# Facility location under uncertainty: a review

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Plants, distribution centers, and other facilities generally function for years or decades, during which time the environment in which they operate may change substantially. Costs, demands, travel times, and other inputs to classical facility location models may be highly uncertain. This has made the development of models for facility location under uncertainty a high priority for researchers in both the logistics and stochastic/robust optimization communities. Indeed, a large number of the approaches that have been proposed for optimization under uncertainty have been applied to facility location problems. This paper reviews the literature on stochastic and robust facility location models. Our intent is to illustrate both the rich variety of approaches for optimization under uncertainty that have appeared in the literature and their application to facility location problems. In a few instances for which examples in facility location are not available, we provide examples from the more general logistics literature.

## 1. Introduction

Facility location decisions are costly and difficult to reverse, and their impact spans a long time horizon. During the time when design decisions are in effect, any of the parameters of the problem (costs, demands, distances) may fluctuate widely. Parameter estimates may also be inaccurate due to poor measurements or to tasks inherent in the modeling process such as aggregating demands points and choosing a distance norm. Recognizing this, researchers have been developing models for facility location under uncertainty for several decades.

The two-stage nature of facility location problems (choose locations now, before we know what the future holds, and react once the uncertainty has been resolved, say, by assigning customers to facilities) has made these problems very attractive to researchers exploring approaches to decision making under uncertainty. A large number of these approaches have been applied to facility location problems.

This paper aims to illustrate the rich variety of approaches to optimization under uncertainty by examining their application to facility location problems. To this end, we have chosen to categorize papers first by their approach to uncertainty, and then by the nature of the facility location problem they discuss. A few of the measures we discuss have not, to the best of our knowledge, been applied to facility location problems. We include these measures in our survey because of their relationship to measures that we do discuss, or simply because of the novelty of their approaches. In these cases, we provide examples from the literature on capacity planning, network design, or other logistics problems that, similar to facility location, involve a strategic

phase during which capital investments are made, followed by a tactical phase, with uncertainty being resolved between the two.

In the interest of brevity, we have opted not to discuss the large body of literature on facility location problems with congested facilities, which attempt to capture the possibility that a customer may need service from a facility that is occupied with another customer. Such models are commonly used in the siting of ambulances, fire stations, and other emergency services. They attempt to guarantee adequate service either by requiring redundant coverage (as in Daskin (1982, 1983), ReVelle and Hogan (1989), and Ball and Lin (1993)) or by explicitly considering the queuing aspect of the problem (as in Larson (1974, 1975), Berman *et al*. (1985), and Marianov and ReVelle (1996)). The reader is referred to the surveys by Daskin *et al*. (1988) and Berman and Krass (2002) for thorough discussions of these models. A related branch of literature considers models in which the facilities may be unable to provide service due to facility disruptions (Bundschuh *et al*., 2003; Berman *et al*., 2004; Snyder and Daskin, 2005), or link failures (Nel and Colbourn, 1990; Eiselt *et al*., 1992).

Throughout this paper we assume that the reader has some familiarity with deterministic facility location theory. For an introduction to this topic, the reader is referred to the texts by Daskin (1995), Dresner (1995), or Drezner and Hamacher (2002). Louveaux (1993) reviews models for stochastic (but not robust) facility location. See Owen and Daskin (1998) for a survey on strategic aspects of facility location, including both dynamic problems and problems under uncertainty. Brandeau and Chiu (1989), Louveaux (1993), Daskin and Owen (1999), and Current *et al*. (2002)

review both deterministic and stochastic facility location. Nikulin (2004) provides an annotated bibliography of papers dealing with robust combinatorial optimization problems, including facility location. See Birge and Louveaux (1997) for a textbook treatment of stochastic programming theory. A summary of the papers cited in this article can be found in spreadsheet form on the author's website, www.lehigh.edu/~lvs2/research.html.

## 2. Decision making under uncertainty

Rosenhead *et al.* (1972) divide decision-making environments into three categories: (i) certainty; (ii) risk; and (iii) uncertainty. In *certainty* situations, all parameters are deterministic and known, whereas risk and uncertainty situations both involve randomness. In *risk* situations, there are uncertain parameters whose values are governed by probability distributions that are known by the decision maker. In *uncertainty* situations, parameters are uncertain, and furthermore, no information about probabilities is known. Problems in risk situations are known as *stochastic optimization problems*; a common goal is to optimize the expected value of some objective function. Problems under uncertainty are known as *robust optimization problems* and often attempt to optimize the worst-case performance of the system.

The goal of both stochastic and robust optimization is to find a solution that will perform well under any possible realization of the random parameters. The definition of "performing well" varies from application to application, and choosing an appropriate performance measure is part of the modeling process. The random parameters can be either continuous or described by discrete scenarios. If probability information is known, uncertainty is described using a (continuous or discrete) probability distribution on the parameters. If no probability information is known, continuous parameters are generally restricted to lie in some prespecified intervals.

The scenario approach has two main drawbacks. One is that identifying scenarios (let alone assigning probabilities to them) is a daunting and difficult task; indeed, it is the focus of a large body of stochastic programming literature. The second disadvantage is that one generally wants to identify a relatively small number of scenarios for computational reasons, but this limits the range of future states under which decisions are evaluated. However, the scenario approach generally results in more tractable models, and furthermore, it has the advantage of allowing parameters to be statistically dependent, which is often not practical when parameters are described by continuous probability distributions (although there are exceptions). Dependence is often necessary to model reality, since, for example, demands are often correlated across time periods or geographical regions and costs are often correlated among suppliers.

Most of the stochastic and robust facility location problems discussed in this paper are NP-hard since they of-

ten have classical facility location problems, which are themselves NP-hard, as special cases. Min-expected-cost extensions of "minisum" models such as the *P*-Median Problem (PMP) (Hakimi, 1964) and the Uncapacitated Fixed-charge Location Problem (UFLP) (Balinski, 1965) (for example, those discussed in Section 3.2.1) are relatively easy to solve since they can often be treated as larger instances of deterministic problems, for which good algorithms exist despite their NP-hard nature. For example, a problem with 100 nodes and ten scenarios can be solved in approximately the time required to solve a deterministic problem with 1000 nodes (a minute or two using today's state-of-the-art algorithms on a desktop computer). On the other hand, problems with a minimax structure such as those described in Section 4.1 are more difficult to solve to optimality; today's best algorithms are able to solve problems perhaps an order of magnitude smaller than corresponding stochastic problems in the same amount of time. This discrepancy parallels the difference in difficulty between deterministic minisum and minimax problems. For example, relatively large instances of the UFLP and PMP may be solved quickly, but the *P*-center problem, which has a minimax structure, is generally solved by embedding a set-covering problem (which is itself NP-hard) inside a binary search routine.

We next discuss stochastic location problems, then turn our attention to robust location problems in Section 4.

## 3. Stochastic location problems

In this section, we discuss stochastic models for facility location. Many of these models have as an objective the minimization of the expected cost or maximization of the expected profit of the system. Others take a probabilistic approach: for example, maximizing the probability that the solution is in some sense "good". Some models are solved using algorithms designed specifically for the problem, whereas others are solved using more general Stochastic Programming (SP) techniques. In any SP problem, one must decide which decision variables are first stage and which are second stage; that is, which variables must be set now and which may be set after the uncertainty has been resolved. In stochastic location modeling, locations are generally first-stage decisions whereas assignments of customers to facilities are second-stage, i.e., recourse, decisions. (If both decisions occur in the first stage, most problems can be reduced easily to deterministic problems in which uncertain parameters are replaced by their means.)

### 3.1. *The Hakimi property*

Several early papers on stochastic location were devoted to establishing whether the Hakimi property holds. The Hakimi property (Hakimi, 1964, 1965) states that there exists an optimal solution to a network location problem in which the facilities are located on the nodes of the network, not along the edges; it holds for minisum problems such as

the PMP and UFLP. Mirchandani and Odoni (1979) prove that the Hakimi property holds for a PMP on a network with shortest-path travel costs in which the cost of a path may be any concave, nondecreasing function of its length. In their problem, both demands and transportation costs may be uncertain. As a result, the least-expected-cost path between two points may change depending on the scenario, as may the optimal assignments of customers to facilities. Mirchandani (1980) uses a similar analysis to determine whether the Hakimi property applies to stochastic versions of the PMP and UFLP under a variety of assumptions. Louveaux and Thisse (1985) maximize the expected utility of profit in a production-distribution system in which they locate a single facility and set production levels in the first stage and make distribution decisions in the second. They show that the Hakimi property applies when the firm is risk neutral (i.e., the utility function is linear) but not when it is risk averse.

## 3.2. *Mean-outcome models*

### 3.2.1. *Minisum location problems*

The most common objective in SP is to optimize the mean outcome of the system; e.g., minimize the expected cost or maximize the expected profit. Cooper (1974) considers the Weber problem in which the locations of the demand points may be random. He assumes a bivariate normal distribution for these locations. The objective is to choose a point for the single facility location to minimize the expected demand-weighted distance to the customers. Cooper proves that the objective function is convex with respect to the location chosen and develops an iterative algorithm that solves the first-order conditions; the algorithm is shown to be globally convergent by Katz and Cooper (1974).

Sheppard (1974) was one of the first authors to propose a scenario approach to facility location. He suggests selecting facility locations to minimize the expected cost, although he does not discuss the issue at length. The first rigorous attempt to choose facility locations to minimize expected cost under scenario-based uncertainty was offered by Mirchandani and Oudjit (1980), who discuss the 2-median problem on a tree with stochastic edge lengths described by discrete scenarios. The objective is to minimize the expected demand-weighted distance. The authors first show that the stochastic l-median problem is equivalent to the deterministic problem, which can be solved in linear time using the algorithm proposed by Goldman (1971). They then derive analytical results for the 2-median problem indicating, for example, that the stochastic 1-median lies on the path connecting the two 2-medians if and only if the demands at the 1-median are served by the same 2-median in all scenarios. They suggest an algorithm based on partial enumeration.

Weaver and Church (1983) present a Lagrangian relaxation algorithm for the stochastic PMP on a general network discussed by Mirchandani and Odoni (1979). Their approach illustrates how some scenario-based stochastic problems can be treated simply as larger versions of the deterministic problem. We are given a set of scenarios, each of which specifies a realization of the demands and travel costs and has a fixed probability of occurrence. The objective is to minimize the expected travel cost, subject to the standard PMP constraints. Customers may be assigned to different facilities in different scenarios. (If customers must be assigned to the same facility in every scenario, the problem reduces to a deterministic PMP in which the uncertain parameters are replaced by their means.) Weaver and Church implicitly treat this problem with $n$ customers and $s$ scenarios as though it were a deterministic problem with $ns$ customers. They solve the model using the standard Lagrangian relaxation method for the PMP (Cornuejols *et al.*, 1977), relaxing the assignment constraints. The Lagrangian subproblem decouples by facility in the same way as does the subproblem for the classical PMP.

Mirchandani *et al.* (1985) begin with the same formulation as Weaver and Church (1983), explicitly reformulating it as a deterministic PMP with $ns$ customers, each corresponding to a customer-scenario pair in the original problem. Similar to Weaver and Church, they also suggest a Lagrangian relaxation method, but instead of relaxing the assignment constraints, they relax the single constraint requiring $P$ facilities to be opened. The resulting subproblem is equivalent to the UFLP in which all facilities have the same fixed cost equal to the single Lagrange multiplier. The authors solve this subproblem using the DUALOC algorithm of Erlenkotter (1978) and update the multiplier using a subgradient method. Since the UFLP does not have the integrality property, their Lagrangian bound is at least as strong as that in Weaver and Church (1983). Of course, this comes at a cost since the subproblem is more difficult to solve, but the authors prove that the procedure is guaranteed to converge to the optimal multiplier in no more than $n-1$ iterations. This model can be used to model multicommodity or multiobjective problems instead of multiscenario ones; the authors use the term "multidimensional" to describe this type of framework.

Louveaux (1986) presents stochastic versions of the Capacitated *P*-Median Problem (CPMP) and Capacitated Fixed-charge Location Problem (CFLP) in which demands, production costs, and selling prices are random. The goal is to choose facility locations, determine their capacities, and decide which customers to serve and from which facilities to maximize the expected utility of profit. Since demands are random and facilities are capacitated, the facilities chosen in the first stage may be insufficient to serve all of the demands in the second stage; hence a penalty for unmet demand is included in the models. To formulate the stochastic CPMP, the constraint requiring $P$ facilities to be opened is replaced by a budget constraint on the total cost; this constraint must be satisfied under any realization of the demand. The budget can be used to determine $P$. The author shows that under a particular type of budget constraint,

the two stochastic models (CFLP and CPMP) are equivalent. Louveaux and Peeters (1992) present a dual-based heuristic for Louveaux's CFLP model with scenario-based uncertainty, and Laporte *et al.* (1994) present an optimal algorithm based on the *L*-shaped method of SP. Louveaux (1993) reviews modeling approaches for these and related problems.

Ravi and Sinha (2004) develop an approximation algorithm for a stochastic version of the UFLP in which facilities may be opened in either the first or second stage, incurring different fixed costs in each. Their algorithm, which has a performance guarantee of eight, is based on the LP-rounding algorithm of Shmoys *et al.* (1997) for the deterministic UFLP.

Listeş and Dekker (2005) introduce an SP model for choosing facility locations in a reverse logistics network and apply it to a case study involving the collection, recycling, and reuse of sand from demolition sites in the Netherlands. A reverse logistics network differs from a traditional (forward) logistics network in that there are many more supply points than demand points. In this model, the amount and location of supply (of waste sand) is uncertain, as is the demand (for recycled sand). The authors formulate a three-stage mixed-integer stochastic programming model for maximizing the expected profit of this system. In the first stage, initial location decisions are made, then supply uncertainty is resolved. In the second stage, additional facilities are opened if necessary to satisfy capacity constraints. Then demand uncertainty is resolved and, in the third stage, product flow decisions are made. The model is formulated as a large-scale Mixed Integer Programming (MIP) problem and solved using CPLEX.

Chan *et al.* (2001) consider "stochastically processed demands" i.e., demands that arise from a queuing process at the customer. For example, a manufacturing plant uses raw materials stochastically and would like "just-in-time" replenishments as the raw material inventory is depleted. The authors develop a stochastic location-routing model for a special case of this problem in which demand probabilities are estimated using the queuing process at each site before optimization takes place. The objective is to minimize the expected cost. The problem is solved heuristically using stochastic decomposition (an extension of Benders decomposition) and space-filling curves. The authors discuss a case study in which they apply their model to a wartime medical evacuation problem using data supplied by the US Army.

A somewhat different approach is taken by Ricciardi *et al.* (2002), who consider a facility location model with random throughput costs at the Distribution Centers (DCs). The objective is to minimize the deterministic transportation cost (plant–DC and DC–customer) plus the expected throughput cost at the DCs. The authors first consider the network flow aspect of the problem (assuming the DC locations are given), developing a multinomial logit model for the expected flows. They then embed the expected cost model into a nonlinear integer program; this model is solved heuristically since for each candidate solution to the location problem, a Lagrangian problem must be solved to compute the expected flows.

Daskin *et al.* (2002) introduce a location–inventory model that minimizes the expected cost of locating facilities, transporting material, and holding inventory under stochastic daily demand. Each facility is assumed to follow a $(Q, R)$ inventory policy whose cost is approximated as the cost of an Economic Order Quantity (EOQ) policy plus that of safety stock (see, e.g., Nahmias (2004)). The model thus reduces to a deterministic model whose objective function is a function of the means and variances of the random parameters. The model's form is identical to that of the UFLP with two additional concave terms in the objective function representing cycle and safety stock. Daskin *et al.* (2002) solve this model using Lagrangian relaxation, whereas Shen *et al.* (2003) solve it using column generation.

Snyder *et al.* (2004) use an approach similar to that of Weaver and Church (1983) to solve a scenario-based stochastic version of the joint location–inventory model of Daskin *et al.* (2002), allowing demand means and variances themselves to be stochastic, as well as costs, lead times, and other parameters. The nonlinear objective function prevents the Lagrangian subproblem from decoupling by facility as in the Weaver–Church algorithm. Rather, it decouples by facility-scenario pair, and the scenarios are linked in a subsequent step of the algorithm.

### 3.2.2. *Location–transportation problems*

Balachandran and Jain (1976) present a capacitated facility location model with piecewise linear production costs that need not be either concave or convex. Demands are random and continuous, described by a joint probability distribution. There are penalty and holding costs for producing too little or too much relative to the realized demand. The objective is to minimize the expected cost of location, production, transportation, and underage and overage. The authors present a branch-and-bound algorithm in which the production functions are replaced by linear underestimates; the resulting problem is a capacitated convex transportation problem whose optimal solution can be used to obtain both lower and upper bounds on the optimal objective value for the original problem. Branching corresponds to partitioning the production cost function and estimating each portion separately.

LeBlanc (1977) considers a similar problem, but with linear production (or transportation) costs. For fixed facility locations, the problem reduces to the stochastic transportation problem (Williams, 1963). LeBlanc presents a Lagrangian heuristic, whereas França and Luna (1982) solve the problem optimally using generalized Benders decomposition. França and Luna also permit additional linear configuration constraints, for example to limit the number of facilities to be constructed.

A simpler but more tractable model is presented by Gregg *et al.* (1988), who minimize a weighted sum of the production cost (a one-time cost for establishing capacity), the transportation cost, and expected overage and underage costs. By varying the weights, the modeler can express different preferences among the objectives. The model is solved using MINOS and is illustrated by a case study involving the Queens Borough of New York City public library system.

In all of these problems, only first-stage decisions are available; there are no recourse decisions. Once production and transportation levels are set, they cannot be changed after the uncertainty is resolved. The objectives therefore include the expected recourse cost: the cost of holding and stockouts.

### 3.2.3. *Dynamic location problems*

Berman and Odoni (1982) study a single-facility location problem in which travel times are stochastic and the facility (e.g., an ambulance) may be relocated at a cost as conditions change. Travel times are scenario-based, and scenario transitions occur according to a discrete-time Markov process. The objective is to choose a facility location for each scenario to minimize expected transportation and relocation costs. The authors show that the Hakimi property applies to this problem and that the problem on a tree is equivalent to the deterministic problem; any scenario can be used to determine the optimal location since the 1-median on a tree is independent of the edge lengths. They then present a heuristic for the problem on a general network that involves iteratively fixing the location in all but one scenario and solving what amounts to a 1-median problem. They also discuss simple bounds on the optimal objective value of the multi-facility problem. Berman and LeBlanc (1984) introduce a heuristic for this problem that loops through the scenarios, performs local exchanges within each, and then performs exchanges to link the scenarios in an effort to reduce relocation costs.

Carson and Batta (1990) present a case study of a similar problem in which a single ambulance is to be relocated on the Amherst campus of SUNY Buffalo as the population moves about the campus throughout the day (from classroom buildings to dining halls to dormitories, etc.). Given the difficulties inherent in identifying probability distributions and estimating relocation costs in practice, Carson and Batta simply divide the day into four unequal time periods and solve a 1-median problem in each. Relocation costs are not explicitly considered, but the decision to use four time periods was arrived at in consideration of the tradeoff between frequent relocation and increased response times.

Another dynamic, stochastic facility location problem was studied by Jornsten and Bjorndal (1994), who choose where and when to locate facilities over time to minimize the expected time-discounted cost; production and distribution costs are random. Their algorithm uses scenario aggregation and an augmented Lagrangian approach. A closely related model is the capacity planning problem studied by Eppen *et al.* (1989). Although their model is not a facility location model, it and other capacity planning models have a similar flavor to facility location models since capacity expansion entails both an up-front fixed cost and an on-going processing cost. In their multiperiod model, demands and selling prices are random; capacity levels (for all time periods) are first-stage decisions and production levels are recourse decisions. Their model chooses capacity configurations at each plant in each time period, subject to a re-tooling cost for changing capacity. The objective is to maximize the time-discounted profit subject to a limit on Expected Downside Risk (EDR). Their algorithm involves successively tightening the EDR constraint and resolving, resulting in multiple solutions; the decision maker can choose among these solutions based on the tradeoff between expected profit and EDR. The formulation presented in the paper is solved by a general-purpose MIP solver but is very large, with as many as five subscripts on some parameters and variables, making it impractical for large problems.

An earlier stochastic capacity planning problem was considered by Manne (1961), who assumes that future demands follow a random walk with an upward trend. The objective is to minimize the expected discounted cost. He shows that it is sufficient to consider a deterministic problem in which the interest rate is replaced with one that depends on the demand variance. Bean *et al.* (1992) relax some of Manne's assumptions about the demand process and cost structures and show similar results using SP.

### 3.2.4. *Competitive location problems*

Ghosh and McLafferty (1982) introduce a model for locating multiple stores so as to maximize market share in a competitive environment with demand uncertainty (actually, uncertainty as to which stores a competitor plans to close, but in this setting they amount to the same thing). The authors discuss a model from the marketing literature for estimating market share given fixed store locations. The location model itself is formulated as a multiobjective model, with each objective representing the market-share-maximization objective in a given scenario. Ultimately, the objectives are combined into a weighted sum to be minimized. If the weights represent scenario probabilities, the objective is equivalent to minimizing the expected cost; otherwise, the weights can be adjusted systematically to find nondominated solutions (solutions for which no objective can be improved without degrading another objective). For a given set of weights, the problem is solved using an exchange heuristic. On a small sample problem, three noninferior solutions were found, and the authors provide some discussion as to how to choose among them.

A similar model introduced by Ghosh and Craig (1983) uses competitive equilibrium theory to estimate the expected profit to be enjoyed by a given facility location in a competitive and uncertain environment. Uncertainty

comes both from changing customer demographics and from the actions of the firm's competitors. After the authors estimate the expected profit, they solve a single-facility location problem heuristically.

De Palma *et al.* (1989) study a multifirm competitive facility location with random consumer utilities. A consumer's utility for firm $i$ is expressed as a constant $a_i$ (the mean utility for the firm) minus the distance from the consumer to the firm's nearest facility minus a random error term. After choosing its maximum-utility firm, each consumer will choose the nearest facility within that firm. Firm $i$ will open $m_i$ facilities to maximize its expected sales (market share). The authors prove that if the $m_i$-median solution is unique for all $i$ and if the consumers' tastes are sufficiently diverse, then there exists a unique location equilibrium, and in that equilibrium firm $i$ locates its facilities at the $m_i$-median solution. The problem therefore reduces to solving a separate PMP for each firm.

### 3.2.5. *Multiechelon facility location problems*

Because of the difficulty in solving stochastic facility location problems, research on more complex multiechelon location models under uncertainty has only begun to appear in the literature in the past 5 to 10 years. Many of these models, which are often referred to as supply chain network design models, may be viewed as stochastic extensions of the seminal model by Geoffrion and Graves (1974). The objective in these papers is generally to optimize the expected outcome; to the best of our knowledge, no published research has tackled multiechelon facility location under the more difficult robust objectives discussed in Section 4.

A qualitative discussion of global supply chain design is given by Vidal and Goetschalckx (1997, 2000); the latter paper also presents a large-scale MIP for choosing plant locations and suppliers that incorporates the suppliers' reliability into the constraints.

MirHassani *et al.* (2000) formulate a supply chain network design problem as a stochastic program with fixed recourse; the SP has binary first-stage variables and continuous second-stage variables. The objective function coefficients are deterministic; uncertainty is present only in the right-hand sides of the recourse constraints, which may represent, for example, demands or capacities. The authors focus especially on parallel implementation issues for their proposed Benders decomposition algorithm.

Tsiakis *et al.* (2001) consider a multiproduct, multiechelon supply chain under scenario-based demand uncertainty. The goal is to choose middle-echelon facility locations and capacities, transportation links, and flows to minimize expected cost. Transportation costs are piecewise linear concave. The model is formulated as a large-scale MIP and solved using CPLEX.

Alonso-Ayuso *et al.* (2003) introduce a more general model that makes decisions about plant capacities, product mix, vendor selection, and assignments of products to plants. The problem is formulated as a multi-stage SP. First-stage variables are binary and represent capacity, product mix, and sourcing decisions. Decisions in subsequent stages are tactical (production, inventory, transportation, etc.) and are represented by continuous variables. The proposed solution algorithm is based on "branch-and-fix coordination"; it was applied with mixed success to moderate-size test problems.

Whereas the preceding three papers consider scenario-based uncertainty, Santoso *et al.* (2003) study a global supply chain network design problem with continuously random parameters, and thus an infinite number of scenarios (their technique, however, can also be used in problems with a large but finite number of scenarios). Costs, demands, and capacities are random. The problem is to decide where to build facilities and what machines to build at each facility to minimize the total expected cost, which includes a shortfall penalty in case the constructed capacity is insufficient to meet the realized demand. The problem is formulated as a two-stage SP with binary first-stage variables and continuous recourse variables and is solved using accelerated Benders decomposition. For each candidate location vector, the expected second-stage cost is estimated using sample average approximation, which estimates the expected cost using a random sample of the uncertain parameters.

Finally, Butler *et al.* (2004) present a model for designing the supply chain for a new product launch under uncertain costs, demands, capacities, and other parameters. Their objective function incorporates both expected profit and regret; it can, however, be viewed as a mean-outcome-type objective with some scenarios weighted more than others based on their optimal objective function values. They also explore the relationship between mean outcome and some of the robustness measures discussed in Section 4. They illustrate their model using a case study involving data from a large consumer electronics company.

### 3.3. *Mean–variance models*

The mean-outcome models discussed above consider only the expected performance of the system, ignoring the variability in performance and the decision maker's possible risk aversion. However, a small body of literature has incorporated the firm's level of risk aversion into the decision-making process, typically by using a mean–variance objective function. Jucker and Carlson (1976) use such an objective in a stochastic formulation of the UFLP in which selling price (and hence demand) may be random. They develop solution methods for four types of firms, all risk averse, characterized by which variables (e.g., price) they set and which others (e.g., demand) they accept as a result. Hodder and Jucker (1985) extend the Jucker–Carlson model to allow for correlation among the random prices. Their model is a quadratic programming model but can be solved easily. Hanink (1984) and Hodder (1984) incorporate the capital asset pricing model (CAPM) into facility location problems and compare it to mean–variance objectives.

Hodder and Dincer (1986) consider the location of capacitated facilities globally under exchange rate uncertainty. The model incorporates the financing aspects of plant construction by endogenously deciding how much of each plant's total cost to borrow from each country; the per-period cost of this financing is a random variable since the exchange rates are uncertain. In addition, costs and per-unit profits are uncertain. The model maximizes a mean–variance expression concerning the total profit. This objective is quadratic and involves a large variance–covariance matrix, each off-diagonal term of which requires a bilinear term in the objective function. Therefore, the authors propose an approximation scheme that effectively diagonalizes the variance–covariance matrix so that the objective function contains only squared terms and no bilinear terms. The resulting model is solved using an off-the-shelf quadratic programming solver for small problems and using a gradient search method for larger ones. No discussion is provided concerning the form of uncertainty (discrete or continuous) or the probability distributions governing it, but in theory any approach could be used as long as the random parameters can be expressed adequately in the form needed for the approximation.

Verter and Dincer (1992) review the literature on stochastic facility location and capacity expansion problems, focusing on global manufacturing problems.

Velarde and Laguna (2004) formulate an extension of the robust international sourcing model of Gutiérrez and Kouvelis (1995) (which is discussed further in Sections 3.5 and 4.2.3) in which demands and exchange rates are stochastic. The objective function minimizes a weighted sum of the expected cost and a variance term that includes only positive deviations from the expected cost. The authors propose a Benders-based heuristic in which cuts are generated via tabu search using neighborhoods that are defined based on the dual variables from the subproblems.

### 3.4. *Probabilistic approaches*

In contrast to the models discussed above, which consider the expected value and/or variance of the stochastic objective function, there is a substantial body of literature that considers probabilistic information about the performance of the system; for example, maximizing the probability that the performance is good or constraining the probability that it is bad, under suitable definitions of "good" and "bad". We discuss three such approaches: (i) max-probability locations; (ii) chance-constrained programming; and (iii) distribution maps.

#### 3.4.1. *Max-probability locations*
Probably the first attempt to solve stochastic location problems of any type is that of Frank (1966), who considers probabilistic centers and medians on a network with arbitrarily distributed independent random demands. He presents methods for finding "max-probability" centers (points that maximize the probability that the maximum-weighted distance from the point is within a given limit) and medians (points that maximize the probability that the total demand-weighted distance from the point is within a given limit). Finding the max-probability center is equivalent to a nonlinear minimization problem, which Frank shows to be easy if the demand distribution is discrete. For medians, he approximates the demand distribution using the normal distribution; the problem is then easy. Frank proves that if the probability distributions are not known but are estimated using sampling, the resulting centers and medians are "good enough" in some sense. He later extends his analysis to jointly distributed normal demands (Frank, 1967).

Berman and Wang (2004) revisit Frank's problem, finding max-probability medians and anti-medians for a network with independent uniformly distributed demands (an anti-median is a location that *maximizes* the demand-weighted distance to the customers). For a given location, the relevant probability is difficult to calculate; the authors suggest both an exact and an approximate method for the calculations. The optimal location is then found using an enumerative approach. Berman *et al.* (2003a) consider the same problems using center-type objectives instead of median-type ones. They, too, propose an enumerative approach. As the Hakimi property does not apply in either of these papers, the optimal solutions may occur on the links or the nodes. Enumeration is possible because the links can be divided into regions on which the objective function is constant; finding the optimal solution thus reduces to enumerating all $O(n^3)$ of these regions. The same probabilistic center problem is considered on the plane by Berman *et al.* (2003b). The authors prove that the objective function is convex under a certain assumption about the uniform distribution parameters and present an exact algorithm, based on dual ascent, for this case. If the assumption does not hold, the authors propose using the dual-ascent algorithm as a heuristic and embedding it into a branch-and-bound scheme based on the triangulation method introduced by Drezner and Suzuki (2004). Problems with 100 nodes or so can be solved in negligible time using Excel's solver.

#### 3.4.2. *Chance-constrained programming*
Chance-constrained programming involves requiring the probability of a certain constraint holding to be sufficiently high. For example, Carbone (1974) chooses $P$ facilities under normally distributed, possibly correlated, demands to minimize an upper bound on the total demand-weighted distance that can be achieved with probability $\alpha$; that is, to minimize $K$ such that $\Pr\left(\sum_i \sum_j h_i d_{ij} Y_{ij} \leq K\right) \leq \alpha$, where $h_i$ is the (random) demand of customer $i$, $d_{ij}$ is the distance from facility $j$ to customer $i$, $Y_{ij}$ has a value of one if customer $i$ is assigned to facility $j$, and $0 \leq \alpha \leq 1$ is a constant. Carbone transforms this problem into its deterministic equivalent, which is a convex minimization problem.

Shiode and Drezner (2003) use a similar approach for a competitive facility location problem on a tree. There are two players, each of whom locates a single facility in turn. Demands are stochastic to the leader but deterministic to the follower. The objective of the paper is to characterize both players' optimal solutions. In the deterministic version, the follower's problem turns out to be a median-type problem whereas the leader's problem is a center-type problem. In the stochastic version, the leader's problem is to choose a location to maximize $K$ such that the probability that the follower's market share is no more than $K$ is at least $\alpha$. (The follower's problem is the same as in the deterministic version.) The authors show that the Hakimi property applies to the leader's problem and propose an efficient solution method that involves solving a one-dimensional unconstrained convex minimization problem using a line search for each node, then choosing the best node.

### 3.4.3. *Distribution maps*

Whereas max-probability models attempt to choose a location that maximizes the probability of optimality, the papers discussed in this section evaluate the probability that any given solution will be optimal once the uncertainty is resolved. A complete description of the probability that any point is optimal is called a "distribution map." Wesolowsky (1977) first used the distribution map concept in facility location, considering the Weber problem on a line with demands whose weights are multivariate normal. He shows that only the demand nodes have a nonzero probability of being optimal, a result analogous to the Hakimi property. He also computes the Expected Value of Perfect Information (EVPI): the difference between the expected cost of the optimal solution and the expected cost if the scenario could always be predicted perfectly.

Drezner and Wesolowsky (1981) extend Wesolowsky's model to include general $l_p$ distances on the plane. They present a procedure for finding approximate probabilities under the rectilinear metric ($l_1$-norm), and another for the general $l_p$-norm when locations are solely restricted to the nodes. Drezner and Wesolowsky (1980) compute the EVPI in the same setting. They show that the EVPI can be separated into individual one-dimensional EVPIs and derive its general expression. They develop exact expressions for the rectilinear case and approximate expressions for the Euclidean case. In addition, they use simulation to show that if the demands are lognormal instead of normal, the expressions are still approximately correct.

### 3.5. *Evaluating options*

Constructing facilities in several countries gives a firm a degree of operational flexibility since it can shift production to countries with favorable exchange rates, local costs, labor availability, etc. Most papers that discuss this strategy are concerned with evaluating the "option value" of a set of global facility locations (generally quite a complex task)

rather than with choosing the optimal set. One can think of these papers as computing the objective value for a given solution to a location problem, as opposed to finding the optimal solution.

Huchzermeier and Cohen (1996) evaluate operational options over multiple time periods under uncertainty in exchange rates. They build a Markov model of exchange rates and solve a supply chain design problem for each scenario, then use stochastic dynamic programming to determine the value of each option. Kogut and Kulatilaka (1994) similarly use dynamic programming to evaluate options when there is a cost for switching production between sites. They discuss the threshold at which switching becomes advantageous and make the observation that the model favors countries with volatile exchange rates since they provide greater opportunity to take advantage of fluctuations. This counter-intuitive result illustrates the difference between financial and operational hedging: whereas financial hedging seeks to eliminate volatility, operational hedging seeks to exploit it. This issue is discussed in a more qualitative setting by Kogut (1985). Lowe *et al.* (2002) provide a decision-analysis approach for the operational hedging concept, illustrating its use with a popular Harvard Business Review case.

Gutiérrez and Kouvelis (1995) present a model to choose suppliers internationally to hedge against changes in exchange rates and local costs. The model reduces to a robust version of the UFLP with uncertain costs. Unlike the models discussed in the preceding paragraph, which are descriptive, the Gutiérrez-Kouvelis model is normative, although necessarily less rich than the descriptive models. We discuss their model further in Section 4.2.3.

## 4. Robust location problems

When no probability information is known about the uncertain parameters, the expected cost and other objectives discussed in Section 3 are irrelevant. Many measures of robustness have been proposed for this situation. The two most common are minimax cost and minimax regret, which are closely related to one another and are discussed in Section 4.1. Other less-common measures are discussed in Section 4.2. Several of these robustness measures are discussed in the text on robust optimization by Kouvelis and Yu (1997); however, they use somewhat different terminology than we do.

As in the stochastic optimization case, uncertain parameters in robust optimization problems may be modeled as being either discrete or continuous. Discrete parameters are modeled using the scenario approach. Continuous parameters are generally assumed to lie in some prespecified interval, since it is often impossible to consider a "worst-case scenario" when parameter values are unbounded. We will describe this type of uncertainty as "interval uncertainty" and describe parameters modeled this way as "interval-uncertain" parameters. Some authors use the

term "scenario" in the continuous case to refer to a particular realization of the uncertain parameters.

For a given problem under uncertainty with no probability information, the *minimax cost solution* is the one that minimizes the maximum cost across all scenarios. This measure is, on the one hand, overly conservative, emphasizing the worst possible scenario, and on the other hand, somewhat reckless, since it may produce quite poor solutions for scenarios other than the one with maximum cost, especially if the scenarios have a form such as "small demand/moderate demand/large demand". Minimax cost may be an appropriate measure for situations in which it is critical for the system to function well even in the worst case: for example, the location of fire stations (Serra and Marianov, 1998) or the case where a firm's competitors are likely to make decisions that make the worst scenario come to pass for the firm.

The other two most common robustness measures consider the *regret* of a solution, which is the difference (absolute or percentage) between the cost of a solution in a given scenario and the cost of the optimal solution for that scenario. Regret is sometimes described as opportunity loss: the difference between the quality of a given strategy and the quality of the strategy that would have been chosen had one known what the future held. Models that seek to minimize the maximum (absolute or relative) regret across all scenarios are called *minimax* (*absolute* or *relative*) *regret* models. Minimax cost problems can often be transformed into equivalent minimax regret problems, and *vice versa*, since the cost and regret of a given scenario differ only by a constant. Solution approaches for one criterion are often also applicable to the other one.

The primary attraction of minimax measures is that they do not require the planner to estimate scenario probabilities, or even to formulate scenarios if data are described using intervals. However, minimax cost and regret seem to be employed more in the academic literature than in practice. In many situations, it is more practical to plan based on a fractile target than on the worst case. For example, a hospital might be sized to meet all demand 95% of the time at the risk of turning away patients during extreme catastrophes. Of course, if the worst-case scenarios can be identified *a priori*, these can simply be excluded from the model.

Although the regret criterion is usually applied in uncertainty situations, in which no probability information is known, it has also been discussed in the context of risk situations. However, minimizing expected regret is equivalent to minimizing expected cost. To see this, consider a general min-expected-absolute-regret problem with variables $x_1, \ldots, x_n$, feasible set $X$, scenarios $s \in S$, objective function coefficients $c_{is}$, scenario probabilities $q_s$, and optimal scenario objective values $z_s^*$:

$$\min \qquad \sum_{s \in S} q_s R_s, \qquad (1)$$

$$\text{subject to} \qquad R_s = \sum_{i=1}^{n} c_{is} x_i - z_s^* \qquad \forall s \in S, \quad (2)$$

$$x \in X. \qquad (3)$$

Substituting the regret variables $R_s$ into the objective function, we get:

$$\min \qquad \sum_{s \in S} q_s \left( \sum_{i=1}^{n} c_{is} x_i - z_s^* \right), \qquad (4)$$

$$\text{subject to} \qquad x \in X. \qquad (5)$$

The objective function of this revised problem is the min-expected-cost objective function minus a constant. The relative regret case is similar. This equivalence is sometimes overlooked in the literature.

### 4.1. *Minimax models*

Minimax regret models are commonly employed in the literature. Generally such problems are solved using problem-specific algorithms; however, general-purpose algorithms for minimax regret linear programs with interval-uncertain objective function coefficients were proposed by Mausser and Laguna (1998) for problems with absolute regret and by Mausser and Laguna (1999a) for problems with relative regret. The algorithms rely on the fact that for a given solution, each uncertain parameter is set either to its lower or its upper endpoint in the regret-maximizing scenario. To identify this scenario, the authors solve a MIP that adds one binary variable and a few constraints to the original model for each uncertain parameter. This approach is practical for small- to moderate-size LPs. Mausser and Laguna (1999b) propose a greedy heuristic for the absolute regret problem that contains some methods for diversification to avoid local optima. Its results can be used on their own or in place of the exact solution to the MIP formulated by Mausser and Laguna (1998).

The general strategy of the algorithm in many minimax regret papers can be described as follows:

*Step 1.* Choose a candidate solution $x$.
*Step 2.* Determine the maximum regret across all scenarios if solution $x$ is chosen. For discrete scenarios, this is easy: just compute the cost of the solution under each scenario and compare it to the optimal cost for the scenario, then choose the scenario with the greatest regret. For interval uncertainty, techniques for finding the regret-maximizing scenario rely on the fact that this scenario typically has each parameter set to an endpoint of its interval. Still, the problem of identifying this scenario can be quite difficult. Solving this problem is the crux of the algorithms by Mausser and Laguna (1998, 1999a, 1999b) discussed above. On the other hand, Averbakh and Berman (2000a) develop an $O(n^2)$ algorithm to determine the

regret-maximizing scenario for their problem, the 1-median on a general network.

*Step 3.* Either repeat Steps 1 and 2 for all possible solutions (as in Averbakh and Berman (2000a)), or somehow find a new candidate solution whose regret is smaller than the regret determined in Step 2 (as in Mausser and Laguna (1998, 1999a)).

Regret-based problems tend to be more difficult than stochastic problems because of their minimax structure. In fact, many deterministic problems that are polynomially solvable have robust versions that are not. For example, the EOQ model is still easy in its robust form (Yu, 1997), but the minimax regret shortest path problem is NP-hard (Yu and Yang, 1998). As a result, many papers restrict their attention to problems with some special structure, such as 1-median problems or *P*-medians on tree networks. The focus in such papers is generally to develop analytical results or polynomial-time algorithms. Kouvelis and Yu (1997) provide a detailed analysis of many of these problems. We review papers on specially structured problems in Section 4.1.1. Then, in Section 4.1.2, we examine more general problems, for which heuristics are generally required.

### 4.1.1. *Specially structured problems*

Given the difficulties introduced by minimax objectives, analytical results and exact algorithms for minimax location problems have generally only been attained for special cases such as locating a single facility or locating facilities on specialized networks. These models are unlikely to be applied directly in practice. Instead, the goal of this research is to provide a foundation for the further study of minimax problems, as well as to investigate interesting combinatorial aspects of these problems.

Chen and Lin (1998) consider the minimax-regret 1-median problem on a tree with interval-uncertain edge lengths and node weights. As with many minimax problems, the Hakimi property does not apply to this problem. Chen and Lin present an $O(n^3)$ algorithm. They require node weights to be positive; however, Burkard and Dollani (2001) present a polynomial-time algorithm for the case in which node weights may be positive or negative. Vairaktarakis and Kouvelis (1999) similarly consider 1-medians on a tree, but in their problem, edge lengths and node weights may be linear over time (i.e., not stochastic but deterministic and dynamic) or random and scenario based. They trace the path of the solution over time (in the dynamic case) and present low-order polynomial algorithms for both cases.

The problem on a general network is significantly harder. In fact, Averbakh (2003a) proves that if edge weights are interval-uncertain, then both the 1-median and weighted 1-center problems on a general network are NP-hard. (The (deterministic) weighted *P*-center problem is to locate *P* facilities to minimize the maximum weighted distance traveled by any customer to its nearest facility.) His proof involves reducing COVER to the problems under considera-

tion. Both problems are polynomially solvable if only the node weights are uncertain (Averbakh and Berman, 1997b, 2000a).

Averbakh and Berman (2000a) consider the minimax-regret 1-median problem on a general network with interval-uncertain node weights. They present the first polynomial-time algorithm for this problem, with complexity $O(mn^2 \log n)$, where $m$ is the number of edges and $n$ is the number of nodes. For the nodal problem, in which facilities may only be located on the nodes, they present an $O(n^3)$ algorithm. They also consider the problem on a tree (with node or edge locations), reducing the known complexity of that problem to $O(n^2)$. The same authors later reduced the complexity of the tree problem further to $O(n \log^2 n)$ (Averbakh and Berman, 2003).

Minimax-regret center problems are even more difficult than median problems since center problems already have a minimax structure in their deterministic forms. Averbakh and Berman (2000b) present an $O(n^6)$ algorithm for the minimax-regret 1-center problem on a tree with interval-uncertain node weights and edge lengths. Part of the reason for the high order of complexity even in this restrictive setting is that determining the maximum regret for a given solution is nontrivial. They present an $O(n^2 \log n)$ algorithm for the problem with unweighted nodes. The complexities were reduced to $O(n^3 \log n)$ and $O(n \log n)$ for the weighted and unweighted cases, respectively, by Burkard and Dollani (2002), whose algorithms are based on computational geometry and make use of the results presented by Averbakh and Berman (2000b) that an optimal edge (although not an optimal location on it) can be found in $O(n^2 \log n)$ time.

Labbé *et al.* (1991) consider the 1-median problem on a network with uncertain node weights. We are given an estimate for each weight, as well as a range on the possible perturbation for each weight; in other words, node weights are interval uncertain, but we also have an estimate of them, possibly the midpoint of the interval. Given an optimal solution $x^*$ for the deterministic problem, the authors define the "degree of optimality for tolerance $\tau$", denoted $\alpha^*(\tau)$, as the maximum possible difference between the cost of $x^*$ under a perturbed objective and the optimal objective value under the perturbed objective, for all perturbations of size no more than $\tau$ for each uncertain weight. In other words, $\alpha^*(\tau)$ is the maximum regret over all "scenarios" in which the weights differ from their estimates by no more than $\tau$. The authors are not interested in finding an $x$ to minimize this regret measure, but rather in performing a sensitivity analysis by characterizing the tradeoff between the uncertainty in the weights ($\tau$) and the worst-case performance of the optimal solution ($\alpha^*(\tau)$). This analysis can be performed for any solution, not just $x^*$.

Carrizosa and Nickel (2003) consider a measure closely related to that of Labbé *et al.* (1991). Rather than defining a solution's robustness based on how much the cost changes for a given magnitude of change in the parameters, Carrizosa and Nickel define robustness as the minimum

magnitude of the parameter perturbation required to violate some desired limit on the cost. If the parameters can change substantially from their estimates without violating the cost cap, the solution is very robust. They consider this measure in the context of the Weber problem with uncertain demands, proposing an iterative solution method that solves, at each iteration, a nondifferentiable concave maximization problem.

### 4.1.2. *General problems*

We now turn our attention to multiple-facility problems on general networks under minimax objectives. Naturally, these problems are significantly harder than those discussed in Section 4.1.1, and as a result, they are usually solved heuristically. To our knowledge, the only papers to present analytical results for such problems are those of Averbakh and Berman (1997a) and Averbakh (2003a). Averbakh and Berman (1997a) consider the minimax-regret weighted *P*-center problem on a general network with interval-uncertain demands. They show that the minimax regret problem can be solved by solving $n + 1$ deterministic weighted *P*-center problems: $n$ of them on the original network and one on an augmented network, where $n$ is the number of nodes in the problem. Since the weighted *P*-center problem can be solved in polynomial time for the special cases in which $P = 1$ or the network is a tree, this leads to a polynomial-time algorithm for the minimax problem in these special cases.

Averbakh (2003b) extends this concept to more general robust (minimax) combinatorial optimization problems whose deterministic forms already have minimax objectives, including certain facility location, scheduling, and other combinatorial problems. Robustness may be defined as the minimax cost or minimax absolute or relative regret, and uncertainty is described by intervals. As in Averbakh and Berman (1997a), the main idea is to reduce the problem to a series of deterministic minimax problems. Averbakh proves three main results: (i) the minimax cost problem can be solved by setting all uncertain parameters to their upper bounds and solving the resulting deterministic problem; (ii) the minimax absolute and relative regret problems can be solved by solving $m$ deterministic problems, each of which involves setting one parameter to its upper bound and the others to their lower bounds, plus one more deterministic problem ($m$ is the number of uncertain parameters); and (iii) a polynomial-time algorithm for the deterministic problem implies a polynomial-time algorithm for the minimax cost and minimax relative regret problems, but not necessarily for the minimax absolute regret problem. However, since absolute regret problems can often be transformed into relative regret problems, a polynomial-time algorithm for the relative regret problem will often imply one for the absolute regret problem.

More commonly, heuristic approaches are used to solve these general problems. Serra *et al.* (1996) solve the maximum capture problem (to locate *P* facilities in order to capture the maximum market share, given that the firm's competitors have already located their facilities) under scenario-based demand uncertainty. They consider both maximizing the minimum market share captured (the maximization analog of the "minimax cost" criterion) and minimizing maximum regret. They present a heuristic that involves solving the deterministic problem for each scenario, choosing an initial solution based on those results, and then using an exchange heuristic to improve the solution. A similar approach is used by Serra and Marianov (1998), who solve the minimax cost and minimax regret problems for the PMP, also under scenario-based demand uncertainty. They present a case study involving locating fire stations in Barcelona. Heuristics for minimax-regret versions of the PMP and UFLP are also discussed by Snyder and Daskin (2005); this paper will be discussed further in Section 4.2.3. Current *et al.* (1997) present a model in which facilities are located over time, but the number of facilities that will ultimately be located is uncertain. They call their model NOFUN ("Number of Facilities Uncertain"). The approach to uncertainty is scenario based (scenarios dictate the number of facilities to open), and the authors discuss minimizing either expected or maximum regret. Their proposed formulation is based on the PMP and is solved using a general-purpose MIP solver.

## 4.2. *Other robustness measures*

We now discuss several other robustness measures in roughly chronological order. Most of these measures have been applied to facility location problems; for those that have not, we provide examples from the broader logistics literature.

### 4.2.1. *Robustness and stability*

One of the earliest robustness measures was proposed by Gupta and Rosenhead (1968) and Rosenhead *et al.* (1972). In these papers, decisions are made over time, and a solution is considered more robust if it precludes fewer good outcomes for the future. An example in the latter paper concerns a facility location problem in which a firm wants to locate five facilities over time. Suppose all possible five-facility solutions have been enumerated, and $N$ of them have cost less than or are equal to some prespecified value. If facility $j$ is included in $p$ of the $N$ solutions, then its "robustness" is $p/N$. One should construct the more robust facilities first, then make decisions about future facilities as time elapses and information about uncertain parameters becomes known. Now suppose that the first facility has been constructed and the firm decides (because of budget, politics, shrinking demand, etc.) not to build any of the other facilities. The "stability" of a facility is concerned with how well the facility performs if it is the only one operating. Stability should be used to distinguish among facilities that are nearly equally robust. Note that these definitions of robustness and stability refer to individual facilities, not

to solutions as a whole. An obvious disadvantage of this robustness measure is that computing it requires enumerating all possible solutions, which is generally not practical.

Schilling (1982) presents two location models that use this robustness measure (although not explicitly), both of which use stochastic, scenario-based demands. The first model is a set-covering-type model that maximizes the number of facilities in common across scenarios subject to all demands being covered in all scenarios and a fixed number of facilities being located in each scenario. By varying this last parameter, one can obtain a tradeoff curve between the total number of facilities constructed and the number of facilities that are common across scenarios. If the firm is willing to build a few extra facilities, it may be able to substantially delay the time until a single solution must be chosen, since the common facilities can be built first. The second model is a max-covering-type model that maximizes the coverage in each scenario subject to the number of common facilities exceeding some threshold. In this case the tradeoff curve represents the balance between demand coverage and common facilities. Unfortunately, Schilling's models were shown by Daskin *et al.* (1992) to produce the worst possible results in some cases. For example, imagine a firm that wants to locate two DCs to serve its three customers in New York, Boston, and Seattle. New York has either 45% or 35% of the demand and Boston has 35% or 45% of the demand, depending on the scenario. The remaining 20% of the demand is in Seattle, in either scenario. If the transportation costs are sufficiently large, the optimal solution in scenario 1 is to locate in New York and Seattle, whereas the optimal solution in scenario 2 is to locate in Boston and Seattle. Schilling's method would instruct the firm to build a DC in Seattle first, since that location is common to both solutions, then wait until some of the uncertainty is resolved before choosing the second site. However, then all of the east-coast demand is served from Seattle for a time, a suboptimal result.

Rosenblatt and Lee (1987) use a similar robustness measure to solve a facility layout problem. Unlike the measure in Rosenhead *et al.* (1972), which considers the percentage of good solutions that contain a given *element* (e.g., facility), Rosenblatt and Lee consider the percentage of scenarios for which a given *solution* is "good", i.e., has its regret bounded by some prespecified limit. Like the previous measure, the Rosenblatt–Lee measure requires enumerating all solutions and evaluating each solution under every scenario, making this approach practical only for very small problems.

### 4.2.2. *Sensitivity analysis*

Sensitivity analysis does not provide an optimization criterion but is used in an evaluative context. For example, Hodgson (1991) uses simulation to estimate the relative regret of the deterministic $P$-median solution when the demands and distances are randomly perturbed by scaling each data element by a normally distributed percentage. He finds that the optimal $P$-median solution is relatively insensitive to errors in the distances and especially to errors in the demands. Labbé *et al.* (1991) provide a more theoretical sensitivity analysis for the 1-median problem with random demands. The goal is to derive a tradeoff between the deterministic solution's maximum regret and the size of the perturbations. This paper was discussed in detail in Section 4.1.1.

Rather than evaluating the sensitivity of the deterministic solution to changes in the data, Cooper (1978) considers the sensitivity of the problem itself by evaluating the minimum and maximum possible cost of the optimal solution over all possible scenarios, assuming implicitly that a solution can be chosen after the uncertainty has been resolved. As in Cooper (1974) (discussed in Section 3.2.1), Cooper (1978) considers the Weber problem when the locations of the demand points are not known. The difference is that whereas Cooper (1974) assumes a probability distribution on the demand points, Cooper (1978) assumes only that the points lie within certain "uncertainty circles". He argues that the minimum (maximum) cost can be found by solving the deterministic problem and subtracting (adding) the sum of the circle radii from the optimal cost. However, Juel (1980) points out an error in Cooper's proof and Juel (1981) shows that Cooper's result holds for the minimum (maximum) cost problem only if the optimal solution does not lie within (at the center of) any of the uncertainty circles. Drezner (1989) provides an asymptotic analysis for the Weber problem on a sphere with random customer locations; his main result is that as the number of demand points approaches infinity, the relative difference between the minimum and maximum costs approaches zero.

### 4.2.3. *p-robustness*

Rather than minimizing regret, several papers place constraints on the maximum regret that may be attained by the solution. This idea was first used by Kouvelis *et al.* (1992), who impose a constraint dictating that the relative regret in any scenario must be no greater than $p$, where $p \geq 0$ is an external parameter. In other words, the cost under each scenario must be within $100(1 + p\%)$ of the optimal cost for that scenario. Snyder and Daskin (2005) refer to this measure as "$p$-robustness". For small $p$, there may be no $p$-robust solutions for a given problem. Thus, $p$-robustness adds a feasibility issue not present in most other robustness measures.

The problem considered by Kouvelis *et al.* (1992) is a facility layout problem in which the goal is to construct a list of $p$-robust solutions, if any exist. The approach used is heuristic in the sense that, although it optimally solves the layout problem for each scenario, there is no guarantee that the resulting list of $p$-robust solutions is exhaustive. The $p$-robustness criterion is also used by Gutiérrez and Kouvelis (1995) in the context of an international sourcing problem. They present an algorithm that, for a given $p$ and $N$, returns either all $p$-robust solutions (if there are fewer than $N$ of them) or the $N$ solutions with smallest maximum

regret. The sourcing problem involves choosing suppliers worldwide so as to hedge against changes in exchange rates and local prices. The problem reduces to the UFLP, so the authors are essentially solving a *p*-robust version of the UFLP. Their algorithm maintains separate branch-and-bound trees for each scenario, and all trees are explored and fathomed simultaneously. Unfortunately, their algorithm contains an error that makes it return incomplete, and in some cases incorrect, results (Snyder, 2005). Gutiérrez *et al.* (1996) use Benders decomposition to search for *p*-robust solutions to the uncapacitated network design problem. For each scenario, the authors solve a separate network design problem; these problems are linked by feasibility cuts that are added simultaneously to all problems.

The three models discussed in the preceding paragraph attempt to find *p*-robust solutions but do not provide a way to differentiate among those found, which may be numerous. Snyder and Daskin (2004) address this issue by combining the min-expected-cost and *p*-robustness measures for the PMP and UFLP: the goal is to find the minimum-expected-cost solution that is *p*-robust. They solve their models using variable splitting (or Lagrangian decomposition). Their method can be used as a heuristic for the minimax-regret PMP or UFLP by systematically varying *p* until one obtains the smallest *p* for which the problem is feasible. However, finding a feasible solution, or even determining whether the problem is feasible, can be difficult, especially for small *p*. Therefore, the authors discuss a mechanism for detecting infeasibility that involves testing the Lagrangian lower bound against an *a priori* upper bound.

### 4.2.4. *Model and solution robustness*

Mulvey *et al.* (1995) introduce a framework for Robust Optimization (the "RO framework") that involves two types of robustness: (i) "solution robustness" (the solution is nearly optimal in all scenarios); and (ii) "model robustness" (the solution is nearly feasible in all scenarios). The definition of "nearly" is left up to the modeler; their objective function has very general penalty functions for both model and solution robustness, weighted by a parameter intended to capture the modeler's preference between the two. The solution robustness penalty might be the expected cost, maximum regret, or von Neumann-Morgenstern utility function. The model robustness penalty might be the sum of the squared violations of the constraints. Uncertainty may be represented by scenarios or intervals, with or without probability distributions. The authors discuss a number of applications to which the RO framework has been applied. In one example, a power company wants to choose the capacities of its plants to minimize cost while meeting customer demand and satisfying certain physical constraints. In the RO model for this problem, the objective function has the form:

$$\min E(\text{cost}) + \lambda \text{Var}(\text{cost})$$
$$+ \omega(\text{sum of squares of infeasibilities}).$$

The first two terms represent solution robustness, capturing the firm's desire for low costs and its degree of risk aversion, whereas the third term represents model robustness, penalizing solutions that fail to meet demand in a scenario or violate other physical constraints such as capacity. The RO framework has explicitly been employed in applications as varied as parallel-machine scheduling with stochastic interruptions (Laguna *et al.*, 2000), the relocation of animal species under uncertainty in population growth and future funding (Haight *et al.*, 2000), production planning (Trafalis *et al.*, 1999), large-scale logistics systems (Yu and Li, 2000), and chemical engineering (Darlington *et al.*, 2000).

Killmer *et al.* (2001) use the RO framework to find solution- and model-robust solutions to a stochastic noxious facility location problem. (Although the authors discuss their model solely in the context of noxious facility location, it is similar to the UFLP and so could be applied to much more general problems.) The RO model for this problem minimizes the expected cost plus penalties for regret, unmet demand, and unused capacity. The expected cost and regret penalty are the solution robustness terms (encouraging solutions to be close to optimal), whereas the demand and capacity variation penalties are model robustness terms (encouraging solutions to be close to feasible). The nonlinear programming model is applied to a small case study involving the location of hazardous waste treatment facilities in Albany, NY and is solved using MINOS.

### 4.2.5. *Restricting outcomes*

One use of the model robustness term in the RO framework is to penalize solutions for being too different across scenarios (in terms of variables, not costs), thus encouraging the resulting solution to be insensitive to uncertainties in the data. Vladimirou and Zenios (1997) formulate several models for solving this particular realization of the RO framework, which they call "restricted recourse". Restricted recourse might be appropriate, for example, in a production planning context in which re-tooling is costly. However, there may be a substantial tradeoff between robustness (in this sense) and cost. The authors present three procedures to solve such problems, each of which begins by forcing all second-stage decisions to be equal, and then gradually loosens that requirement until a feasible solution is found. The SP problems are solved using standard integer SP algorithms. The authors analyze the tradeoff between robustness and cost, and often find large increases in cost as the restricted recourse constraint is tightened.

In contrast, Paraskevopoulos *et al.* (1991) present a model for robust capacity planning in which they restrict the sensitivity of the *objective function* (rather than the variables) to changes in the data. Instead of minimizing expected cost, they minimize expected cost plus a penalty on the objective's sensitivity to changes in demand. The penalty is weighted based on the decision maker's level of risk aversion. The advantage of this robustness measure is that the resulting problem resembles the deterministic

problem with the uncertain parameters being replaced by their means and with an extra penalty term added to the objective. Scenarios and probability distributions do not enter the mix. The down-side is that computing the penalty requires differentiating the cost with respect to the error in the data. For realistic capacity planning problems, even computing the expected cost (let alone its derivative) is difficult and in some cases must be done using Monte Carlo simulation. For linear models, including most location models, computing the expected cost is easy, but the penalty becomes a constant and the problem reduces to the deterministic problem in which uncertain parameters are replaced by their means; this generally yields poor results. Therefore, the robustness measure of Paraskevopoulos *et al.* (1991) has not been applied to location problems. On the other hand, the requirement that solutions have similar costs across scenarios has some resemblance to the notion of *p*-robustness, discussed in Section 4.2.3, which has been applied to location problems.

### 4.2.6. *Restricting the scenario space*

Daskin *et al.* (1997) and Owen (1999) introduce the notion of "$\alpha$-reliable minimax regret", the idea behind which is that the traditional minimax regret criterion tends to focus on a few scenarios that may be catastrophic but are unlikely to occur. In the $\alpha$-reliable framework, the maximum regret is computed only over a subset of scenarios, called the "reliability set", whose total probability is at least $\alpha$. Therefore, the probability that a scenario that was not included in the objective function comes to pass is no more than $1 - \alpha$. The parameter $\alpha$ is specified by the modeler but the reliability set is chosen endogenously. The authors apply the $\alpha$-reliable concept to the minimax-regret *P*-median problem, but it could be applied to other problems, and even to other robustness measures. For instance, one could just as easily formulate, say, the $\alpha$-reliable minimax cost UFLP. Daskin *et al.* (1997) solve the problem using standard LP/branch-and-bound techniques, whereas Owen (1999) develops a genetic algorithm for its solution. Daskin *et al.* (1997) also discuss the tradeoff between reliability (a large $\alpha$) and regret (a large objective function).

The essence of the $\alpha$-reliability approach is that the robustness of a solution is evaluated only over a subset of the scenario space, i.e., the reliability set. In that sense, $\alpha$-reliability is similar to two recently developed approaches to general robust optimization: the "ellipsoidal" model formulated by Ben-Tal and Nemirovski (2000, 2002), in which robustness is evaluated only on an ellipsoid in the scenario space, and the approach advocated by Bertsimas and Sim (2003, 2004), in which robustness is evaluated only over those scenarios in which at most a fixed number of the uncertain parameters are allowed to deviate from their estimates. In the Ben-Tal–Nemirovski approach, the ellipsoid (the analog to the reliability set) is chosen exogenously, whereas in both the Bertsimas–Sim approach and the $\alpha$-reliability approach, the set is chosen endogenously and

the solution must be robust over all possible realizations of that set.

The $\alpha$-reliable approach is extended by Chen *et al.* (2003), who introduce a measure called "$\alpha$-reliable mean-excess regret", which is closely related to the conditional value-at-risk (CVaR) objective of portfolio optimization (Rockafeller and Uryasev, 2000). The objective function consists of a weighted sum of the maximum regret over the reliability set (as in $\alpha$-reliable minimax regret) and the conditional expectation of the regret over the scenarios excluded from the reliability set. Unlike the $\alpha$-reliable minimax regret model, the mean-excess model ensures that solutions perform reasonably well even in the "ignored" scenarios. The mean-excess model is also significantly more tractable; in fact, Chen *et al.* (2003) suggest a heuristic for solving the $\alpha$-reliable minimax regret model by solving a series of mean-excess models.

### 4.2.7. *A game-theoretic approach*

Blanchini *et al.* (1997) consider capacitated network flow and network design problems under uncertainty. Demands are uniformly distributed. They address two questions: (i) does there exist a flow assignment strategy that is guaranteed to be feasible, regardless of the realization of the demands? and (ii) what is the minimum-cost network design that allows a guaranteed-feasible flow assignment strategy? They model the problem as a two-person game in which each player's moves correspond to setting flows in the network. The first player is the network manager, who must obey the system constraints. The second player represents the demand; this player acts malevolently to push the demands so that the constraints are violated. Finding a winning strategy reduces to solving a flow problem on a line.

## 5. Discussion

The literature on facility location under uncertainty has been growing steadily. Roughly half of the papers cited in this article were published in the past 10 years and roughly a third were published in the past five years. The growing interest in these problems is due to the increased recognition of the uncertainties faced by most firms, as well as to improvements in both optimization technology and raw computing power.

When we think of facility location under uncertainty, many of us think only of the two most common objectives: (i) minimizing expected cost (for stochastic problems); and (ii) minimizing maximum regret (for robust problems). Yet a wide variety of other approaches has been proposed; this paper discusses at least a dozen such measures. Many of these approaches have modeling, analytical, and computational advantages over the traditional objectives. We have explored these alternative measures with the intention of providing a foundation for researchers doing work in this and related fields.

**Table 1.** Reported applications of stochastic and robust facility location models

| Reference | Application |
| --- | --- |
| Butler *et al.* (2004) | Consumer electronics company |
| Carson and Batta (1990) | Ambulance location on SUNY Buffalo campus |
| Chan *et al.* (2001) | Military medical evacuation |
| Gregg *et al.* (1988) | Queens Borough of New York City public library system |
| Killmer *et al.* (2001) | Hazardous waste treatment facilities in Albany, NY |
| Listeş and Dekker (2005) | Recycling of sand from demolition sites |
| Lowe *et al.* (2002) | Harvard Business School case (Applichem) |
| Paraskevopoulos *et al.* (1991) | PVC industry |
| Santoso *et al.* (2003) | Cardboard packaging company, global supply chain |
| Serra and Marianov (1998) | Fire station location in Barcelona |
| Tsiakis *et al.* (2001) | European supply chain |

Relatively few of the models discussed in this paper seem to have been applied to real-world problems. Table 1 lists the papers that provide a detailed case study in which the proposed model is applied to a real problem, whether or not the resulting solutions were implemented. Although the list of applications is varied, it is substantially shorter than lists of successful applications of deterministic facility locations models (see, e.g., Current *et al.* (2002)). In our opinion, the lack of successful applications can be explained, at least in part, by the cumbersome data requirements of many stochastic models, which often require estimates of many parameters over a range of hypothetical scenarios. Robust optimization reduces the data burden by hedging against a set of scenarios whose probabilities, or even whose composition, need not be known explicitly. On the other hand, robust location problems have proven difficult to solve for realistic instances. Reducing the data and computational burden will be critical for this active body of research to become practical. To that end, we have identified four research avenues that we believe are within the grasp of today's operations research technology.

1. *Exact algorithms for minimax problems*. Exact algorithms for problems with minimax objectives are still in their infancy. Very few exact algorithms are available for general problems on general networks; most apply only to specially structured problems with limited potential for direct application. Because of the popularity of these models in the literature, exact algorithms for solving them would be welcomed by researchers and practitioners.
2. *Multiechelon models*. Only recently have researchers begun to study multiechelon location or supply chain network design problems under uncertainty. The initial attempts have been promising; however, many avenues for further research remain. For example, these problems have been solved using stochastic objectives but not robust ones. Furthermore, there is a need for models that capture the costs of tactical and/or operational functions of the supply chain under uncertainty, including inventory, transportation, and scheduling.
3. *SP technology*. The optimization technology developed by the SP community has become extremely powerful, but it has only recently begun to be used in models for facility location and other logistics problems. There is great potential for solving complex, realistic problems by leveraging the available and emerging SP technology.
4. *Meta-heuristics for general problems*. Meta-heuristics have been applied successfully to deterministic location problems, but few, if any, have been developed for their stochastic and robust counterparts. Given the array of approaches discussed in this paper, meta-heuristics would be particularly valuable since they allow changes to the objective function and constraints with relative ease. Modelers would be able to choose from solutions generated using a number of objectives without requiring special-purpose algorithms for each.

Undoubtedly, there are many other research avenues that will prove to be productive; we hope that this survey paper helps to facilitate future research in this area.

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