Multivariate Analysis of Variance MANOVA

Multivariate Statistic

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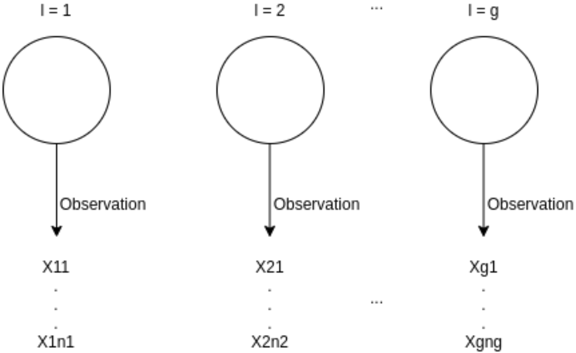
Agenda

- Model
- MANOVA
- Model Check
- MANOVA2

Model



I = 1
I = 2
...
I = g
Observation
Observation
Observation
X11
X1n1
X21
X2n2
...
Xg1
Xgng

Assumptions:

- $$X_{\ell,j} \sim N\left(\mu_\ell, \Sigma_\ell\right)$$
- $$\Sigma_1 = \Sigma_2 = \cdots = \Sigma_g$$
- $$\mu_\ell = \mu + \tau_\ell, \qquad \sum_{\ell=1}^{g} n_\ell \tau_\ell = 0$$

Hypothesis

$$H_0 : \forall \mu_\ell = \mu \qquad \leftrightarrow \qquad \forall \tau_\ell = 0$$
$$H_1 : \exists \tau_\ell \neq 0$$

Decomposition of variance

$$SST = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} \left(X_{\ell,j} - \bar{X}\right) \left(X_{\ell,j} - \bar{X}\right)^T$$

$$SSW = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} \left(X_{\ell,j} - \bar{X}_\ell\right) \left(X_{\ell,j} - \bar{X}_\ell\right)^T$$

$$SSB = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} \left(\bar{X}_\ell - \bar{X}\right) \left(\bar{X}_\ell - \bar{X}\right)^T$$

Decomposition of variance

$$SST = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} \left(X_{\ell,j} - \bar{X}\right) \left(X_{\ell,j} - \bar{X}\right)^T$$

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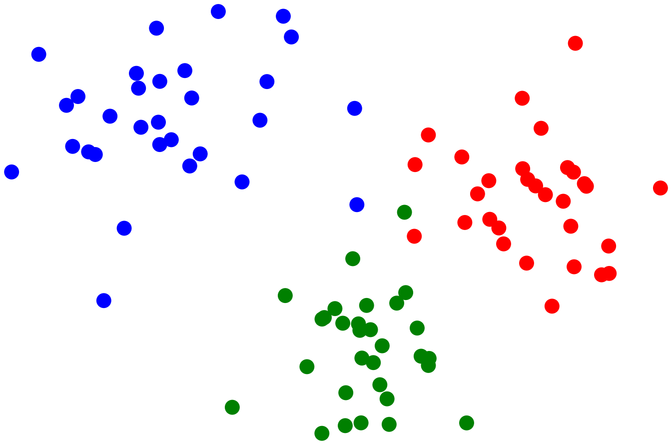
$$\Lambda^* = \frac{|SSW|}{|SSB + SSW|}$$

Illustration



Exact Distributions of Wilks Lambda.

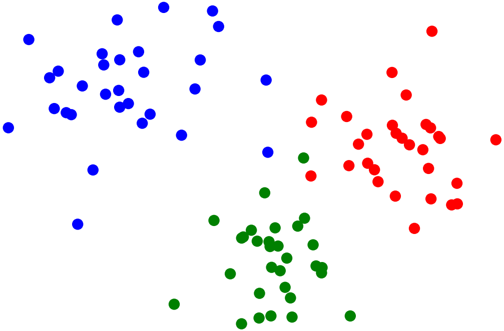
No. of variables	No. of groups	Sampling distribution for multivariate normal data
$p = 1$	$g \geq 2$	$\left(\frac{\Sigma n_\ell - g}{g-1}\right)\left(\frac{1-\Lambda^*}{\Lambda^*}\right) \sim F_{g-1,\Sigma n_\ell - g}$
$p = 2$	$g \geq 2$	$\left(\frac{\Sigma n_\ell - g - 1}{g-1}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2(g-1),2(\Sigma n_\ell - g - 1)}$
$p \geq 1$	$g = 2$	$\left(\frac{\Sigma n_\ell - p - 1}{p}\right)\left(\frac{1-\Lambda^*}{\Lambda^*}\right) \sim F_{p,\Sigma n_\ell - p - 1}$
$p \geq 1$	$g = 3$	$\left(\frac{\Sigma n_\ell - p - 2}{p}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2p,2(\Sigma n_\ell - p - 2)}$

Approximate distribution of Wilks Lambda.

$$-\left(n-1-\frac{p+g}{2}\right) ln\left(\Lambda^{*}\right) \sim \chi^{2}_{p(g-1)}$$

Pairwise CI's

$$\left[(\bar{X}_{ai} - \bar{X}_{bi}) \pm t_{n-g,\frac{\alpha}{\frac{pg(g-1)}{2}}} \cdot \sqrt{S_{wii} \cdot \left(\frac{1}{n_a} + \frac{1}{n_b}\right)}\right], i = 1, ..., p \wedge a, b = 1, ..., g$$



Model Check

Hypothesis:

$$H_0 : \forall \Sigma_\ell = \Sigma$$
$$H_1 : \exists \Sigma_\ell \neq \Sigma$$

Likelihood Ratio Test

$$\Lambda = \frac{\underset{\Sigma}{max} L(\Sigma)}{\underset{\Sigma_1, \cdots, \Sigma_g}{max} L(\Sigma_1, \cdots, \Sigma_g)}$$

$$-2 \log \Lambda = \left((n-g) \log |S_p| - \sum_{l=1}^{g} (n_l - 1) \log |S_l| \right)$$

Reject:

$$-2c \cdot \log \Lambda > \chi^2_{p\frac{p+1}{2}, \alpha}$$

MANOVA2

Assumptions:

- $$X_{\ell k,j} \sim N_p\left(\mu_{\ell k}, \Sigma_{\ell k}\right)$$
- $$\forall \Sigma_{\ell k} = \Sigma$$
- $$\mu_{\ell k} = \mu + \tau_\ell + \beta_k + \gamma_{\ell k}$$

Hypothesis

$H_{0,\gamma} : \forall \gamma_{\ell k} = 0 \quad H_{0,\tau} : \forall \tau \ell = 0 \quad H_{0,\beta} : \forall \beta_k = 0$

$H_{1,\gamma} : \exists \gamma_{\ell k} \neq 0 \quad H_{1,\tau} : \exists \tau \ell \neq 0 \quad H_{1,\beta} : \exists \beta_k \neq 0$