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Invited Review

A bibliography for some fundamental problem categories in discrete location science

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Charles ReVelle originated the idea for this review and compiled the initial list of papers to be included in the review. His untimely death last summer did not allow him to see this paper through to completion. We dedicate this review to his memory in recognition of the profound impact that he and his students have had on the field of location modeling, theory and applications.

Abstract

Following a brief taxonomy of the broad field of facility location modeling, this paper provides an annotated bibliography of recent papers in two branches of discrete location theory and modeling. In particular, we review papers related to (1) the median and plant location models and (2) to center and covering models. We show how the contributions of the papers we review are embedded in the field. A summary and outlook conclude the paper.

Keywords: Location modeling; Bibliography; Median problems; Center problems; Covering; Plant location

1. Introduction

Location models have been studied in various forms for hundreds of years. Even though the contexts in which these models are situated may differ, their main features are always the same: a *space* including a *metric*, *customers* whose locations in the given space are known, and *facilities* whose locations have to be determined according to some objective function. For more detailed and systematic introductions to the field, see Daskin (1995), Eiselt and Sandblom (2004) and ReVelle and Eiselt (2005). Many classification schemes of the field exist, most use the decision maker's objective or the space of the model as their main criterion and this paper is no exception. We divide the contributions in the literature on location modeling into four broad categories:

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(1) **Analytic models** are based on a large number of simplifying assumptions. For example, a typical analytic model assumes that demands are uniformly distributed with density ρ over the service region of size a. Models also often assume that the fixed cost of locating a facility is f, no matter where in the service region the facility is located and the cost per unit shipment per unit distance is c. Analytic models admit closed-form expressions for the total cost, typically as a function of the total number of facilities being located. For example, under the assumptions above and with the additional assumption that travel distances follow the Manhattan metric, the total cost associated with using n facilities is $fn + \frac{\rho ca}{3}\sqrt{\frac{2a}{n}}$, which is minimized for $n = a\left(\frac{c\rho\sqrt{2}}{6f}\right)^{2/3}$. At this point, the total cost is approximately $1.1447Af^{1/3}(c\rho)^{2/3}$. While such models provide valuable insight into the relationship between the optimal total cost and number of facilities on the one hand and the key input parameters on the other hand, the stringent assumptions made by this class of models make them of limited value for decision-making for practical location problems. The reader is referred to Leamer (1968) and Geoffrion (1979) for a discussion of this class of models.

(2) **Continuous models** typically assume that facilities can be located anywhere in the service area, while demands are often taken as being at discrete locations. The classic model in this area is the Weber problem of locating a single facility to serve m demands with coordinates (x_i, y_i) with $i = 1, \ldots, m$ and demands (weights) w_i, $i = 1, \ldots, m$. Distances in the Weber problem are often taken to be straight-line or Euclidean distances. The problem is to locate a single facility with coordinates (x_0, y_0) to minimize the demand-weighted total distance. Drezner et al. (2001) provide a review of this model, solution algorithms for the problem and a discussion of important extensions to the problem. Continuous models of this sort can be applied in limited contexts in which it is possible to locate facilities anywhere in the space being considered. For example, it may be possible to use such models in locating video cameras or pollution censors to monitor certain environments.

(3) **Network models** assume that the location problem is embedded in a network composed of links and nodes. Demands typically arise on the nodes, though some research has been done on the case in which demands arise on the links and nodes. One practical example in which demands arise on both the nodes and the links is the demand for emergency highway services. Much of the literature in this area is concerned with finding special structures that can be exploited to derive low-order polynomial time algorithms for particular cases of various problems. Goldman's (1971) O(n) algorithm for finding the 1-median on a tree composed of n nodes (and the often disregarded paper by Hua Lo-Keng et al. (1962) that predated Goldman's work) is typical of this class of research. In a subsequent paper, Goldman (1972) proposes an O(n) algorithm for the 1-center on a tree.

(4) **Discrete location** models assume that there is a discrete set of demands, I, and a discrete set of candidate locations, J. Such problems are often formulated as integer or mixed integer programming problems as shown below. Most such problems are **NP**-hard on general networks. (Again, much of the research in the network modeling area entails finding special instances of such problems or special graph structures under which the problems admit polynomial time solution algorithms.) As such, there is a sizable body of literature (some of which is summarized below) devoted to finding effective and efficient heuristic algorithms for many discrete problems. Discrete location models have been used in many practical contexts. In this paper, we provide an annotated bibliography of papers that have appeared recently in two major areas of discrete location modeling: plant and median problems on the one hand and center and covering problems on the other hand. Section 2 focuses on median and plant location problems, which are both based on minimizing the average demand-weighted distance between a demand node and the facility to which the demand node is assigned. Section 3 addresses covering based models in which there is (are) one (or more) service standard(s) that are to be met or partially met by the facilities being located. In some cases, papers discussed both median and covering objectives. In those cases, the papers are listed in Section 3 after we present a brief overview of the class of covering problems. We present conclusions in Section 4. We do not pretend that this annotated bibliography is complete in any sense even within the class of discrete location modeling. First, there are entire classes of papers that we intentionally omitted including much of the literature on obnoxious facility location modeling as well as some of the literature on the issue of aggregation. Also, as is the case in any study of this sort, many papers were published between the time

we initially drew up the list of papers to be included in the bibliography and the time we completed the review of the papers. Some of the more recent papers have been included below though there are many that we have undoubtedly omitted. Despite these omissions, we believe that the bibliography below will be a valuable summary of recent research in this clearly dynamic and evolving area.

2. Median and plant location problems

Plant location and median problems are both concerned with minimizing the demand weighted total distance between demand nodes in the set I, and facilities in the set J. Typically, demands are assigned to the nearest open facility, though in the case of capacities, economies of scale, or other restrictions or cost structures, demands may be assigned to more remote facilities. Additionally, the demands at nodes may be split between different facilities in the optimal solution to some problems.

We begin with the p-median problem. This model takes as input the demands (or weights) w_i, at each node $i \in I$, the distances d_{ij} between each demand node $i \in I$ and each candidate facility site $j \in J$ and p, the number of facilities to be located. The key decisions are where to locate the p facilities and which facility should serve each demand node. We define the following decision variables:

$x_j = 1$ if a facility is located at candidate node $j \in J$ and 0 otherwise,
$y_{ij} = 1$ if demand node $i \in I$ is assigned to facility at candidate node $j \in J$ and 0 otherwise.

With this notation, ReVelle and Swain (1970) formulated the p-median problem as follows:

$$\text{Minimize} \quad \sum_{j \in J} \sum_{i \in I} w_i d_{ij} y_{ij} \tag{1}$$

$$\text{s.t.} \quad \sum_{j \in J} y_{ij} = 1 \quad \forall i \in I, \tag{2}$$

$$y_{ij} - x_j \leqslant 0 \quad \forall i \in I, \ \forall j \in J, \tag{3}$$

$$\sum_{j \in J} x_j = p, \tag{4}$$

$$x_j \in \{0, 1\} \quad \forall j \in J, \tag{5}$$

$$y_{ij} \in \{0, 1\} \quad \forall i \in I, \ \forall j \in J. \tag{6}$$

The objective function (1) minimizes the demand-weighted total distance. Since the demands are known and the total demand is fixed, this is equivalent to minimizing the demand-weighted average distance. Constraints (2) ensure that each demand node is assigned, while constraints (3) stipulate that the assignments can only be made to open facilities. Constraint (4) states that exactly p facilities are to be opened. Constraints (5) and (6) are standard integrality constraints. We note that constraints (6) can be relaxed to non-negativity constraints since, if an optimal solution results in a demand node being assigned to multiple facilities, then the demand node must be equidistant to all facilities to which it is partially assigned. If integrality of the assignment variables is required, the demand node can be arbitrarily assigned completely to one of the facilities to which it was partially assigned.

In a follow-up of his classical work in 1964, Hakimi's (1965) seminal paper proved that at least one optimal solution to the p-median problem on a network consists of locating only on the nodes when the problem is cast in terms of an underlying network. This reduces the search for the set of optimal solutions to the nodes and led ReVelle and Swain (1970) to the formulation above. Kariv and Hakimi (1979a,b) showed that the p-median problem is **NP**-hard.

The p-median model implicitly assumes that the cost of locating a facility at each candidate site is the same for all sites. Often this is not the case. Balinski (1965) formulated an extension of the p-median model which minimizes the sum of the facility location costs and the transportation costs. A facility located at candidate site $j \in J$ costs f_j per unit time, where the demands are measured relative to the same unit of time (e.g., cost and demand per year). We introduce a parameter α which converts the demand-weighted total distance to cost units. For example, if the demands are measured in terms of tons/year, then α would be the cost per ton-mile.

With this additional notation, the plant location model, sometimes referred to as the uncapacitated facility location model, can be formulated as follows:

$$\text{Minimize} \quad \sum_{j\in J} f_j x_j + \alpha \sum_{j\in J} \sum_{i\in I} w_i d_{ij} y_{ij} \tag{7}$$

$$\text{s.t.} \quad \sum_{j\in J} y_{ij} = 1 \quad \forall i \in I, \tag{2}$$

$$y_{ij} - x_j \leqslant 0 \quad \forall i \in I, \ \forall j \in J, \tag{3}$$

$$x_j \in \{0,1\} \quad \forall j \in J, \tag{5}$$

$$y_{ij} \in \{0,1\} \quad \forall i \in I, \ \forall j \in J. \tag{6}$$

The objective function (7) minimizes the sum of the fixed facility costs and the transportation costs. The constraints are identical to those of the p-median problem, except that we drop constraint (4) on the number of facilities to locate. This constraint is no longer needed since the objective function penalizes the use of unnecessarily large numbers of facilities. As in the p-median model, constraints (5) can be relaxed to non-negativity constraints since each demand node will naturally be assigned to the closest open facility.

The remainder of this section summarizes recent papers related to the p-median and plant location problems. Papers are listed alphabetically by the authors.

Agar, M.C., Salhi, S. (1998) "Lagrangean Heuristics Applied to a Variety of Large Capacitated Plant Location Problems," *Journal of the Operational Research Society* **49**, 1072–1084.

The focus of the paper concerns the performance of Lagrangean heuristics applied to a variety of capacitated plant location problems. In addition to the standard capacitated plant location problem and its single-source version (i.e., the variety in which all customers are served from a single facility, a problem that, once the facilities are located, reduces to a generalized assignment problem), the authors also consider a problem, in which a facility can assume a finite number of different capacities, each of which having its own fixed cost. This problem can be solved with and without the single-source constraint. The authors start with a generic Lagrangean relaxation method and then devise various specific features, e.g., step size, interchange heuristics for better feasible solutions, local search, etc. In order to evaluate the proposed algorithm, problems available in the literature are solved as well as new problems, generated according to principles outlined by Christofides for vehicle routing problems. The problems have up to 1000 customer locations and 100 potential facility locations. The duality gaps are less than 7–8% in the worst case for the capacitated plant location problems and less than 12% for problems with multiple potential capacities. The average duality gaps are much smaller, and the computations times for most multi-capacity problems are within half an hour. Contrary to conventional wisdom, for the standard capacitated problems, the single-source problems do not appear harder than those in which this additional constraint is not imposed.

Alfieri, A., Brandimarte, P., D'Orazio, S. (2002) "LP-Based Heuristics for the Capacitated Lot-Sizing Problem: the Interaction of Model Formulation and Solution Algorithm," *International Journal of Production Research* **40**, 441–458.

The problem under investigation is the capacitated lot-sizing problem, in which the objective function minimizes the sum of inventory holding and production setup costs, subject to inventory balancing constraints and raw material availability constraints. The production level variables and the inventory variables are continuous, while the setup variables are binary. It is known that this formulation produces weak bounds. Two disaggregated formulations are offered: both redefine the production variables and the former is a version of a plant location problem (with locations referring to time rather than space), and the second is a version of a shortest path problem. Computational experiments with extensive finetuning are provided.

Alp, O., Erkut, E., Drezner, Z. (2003) "An Efficient Genetic Algorithm for the p-Median Problem," *Annals of Operations Research* **122**, 21–42.

The paper proposes a simple genetic algorithm for the p-median problem. The initial population is selected such that each candidate node is represented at least once. A new solution is generated by selecting two parents at random and using a greedy drop algorithm to delete from the union of the locations represented by the two parents those nodes that are not in both parents, until p locations remain. If the solution is better than the worst solution in the parent population, it replaces that solution; otherwise it is discarded. The authors test the algorithm on 80 test problems ranging from 100 to 1000 nodes and found that the algorithm performs well when compared with an older genetic algorithm, simulated annealing, a vertex substitution algorithm and an optimal code.

Al-Sultan, K.S., Al-Fawzan, M.A. (1999) "A Tabu Search Approach to the Uncapacitated Facility Location Problem," *Annals of Operations Research*, **86**, 91–103.

The authors present a tabu search algorithm for the uncapacitated fixed charge location problem. The algorithm is tested on small problems with up to 57 demand nodes and candidate locations. In all instances, the tabu search algorithm found the optimal solution.

Avella, P., Sassano, A. (2001) "On the *p*-Median Polytope," *Mathematical Programming*, Series A **89**, 395–411.

The authors first demonstrate that the *p*-median polytope is a subset of the stable set polytope. They then derive so-called *W*-2 inequalities as well as I^*-cover inequalities. The latter are not valid inequalities, but it can be guaranteed that they do not cut off the optimal solution of the problem. Based on these types of inequalities, problems from Beasley's OR library are solved. Most of the problems could be solved by using only cutting planes based on the inequalities derived in this paper.

Avella, P., Sforza, A. (1999) "Logical Reduction Tests for the *p*-Median Problem," Annals of Operations Research **86**, 105–115.

The paper examines properties of optimal solutions of the *p*-median problem. Based on these properties, logical constraints are derived that allow users to fix variables at zero or one and subsequently delete the variables from the problem. The two procedures described here are of complexity $O(n^2 \log)$ and $O(n^3)$, respectively. In addition, the authors also fix variables while solving the Lagrangean relaxation. In their computational tests on random problems (20 to 150 nodes), 30–99% of the arcs can be deleted, while in problems from the OR library (100–200 nodes), time savings from 20% to 90% are reported.

Averbakh, I., Berman, O. (1999) "Parallel Complexity of Additive Location Problems," *INFORMS Journal on Computing* **11**, 292–298.

The purpose of the paper is to devise efficient parallel algorithms (so-called NC-algorithms) for multifacility problems with identical facilities and minisum objective. In particular, the objective is to minimize the setup cost of a facility plus a "service cost," which is a monotone nondecreasing function of the facility - customer distance. This problem, called 1a, is an uncapacitated plant location problem on a tree. Problem 1b is the same as 1a, except that a constraint is added that restricts the number of facilities to at most *p* (an exogenous parameter). Well-known problems result as special cases of problem 1b: If the cost functions are linear and the setup costs are zero, then the problem reduces to the *p*-median problem on a tree. If the costs can be expressed as step functions, problem 1b is the *p*-coverage location problem.

It is well known that the uncapacitated plant location problem on a tree can be solved by serial processors in $O(n^2)$ time, the *p*-median problem requires $O(p^2n^2)$ time, and the *p*-coverage problem was shown to be solvable in $O(pn^2)$ time. However, none of these algorithms can be parallelized. In this paper, the authors demonstrate that both problems can be solved in $O(\log^3 n)$ time given $O(n^3/\log n)$ processors for problem 1a and $O(p^2n^3/\log n)$ processors for problem 1b.

Averbakh, I., Berman, O. (2000) "Minmax Regret Median Location on a Network Under Uncertainty," *INFORMS Journal on Computing* **12**, 104–110.

This paper considers the single facility location problem which minimizes the maximum regret associated with the location over a set of scenarios. The uncertainty is in the demand at each node which is characterized only by lower and upper bounds on the demand. They show that the problem on a general graph can be solved in $O(n^3)$ if the solution is restricted to the nodes. When the facility can be on the nodes and links, they develop an $O(mn^2 \log n)$ algorithm. They also present an $O(n^2)$ algorithm for both variants of the problem on a tree.

Averbakh, I., Berman, O., Drezner, Z., Wesolowsky, G.O. (1998) "The Plant Location Problem with Demand-Dependent Setup Costs and Centralized Allocation," *European Journal of Operational Research* **111** (3), 543–554.

The authors introduce a variant of the uncapacitated fixed charge facility location problem in which the fixed setup cost for a facility depends on the number of customers assigned to the facility, in a potentially non-linear manner. Customer assignments are done centrally to minimize the sum of the transportation and fixed facility location costs. For a tree network, the paper provides an $O(n^3)$ dynamic programming algorithm.

Averbakh, I., Berman, O., Drezner, Z., Wesolowsky, G.O. (2007) "The Uncapacitated Facility Location Problem with Demand-Dependent Setup and Service Costs and Customer-Choice Allocation," *European Journal of Operational Research* **179** (3), 956–967.

The authors consider the problem of locating a set of facilities on a tree such that the demand-dependent facility cost and a portion of the customer transportation cost is minimized. The facilities also set prices for service and customers choose to patronize the facility that provides the smallest sum of the service price and the portion of the transport costs that are customer-paid. An $O(n^5)$ dynamic programming algorithm is given.

Baldacci, R., Hadjiconstantinou, E., Maniezzo, V., Mingozzi, A. (2002) "A New Method for Solving Capacitated Location Problems Based on a Set Partitioning Approach," *Computers & Operations Research* **29**, 365–386.

In this work, the authors first formulate the capacitated p-median problem in the usual form, followed by an alternative formulation as a set partitioning problem, in which zero–one variables are assigned to all feasible clusters of customers. This formulation has a very simple structure as its constraints are only the usual set partitioning constraints (allowing each customer to be associated with exactly one cluster that is included in the solution) and a single constraint that forces exactly p clusters to be chosen. However, the formulation has a huge number of variables. The main contribution of the paper is the construction of lower bounds to the problem. These bounds are generated by way of heuristics, one of which uses Lagrangean relaxations that, for fixed multipliers, result in independent knapsack problems, and another that is based on linear programming problems which derive from the dual of the set partitioning formulation of the problem. Extensive tests are reported with problems having up to $n = 200$ nodes. The running times to generate the bounds are reasonable, and the percentage error of the lower bound is almost always in the 90 percent range.

Barahona, F., Jensen, D. (1998) "Plant Location with Minimum Inventory," *Mathematical Programming* **83**, 101–111.

The problem under consideration is an uncapacitated plant location problem with multiple commodities, e.g., computer spare parts. The objective is to minimize warehouse location costs, transportation costs, and inventory holding costs. The constraints of the problem require that a 95% service requirement is satisfied for a 2-hour service standard, and there is an upper bound on the number of warehouses that can be opened. The remaining constraints link classes of variables to each other and require that each customer is allocated to one warehouse. The authors then apply Dantzig-Wolfe decomposition as a solution method with subproblems that turn out to be minimum cut network problems. They then use a subgradient technique to generate good multipliers. A small number of problems is solved. These problems have up to 300 potential warehouse locations, up to 350 customers, and up to 250 different spare parts, i.e., commodities. Most problems are solved within half an hour, many of the solutions have objective values within a few percentage points of those of the linear programming relaxation.

Berman, O., Drezner, Z. (2006) "Location of Congested Capacitated Facilities with Distance-Sensitive Demand," *IIE Transactions* **38**, 213–221.

The authors consider the problem of locating a fixed number of facilities on a network to maximize the expected number of demands that can be served subject to a constraint on the mean service time at a facility. Facilities are modeled as $M/M/1$ queues with an endogenously determined arrival rate. The demand at a node decays exponentially with the distance between the node and the facility to which it is assigned. Demands need not be assigned to the nearest facility. Steady state requirements limit the demand assigned to any facility. The authors provide an $O(n^3)$ algorithm for the problem of locating a single facility on the network. Vertex substitution, simulated annealing and tabu search algorithms were tested on 40 problems for the multi-facility case. The simulated annealing algorithm found the best known solution in 35 of the 40 cases.

Berman, O., Drezner, Z., Wesolowsky, G.O. (2001) "Location of Facilities on a Network with Groups of Demand Points," *IIE Transactions* **33**, 637–648.

The authors consider the problem of locating a given number of facilities to serve groups of demands. The objective function for each group of demand nodes can be a function of one of three distance criteria: the minimum distance to any node in the group, the average distance, and the maximum distance. Similarly three objective functions are considered – the minisum, the minimax and the maximin objectives – resulting in nine possible combinations of distance metrics to groups and objective functions. The authors identify dominant sets of nodes for each problem. A modified greedy algorithm, a tabu search, and a simulated annealing algorithm were tested for selected problems. The greedy and tabu search algorithms performed well, while the simulated annealing algorithm was inferior to those for the problems and instances tested.

Berman, O., Drezner, Z., Wesolowsky, G.O. (2002) "Satisfying Partial Demand in Facilities Location," *IIE Transactions* **34**, 971–978.

This paper presents a partial assignment p-median problem. The objective is to minimize the weighted demand that is actually allocated subject to a constraint that at least α percent of the total demand is assigned. The authors initially formulate the problem of locating a single facility on the plane. They propose a heuristic that alternates between solving a knapsack problem (to determine the set of nodes to serve) and the Weber problem (to locate the facility). This is tested on problems with up to 10,000 demand points, all of which could be solved in under 3 seconds. They also formulate an integer programming version of the problem on a network and propose a Lagrangean relaxation/branch and bound algorithm for its solution. This is tested on problems of up to 120 nodes. The algorithm is compared to CPLEX which performed better for problems with more than 5 facilities.

Berman, O., Drezner, Z., Wesolowsky, G.O. (2005) "The Facility and Transfer Plant Location Problem," *International Transactions in Operational Research* **12**, 387–402.

The paper formulates the problem in which a single new facility and p transfer facilities are to be located to serve demands. The unit cost of transportation between the transfer sites and the facility is a fraction α of the unit cost of direct transport between the demand node and the facility (or the transfer site). Three heuristics are tested: a vertex substitution algorithm, simulated annealing and tabu search. All three are shown to perform well on a variety of test problems with 100 to 900 nodes. When the problem is simplified so that the location of the single facility is known, the simulated annealing algorithm performed the best.

Berman, O., Drezner, Z., Wesolowsky, G.O. (2006) "The Maximal Covering Problem with Some Negative Weights," working manuscript.

The authors formulate a maximal covering problem with negative weights and show that the standard linkage constraint between the coverage variables and the location variables does not guarantee that negatively weighted nodes which are, in fact covered, will be counted as such. They formulate three additional constraints to handle the problem: a non-linear constraint, a disagreggate forcing constraint, and an aggregate forcing constraint. Tests with CPLEX on the latter two constraints, using problems with up to 900 nodes, suggest that the aggregate constraints are better. The authors then develop and test a variant of the Teitz and Bart substitution algorithm, a tabu search and a simulated annealing algorithm. The simulated annealing algorithm achieved the best results as measured by the fraction of times that the optimum solution was found and in terms of the average computing time required to attain the solution.

Brimberg, J., ReVelle, C. (2000) "The Maximum Return-on-Investment Plant Location Problem," *Journal of the Operational Research Society* **51**, 729–735.

The paper considers bicriterion plant location problems. The authors demonstrate that the bicriterion problem that minimizes the objectives investment cost and total cost, is equivalent to a bicriterion problem that maximizes profit and minimizes investment cost. The paper first determines some results concerning the efficient set, and then solves a series of randomly generated test problems with the weighting method. One of these efficient solutions is shown to maximize the return on investment. The problems have up to 100 potential plant sites and 300 customer locations. The problems tend to get less integer-friendly as the problem size increases. Some of the problems required in excess of 15,000 simplex iterations.

Burkard, R.E., Cela, E., Dollani, H. (2000) "2-Medians in Trees with Pos/Neg Weights," *Discrete Applied Mathematics* **105**, 51–71

The paper investigates 2-medians problems on trees, in which the nodes have positive or negative weights. Two problems are studied: P1 minimizes the sum of minimal weighted distances, while P2 minimizes the sum of weighted minimal distances. The two objectives are identical as long as weights are nonnegative. The main contribution of the paper are complexity results: the problem P1 has the node property, while P2 does not. P1 can be solved in $O(n^2)$ on trees, in linear time on a path, and in $O(n \log n)$ on an extended star. The problem P2, which has at least one of two medians at a node, can be solved in $O(n^3)$ on a tree and $O(n^2)$ on a path.

Cañavate-Bernal, R., Landete-Ruiz, M., Marín-Pérez, A. (2000) "On the Resolution of the Single Product Capacitated Machine Siting Problem," *Journal of the Operational Research Society* **51**, 982–992.

The problem considered in this paper is a simple plant location problem, in which facilities are to be opened at discrete sites, so that each facility may house a number of machines that are manufacturing a given product. The problem is first formulated as suggested by ReVelle and Laporte. Then the authors provide a somewhat tighter formulation. Lagrangean decomposition (variable splitting) and Lagrangean relaxation provide lower bounds and a heuristic determines upper bounds and feasible solutions. The heuristic involves subgradient optimization. Computational experiments compare different relaxations. The test problems are adaptations of simple plant location problems found in Beasley's OR library. While some relaxations consistently fail to provide good solutions, two Lagrangean heuristics performed well. For larger problems, the Lagrangean relaxation appears to outperform the decomposition.

Canós, M.J., Ivorra, C., Liern, V. (2001) "The Fuzzy *p*-Median Problem: A Global Analysis of the Solutions," *European Journal of Operational Research* **130**, 430–436.

The paper considers fuzzy *p*-median problems, in which it is no longer required to serve the demand at all nodes. A formulation was provided by the same authors in an earlier paper in the same journal 2 years earlier. The key is a sensitivity analysis that determines cost functions that arise from partial coverage of the demand. Each reduction of covered demand is associated with a profit interval whose bounds are optimistic and pessimistic estimates. Given the usual fuzzy approach with a membership function, profit bands result that provide additional insight into the problem and suggest which demand may not be covered.

Cánovas, L., Landete, M., Marín, A. (2000) "New Facets for the Set Packing Polytope," *Operations Research Letters* **27**, 153–161.

The paper describes facets of the set packing problem that are associated with a graph structure called grille. Given that the simple plant location problem can be formulated as a set packing problem, these valid inequalities can be computationally advantageous when solving these location problems with branch-and-bound procedures.

Cánovas, L., Landete, M., Marín, A. (2002) "On the Facets of the Simple Plant Location Packing Polytope," *Discrete Applied Mathematics* **124**, 27–53.

The paper first formulates the simple plant location problem as a set packing problem. Given this formulation, the authors determine subgraphs of the intersection graph that define facets of the problem. Some of the graph structures have known shapes, such as fans or wheels. The resulting valid inequalities can be used to reduce the computational effort in branch-and-bound procedures.

Chardaire, P., Lutton, J.-L. Sutter, A. (1999) "Upper and Lower Bounds for the Two-Level Simple Plant Location Problem," *Annals of Operations Research* **86**, 117–140.

The paper discusses a two-level problem common in the telecommunications industry. Each terminals (or customer) is connected to a concentrator, which, in turn, is connected to first-level concentrators, which are then connected to the central site. The authors first formulate the problem as a standard two-subscript flow formulation and then offer another formulation with three subscripts. While the latter formulation is considerably larger, it offers the possibility to add facet-defining polyhedral cuts to it. Lagrangean relaxation is applied to generate lower bounds and feasible solutions. In addition, a simulated annealing technique is devised to improve the upper bounds. Computational experience with up to 150 terminals is very encouraging.

Chen, B., Lin, C.S. (1998) "Minimax-Regret Robust 1-Median Location on a Tree," *Networks*, **31**, 93–103.

The paper develops low-degree polynomial algorithms for the 1-median location on a tree, given a minimax regret objective. Throughout the paper, the authors assume that edge distances as well as node weights are unknown, but must lie in a known interval. Each specific realization of a distance and a weight within these intervals is called a scenario, and, based on the well-known node property, each scenario median is located at a node. Defining M as the set of all scenario medians, the authors prove that M is a connected set. It was shown by Kouvelis et al. (1993) in a working paper that the 1-median in a tree, given a minimax regret objective, does not have to be located at a node. The algorithm proposed by the authors of this paper determines the 1-median x^* and the node 1-median v^* in $O(\alpha|M||V|)$ time, where α denotes the maximal number of scenario medians on a path among all paths of a tree. This contrasts positively with the $O(|V|^4)$ exhaustive search algorithm by Kouvelis *et al.* Added in proof: Averbakh and Berman have presented an $O(|V|\log^2|V|)$ algorithm for the same problem at a conference.

Chiou, Y., Lan, L. (2001) "Genetic Clustering Algorithms," *European Journal of Operational Research* **135**, 413–427.

This paper compares three different encoding/decoding approaches to solving a clustering problem using genetic algorithms with an agglomerative hierarchical clustering method. The latter approach begins with each element forming its own cluster and merges them until only one mega-cluster remains. For a problem with N items, this process generates in N solutions. The best of these is returned as the final solution. The objective in the test problems is to maximize the ratio of the between cluster variability to the within cluster variability. Of the three genetic algorithms, the most effective is the one which finds cluster seed points (which are analogous to facilities in the p-median problem). The authors also test the algorithms on a fixed charge location problem.

Chiyoshi, F., Galvão, R.D. (2000) "A Statistical Analysis of Simulated Annealing Applied to the p-Median Problem," *Annals of Operations Research* **96**, 61–74.

The paper combines a variant of Teitz and Bart's vertex substitution method and simulated annealing to heuristically solve p-median problems. The authors pay special attention to the cooling schedules and the probability of moves that deteriorate the solution. Here, a ratio of 1:1 for accepted to rejected moves is employed. Their computational tests were performed on p-median problems from Beasley's OR library that feature 100–900 nodes. The solution times are all within 21/2 minutes and the worst gap to optimality is 1.6%.

Dai, Z., Cheung, T.Y. (1997) "A New Heuristic Approach for the *p*-Median Problem," *Journal of the Operational Research Society* **48**, 950–960.

The paper deals with a standard multi-Weber problem (i.e., the *p*-median problem in the plane) with Euclidean distances. Based on Rosing's geometric partitioning approach of non-overlapping areas associated with facility locations, the authors devise two heuristic algorithms. These improvement heuristics temporarily and sequentially delete each pair (triple) of facilities and determines the optimal pair (triple) of facility locations. A replacement is made whenever there is an improvement. In the computational tests, the triple replacement is very slow, while the pairwise replacements work well if compared with a number of other heuristics reported in the literature by various authors.

Dasci, A., Verter, V. (2001a) "A Continuous Model for Production–Distribution System Design," *European Journal of Operational Research* **129**, 287–298.

The paper first surveys analytical contributions to production–distribution system design problems, defined as systems that include layers for each stage of the supply chain. Most models in this class are defined on networks and formulated as mixed integer programming problems. The size of such models is typically very large for realistic situations. This contribution attempts to overcome this problem by considering continuous approximations. The model has one facility open an endogenous number of branches given a variety of costs: in addition to transportation costs, the authors consider site-dependent fixed cost and also site-dependent capacity acquisition costs, in which a factor α determines of the economies of scale. It is assumed that site-dependent parameters vary only slightly within each region. The authors calculate the average distance between a facility and the customers within its trading region, assuming that the facility is located at the center of the area it serves. The procedure determines service regions first, a subsequent analysis can then locate the facilities within the areas. Some approximations lead to closed-form solutions which are further analyzed for specific values of α.

Dasci, A. Verter, V. (2001b) "The plant location and technology acquisition problem," *IIE Transactions* **33**, 963–973.

This paper extends the traditional uncapacitated fixed charge location model to incorporate multiple products and multiple technologies. The cost of using a technology at a plant increases in a concave manner with the total production of the product at the plant. The model is solved for relatively small problems with 25 demand nodes, 8 facility sites, 3 products and 2 technologies using a progressive piecewise linear underestimation algorithm suggested by Verter and Dincer (1995). Computation times and objective function values are given for somewhat larger problems as well, though details of the solutions for the larger problems are omitted. The formulation results in each demand node being served by only one facility and, for each product, at most one technology being used at any facility.

Daskin, M.S., Coullard, C., Shen, Z.-J.M. (2002) "An Inventory-Location Model: Formulation, Solution Algorithm and Computational Results," *Annals of Operations Research* **110**, 83–106.

This paper is a companion to the Shen et al. (2003) paper. The underlying model is the same extension of a fixed charge location model to incorporate a non-linear inventory term. Whereas Shen, Coullard and Daskin employ column generation to solve a recast set partitioning variant of the problem, this paper uses Lagrangian relaxation, relaxing the assignment constraint. The Lagrangian approach is shown to be somewhat more efficient than the set partitioning approach.

de Farias, I. (2001) "A Family of Facets for the Uncapacitated *p*-Median Polytope," *Operations Research Letters*, **28**, 4, 161–167.

This paper presents a facet-defining constraint for the *p*-median problem. The authors provide a separation heuristic for this constraint. The efficacy of using the constraint is demonstrated using the capacitated *p*-median problem with unsplit demands for problems with up to 70 candidate sites and 100 demand nodes. The average number of branch and bound nodes was reduced by 92% and the computation time by 71% when the new constraint was employed.

Delmaire, H., Díaz, J.A., Fernández, E., Ortega, M. (1998) "Reactive GRASP and Tabu Search Based Heuristics for the Single Source Capacitated Plant Location Problem," *INFOR* **37**, 194–225.

The paper deals with the capacitated single-source plant location problem. This problem is known to be difficult, as even for a given set of open plants, the allocation problem is a version of the generalized assignment problem, which is **NP**-hard (while the multiple-source problem in the allocation phase is just a transportation problem). The paper describes four heuristic algorithms for the problems: one GRASP (previously developed by the authors), a tabu search, and two heuristics. As all heuristics, they work in construction and improvement phases. There is a random component in the construction phase (i.e., the method does not necessarily pick the "next best" node as greedy does), and the improvement phase includes various moves, such as shift, swap, open and close facilities. The heuristics are tested on problems of up to 90 customers and 30 facilities. In general, the hybrids come out best with very moderate computational requirements (a few minutes) and in terms of robustness, i.e., independence of the result from the specific run of the experiment.

Díaz, J.A., Fernández, E. (2002) "A Branch-and-Price Algorithm for the Single Source Capacitated Plant Location Problem," *Journal of the Operational Research Society* **53**, 728–740.

The paper considers one version of the capacitated plant location problem, in which the decision maker has to choose how many and where to open facilities in order to minimize construction and transportation costs. There are two version of the problem: one, in which a customer's demand can be satisfied by multiple facilities, and another, in which each customer's demand is satisfied by a single supplier. While both problems are **NP**-hard, the latter version is much harder. Actually, the demand allocation subproblem is known to be a special case of the well-known generalized assignment problem. The single-source version is discussed in this paper. In their branch-and-price method, the authors perform bounding as an iterative process with two phases: while phase 1 chooses a subset of facilities to be located, phase 2 employs a column generation procedure for the allocation of customers to facilities; this process generates upper and lower bounds. The computational results for problems with up to 90 customers and 30 potential locations show that the suggested method, requiring mostly moderate computation times, outperforms Lagrangean relaxation.

Drezner, T., Drezner, Z., Salhi, S. (2006) "A Multi-Objective Heuristic Approach for the Casualty Collection Points Location Problem," *Journal of the Operational Research Society* **57**, 727–734.

The paper considers the problem of locating a fixed number of facilities in a high dimension multi-objective environment. To identify a compromise solution, the authors propose the minimax regret multiobjective approach which minimizes the maximum percent deviation of any objective from its optimal value. Five objectives are considered: the median objective, the center objective, covering with two different distance values and a variance (or equity) objective. Optimal solutions to the individual objective problems as well as the minimax multiobjective problem were found using a vertex substitution algorithm as well as tabu search. The model was tested on a dataset from Orange County, CA with 577 demand points and 143 candidate locations.

Drezner, Z., Shiode, S. (2007) "A Distribution Map for the One-Median Location Problem on a Network," *European Journal of Operational Research* **179** (3), 1266–1273.

The problem of locating a single median on a network and on a tree is considered when the demands at each node are jointly distributed normal random variables. For any realization of the random demands, a different node might be optimal. The distribution map gives the probability that each node is the optimal solution across all realizations. Computational procedures for finding the distribution map for small networks are given and compared to simulation results.

Drezner, Z., Wesolowsky, G.O. (1999) "Allocation of Discrete Demand with Changing Costs," *Computers & Operations Research* **26**, 1335–1349.

The paper studies primarily the allocation of demand to customers. Customers and their demand are distributed at discrete points in the Euclidean plane. There exist already p facilities and customers will patron-

ize the facility for which the delivered price is lowest. The price includes the (linear) transportation cost, the cost plus a markup, and a proportion of the setup cost. The latter depends on the number of customers who patronize the facility. The authors outline a simple iterative procedure and report extensive computational results. They test 1000 randomly generated problems with 100 demand points and 10 existing facilities each. In more than 90% of the cases, 3, 4, or 5 facilities are active while the others serve no customers. In all cases at least 3 facilities were inactive. It was observed that as the fixed costs increase, fewer facilities are active. The authors then attempt to locate a new facility at one of 10,000 potential locations and then start their iterative procedure. It appears that the average number of customers attracted to the new facility is less than average.

Erlebacher, S.J., Meller, R.D. (2000) "The Interaction of Location and Inventory in Designing Distribution Systems," *IIE Transactions* **32,** 155–166.

The authors formulate a joint location inventory model. They assume (1) customer demand in each square subregion is uniformly distributed, (2) distribution centers can be located anywhere in the region, (3) capacitated plants, (4) rectilinear distance and (5) a continuous review inventory model. They then develop an approximate model assuming uniform demand everywhere in diamond-shaped regions. They use this to solve for the optimal number of distribution centers. They propose an iterative heuristic location-allocation model to solve the original model. The approach is tested on small (3×4 and 4×4) regions and one larger (20×30) region. They found that a heuristic that progressively adds customers to distribution centers, updates the cost, and then assigns the customer with the next largest demand and finally solves local location problems, supplemented by a 2-opt customer exchange procedure, worked well. They found that as customer demand becomes more skewed (concentrated), fewer distribution centers are located.

Estivill-Castro, V., Houle, M. (2001) "Robust Distance-Based Clustering with Applications to Spatial Data Mining," *Algorithmica* **30** (2), 216–242.

The authors present a fast heuristic for the p-median problem in the context of data mining in which the problem is to find p representative points from among the sampled values in 2-dimensional space. The algorithm modifies the Teitz and Bart algorithm in two ways. First, the proposed algorithm determines a starting solution by using Delaunay triagulations, while a variant of Kruskal's minimal spanning tree algorithm is used to find the initial clusters. Second, the algorithm only considers the u closest points to a center are considered in the Teitz and Bart algorithm. The authors compared their approach with a standard Teitz and Bart approach as well as the k-means approach which minimizes the squared distance and does not restrict the centers to being sampled points. They found that the k-means approach is competitive with the others when there is no noise in the data. When there is noise, the modified Teitz and Bart algorithm proposed in the paper performs well if the noise is additive, but not if it is multiplicative.

Fernandez, E., Puerto, J. (2003) Multiobjective Solution of the Uncapacitated Plant Location Problem," *European Journal of Operational Research* **145** (3), 509–529.

The authors consider an extension of the uncapacitated fixed charge location problem in which there may be uncertainty regarding the fixed costs and/or the demands or shipment costs. The problem becomes that of finding a set of non-dominated locations and assignments of demands to facilities. A dynamic programming based algorithm is proposed for generating the exact and approximate tradeoff curves, where the approximate curve does not consider all possible assignments of demands to facilities. The problem was tested on problem instances of between 10 candidate sites and 20 demand nodes to 20 candidate sites and 50 demand nodes.

Fischer, K. (2002) "Sequential Discrete p-Facility Models for Competitive Location Planning," *Annals of Operations Research* **111**, 253–270.

The paper discusses a competitive location model in which duopolists locate in a discrete space. The two competitors locate q and p facilities, respectively, so as to maximize their profit given delivered prices and (linear) transport costs. Both duopolists employ spatial price discrimination. Customers are located at discrete points as well. The model concerns a homogeneous product and the facilities engage in location and price competition. The demand is elastic, depending on the price charged at the site, and customers are assumed to satisfy their demand at the cheapest supplier. It is assumed that the demand must be completely satisfied. Furthermore, the facilities are assumed to have sufficiently large capacities. Both players are assumed to have identical cost structures, so that the variable costs are here normalized to zero. Both competitors are assumed to have perfect foresight, i.e., complete information. The sequential von Stackelberg game concept is applied. The result is a bilevel nonlinear binary optimization problem, in which locations and prices are chosen once. An alternative is a 3-stage game in which the third phase determines the prices at Nash equilibrium. They can easily be determined. The resulting model is a simpler linear binary bilevel optimization problem. The author then suggests heuristic approaches for the solution of the models.

Gamal, M., Salhi, S. (2001) "Constructive Heuristics for the Uncapacitated Continuous Location-Allocation Problem," *Journal of the Operational Research Society* **52**, 821–829.

The multisource Weber problem – locating multiple sites in continuous space to serve discrete demands – is considered. Two heuristic approaches are proposed. The first heuristic entails generating the initial solution for Cooper's algorithm by finding points that are far from each other, as opposed to starting with a purely random starting solution. The algorithm can be repeated numerous times since the current solution is used to generate the next starting set of p points that are far from each other. The authors also introduce the notion of forbidden regions around points and a freeing strategy for relaxing the forbidden regions. The second heuristic entails using the solution to the discrete p-median problem (solved via Salhi's perturbation heuristic) as the input to Cooper's algorithm. The algorithms were tested on problems with 50, 287, 654 and 1060 points. The perturbation method coupled with Cooper's algorithm performs the best on average and is the most stable over a range of p values.

García-Lopez, F., Melián-Batista, B., Moreno-Pérez, J.A., Moreno-Vega, J.M. (2002) "The Parallel Variable Neighborhood Search for the p-Median Problem," *Journal of Heuristics* **8**, 375–388.

The paper devises heuristics that utilize parallel processors for the standard p-median problem. The basis for the heuristics is Hansen and Mledenovic's variable neighborhood search. In general, variable neighborhood search is a metaheuristic that examines a given neighborhood until it has found a local optimum, then changes the neighborhood in a "shake procedure" that finds a new starting solution. Then a new local search commences. Three parallelizations are considered: the first uses parallel processors to reduce the running time, while the others employ the parallel processors to explore more neighborhoods. In a set of computational tests with 1400 node problems from the TSPLIB problem library with p between 20 and 100, it turns out that the time reduction with the first parallel procedure is significant (8 processors cut the processing time down to about 1/6), while the additional processors do not provide much improvement in the two parallel algorithms that explore more neighborhoods.

García-López, F., Melián-Batista, B., Moreno-Pérez, J.A., Moreno-Vega, J.M. (2003) "Parallelization of the Scatter Search for the p-Median Problem," *Parallel Computing* **29**, 575–589.

The paper uses parallel variants of scatter search to solve the p-median problem. The three versions parallelize the local search, the determination of combinations of subsets of the reference set, and a multistart procedure. Computational tests on 1400-node problems with $p = 10, \ldots, 100$ reveals that the first version shows the highest degree of time reduction as more processors are added (up to 8), and it has the shortest overall computation times.

Ghiani, G., Guerriero, F., Musmanno, R. (2002) "The Capacitated Plant Location Problem with Multiple Facilities in the Same Site," *Computers & Operations Research* **29**, 1903–1912.

The paper discusses the standard capacitated plant location problem, in which multiple facilities are permitted to locate at the same site. The main contribution of the paper is a new heuristic based on

Lagrangean relaxation of the problem. First, the relaxation dualizes the constraints that require that each customer's entire demand is satisfied. Given some multipliers, for each site the subproblem is a fractional multi-knapsack problem, for which a procedure is described. The lower and upper bounds determined in the process are reasonably tight. The authors then solve a real application that locates polling stations and assigns voters to them. While LINGO does not solve the problem within 4 days, the heuristic readily finds a solution that is more than one third less expensive than the solution presently used by the authorities.

Ghosh, D. (2003) ''Neighborhood Search Heuristics for the Uncapacitated Facility Location Problem,'' *European Journal of Operational Research* **150**, 150–162.

In this paper, the author examines heuristic methods for the uncapacitated facility location problem. In the first stage, he tests three methods with different neighborhoods: add-and-swap, add-delete-swap, and greedy. The performance of these methods is very similar, but as the add-delete-swap is fastest, it is chosen as the basis for the two metaheuristics examined in this paper. The first is tabu search, and the second is complete local search with memory. After some finetuning, the two heuristics are first tested against the optimal solution on medium-sized instances with 75, 100, and 125 nodes. The time requirement for the exact solution is below 11 minutes for all instances, while it is less than two seconds for the heuristics. Furthermore, the heuristics are within less than a tenth of a percent off the optimal solution, with tabu search performing very slightly better. On problems with 250, 500, and 750 nodes, the deviation from the optimal objective value is again within 0.1 percent with tabu search being again superior. Computation times are within 14 minutes for all problems.

Goldengorin, B., Ghosh, D., Sierksma, G. (2003) ''Branch and Peg Algorithms for the Simple Plant Location Problem,'' *Computers & Operations Research* **30**, 967–981.

The paper (re-)formulates the simple plant location problem as a pseudo-Boolean problem. From that formulation, it derives a pegging rule for a branch-and-peg algorithm. The authors then suggest three branching rules. Based on these rules, a number of test problems is solved. The test series consists in part of problems taken from Beasley's OR library and in part they are randomly generated. The benchmark problems from the OR library have 50 nodes and 16, 25, and 50 potential sites, respectively, while the randomly generated problems have 50 nodes, 30, 40, and 50 potential sites and arc densities of .25, .5, .75, and 1.0. The authors employ three branching rules, one ''min subscript'' rule, and two rules that employ fitness functions. The test series reveals the advantage of the suggested branch and peg approach, whose computation times are significantly lower than that of comparable branch and bound techniques. For lower densities, one of the fitness functions has an advantage, while the branching rules are very similar as the density increases towards one.

Goldengorin, B., Tijssen, B.A., Ghosh, D., Sierksma, G. (2003) ''Solving the Simple Plant Location Problems Using a Data Correcting Approach,'' *Journal of Global Optimization* **25**, 377–406.

The paper applies Goldengorin's data correction approach to the simple plant location problem. First, in a preprocessing phase, the authors present reduction rules that are significantly more powerful than those suggested by Khumawala (1972). The main solution method is a version of the data correction method based on a pseudo-Boolean formulation of the problem. The basic idea is to set up a problem that is similar to the given problem that is (a) of the same size, and that (b) can be solved in polynomial time. Then it is possible to calculate in polynomial time an upper bound for all problem instances of the same size. The proposed algorithm first applies the reduction rules. This is followed by the data correction method which solves an instance of the problem of some distance of α from the original problem. The extensive computational results include problems from Beasley's OR library, Körkel- and Körkel-type problems, Galvão and Raggi problems, and Bilde and Krarup-type problems. In general, it was observed that the new preprocessing phase solved to optimality no less than 3/4 of the problems in the OR library, while Khumawala's method did not solve any of them. The authors also compared different bounds, and were able to solve problems with size up to 200 nodes to optimality in very modest computing time.

Hansen, P., Mladenović, N., Perez-Brito, D. (2001) "Variable Neighborhood Decomposition Search," *Journal of Heuristics* **7**, 335–350.

The authors describe a variant of the well-known variable neighborhood search. The method is described in general, and then applied to the *p*-median problem. Extensive computational tests on medium to large-scale problems of up to almost 6000 nodes are solved, and results, especially on larger problems, are encouraging with reasonably running times and very small errors.

Harkness, J., ReVelle, C. (2003) "Facility Location with Increasing Production Costs," *European Journal of Operational Research* **145**, 1–13.

The authors present an extension of the fixed charge location problem in which the cost of production at each site is a piecewise linear convex function of the demand served at the site. Four different integer linear programming formulations of the problem are presented; three allow any number of linear segments on the production cost curve, while the fourth allows only two segments. The formulations are tested on 216 randomly generated problems with 75 or 100 demand nodes and 30 or 50 candidate sites. All problems are solved using CPLEX 3.0. While adding redundant constraints of the form $x_{ij} \leqslant y_i$, (where x_{ij} denotes the fraction of demand at node j that is served by candidate site i and y_i equals 1 if a facility is sited at candidate site i and 0 if not) significantly reduced the number of branch and bound nodes, its inclusion increased the solution times significantly.

Hindi, K.S., Pieńkosz, K. (1999) "Efficient Solution of Large Scale, Single-Source, Capacitated Plant Location Problems," *Journal of the Operational Research Society* **50**, 268–274.

The authors discuss the single-source capacitated plant location problem. They first dualize the "one-supplier-per-customer" constraint. The resulting Lagrangean relaxation produces a lower bound to the objective value of the original problem. Feasible solutions are generated by a three-step procedure that includes a greedy heuristic, steepest descent search, and restricted neighborhood search. The procedure is tested on sample problems from Beasley's OR library with sizes of up to 100 potential plant sites and 1,000 customers. The computation times are less than 40 minutes for the largest problem in the set.

Hinojosa, Y., Puerto, J., Fernandez, F. (2000) "A Multi-Period Two-Echelon Multi-Commodity Capacitated Plant Location Problem," *European Journal of Operational Research* **123** (2), 271–29.

The authors consider a multi-echelon problem in which material is produced at plants, shipped to warehouses and from there to customers. They formulate a multi-period version of the problem, which finds the locations and capacities of plants and warehouses as well as the shipping pattern over time. Plants or warehouses which are initially open (closed) can be closed (opened) once during the planning period. The authors develop a Lagrangean relaxation for the problem as well as a heuristic to derive upper bounds on the cost minimization problem. They test the algorithms on problems with up to 4 time periods, 75 customers, 25 warehouses, 25 plants and 2 products. (Two smaller problem used 3 products). The gaps between the bounds averaged less than 5 percent.

Holmberg, K. (1999) "Exact Solution Methods for Uncapacitated Location Problems with Convex Transportation Costs," *European Journal of Operational Research* **114**, 127–140.

The paper first considers a standard simple plant location problem with concave transportation costs. As in this case the transportation quantities are no longer necessarily integer, integrality is enforced via a set of constraints. The author applies the standard transformation from integer to zero–one variables, linearizing the objective function at the integer points. As integrality is required, there is no error. Furthermore, given convex objectives the zero–one variables are assigned in the proper order (from small to large) without specifically requiring it. The paper then progresses to the simple plant location problem with spatial interaction, in which the objective function comprises one part with costs and another with a slightly superlinear convex combinatorial function. Again, the errorless linearization is applied. The paper then discusses various solution techniques, e.g., dual ascent methods, Benders' decomposition, and hybrid methods. Computational tests on randomly generated problems with up to 50 potential plant locations and up to 200 customer locations show that problems are solved in less than 1 1/2 hours.

Hribar, M., Daskin, M. (1997) "A Dynamic Programming Heuristic for the p-Median Problem," *European Journal of Operational Research* **101**, 499–508.

This paper presents a dynamic programming based heuristic for the p-median problem. The stage variable is the number of facilities located so far in the progress of the algorithm and the state variable is the configuration of facility sites. The algorithm is heuristic in that only the best H solutions at any stage are retained in going to the next stage, thereby avoiding the curse of dimensionality. For $H = 1$ the algorithm reduces to the greedy algorithm and for infinitely large H, the algorithm is optimal. The paper shows that increasing H does not always improve the solution quality. Computational results are presented for 55 and 88 node problems. Using careful data structures, the execution time of the algorithm increases roughly linearly with H.

Jayaraman, V., Pirkul, H. (2001) "Planning and Coordination of Production and Distribution Facilities for Multiple Commodities," *European Journal of Operational Research*, **133** (2), 394–408.

This paper presents an integrated model for locating production plants and warehouses to minimize the sum of the plant and warehouse location costs, raw material acquisition costs, production costs, and shipment costs of (1) raw materials from vendors to plants, (2) finished products from plants to warehouses and (3) from warehouses to customer zones in a multi-product environment. The model is formulated as a mixed integer programming problem and solved heuristically using Lagrangean relaxation. Randomly generated problems with up to 75 customer zones, 15 candidate warehouse sites, 10 possible plant locations, 3 products and 2 suppliers had gaps under 3 percent and solution times under 1 minute. For larger problems with 150 customers, 30 candidate warehouses, 10 plants, 5 products, 3 vendors and 2 raw materials displayed gaps under 4 percent and solution times under 3 minutes.

Jayaraman, V., Ross, A. (2003) "A Simulated Annealing Methodology to Distribution Network Design and Management," *European Journal of Operational Research* **144**, 629–645.

The paper considers a multi-echelon PLOT (Production, logistics, outbound, transportation) distribution system that is modeled as a 4-partite graph: a central manufacturing plant on level 1, a variety of distribution centers on level 2, cross-docking stations on level 3, and customer outlets on level 4. The authors devise two problems: the strategic problem P1 that locates distribution centers and cross-docking stations among a finite set so as to minimize fixed costs to operate distribution centers and cross-docks, transport costs from distribution centers to cross-docs, and costs to ship units from the cross-docks to customer outlets. The operations problem P2 determines the quantities shipped through the network so as to minimize transportation costs. In the PLOT system, the problem P1 is solved first, and the solution is then incorporated in P2. The simulated annealing approach is then tested on a number of problems and its solutions are compared to the optimal solutions obtained with LINGO. The loss in objective value is about 4%, whereas the gain in terms of computation time is several hundred times.

Kalcsics, J., Nickel, S., Puerto, J. (2003) "Multifacility Ordered Median Problems on Networks: A Further Analysis," *Networks* **41**, 1–12.

The paper considers multifacility ordered median problems. These problems are known to have well-known models, e.g., p-medians, p-centers, and p-centdians, as special cases. The authors first determine a finite dominating set for the problem and then describe a polynomial algorithm for the problem on tree networks. The paper concludes by using this result in conjunction with approximation algorithms to state an efficient approximate solution on general networks.

Ku, S.-C., Lu, C.-J. Wang, B.-F., Lin, T.-C. (2001) "Efficient Algorithms for Two Generalized 2-Median Problems on Trees," *Lecture Notes in Computer Science* **2223**, 768–778.

The paper presents $O(n \log n)$ algorithms for two variants of the 2-median problem on a tree. The first variant is one in which the distance between the two medians is constrained to be less than some value. The second variant is one in which the eccentricity of the solution is constrained to be less than some value, where the eccentricity is the maximum distance between a demand node and the facility assigned to serve

the node. The authors show that the two variants can also be solved in $O(n \log n)$ time if the facilities can be located on the links as on the nodes. Finally, when the edge lengths and vertex weights are polynomially bounded, the problems can be solved in linear time.

Lau, F.C.M., Cheng, P.K.W., Tse, S.S.H. (2001) "An Algorithm for the 2-Median Problem on Two-Dimensional Meshes," *The Computer Journal* **44**, 101–108.

The paper discusses 1-median and 2-median problems on mesh networks in which customers have fixed locations and a demand of one. The 1-median problem is the same as the simple problem of finding 1-medians in the plane under the ℓ_1 metric. The authors then describe an algorithm for the 2-median problem. The computational complexity of the problems is $O(q)$ and $O(mn^2q)$, where m and n are the numbers of rows and columns with demand nodes, while q is the number of demand points in the network.

Lim, S.K., Kim, Y.-D. (2001) "Plant Location and Procurement Planning in Knockdown Production Systems," *Journal of the Operational Research Society* **52**, 271–282.

The paper considers a dynamic problem for a global manufacturing system that includes a home production base as well as local production bases anywhere on the globe. The authors formulate a large-scale mixed integer programming problem that includes fixed location costs, transportation costs, subcontracting costs, machine acquisition costs, and processing costs on new and existing machines. The large-scale dynamic problem is solved by means of a two-stage procedure. In the first stage, the dynamic plant location problem, including location costs, transportation costs, and subcontracting costs, is solved by means of a branch and cut procedure. On the basis of the resulting solution, the remaining multiperiod capacity planning problem is solved by means of a heuristic that includes a branch and cut procedure and a variable reduction procedure. The methods were tested on 120 randomly generated moderately-sized problems. Computation times are within 26 hours, and the average duality gap is 0.2%.

Marín, A., Pelegrín, B. (1997) "A Branch-and-Bound Algorithm for the Transportation Problem with Location of p Transshipment Points," *Computers & Operations Research* **24**, 659–678.

The paper investigates a standard transshipment problem in which supplies equal demands. The task is to choose p transshipment points so as to minimize transportation costs plus costs for establishing the transshipment points. The problem is shown to be an extension of the p-median problem. The authors first apply Lagrangean decomposition, keeping the dualized constraints in the problem that provides lower bounds. Sensitivity analyses are used to establish a dual ascent procedure to generate bounds. A branch and bound procedure is described, and a series of fairly small test problems from the literature is solved.

Melkote, S., Daskin, M. (2001) "An Integrated Model of Facility Location and Transportation Network Design," *Transportation Research A – Policy* **35**, 515–538.

This paper extends the classical uncapacitated fixed charge location problem on a network to include the possibility of link additions. The problem is formulated as a mixed integer programming problem. The problem is then recast as an uncapacitated fixed charge network design problem. Test cases are solved using a standard mixed integer programming solver (CPLEX 3.0) for problems with up to 40 nodes, 10 facilities, and 160 links. The results suggest that for small budgets, it is more important to invest in the network, whereas with large budgets greater investment should be made in the facilities.

Melkote, S., Daskin, M.S. (2001) "Capacitated Facility Location/Network Design Problems," *European Journal of Operational Research* **129**, 481–495.

This paper extends the capacitated fixed charge location problem to incorporate link additions. The problem is solved on randomly generated networks with up to 40 nodes and 160 links. Somewhat counterintuitively, the paper finds that both link investment expenditures and transport costs can increase as the facility capacity increases.

Mercer, A. (2001) "A Comment on Brimberg and ReVelle (2000): The Maximum Return-on-Investment Plant Location," *Journal of the Operational Research Society* **52**, 240–241 (and response by Brimberg, ReVelle, and Rosing, pp. 241–242).

The author of the comment draws Brimberg and ReVelle's attention to some work he did back in the late 1960s and early 1970 for firms in England. Also, he points out that he observed the occurrence of the "additivity property," i.e., the fact that the optimal locations of r facilities are part of the solution of the problem of locating $(r+1)$ facilities. In their reply, Brimberg, ReVelle, and Rosing point out that this "nesting" is an observed, but not a provable property. Furthermore, the property would allow the greedy algorithm to be used to find optimal solutions, which is not possible in general, as the problems under consideration are **NP**-hard. Still, greedy approaches can be useful if quick answers are required.

Nozick, L.K., Turnquist, M.A. (2001a) "Inventory, Transportation, Service Quality and the Location of Distribution Centers," *European Journal of Operational Research* **129**, 362–371.

The authors extend the inventory location model developed in Nozick and Turnquist (1998) to include coverage as a proxy for service. This results in a weighted multi-objective model similar to the median center tradeoff model discussed in Daskin (1995). The inventory model considers the safety stock required in an $(s-1, s)$, one-for-one replenishment policy to attain a given probability of stockouts. As in Nozick and Turnquist (1998), the authors assume that the safety stock inventory is a linear function of the number of distribution centers. To achieve a given level of service, as measured by the number of covered demands, additional distribution centers are required with the concomitant increase in inventory costs.

Nozick, L. K., Turnquist, M.A. (2001b) "A Two-Echelon Inventory Allocation and Distribution Center Location Analysis," *Transportation Research Part E* **37**, 421–441.

The authors present an integrated location-inventory model. Multi-product inventory is managed according to a one-for-one replacement policy. When minimizing the stockout penalties at a plant and at the distribution centers, they show that, for low annual demand rates, it is optimal to use a build-to-order strategy; for intermediate demand rates, inventory should be held centrally; and for large demand rates, inventory should be held at the distribution centers and the plant. Using this inventory model, which takes the number of distribution centers as an input, the authors estimate an additional fixed cost representing the average safety stock cost. This is used in a fixed charge location model which is solved for the facility locations. This results in a new estimate of the number of facilities, which can then be used in the inventory model to refine the estimate of the fixed cost for the location model. The process is iterated until a consistent location and inventory solution is identified. The process is illustrated using hypothetical data representing an auto manufacturer wishing to locate distribution centers for finished vehicles. A key assumption is that "the relationship between safety stock and the number of distribution centers is approximately linear."

Rao, V. Sridharan, R. (2002) "Minimum-Weight Rooted Not-Necessarily-Spanning Arborescence Problem," *Networks* **39**, 2 77–87

The paper addresses the problem of finding a minimum weight rooted, not-necessarily spanning, arborescence of a graph with arc weights which may be negative, zero, or positive. The paper shows that the uncapacitated facility location problem is a special case of the problem at hand. The paper provides an upper bounding heuristic for a variant of the problem in which some links are forced into the solution as well as a Lagrangean relaxation of the integer programming formulation that calls the upper bounding heuristic to convert the Lagrangean solution into a feasible solution. Computational tests on 50 problems with 10–55 nodes and 12–95 arcs showed that the procedure found the optimal solution 88% of the time. The average gap for the remaining 6 problems was 14%. In 12 problems with 26–301 nodes, the algorithm found the optimal solution 9 times in 500 iterations and in all 12 cases when the iteration limit was relaxed. CPLEX found the optimal solution in all 12 instances but could only verify optimality in the 7 smallest problems. The CPLEX solution time was approximately 50 times greater than that of the Lagrangean algorithm for the larger problems.

Rolland, E., Schilling, D., Current, J. (1997) "An Efficient Tabu Search Procedure for the *p*-Median Problem," *European Journal of Operational Research* **96**, 329–342.

The authors develop a tabu search algorithm for the *p*-median problem. The algorithm includes random tabu times (the value of which is changed and then held constant for some number of iterations within the algorithm), a standard aspiration criterion, a diversification strategy based on the frequency of selection of a node, and strategic oscillation to allow the number of selected sites to deviate from *p* for short periods of time. The algorithm is compared to a standard neighborhood search algorithm and the global/regional interchange algorithm of Densham and Rushton (1992) for randomly generated problems with between 13 and 500 nodes. The tabu search algorithm found the optimal solution for the small problems (up to 100 nodes) more frequently than did the other algorithms and had a smaller optimality gap when it did not find the optimal solution. Also, the tabu search algorithm took less time than did the other algorithms tested. For the larger problems, the tabu search algorithm found the best solution in all but one case and was, on average, better than the other two algorithms in terms of both solution quality and solution time.

Rosing, K. (2000) "Heuristic Concentration: A Study of Stage One," *Environment and Planning B* **27** (1) 137–150.

The paper examines the impact of (1) the power of the heuristic used to generate alternative solutions as inputs to stage 2 of a heuristic concentration (HC) algorithm, (2) the number of solutions generated and (3) the number used to identify the concentration set. In particular, the paper compares the efficacy of the Teitz and Bart (TB) algorithm with that of Maranzana (M). Concentration sets generated using the TB algorithm are more likely to result in optimal solutions than are those generated using the M algorithm. Also, the size of the concentration set required to ensure optimality grows with the problem size when the M algorithm is employed but not when the TB algorithm is used. Both findings indicate that the TB algorithm is preferred in generating the concentration set. The author suggests using a large number of runs of the initial heuristic. Both the size of the second phase optimization problem and the likelihood of finding an optimal solution increase with the number of solutions used to generate the concentration set.

Rosing, K., Hodgson, M. (2002) "Heuristic Concentration for the *p*-Median: an Example Demonstrating How and Why it Works," *Computers & Operations Research* **29**, 1317–1330.

This paper tests and discusses heuristic concentration on the *p*-median problem. The authors suggest that it works well because the cases in which the stable partitioning pattern obtained from Teitz and Bart differs from the optimal solution typically have geographically localized differences. Furthermore, some of the solutions that are used as inputs into the heuristic concentration algorithm will compensate for these differences, i.e., the other solutions will include the optimal nodes excluded by any stable partitioning pattern in question.

Rosing, K.E., ReVelle, C.S. (1997) "Heuristic Concentration: Two Stage Solution Concepts," *European Journal of Operational Research* **97**, 75–86.

The authors investigate different strategies for heuristic concentration. They first solve a total of 90 problems each 200 times with Teitz and Bart's vertex substitution method, starting with random initial solutions. Comparing the solutions obtained with this procedure with an optimal solution generated with an exact integer programming solution procedure, it is determined that the union of the five best heuristic solutions is quite likely to contain the optimal solution. This union can then form the concentration set, which is used in a standard *p*-median formulation, in which facilities are restricted to the nodes in the concentration set. A second possibility is to fix all those locations that appear in each heuristic solution and use as potential locations only those that appear in some, but not all, heuristic solutions. The *p*-median formulation that results from this process is very small and can easily be solved to optimality. The authors report that about 80% of the problems could be solved to optimality. Furthermore, the performance was better on larger problems.

Rosing, K., ReVelle, C., Schilling, D. (1999) "A Gamma Heuristic for the *p*-Median Problem," *European Journal of Operational Research* **117**, 522–532.

The authors propose a three-phased heuristic for solving the *p*-median problem. The first phase involves using a random start heuristic (e.g., the Teitz and Bart exchange heuristic based on random starting solutions) to form a concentration set, or a set of facility locations that occur in the top m solutions. In the second stage, the authors propose using a 2-opt exchange heuristic on the sites that are in a solution set but that were not in *all* of the top m solutions (i.e., the set they denote as C_f). This is followed by a 1-opt (Teitz and Bart) procedure using the entire solution space. The algorithm was tested on problems with between 100 and 300 nodes and between 5 and 50 facility sites, resulting in 81 test problems, each of which was solved 5 times using the heuristic. In 23 of the cases, the heuristic found the optimal solution every time; in only 9 of the cases, did the heuristic fail to find the optimal solution at all. In only one case was the worst-case gap greater than 1%.

Salhi, S. (2002) "Defining Tabu List Size and Aspiration Criterion within Tabu Search Methods," *Computers & Operations Research* **29**, 67–86.
The paper describes variants of tabu search as applied to p-median problems. An associated computational study of problems up to 1000 nodes and up to 100 facilities reveals that the performance of the suggested variants is very similar to those of the usual versions of tabu search.

Schmidt, G., Wilhelm, W.E. (2000) "Strategic, Tactical and Operational Decisions in Multi-National Logistics Networks: A Review and Discussion of Modelling Issues," *International Journal of Production Research* **38**, 1501–1523.
The paper surveys approaches to the solution of multi-echelon, multinational logistics problems. The authors consider the strategic, tactical, and operational levels separately. The strategic level includes the choice of plant locations, production technologies, and plant capacities, the tactical level features production levels, material flows, and inventory levels, and the operational level consists of mainly scheduling decisions. Each item is discussed in detail. A literature review is provided for each level separately. A profit-maximizing model is formulated. It includes revenues, fixed location costs, production costs, as well as transportation cost.

Shaw, D. (1999) "A Unified Limited Column Generation Approach for Facility Location Problems on Trees," *Annals of Operations Research* **87**, 363–382.
This paper shows that a variety of facility location problems on a tree, including the uncapacitated fixed charge location problem, the maximal covering problem and a number of other covering problems, can be structured as tree partitioning problems. The paper provides a polynomial time algorithm for the tree partitioning problem. When applied to each of these problems, the time complexity of the generic tree partitioning algorithm matches that of the best known specialized algorithm for the problems. The paper also shows that the p-median problem can be structured as a generic tree partitioning problem which can be solved in polynomial time using a bottom-up dynamic programming algorithm outlined in the paper.

Shen, Z.-J.M., Coullard, C., Daskin, M.S. (2003) "A Joint Location-Inventory Model," *Transportation Science* **37**, 40–55.
This paper presents an extension of the fixed charge location model to include a (Q, R) inventory model. The inventory model is approximated using an *EOQ* model with a safety stock term. The resulting integer programming problem adds a square root term to the fixed charge location model. The problem is recast as a set partitioning problem and solved using column generation. The non-linear pricing problem that results can be solved in low-order polynomial time. The approach is tested on problems with 33, 49, 88 and 150 cities.

Sherali, H., Park, T. (2000) "Discrete Equal-Capacity p-Median Problem," *Naval Research Logistics* **47**, 2, 166–183.
This paper formulates a capacitated p-median problem in which the capacity at each site is an endogenously determined integer multiple of some common base capacity, U. The authors propose an enhanced formulation based on a convex hull representation of a subset of the constraints. They propose an upper bounding heuristic based on solving a sequence of linear programming relaxations of the problem. The approach is tested on small

problems as well as larger problems with up to 47 customers and candidate sites. For an exact solution they recommend using the original formulation enhanced by a class of residual inequality constraints.

Teo, C.-P., Ou, J., Goh, M. (2001) "Impact on Inventory Costs with Consolidation of Distribution Centers," *IIE Transactions* **33**, 99–110.

This paper explores the impact of consolidating demand at distribution centers. The authors ignore the effects of the outbound transportation costs and focus only on the tradeoff between the facility investment costs and the inventory costs. They show that if demands are identically and independently distributed, then consolidation is an effective strategy. When demand follows a more general stochastic process, consolidation may not be effective. In a set of experiments they found that consolidation is likely to be effective as long as the mean to variance ratios of the demand locations do not differ by more than an order of magnitude.

Vairaktarakis, G.L., Kouvelis, P. (1999) "Incorporation Dynamic Aspects and Uncertainty in 1-Median Location Problems," *Naval Research Logistics* **46**, 147–168.

The paper considers 1-median locations on tree networks, in which node demand as well as edge distances vary over time according to a linear function. Furthermore, there is a set of scenarios that models the uncertainty of the parameters. The objectives considered in this paper are robust deviation and relative robustness, both criteria that are closely related to the usual minimax regret criterion. Most combinations of uncertainty, linear trend parameters, and objective result in low-degree polynomial algorithms that are linear in the number of scenarios and linear or quadratic in the number of nodes.

Zhao, P. Batta, R. (2000) "An Aggregation Approach to Solving the Network *p*-Median Problem with Link Demands," *Networks* **36** (4), 233–241.

The authors consider the *p*-median problem on a network with link demands. The well-known nodal optimality property does not hold in the presence of link demands. The authors provide an error bound, however, for locating only on the nodes. The bound depends only on the length and demand on a single link. With this insight they develop a scheme for aggregating the demands on the links to aggregation points so that the type C error (Hillsman and Rhoda, 1978) is avoided. They discuss how their aggregation approach can be embedded in both the Maranzana (1964) and Teitz and Bart (1968) heuristic algorithms.

3. Center and covering problems

The median and plant location models focus on the demand-weighted average (or total) distance. Such models are typically useful when a cost-based or profit-based objective is appropriate. In many contexts, however, the sum of distances is not an appropriate measure of the quality of the solution. This is particularly true when designing systems for emergency services. Following Rawls's "theory of justice" (Rawls, 1971) that the quality of a solution is no better than the worst-served entity, an objective emerges that attempts to maximize the lowest service standard to any of the customers in the problem.

To formalize, denote again I as the set of demand nodes and let J symbolize all candidate sites at which facilities may be located. The model that allows locations only at the vertices of a network is referred to as the *vertex-center problem* and it can be formulated as follows:

$$\begin{aligned}
\text{Min}\quad & z \\
\text{s.t.}\quad & \sum_{j\in J} y_{ij} = 1 \quad \forall i \in I && (2)\\
& y_{ij} - x_j \leqslant 0 \quad \forall i \in I,\ \forall j \in J, && (3)\\
& \sum_{j\in J} x_j = p, && (4)\\
& z - \sum_{j\in J} d_{ij} y_{ij} \geqslant 0 \quad \forall j \in J, && (8)\\
& x_j \in \{0,1\} \quad \forall j \in J, && (5)\\
& y_{ij} \in \{0,1\} \quad \forall i \in I,\ \forall j \in J. && (6)
\end{aligned}$$

The objective, in conjunction with constraints (8), minimizes the maximal facility customer distance. Constraints (3) again ensure that assignments can only be made to facilities that are actually open, while constraint (4) guarantees that p facilities are located. Constraints (5) and (6) are the usual integrality conditions.

Another possibility that ensures a predefined service standard is to choose the largest allowable distances between any customer and his closest facility. The objective of the *location set covering model* is to minimize the number of facilities that are needed to provide service to all customers. In fact, service standards are often stipulated as part of the legislation associated with the provision and funding of such services. In such cases, a covering-based model is often the most appropriate choice for location modeling.

Formally, demands at a node $i \in I$ are typically said to be covered by a facility at candidate site $j \in J$ if the distance (or travel time) between the nodes is less than some critical distance D_c. Other definitions of coverage are also possible. If node $i \in I$ can be covered by a facility at candidate site $j \in J$, then we set $a_{ij} = 1$, otherwise $a_{ij} = 0$. We also let $N_i = \{j : a_{ij} = 1\}$, i.e., N_i is the set of all candidate sites that can cover demand node $i \in I$.

With this additional notation, we can formulate the location set covering model, originally proposed by Toregas et al. (1971), as follows:

$$\text{Minimize} \quad \sum_{j \in J} x_j \tag{9}$$

$$\text{s.t.} \quad \sum_{j \in N_i} x_j \geqslant 1 \quad \forall i \in I, \tag{10}$$

$$x_j \in \{0,1\} \quad \forall j \in J. \tag{5}$$

The objective function (9) minimizes the number of facilities that are used. Constraints (10) state that for each demand node $i \in I$, at least one facility must be located within the set N_i of candidate facility sites that can cover the node. Constraints (5) are the integrality constraints.

The location set covering problem is **NP**-hard. Fortunately, in many practical instances, large versions of the problem can be solved to optimality using either commercial solvers or other software. In fact, a number of row and column reduction rules can be used to force facilities into or out of the solution and to eliminate redundant rows of a particular problem instance.

From a practical perspective there are a number of problems with the location set covering problem. First, the number of facilities required to cover all demand nodes often exceeds the available budget. Second, the model fails to discriminate between large demand nodes and small demand nodes. When it is impossible to cover all demand nodes within the specified service standard, it is often important to give priority to the nodes with the greater demand. Recognizing these limitations of the location set covering model, Church and ReVelle (1974) formulated the maximum covering problem shown below

$$\text{Maximize} \quad \sum_{i \in I} w_i z_i \tag{11}$$

$$\text{s.t.} \quad z_i - \sum_{j \in N_i} x_j \leqslant 0 \quad \forall i \in I, \tag{12}$$

$$\sum_{j \in J} x_j = p, \tag{4}$$

$$x_j \in \{0,1\} \quad \forall j \in J, \tag{5}$$

$$z_i \in \{0,1\} \quad \forall i \in I, \tag{13}$$

where z_i equals 1 if demand node $i \in I$ is covered and 0 if not. The objective function (11) maximizes the number of covered demands. Constraints (12) link the coverage and location variables and states that demand node $i \in I$ cannot be counted as being covered unless we locate at least one facility at one of the candidate sites that can cover the node. Constraint (4) limits the number of facilities to p, while constraints (5) and (13) are standard integrality constraints.

If the number of facilities needed to cover all demands exceeds the available resources, relaxing the requirement for 100% coverage is one option. Another option is to relax the service standard until a standard is found that allows for total coverage with the available resources. This is the approach adopted by the p-center model (Hakimi, 1964) which minimizes the maximum distance between a demand node and the nearest sited facility. With the notation defined in Section 2, this problem can be formulated as follows, where Q is the maximum distance to be minimized:

$$\text{Minimize} \quad Q \tag{14}$$

$$\text{s.t.} \quad \sum_{j \in J} y_{ij} = 1 \quad \forall i \in I, \tag{2}$$

$$y_{ij} - x_j \leqslant 0 \quad \forall i \in I, \ \forall j \in J, \tag{3}$$

$$\sum_{j \in J} x_j = p, \tag{4}$$

$$\sum_{j \in J} d_{ij} y_{ij} - Q \leqslant 0 \quad \forall i \in I, \tag{15}$$

$$x_j \in \{0, 1\} \quad \forall j \in J, \tag{5}$$

$$y_{ij} \in \{0, 1\} \quad \forall i \in I, \ \forall j \in J. \tag{6}$$

Note that this formulation is very similar to the formulation of the p-median problem except that the objective minimizes the maximum distance between a demand node and the facility to which it is assigned, and constraints (15) define the maximum distance in terms of the assignment variables. These constraints stipulate that Q must be greater than or equal to the assigned distance for each demand node.

In the remainder of this section, we summarize recent papers related to the covering problems. Papers are again listed alphabetically by the authors.

Adenso-Díaz, B., Rodríguez, F. (1997) "A Simple Search Heuristic for the MCLP: Application to the Location of Ambulance Bases in a Rural Region," *Omega* **25**, 181–187.

The paper formulates a max cover location problem for the location of ambulances in the province of León, Spain. The province has about 500,000 people and 213 counties, the smallest unit considered in this work. The solution method of choice is tabu search. With 25 ambulances, almost 99.5% of the population can be covered within 25 minutes. This compares favorably with 36 ambulances that would be needed to cover the entire population. Computation times are negligible.

Agarwal, P.K., Sharir, M. (1998) "Efficient Algorithms for Geometric Optimization," *ACM Computing Surveys* **30**, 412–458.

The paper surveys methods for geometric optimization problems under special consideration of two methods suggested by Megiddo: the parametric searching technique and the prune-and-search technique. The paper first describes the methods in general and then surveys their application to a large number of optimization problems under special consideration of complexity results. Among the location problems that are surveyed are Euclidean 1- and 2-centers, rectilinear p-centers, Euclidean p-line centers (in which given points have to be covered by p strips whose width is to be minimized), Euclidean p-medians, and segment centers (in which a given line segment is to be rotated, so as to minimize the maximum distance between any given point and the line) is to be minimized.

Agarwal, P.K., Procopiuc, C.M. (2002) "Exact and Approximation Algorithms for Clustering," *Algorithmica*, **33**, 2, 201–226.

The authors present a sub-exponential algorithm for the K-center problem in $\mathbb{R}^d$. The dynamic programming-based algorithm runs in $n^{O(k^{1-1/d})}$ time for any ℓ_p metric. They also present a $(1+\varepsilon)$ approximation algorithm for the K-center problem with running time $O(n \log k) + (\frac{k}{\varepsilon})^{O(k^{1-1/d})}$. Finally, the authors present an $n^{O(\sqrt{k})}$-time algorithm for the capacitated K-center problem on the plane under any ℓ_p metric, provided the capacities of the centers are not too small. In particular, the capacities must be either $O(1)$ or $\Omega(n/\sqrt{k})$.

Almiñana, M., Pastor, J.T. (1997) "An Adaptation of SH Heuristic to the Location Set Covering Problem," *European Journal of Operational Research* **100**, 586–593.

The paper describes and tests a new heuristic method for the location set covering problem. The method presented here is an improvement of Lopes and Lorenas' (1994) heuristic method for set covering problems. It combines Lagrangean relaxation and surrogate relaxations. In a series of tests of up to 500 columns, the new algorithm proves to be faster and better than its predecessor. Almost all problems are solved within 10 seconds.

Averbakh, I., Berman, O. (1998) "Location Problems with Grouped Structure of Demand: Complexity and Algorithms," *Networks* **31**, 81–92.

The authors study location problems in which demands by individual (or local) customers are grouped. The objective is then to optimize the needs of the resulting "global customers." Assuming that all facilities provide undistinguishable service, complexity results are derived for grouped covering, median, and center problems on trees. In all cases, the problems are **NP**-hard, in some cases even on path trees. The case of distinguishable facilities that may also include facility–facility interactions has polynomial algorithms for grouped covering, median, and center problems.

Averbakh, I., Berman, O. (2002) "Parallel NC-Algorithms for Multifacility Location Problems with Mutual Communication and Their Applications," *Networks* **40**, 1–12.

The paper investigates a class of location problems on trees and devise parallel algorithms to solve them. All problems have in common that they locate p distinguishable facilities on the tree, so that each facility is located within its feasible region (a subtree of the tree) and that there are upper bounds on the distances between the facilities. The authors then describe a variety of subproblems: First, let each node represent a customer and assume that the feasible region of each facility is defined by some upper bound on the facility–customer distances. The second subproblem is similar, except that the nodes are now partitioned into groups and there is an upper bound on the total distance between the nodes of that group and a facility. The next two subproblems are the minimax versions of the former problems, and the last subproblem is a structural representation of optimal sets of locations. The authors then describe parallel algorithms of complexity $O(\log(n+p))$ for the first two problems and $O(\log^4 p + \log^2 n)$ and $O(\log^2(n+p))$ algorithms for the minimax versions, where n is the number of nodes of the tree. Furthermore, the authors show serial algorithms of complexity $O(p^3 \log p + n^2)$ (the best known algorithm to date) and $O(p^3 + (n^2 + ntp + p^2)\log(n+p))$ (the first strongly polynomial algorithm known for the problem) for the two minimax problems, respectively, with t denoting the number of groups of nodes in the tree. All algorithms require a polynomial number of processors.

Berman, O., Drezner, Z., Wesolowsky, G.O. (2002) "The Collection Depots Location Problem on Networks," *Naval Research Logistics* **49**, 15–24.

This paper considers the location of a single facility with distances being the demand-weighted sum of the (a) distance from the facility to a demand node, (b) from the demand node to one of a number of known collection points (e.g., landfills), and (c) from the collection point back to the facility. Two variants of the problem are considered: the minisum version in which the objective is to minimize the demand weighted distance over all demand nodes and the minimax version which minimizes the maximum distance. The authors show that if there is only one collection point, the optimal solution is to locate the facility there. They also identify dominant sets of nodes for both variants of the problem. Finally, they analyze the problem on trees and develop additional solution properties for this special case.

Berman, O., Krass, D. (2002) "The Generalized Maximal Covering Location Problem," *Computers & Operations Research* **29**, 563–581.

This paper is similar to Berman et al. (2003). The authors formulate a model in which coverage at node i decreases in a step function manner with the distance between i and the facility assigned to serve node i. They formulate a strong and a "weak" formulation (weak in that it aggregates constraints of the strong formulation) and then show that in this case they give the same LP bounds. Extensive computational tests on problems between 20 and 400 nodes with up to 5 levels of coverage show that the greedy algorithm performs very well but that CPLEX has trouble solving problems, particularly with large numbers of nodes and moderate (10–50) facilities.

Berman, O., Krass, D., Drezner, Z. (2003) "The Gradual Covering Decay Location Problem on a Network," *European Journal of Operational Research* **151**, 474–480.

This paper is similar to Berman and Krass (2002). The authors formulate a model in which the coverage of a node i depends on the distance between i and the covering node. If the distance $d_{ij} \leqslant \ell_i$, the coverage function is 1; if $d_{ij} \geqslant u_i$, the function is 0; if $\ell_i \leqslant d_{ij} \leqslant u_i$, it is a non-increasing convex function of the distance. They show that the solution consists of locating on the nodes or on the set of points that are ℓ_i or u_i from a node. They formulate two variants of the problem which are shown to be special cases of the uncapacitated fixed charge location problem. The p-median and maximal covering models are also special cases of the problem they formulate. A small numerical example illustrates the two formulations. No computational experience is provided.

Bespamyatnikh, S., Bhattacharya, B., Keil, M., Kirkpatrick, D., Segal, M. (2002) "Efficient Algorithms for Centers and Medians in Interval and Circular-Arc Graphs," *Networks* **39**, 144–152.

Assuming unit lengths of edges and all demand located at the nodes, the paper derives efficient algorithms for p-median problems and p-center problems in interval graphs and circular-arc graphs. Starting with a linear-time algorithm (with preprocessing) for the 1-median in interval graphs, the paper then derives an $O(pn\log n)$ algorithm for p-median problems in interval graphs, an improvement over the previously known $O(pn^2)$ algorithm. Using these results for interval graphs, the authors then solve p-median problems on circular arc graphs in $O(pn^2 \log n)$ time. The paper concludes with an $O(pn)$ algorithm for p-center problems on circular arc graphs.

Bespamyatnikh, S., Segal, M. (1999) "Rectilinear Static and Dynamic Discrete 2-Center Problems," *Lecture Notes in Computer Science* **1663**, 276–287.

The paper presents polynomial time algorithms for the problem of locating 2 squares or rectangles that are parallel to the axes and centered at one of a finite set of n points so that each of m demand points is covered at least once and the size of the larger square or rectangle is minimized. For the case of a square, the authors present an $O(\max(n\log n, m\log n(\log n + \log^{d-1} m)))$ algorithm, where d is the dimension of the space. For the case of rectangles, the authors give an $O(mn^{d-1}\log^{d-1} m\log n)$ algorithm. The paper also considers the dynamic case in which nodes can be added or removed from the set of demand points. After each addition or remove, one wants to ask what is the minimal cover. For the case of squares, the authors provide an $O(m\log n(\log n + \log^{d-1} m))$ algorithm, while for the case of rectangles, the algorithm has complexity $O(mn^{d-1}\log n(\log n + \log^{d-1} m))$.

Bhatia, R., Guha, S. Khuller, S., Sussmann, Y.J. (1998) "Facility Location with Dynamic Distance Functions," *Journal of Combinatorial Optimization* **2**, 199–217.

In this work, the authors investigate dynamic versions of the basic p-center problem and a number of its variants. In particular, the length of the edges (a proxy for travel time) is assumed to be dynamic, e.g., due to traffic fluctuations. The problem is discretized into T time slots, so that within each such slot the problem is static. The main contribution of the paper is to devise approximation algorithms and determine their factors, i.e., the number of times the objective value is that of an optimal solution. For 2 time slots, factors are 3 for most variants, while for T time slots, no fixed factors are possible (except if $\mathbf{P} = \mathbf{NP}$), and all factors determined here are functions of the ratio between the longest and the shortest edge length of any edge in the graph.

Borras, F., Pastor, J. (2002) "The Ex-Post Evaluation of the Minimum Local Reliability Level: An Enhanced Probabilistic Location Set Covering Model," *Annals of Operations Research* **111** (1–4), 51–74.

The authors evaluate the accuracy of a number of stochastic location set covering models that have been proposed in the literature and introduce a new model, the queuing reliability location set covering

model. The models considered include the Ball and Lin (1993) model (which uses busy estimates around each facility and allows queuing) and the model of ReVelle and Hogan (in which the busy estimates are based on activity at a demand node). The new model is similar to that of Ball and Lin but does not allow queuing. The models are evaluated in terms of (1) the accuracy of the predicted reliability compared with that obtained by simulation and (2) the number of vehicles required to attain a given level of vehicle availability. The models were evaluated using a 55 node and a 79 node dataset. The new formulation attains the minimum local reliability more often than do either of the other models and does so with fewer facilities.

Burkard, R.E., Dollani, H. (2003) "Center Problems with Pos/Neg Weights on Trees," *European Journal of Operational Research* **145**, 483–495.

The paper investigates center problems in which nodes can have positive or negative weights, which are designed to indicate "friendly" and "obnoxious" facilities. The focus of the paper are algorithms with low computational complexity. While the problem can be solved in linear time on paths and star graphs, it takes $O(mn \log n)$ on a general network and $O(kn\log^2 n)$ on a tree, where k is a parameter that depends on the structure of the tree. The unweighted problem on a cactus can also be solved in linear time, while the discrete p-center problem on trees with pos/neg weights is solvable in $O(n^2 logn)$ time.

Caruso, C., Colorni, A., Aloi, L. (2003) "Dominant, an Algorithm for the p-Center Problem," *European Journal of Operational Research* **149**, 53–64.

The paper presents four algorithms that solve p-center problems; two to optimality and two heuristics. The results of the first heuristic are used as starting points for the other techniques. All methods are based on the solution of a sequence of set covering problems. The two heuristics, whose worst-case complexities are $O(p \ln n)$ and $O(p^2 n)$, respectively, solve problems up to 900 nodes in a few seconds, while the two exact methods, whose worst-case complexities are $O(pn(n/p)^p)$ and $O(pn^2(n/p)^p)$, respectively, solved many, but not all, of the larger problems within the 15-minute limit specified by the authors.

Daskin, M., Owen, S. (1999) "Two New Location Covering Problems: The Partial p-Center Problem and the Partial Set Covering Problem," *Geographical Analysis* **31** (3), 217–235.

This paper introduces two location covering problems: (1) the partial covering p-center problem minimizes a coverage distance such that a given fraction of the population is covered, and (2) the partial set covering problem seeks the minimum number of facilities needed to cover an exogenously specified fraction of the population within a given coverage distance. The problems are formulated as integer programming problems. Both problems are **NP**-hard. Also, node optimality does not hold for either problem. For the vertex-based variants of the problem, the paper proposes a bisection search approach with an embedded maximal covering algorithm. The algorithm was tested on two variants of a 150 node datasets. The results indicate that significant improvements in both objective function values can be achieved by relaxing the requirements for total coverage.

Espejo, L.G.A., Galvão, R.D., Boffey, B. (2003) "Dual-Based Heuristics for a Hierarchical Covering Location Problems," *Computers & Operations Research* **30**, 165–180.

The paper considers a hierarchical covering problem. There are two levels of facilities, e.g., clinics and hospitals. The model is based on the work by Moore and ReVelle (1982) in which the hierarchy is successively inclusive, i.e., the higher-level facility covers everything that facilities on lower levels cover. Given two levels, there are three types of coverage distances, one for the low-level service at the low-level facility, and one each for the low- and the high-level service at the high level facility. The authors formulate the problem as an integer programming problem and set up a combined Lagrangean and surrogate relaxation, resulting in a zero–one knapsack problem. A subgradient optimization method is then described to generate bounds and solve the problem. Computational experiments are performed on the 55-node Swain graph, randomly generated graphs, as well as graphs from Beasley's OR library for the p-median problem, the last having up to 700 nodes. Computation times were very reasonably throughout, as was the percentage gap with respect

to the optimal solution or, in case of larger problems where optimal solutions were not available, in relation to the upper bound.

Galvao, R., Espejo, L., Boffey, B. (2000) "A Comparison of Lagrangean and Surrogate Relaxations for the Maximal Covering Location Problem," *European Journal of Operational Research* **124**, 377–389.

The authors compare a Lagrangean heuristic (relaxing the linkage constraints) and a surrogate relaxation (adding weighted linkage constraints) for the maximal covering problem. They show that if you solve the integer programming subproblem (a knapsack problem) optimally for the surrogate relaxation, the resulting bound may be better than that of either Lagrangean relaxation or the linear programming relaxation of the maximal covering problem. When the subproblem is solved as a linear programming relaxation itself, the bounds are no better. Computational results bear out these theoretical results. The authors prove that the optimal Lagrange multiplier for any linkage constraint cannot exceed the demand at the node whose linkage constraint is relaxed.

Grubesic, T.H., Murray, A.T. (2002) "Constructing the Divide: Spatial Disparities in Broadband Access," *Papers in Regional Science* **81**, 197–221.

The paper studies the local impact of local socioeconomic factors on broadband network access. Residences and businesses are supplied with broadband access though central switching offices, of which there are more than 22,000 in the United States. Reliable service can only be provided to customers within about 12,000 feet of a central switching office. In order to maximize profit, the idea is to locate appropriate equipment at central switching offices so as to maximize coverage, resulting in a max cover problem. The objective is to minimize non-coverage, subject to the constraints that (1) ensure that each customer is either covered at least once, or is not covered (i.e., this is where non-coverage is defined), and the requirement that exactly p facilities are located. The authors then investigate Franklin County in Central Ohio. The county covers about 540 square miles with about one million people living in it. There are 39 central switching offices. Daytime population is used as a proxy expression for commercial demand, while residential demand is calculated as a function of the number of households, household income, and the level of education. The appropriate data are collected via the GIS system Arcview. Some of the results are surprising at first glance, but may be explained by the fact that low income – high density (typically inner city) areas are just as attractive to the model as are high income – low density (e.g., suburban) areas.

Harewood, S.I. (2002) "Emergency Ambulance Deployment in Barbados: a Multi-Objective Approach," *Journal of the Operational Research Society* **53**, 185–192.

The paper formulates a biobjective programming problem to locate ambulances on the island of Barbados. One objective minimizes the cost of serving customers, while the other maximizes multiple coverage, given a certain distance standard. The road network of Barbados as modeled has 382 nodes, 50 of which are potential ambulance locations. The problem is solved with one objective at a time, and a variety of sensitivity analyses is performed.

Hochbaum, D.S., Pathria, A. (1997) "Generalized p-Center Problems: Complexity Results and Approximation Algorithms," *European Journal of Operational Research* **100**, 594–607.

The paper examines two closely related problems, the p-SetSupplier problem and the p-SetCenter problem. In both problems, the nodes appear in two disjoint sets, V and W. The set of nodes in V is subdivided into k subsets, and there are given lower and upper bounds for each of these subsets. The p-SetSupplier Problem is then to determine centers, such that in each subset of nodes the number of chosen centers is between the prespecified bounds, and the maximum distance between any of the customers in W and its nearest center in any of the subsets of V is minimized. The p-Set Center Problem is defined as minimizing the maximal distance between any node in $V \cup W$ and its nearest center. The paper provides **NP**-hardness results for the two aforementioned problems as well as the related p-pairSupplier and p-pairCenter problems, which are defined as problems in which the p supplier or center nodes are each chosen from exactly one of p given pairs of nodes. Approximation algorithms with proven error bounds are described for

p-PairSupplier and p-PairCenter as well as p-SetSupplier and p-SetCenter problems, given that the triangle inequality holds.

Hu, C.-H., Egbelu, P.J. (2000) "A Framework for the Selection of Idle Vehicle Home Locations in an Automated Guided Vehicle System," *International Journal of Production Research* **38**, 543–562.

The paper describes the problem of locating automated guided vehicles at "dwell points" in an automated manufacturing environment. The problem is formulated as the standard p-center and p-median problems. For the p-center problem, the authors describe the well-known exact solution technique that solves a sequence of set covering problems. The heuristic starts with the given network and sequentially removes arcs until a solution is found. For the p-median formulation, a best addition greedy heuristic is described. The methods are illustrated by an example, followed by some thoughts on dynamic versions of the problems.

Karaata, M.H., El-Rewini, H. (2001) "On Minimizing the Cost of Location Management in Mobile Environments," *Journal of Parallel and Distributed Computing* **61**, 950–966.

The paper discusses the location of proxy agents in order to minimize the cost of search. This problem can be formulated as a p-center or a p-median problem, depending on the definition of costs. The authors present a simple heuristic for the problem on general graphs and an exact algorithm for p-center problems on networks. The solution strategy presented here is amenable to distributed implementation.

Khuller, S., Pless, R., Sussmann, Y. (2000) "Fault Tolerant K-Center Problems," *Theoretical Computer Science* **242** (1–2), 237–245.

This paper considers three variants of the K-center problem. In the first variant, every vertex must have at least α centers close to it, for $\alpha \leqslant K$. In particular, the problem tries to find K vertex centers such that the maximum distance from a vertex to the farthest of α centers is minimized. In the second variant, the minimization is taken only over vertices at which a center is not located. The third problem, termed the α-neighbor K-suppliers problem, is similar except that there is a set U of candidate centers and a set V of demand vertices; every vertex in V must have α centers close to it (e.g., the minimization is taken over the vertices in V). For the first problem, the paper provides a polynomial time approximation algorithm that achieves an approximation factor of 3 for any α and a factor of 2 for $\alpha < 4$. For the second problem, the paper provides an algorithm which achieves a factor of 2, and for the α-neighbor K-suppliers problem, the paper gives an algorithm which achieves a factor of 3.

Khuller, S., Sussmann, Y.J. (2000) "The Capacitated K-Center Problem," *SIAM Journal on Discrete Mathematics* **13**, 403–418.

The paper discusses p-center problems on graphs, in which each center has an upper bound on the number of customers it can serve. Clearly, the problem is **NP**-hard, and the purpose of the paper is to design approximation algorithms that run in polynomial time and that have an approximation factor ρ, where $100(1 - \rho)$ is the percentage deviation of the objective value of the approximation objective in comparison to the optimal objective. For the version of the problem that allows multiple centers to locate at a single vertex the authors devise an algorithm with factor 5. Disallowing multiple centers at the same node results in an algorithm with factor 6. For the version with fixed location costs and a budget constraint, an algorithm with factor 13 is devised.

Konemann, J., Li, Y., Parekh, O., Sinha, A. (2002) "Approximation Algorithms for Edge-Dilation k-Center Problems," *Lecture Notes in Computer Science* **2368**, 210– 219.

This paper considers the problem of locating k facilities (centers) on the vertices of a graph so that the maximum stretch of the graph is minimized. The stretch for a pair of nodes (u, v) is defined as the distance between nodes u and v on the path passing through the centers assigned to the nodes in relation to the direct distance between the pair of nodes on the graph. When each node is assigned to a unique center, the authors provide an approximation algorithm with a worst case bound of $4opt+3$, where opt is the value of the optimal solution. They also show that there can be no $5/4 - \varepsilon$ approximation algorithm (for $\varepsilon > 0$) unless $\mathbf{P} = \mathbf{NP}$. For the case in which each node can be assigned to multiple centers, they show that there exists an approximation algorithm whose stretch is bounded by $2opt + 1$. Finally for the case in which each center can have a maximum number of nodes assigned to it, they provide an algorithm whose worst case bound is $12opt + 1$.

Marianov, V., Serra, D. (2002) "Location-Allocation of Multiple-Server Service Centers with Constrained Queues or Waiting Times, *Annals of Operations Research* **111** (1–4), 35–50.

The paper formulates two variants of the location set covering problem that account for congestion at the servers. In the first model, exactly m servers are to be located at each selected site so that the probability of finding b or fewer customers in the queue is at least α. In the second model, the number of servers to be located at a site is endogenously determined. The models are solved using heuristic concentration with a Teitz and Bart exchange algorithm used in each of the two phases of the algorithm. The algorithm is tested on Swain's 55 node network. In all cases, the heuristic found a solution that minimized the number of sites and the number of servers. In cases for which an alternate optimum existed the algorithm did not always find the solution that also minimizes the demand-weighted distance.

Panigrahy, R., Vishwanathan, S. (1998) "An O(log $*n$) Approximation Algorithm for the Asymmetric p-Center Problem," *Journal of Algorithms* **27**, 259–268.

It is well known that the (asymmetric) p-center problem is **NP**-hard. Defining an α-approximation algorithm for a minimization problem to have an absolute performance ratio of α if the objective value of the approximation in relation the optimal objective value never exceeds the value of α, it is known that symmetric p-center problems have $\alpha = 2$, and, unless $\mathbf{P} = \mathbf{NP}$, this is the best bound possible. There is, however, no simple extension to asymmetric problems. Similar to traveling salesman problems that have $\alpha = 3/2$ for symmetric problems, while asymmetric problems have α as a function of $O(\log n)$, the authors repeatedly employ a greedy set covering algorithm to arrive at an $O(\log * n)$ complexity, given the optimum radius R as an input parameter in the problem.

Pelegrin, B., Canovas, A. (1998) "A New Assignment Rule to Improve Seed Points Algorithms for the Continuous k-Center Problem," *European Journal of Operational Research* **104**, 366–374.

The authors present new versions of seed point algorithms, whose first step is to generate p points for the p-center problem, and step 2 generates a partition of the demand points. The authors present three new ways to locate the seeds: the first two rules are based on farthest distances between a new point and already existing points, while the third rule is based on partitions. A new rule other than the usual "closest" rule is also outlined for assigning customers to seed points. The authors report on successfully solving some larger problems with their heuristics.

Rahman, S.-U., Smith, D.K. (1999) "Deployment of Rural Health Facilities in a Developing Country," *Journal of the Operational Research Society* **50**, 892–902.

The paper attempts to locate 10 health and family welfare centers and up to forty community clinics in a region in Bangladesh. While the objective is to maximize accessibility, existing facilities must be taken into consideration. Closing or relocating existing facilities is generally not an option. The authors formulate the problem as maximum covering problems, in which family welfare centers and community clinics are dealt with separately. They then perform a variety of sensitivity analyses, particularly on the largest allowable customer-facility distance and the number of facilities. It was shown that the existing locations of the family welfare centers were quite poor as compared to their optimal solutions.

Rayco, M.B., Francis, R.L., Tamir, A. (1999) "A p-Center Grid-Positioning Aggregation Procedure," *Computers & Operations Research* **26**, 1113–1124.

The purpose of the paper is to devise an algorithm that locates diamonds in a plane so that the center of each diamond represents a cluster of given demand points. When all diamonds are of equal size, the location of a single diamond will determine the locations of all of them; this is referred to as the diamond grid positioning problem. The positioning for p-center problems with rectilinear distances is determined, such that the upper bound of the error is minimized. Procedures for unweighted and weighted problems are developed. The unweighted and weighted problems can be solved in $O(n \log n)$ and $O(n \log^2 n)$ time, respectively, where n denotes the number of demand points. The procedure is subsequently tested on actual data from South Florida as well as on randomly generated point patterns. Furthermore, it is compared to an equal partitioning heuristic, which first determines the smallest diamond that covers all given points and subsequently subdivides that diamond into identical diamonds. The results indicate that both procedures have similar error bounds.

Tamir, A. (2001) "The k-Centrum Multi-Facility Location Problem," *Discrete Applied Mathematics* **109**, 293–307.

The author defines p-facility k-centrum problems as location problems that minimize the sum of the k largest customer-facility distances in a graph. Discrete versions of these problems were first described by Slater (1978) and later by Andreatta and Mason (1985). These problems reduce to p-center problems for $k = 1$, while they become p-median problems for $k = n$. While the problem on general graphs is obviously **NP**-hard, this paper derives new complexity results on paths and trees. The (absolute and discrete) single-facility problem on a path can be solved in $O(n \log n)$ time (the unweighted version in linear time), while on trees, the problem is solvable in $O(n\log^2 n)$ time. The discrete version of the multi-facility problem on a path can be solved in $O(kpn^4)$, while on trees it takes $O(\min\{k, p\}kpn^5)$. The continuous problems need to examine $O(n^2)$ and $O(n^3)$ potential sites, respectively. The single-facility problem on general graphs has a complexity of $O(mnk^{1/3}\alpha(n/k)\log^2 n)$ with $\alpha(n/k)$ being a very slowly growing function.

Tamir, A., Perez-Brito, D., Moreno-Perez, J.A. (1998) "A Polynomial Algorithm for the p-Centdian Problem on a Tree," *Networks* **32**, 255–262.

The paper deals with p-centdians on tree networks, i.e., linear convex combinations of p-median and p-center objectives, in which medians and centers are determined on the basis of (generally different) weights. The main tool is an r-restricted p-median problem, which is a conditional problem whose objective is of the minisum type, subject to the condition that no weighted node-facility distance exceeds a given parameter r. The authors then determine a finite dominating set of $O(n^3)$ points which is guaranteed to include an optimal solution of the p-centdian problem. While the p-median problem on trees can be solved in $O(pn^2)$ and p-center problems on trees are solvable in $O(n\log^2 n)$ time, p-centidan problems are considerably more difficult. The continuous version in which facilities can be located anywhere on the network was shown to be solvable in $O(pn^6)$ for $p \geqslant 6$ and $O(n^p)$ for <6, and the discrete version with facility locations restricted to the nodes of the tree can be solved in $O(pn^4)$ for $p \geqslant 4$ and $O(n^{\mathrm{p}})$ for $p < 4$.

4. Conclusions and outlook

The field of location theory and modeling, an area that traces its roots back to the first half of the 20th century with the seminal work by Weber (1909); Hotelling (1929); Christaller (1933) and Lösch (1954), remains an active and vibrant field. The annotated bibliography presented here has surveyed recent contributions to the field and put them in the context of classical location theory.

Extensions of the original problems include the location of undesirable facilities (such as polluting or hazmat facilities) as pioneered by Church and Garfinkel (1978), location models that take into account the reactions of competing facilities, probabilistic models, and many others.

Some new models appear to be particularly promising and we would like to outline three of them. The first considers different structures of customer demand. Arguably, the modeling of demand, while much discussed in the pertinent literature in marketing, has proved to be extremely difficult. Models ranging from gravity types to logit functions appear to be most promising in the context of location modeling. As an example, Ng and Tuan (2003) use gravity models in the location of production facilities in China. Much remains to be done in this context.

Another strand of research includes congestion along the roads and at the facilities. The inclusion of such queuing features is of growing importance in many practical models, e.g., the operation of fast food outlets where customers arrive in bulk during lunch hour and, to a lesser extent, in the morning and in the evening. Such uneven arrival patterns necessitate the inclusion of proper scheduling rules in the models. The paper by Berman and Drezner (2006) is a recent example of a contribution in this field.

Finally, we would like to mention models that, in addition to customers and facilities, include transshipment points. Typical examples are transfer points in the management of municipal solid waste systems, see, e.g., Eiselt (in press). Such transfer points were shown to be able to significantly reduce the costs of operating such management systems.

While some well-structured location problems are "integer-friendly," many realistic problems are not, necessitating the use of heuristic algorithms. In the context of location models, heuristics were first put forward by Cooper (1964) and Teitz and Bart (1968). Their usage has proliferated tremendously. Metaheuristics (see, e.g., Eiselt and Sandblom, 2004) are increasingly employed to solve realistic location problems in health care, national security, and other critical domains. Most prominently among them are genetic algorithms and tabu search. Some new references are Alp et al. (2003) and Mladenovic et al. (2003).

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