XIII. Signal to Noise (S/N) [v1.3.4]

A. Overview

- It is relatively easy to generate a good idea for an observational experiment in astronomy
  - ⋄ Imaging an exoplanet
  - ⋄ Detecting the black hole at our Galaxy's center
  - ⋄ Observing the gas that fuels stars in distant galaxies
- It is much harder to generate a good idea that can be performed/tested with a telescope!
  - ⋄ Finite collecting power (D^2)
  - ⋄ Finite time (t)
  - ⋄ Finite angular resolution (D^{-1})
  - ⋄ Finite sky background (t^2)
  - ⋄ Finite universe (age+size; t)
- Key: Is our source sufficiently bright to yield a signal that deviates significantly from a null detection?
  - ⋄ Compare expected signal to the predicted/measured noise
  - ⋄ For a confident detection, desire signal-to-noise $S/N > 3$
  - ⋄ And, because this exercise is highly idealized, best to aim for $S/N > 5$ (or 10!)
- References
  - ⋄ ??

B. Signal

- For many experiments, we are limited by the number of photons from the source that we can collect
  - ⋄ Faint, nearby sources (planets)
  - ⋄ Bright, but very distant sources (galaxies)
- Signal is directly proportional to the source flux
  - ⋄ Bolometric flux F_B
    - ▲ Energy per second per area
    - ▲ No telescope observes photons at <u>all</u> energies
    - ▲ Generally only one to a few octaves in energy
  - ⋄ Specific flux
    - ▲ Energy per second per area per energy interval (recall, $\Delta E = h\Delta\nu$)
    - ▲ f_ν: erg/s/cm^2/Hz

- ▲ f_λ: erg/s/cm^2/Å
- ▲ These are interchangeable: $f_\lambda d\lambda = f_\nu d\nu$

- Photons <u>collected</u> per unit time: ϕ
  - ◇ Consider a circular primary mirror with diameter D_p
    - ▲ In theory, its collecting area is $A = \pi(D_p/2)^2$
    - ▲ But, it is always obstructed by optics (e.g. the secondary) and/or instrumentation (prime focus camera)
  - ◇ Express its effective area as $A_{\text{eff}} < A$
  - ◇ Photons collected per second per unit wavelength ϕ_λ

$$\phi_\lambda = A_{\text{eff}} \frac{f_\lambda}{h\nu} \tag{1}$$

    - ▲ Units of ϕ_λ: photons/s/Å
    - ▲ Example
      - ○ Betelgeuse has $V = 0.5\,\text{mag}$ or $f_\lambda = 2.7 \times 10^{-9}$ ergs/s/cm^2/Å
      - ○ Keck telescope has $D_p = 10\,\text{m}$ and $A_{\text{eff}} = 7.2 \times 10^5\,\text{cm}^2$
      - ○ $\phi_\lambda = 5 \times 10^8$ photons/s/Å
      - ○ Maybe Keck is overkill for this star!

- Signal: Photons detected (depends on η)
  - ◇ While ϕ_λ photons are being collected every second, not all will be recorded
    - ▲ A small percentage are lost to imperfect mirror coatings ($\sim 1\%$ per bounce)
    - ▲ Camera optics don't have perfect transmission
    - ▲ Detector has a QE less than 100%
  - ◇ Define η as the end-to-end efficiency of the system
    - ▲ (Camera) x (detector) x (telescope) [ignoring the atmosphere]
    - ▲ Typically, $\eta \approx 10 - 30\%$
    - ▲ i.e. $70 - 90\%$ of the photons are lost!
  - ◇ And it is wavelength dependent,
    - ▲ e.g. we will use η_λ

- The Signal is also proportional to the integration time t
  - ◇ Signal per unit wavelength: S_λ

$$S_\lambda = \phi_\lambda \, \eta_\lambda \, t \tag{2}$$

  - ◇ This is counts per Angstrom!
    - ▲ For imaging, you need to integrate (sum) over the filter
    - ▲ A typical broad band filter (V) has
    - ▲ $\Delta\lambda \approx 1200$Å
    - ▲ Total signal: $S = S_\lambda \, \Delta\lambda$

- ◇ Measure the total signal in electrons
  - ▲ These are not the counts recorded in images!
  - ▲ Need to include the gain..
  - ▲ But these follow Poisson stats

C. Noise

- In the absence of noise, one could always achieve a detection
  - ◇ Provided you could wait long enough to detect the signal
  - ◇ This might require t approaching ∞!
- In practice, there is uncertainty in our measurements from a wide range of sources
  - ◇ These limit the S/N that we can achieve
  - ◇ We ignore, for now, systematic uncertainties
- Detector noise
  - ◇ Read noise (RN)
    - ▲ An unavoidable, nearly Gaussian source of counts that we associate with the detector read-out electronics
    - ▲ It affects our measurements once per exposure, independent of exposure time
    - ▲ Typically, $RN \approx 3 - 10$ electrons
    - ▲ Variance: $\sigma_{RN}^2 = RN^2$
  - ◇ Dark current
    - ▲ Noise associated with thermal motions of electrons in the detector
      - ○ Proportional to the exposure time
      - ○ Poisson process
    - ▲ Express the count rate as DC
      - ○ Then, the variance is $\sigma_{\text{dark}}^2 = DC \cdot t$
      - ○ Typical values are $DC = 1 - 10$ electrons per hour
      - ○ Often ignored because it is such a small contribution
- Object "noise"
  - ◇ The detection of photons from our source is a Poisson process
    - ▲ Expect an average signal of S photons
    - ▲ Variance in Poisson equals the mean:

$$\sigma_{\text{obj}}^2 = S \tag{3}$$

  - ◇ Noise from the source: $\sigma_{\text{obj}} = \sqrt{S}$
- Background
  - ◇ Recall that the night sky is far from dark
    - ▲ Airglow, moon light

- ▲ Zodiacal light
- ▲ EBL

◇ At mid-IR wavelengths, the dome, air and telescope are all shining too

◇ Generally, the background is uniform on the sky

- ▲ Not localized like a star or a distant galaxy
- ▲ Every pixel in the image includes sky background

◇ Characterized as a surface brightness (μ_λ)

- ▲ Specific flux per unit area
- ▲ Usually an angular area
  - ○ per square arcsec ($\Box''$)
  - ○ $\mu \approx 19 - 24$mag $/\Box''$ for optical (wavelength, moon dependent)

◇ Calculating the noise from the sky background

- ▲ Similar to our calculations for the Source
- ▲ But we need to integrate over an effective background area A_B
  - ○ For imaging, this is usually the angular size of our source
  - ○ For spectroscopy, the aperture (e.g. slit width) matters
- ▲ Mean number of electrons recorded per wavelength

$$B_\lambda = \mu_\lambda A_B A_{\text{eff}} \eta_\lambda t \tag{4}$$

- ▲ Assume some wavelength interval

$$B = B_\lambda \, \Delta\lambda \tag{5}$$

- ▲ Again, expect Poisson statistics
  - ○ Variance: $\sigma^2_{\text{sky}} = B$
  - ○ Noise $\sigma_{\text{sky}} = \sqrt{B}$

- Total Noise (N)

  ◇ Add it all up in quadrature

$$N = \sqrt{\sigma^2_{\text{obj}} + \sigma^2_{\text{sky}} + \sigma^2_{RN} + \sigma^2_{DC}} \tag{6}$$

  ◇ Note: All but σ_{RN} are proportional to time

  ◇ For this reason, one usually integrates long enough so that σ^2_{RN} may be ignored

D. Read Noise limited S/N

- Extremely faint sources *and* background

  ◇ Or a very poor detector

- Read noise dominated

  ◇ $N \approx \sigma_{\text{RN}}$

- ◇ S/N is proportional to t!

$$\left.\frac{S}{N}\right|_{RN} = \frac{(f_\lambda A_{\rm eff} \Delta\lambda \eta_\lambda / h\nu) t}{\sigma_{\rm RN}} \tag{7}$$

- ◇ Integrate twice as long and achieve twice the signficance!
- ◇ This sounds good, but it is almost always a bad sign
- ◇ It is primarily the result of your very faint signal or inefficient telescope!

- Note: $S/N|_{RN}$ is also proportional to $A_{\rm eff}$ (i.e. D^2) which motivates the construction of ever larger telescopes

E. Source-dominated S/N

- Source count rate significantly exceeds that of all others
- Typical case for bright sources
  - ◇ $N \approx \sigma_{\rm obj}$
  - ◇ S/N is proportional to $(f_\lambda t)^{1/2}$

$$\left.\frac{S}{N}\right|_{obj} = \sqrt{\frac{f_\lambda A_{\rm eff} \eta_\lambda t \Delta\lambda}{h\nu}} \tag{8}$$

- Consider the time to achieve a given S/N

$$t \propto \left(\left.\frac{S}{N}\right|_{obj} \right)^2 f_\lambda^{-1} \, \Delta\lambda^{-1} \tag{9}$$

  - ◇ It is inversely proportional to f_λ and $\Delta\lambda$
  - ◇ But the square of the desired S/N!
  - ◇ Takes $4\times$ longer exposure to double the statistical significance

F. Background dominated S/N

- Common for faint sources
  - ◇ Or very bright backgrounds
- $N \approx \sigma_{\rm sky}$
  - ◇ Once again, $S/N \propto t^{1/2}$

$$\left.\frac{S}{N}\right|_{sky} = \frac{f_\lambda A_{\rm eff} \eta_\lambda t \Delta\lambda}{\sqrt{\mu_\lambda A_B A_{\rm eff} \eta \Delta\lambda t}} \propto \frac{t^{1/2} f_\lambda \sqrt{A_{\rm eff} \Delta\lambda}}{\sqrt{\mu_\lambda A_B}} \tag{10}$$

- Time to achieve a desired S/N

$$t \propto \frac{(S/N)^2 \mu_\lambda A_B}{f_\lambda^2 A_{\rm eff} \Delta\lambda} \tag{11}$$

- ◇ t is proportional to μ_λ
  - ▲ Avoid the moon (at wavelengths where it matters)
  - ▲ Avoid bright sky lines
  - ▲ Reduce thermal background (if it matters)
- ◇ t is proportional to A_B
  - ▲ Reduce the size of the background region
  - ▲ Imaging of point sources (e.g. stars)
    - ◦ $A_B \propto \theta^2 \propto (\lambda/D)^2$
    - ◦ $A_B \propto (FWHM_{\rm seeing})^2$
    - ◦ Big gains in improving seeing and/or the diffraction limit
  - ▲ Spectroscopy
    - ◦ $A_B \propto$ (Slit width $\times$ $FWHM_{\rm seeing}$)
    - ◦ Narrow the slit width to reduce the background
- ◇ t is proportional to $A_{\rm eff}^{-1}$, i.e. D^{-2}
- ◇ Increase telescope diameter
  - ▲ $t \propto A_B/A_{\rm eff}$
  - ▲ In diffraction limit, $t \propto D^{-4}$!
  - ▲ Say hello to my friend TMT!!

G. Exposure Time Calculators

- Many observatories provide software to estimate the S/N of an observation
  - ◇ These are intended to capture all of the factors (e.g. $\eta_\lambda, A_{\rm eff}$) for a given instrument+telescope
  - ◇ They generally include a range of tools for the source and background too
- Examples
  - ◇ HST
    - ▲ COS spectrograph
    - ▲ WFC3 imager
  - ◇ Keck
    - ▲ ESI (`esi_s2ngui`)
    - ▲ HIRES (`hires_s2ngui`)
  - ◇ These are useful, but...
    - ▲ Highly idealized
    - ▲ Simply sky background models
    - ▲ Occasionally erroneous!

H. S/N Considerations for Imaging

- The PSF is critical (for small objects)

- $\Delta\lambda$ matters too

I. S/N Considerations for Spectroscopy

- Resolution (R)
  - ◇ Definition
    - ▲ Width of an unresolved spectral line
    - ▲ e.g. the line-emission from an arc lamp (Ne)
    - ▲ Often expressed as a $FWHM$
      - ○ In km/s or Å
      - ○ Recall $FWHM = 2\sigma\sqrt{2\ln 2}$
- Dispersion
  - ◇ Size (in km/s or Å) of a pixel
  - ◇ $\Delta\lambda$
  - ◇ Occasionally, and confusingly, one may report the "resolution" of the spectrograph as $\lambda/\Delta\lambda$
- Sampling
  - ◇ $FWHM/\Delta\lambda$
  - ◇ Nyquist limit: ($FWHM/\Delta\lambda = 2$)
- Equivalent Width (EW)
  - ◇ Absorption: Measure of the fracton of photons that were absorbed by a medium (gas, dust)
  - ◇ Let f_λ^{int} be the unabsorbed light
  - ◇ Definition
    - ▲ EW or W_λ

$$W_\lambda = \int \frac{f_\lambda^{obs}}{f_\lambda^{int}}\, d\lambda \tag{12}$$

    - ▲ Integral is evaluated over the feature or region of interest
    - ▲ Calculating the variance

$$\sigma^2(W_\lambda) = \sum_{i=1}^{n} \Delta\lambda_i^2\ \sigma_i^2(f^{obs}/f^{int}) \tag{13}$$

$$\approx n\ \Delta\lambda^2\ \sigma^2(f^{obs}/f^{int}) \tag{14}$$

    - ▲ Where the latter approximation assumes that $\Delta\lambda$ and the error σ_i vary little over the n pixels of integration
    - ▲ Recognizing that $\sigma(f^{obs}/f^{int})$ is simply the inverse of the S/N, we finish with:

$$\sigma(W_\lambda) \approx \frac{\Delta\lambda\ n^{1/2}}{S/N} \tag{15}$$