Report on

**A boundary integral equation approach to computing eigenvalues of the Stokes operator**
by Travis Askham and Manas Rachh

The paper is concerned with the numerical solution of the eigenvalue problem for the Stokes operator in 2D by using a boundary integral equation method. The proposed approach is based on the same ideas and the same numerical procedures as the presented approach in [54] for the eigenvalue problem of the Laplacian. As basis for the numerical approximation of the Stokes eigenvalue problem two boundary integral equations of the second kind, a standard double layer potential ansatz in the case of simple connected domains and a combined potential ansatz for the case of multiple connected domains, are used. The main contribution of the paper consists in providing the theoretical results (mainly results on the solvability of interior and exterior boundary value problems for the corresponding source problem of the eigenvalue problem) which justify the numerical approximation of the proposed boundary integral formulations of Stokes eigenvalue problem by the procedures presented in [54] (Nyström discretization, root-finding algorithm for the determinants of the discretized boundary integral equations). For simply connected domains which have an analytical boundary the convergence of simple eigenvalues of the discretization of the double layer potential formulation follows from the results in [54]. For multiple connected domains (with analytical boundary and simple eigenvalues) the convergence of the eigenvalues of the discretization of the combined potential ansatz may be expected with the same reasoning as in [54]. For the computations of the eigenvalues of the discretized problems a root-finding algorithm as in [54] is suggested and applied. This procedure may cause spurious eigenvalues which may be identified by given post-processing routines. Three numerical examples are presented which demonstrates the behavior of the method.

Although there are no new ideas, concepts or techniques presented in the paper I recommend it for a publication since the paper well illustrates the pros and cons of the chosen approach (in my opinion better as it is the case in [54] for the Laplacian). Moreover, for sufficiently smooth problems (in 2d) this approach seems to be the method of choice.

The following remarks should be considered in the revised version:

1. In different parts of the paper the connection between the eigenvalue problem of the Stokes operator and the biharmonic eigenvalue problem

is addressed and utilized. Therefore for me it is strange that just in the last section of the paper ("Conclusions") this connection is elaborated. It would be better that this elaboration is done somewhere at the beginning of the paper and not in the end.

2. In the abstract the Stokes operator is denoted as fourth-order operator. This is unusual and in my opinion not the case (even if there is an equivalence to a forth-order operator).

3. I would appreciate it if the following issue would be addressed in the paper: under which weaker conditions (not analytic boundary, non-simple eigenvalues) convergence of the discretization of the eigenvalues can be proven. (Are there results concerning the convergence of the eigenfunctions?)

4. In the paper it is not explicitly mentioned that both presented boundary integral equations may have non-trivial null-spaces for certain $k$ with $\mathrm{Im}(k) < 0$. These $k$s correspond to the eigenvalues of some related eigenvalue problems in the exterior domain. It is possible that these $k$s are also be detected by the root search method and that these eigenvalues may be some of those "spurious eigenvalues too far from real-valued" which are blamed to be caused by "noisy numerical determinant evaluation" but are actually caused by the used boundary integral formulation.

5. In general it would be of interest which kind of spurious eigenvalues may occur from a"noisy numerical determinant evaluation". In the paper only two kinds are specified, namely spurious eigenvalues with "large" imaginary part (which may result also from the boundary integral formulation itself as pointed out above) and spurious "double roots" on the real line. Isn't it possible that for example spurious single roots occur on the real line?

6. For me it is not clear why with the suggested procedure spurious "double roots' may be detected. What is actually the argument that this procedure detects spurious double roots?

7. In the caption of figure 5 the given interval for $k$ does not match with the plots. Moreover, in each plot of figure 5 two functions are given but in the caption only one function is described.

8. Subsection 1.1 ("Relation to other work") may lead to the false impression that boundary integral equations of the second kind combined

with the proposed root finding method is the only profound and reasonable boundary integral equation approach for the Stokes eigenvalue problem. The authors seem not be aware of works which deal with boundary integral equation methods based on integral equations of the first kind (for several types of eigenvalue problems(Laplacian, Maxwell, fluid-solid interaction, plasmonics, ....)) which are well analyzed and for which a rigorous and comprehensive convergence theory is established. Such kind of an approach could be also applied to the Stokes eigenvalue problem. Moreover, in these works instead of the root search for the determinant the contour integral method for nonlinear eigenvalue problem is used for which multiple and spurious eigenvalues seem not to be a problematic issue. In some way these alternatives should be briefly mentioned in the paper.

9. I cannot follow the ”formal“ derivation of the fundamental solution from equation (8) to (10). Is it possible to describe the derivation in more detail? (It would be also good to mention the reference [7] between equation (8) and (10)).

10. Lemma 4 only seems to hold for $k$ with $\mathrm{Im}(k) \geq 0$, see the asymptotic expansion of the Hankel function.

11. In the proof of Theorem 3 and 4 only the uniqueness of the solution is shown but not the existence. (The existence would follow for example from the Fredholm theory.)

12. Theorem 8, 9 and 10 only seem to be valid for $k$ with $\mathrm{Im}(k) \geq 0$.

Typos:

1. p.8, l.22: Theorem -> theorem.

2. p.14, l.39: $Re(k)$-> $\mathrm{Re}(k)$.

3. p.14, l.39: $Re(\eta)$, $Re(k)$-> $\mathrm{Re}(\eta)$, $\mathrm{Re}(k)$.