Д/З Аппроксимация на решении
8.7; 8.8; 8.2; 8.3; 8.4; 8.5

Найти порядок аппрокс.

① ВМ;

$$\begin{cases} \dfrac{y_{k+1} - 2y_k + y_{k-1}}{h^2} = -\lambda y_k \\ 2\dfrac{y_1 - y_0}{h^2} = -\lambda y_0 \\ y_N = y_{N-1} \end{cases} \qquad h = \frac{1}{N - 1/2}$$

$$y_{k+1} - 2y_k \underbrace{\left(1 - \frac{\lambda h^2}{2}\right)}_{p} + y_{k-1} = 0$$

$$\Rightarrow y_{k+1} - 2p y_k + y_{k-1} = 0.$$

$y_k = \mu^k$, $\mu^{k+1} - 2p\mu^k + \mu^{k-1} = 0$

$$\mu^{k-1}(\mu^2 - 2p\mu + 1) = 0$$

$$\mu_{1,2} = p \pm \sqrt{p^2 - 1}$$

I случай $\mu_1 \neq \mu_2$

$$\Rightarrow y_k = c_1 \mu_1^k + c_2 \mu_2^k$$

$$y_1 - y_0 = -\frac{\lambda h^2}{2} y_0$$

$$y_1 - y_0\left(1 - \frac{\lambda h^2}{2}\right) = 0$$

$$\begin{cases} y_1 - p y_0 = 0. \\ y_N = y_{N-1} \end{cases} \Rightarrow \begin{cases} c_1\mu_1 + c_2\mu_2 = p(c_1 + c_2) \\ c_1\mu_1^N + c_2\mu_2^N = c_1\mu_1^{N-1} + c_2\mu_2^{N-1} \end{cases}$$

$\mu_1\mu_2 = 1$, $\mu_1 + \mu_2 = 2p$

$$\Rightarrow \begin{cases} \left(\frac{c_1}{c_2}\right)\mu_1 + \mu_2 = p\left(\frac{c_1}{c_2} + 1\right) \\ \left(\frac{c_1}{c_2}\right)\mu_1^N + \mu_2^N = \frac{c_1}{c_2}\mu_1^{N-1} + \mu_2^{N-1} \end{cases}$$

$$\Rightarrow k := \frac{c_1}{c_2} \Rightarrow \begin{cases} k\mu_1 + \mu_2 = p(k+1) \\ k\mu_1^N + \mu_2^N = k\mu_1^{N-1} + \mu_2^{N-1} \end{cases}$$

$$\Rightarrow k\mu_1^2 - p(k+1)\mu_1 + 1 = 0$$

$$k\mu_1^{2N} + 1 = k\mu_1^{2N-1} + \mu_1$$

$$k\mu_1^{2N-1}(\mu_1 - 1) = \mu_1 - 1$$

$$k\mu_1^{2N-1} = 1$$

$$D = p^2(k+1)^2 - 4k;$$

$$\mu_{1,2} = \frac{p(k+1) \pm \sqrt{p^2(k+1)^2 - 4k}}{2k}$$

$$c_1\mu_1 + c_2\mu_2 - p(c_1 + c_2) = 0$$

$$p = \frac{\mu_1 + \mu_2}{2}$$

$$\Rightarrow c_1\mu_1 + c_2\mu_2 - \frac{\mu_1 + \mu_2}{2}(c_1 + c_2) = 0$$

$$\frac{1}{2}c_1\mu_1 + \frac{1}{2}c_2\mu_2 - \frac{1}{2}c_1\mu_2 - \frac{1}{2}c_2\mu_1 = 0$$

$$\frac{1}{2}c_1(\mu_1 - \mu_2) + \frac{1}{2}c_2(\mu_2 - \mu_1) = 0$$

$$\Rightarrow c_1 = c_2$$

$$\Rightarrow \mu_1^N + \mu_2^N = \mu_1^{N-1} + \mu_2^{N-1}$$

$$\mu_1^{2N} + 1 = \mu_1^{2N-1} + \mu_1$$

$$\mu_1^{2N-1}(\mu_1 - 1) - (\mu_1 - 1) = 0$$

$$(\mu_1 - 1)(\mu_1^{2N-1} - 1) = 0 \qquad \mu_1\mu_2 = 1, \quad \mu_1 + \mu_2$$

$$\Rightarrow \mu_1^{2N-1} = 1$$

$$\Rightarrow \mu_1 = e^{i\frac{2\pi k}{2N-1}}, \quad k = 1, \dots, N-1$$

$$\Rightarrow y_k = c_1\left(e^{i\frac{2\pi k}{2N-1}} + e^{-i\frac{2\pi k}{2N-1}}\right) =$$

$$= 2c_1 \cos\frac{2\pi k}{2N-1}, \quad k = 1, \dots, N-1$$

$$p = \frac{\mu_1 + \mu_2}{2} = \cos\frac{\pi k}{N - 1/2} = 1 - \frac{\lambda h^2}{2}$$

$$\frac{\lambda h^2}{2} = 1 - \cos\frac{\pi k}{N - 1/2}$$

$$\Rightarrow \cancel{\lambda h^2 = 4 \sin^2}$$

$$\lambda = \frac{2}{h^2}\left(1 - \cos\frac{\pi k}{N - 1/2}\right)$$

у однородной

$$\Rightarrow \quad \lambda = 2(N - 1/2)^2 \left(1 - \cos\frac{\pi k}{N - 1/2}\right)$$

А теперь рассмотрим:

$$-\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + p y_k = \cancel{f_k}\ \lambda y_k$$

$$\Rightarrow \lambda = \boxed{p + 2(N - 1/2)^2 \left(1 - \cos\frac{\pi k}{N - 1/2}\right)}$$

$$y_k = \cos\frac{\pi k k}{N - 1/2}, \quad k = 1, \ldots, N-1$$