DM536/DM574: INTRODUCTION TO PROGRAMMING
Exercise list (Autumn 2022)

1 Operators and expressions

1. For each of the following expressions, write the order in which it is evaluated.

   (a) `a - b - c - d`
   (b) `a - b + c - d`
   (c) `a + b / c / d`
   (d) `a / b * c * d`
   (e) `a / b / c ** d`
   (f) `(a - (b - c)) - d`
   (g) `a % (b % c) * d * e`
   (h) `(a + b) * c + d * e`
   (i) `(a + b) * (c - d) % e`

2. Suppose that `i` and `j` are two numerical variables and that `b` is a logical value. Remove unnecessary parentheses from each of the following expressions.

   (a) `((3 * i) + 4) / 2`
   (b) `((3 * j) / (7 - i)) * (i + (-23 * j))`
   (c) `((((i + j) + 3) + j) * (((i - 4) / j) + -323))`
   (d) `(3 >= (j - 3)) == ((323 - (j * -7)) != (43))`
   (e) `((3 >= 5) == ((not b) or b))`
   (f) `(b or (not (b and (3 == (i * 2)))))`
   (g) `(not (not b) or (b and ((4 >= i+j) or (False))))`

   Could some of these expressions have been written in a simpler form?

3. For each of the following code snippets, find the value stored in each variable at the end of execution.

   (a)
   ```
   y = 4
   y = y + y
   ```
   (b)
   ```
   x = 5
   y = x
   x = x + y
   ```
   (c)
   ```
   y = 4
   z = 3
   x = y // z;
   ```
   (d)
   ```
   b = 3.1
   c = 0
   c = c + 2
   b = b * (c + 3)
   ```
   (e)
   ```
   j = 2
   i = 1
   j = 3 + i * 2
   i = j // 2 * i + 3
   i = i + 1
   ```

4. Suppose we need to work with the following data:

   - an age;
   - a weight;
   - the number of a lottery ticket;
   - a salary;
   - a person's gender (male or female);
   - a person's marital status (single, married, divorced, widowed);
   - a distance between stars, measured in light-years;
   - a distance on the Earth's surface, measured in meters.

   Propose names and types for variables to store these data.

5. *Solving equations.* Write a simple program to solve second-degree equations. The program should start by asking the coefficients of the equation, and afterwards print the solutions on the screen, if there are any, or a warning, otherwise.

   Recall that a second-degree equation has the general form $ax^2 + bx + c = 0$, where a, b and c are real numbers with $a \neq 0$. The solutions of this equation are $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, assuming that $b^2 - 4ac > 0$. If $b^2 - 4ac = 0$, then there is only one solution ($x = -\frac{b}{2a}$), and if $b^2 - 4ac < 0$ then the equation has no (real-valued) solutions.

2 Recursive programming with numbers

1. Write a function `sum_up_to(n: int) -> int` that returns the sum of the natural numbers up to `n`.
2. Write a function `sum_even(n: int) -> int` that returns the sum of all even numbers up to `n`.
3. Write a function `sum_between(m: int, n: int) -> int` that returns the sum of the numbers between `m` and `n`.
4. Write a function `factorial(n: int) -> int` that returns the factorial of `n`.
5. Write a function `double_factorial(n: int) -> int` that returns `n`!! ($n!! = 1 \times 3 \times 5 \times \cdots \times n$, if n is odd, and $n!! = 2 \times 4 \times 6 \times \cdots \times n$, if n is even).
6. Write a function `logarithm(n: int) -> int` that returns the integer base-2 logarithm of `n`.
7. Write a function `gcd(m: int, n: int) -> int` that computes the greatest common divisor of `m` and `n` using Euclides' algorithm:
$$\begin{cases} \gcd(m,m) = m \\ \gcd(m,n) = \gcd(m,n-m) & m < n \\ \gcd(m,n) = \gcd(m-n,n) & m > n \end{cases}$$
8. Write a function `lcm(m: int, n: int) -> int` that returns the least common multiple of `m` and `n`.
9. Write a function `first_digit(n: int, k:int) -> int` that returns the first digit of the decimal representation of `n` in base `k`. If unspecified, `k` should take the value 10.
10. Write a function `print_multiples() -> None` that prints on the screen the multiples of 7 that are than 500 (in ascending order).

    Generalize your solution to a function `print_multiples(k: int, n: int) -> None` that prints on the screen the multiples of `k` that are less than `n` (in ascending order).

    Hint: use an auxiliary function.
11. Write a function `count_divisors(n: int) -> int` that returns the number of divisors of `n`.
12. A perfect number is a number that equals the sum of its divisors (excluding itself). For example, 6 is a perfect number: its divisors are $\{1, 2, 3, 6\}$, and $1 + 2 + 3 = 6$.

    Write a function `is_perfect(n: int) -> bool` that checks whether `n` is a perfect number.
13. Write a function `count_perfect(n: int) -> int` that returns the number of perfect numbers smaller than `n`.
14. Write a function `is_prime(n: int) -> bool` that checks whether `n` is prime.
15. Write a function `count_primes(n: int) -> int` that returns the number of primes smaller than `n`.
16. Write a function `sum_beyond(k: int) -> int` that finds the least `n` such that the sum of the natural numbers up to `n` is at least `k`.
17. Write a function `is_palindrome(n: int) -> bool` that checks whether `n` is a palindrome.
18. Write a function `find_power(k: int) -> int` that returns the smallest number `n` such that $2^{\mathtt{n}}$ starts with `k`. What do you have to assume about `k`?

3 Recursive programming with lists

1. Write a function `length(v: list) -> int` that returns the length of `v`.
2. Write a function `count(x:Any, v: list) -> int` that counts the number of times that `x` appears in `v`.
3. Write a function `member(x:Any, v: list) -> bool` that checks whether `x` appears in `v`.
4. Write a function `subset(v: list, w: list) -> bool` that checks whether all elements of `v` occur in `w`.

5. Write a function `set_equals(v: list, w: list) -> bool` that determines whether `v` and `w` represent the same set. Recall that a set does not have order and does not count duplicate elements.

6. Write a function `intersection(v: list, w: list) -> list` returning a list containing the elements that occur both in `v` occur in `w`.

7. Write a function `sum(v: list[int]) -> int` that sums all the values in `v`.

8. Write a function `max(v: list[int]) -> int` that returns the largest element in the nonempty list `v`.

9. Write a function `smaller_than(n: int, v: list[int]) -> int` that counts how many elements of `v` are strictly smaller than `n`.

10. Write a function `two_zeros(v: list[int]) -> bool` that checks whether `v` contains two consecutive zeros.

11. Write a function `even_after_7(v: list[int]) -> int` that computes the number of even elements in `v` occurring after the first 7.

12. Write a function `is_sorted(v: list[int]) -> bool` that checks whether the list `v` is sorted.

13. Write a function `squares(n: int) -> list[int]` that returns a list with the squares of all natural numbers from 1 to `n`.

14. Write a function `decreasing_squares(n: int) -> list[int]` that returns a list with the squares of all natural numbers from `n` to 1.

15. Write a function `divisors(n: int) -> list[int]` that returns a list containing the divisors of `n`.

16. Write a function `square_it(v: list[int]) -> list[int]` that returns a list containing the squares of all elements in `v`.

17. Write a function `reverse(v: list) -> list` that returns a list containing the elements of `v` in reverse order.

18. Write a function `compare(v: list[int], n: int) -> tuple[int,int,int]` that returns a tuple containing: as first element, the number of elements of `v` larger than `n`; as second element, the number of elements of `v` equal to `n`; and, as third element, the number of elements of `v` smaller than `n`.

19. Write a function `join(v: list, w: list) -> list` that returns a list containing the elements of `v` followed by the elements of `w` (in the original order).

20. Write a function `sorted_join(v: list[int], w: list[int]) -> list[int]` that takes two ordered lists `v` and `w` as input and returns an ordered list containing all elements from either `v` or `w`.

21. Write a function `shuffle(v: list, w: list) -> list` that takes two lists `v` and `w` and constructs a list by taking alternately one element from each of `v` and `w`.

22. Write a function `remove(x: Any, v: list) -> list` that returns a list containing all the elements of `v` that are different from `x`.

4 Recursive programming with strings

Some exercises require identifying letters, or converting characters from/to lowercase. The Python functions `chr(n: int) -> str` and `ord(s: str) -> int` convert between characters and their ASCII code. The lowercase alphabet uses ASCII codes 97 to 122, and the uppercase alphabet uses ASCII codes 65 to 90.

1. Write a function `count(c: str, s: str) -> int` that counts the number of occurrences of the character `c` in the string `s`.

2. Write a function `member(c: str, s: str) -> bool` that checks whether the character `c` appears in the string `s`.

3. Write a function `is_prefix(s1: str, s2: str) -> bool` that checks whether `s1` is a prefix of `s2`.

4. Write a function `is_suffix(s1: str, s2: str) -> bool` that checks whether `s1` is a suffix of `s2`.

5. Write a function `is_substring(s1: str, s2: str) -> bool` that checks whether `s1` is a substring of `s2`.

6. Write a function `contains(s1: str, s2: str) -> bool` that checks whether `s2` can be obtained from `s1` by deleting some characters.

7. Write a function `caesar_code(s: str, n: int) -> str` that increases the ASCII code of each character in `s` by `n`. What is the simplest way to implement the inverse function `decode`?

8. Write a function `to_uppercase(s: str) -> str` that converts the string `s` to uppercase (ignoring all non-alphabetic characters).

9. Write a function `to_lowercase(s: str) -> str` that converts the string `s` to lowercase (ignoring all non-alphabetic characters).

10. Write a function `toCamelCase(s: str) -> str` that converts a string of text into camel notation (i.e.: removes spaces and changes the first character after each space into uppercase, if it is a letter).

11. Write a function `equals_ignore_case(s1: str, s2: str)` that determines whether `s1` and `s2` are equal up to changes of case.

12. Write a function `first_position(c: str, s: str)` that returns the index of the first occurrence of the character `c` in `s`, or -1 if `c` does not occur in `s`.

13. Write a function `last_position(c: str, s: str)` that returns the index of the last occurrence of the character `c` in `s`, or -1 if `c` does not occur in `s`.

14. Write a function `positions(c: str, s: str) -> list[int]` that returns a list containing the indices of the occurrences of `c` in `s`.

15. Write a function `is_permutation(s1: str, s2: str) -> bool` that determines whether `s1` and `s2` contain exactly the same characters (counting repetitions).

16. Write a function `reverse(s: str) -> str` that reverses a string.

17. Write a function `reverse_words(s: str) -> str` that reverses the individual words inside a given string (preserving their order).

    Hint: write an auxiliary function `split(s: str) -> list[str]` that splits a string at every occurrence of a particular character.

18. Write a function `remove_vowels(s: str) -> str` that takes a string as an argument and returns the result of removing all vowels in it.

19. Write a function `respace(s: str, n: int) -> str` that, given a string `s` and a positive integer `n`, returns the string obtained by first removing all spaces from `s` and afterwards adding a space after every `n` characters.

20. Write a function `encode_with_key(s: str, code: dict[str,str]) -> str` that encodes the string `s` character-by-character. The dictionary `code` maps each uppercase letter to its encoding; lowercase characters should be encoded accordingly, and all remaining characters left unchanged.

21. Write a function `histogram(s: str) -> dict[str,int]` that receives a string and returns a dictionary mapping each uppercase letter to the number of times it occurs (in either lower- or uppercase) in `s`. Non-alphabetic characters are not counted.

22. Write a function `replicate(s: str, v: list[int]) -> str` that receives a string and a list of the same length and returns the string containing `v[i]` copies of the character `s[i]`.

5 Functional programming

1. Write a function `sum(v: list[int]) -> int` that sums all the values in `v`.

2. Write a function `length(v: list) -> int` that returns the length of `v`.

3. Write a function `remove(x: Any, v: list) -> list` that returns a list containing all the elements of `v` that are different from `x`.

4. Write a function `count(x:Any, v: list) -> int` that counts the number of times that `x` appears in `v`.
5. Write a function `max(v: list[int]) -> int` that returns the largest element in the nonempty list `v`.
6. Write a function `square_it(v: list[int]) -> list[int]` that returns a list containing the squares of all elements in `v`.
7. Write a function `squares(n: int) -> list[int]` that returns a list with the squares of all natural numbers from 1 to `n`.
8. Write a function `decreasing_squares(n: int) -> list[int]` that returns a list with the squares of all natural numbers from `n` to 1.
9. Write a function `reverse(v: list) -> list` that returns a list containing the elements of `v` in reverse order.
10. Write a function `sum_up_to(n: int) -> int` that returns the sum of the natural numbers up to `n`.
11. Write a function `sum_between(m: int, n: int) -> int` that returns the sum of the numbers between `m` and `n`.
12. Write a function `sum_even(n: int) -> int` that returns the sum of all even numbers up to `n`.
13. Write a function `factorial(n: int) -> int` that returns the factorial of `n`.
14. Write a function `double_factorial(n: int) -> int` that returns `n`!! ($n!! = 1 \times 3 \times 5 \times \cdots \times n$, if n is odd, and $n!! = 2 \times 4 \times 6 \times \cdots \times n$, if n is even).
15. Write a function `member(x:Any, v: list) -> bool` that checks whether `x` appears in `v`.
16. Write a function `subset(v: list, w: list) -> bool` that checks whether all elements of `v` occur in `w`.
17. Write a function `intersection(v: list, w: list) -> list` returning a list containing the elements that occur both in `v` occur in `w`.
18. Write a function `smaller_than(n: int, v: list[int]) -> int` that counts how many elements of `v` are strictly smaller than `n`.
19. Write a function `caesar_code(s: str, n: int) -> str` that increases the ASCII code of each character in `s` by `n`.
20. Write a function `to_uppercase(s: str) -> str` that converts the string `s` to uppercase (ignoring all non-alphabetic characters).
21. Write a function `to_lowercase(s: str) -> str` that converts the string `s` to lowercase (ignoring all non-alphabetic characters).
22. Write a function `count_divisors(n: int) -> int` that returns the number of divisors of `n`.
23. Write a function `is_perfect(n: int) -> bool` that checks whether `n` is a perfect number.
24. Write a function `count_perfect(n: int) -> int` that returns the number of perfect numbers smaller than `n`.
25. Write a function `is_prime(n: int) -> bool` that checks whether `n` is prime.
26. Write a function `count_primes(n: int) -> int` that returns the number of primes smaller than `n`.
27. Write a function `two_zeros(v: list[int]) -> bool` that checks whether `v` contains two consecutive zeros.
28. Write a function `is_sorted(v: list[int]) -> bool` that checks whether the list `v` is sorted.
29. Write a function `compare(v: list[int], n: int) -> tuple[int,int,int]` that returns a tuple containing: as first element, the number of elements of `v` larger than `n`; as second element, the number of elements of `v` equal to `n`; and, as third element, the number of elements of `v` smaller than `n`.
30. Write a function `positions(c: str, s: str) -> list[int]` that returns a list containing the indices of the occurrences of `c` in `s`.

31. Write a function replicate(s: str, v: list[int]) -> str that receives a string and a list of the same length and returns the string containing v[i] copies of the character s[i].

32. Write a function remove_vowels(s: str) -> str that takes a string as an argument and returns the result of removing all vowels in it.

33. Write a function encode_with_key(s: str, code: dict[str,str]) -> str that encodes the string s character-by-character. The dictionary code maps each uppercase letter to its encoding; lowercase characters should be encoded accordingly, and all remaining characters left unchanged.

34. Write a function gcd(m: int, n: int) -> int that computes the greatest common divisor of m and n using Euclides' algorithm:

$$\begin{cases} \gcd(m,m) = m & \\ \gcd(m,n) = \gcd(m,n-m) & m < n \\ \gcd(m,n) = \gcd(m-n,n) & m > n \end{cases}$$

35. Write a function first_digit(n: int, k:int) -> int that returns the first digit of the decimal representation of n in base k. If unspecified, k should take the value 10.

36. Write a function trim(s: str) -> str that removes leading spaces in s.

37. Write a function dimensions(m: list[list]) -> list[int] that returns a list with the lengths of all elements of m.

38. Write a function is_matrix(m: list[list]) -> bool that checks whether m is a matrix.

39. Write a function is_square_matrix(m: list[list]) -> bool that determines whether m is a square matrix.

40. Write a function zeros(m: int, n: int) -> list[list[int]] that returns a matrix with m rows and n columns whose entries are all 0.

41. Write a function identity(n: int) -> list[list[int]] that returns a matrix with n rows and n columns whose entries are 1 in the diagonal and 0 elsewhere.

    For example, identity(3) should return [[1,0,0],[0,1,0],[0,0,1]].

42. Write a function triangle(n: int) -> list[list[int]] that returns a triangular array of 1s where the first row has one element and each row afterwards contains one more element than the previous one.

6 List comprehension

All these functions can be written as one-liners using list comprehension. (Re)implement them.

1. Write a function squares(n: int) -> list[int] that returns a list with the squares of all natural numbers from 1 to n.

2. Write a function decreasing_squares(n: int) -> list[int] that returns a list with the squares of all natural numbers from n to 1.

3. Write a function divisors(n: int) -> list[int] that returns a list of the divisors of n.

4. Write a function square_it(v: list[int]) -> list[int] that returns a list containing the squares of all elements in v.

5. Write a function reverse(v: list) -> list that returns a list containing the elements of v in reverse order.

6. Write a function remove(x: Any, v: list) -> list that returns a list containing all the elements of v that are different from x.

7. Write a function positions(c: str, s: str) -> list[int] that returns a list containing the indices of the occurrences of c in s.

8. Write a function count(x:Any, v: list) -> int that counts the number of times that x appears in v.

9. Write a function `member(x:Any, v: list) -> bool` that checks whether `x` appears in `v`.

10. Write a function `smaller_than(n: int, v: list[int]) -> int` that counts how many elements of `v` are strictly smaller than `n`.

11. Write a function `zeros(m: int, n: int) -> list[list[int]]` that returns a matrix with `m` rows and `n` columns whose entries are all 0.

12. Write a function `identity(n: int) -> list[list[int]]` that returns a matrix with `n` rows and `n` columns whose entries are 1 in the diagonal and 0 elsewhere.

13. Write a function `triangle(n: int) -> list[list[int]]` that returns a triangular array of `1`s where the first row has one element and each row afterwards contains one more element than the previous one.

14. Write a function `multiplication_table(n: int) -> list[list[int]]` that returns a multiplication table up to `n`.

15. Write a function `transpose(m: list[list]) -> list[list]` that returns the matrix obtained from `m` by interchanging rows and columns.

7 Small projects I

The next exercises are slightly more complicated than the previous ones, and you should try to solve them using different programming styles.

1. *Finding zeros.* Given a function f on the natural numbers, a simple way to find the least n for which $f(n) = 0$ is by testing: we compute $f(0)$, $f(1)$, etc, until we find the right value.

   Write a function `find_zero(f) -> int` that implements this algorithm. What happens if you call your function with argument `lambda n: n+1`?

2. *Solving equations.* A possible way to solve an equation $f(x) = 0$, if f is a continuous function, is the (*bisection method*). Given $a < b$ such that $f(a) \times f(b) < 0$, we first compute the midpoint $c = \frac{a+b}{2}$ and the value of $f(c)$; if $f(a) \times f(c) < 0$, we repeat the procedure with $b = c$; otherwise, we repeat it with $a = c$. This procedure terminates when the difference between a and b is smaller than a given value (the error), in which case the current value of a (or b, or c) is an approximate solution of $f(x) = 0$.

   Implement this method as a function `bisection(f, a: float, b: float, eps: float) -> float`.

3. Write a functional implementation of the function `gcd(m: int, n: int) -> int` that computes the greatest common divisor of `m` and `n` using Euclides' algorithm.

   Hint: use `fixpoint`.

4. Pascal's triangle is defined as follows: the first line contains simply the number 1; each line is computed from the previous by adding each pair of consecutive numbers, and including an additional 1 at the beginning and at the end.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & & \searrow & & \swarrow & & \\ & 1 & & 2 & & 1 & \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

   Write a function `pascal(n: int) -> list[int]` that returns the `n`-th line of Pascal's triangle.

   Challenge: be brave, and use `iterate`.

5. Write a function `differences(v: list[int]) -> list[list[int]]` that takes an array `v` and returns an array of arrays such that: its first line is `v`; and each other line contains the differences between consecutive elements of the previous lines. For example, for `v=[2,1,5,-2]` the expected result of `differences(v)` is `[[2,1,5,-2],[1,-4,7],[5,-11],[16]]`.

6. The sequence of Fibonacci numbers f_n is defined by $f(0) = f(1) = 1$ and $f(n+2) = f(n)+f(n+1)$. Write a function `int fibonacci(int n)` that returns the `n`-th Fibonacci number.

   Note: the obvious recursive solution is easy, but it won't work.

7. Write a function `reverse_words(s: str) -> str` that reverses the individual words inside a given string (preserving their order).
8. Write a function `shift_words(s: str, n: int) -> str` that shifts each individual word inside the argument string by the given number of characters.

   Note: use the function `split` defined previously.
9. Write a function `product(m1: list[list[int]], m2: list[list[int]]) -> list[list[int]]` that returns the matrix multiplication of `m1` and `m2`.

8 Imperative programming on numbers

1. Write a function `print_multiples() -> None` that prints on the screen the multiples of 7 that are than 500 (in ascending order).

   Generalize your solution to a function `print_multiples(k: int, n: int) -> None` that prints on the screen the multiples of `k` that are less than `n` (in ascending order).
2. Write a function `sum_up_to(n: int) -> int` that returns the sum of the natural numbers up to `n`.
3. Write a function `sum_even(n: int) -> int` that returns the sum of all even numbers up to `n`.
4. Write a function `sum_between(m: int, n: int) -> int` that returns the sum of the numbers between `m` and `n`.
5. Write a function `sum_beyond(k: int) -> int` that finds the least `n` such that the sum of the natural numbers up to `n` is at least `k`.
6. Write a function `factorial(n: int) -> int` that returns the factorial of `n`.
7. Write a function `double_factorial(n: int) -> int` that returns `n`!!
8. Write a function `int fibonacci(int n)` that returns the `n`-th Fibonacci number.
9. Write a function `logarithm(n: int, m: int) -> int` that returns the integer base-`m` logarithm of `n`. If unspecified, `m` should be 2.
10. Write a function `count_divisors(n: int) -> int` that returns the number of divisors of `n`.
11. Write a function `is_perfect(n: int) -> bool` that checks whether `n` is a perfect number.
12. Write a function `count_perfect(n: int) -> int` that returns the number of perfect numbers smaller than `n`.
13. Write a function `is_prime(n: int) -> bool` that checks whether `n` is prime.
14. Write a function `count_primes(n: int) -> int` that returns the number of primes smaller than `n`.
15. Write a function `nth_prime(n: int) -> int` that returns the `n`th prime number.
16. Write a function `gcd(m: int, n: int) -> int` that computes the greatest common divisor of `m` and `n` using Euclides' algorithm:

$$\begin{cases} \gcd(m,m) = m \\ \gcd(m,n) = \gcd(m,n-m) & m < n \\ \gcd(m,n) = \gcd(m-n,n) & m > n \end{cases}$$

17. Write a function `first_digit(n: int, k:int) -> int` that returns the first digit of the decimal representation of `n` in base `k`. If unspecified, `k` should take the value 10.
18. Write a function `is_palindrome(n: int) -> bool` that checks whether `n` is a palindrome.
19. Write a function `find_power(k: int) -> int` that returns the smallest number `n` such that $2^{\mathtt{n}}$ starts with `k`.

9 Imperative programming on lists

1. Write a function `sum(v: list[int]) -> int` that sums all the values in `v`.
2. Write a function `count(x:Any, v: list) -> int` that counts the number of times that `x` appears in `v`.
3. Write a function `max(v: list[int]) -> int` that returns the largest element in the nonempty list `v`.
4. Write a function `smaller_than(n: int, v: list[int]) -> int` that counts how many elements of `v` are strictly smaller than `n`.
5. Write a function `squares(n: int) -> list[int]` that returns a list with the squares of all natural numbers from 1 to `n`.
6. Write a function `decreasing_squares(n: int) -> list[int]` that returns a list with the squares of all natural numbers from `n` to 1.
7. Write a function `divisors(n: int) -> list[int]` that returns a list containing the divisors of `n`.
8. Write a function `two_zeros(v: list[int]) -> bool` that checks whether `v` contains two consecutive zeros.
9. Write a function `is_sorted(v: list[int]) -> bool` that checks whether the list `v` is sorted.
10. Write a function `member(x:Any, v: list) -> bool` that checks whether `x` appears in `v`.
11. Write a function `subset(v: list, w: list) -> bool` that checks whether all elements of `v` occur in `w`.
12. Write a function `intersection(v: list, w: list) -> list` returning a list containing the elements that occur both in `v` occur in `w`.
13. Write a function `first_position_max(v: list[int]) -> int` that returns the index of the first occurrence of `s`'s maximum element, or -1 if `s` is empty.
14. Write a function `last_position_max(v: list[int]) -> int` that returns the index of the last occurrence of `s`'s maximum element, or -1 if `s` is empty.
15. Write a function `add_positions_max(v: list[int]) -> int` that returns the sum of the indices of the positions where the maximum of `s` appears.
16. Write a function `positions_max(v: list[int]) -> list[int]` that returns a list containing the indices of the positions where the maximum of `s` appears.
17. Write a function `square_it(v: list[int]) -> None` that replaces each element of `v` with its square.
18. Write a function `reverse(v: list) -> list` that returns a list containing the elements of `v` in reverse order.
19. Write a function `compare(v: list[int], n: int) -> tuple[int,int,int]` that returns a tuple containing: as first element, the number of elements of `v` larger than `n`; as second element, the number of elements of `v` equal to `n`; and, as third element, the number of elements of `v` smaller than `n`.
20. Write a function `even_after_first_7(v: list[int]) -> int` that computes the number of even elements in `v` occurring after the first 7.
21. Write a function `even_after_last_7(v: list[int]) -> int` that computes the number of even elements in `v` occurring after the last 7.
22. Write a function `sorted_join(v: list[int], w: list[int]) -> list[int]` that takes two ordered lists `v` and `w` as input and returns an ordered list containing all elements from either `v` or `w`.
23. Write a function `shuffle(v: list, w: list) -> list` that takes two lists `v` and `w` and constructs a list by taking alternately one element from each of `v` and `w`.
24. Write a function `largest_increasing_sequence(v: list) -> int` that returns the length of the largest increasing sequence of consecutive elements of `v`.

The next exercises use two-dimensional lists. Two-dimensional lists whose elements all have the same length are often called matrices.

25. Write a function `dimensions(m: list[list]) -> list[int]` that returns a list with the lengths of all elements of `m`.

26. Write a function `is_matrix(m: list[list]) -> bool` that checks whether `m` is a matrix.

27. Write a function `is_square_matrix(m: list[list]) -> bool` that determines whether `m` is a square matrix.

28. Write a function `zeros(m: int, n: int) -> list[list[int]]` that returns a matrix with `m` rows and `n` columns whose entries are all 0.

29. Write a function `identity(n: int) -> list[list[int]]` that returns a matrix with `n` rows and `n` columns whose entries are 1 in the diagonal and 0 elsewhere.

30. Write a function `triangle(n: int) -> list[list[int]]` that returns a triangular array of `1`s where the first row has one element and each row afterwards contains one more element than the previous one.

31. Write a function `multiplication_table(n: int) -> list[list[int]]` that returns a multiplication table up to `n`.

32. Write a function `sum_all(m: list[list[int]]) -> int` that sums all the values in the list of lists `m`.

33. Write a function `max_all(m: list[list[int]]) -> int` that returns the largest element in the nonempty list of lists `m`.

34. Write a function `parity(m: list[list[int]]) -> None` that replaces each even element of `m` by `0` and each odd element of `m` by `1`.

35. Write a function `trace(m: list[list[int]]) -> int` that returns the sum of all elements in the diagonal of `m` (the *trace* of `m`), if `m` is a square matrix.

36. Write a function `column(m: list[list], j: int) -> list` that returns the `j`-th column of the matrix `m`. Which expression gives us the `i`-th row of `m`?

37. If two matrices have the same number of rows and columns, we can add them entry-by-entry. Write a function `add(m1: list[list[int]], m2: list[list[int]]) -> list[list[int]]` that implements this operation.

38. Write a function `multiply(a: int, m: list[list[int]]) -> None` that multiplies all entries of `m` by `a`.

39. Write a function `del_row_and_col(m: list[list], i: int, j: int) -> list[list]` that returns the matrix obtained by removing the `i`-th row and the `j`-th column of `m`.

40. Write a function `differences(v: list[int]) -> list[list[int]]` that takes an array `v` and returns an array of arrays such that: its first line is `v`; and each other line contains the differences between consecutive elements of the previous lines. For example, for `v=[2,1,5,-2]` the expected result of `differences(v)` is `[[2,1,5,-2],[1,-4,7],[5,-11],[16]]`.

41. Write a function `transpose(m: list[list]) -> list[list]` that returns the matrix obtained from `m` by interchanging rows and columns.

42. Write a function `product(m1: list[list[int]], m2: list[list[int]]) -> list[list[int]]` that returns the matrix multiplication of `m1` and `m2`.

10 Imperative programming on strings

1. Write a function `count(c: str, s: str) -> int` that counts the number of occurrences of the character `c` in the string `s`.

2. Write a function `member(c: str, s: str) -> bool` that checks whether the character `c` appears in the string `s`.

3. Write a function `is_prefix(s1: str, s2: str) -> bool` that checks whether `s1` is a prefix of `s2`.

4. Write a function `is_suffix(s1: str, s2: str) -> bool` that checks whether `s1` is a suffix of `s2`.
5. Write a function `is_substring(s1: str, s2: str) -> bool` that checks whether `s1` is a substring of `s2`.
6. Write a function `contains(s1: str, s2: str) -> bool` that checks whether `s2` can be obtained from `s1` by deleting some characters.
7. Write a function `to_uppercase(s: str) -> str` that converts the string `s` to uppercase (ignoring all non-alphabetic characters).
8. Write a function `to_lowercase(s: str) -> str` that converts the string `s` to lowercase (ignoring all non-alphabetic characters).
9. Write a function `toCamelCase(s: str) -> str` that converts a string of text into camel notation (i.e.: removes spaces and changes the first character after each space into uppercase, if it is a letter).
10. Write a function `equals_ignore_case(s1: str, s2: str)` that determines whether `s1` and `s2` are equal up to changes of case.
11. Write a function `first_position(c: str, s: str)` that returns the index of the first occurrence of the character `c` in `s`, or -1 if `c` does not occur in `s`.
12. Write a function `last_position(c: str, s: str)` that returns the index of the last occurrence of the character `c` in `s`, or -1 if `c` does not occur in `s`.
13. Write a function `positions(c: str, s: str) -> list[int]` that returns a list containing the indices of the occurrences of `c` in `s`.
14. Write a function `is_permutation(s1: str, s2: str) -> bool` that determines whether `s1` and `s2` contain exactly the same characters (counting repetitions).
15. Write a function `reverse(s: str) -> str` that reverses a string.
16. Write a function `reverse_words(s: str) -> str` that reverses the individual words inside a given string (preserving their order).
17. Write a function `remove_vowels(s: str) -> str` that takes a string as an argument and returns the result of removing all vowels in it.
18. Write a function `respace(s: str, n: int) -> str` that, given a string `s` and a positive integer `n`, returns the string obtained by first removing all spaces from `s` and afterwards adding a space after every `n` characters.
19. Write a function `shift_words(s: str, n: int) -> str` that shifts each individual word inside the argument string by the given number of characters.
20. Write a function `caesar_code(s: str, n: int) -> str` that increases the ASCII code of each character in `s` by `n`.
21. Write a function `encode_with_key(s: str, code: dict[str,str]) -> str` that encodes the string `s` character-by-character. The dictionary `code` maps each uppercase letter to its encoding; lowercase characters should be encoded accordingly, and all remaining characters left unchanged.
22. Write a function `histogram(s: str) -> dict[str,int]` that receives a string and returns a dictionary mapping each uppercase letter to the number of times it occurs (in either lower- or uppercase) in `s`. Non-alphabetic characters are not counted.
23. Write a function `replicate(s: str, v: list[int]) -> str` that receives a string and a list of the same length and returns the string containing `v[i]` copies of the character `s[i]`.

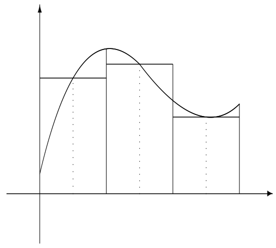
11 Small projects II

This exercises are all meant to be solved imperatively, possibly with some recursive/functional implementations of helper functions.

1. The *sieve of Eratosthenes* is one of the oldest algorithms to find all prime numbers up to a given n. First, one writes down a list containing all numbers from 1 to n, and crosses out the 1. Then, one repeatedly picks the next number k from the list that has not been crossed out, and crosses out all larger multiples of k. When the end of the list is reached, the numbers not crossed out are precisely the prime numbers smaller than or equal to n.

   Implement this algorithm as a function `eratosthenes(n: int) -> list[int]`.

2. *Computing areas.* Suppose f is a continuous and positive function in an interval $[a, b]$. The area between the horizontal axis and the graph of f in the interval $[a, b]$ (also called the *integral* of f in $[a, b]$) can be computed as precisely as required by the following method: we divide the interval $[a, b]$ in n subintervals of equal width, and approximate the integral of f in each subinterval by the area of the rectangle whose height is given by the value of f value in the midpoint of the interval (see the figure below).

   Implement this method as a function `integral(f, a: float, b: float, n: int) -> float` that returns the computed approximate value of the integral of f in the interval $[a, b]$.

3. Reimplement the bisection method presented earlier imperatively.

4. Reimplement the function `pascal(n: int) -> list[int]` that returns the `n`-th line of Pascal's triangle imperatively.

12 Implementing datatypes

1. *Time management.* Implement a module `timestamp` defining a datatype `TimeStamp` that allows working with points in time, characterized by hours, minutes and seconds. Your module should also provide the following functions:

   - a function `make_timestamp(hours: int, minutes: int, seconds: int) -> TimeStamp` returning a new instance of `TimeStamp` representing the given time, where all arguments are optional and default to 0 when absent;
   - a function `valid(hours: int, minutes: int, seconds: int) -> bool` that checks whether its given arguments are in the valid range;
   - a function `skip_second(t: TimeStamp) -> None`, a function `skip_minute(t: TimeStamp) -> None`, and a function `skip_hour(t: TimeStamp) -> None` that add one second, one minute and one hour, respectively, to the timestamp `t` (assume that 23:59:59 is followed by 0:00:00);
   - a function `skip(t1: TimeStamp, t2: TimeStamp) -> None` that adds the amount of time described in `t2` to `t1`;
   - a function `equals(t1: TimeStamp, t2: TimeStamp) -> bool` that determines whether `t1` and `t2` represent the same timestamp;
   - a function `copy(t: TimeStamp) -> TimeStamp` that returns a copy of `t`;
   - a function `to_string(t: TimeStamp) -> str` that returns a textual representation of `t`.

2. On top of the previous module, implement a module `date` defining a datatype `Date` whose instances include information about the year, month, day and timestamp. Your module should exploit `timestamp` as much as possible, and define the following functions:

- a function `make_date(year: int, month: int, day: int, time: TimeStamp) -> Date` that creates an instance of `Date` corresponding to the given time on the given date; if the time is omitted, it should be set to midnight;
- a function `valid(year: int, month: int, day: int) -> bool` that checks whether its arguments represent a valid date;
- a function `skip_day(d: Date) -> None`, a function `skip_month(d: Date) -> None`, and a function `skip_year(d: Date) -> None` that skip the date `d` forward by one day, one month or one year, respectively;
- a function `skip_time(d: Date, t: TimeStamp) -> None` that skips `d` forward by the indicated amount of time;
- a function `equals(d1: Date, d2: Date) -> bool` that determines whether `d1` and `d2` represent the same date;
- a function `copy(d: Date) -> Date` that returns a copy of `d`;
- a function `to_string(d: Date) -> str` that returns a textual representation of `d`.

3. A point on a flat surface (such as a computer screen) is defined by two coordinates, also called its horizontal and vertical components. Implement a module `point2d` defining a datatype `Point2D` whose instances represent two-dimensional points. Your module should also provide the following methods:

   - a function `make_point(x: float, y: float) -> Point2D` that returns a new instance of `Point2D` with the given coordinates;
   - a function `move(p: Point2D, dx: float, dy: float) -> None` that moves `p` according to the vector (`dx`,`dy`);
   - a function `distance_to_origin(p: Point2D) -> float` that returns `p`'s distance to the origin;
   - a function `distance(p1: Point2D, p2: Point2D) -> float` that returns the distance between `p1` and `p2`;
   - a function `equals(p1: Point2D, p2: Point2D) -> bool` that determines whether `p1` and `p2` represent the same point;
   - a function `copy(p: Point2D) -> Point2D` that returns a copy of `p`;
   - a function `to_string(p: Point2D) -> str` that returns a textual representation of `p`.

   Consider the following client for your module.

```
p1 = make_point(0, 0)
p2 = p1

move(p1, 1, 1)
print(to_string(p2))

move(p2, 3, 3)
print(to_string(p2))
```

   What output is produced by this program?

4. A polygon is a region on the plane limited by straight line segments (its sides).

   Implement a module `polygon` defining a datatype `Polygon` whose instances are polygons, represented as a sequence of points (its vertices) such that there is a line between each two consecutive points, as well as between the first and the last. Exploit module `point2d` as much as possible. Your module should also provide the following methods:

   - a function `make_polygon(v: list[Point2D]) -> Polygon` that creates a polygon from the list of its vertices;
   - a function `perimeter(p: Polygon) -> float` returning the perimeter of `p`;
   - a function `nearest(p: Polygon) -> Point2D` that returns the vertex of `p` that is closest to the origin;
   - a function `longest_side(p: Polygon) -> float` returning the length of `p`'s longest side;

- a function `move(p: Polygon, dx: float, dy: float) -> None` that moves `p` according to the vector (`dx`,`dy`);
- a function `vertices_in_quadrant(p: Polygon, n: int) -> int` counting how many of `p`'s vertices lie on the `n`-th quadrant;
- functions `is_triangle(p: Polygon) -> bool` and `is_rectangle(p: Polygon) -> bool` that determine whether `p` is a triangle or a rectangle, respectively;
- a function `equals(p1: Polygon, p2: Polygon) -> bool` that determines whether `p1` and `p2` represent the same polygon (note that the polygon's vertices need not be given in the same order); ;
- a function `copy(p: Polygon) -> Polygon` that returns a copy of `p`;
- a function `to_string(p: Polygon) -> str` that returns a textual representation of `p`.

5. A discrete representation of a function f is a list of `Point2D` such that $f(x) = y$ holds for each point (x, y) of the list, and the elements of the list are sorted by their first coordinate.

   Implement a module `graph` defining a datatype `Graph` whose instances are discrete representations of functions. Your module should also provide the following functions:

   - a function `make_graph(p: list[Point2D]) -> Graph` that receives a list of the form described and returns the corresponding function representation;
   - a function `valid(graph: Point2D) -> bool` that checks whether its argument can be passed along to the previous function;
   - a function `max(f: Graph) -> float` that finds the maximum value of `f`, according to its discrete representation;
   - a function `increasing(f: Graph) -> bool` that checks whether `f` is increasing or not;
   - a function `changeRate(f: Graph) -> list[float]` that returns a list with one less element than the graph of `f` containing this function's average rate of change in each interval;
   - a function `equals(f1: Graph, f2: Graph) -> bool` that determines whether `f1` and `f2` are two equal discrete representations of the same function;
   - a function `copy(f: Graph) -> Graph` that returns a copy of `f`;
   - a function `to_string(f: Graph) -> str` that returns a textual representation of `f`.

6. *Shopping carts.* An online supermarket is building a backoffice system, consisting of several interacting datatypes.

   Your task is to implement a module `shopping_cart` defining a datatype `ShoppingCart` to represent shopping carts containing the products selected by a client. The products in the shopping cart are themselves instances of a datatype `Product` that you will not develop, and which provides (among others) the following functions:

   - `price(p: Product) -> float`: returns the price of product `p`;
   - `equals(p1: Product, p2: Product) -> bool`: returns `True` if `p1` and `p2` represent the same product;
   - `copy(p: Product) -> Product`: returns a new product that is equal to `p`.

   Implement module `shopping_cart`. Besides defining the datatype `ShoppingCart`, your module should also provide the following methods:

   - a function `make_shopping_cart() -> ShoppingCart` that returns an empty shopping cart;
   - a function `add(p: Product, s: ShoppingCart) -> None` that adds product `p` to shopping cart `s`;
   - a function `number_of_items(s: ShoppingCart) -> int` returning the number of produts in `s`;
   - a function `free_delivery(s: ShoppingCart) -> bool` indicating whether `s` is eligible for free delivery (only for shopping carts with more than 50 items);
   - a function `total_price(s: ShoppingCart) -> float` returning the cost of `s`;
   - functions `most_expensive(s: ShoppingCart) -> Product`, returning the most expensive product in `s`, and `highest_price(s: ShoppingCart) -> float`, returning its price;
   - a function `how_many(p: Product, s: ShoppingCart) -> int` returning the number of items equal to `p` in `s`;
   - a function `remove_most_expensive(s: ShoppingCart) -> None` removing all copies of the most expensive product in `s`.