GENERALIZED RYBICKI PRESS SCHEME

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Abstract. This article introduces the generalized Rybicki Press scheme for covariance matrices of the form

$$K(t_i, t_j) = \sum_{k=1}^{m} \alpha_k \exp(-r_k|t_i - t_j|)$$

where the points x, y lie on an interval. The computational cost scales as $\mathcal{O}(m^2 n)$ for inverting and computing the determinant of this linear system. The new algorithm relies on embedding the covariance matrix into an extended banded sparse matrix. The result holds true in exact arithmetic.

Key words.

AMS subject classifications. 15A23, 15A15, 15A09

1. Reinterpretation of Rybicki Press scheme in terms of sparse embedding. We will first reinterpret the Rybicki Press scheme in terms of the extended sparse matrix algebra. We will work with a specific example of a 4×4 matrix and the kernel is $Kr) = \exp(r)$ at the data points $t_1 < t_2 < t_3 < t_4$.

$$\begin{bmatrix} 1 & \exp(t_1 - t_2) & \exp(t_1 - t_3) & \exp(t_1 - t_4) \\ \exp(t_1 - t_2) & 1 & \exp(t_2 - t_3) & \exp(t_2 - t_4) \\ \exp(t_1 - t_3) & \exp(t_2 - t_3) & 1 & \exp(t_3 - t_4) \\ \exp(t_1 - t_4) & \exp(t_2 - t_4) & \exp(t_3 - t_4) & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Introduce the following additional variables:

$$r_4 = \exp(-t_4)x_4 \tag{1.1}$$
$$r_3 = \exp(-t_3)x_3 + r_4 \tag{1.2}$$
$$r_2 = \exp(-t_2)x_2 + r_3 \tag{1.3}$$

$$l_1 = \exp(t_1)x_1 \tag{1.4}$$
$$l_2 = \exp(t_2)x_2 + l_1 \tag{1.5}$$
$$l_3 = \exp(t_3)x_3 + l_2 \tag{1.6}$$

Now using all these variables, rewriting the equations we get the 10 equations for these 10 unknowns. Suitable ordering of the equations and unknowns have to be done to exhibit the banded property of the extended system.

$$\begin{bmatrix} 1 & \exp(t_1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \exp(t_1) & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \exp(-t_2) & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \exp(-t_2) & 1 & \exp(t_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \exp(t_2) & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \exp(-t_3) & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \exp(-t_3) & 1 & \exp(t_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \exp(t_3) & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \exp(-t_4) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \exp(-t_4) & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ r_2 \\ l_1 \\ x_2 \\ r_3 \\ l_2 \\ x_3 \\ r_4 \\ l_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ 0 \\ b_2 \\ 0 \\ 0 \\ b_3 \\ 0 \\ 0 \\ b_4 \end{bmatrix} \tag{1.7}$$

which is written in the form $A_{ex}x_{ex} = b_{ex}$. Note that the solution vector x to the original set of equations is a subset of the vector x_{ex}.

Equation 1.7 is now an extended sparse matrix, whose bandwidth is 2. This extends in general for a $N \times N$ (again with a suitable ordering of unknowns and equations). The bandwidth is always 2, since our kernel is just $Kr) = \exp(r)$. The other *nice* property is that $\boxed{\det(A_{ex}) = \det(A)}$. Also, note that the extended matrix A_{ex} is also symmetric.

2. Generalized Rybicki Press. The same idea carries over the generalized Rybicki Press scheme, i.e., if $K(r) = \sum_{k=1}^{m} \alpha_k \exp(-\beta_k r)$, then if we introduce new variables and form the extended sparse system, we observe that the extended sparse system is again a banded matrix.