Predictive Distribution of Light Curves as Gaussian Processes
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Minimum Variance Estimate. Ignoring any large time trend in the light curve, we can write any realization as

$$\mathbf{y} = \mathbf{s} + \mathbf{n} \tag{1}$$

and define

$$\mathbf{S} \equiv \left\langle \mathbf{s}\mathbf{s}^T \right\rangle, \ \mathbf{N} \equiv \left\langle \mathbf{n}\mathbf{n}^T \right\rangle, \ \mathbf{S}_* \equiv \langle (s_*)\mathbf{s} \rangle \tag{2}$$

We then have the minimum variance estimate (MVE) $\widehat{s_*}$ for s_* as

$$\widehat{s_*} = \mathbf{S}_*^T [\mathbf{S} + \mathbf{N}]^{-1} \mathbf{y} \tag{3}$$

with the variance of s_* about $\widehat{s_*}$ as

$$\left\langle (s_* - \widehat{s_*})^2 \right\rangle = \left\langle s_*^2 \right\rangle - \mathbf{S}_*^T [\mathbf{S} + \mathbf{N}]^{-1} \mathbf{S}_* \tag{4}$$

Unconstrained Realizations of the Underlying Process. Given the symmetric, postive-definite covariance matrix $\mathbf{C} \equiv \mathbf{S} + \mathbf{N}$, generating a random realization is straightforward.

- Cholesky decompose $\mathbf{C} = MM^T$, where M is the lower triangular matrix.
- Generate $\mathbf{r} \sim \mathcal{N}(\mathbf{0}, I)$
- Compute $\mathbf{x} = M\mathbf{r}$ so that $\left\langle \mathbf{x}\mathbf{x}^T \right\rangle = \left\langle M\mathbf{r}(M\mathbf{r})^T \right\rangle = \left\langle M\mathbf{r}\mathbf{r}^T M^T \right\rangle = \mathbf{C}$

Predictive Distribution from the Observed Light Curve. Given $\mathbf{C}$ and the observed light curve vector $\mathbf{y}$, we can compute Equation (3) and (4) by

- Cholesky decompose $\mathbf{C} = MM^T$, where M is the lower.
- Solve $MA = \mathbf{y}$ for A and then $M^T\alpha = A$ for α, so that $\mathbf{C}\alpha = (MM^T)\alpha = \mathbf{y}$, thus $\alpha = \mathbf{C}^{-1}\mathbf{y}$.
- $\widehat{s_*} = \mathbf{S}_*^T \alpha$ is the MVE.
- Solve $M\mathbf{v} = \mathbf{S}_*$ for $\mathbf{v}$
- $\langle (s_* - \widehat{s_*})^2 \rangle = \langle s_*^2 \rangle - \mathbf{v}^T\mathbf{v}$, given $\mathbf{S}_*^T \mathbf{C}^{-1} \mathbf{S}_* = \mathbf{S}_*^T (M^T)^{-1} M^{-1} \mathbf{S}_* = \mathbf{v}^T\mathbf{v}$

Constrained Realizations of the Underlying Process by Observed Light Curves. A typical realization contrained by the data at unmeasured epoch can be produced as s_* by

$$s_* = u_* + \widehat{s_*} \tag{5}$$

where u_* is a Gaussian process with zero mean and correlation matrix $\mathbf{Q}$

$$\mathbf{Q} = [\mathbf{S}^{-1} + \mathbf{N}^{-1}]^1 = \mathbf{S}[\mathbf{S} + \mathbf{N}]^{-1}\mathbf{N} = \mathbf{N}[\mathbf{S} + \mathbf{N}]^{-1}\mathbf{S} \tag{6}$$

u_* can be generated using the same way as in the unconstrained prediction case using the Cholesky decomposition of $\mathbf{Q}$.

References

[1] Written in Oct 20, 2011