the applicability of the model for a physical problem!

# A. Material Model

- Helmholtz-Energy Function:

$$\rho_0\Psi = \frac{\lambda}{2}\ln(J)^2 + \mu\left[\ln(J) + \frac{\mathrm{I}_{\mathbf{C}} - 3}{2}\right] - \beta\,\ln(J)\,(\theta - \theta_0) - \rho_0 c_\varepsilon\left[\theta\,\ln\left(\frac{\theta}{\theta_0}\right) - \theta + \theta_0\right]$$

- Second Piola-Kirchhoff stress

$$\mathbf{S} = \mu\left(\mathbf{I} - \mathbf{C}^{-1}\right) + \lambda\ln(J)\mathbf{C}^{-1} - \beta\,(\theta - \theta_0)\,\mathbf{C}^{-1}$$

- Residuum - Balance of linear momentum

$$R_{\mathbf{u}}(\mathbf{u}, \theta, \mathbf{v}_u) = \int_{\Omega_0}\mathbf{S}\cdot\delta\mathbf{E}\;\mathrm{d}V = 0 \quad \text{with} \quad \delta\mathbf{E}(\mathbf{u}, \mathbf{v}_u) = \operatorname{sym}\left(\mathbf{F}^{\mathrm{T}}\nabla_{\boldsymbol{X}}(\mathbf{v}_u)\right)$$

- Linearization of balance of linear momentum

$$\begin{aligned}\mathrm{LIN}\left[R_{\mathbf{u}}\right] = R_{\mathbf{u}}(\tilde{\mathbf{u}}, \tilde{\theta}, \mathrm{v}_{\mathbf{u}}) &+ \int_{\Omega_0}\delta\tilde{\mathbf{E}}\cdot\tilde{\mathbb{C}}\cdot\Delta\mathbf{C}\;\mathrm{d}V \\ &+ \int_{\Omega_0}\nabla_{\boldsymbol{X}}(\Delta\mathbf{u})\tilde{\mathbf{S}}\cdot\nabla_{\boldsymbol{X}}(\mathbf{v}_u)\;\mathrm{d}V \\ &- \int_{\Omega_0}\beta\delta\tilde{\mathbf{E}}\cdot\tilde{\mathbf{C}}^{-1}\,\Delta\theta\;\mathrm{d}V\end{aligned}$$

with the following terms:

$$\begin{aligned}\mathbb{C}(\mathbf{u}, \theta) &= \left(\lambda\ln(J) - \mu - \beta\left[\theta - \theta_0\right]\right)\frac{\partial\mathbf{C}^{-1}}{\partial\mathbf{C}} + \frac{\lambda}{2}\,\mathbf{C}^{-1}\otimes\mathbf{C}^{-1} \\ \frac{\partial\mathbf{C}^{-1}}{\partial\mathbf{C}} &= -\frac{1}{2}\left(\left(\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\right)^{\overset{23}{\mathrm{T}}} + \left(\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\right)^{\overset{24}{\mathrm{T}}}\right) \\ \Delta\mathbf{C} &= 2\operatorname{sym}\left(\tilde{\mathbf{F}}^{\mathrm{T}}\nabla_{\boldsymbol{X}}(\Delta\mathbf{u})\right)\end{aligned}$$

- Residuum - Balance of energy

$$R_\theta(\mathbf{u},\theta,\mathrm{v}_\theta) = \int_{\Omega_0} \rho_0 c_\varepsilon\, \theta' \mathrm{v}_\theta + \beta\theta \nabla_{\boldsymbol{X}}(\mathbf{x}') \cdot \mathbf{F}^{-\mathrm{T}} \mathrm{v}_\theta - \mathbf{Q} \cdot \nabla_{\boldsymbol{X}}(\mathrm{v}_\theta)\,\mathrm{d}V + \int_{\Gamma_{\theta,vN}} \mathbf{Q}\cdot\mathbf{N}\,\mathrm{d}A$$

where

$$\mathbf{Q} = -\kappa J \mathbf{C}^{-1} \nabla_{\boldsymbol{X}}(\theta)$$

- Linearization of balance of energy

$$\begin{aligned}\mathrm{LIN}\,[R_\theta] = R_\theta(\tilde{\mathbf{u}},\tilde{\theta},\mathrm{v}_\theta) + \int_{\Omega_0} \Bigg[ \frac{\partial R_\theta}{\partial \theta}\bigg|_{\tilde{\theta}} \cdot \Delta\theta + \frac{\partial R_\theta}{\partial \theta'}\bigg|_{\tilde{\theta}} \cdot \Delta\theta' + \frac{\partial R_\theta}{\partial \nabla_{\boldsymbol{X}}(\theta)}\bigg|_{\tilde{\theta}} \cdot \Delta\nabla_{\boldsymbol{X}}(\theta) \\ + \frac{\partial R_\theta}{\partial \nabla_{\boldsymbol{X}}(\mathbf{x}')}\bigg|_{\tilde{\mathbf{u}}} \cdot \Delta\nabla_{\boldsymbol{X}}(\mathbf{x}') + \frac{\partial R_\theta}{\partial \mathbf{C}}\bigg|_{\tilde{\mathbf{u}}} \cdot \Delta\mathbf{C} + \frac{\partial R_\theta}{\partial \mathbf{F}}\bigg|_{\tilde{\mathbf{u}}} \cdot \Delta\mathbf{F} \Bigg]\,\mathrm{d}V\end{aligned}$$

with the following terms:

$$\frac{\partial R_\theta}{\partial \theta} \cdot \Delta\theta = \mathrm{v}_\theta\, \beta \nabla_{\boldsymbol{X}}(\mathbf{x}') \cdot \mathbf{F}^{-\mathrm{T}}\, \Delta\theta$$

$$\frac{\partial R_\theta}{\partial \theta'} \cdot \Delta\theta' = \mathrm{v}_\theta\, \rho_0 c_\varepsilon \frac{\Delta\theta}{\Delta t}$$

$$\frac{\partial R_\theta}{\partial \nabla_{\boldsymbol{X}}(\theta)} \cdot \nabla_{\boldsymbol{X}}(\Delta\theta) = \nabla_{\boldsymbol{X}}(\mathrm{v}_\theta) \cdot \kappa J \mathbf{C}^{-1} \cdot \nabla_{\boldsymbol{X}}(\Delta\theta)$$

$$\frac{\partial R_\theta}{\partial \nabla_{\boldsymbol{X}}(\mathbf{x}')} \cdot \nabla_{\boldsymbol{X}}(\Delta\mathbf{x}') = \mathrm{v}_\theta\, \beta\theta \mathbf{F}^{-\mathrm{T}} \cdot \frac{\nabla_{\boldsymbol{X}}(\Delta\mathbf{u})}{\Delta t}$$

$$\begin{aligned}\frac{\partial R_\theta}{\partial \mathbf{C}} \cdot \Delta\mathbf{C} = \kappa \frac{J}{2} \left(\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}\right) \cdot \nabla_{\boldsymbol{X}}(\theta) \cdot \nabla_{\boldsymbol{X}}(\mathrm{v}_\theta) \cdot \Delta\mathbf{C} \\ - \kappa \frac{J}{2} \left( \left(\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}\right)^{\overset{23}{\mathrm{T}}} + \left(\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}\right)^{\overset{24}{\mathrm{T}}} \right) \cdot \nabla_{\boldsymbol{X}}(\theta) \cdot \nabla_{\boldsymbol{X}}(\mathrm{v}_\theta) \cdot \Delta\mathbf{C}\end{aligned}$$

$$\frac{\partial R_\theta}{\partial \mathbf{F}} \cdot \Delta\mathbf{F} = -\mathrm{v}_\theta\, \beta\theta \nabla_{\boldsymbol{X}}(\mathbf{x}') \cdot (\mathbf{F}^{-\mathrm{T}} \otimes \mathbf{F}^{-\mathrm{T}})^{\overset{24}{\mathrm{T}}} \cdot \nabla_{\boldsymbol{X}}(\Delta\mathbf{u})$$

- Approximating time derivatives by:

$$\mathbf{x}' = \frac{\mathbf{u} - \mathbf{u}^{\mathrm{n}}}{\Delta t} \quad \text{and} \quad \theta' = \frac{\theta - \theta^{\mathrm{n}}}{\Delta t}$$

# B. Cantilever Beam

Given is a purely mechanical cantilever beam with square cross-section, length $L$, side $a$ and material parameters $E$ and $\nu$. On the free end, there is a transverse force $Q$ and normal force $P$ as shown in Figure 1.

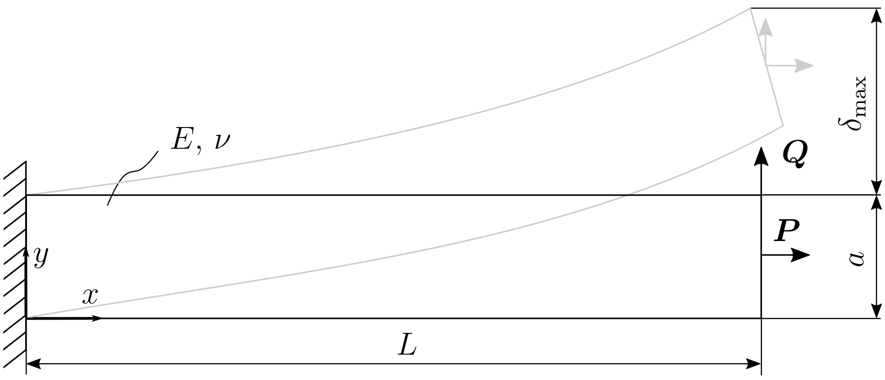


**Figure 1:** Cantilever beam model

Use the following values for the individual parameters:

| Description | Variable | Value | Unit |
|---|---|---|---|
| Length | $L$ | 1.5 | m |
| Side length | $a$ | 0.1 | m |
| Young's modulus | $E$ | $70 \times 10^9$ | $\mathrm{N\,m^{-2}}$ |
| Poisson's ratio | $\nu$ | 0.3 | – |
| Transverse forces | $\boldsymbol{Q}$ | $0 \ldots 100 \times 10^3$ | N |
| Longitudinal force | $\boldsymbol{P}$ | $120 \times 10^3$ | N |

Use the following analytical solutions of maximum deflection $\delta_{\max}$ for validation in the linear case

$$\delta_{\max}^{\mathrm{L}} = \frac{QL^3}{3EI},$$

whereas for the non-linear case

$$\delta_{\max}^{\mathrm{NL}} = \frac{QL^3}{EI} \frac{(\alpha - \tanh(\alpha))}{\alpha^3},$$

with $\alpha = \sqrt{\frac{PL^2}{EI}}$, where $I$ is the area moment of inertia.

# C. Abstracted brake disc

For the abstracted brake disc, use the following boundary conditions: (the dimensions are the same as in E6_E7) are considered:

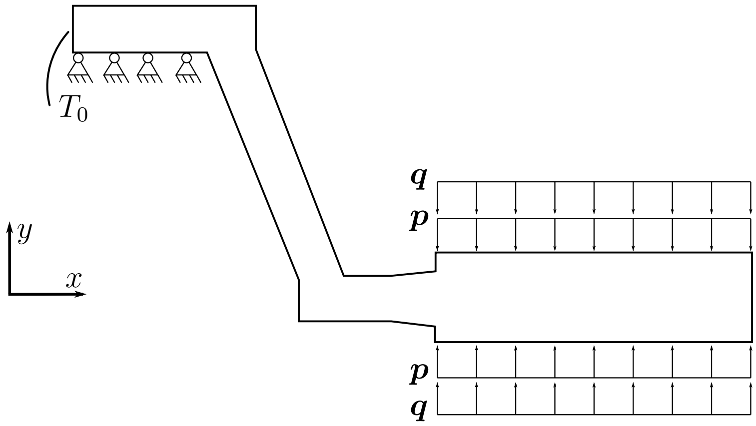


**Figure 2:** 2D Brake Disc

The required material parameters are specified in the table below:

| Description | Variable | Value | Unit |
|---|---|---|---|
| Depth | $b$ | 1000 | mm |
| Young's modulus | $E$ | $210 \times 10^3$ | MPa |
| Poisson's ratio | $\nu$ | 0.33 | – |
| Initial temperature | $T_0$ | 20 | °C |
| Spec. heat capacity | $c_\varepsilon$ | $452 \times 10^6$ | $\mathrm{mJ\,t^{-1}\,K}$ |
| Thermal expansion coeff. | $\alpha$ | $11 \times 10^{-6}$ | $\mathrm{K^{-1}}$ |
| Thermal conductivity | $\kappa$ | 48 | $\mathrm{mW\,K^{-1}\,mm^{-1}}$ |
| Density | $\rho$ | $7.8 \times 10^{-9}$ | $\mathrm{t\,mm^{-3}}$ |

Make sure to use a consistent notation throughout your derivation. Use the table below as an example. Refer to additional formulary (Formelsammlung Tensorrechnung und Kontinuumsmechanik) uploaded to ILIAS for further examples.

| **Description** | **Output** | **Syntax** |
|---|---|---|
| scalars - not bold, low | $u$ | u |
| 1. order tensors - bold, straight, low | $\mathbf{f} = \mathrm{f}_i \, \mathbf{e}_i$ | `\bm{\mathrm{f}}` |
| 2. order tensors - bold, straight, high | $\mathbf{F} = F_{ij} \, \mathbf{e}_i \otimes \mathbf{e}_j$ | `\bm{\mathrm{F}}` |
| 3. order tensors | n/A | - |
| 4. order tensors - bold, straight, high | $\mathbb{C}$ | `\mathbbm{C}` |
| 1. order tensor component | $\mathrm{f}_i$ | `\mathrm{f}_i` |
| 2. order tensor component | $\mathrm{F}_{ij}$ | `\mathrm{F}_{ij}` |
| Divergence - Current | $\nabla \cdot \mathbf{F}$ | `\nabla\cdot\bm{\mathrm{F}}` |
| Divergence - Reference | $\nabla_{\boldsymbol{X}} \cdot \mathbf{F}$ | `\nabla_{\bm {X} }\bm{\mathrm{F}}` |
| Gradient - Current | $\nabla \mathbf{F}$ | `\nabla\bm{\mathrm{F}}` |
| Gradient - Reference | $\nabla_{\boldsymbol{X}} \mathbf{F}$ | `\nabla_{\bm {X} }\bm{\mathrm{F}}` |
| Vector - bold, straight, low | $\mathbf{f}$ | `\bm{\mathrm{f}}` |
| Matrix - bold, straight, high | $\mathbf{F}$ | `\bm{\mathrm{F}}` |