Detection and classification (LDA and QDA)

Multivariate Statistic

Made by:

Lasse Gøransson, Marc Evald, Anne-Charlotte Poulsen & Aske Møller

SDU Robotics
The Maersk Mc-Kinney Moller Institute
University of Southern Denmark

Agenda

Detection and classification

- Motivation
- Assumptions
- Definitions
- Classification with LDA and QDA

Purpose:

- Sort new observations into one of two or more populations

- π_1: pdf $f_1(x)$, μ_1, Σ_1
- π_2: pdf $f_2(x)$, μ_2, Σ_2
- $X_0 \in \mathbb{R}^p$

$$d(X_0) = \begin{cases} d1, & X_0 \in R_1 \\ d2, & X_0 \in R_2 \end{cases}$$

General Criteria
Definitions

- Cost Matrix:

Classify as $\rightarrow$	d_1	d_2
True π_1	0	$c[d_2\|\pi_1]$
π_2	$c[d_1\|\pi_2]$	0

- Prior Probabilities:
  - $p_1 = P[X_0 \in \pi_1]$
  - $p_2 = P[X_0 \in \pi_2]$
  - $p_1 + p_2 = 1$

- General Criteria (using ECM)
  - $p_1 f_1(x_0) c[d_2|\pi_1] \underset{d_2}{\overset{d_1}{\gtrless}} p_2 f_2(x_0) c[d_1|\pi_2]$
  - $\frac{f_1(x_0)}{f_2(x_0)} \underset{d_2}{\overset{d_1}{\gtrless}} \frac{p_2}{p_1} \frac{c[d_1|\pi_2]}{c[d_2|\pi_1]}$

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- Equal Cost ($min(TPM)$)
  - $\frac{f_1(X_0)}{f_2(x_0)} \underset{d_2}{\overset{d_1}{\gtrless}} \frac{p_2}{p_1}$

General Criteria

Two population Classification

- General Criteria (using ECM)
  - $p_1 f_1(x_0) c[d_2|\pi_1] \underset{d_2}{\overset{d_1}{\gtrless}} p_2 f_2(x_0) c[d_1|\pi_2]$
  - $\frac{f_1(x_0)}{f_2(x_0)} \underset{d_2}{\overset{d_1}{\gtrless}} \frac{p_2}{p_1} \frac{c[d_1|\pi_2]}{c[d_2|\pi_1]}$
- Equal Cost ($min(TPM)$)
  - $\frac{f_1(X_0)}{f_2(x_0)} \underset{d_2}{\overset{d_1}{\gtrless}} \frac{p_2}{p_1}$
- Equal Cost & Priors
  - $f_1(x_0) \underset{d_2}{\overset{d_1}{\gtrless}} f_2(x_0)$

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  - $\frac{f_1(x_0)}{f_2(x_0)} \underset{d_2}{\overset{d_1}{\gtrless}} \frac{p_2}{p_1} \frac{c[d_1|\pi_2]}{c[d_2|\pi_1]}$
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- Equal Cost & Priors
  - $f_1(x_0) \underset{d_2}{\overset{d_1}{\gtrless}} f_2(x_0)$
- MAP
  - $P[\pi_1|x_0] = \frac{p_1 f_1(x_0)}{p_1 f_1(x_0)+p_2 f_2(x_0)}$
  - $P[\pi_2|x_0] = \frac{p_2 f_2(x_0)}{p_1 f_1(x_0)+p_2 f_2(x_0)}$

Classification for 2 MVN populations

Having $X|_{\pi_i} \sim \mathcal{N}_p(\mu_i, \Sigma_i)$ for $i = 1, 2$
If $\Sigma_1 = \Sigma_2 \overset{def}{=} \Sigma$:

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- LDA
  - $(\mu_1 - \mu_2)^T \Sigma^{-1} x_0 - \frac{1}{2}(\mu_1 - \mu_2)^T \Sigma^{-1}(\mu_1 - \mu_2) \underset{d_2}{\overset{d_1}{\gtrless}} \log(\frac{p_2}{p_1} \frac{c(d_1|\pi_2)}{c(d_2|\pi_1})$

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If $\Sigma_1 \neq \Sigma_2$:

- QDA
  - $\frac{1}{2} x_0^T (\Sigma_1^{-1} - \Sigma_2^{-1}) x_0 + \left(\mu_1^T \Sigma_1^{-1} - \mu_2^T \Sigma_2^{-1}\right) x_0 - k \underset{d_2}{\overset{d_1}{\gtrless}} \log(\frac{p_2}{p_1} \frac{c(d_1|\pi_2)}{c(d_2|\pi_1})$
  - $k = \frac{1}{2} \log \frac{|\Sigma_1|}{|\Sigma_2|} + \frac{1}{2} \left(\mu_1^T \Sigma_1^{-1} \mu_1 - \mu_2^T \Sigma_2^{-1} \mu_2\right)$

Performance

Classification for 2 MVN populations

	π_1	π_2
d_1	n_{1C}	n_{2M}
d_2	n_{1M}	n_{2C}

Performance
Classification for 2 MVN populations

	π_1	π_2
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- $APER = \frac{n_{1M}+n_{2M}}{n_{1M}+n_{1C}+n_{2M}+n_{2C}}$
- $A\hat{E}R$