Сулейманов Асхаб, 409 группа

1-d дискретное разложение Фурье.

1 Исходная задача.

1)Для функции $u(x) \in C^\infty[0,1]$, удовлетворяющей краевым условиям:

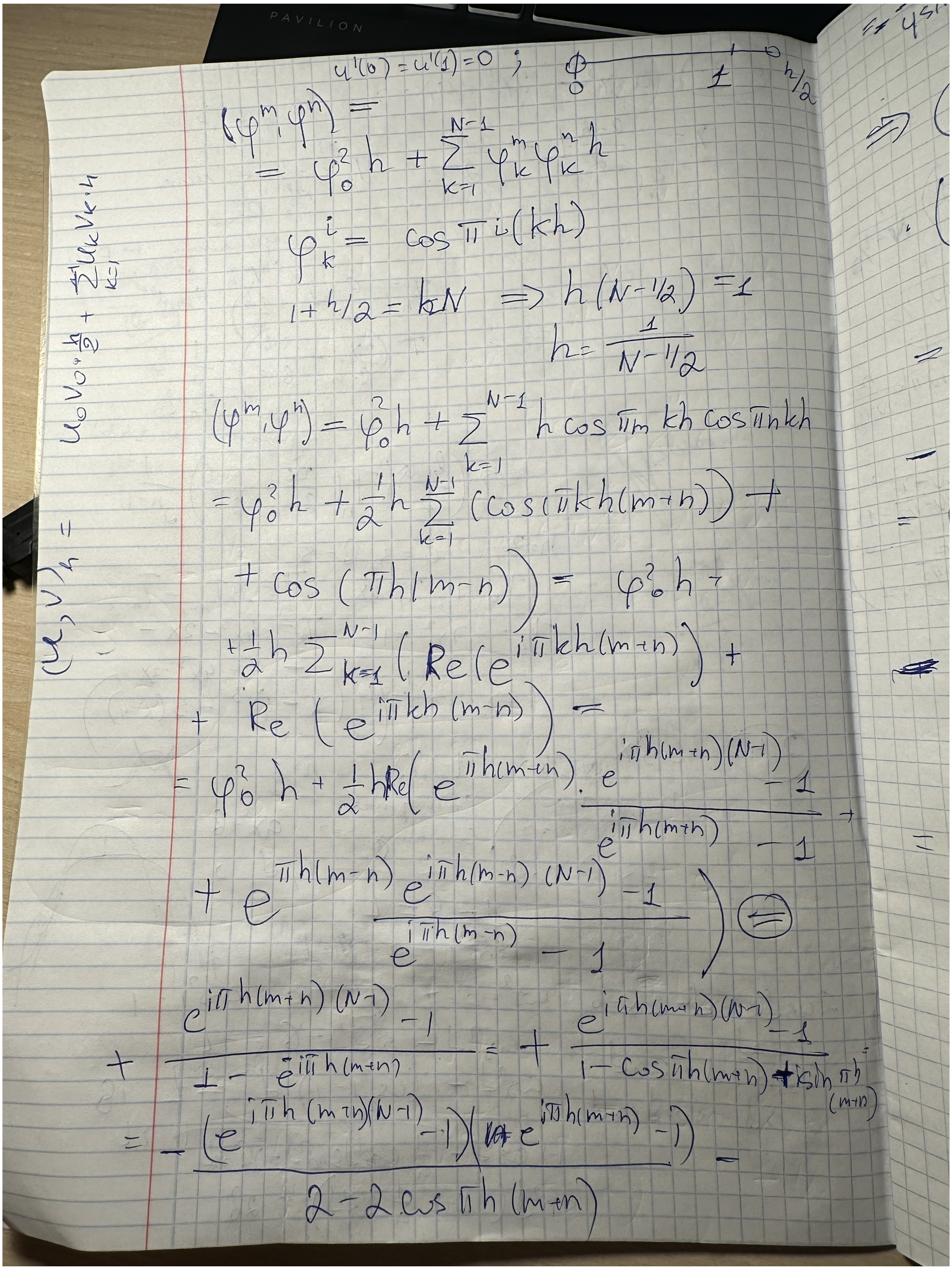
$$u'(0) = u'(1) = 0,$$

необходимо выписать тригонометрический ряд Фурье и сформулировать теорему сходимости.
2)На заданной сетке:

$$\begin{aligned} &x_0 = 0, \\ &x_N = 1 + \frac{h}{2}, \\ &h = \frac{1}{N - 0.5}, \end{aligned}$$

выписать дискретный тригонометрический ряд Фурье. Найти дискретное скалярное произведение, сохраняющее ортогональность базисных функций. Нормировать базисные функции.
3)Для некоторой тестовой функции из указанного класса численно найти порядок сходимости её дискретного ряда Фурье.

2 Тригонметрический ряд Фурье. Ортогональность. Вычисление коэффициентов.



$u'(0)=u'(1)=0$;

$(u,v)_h = u_0 v_0 \frac{h}{2} + \sum_{k=1}^{N-1} u_k v_k h$

$(\varphi^m,\varphi^n) =$

$= \varphi_0^2 h + \sum_{k=1}^{N-1} \varphi_k^m \varphi_k^n h$

$\varphi_k^i = \cos \pi i (kh)$

$1 + h/2 = hN \Rightarrow h(N-1/2) = 1$

$h = \frac{1}{N-1/2}$

$(\varphi^m,\varphi^n) = \varphi_0^2 h + \sum_{k=1}^{N-1} h \cos \pi m kh \cos \pi n kh$

$= \varphi_0^2 h + \frac{1}{2} h \sum_{k=1}^{N-1} (\cos(\pi k h(m+n)) + \cos(\pi h(m-n))) = \varphi_0^2 h +$

$+ \frac{1}{2} h \sum_{k=1}^{N-1} (Re(e^{i\pi kh(m+n)}) + Re(e^{i\pi kh(m-n)})) =$

$= \varphi_0^2 h + \frac{1}{2} h Re\left(e^{i\pi h(m+n)} \cdot \frac{e^{i\pi h(m+n)(N-1)}-1}{e^{i\pi h(m+n)}-1} + e^{i\pi h(m-n)} \frac{e^{i\pi h(m-n)(N-1)}-1}{e^{i\pi h(m-n)}-1}\right) =$

$+ \frac{e^{i\pi h(m+n)(N-1)}-1}{1-e^{i\pi h(m+n)}} = + \frac{e^{i\pi h(m+n)(N-1)}-1}{1-\cos\pi h(m+n) + i\sin \pi h(m+n)}$

$= -\frac{(e^{i\pi h(m+n)(N-1)}-1)(e^{i\pi h(m+n)}-1)}{2-2\cos\pi h(m+n)} -$

$$= \frac{1}{4\sin^2\frac{\pi h(m+n)}{2}}\left(e^{i\pi h(m+n)N} - e^{i\pi h(m+n)(N-1)} - e^{i\pi h(m+n)} + 1\right)$$

$$\Rightarrow = \varphi_0^2 h \cancel{-} \frac{1}{2}h \cdot \frac{1}{4\sin\frac{\pi h(m+n)}{2}} \cdot$$

$$\cdot\left(\cos(\pi h(m+n)N) + 1 - \cos(\pi h(m+n)(N-1)) - \right.$$

$$\left. - \cos(\pi h(m+n))\right) =$$

$$= \varphi_0^2 h - \frac{h}{4\sin^2\frac{\pi h(m+n)}{2}} \cdot \left(\sin^2\frac{\pi h(m+n)}{2} - \right.$$

$$\left. - \sin\frac{\pi h(m+n)}{2}\sin\frac{\pi h(2N-1)(m+n)}{2} + (-\|-)(m-n)\right)$$

$$= \varphi_0^2 h - \frac{h}{4}\left(+2 \cancel{-} \frac{\sin\left(\frac{\pi h(m+n)(2N-1)}{2}\right)}{\sin\frac{\pi h(m+n)}{2}} + \right.$$

$$\left. \cancel{-} \frac{\sin\left(\frac{\pi h(m-n)(2N-1)}{2}\right)}{\sin\left(\frac{\pi h(m-n)}{2}\right)}\right) =$$

$$= \varphi_0^2 h + \frac{h}{4}\left(-2 + \frac{\sin\pi(m+n)}{\sin\frac{\pi h(m+n)}{2}} + \right.$$

$$\left. + \frac{\sin\pi k(m-n)}{\sin\frac{\pi h(m-n)}{2}}\right) = \varphi_0^2 h - \frac{h}{2} = 0$$

$$\Rightarrow \varphi_0^2 = 1/2$$

$$\varphi_0 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow (u, v) = \left(\frac{u_0 v_0}{\sqrt{2}} + \sum_{k=1}^{N-1} u_k v_k\right)h$$

$$(\varphi^n, \varphi^n) = \left(\sum_{k=1}^{N-1} \cos^2 \pi k h n + 1/2\right) h =$$

$$= \left(\sum_{k=1}^{N-1} \cos^2 \pi k \frac{n}{N-1/2} + 1/2\right) \frac{1}{N-1/2} =$$

$$= \left(\frac{N-1}{2} + \frac{1}{2}\sum_{k=1}^{N-1} \cos 2\pi k \frac{n}{N-1/2} + 1/2\right) \frac{1}{N-1/2}$$

$$= \frac{N}{2(N-1/2)} + \frac{1}{N-1/2}\frac{1}{2}\underbrace{\sum_{k=1}^{N-1} \cos 2\pi k \frac{n}{N-1/2}}$$

$$\underbrace{Re\left(\sum_{k=1}^{N-1} e^{i 2\pi k \frac{n}{N-1/2}}\right)}$$

$$\underbrace{Re\left(e^{i 2\pi \frac{n}{N-1/2}} \cdot \frac{e^{i 2\pi n \frac{N-1}{N-1/2}} - 1}{e^{i 2\pi \frac{n}{N-1/2}} - 1}\right)}$$

$$\frac{N}{2(N-1/2)} - \frac{1/2}{2(N-1/2)} =$$

$$= 1/2$$

$$C_m = \frac{(u, \varphi^m_*)}{(\varphi^m, \varphi^m)} = 2(u, \varphi^m)$$

$$\frac{e^{i\varphi(N-1)} - 1}{1 - e^{-i\varphi}} =$$

$$= \frac{e^{i\varphi(N-1)} - 1}{1 - \cos\varphi + i\sin\varphi} =$$

$$= \frac{(e^{i\varphi(N-1)} - 1)(1 - e^{i\varphi})}{2 - 2\cos\varphi} =$$

$$= \frac{1}{4\sin^2\varphi/2}\left(-e^{i\varphi N} + e^{i\varphi(N-1)} + e^{i\varphi} - 1\right)$$

$$\frac{1}{4\sin^2\varphi/2}\left(-\cos\varphi N + \cos\varphi(N-1) + \cos\varphi - 1\right)$$

$$\frac{1}{4\sin^2\varphi/2}\left(\sin\varphi\frac{2N-1}{2}\sin\frac{\varphi}{2} - 2\sin^2\varphi/2\right)$$

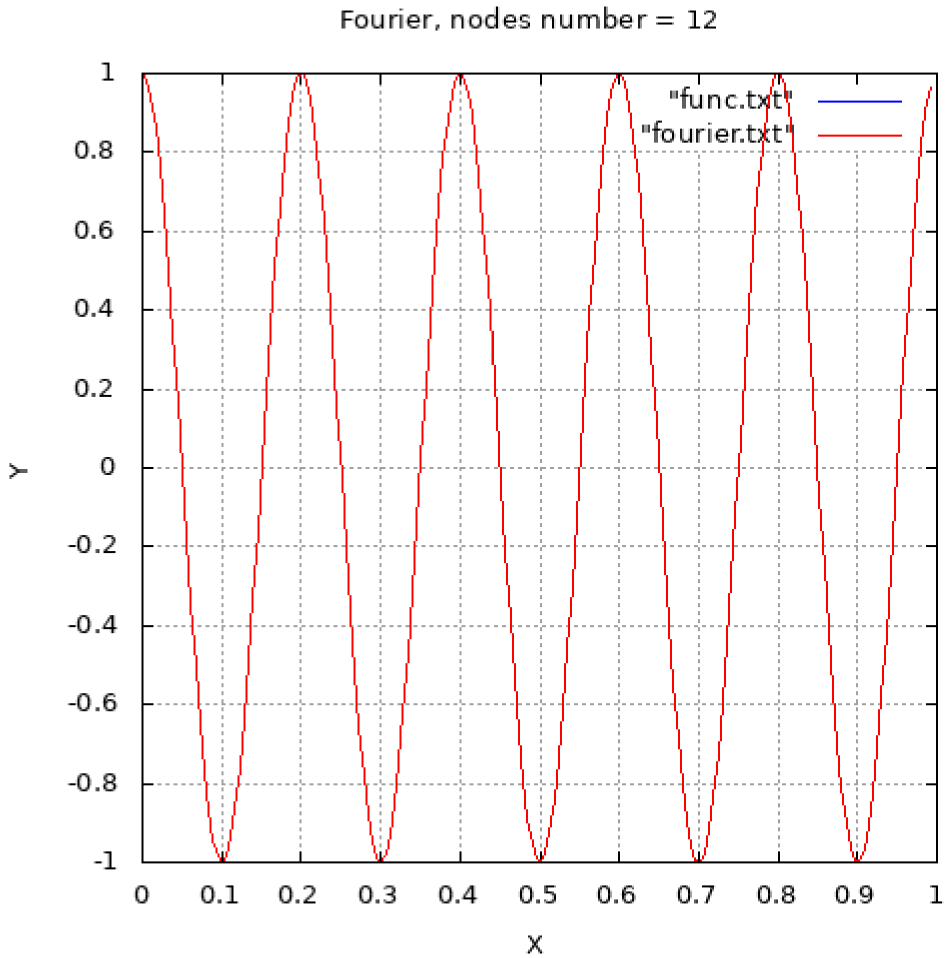
$$\frac{1}{4}\left(\frac{\sin\varphi(N-1/2)}{\sin\varphi/2} - 2\right) = -1/2$$

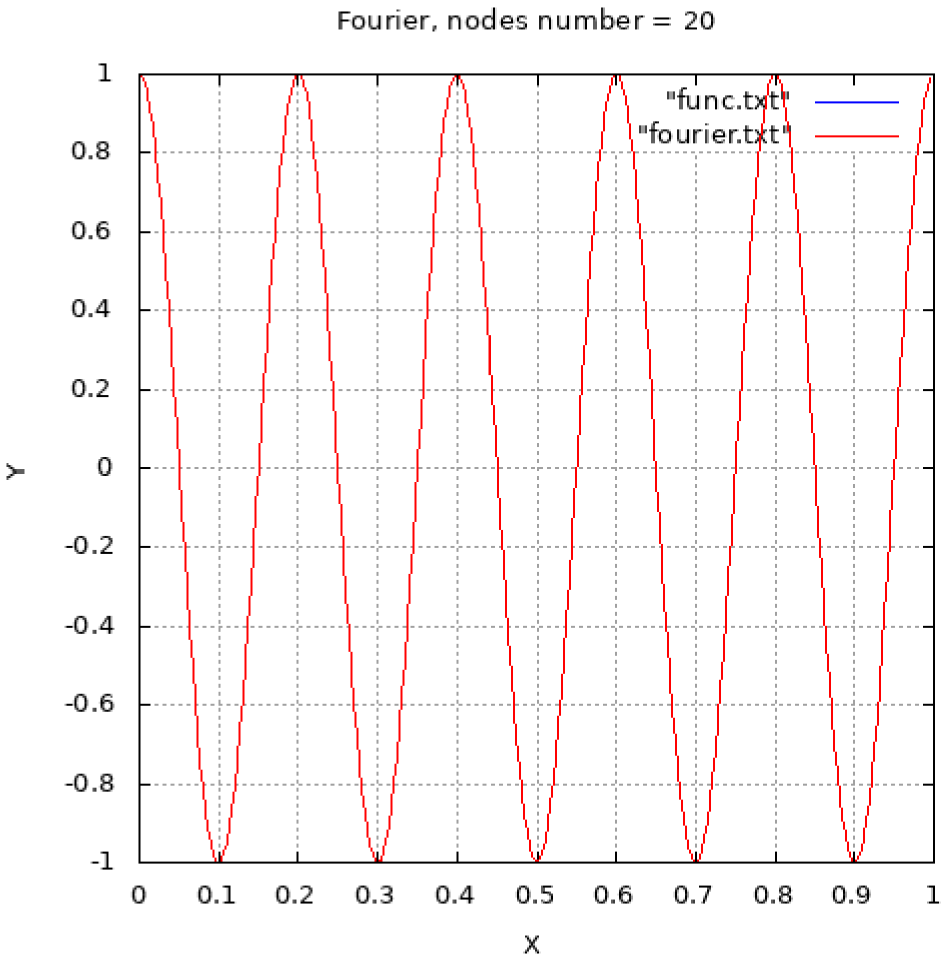
$$\int_a^b$$

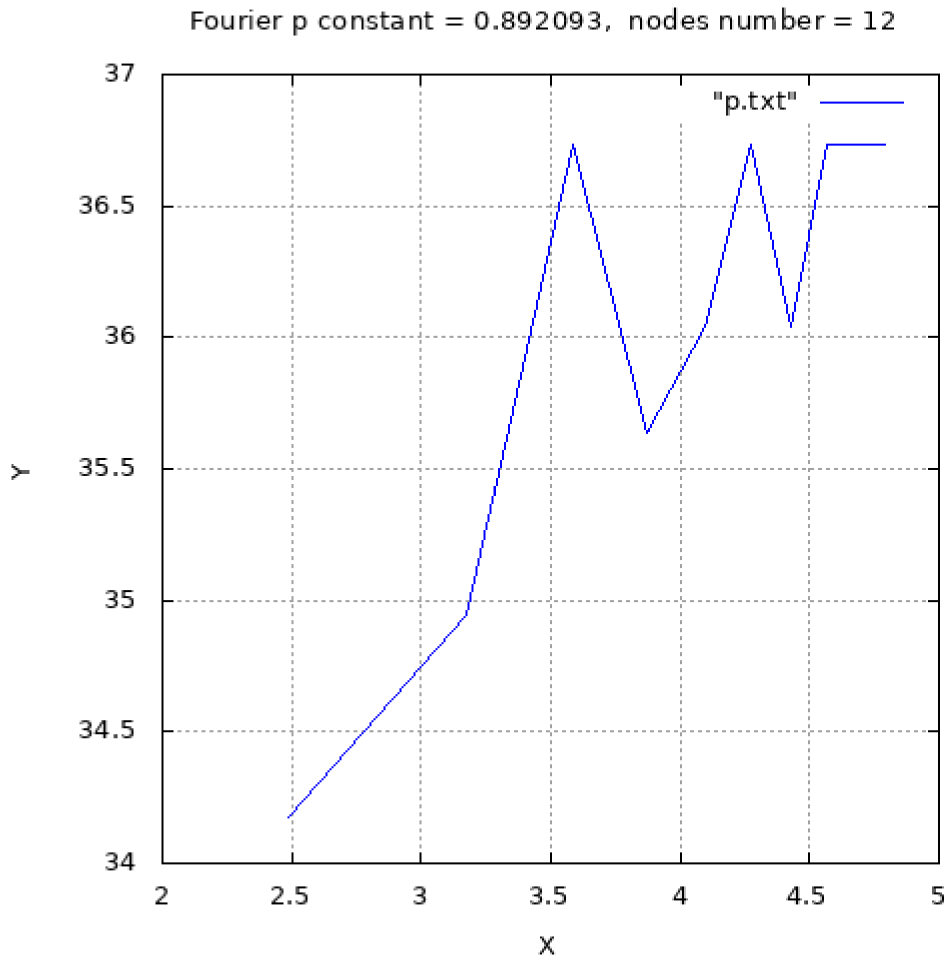
$$I(\varphi) =$$

3 Примеры работы.

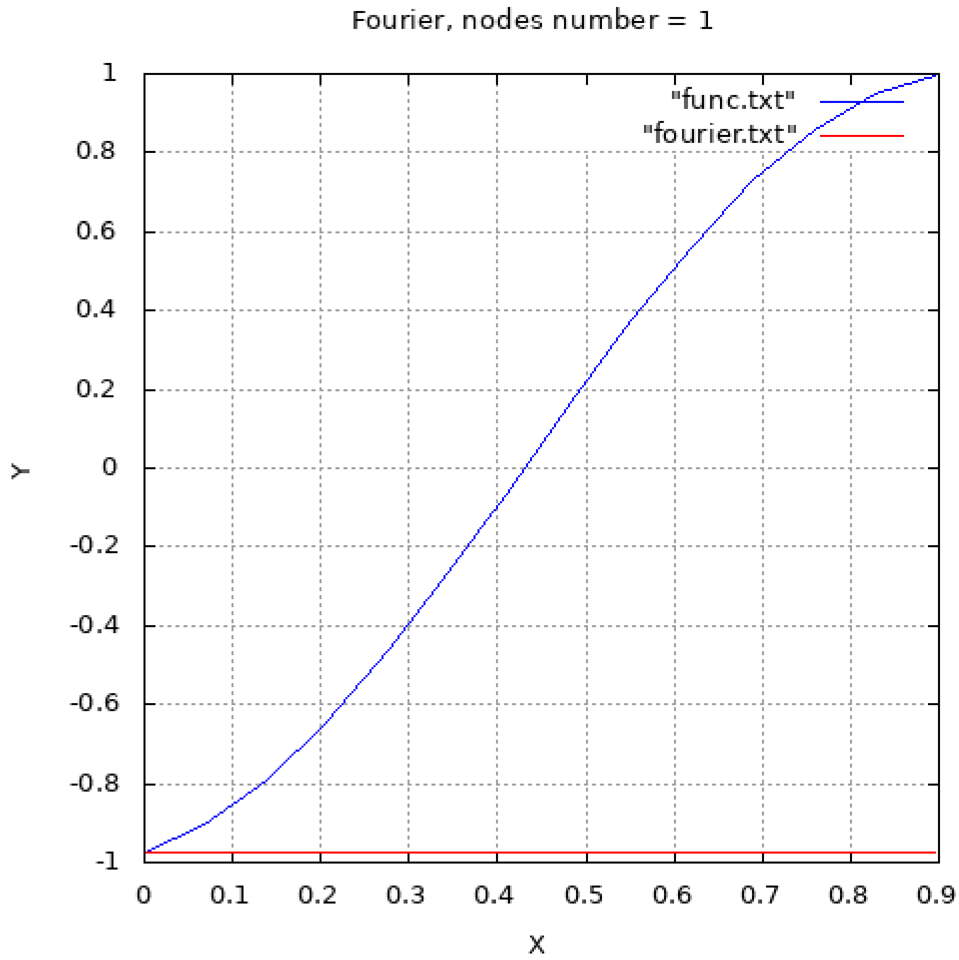
1) $\cos\left(10\Pi x\right)$

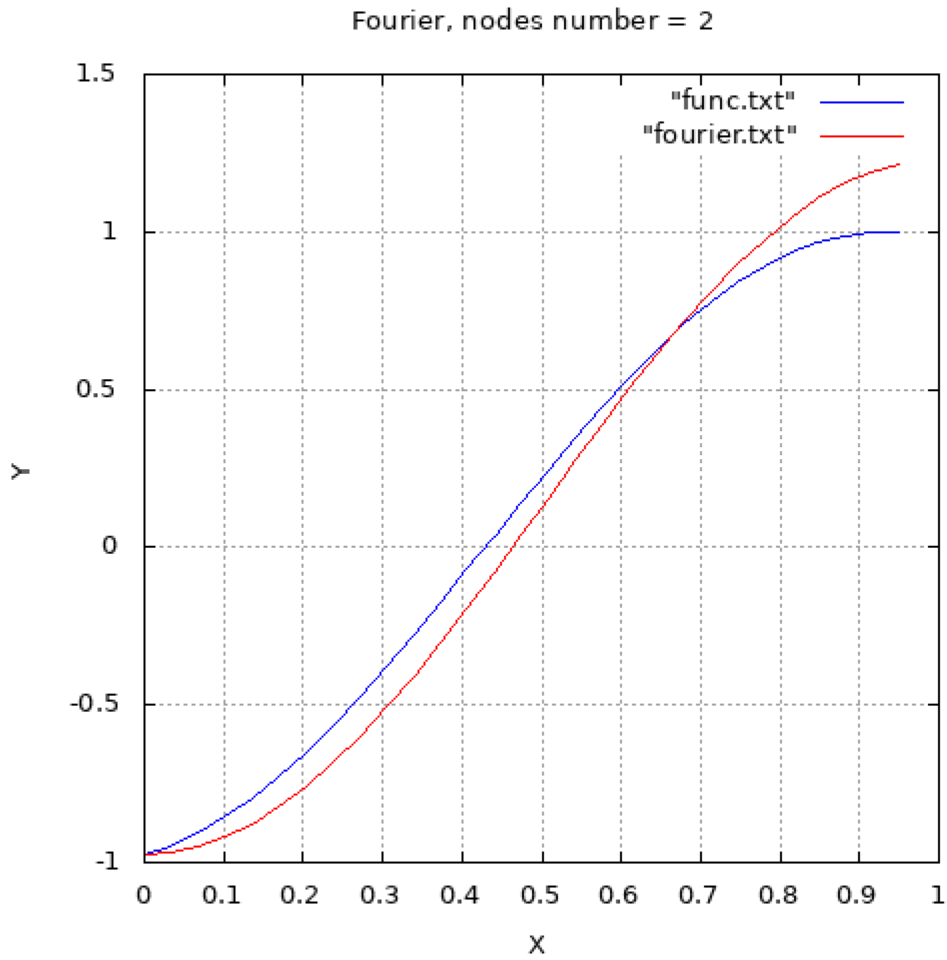



Fourier, nodes number = 20
"func.txt"
"fourier.txt"
Y
X
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

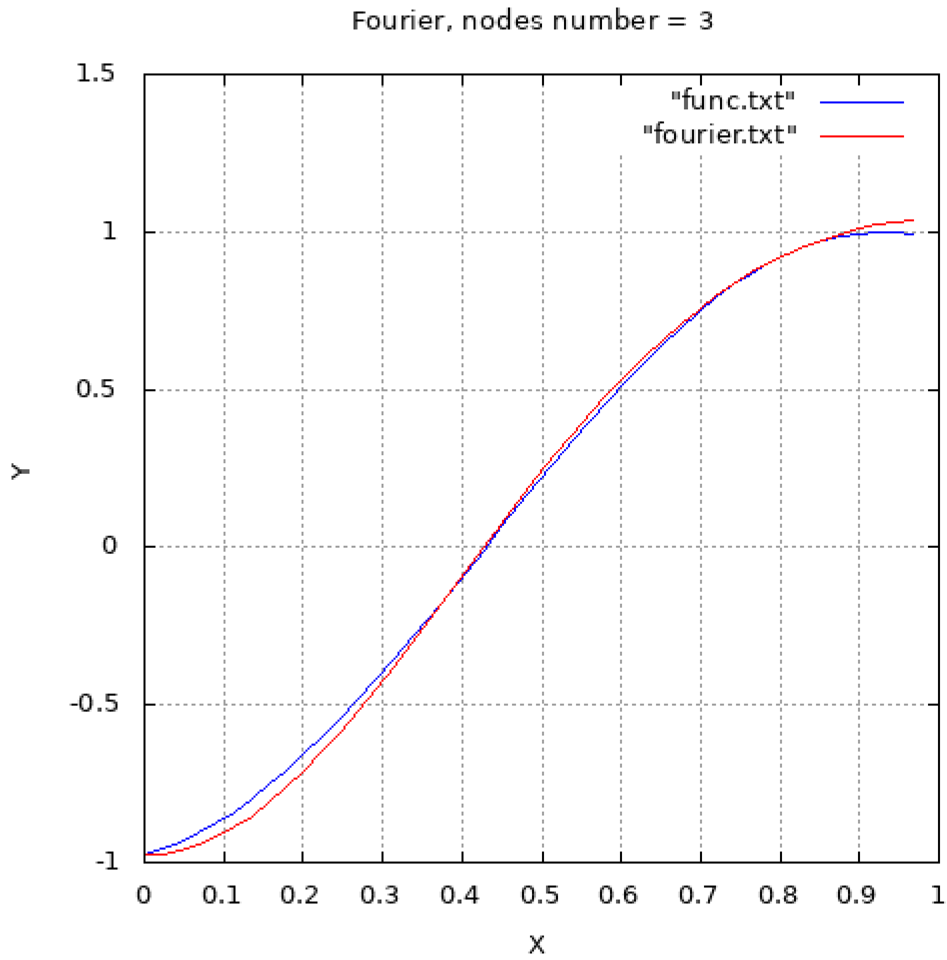

Fourier p constant = 0.892093, nodes number = 12
"p.txt"
Y
X
37
36.5
36
35.5
35
34.5
34
2
2.5
3
3.5
4
4.5
5

$2)\ \sin\left(\Pi\left(x+\frac{\Pi}{2}\right)\right)$

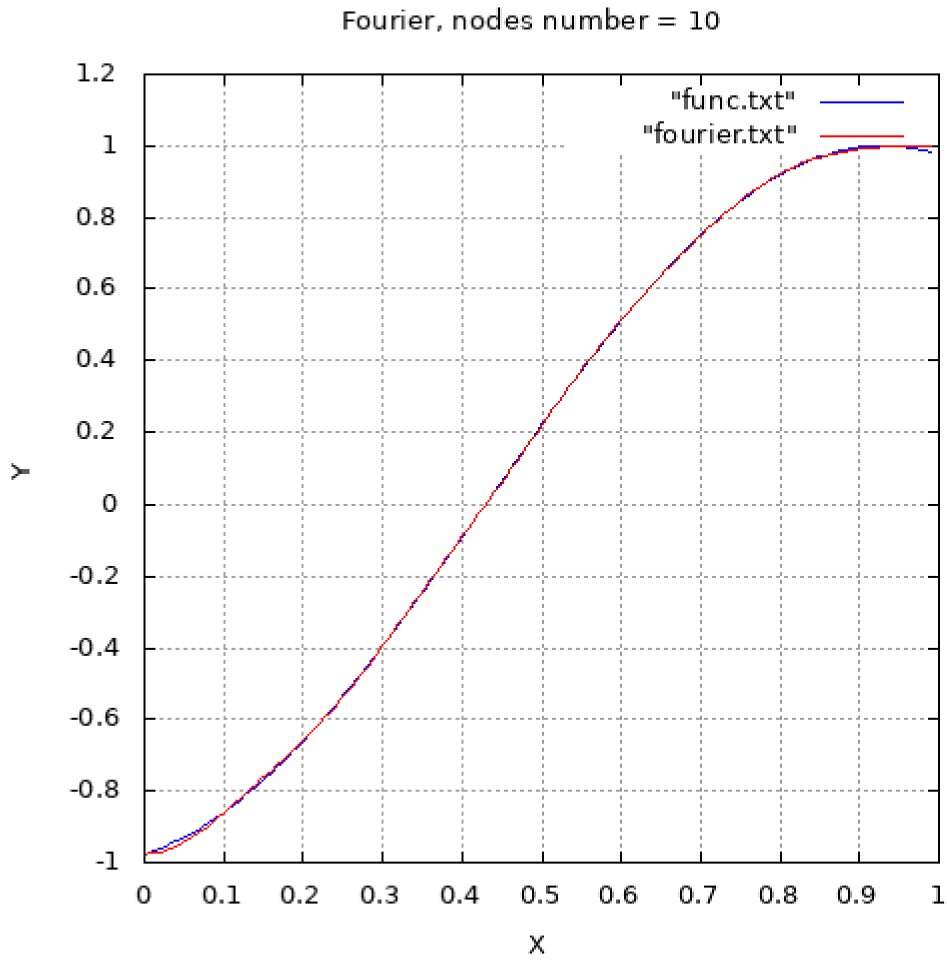


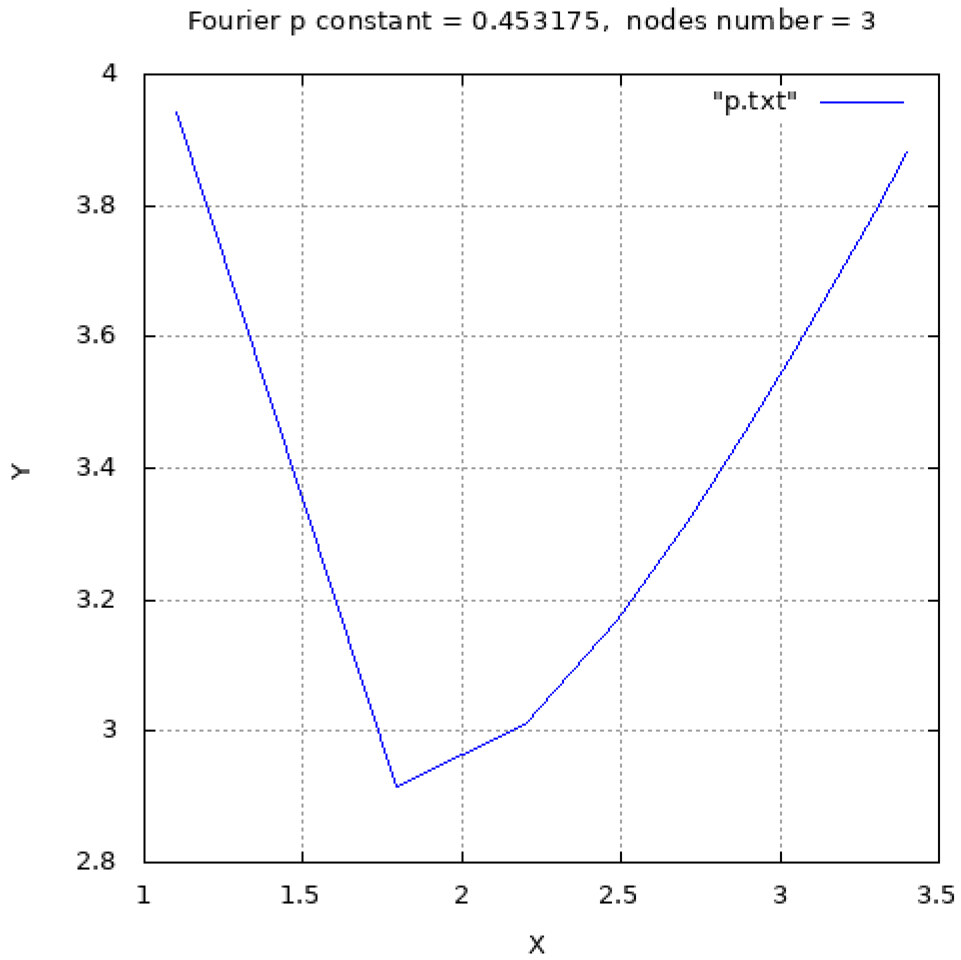


Fourier, nodes number = 2
"func.txt"
"fourier.txt"
1.5
1
0.5
0
-0.5
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
X
Y

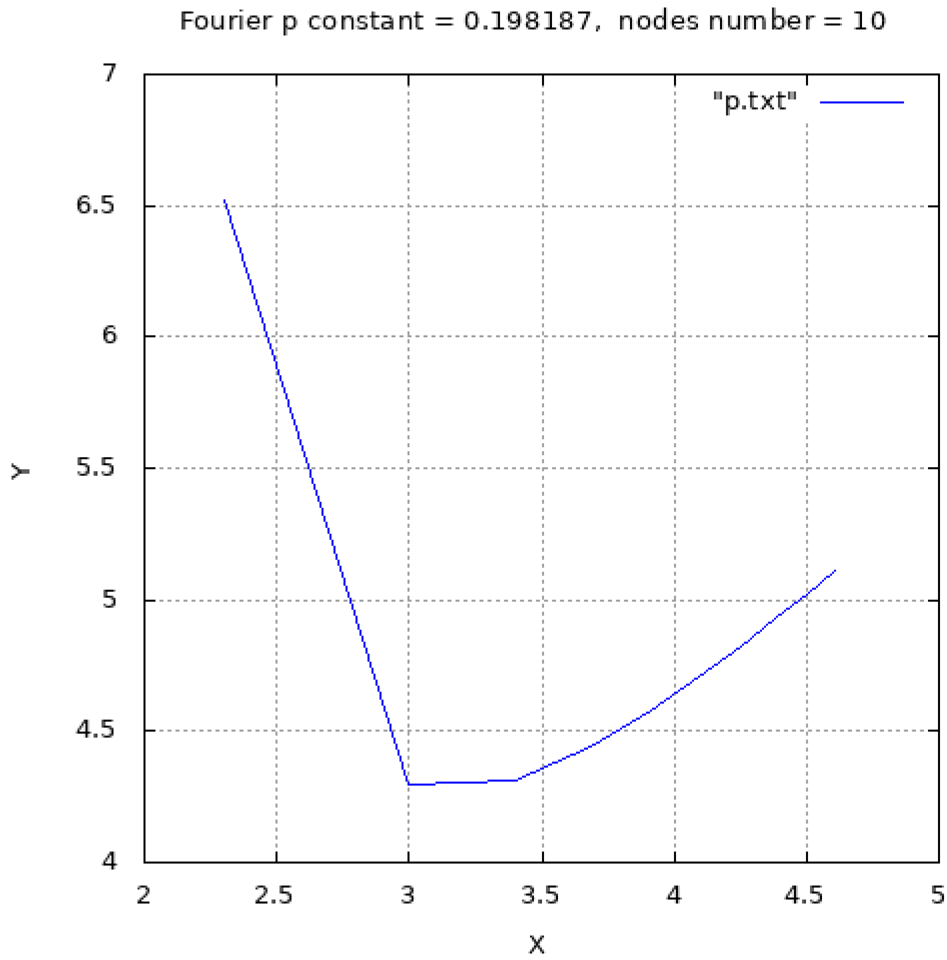


Fourier, nodes number = 3
"func.txt"
"fourier.txt"
Y
X
1.5
1
0.5
0
-0.5
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

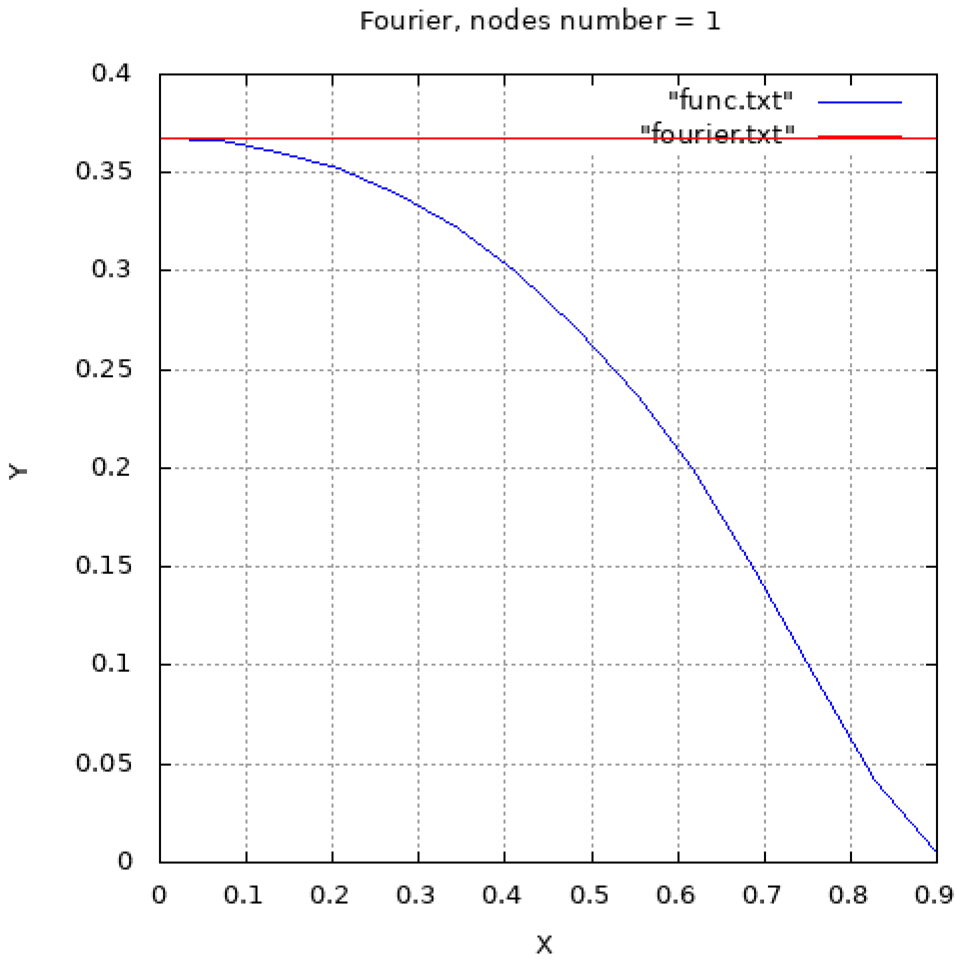

Fourier, nodes number = 10
"func.txt"
"fourier.txt"
Y
X
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Fourier p constant = 0.453175, nodes number = 3
"p.txt"
Y
X
4
3.8
3.6
3.4
3.2
3
2.8
1
1.5
2
2.5
3
3.5

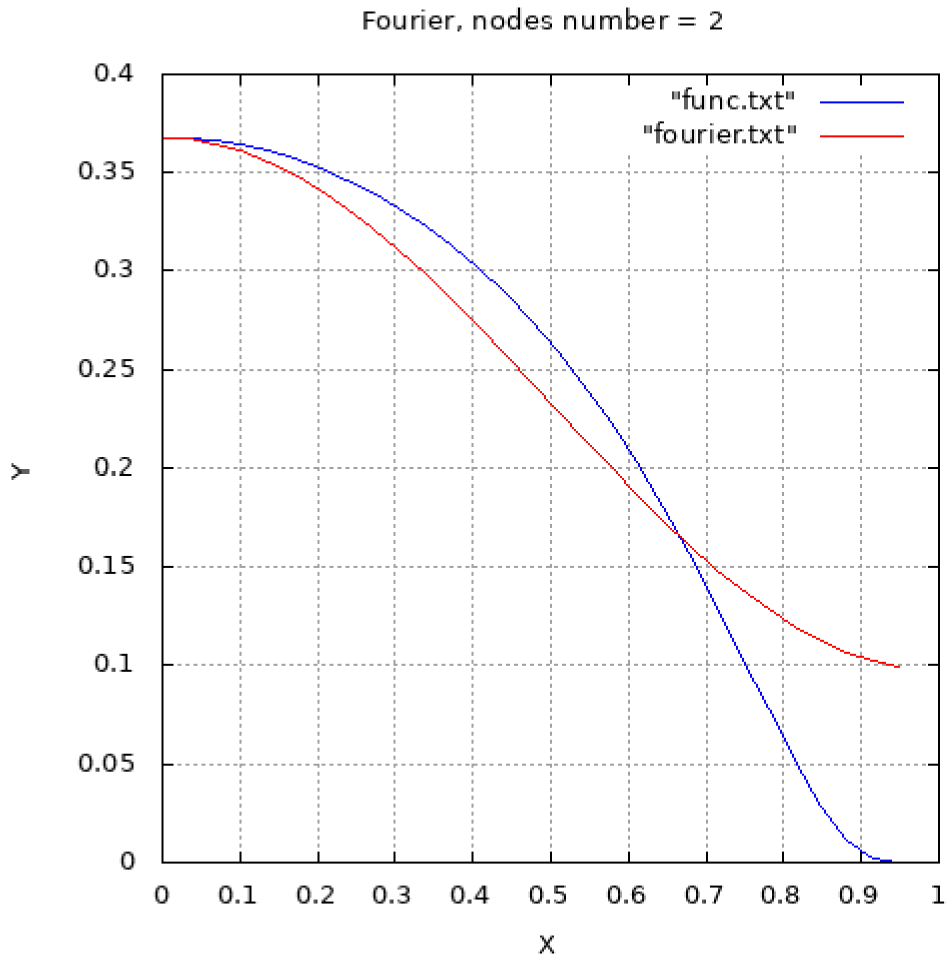


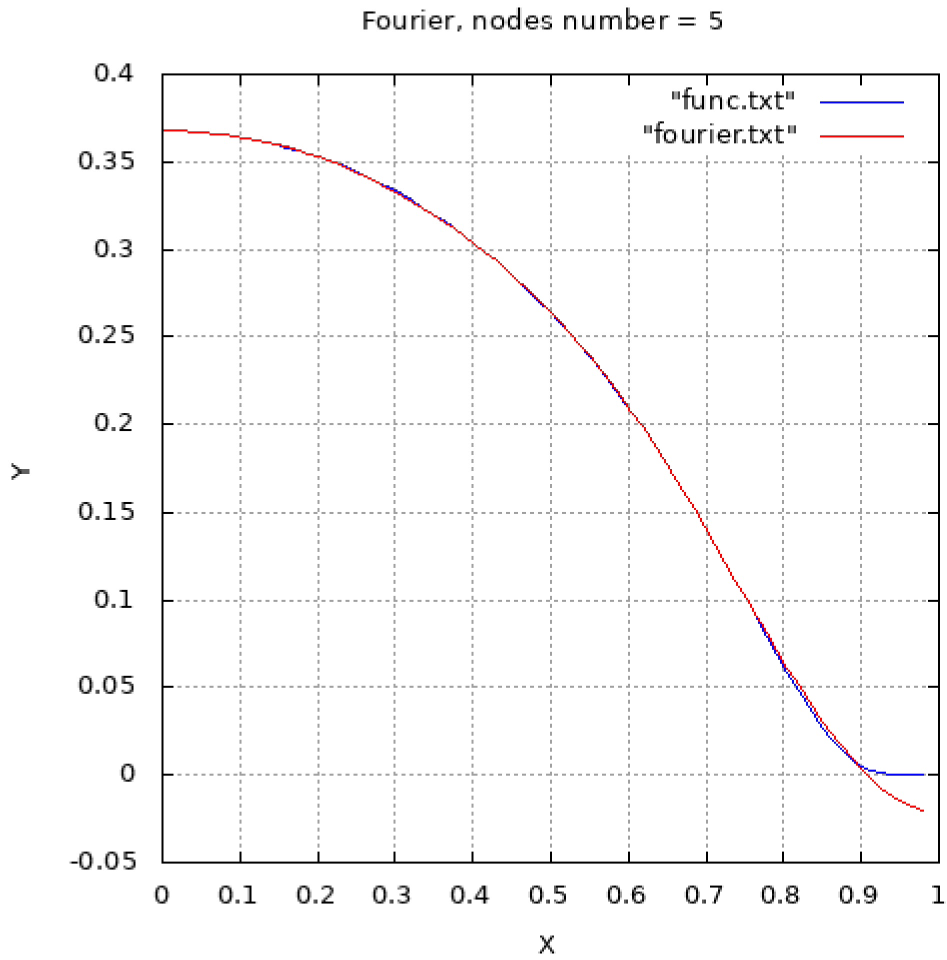
Fourier p constant = 0.198187, nodes number = 10
"p.txt"
7
6.5
6
5.5
5
4.5
4
Y
2
2.5
3
3.5
4
4.5
5
X

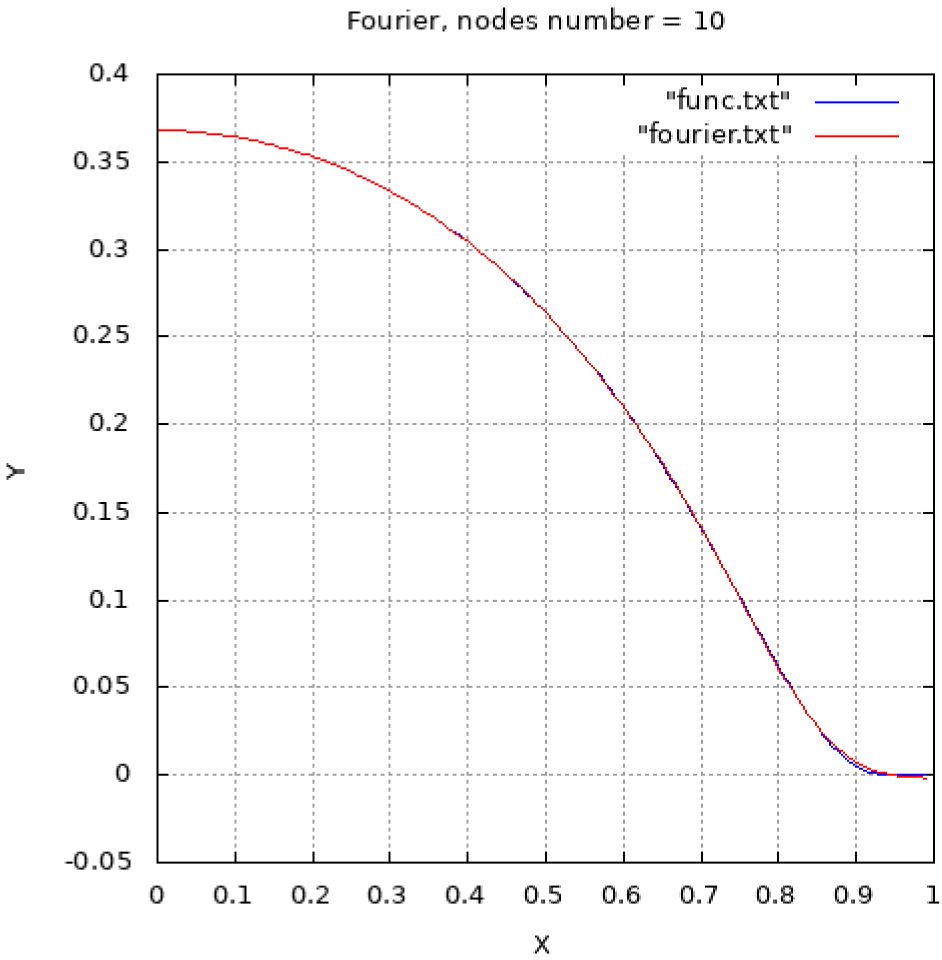


Fourier, nodes number = 1
"func.txt"
"fourier.txt"
Y
X
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

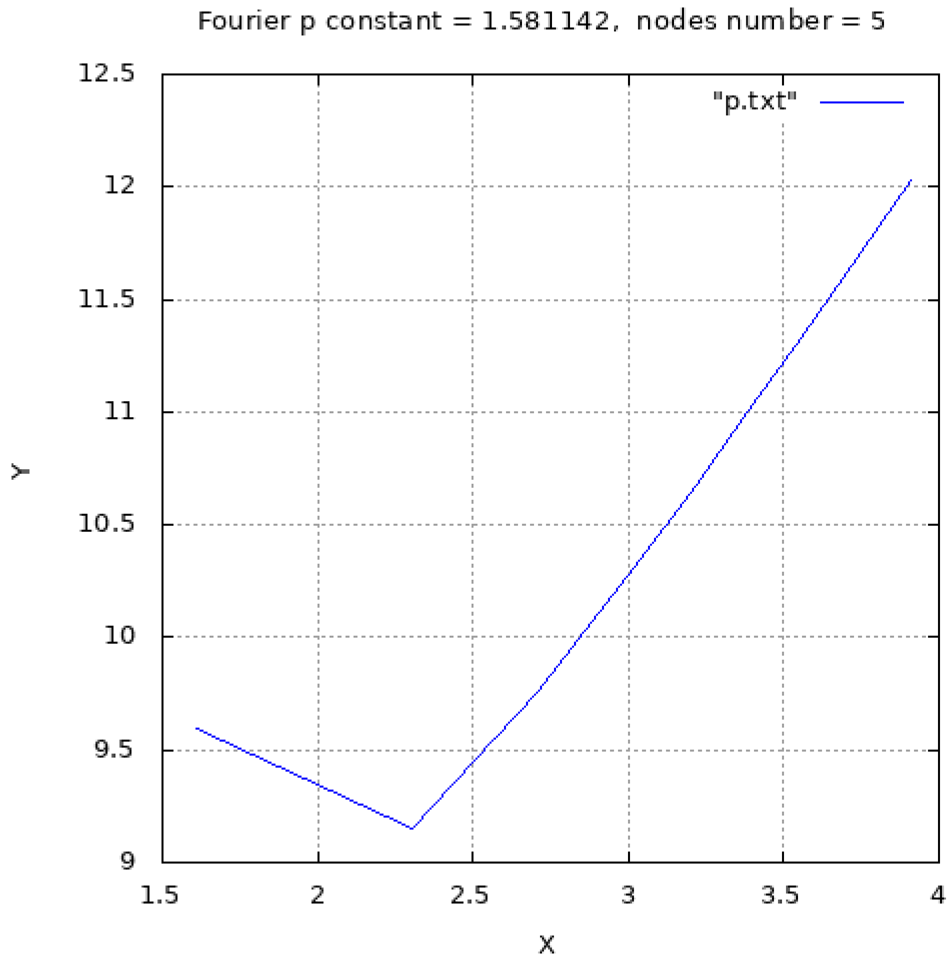
3) $e^{\frac{1}{x^2-1}}$

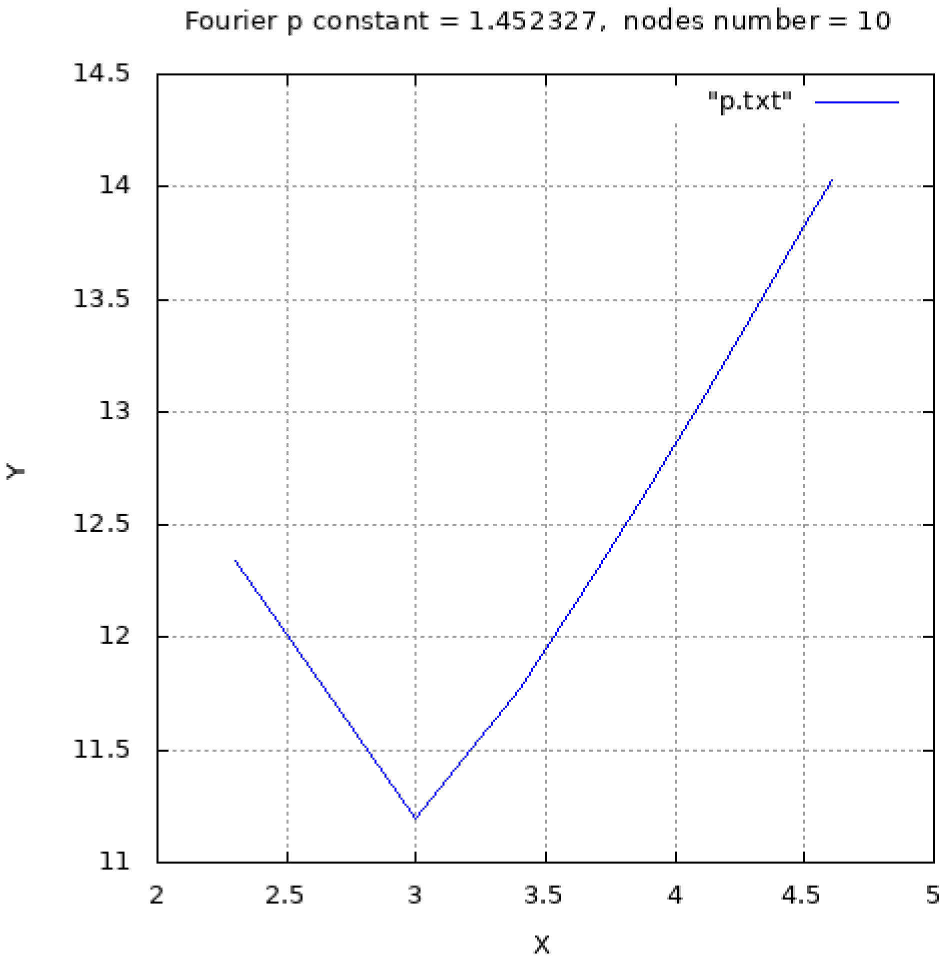

Fourier, nodes number = 2
"func.txt"
"fourier.txt"
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
X


Fourier, nodes number = 5
"func.txt"
"fourier.txt"
Y
X
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

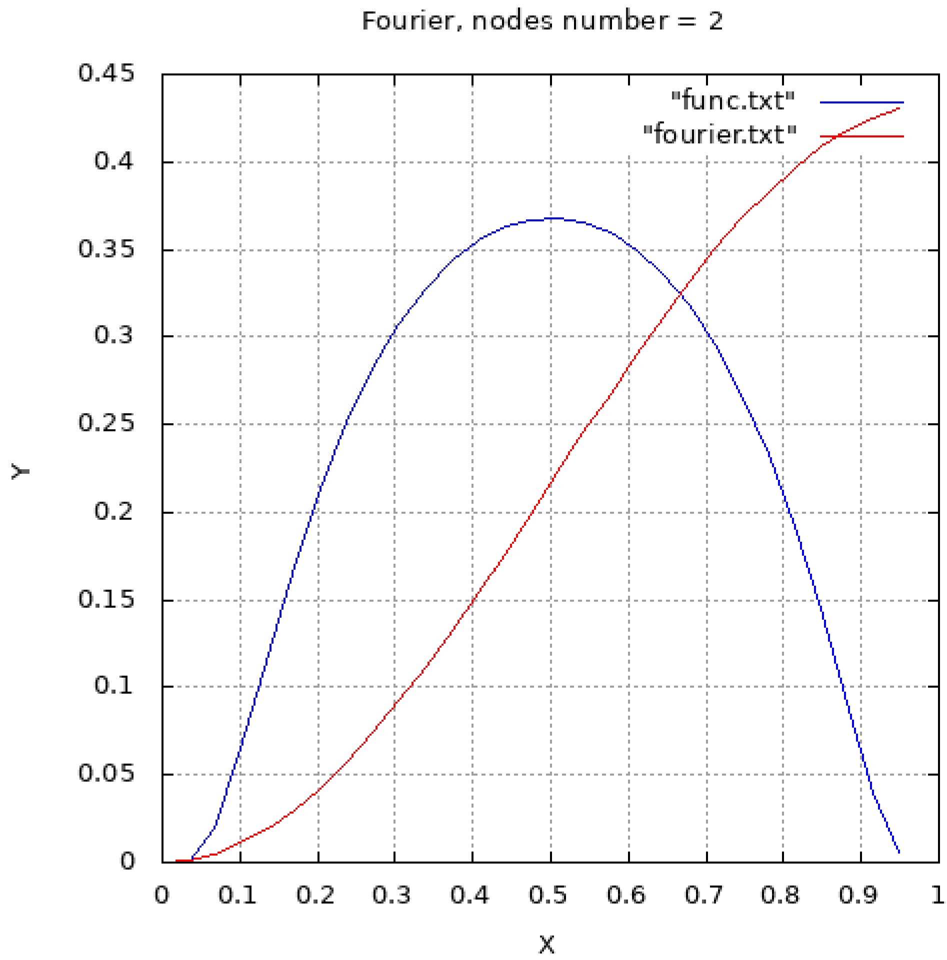


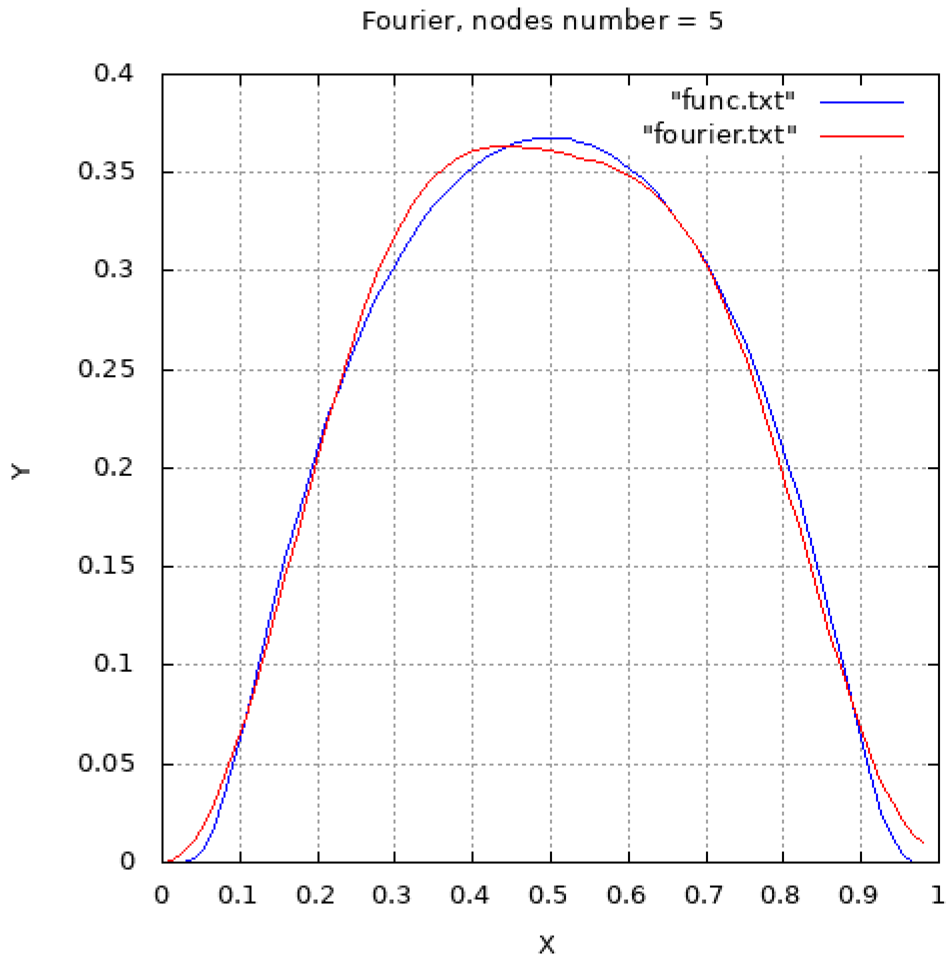
Fourier, nodes number = 10
"func.txt"
"fourier.txt"
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Y
X

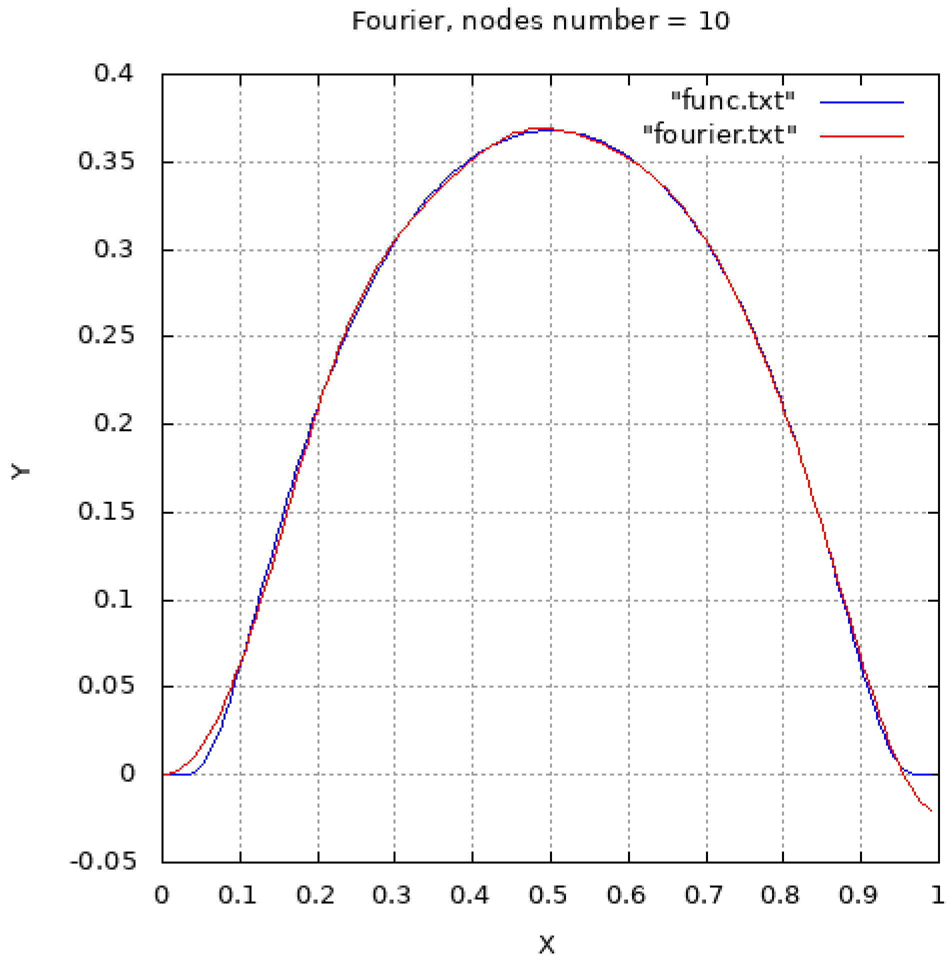

Fourier p constant = 1.581142, nodes number = 5
"p.txt"
12.5
12
11.5
11
10.5
10
9.5
9
1.5
2
2.5
3
3.5
4
X
Y

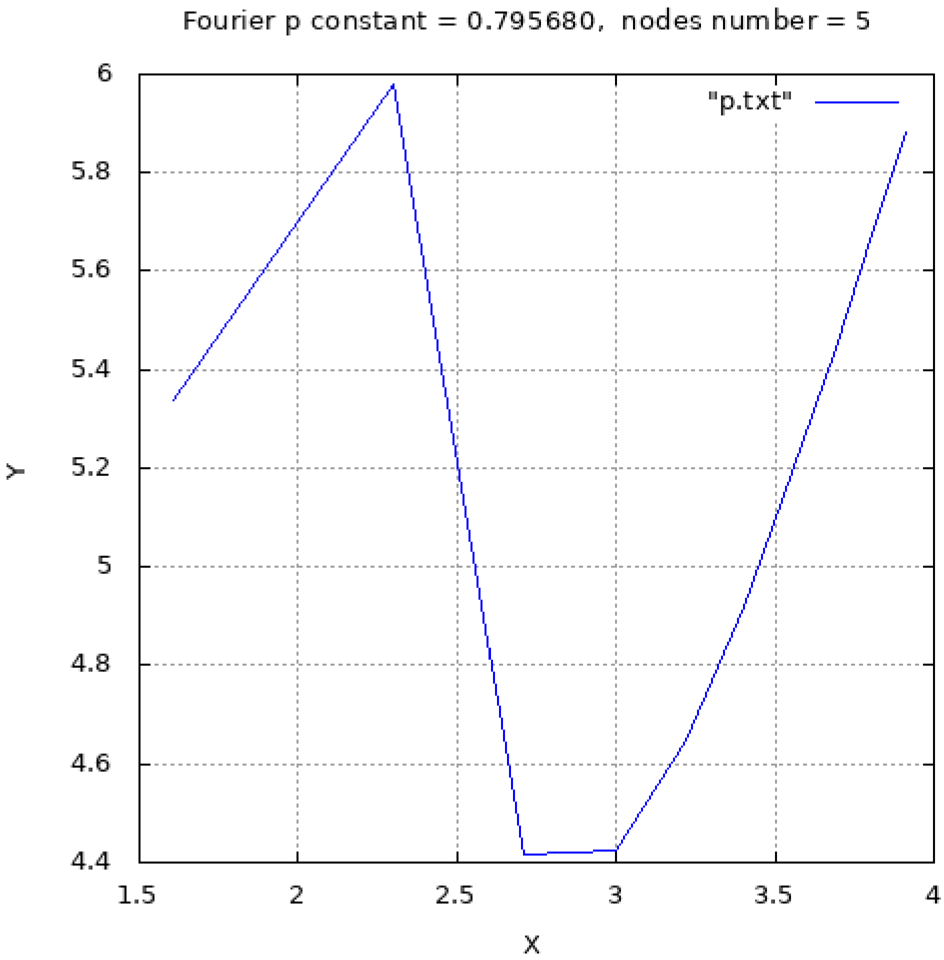


4) $e^{\frac{1}{\left(2x-1\right)^2-1}}$

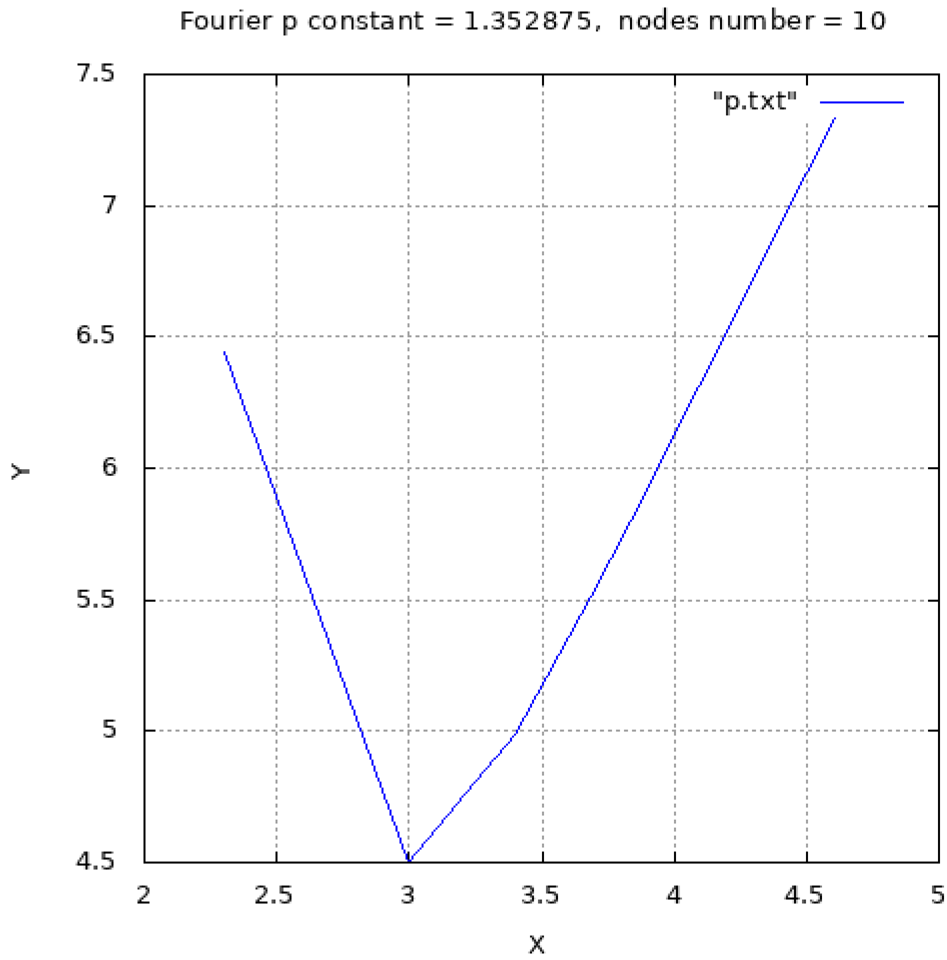

Fourier, nodes number = 2
"func.txt"
"fourier.txt"
Y
X
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Fourier, nodes number = 5
"func.txt"
"fourier.txt"
Y
X
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Fourier, nodes number = 10
"func.txt"
"fourier.txt"
Y
X
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Fourier p constant = 0.795680, nodes number = 5
"p.txt"
6
5.8
5.6
5.4
5.2
5
4.8
4.6
4.4
Y
1.5
2
2.5
3
3.5
4
X


Fourier p constant = 1.352875, nodes number = 10
"p.txt"
7.5
7
6.5
6
5.5
5
4.5
Y
2
2.5
3
3.5
4
4.5
5
X