QAD

Fabian Oberreiter, Konrad Medicus, Tobias Hilgart

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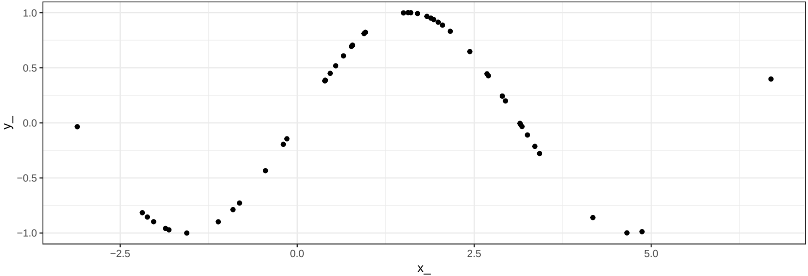
Motivation - Asymetric Dependence

Motivation

Bivariate sample:

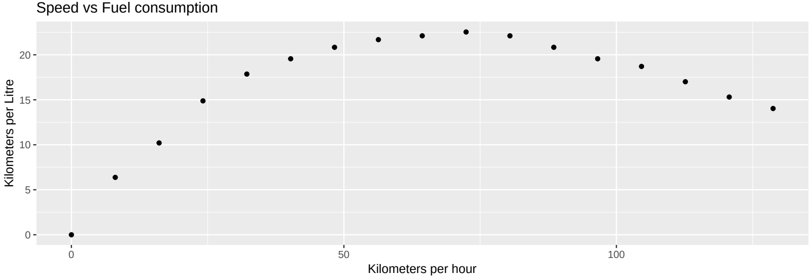


$X \sim \mathcal{N}(1.42, 2),\ Y = \sin(X).$

Would it be easier to predict Y given X, or X given Y?

Motivation

Real life example: Average speed vs. Fuel consumption



Motivation

What is dependence between random variables X and Y?

X ... result of a coin toss,
Y ... result of tossing the same coin a second time.
-> we do not gain information about Y if we know X and vice versa.

X ... result of drawing a card from a deck containing 2 cards,
Y ... result of drawing the remaining card.
-> we know everything about Y if we know X and vice versa.

Motivation

Why could this be a problem?

How would we usually quantify dependence?

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How would we usually quantify dependence?

Correlation coefficient:

```r
cor(x_,y_)
```

```
## [1] 0.2870855
```

```r
cor(y_,x_)
```

```
## [1] 0.2870855
```

Motivation

By definition of correlation (or covariance):

$corr(X, Y) = corr(Y, X)$
The same holds true for spearman or kendall correlation.

Correlation can NOT be used as a meassure of asymmetric dependence.

Motivation

What properties shouldan asymetric dependency meassure q have?

- $q(X, Y) \in [0, 1]$
- $q(X, Y) = 0$ iff X and Y are independent (independence)
- $q(X, Y) = 1$ iff Y is a function of X (complete dependence)
- It may be, that $q(X, Y) \neq q(Y, X)$.
- Scale changes should not affect the outcome.

qad-measure - Construction:

- Start with a bivariate sample
- Use the pseudo-observations to construct the empirical copula E_n.
- Aggregate E_n to the empirical checkerboard copula $\hat{C}_n$.
- Estimate $q(A) = 3D_1(A, \Pi)$ via $q(\hat{C}_n) = 3D_1(\hat{C}_n, \Pi)$.
- For sufficiently large n, we have $q(\hat{C}_n) \approx q(A)$.

(Empirical) Copula

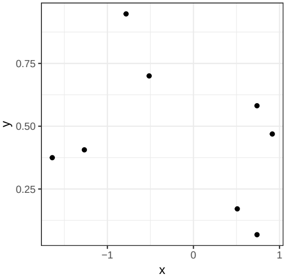
Idea

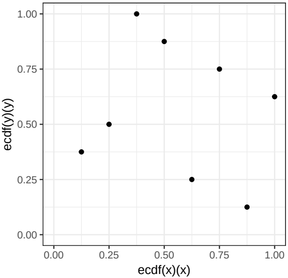
We have something that links univariate and bivariate distributions and contains all the information about mutual dependency: Copula (lat. link").

It's existance is guranteed by Sklar's Theorem, therefore it makes sense to and use it for a dependency measure.

How do we get this copula from a given bivariate sample?

Empirical copula

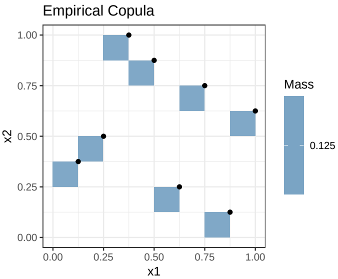




similarly to the empirical distribution:

We start with a sample $(x_1, y_1), \ldots, (x_n, y_n)$ and the pseudo - observations (normalized ranks).

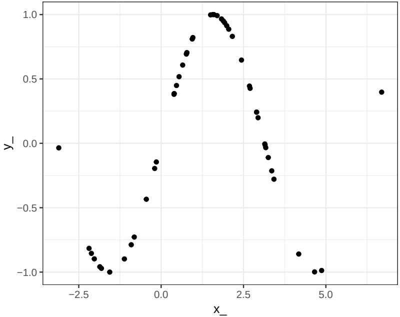
Empirical copula

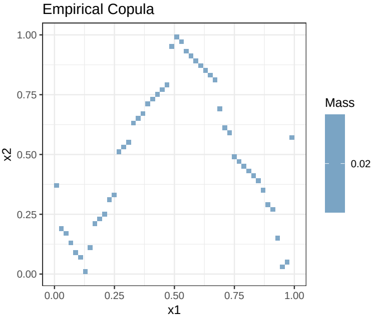


And then proceed to construct an empirical copula as shown above.

Empirical copula of our example

Let's compute the empirical copula for our example:





Limitations

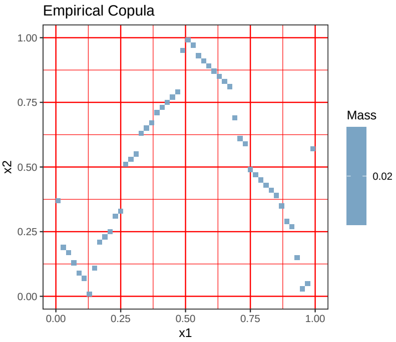
We would wish that the empirical copula E_n was a good estimate for the true underlying copula A.

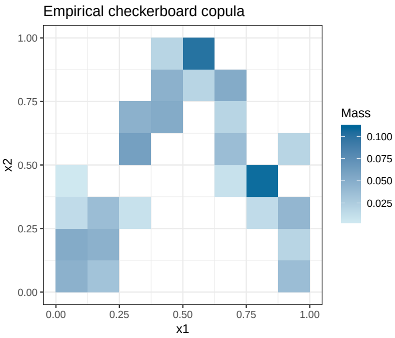
Using the aforementioned metric D_1, this is not always the case.

We need to use a different estimator for A.

Empirical checkerboard copula

We aggregate the empirical copula into what we call empirical checkerboard copula.





emp_c_copula()

The function *emp_c_copula* computes the mass-distribution of the empirical (checkerboard) copula, given a bivariate sample.

Arguments:

- X: a dataframe containing a bivariate sample
- smoothing: a logical indication whether the checkerboard copula is to be calculated
- resolution: an integer indicating the resolution of the checkerboard aggregation (the number of breaks in the grid)

emp_c_copula()

The function *emp_c_copula* computes the mass-distribution of the empirical (checkerboard) copula, given a bivariate sample.

```
n = 50
x = rnorm(n,0,1)
y = runif(n,0,1)
df = data.frame(x,y)
emp_cop = emp_c_copula(df,smoothing = FALSE)
emp_check_cop = emp_c_copula(df,smoothing = TRUE,resolutio
```

plot_density()

The function *plot_density* plots the density/mass of the empirical (checkerboard) copula.

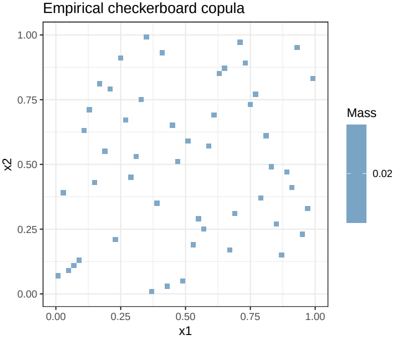
Arguments:

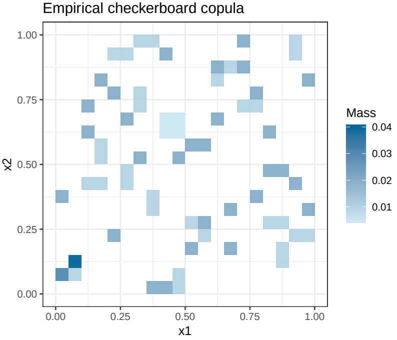
- mass_matrix: A squared matrix containing the mass distribution (output of emp_c_copula())
- density: A logical indication whether density or mass is plotted

plot_density()

The function plot_density allows us to visualize copulae:

```
plot_density(emp_cop, density=FALSE)
plot_density(emp_check_cop, density=FALSE)
```





Exercise 1

Calculate the empirical copula – not the checkerboard copula – while having the argument *smoothing=TRUE*. (Hint: Resolution parameter)

Optional: Visualize this by plotting the pseudo observations of your generated sample into the mass-plot (Compare: slide 13,14).

Explain the differences in the following plots:

```
n=10
df = data.frame(x = rnorm(n), y = runif(n))
emp_cop = emp_c_copula(df, smoothing = FALSE)
emp_check_cop = emp_c_copula(df, smoothing = TRUE, resoluti
plot_density(emp_cop, density = FALSE)
plot_density(emp_check_cop, density = FALSE)
```

The dependency measure

The dependency measure

Based on a metric D_1 for copulae, a dependency measure can be constructed in the following way:

$$q(A) := 3 \cdot D_1(A, \Pi) \;\in [0, 1]$$

Π denotes the product copula, which stems from two completely independet RVs.

In a way, we measure the "distance" to complete independece.

qad()

The function *qad* quantifies the asymmetric dependence of two RVs X and Y. It achieves this by calculating the empirical checkerboard copula to estimate the dependency measure q.

Arguments:

- X: a dataframe containing a bivariate sample in two columns.
- resolution: resolution of the emp. copula
- premutation: a logical indicating, whether a permutated p-value is computed.

qad()

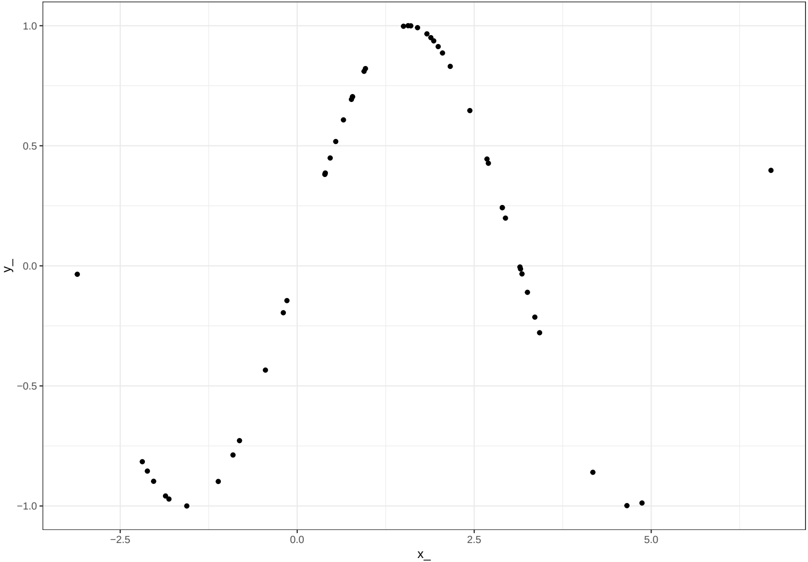
The function *qad* quantifies the asymmetric dependence of two RVs X and Y. It achieves this by calculating the empirical checkerboard copula to estimate the dependency measure q.

There are multiple ways to see the results of the qad calculation:

- summary(qad(X))
- coef(qad(X))
- qad(X)$results

qad Example 1

Back to our initial example:



qad Example 1

We take a look at the mutual dependencies.

```
mod = qad(X, print = FALSE)
coef(mod, select = c("q(x1,x2)","q(x2,x1)","mean.dependence
                     "asymmetry"))
```

```
##         q(x1,x2)         q(x2,x1) mean.dependence          as
##        0.7885979        0.4193818       0.6039898          0.
```

We see that X predicts Y better than Y predicts X.

qad Example 2

```r
n = 200
x2 = runif(n,-2,4)
y2 = exp(x2)
df = data.frame(x2,y2)
model = qad(df, print = FALSE, permutation = TRUE)
model$results
```

```
##                           coef p.values
## 1          q(x1,x2) 0.9608946        0
## 2          q(x2,x1) 0.9608946        0
## 3 mean.dependence 0.9608946        0
## 4 asymmetry       0.0000000        1
```

pairwise.qad

With the function *pairwise.qad*, we can apply the *qad* function on each pair of columns of the given dataframe. Other arguments are the same as for *qad*.

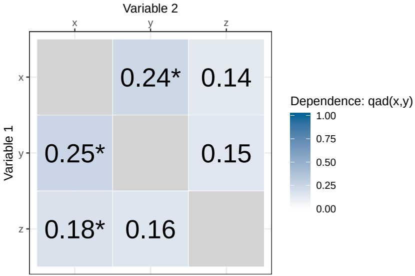
```
n = 1000
x = runif(n,0,1)
y = x^2+rnorm(n,0,1)
z = runif(n,0,1)
df = data.frame(x,y,z)

model = pairwise.qad(df, permutation = TRUE, nperm = 10,
                     DoParallel = TRUE)
```

```
## [1] "Process..."
## [1] "Process: 1 / 3"
## [1] "Process: 2 / 3"
```

heatmap.qad

We can plot the results as a heatmap:

```
heatmap.qad(model, select = "dependence", fontsize = 8,
            significance = TRUE)
```



heatmap.qad

Arguments:

- pw_qad: the output of the function *pairwise.qad()*
- select: dependence value to be plotted: “dependence”, “mean.dependence” or “asymmetry”
- significance: logical; if p-values were calculated, marks the significant dependency values with a star
- sign.level: manually set significance level (default: 0.05)

Exercise 2

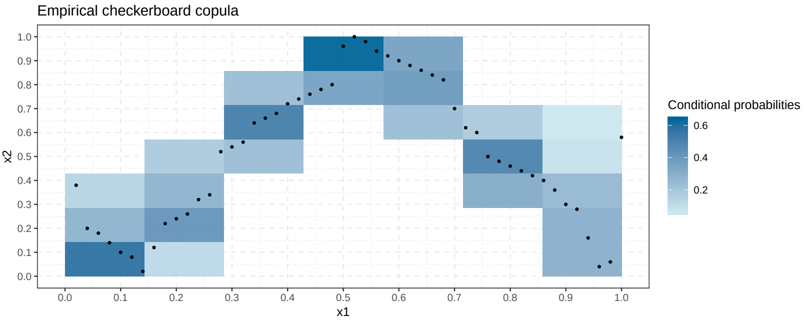
- Download the RTR-dataset

  (via load(url("http://www.trutschnig.net/RTR.RData")))

  and sample 1000 observations.
- Examine which columns could have interesting dependencies.
- This does not makes sense for all columns. (why?)
- Visualize the dependencies for the relevant columns. Argue, why you chose them.

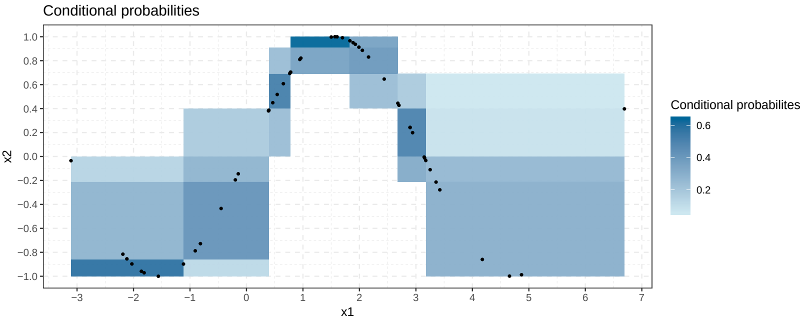
plot.qad

```
plot(mod, addSample = TRUE, copula = TRUE)
```



plot.qad

```
plot(mod, addSample = TRUE)
```



predict.qad

The function *predict.qad()* predicts the probabilities to end up in specific intervals given x or y values.

The prediciton can be computed in the copula or the retransformed data setting.

```
predict(mod, values = 1.42, conditioned = "x1", nr_interval
```

```
##      (-1,-0.79] (-0.79,-0.02] (-0.02,0.43] (0.43,0.82]
## 1.42          0             0            0       0.216
```

Exercise 3

- Create a dummy dataset: $n = 100$ observations of a uniformly distributed random variable and the corresponding square values.
- Create a new qad object and use the plot function to visualize the copula.
- Add the observations to the plot with the addSample parameter.
- Use the predict.qad method and sum up the probabilities for one strip (horizontal or vertical) for one value.

cci - Conditional Confidence Interval

We can also use the qad metric to test hypothesis:

cci provides a confidence interval for independence (for the qad measure).

We can compare the qad-value to the interval boundaries and check if the null hypothesis of independence holds or has to be rejected.

cci

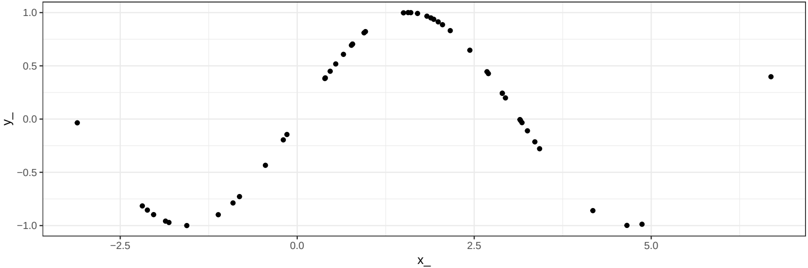
We calculate our boundaries

```
c = cci(n, alternative = "one.sided")
```

and check, whether the calculated dependence for our example lies between them.

```
if(coef(mod, select = 'q(x1,x2)') %in% c){
 print('Accept H0')
}else{
 print('Reject H0')
}
```

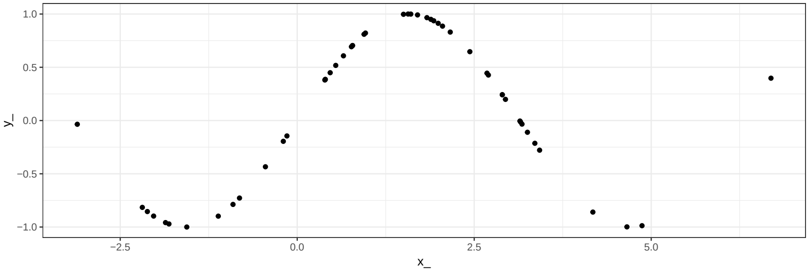
cci



And indeed, for our example, we reject the hypothesis of independence:

```
## [1] "q(x1,x2) not in [ 0 , 0.1813481327652 ]"
## [1] "Reject H0"
```

cci



Similarly for the other direction:

```
## [1] "q(x2,x1) not in [ 0 , 0.1813481327652 ]"
## [1] "Reject H0"
```

Final Exercise 4:

Take a look at the *attitude* dataset:

Find out the two sets with the highest (one sided) dependency. Provide respective plots.

Are those dependencies symmetric? Argue by providing p-values for asymmetry (hint: do not to calculate the p-values for all pairs, only the chosen two)

Reject or Accept the hypothesis of independence for your chosen pairs.