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Location and reliability problems on a line: Impact of objectives and correlated failures on optimal location patterns

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ABSTRACT

In this paper we study a class of locations models where facilities are not perfectly reliable and failures may be correlated. We analyze problems with Median and Center objectives under complete and incomplete customer information regarding the state of facilities. The goal is to understand how failure probabilities, correlations, availability of information, and problem objective affect the optimal location patterns. In particular, we want to find analytical confirmations for location patterns observed in numerical experiments with network location models. To derive closed-form analytical results the analysis is restricted to a simple (yet classic) setting: a 2-facility problem on a unit segment, with customer demand distributed uniformly over the segment (results can be extended to other demand distributions as well). We derive explicit expressions for facility trajectories as functions of model parameters, obtaining a number of managerial insights. In addition we provide the decomposition of the optimal cost into the closed form components corresponding to the cost of travel, the cost of facility unreliability and the cost of incomplete information. Most of the theoretical insights are confirmed via numerical experiments for models with larger (3–5) number of facilities.

1. Introduction

Service facilities can, and do, fail. Failures may occur for a variety of reasons, ranging from natural disasters, to locally disruptive events (strikes, power outages), to temporary shortages of capacity. While the traditional location theory tended to ignore the reliability aspects, there is a growing interest in better understanding the impact of imperfect reliability on the optimal location patterns and the resulting costs.

In the current paper we examine a class of models where service facilities may fail and the failure events may be correlated. We distinguish between two cases based on the information available to customers: the "complete information" (CI) case where customers are aware of the operational status of all facilities before they start their trip and thus travel directly to the closest operating facility (if one exists), and the "incomplete information" (II) case where such advanced information is not available and the customers have to search for an operating facility. We consider two classical objective functions, the *center* objective, where the goal is to minimize the maximum travel cost faced by any customer and the *median* objective, where the decision-maker is interested in minimizing the total travel cost. This leads to four different models describing facility-customer interaction.

Our primary interest in this paper is to investigate the interplay of four important aspects affecting facility reliability and location decisions:

- Probability of failure of a facility.
- Correlation of failure events (i.e., the extent to which the failure of one facility is informative of the operational status of other facilities).
- Information available to customers, i.e., CI vs. II cases.
- The decision-maker's objective function: median vs. center.

To date, most of the research on problems of locating unreliable facilities has focused on the median objective using a network topology and assuming that the facilities have identical probabilities of failure and that failure events are independent. The CI case was considered in Snyder and Daskin [26], as well as Berman et al. [7], while the II case was introduced in Berman et al. [8]. In the latter two references the following optimal location patterns were observed in the computational experiments: facility centralization, where facilities tend to locate closer to each other as the likelihood of failure increases, and facility co-location, where several facilities are placed in the same location to provide effective back-up in case of failure. Moreover, in [8] it

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was observed (also via computational experiments) that the centralization and co-location effects are stronger in the II case. For the case of the median objective and uncorrelated failures these two effects were also substantiated by asymptotic results (i.e., for the probability of failure approaching 1).

Recently Berman and Krass [4] studied the median problem with correlated failure events on a unit line for the CI case, where the centralization and co-location effects were established analytically. Unfortunately, the analysis used in [4] does not appear to be directly extendable to the II case or to the center objective.

In order to derive closed-form expressions capturing these effects under different informational assumptions as well as different model objectives we start with a simplified, yet classical, setting: continuous location of two identical facilities on a unit line segment with uniformly distributed demand. Our goal is to analyze the centralization and co-location patterns analytically (for cases other than the one covered in [4]) and to understand how these effects change as the objective and/or information available to customers changes. Specifically, we would like to know: do centralization and/or co-location effects occur for all four models or just some of the models? At what rate does centralization happen as a function of other model parameters, such as the probability of failure? How do the results differ for the complete/incomplete information cases? What is the effect of the correlation of failures? Do the insights established for the *median* objective remain valid for the *center* objective? After deriving closed-form solutions for the simplified systems, we perform numerical simulations for systems with larger number of facilities to make sure that insights obtained for the simpler systems continue to hold. We also extend our analysis by allowing different failure probabilities of the facilities as well as the non-uniformly distributed demand.

Our results shed new light on the optimal location patterns, including centralization and co-location effects. For problems with the median objective our results provide analytical confirmation to the experimental results of earlier studies: indeed the centralization effect is present and the centralization rate is much faster for the incomplete information case in comparison to the complete information case. Co-location above a certain probability of failure threshold is also confirmed for the incomplete information case (it is not present in the CI case). Surprisingly, both co-location and centralization patterns appear to be primarily due to the median objective: for the models with the center objective we observe the de-centralization pattern where the facilities tend to drift apart as the probability of failure increases (in the incomplete information case, the centralization reappears for sufficiently high probability of failure). The effects of unequal failure probabilities are also insightful: the more reliable facility is generally kept closer to the center, while the more failure-prone facility is kept closer to the periphery of the region; these effects are present for all four models.

In addition to analyzing the optimal location patterns, we provide the decomposition of the optimal cost into the components corresponding to the deterministic travel cost, the facility (un)reliability ("value of reliability"), and the presence of advanced information about the operational status ("value of information")—this decomposition was originally introduced in [8] for the model with the median objective, uncorrelated failures and incomplete information. We derive closed-form expressions for the cost components, and thus are able to determine when the informational/reliability component becomes the dominant cost driver. This enables the decision-maker to focus the attention on the improvements of the underlying system that will provide the largest benefits. Overall, the payoff of limiting our analysis to a stylized (2-facilities, linear demand) setting is the ability to obtain much sharper insights under more general modeling conditions than was previously possible.

The plan for the remainder of the paper is as follows. Our modeling framework is defined in Section 2, which also contains the main analytical results. In particular, the expressions for the optimal location configurations are derived in Section 2.3. Parametric analysis of optimal location patterns is presented in Section 3 (Section 3.3 describes numerical experiments for more than 2 facilities). The optimal cost decomposition and the resulting analysis is contained in Section 4. Section 5 discusses extensions to the case of non-identical failure probabilities and non-uniform spatial distribution of demand. Section 6 includes conclusions and directions for future research.

To close the current section we provide a brief literature review of the analysis of reliability in location models. This literature can be broadly divided into three streams: the "exogenous failures" stream where failures are assumed to be due to the factors exogenous to the model, the "endogenous failures" stream where the system is modeled as a spatially distributed queuing network and failures result from congestion at the facilities and are thus endogenous to the model, and the "intelligent adversary" stream where the failures are due to the actions of an intelligent adversary seeking to maximize the disruption in the system.

The "exogenous failures" stream, to which the current paper belongs, traces its roots to the reliability coverage model developed by Daskin [13] for the analysis of emergency response system (see [12] for an application). A more direct connection to the current work is Drezner [16] who formulated a model similar to our median and center problems with complete information and correlated failures and suggested simple heuristics for it. Drezner's paper assumes, for the center problem, that at most q facilities out of p may fail and makes several other strong assumptions (the CI case is assumed). Other papers in this stream include Lee [20] who developed a heuristic for Drezner's problem on a plane, [26,7,8] mentioned earlier, as well as Cui et al. [15], who study the uncapacitated facility location problem on a network where (uncorrelated) failure probabilities are site dependent. The only treatment of the correlated failure case we are aware off is by Li and Ouyang [21] who analyze a median complete information problem on the plane with correlated (and identical) failure probabilities; the analysis is done under quite stringent assumptions. The reader is referred to Snyder [25] for a more complete review.

The "endogenous failures" stream originated with the descriptive models which treated the locations as given (see Fitzimmons [17] for an early Monte-Carlo simulation model and the hypercube model of Larson [19] which uses queuing-based approximation). The optimal location modeling can be traced to Berman and Larson [9], who introduced the stochastic p-median model (see also Fitzsimmons and Srikar [18] for a heuristic approach based on the Monte-Carlo simulation model). We refer the reader to Berman et al. [5], Berman and Krass [3] and Snyder [25] for extensive reviews. We note that while in most queuing-based models, the facilities reliability is handled implicitly through service-level constraints, more explicit treatment of failures by customers to obtain the desired service (which is at the foundation of the "exogenous stream" described above) can be found in Berman et al. [6] and Berman et al. [2].

The "intelligent adversary" stream contains leader-follower games where the locational decisions are made under an assumption that an intelligent adversary will later disrupt a certain subset of the facilities, this is the so-called "interdiction" problem. While interdiction problems have been extensively studied in connection to the design of communication networks, their introduction to locational analysis is relatively recent and due to Church et al. [11].

Table 1
Joint failure probability distribution table.

		Facility 2		Sum
		Available	Not available	
Facility 1	Available	p_3	p_2	$(1-p)$
	Not Available	p_4	p_1	p
Sum		$(1-p)$	p	1

Further papers in this stream are Church and Scarparra [10], Scarparra and Church [23,24], O'Hanley and Church [22], and Berman et al. [1].

2. Modeling framework

We start by introducing our basic framework consisting of facilities and customers. Next we introduce the four models considered in the current paper. In Section 2.3 we derive explicit formulas for the optimal locations of two facilities.

2.1. Facilities, failures, and customer travel

Our basic framework consists of two facilities located on a unit line segment. Customer demand is assumed to be uniformly distributed over the line segment (this assumption will be relaxed later) with the total available demand assumed to be 1. We will use $x_1,x_2 \in [0,1]$ to represent both, the location of facility $i=1$, 2 and its distance from the origin (left end-point of the unit segment); we will assume $x_1 \le x_2$ i.e., that facility 1 is located to the left of facility 2.

We assume that a facility may "fail" at any point in time (here "failure" means that customers seeking to obtain service from the facility have to look elsewhere). We initially assume that both facilities are identical and thus have the same marginal probability of failure, p. This assumption is made in order for the results to be more transparent, as well as compatible with previous papers (as stated in the introduction, one of the key goals of the analysis is to provide analytical confirmation for the effects previously observed in computational experiments); we will relax this assumption in Section 5 below.

The joint distribution of failure probabilities is shown in Table 1.

Let ρ be the correlation coefficient of failure at the two facilities. It is easy to show that

$$\rho = \frac{p_1 - p^2}{p(1-p)}. \tag{1}$$

Note that the identical marginal failure probabilities assumption implies that the joint probability matrix is symmetrical, i.e., $p_4 = p_2$. Using (1) we obtain the following expressions for the remaining elements in terms of (marginal) failure probability of a given facility p and the correlation coefficient ρ; these expressions will be useful in subsequent analysis

$$p_1 = \rho p(1-p) + p^2, \tag{2}$$

$$p_2 = (1-p)p(1-\rho), \tag{3}$$

$$p_3 = (1-p)(1-p+p\rho). \tag{4}$$

The pair (p,ρ) have to be "compatible" in order for the resulting joint distribution to be valid (the reader is referred to Chaganty and Joe [14] for further discussion of this topic). In particular, it is not hard to show that the following conditions must be satisfied:

$$-\rho \le \min\left\{\frac{p}{1-p}, \frac{1-p}{p}\right\}. \tag{5}$$

We next turn our attention to the customer-facility interactions. We assume that customers travel to the facilities to obtain service, and that the service can only be obtained from an operational facility. For a customer located at $z \in [0,1]$ the travel distance to facility $i \in \{1,2\}$ is given by $|x_i - z|$; without loss of generality we assume unit travel cost per distance traveled. We further assume that customers *know where the facilities are located* before they start their trip, however the customers *may or may not know whether the facilities are operational*. Thus, we distinguish two cases.

In the *Complete Information* case ("CI"), the customers also know the operational status of both facilities before they start their trip. Thus, a customer travels directly to the operating facility that is closest to them, provided at least one facility is operating. If both facilities have failed, a fixed penalty cost β is incurred (this may represent the cost of having to obtain service from another service provider or the opportunity cost for the operator of the two facilities). Consider a customer located at $z \in [0,1]$ and let $x(z) = arg\ \min_{i=1,2} |z - x_i|$ be the closest facility to z. The expected travel cost for the CI case is given by

$$TC_{CI}(z) = p_3 |z - x(z)| + p_2(|z - x_1| + |z - x_2|) + p_1 \beta, \tag{6}$$

where the first term represents the case where both facilities are operational, the second where only one facility is operational, and the third where both facilities are not operational.

In the *Incomplete Information* case ("II") the customer does not have advanced knowledge of the operating status of the facilities. Thus, we assume that the customer always travels to the closest facility first. If that facility is operating, the travel stops there, otherwise the customer travels to the second facility. If this facility is operating, the service is provided; otherwise a penalty cost β is charged. For a customer located at $z \in [0,1]$ the expected travel cost for the II case is given by

$$\begin{aligned} TC_{II}(z) &= |z - x(z)| + p(|x(z) - x_1| + |x(z) - x_2|) + p_1 \beta \\ &= |z - x(z)| + p|x_1 - x_2| + p_1 \beta, \end{aligned} \tag{7}$$

where the first term represents the cost of travel to the closest facility, which is incurred with probability 1, the second term is the cost of travel between the facilities when only one facility is operational, and the third term is the penalty when both facilities have failed.

2.2. Model formulations

We are now ready to formulate the four location models analyzed in the current paper. Two of the classical objectives in location theory are the median and the center (also known as "minimax"). Under the median objective, the decision-maker seeks to minimize the expected travel cost of customers to facilities. The center objective seeks to minimize the maximum customer-to-facility travel distance (thus ensuring best possible service for the worst-off customers). In our setting, these objectives together with the two different informational cases defined earlier lead to the following four models.

Model MPUF-CI: (Median Problem with Unreliable Facilities and Complete Information) $\min_{x_1,x_2 \in [0,1]} \int_0^1 TC_{CI}(z)\, dz$.
Model MPUF-II: (Median Problem with Unreliable Facilities and Incomplete Information) $\min_{x_1,x_2 \in [0,1]} \int_0^1 TC_{II}(z)\, dz$.
Model CPUF-CI: (Center Model with Unreliable Facilities and Complete Information) $\min_{x_1,x_2 \in [0,1]} \max_{z \in [0,1]} TC_{CI}(z)$.

Model CPUF-II: (Center Model with Unreliable Facilities and Incomplete Information) $\min_{x_1,x_2 \in [0,1]} \max_{z \in [0,1]} TC_{II}(z)$.

2.3. Optimal locations

In this section we derive closed-form expressions for the optimal locations of the facilities for the four models stated above. The managerial implications and parametric analysis stemming from these results will be discussed in Section 3.

As noted in the introduction, model MPUF-CI was analyzed in [4]; the optimal locations are given by

$$S^{\text{MPUF-CI}} = \left\{ \frac{1+p(1-\rho)}{4}, \frac{3-p(1-\rho)}{4} \right\}. \tag{8}$$

Note that by (5), $p(1-\rho) \le 1$, ensuring that $x_1 \le \frac{1}{2} \le x_2$ in the solution above. Not surprisingly, the optimal location is symmetric, i.e., $x_2 = 1-x_1$. In fact, symmetry exists for all four models in our paper. When the facility failures are independent, i.e., $\rho = 0$, we obtain the following expressions for the optimal locations:

$$S^{\text{MPUF-CI}} = \left\{ \frac{1+p}{4}, \frac{3-p}{4} \right\}. \tag{9}$$

We next turn our attention to the MPUF-II model. Note that since the penalty term $p_1\beta$ does not affect the location decisions, we will generally omit it from the expressions below. We can derive the following expression for the objective function:

$$Z^{\text{MPUF-II}} = \min_{0 \le x_1 \le x_2 \le 1} \left[\int_0^{x_1} (x_1 - z)\, dz + \int_{x_1}^{(x_1+x_2)/2} (z - x_1)\, dz + \int_{x_2}^{1} (z - x_2)\, dz + \int_{(x_1+x_2)/2}^{x_2} (x_2 - z)\, dz + p \int_0^{(x_1+x_2)/2} (x_2 - x_1)\, dz + p \int_{(x_1+x_2)/2}^{1} (x_2 - x_1)\, dz \right], \tag{10}$$

Taking the integrals and solving the optimality conditions leads to the following result (the proof of this and all subsequent results can be found in Appendix).

Proposition 1. *The optimal location vector for the MPUF-II model is*

$$S^{\text{MPUF-II}} = \left\{ \min\left[\frac{1+2p}{4}, \frac{1}{2}\right], \max\left[\frac{3-2p}{4}, \frac{1}{2}\right] \right\}.$$

For the CPUF-CI model the objective function for this model can be rewritten as follows:

$$Z^{\text{CPUF-CI}} = \min_{x_1,x_2 \in [0,1]} \max_{z \in [0,1]} \left\{ (1-p) \min_{i=1,2} |z - x_i| + p_2 \max_{i=1,2} |z - x_i| \right\}. \tag{11}$$

We will use $Z^{\text{CPUF-CI}}_{x_1,x_2}(z)$ to refer to the expression in the curly brackets above. The first term in this expression corresponds to the case where the closest facility is operational and the second to the case where only the furthest facility is operational. The next result shows that the customer location with the longest expected travel distance must be one of three possible points; the symmetry of the solution is also shown.

Lemma 1. *For any facility locations $x_1, x_2 \in [0,\ 1]$ the maximum in (11) must occur for $z \in \{0, (x_1+x_2)/2, 1\}$, and thus the objective function is given by*

$$Z^{\text{CPUF-CI}}_{x_1,x_2} = \max\{ Z^{\text{CPUF-CI}}_{x_1,x_2}(z) \,|\, z \in \{0, (x_1+x_2)/2, 1\} \}.$$

Moreover, there exists an optimal symmetric solution to CPUF-CI with $x_2 = 1-x_1$.

Table 2
Summary of models and optimal locations.

Model name	Generalization of	Facility status	Optimal solution
MPUF-CI	Median	Known	$\left\{\frac{1+p(1-\rho)}{4}, \frac{3-p(1-\rho)}{4}\right\}$
MPUF-II	Median	Unknown	$\left\{\min\left[\frac{1+2p}{4}, \frac{1}{2}\right], \max\left[\frac{3-2p}{4}, \frac{1}{2}\right]\right\}$
CPUF-CI	Center	Known	$\left\{\frac{1-p+p\rho}{4}, \frac{3+p-p\rho}{4}\right\}$
CPUF-II	Center	Unknown	$\left\{\frac{1}{4}, \frac{3}{4}\right\}$

As a consequence of this lemma, for a given $x_1 \le 0.5$, the customer location which maximizes the righthand side of (11) must be $z \in \{0, 1/2\}$. Thus, the objective value of CPUF-CI can be expressed as follows:

$$Z^{\text{CPUF-CI}}_{x_1,x_2} = \max\{ (1-p)x_1 + p_2(1-x_1), (1-p+p_2)(\tfrac{1}{2} - x_1) \}. \tag{12}$$

The second term above is clearly decreasing in x_1, while the first term can be shown to be non-decreasing in x_1 over the relevant range (note that by (5) we know that $p \le 1/(1-\rho)$ must hold for all valid specifications of p and ρ). Thus, the expression above is minimized by finding x_1 where both terms have the same value. This leads to the following result.

Proposition 2. *An optimal solution to CPUF-CI is given by*

$$S^{\text{CPUF-CI}} = \left\{ \frac{1-p+p\rho}{4}, \frac{3+p-p\rho}{4} \right\}.$$

This solution is unique, unless $1-p+p\rho = 0$ (equivalently, $p_3 = 0$), in which case any location vector yields the same objective function value.

Note that when the facility disruptions are independent, $\rho = 0$ and the unique optimal solution is

$$x_1 = \frac{1-p}{4}, \quad x_2 = \frac{3+p}{4}. \tag{13}$$

Finally, we analyze the CPUF-II (the center problem with incomplete information). The equivalence of Lemma 1 is easy to establish for this model as well. This leads to the following expression for the objective function when facility 1 is located at $x_1 \in [0, \frac{1}{2}]$ and facility 2 at $x_2 = 1-x_1$

$$Z^{\text{CPUF-II}} = \text{Max}\{ x_1 + p_2(1-2x_1), (\tfrac{1}{2} - x_1) + p_2(1-2x_1) \}. \tag{14}$$

Observe that $p_2 = (1-p)p(1-\rho) \le 0.5$ for all values of $p \in [0,1]$ and $\rho \in [-1,1]$, implying that the first term is non-decreasing in x_1 while the second term is decreasing in x_1. This leads to the following result.

Proposition 3. *The unique optimal solution for Problem CPUF-II is given by $S^{\text{CPUF-II}} = \{\frac{1}{4}, \frac{3}{4}\}$. This solution is unique except when $p = \frac{1}{2}$ and $\rho = -1$ (i.e., $p_2 = 0.5$), in which case any location vector $\{x_1, 1-x_1\}$ with $x_1 \in [\frac{1}{4}, \frac{1}{2}]$ is also optimal.*

The models discussed in the current section and the optimal location results are summarized in Table 2. The managerial implications of these results are discussed in the next section.

3. Parametric analysis

In this section we examine how the optimal facility locations change in response to changes in the basic model parameters, p and ρ, focusing particularly on the centralization and co-location

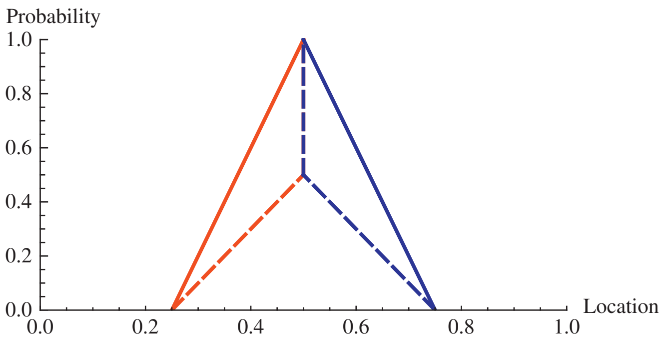

Fig. 1. Optimal locations for MPUF-CI (solid lines) and MPUF-II (dashed lines) as a function of failure probability p ($\rho=0$).

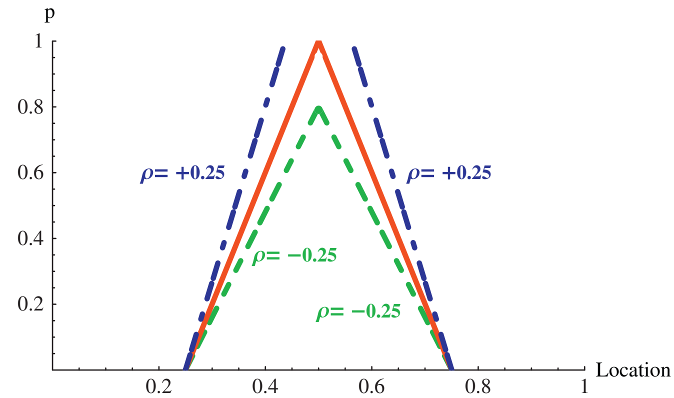

Fig. 2. Optimal locations for the MPUF-CI model for different values of ρ.

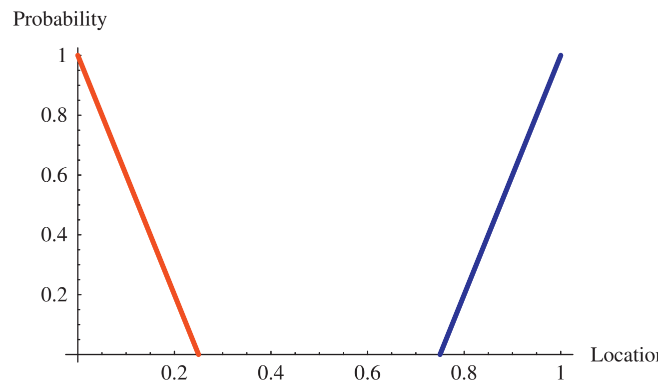

Fig. 3. Optimal locations for the CPUF-CI model with uncorrelated failure probabilities.

effects. We discuss the median-based models first in Section 3.1 and then the center-based models in Section 3.2.

3.1. MPUF-CI and MPUF-II models

First consider the independent (or, more precisely, uncorrelated) failure case where $\rho=0$. The trajectories of the optimal locations of the two facilities as a function of p are depicted in Fig. 1 for models MPUF-CI and MPUF-II, where locations derived from model MPUF-CI are represented by solid lines and those derived by model MPUF-II are represented by dashed lines; expressions for optimal locations are given in (9) and second row of Table 2, respectively (note that for the MPUF-II model the locations are not influenced by ρ).

For the underlying deterministic 2-median problem, the optimal locations are $\{\frac{1}{4},\frac{3}{4}\}$, corresponding to our $p=0$ case. We see that the MPUF-CI model exhibits the centralization effect—as p increases, the facilities move towards each other at the rate 1/2, but not the co-location effect (except in the limiting $p=1$ case). This movement is quite intuitive—as the probability of failure increases, so does the likelihood that a customer will have to seek service from the further facility; moving facilities towards each other decreases the possibility of overly long trips, at the cost of increased travel distance to the "preferred" facility.

The same effect can also be observed in a stronger form for the MPUF-II model. Due to the lack of information about the operational status of the facilities, longer trips are more likely in this case. As p increases, the model responds by moving the facilities towards each other at the rate of 1—twice as fast as in the complete information case. Moreover, we also observe the co-location effect: for $p\geq 0.5$, the two facilities are co-located at the mid-point of the segment. The increased cost of failures in the incomplete information case is quite clear.

We next turn our attention to the correlated failure case (i.e., $\rho\neq 0$). For the MPUF-CI model the relevant equation can be found in the first row of Table 2; the corresponding location trajectories as functions of the failure probability p are depicted in Fig. 2. Compared to the uncorrelated failure case, we see that the effect of the positive correlation of failures is to slow down the centralization effect (i.e., the inter-facility distance is increased for the same value of p compared to the uncorrelated case), while the negative correlation increases the centralization effect. As noted earlier in (5), not all pairs of values of $\rho\in[-1,1]$ and $p\in[0,1]$ are possible when $\rho<0$; for example for $\rho=-0.25$ trajectory displayed on the figure we must have $p\in[0,0.8]$; at $p=0.8$ the two facilities are co-located when $\rho=-0.25$. In general, co-location occurs for $\rho\leq 0$ only when p reaches its maximum value of $1/(1-\rho)$.

The intuition behind these location patterns is fairly clear: when the correlation of failures is positive, the likelihood that both facilities are operational or inoperational simultaneously is increased, making the possibility of a trip to the further facility less likely; hence the facility locations move further apart and closer to the deterministic 2-median pattern. On the other hand, when the correlations of failures are negative, the chances that the service will need to be obtained from the furthest facility is increased, hence the facilities move closer together.

We note that the reason the MPUF-II solution is not affected by ρ is due to our analysis being limited to the 2-facility case only (note that the trip to the closest facility occurs with probability 1 in this model). However, it is interesting to note that the location trajectory observed for MPUF-II in Fig. 1 is the lower envelope of the trajectories for the MPUF-CI case (the $\rho=-1$ case). Thus, from the decision-maker's point of view, the case when customers do not have advance information about the operational status of the facilities is similar to the CI case where only one facility is operational at any given time. This points to the value of the information when facilities are subject to failure; we will analyze this point further in Section 4 below.

3.2. CPUF-CI and CPUF-II models

First note that for the deterministic 2-Center model, the optimal solution is given by $\{\frac{1}{4},\frac{3}{4}\}$, coinciding with the 2-Median solution. For the uncorrelated case, the trajectory of the optimal solutions as a function of p for the CPUF-CI model is given by (13) and depicted in Fig. 3. The behavior of the optimal solution is the opposite of what we observed for the MPUF-CI model earlier: as the failure probability p increases, we observe a "decentralization" effect, since the inter-facility distance grows linearly with p. This behavior may, at first, appear quite counter-intuitive. The reasons behind this behavior are explained by Lemma 1 which

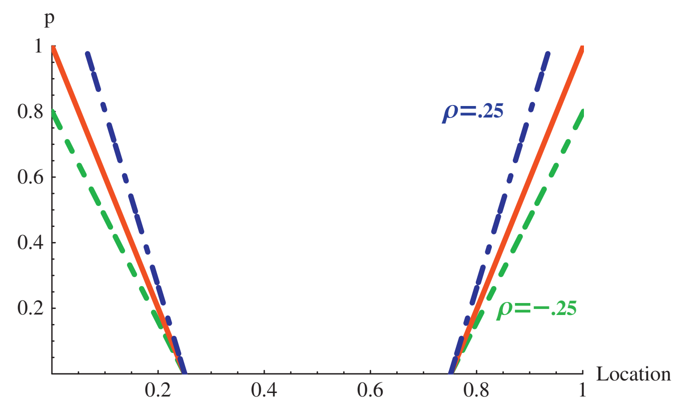

Fig. 4. Optimal locations for the CPUF-CI model for different values of ρ.

shows that for symmetric location patterns, the optimal locations are determined by the expected travel distances to the facilities from just two customer locations: the left endpoint and the midpoint of the segment; at the optimum, the expected travel distances for these customer locations must be the same. This is indeed the case for the deterministic 2-Center solution (i.e., when $p=0$). When p increases the expected travel cost increases for both customer locations, but more so for the endpoint location (because of the much longer travel distance to the further facility). To compensate for that, the facilities are moved closer to the endpoints (recall that the expected travel cost from the left end-point is an increasing function of x_1).

The optimal location trajectories for the CPUF-CI model with correlated failures are depicted in Fig. 4; by Proposition 2, the inter-facility distance for this model is given by $(1+p-p\rho)/2$. Once again we observe the opposite behavior to the MPUF-CI model: for positive correlation of failures, the facilities tend to be closer to each other than for the negative correlation case. The explanation for this behavior is provided by (12). An increase in ρ decreases the likelihood of travel to the further facility (since when the closest facility fails the probability of failure for the second facility is higher for larger ρ). This benefits the customer at the left endpoint more than the customer at the mid-point, due to the larger travel distance to the further facility. Thus, in order to maintain equal costs for both customers, the facilities must move towards the midpoint. When $\rho<0$ the effect is reversed: the likelihood of only one facility being operational is larger, which translates into a larger cost increase for the customer at 0 and a smaller increase for the customer at 0.5. To off-set this, the primary (closer) facility is shifted towards the respective endpoint.

The preceding discussion also yields another insight: for given values of p and ρ, moving the facilities towards the middle benefits the mid-point customer and penalizes the end-point customers, while moving the facilities towards the endpoints does the opposite. Moreover, it is not hard to show that the rate of change (increase or decrease) in the cost for the endpoint customers is smaller than for the midpoint customer. Since the CPUF-CI objective is driven by the maximum cost, the rate of change in the cost function is smaller when facilities are moved towards the middle compared to when they are moved towards the endpoints. This suggests that miscalculating the optimal location is less costly in the direction towards the mid-point of the segment. Thus, if one has to make an error with respect to the optimal location, one would rather be too far from the endpoints than too close.

To close this section we note that the optimal locations for the CPUF-II model are independent of both p and ρ, coinciding with the 2-Center solution. This result is certainly due to the limitations imposed by restricting our analysis to just two facilities. However, as shown in the next section, the results observed for the two-facility case appear to be surprisingly robust, largely extending to the higher number of facilities as well.

3.3. *Numerical results with $K=3, 4, 5$ facilities*

In this section we investigate the extent to which the analytical results derived for the 2-facility case are generalizable for larger number of facilities. As noted earlier, the only analytical solutions that are currently available for the $K>2$ case are for the MPUF-CI problem (see [4]). For other models the results presented in this section are based on heuristic search.

We proceed as follows. First, we specify a given number of facilities K, marginal failure probability p (assumed to be identical for all facilities) and the correlation coefficient ρ, where the correlation matrix is assumed to have all off-diagonal entries equal to ρ. We used $\rho=-0.1,0,0.25$. Note that in order to maintain feasibility, when $\rho=-0.1$ we must have p belong to [0.15, 0.85], [0.2, 0.8] and [0.3, 0.7] for $K=3$, 4, 5, respectively. Next, we generate a joint probability distribution which satisfies the specified marginal distribution and correlation matrix. To do so, we follow the algorithm suggested by Qaqish [27]. This generates a problem instance for each of our four models, which are then solved via the "heuristic search" as follows: we assume (supported by many trial examples) that the optimal solution is symmetric about the midpoint of the interval (for odd K this means locating one of the facilities at the midpoint) and then conduct a nested line search on the half-interval to locate the facilities. This procedure was followed for $K=3$, 4 and 5. The results for the models with median objectives is displayed in Fig. 5—the CI results are in the left column and the II results in the right one. First observe that for the MPUF-CI model, the overall behavior of the optimal solutions is quite similar to the $K=2$ case discussed earlier—a centralization of facilities as the probability of failure increases, with the facilities being more centralized when the correlation is negative (since the probability of having to travel to a further facility increases) and less centralized when the correlation is positive. It is interesting to note that the trajectories followed by facilities are concave, with the degree of concavity increasing with the number of facilities. Thus the model tries to "resist" the centralization effect (which, of course, increases the expected travel cost vs. the median solution) as long as possible; the more facilities there are, the less centralization is required for the same probability of failure. For example, for $p\approx 0.3$ there is barely any centralization for the $K=5$ case, while the centralization is quite significant for the $K=2$ case. We also note that while the "outer" facilities (i.e., the ones closest to the endpoints of the region) move towards each other, the distance between the "inner" facilities tends to increase - this is particularly obvious for the $K=5$ case. This is the "peripheral concentration" first observed in [4]—the outer facilities are backed up by the inner ones to reduce the probability of long trips (the customer is more likely to find an operating facility in close proximity to the first facility they reach).

Turning our attention to the MPUF-II case (the plots in the right column), we see the same general tendencies as for the $K=2$ case: facilities are centralizing at a much greater rate than in the CI case (note that the trajectories of motion are generally convex), the centralization leads to co-location above certain threshold probability of failure (this threshold is increasing with the number of facilities), and the centralization is weaker for the positive correlation case. However, there are some intriguing differences from the $K=2$ case as well, particularly for $K=4$, 5. For the $K=4, \rho=+0.25$ case, the two inner facilities initially start out moving towards each other, but around $p=0.4$, this motion is

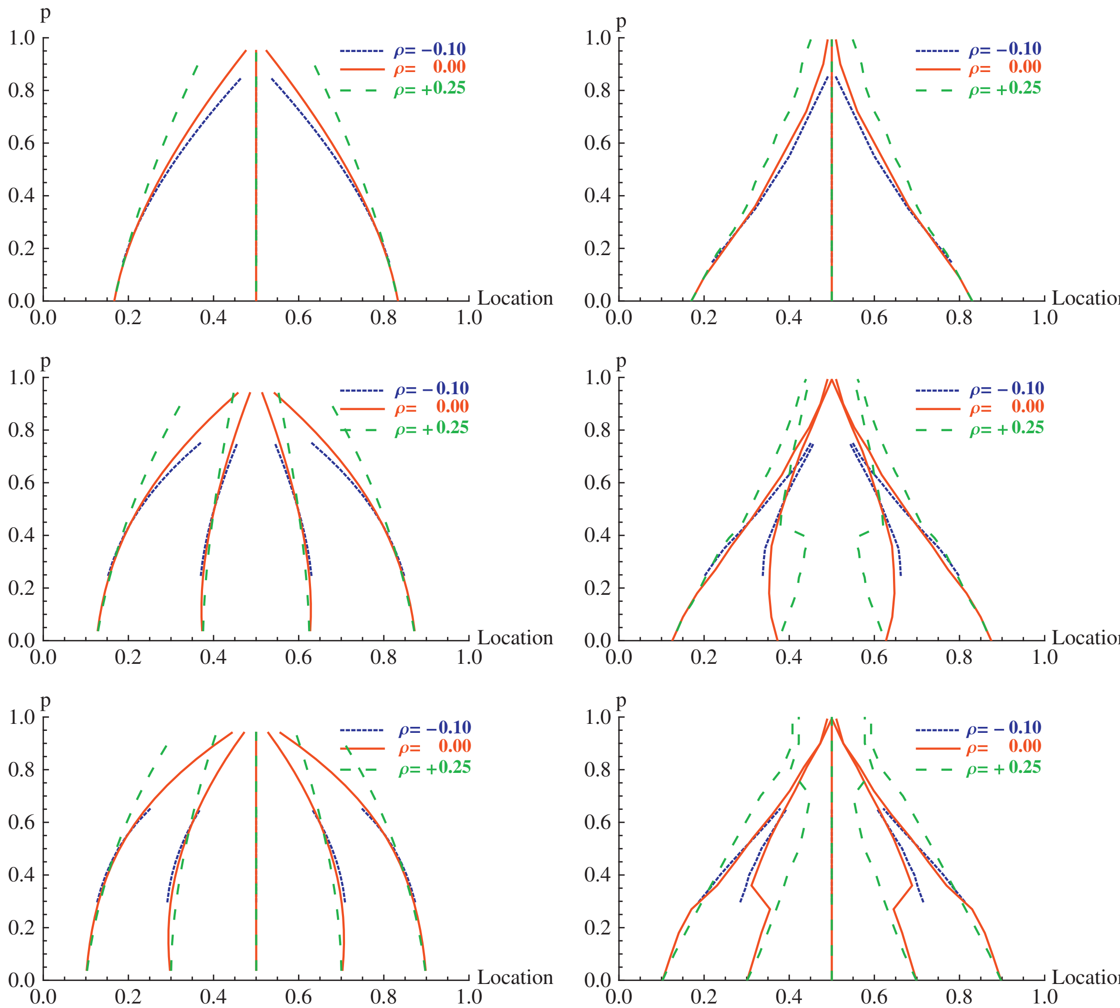

Fig. 5. Optimal locations for MPUF-CI (left column) and heuristic-optimal locations for MPUF-II for $K=3$, 4, 5.

reversed and the facilities quickly move towards the outer facilities (while this might be an artifact of the heuristic solution, a thorough numerical search appears to confirm that this likely reflects the behavior of the optimal solution). This can be interpreted as the "peripheral concentration" effect described above, but it sets in quite sharply, rather than gradually as in the CI case. We see the same behavior for $K=5$: the sharp trajectory reversal is now present at $p=0.8$ for the positive correlation case and $p=0.3$ for the uncorrelated case. Thus, for the II model when the probability of long trips is low (either due to low failure probability or higher positive correlation), the peripheral concentration effect is not present, and then sets in quite suddenly when a certain failure probability threshold is crossed.

The plots for the models with the center objective are presented in Fig. 6 with the plots for the CI case are on the left and the II case on the right. For the CI case the results are generally in line with the analytical results for $K=2$ case: we see a decentralization effect, with all facilities moving towards the endpoints of the interval. It is interesting to note that for $K=4$, 5 cases the movement is non-linear and different for inner and outer facilities—the former have concave trajectories, while the latter convex ones. Thus, while the outer facilities move towards the endpoints quite quickly as the failure probability increases, the inner facilities maintain a more central position until the failure rate is quite high. The concentration of facilities at the endpoints does not happen until the failure rate is quite high. The impact of correlation on optimal locations is generally weak and going in the opposite direction than for median models: higher centralization for positive correlation and lower for negative one.

Finally, the behavior of (heuristic-)optimal locations for the II case with the center objective is quite different. Recall that in the $K=2$ case, neither the failure probability, not the correlation coefficient had any impact on the optimal locations. Surprisingly, this appears to be not just an artifact of limiting the analysis to $K=2$ case: the impact of correlation for the $K=3$–5 cases is very low, and there is a low sensitivity to the failure probability p for large ranges of values. For example, for $K=5$, the initial 5-median solution is essentially unchanged until p approaches 0.7, at which point the two outer facilities collapse to the inner ones, and the solution is unchanged above that. Similar behavior can be seen for $K=3$, except that here the "collapse" happens earlier—at $p=0.6$. Thus the lack of advance information together with the center objective appear to contribute to very stable solution patterns.

In summary, numerical analysis of heuristic-optimal location patterns with $K>2$ largely confirm the intuition obtained for the 2-facility case, however the trajectories of the facilities are more complex in these cases.

4. Cost decomposition: value of reliability and information

In this section we decompose the total cost function into components corresponding to the deterministic travel cost, the cost of facility failures and the cost of not having the advanced

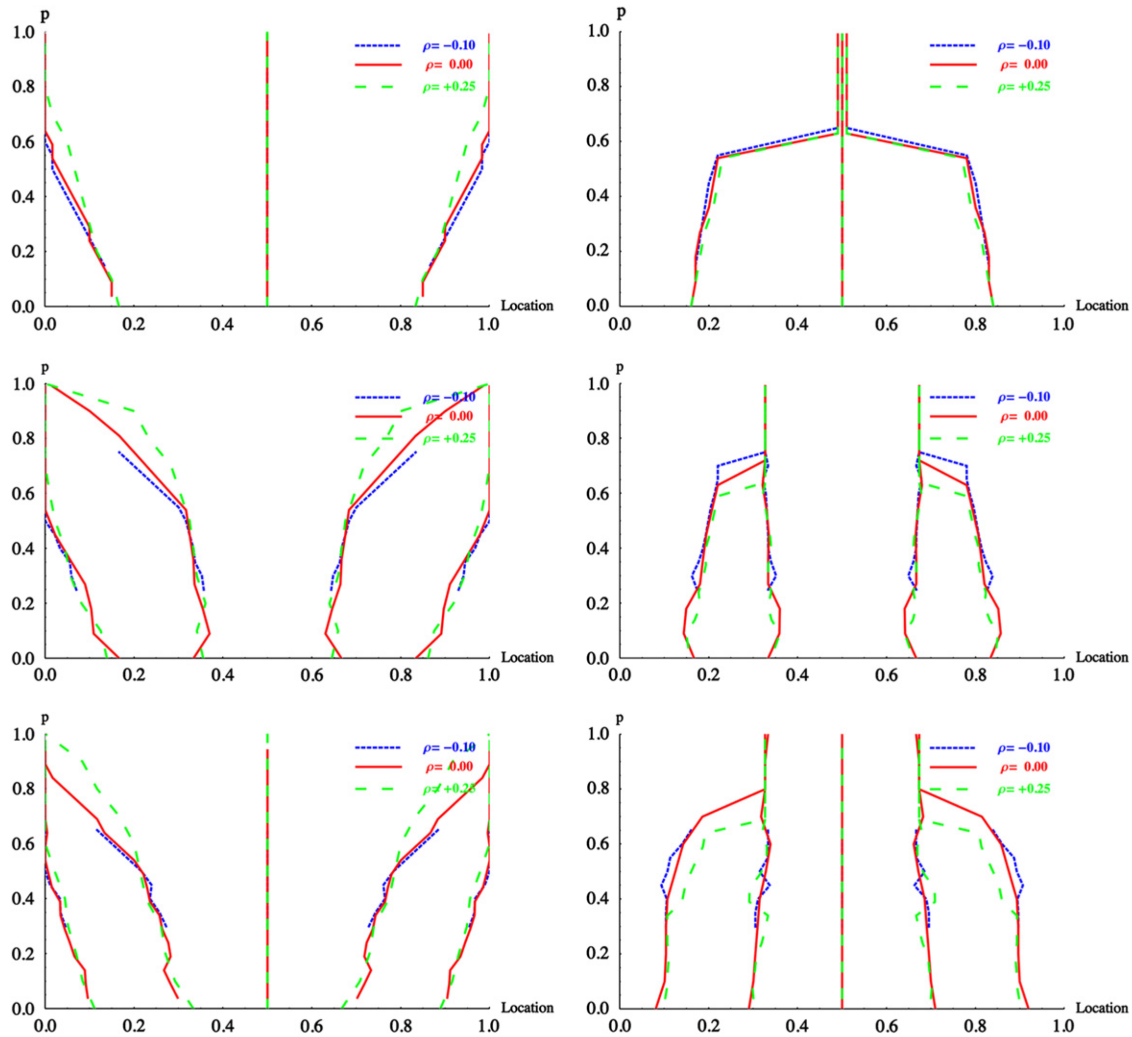

Fig. 6. Heuristic-optimal locations for CPUF-CI (left column) and CPUF-II (right column) for $K=3$, 4, 5.

information. For the MPUF model (median objective), the decomposition takes the following form:

$$Z^{\text{MPUF-II}} = Z^{MEDIAN} + (Z^{\text{MPUF-CI}} - Z^{MEDIAN}) + (Z^{\text{MPUF-II}} - Z^{\text{MPUF-CI}}). \quad (15)$$

Here, the first term, i.e., the value of the median objective represents the deterministic travel cost. The second component, $Z^{\text{MPUF-CI}} - Z^{MEDIAN}$ represents the cost of facility failures (as the failure probability $p \to 0$ this component approaches 0), we will refer to this component as the "Value of Reliability" (VoR). Similarly, the third component, $Z^{\text{MPUF-II}} - Z^{\text{MPUF-CI}}$ represents the value of providing customers with the advanced information about the operational status of the facilities; we refer to it as the "Value of Information" (VoI) component.

For the CPUF model the same decomposition is given by

$$Z^{\text{CPUF-II}} = Z^{CENTER} + (Z^{\text{CPUF-CI}} - Z^{CENTER}) + (Z^{\text{CPUF-II}} - Z^{\text{CPUF-CI}}), \quad (16)$$

with the VoR and the VoI components given by the second and third terms, respectively.

In general, by evaluating the contribution of each component to the solution, a decision-maker can answer the following questions:

- Are we investing enough in opening more facilities? If the first component is dominant, opening additional facilities should be considered.
- Should the focus be on improving the reliability of the facilities? This would be the case when the VoR component is dominant.
- Should the focus be on providing advance information on the status of facilities? This would be signaled by the large value of the VoI component.

While the decomposition (15) was originally introduced in [8] for the MPUF-II model on a network, the analysis of the linear 2-facility models allows us to evaluate the VoR and VoI components analytically and to provide full parametric analysis of their values with respect to the failure probability p and failure correlation ρ parameters.

One important note is that this analysis is not possible without re-introducing the penalty term β (incurred whenever all facilities have failed and no service can be provided) back into the model. While this term does not play a role in determining the optimal location (and thus was omitted in the previous sections), it certainly does play a role in the cost decomposition. Indeed, when $\beta = 0$ it is easy to see that the cost of 0 is achieved in $Z^{\text{MPUF-CI}}$ and $Z^{\text{CPUF-CI}}$ models when the facilities fail with probability $p=1$, since no travel can take place in this case. Moreover, the values of VoR terms in (15) and (16) is negative in this case. To avoid this nonsensical situation, we will set $\beta = kZ^{MEDIAN}$ and $\beta = kZ^{CENTER}$ for the MPUF and CPUF models, respectively, where

$k \geq 1$. This is based on the intuition that the disutility to the customer of not being able to obtain service must be at least as high as the travel cost in the failure-free environment (represented by Z^{MEDIAN} and Z^{CENTER}); in fact this quantity provides a convenient scaling factor—k reflects the relative increase in disutility over the failure-free case; one would expect $k>1$ (as we will see below, $k=1$ does not provide a sufficiently high penalty cost to avoid counter-intuitive results). In practice, when unserved customers must be satisfied by contracting the service from a third-party, k should be easy to estimate. If the unserved demand is lost, the estimation should be based on assessing the foregone future revenue streams.

The cost decomposition for the MPUF and CPUF models are analyzed in Sections 4.1 and 4.2, respectively. As before, for the sake of transparency we assume that the marginal probabilities of failure for the two facilities are the same; while the extension to the unequal failure probability case is certainly do-able, the formulas become messier, with no significant gain in insights.

4.1. Cost decomposition for the median objective

Plugging in the optimal solutions in (8) and Proposition 1 into the objective function of problem CI (see [4]) and expression (10), respectively, and recalling that $Z^{MEDIAN}=\frac{1}{8}$, we obtain, after simplifications, the following expressions:

$$Z^{\mathrm{MPUF-CI}}=\frac{1}{8}((1-\rho)^2p^3+(1-\rho)(k+\rho-5)p^2 +((k-4)\rho+3)p+1) \tag{17}$$

and

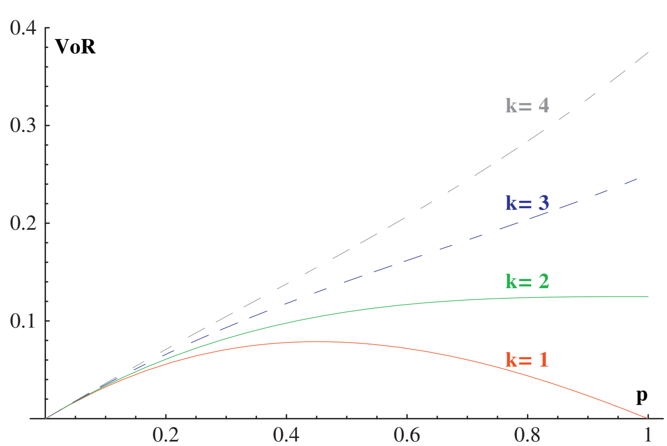

Fig. 7. VoR for the MPUF model as function of p for different values of k (independent failure case).

$$Z^{\mathrm{MPUF-II}}=\begin{cases}\frac{1}{8}\left(1+p\left(4-p\left(4-\frac{k}{8}\right)+\frac{k}{8}(1-p)\rho\right)\right), & \text{if } p\leq\frac{1}{2};\\ \frac{1}{4}\left(1+\frac{kp}{2}(p+\rho(1-p))\right), & \text{otherwise.}\end{cases} \tag{18}$$

These expressions can now be used to derive VoR and VoI terms explicitly. The resulting expressions are particularly simple for the uncorrelated failures case $\rho=0$ and are given below

$$VoR=\frac{1}{8}(p^3-(5-k)p^2+3p) \tag{19}$$

and

$$VoI=\begin{cases}\frac{1}{8}(p+p^2-p^3), & \text{if } p\leq\frac{1}{2};\\ \frac{1}{8}(1-p(3+(p-5)p)), & \text{otherwise.}\end{cases} \tag{20}$$

The VoR term for the independent failures case is depicted in Fig. 7. First, observe that for $k=1$ the VoR is not monotonic with respect to the failure probability p. This is due to the fact that the penalty term is not high enough: when $p\approx 1$, no travel takes place. Since the penalty cost is the same as the Z^{MEDIAN}, the model is indifferent between the failure-prone and failure-free facilities. In fact, it is not hard to show from (19) that in order to guarantee that the VoR is a non-decreasing function of p, we must have $k\geq 2$. It turns out that the same condition is sufficient to ensure that VoR is an increasing function of p for the general case (i.e., when ρ is not necessarily 0)—this is derivable from (17). Thus, we argue that the minimal reasonable value for the penalty term is $k\geq 2$, i.e., $\beta\geq 2Z^{MEDIAN}$.

As seen in Fig. 7, for larger values of k (i.e., $k\geq 3$) the VoR term becomes essentially linear in the failure probability p. Fig. 8 presents the general case of the VoR term for $k=2$ and $k=4$, with the failure correlation coefficient $\rho\in\{-0.25,0.0,0.25\}$. It can be seen that the effect of ρ is relatively mild—the failure probability p and the penalty term coefficient k are far more important drivers of the VoR term than the correlation coefficient ρ.

We next turn our attention to the VoI term given by (20). It is interesting to note that in the independent failure case, this term is independent of the penalty term coefficient k, and is only a function of the failure probability p. Fig. 9 contains the difference $VoI-VoR$ as a function of p for different values of k (the latter affects the VoR but not the VoI part in this case). Observe that the difference is positive for all values of p and for $k=1$, 2, 3, 4—indicating that the value of having advanced information about the status of the facilities may be substantially larger than the value of having more reliability (clearly this is not the case when k is very large). Since VoI is often easier to improve in practice than the VoR (the former may only require installation of some notification service about the status of the facilities, while the latter is likely to require capital investments), providing

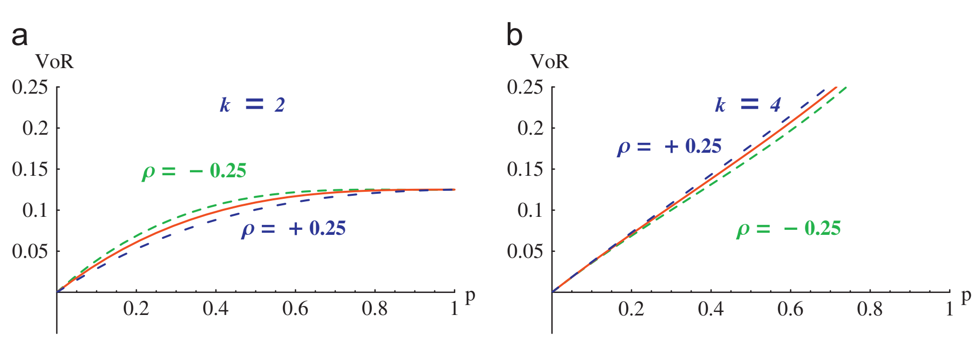

Fig. 8. VoR for the MPUF model as function of p and ρ for (a) $k=2$ and (b) $k=4$.

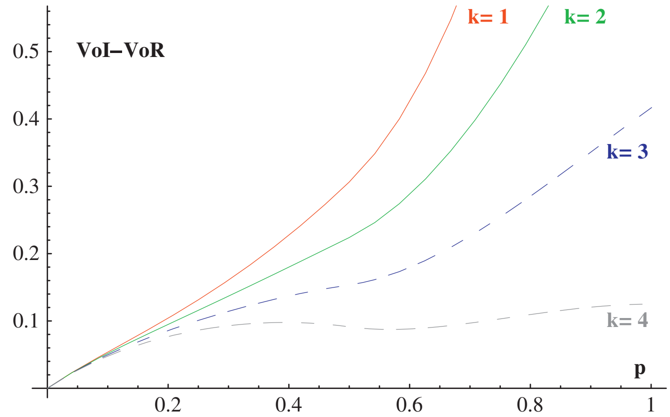

Fig. 9. MPUF model: the difference between the VoI and VoR terms as a function of p and k.

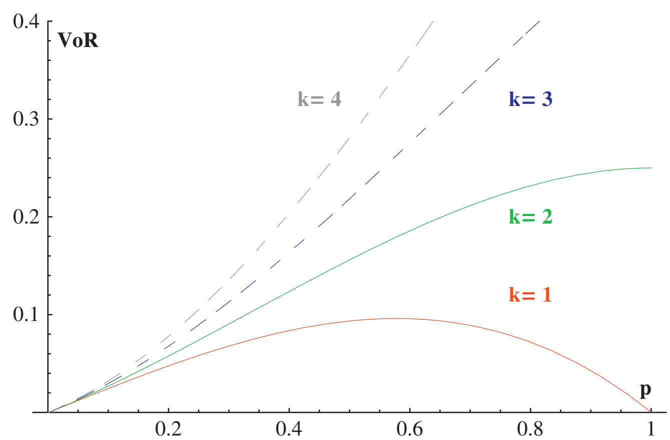

Fig. 10. VoR for the CPUF model as function of p for different values of k (independent failure case).

advanced information to customers may frequently be the best, as well as the most cost-efficient way of improving the system.

4.2. Cost decomposition for the center objective

In this section we present the cost decomposition analysis for the Center Objective. Similar to (17) and (18), by plugging the optimal solutions into the cost functions for the CPUF models we obtain

$$Z^{\text{CPUF-CI}} = \frac{1}{4}[(1-p)(1+2p-2p\rho+p^2-2p^2\rho+p\rho^2) + k(\rho p(1-p)+p^2)] \tag{21}$$

and

$$Z^{\text{CPUF-II}} = \frac{1}{4}+\frac{1}{2}(1-p)p(1-\rho)+[\rho(1-p)+p]\frac{kp}{4}. \tag{22}$$

As in the previous section, simplifications occur when $\rho=0$, leading to the following expressions for the VoR and VoI terms (recall that $Z^{CENTER}=\frac{1}{4}$):

$$VoR = Z^{\text{CPUF-CI}} - Z^{CENTER} = \tfrac{1}{4}(p+(k-1)p^2-p^3) \tag{23}$$

and

$$VoI = Z^{\text{CPUF-II}} - Z^{\text{CPUF-CI}} = \tfrac{1}{4}(p-p^2+p^3). \tag{24}$$

As before, in order to ensure that VoR is an increasing function of the failure probability p, we need to ensure that $k \geq 2$ (this also applies to the correlated failures case). The plot of VoR as a function of p for various values of k is given in Fig. 10. For higher values of k, VoR behaves like a quadratic function in p—increasing rapidly as the probability of failure increases. Thus, the sensitivity to p is higher than for the MPUF model.

The correlated failure case of VoR is depicted in Fig. 11. As in the case of the MPUF model, the VoR term is relatively insensitive to ρ.

The correlated failure case for VoI is shown in Fig. 12. Once again, we see that correlation coefficient ρ has only a minor effect on the objective function value. Note that the value of VoI is independent of k.

Finally, in Fig. 13 we show the difference between the VoI and VoR terms (analytically, this difference is given by $(2p^3-(k-2)p^2)/4$). We see that the difference is negative for all $k>3$ and is growing with k. In fact, even for $k=3$, there is a substantial range of values of p where the difference is negative. This implies that for moderate-to-high values of the penalty term, cost of reliability dwarfs the cost of information in the CPUF model. Thus, in this case it would be more sensible for management to concentrate on reducing the failure rate of the facilities.

5. Model extensions: non-identical facilities and non-uniform demand

In this section we briefly discuss two extensions of our basic analysis: allowing non-uniform failure probabilities of the facilities and non-uniform spatial distribution of customer demand.

First, suppose the two facilities have marginal failure probabilities of α and β, where α is associated with the left-most facility. The joint probability matrix can still be represented by Table 1, with row and column sums adjusted (of course, the distribution is no longer symmetric). For a given ρ, let $u=\rho\sqrt{(1-\alpha)\alpha(1-\beta)\beta}=\rho\sigma_x\sigma_y$ be the covariance. The components of the probability matrix can now be represented as follows:

$$p_1=\alpha\beta+u,\quad p_2=\beta(1-\alpha)-u,\quad p_3=(1-\alpha)(1-\beta)+u,\quad p_4=\alpha(1-\beta)-u,$$

with the consistency conditions on ρ given by

$$\max\left\{-\sqrt{\frac{\alpha\beta}{(1-\alpha)(1-\beta)}},-\sqrt{\frac{(1-\alpha)(1-\beta)}{\alpha\beta}}\right\}\leq\rho \leq \min\left\{\sqrt{\frac{\beta(1-\alpha)}{\alpha(1-\beta)}},\sqrt{\frac{\alpha(1-\beta)}{\beta(1-\alpha)}}\right\}.$$

These probabilities and conditions can now be used in place of (2)–(4) to derive (after some algebraic manipulations) expressions for the four models in question—these are given in Appendix. The resulting expressions are somewhat messy—for one thing, there are three parameters at play now, for another many non-linearities are involved. There are two ways to compare the results with those obtained above for the identical probabilities case. The most direct comparison is by assuming a linear relationship between the failure probabilities of the two facilities (e.g., $\alpha=\beta+0.2$) and letting α (which plays the role of p) vary for different values of ρ. The picture that emerges is essentially equivalent to the one seen for the identical probability case earlier: the centralization and co-location in case of MPUF models, the decentralization in case of CPUF-CI model, and no sensitivity to either α or ρ for CPUF-II model. The motion trajectories one observes are the same as before, with the only difference that the initial location of the more reliable facility is closer to the center of the interval (the detailed plots are not presented due to space limitations).

A different approach is to fix the probability of failure of one of the facilities (e.g. set $\beta=0.25$) and then vary the failure probability of the other facility. The resulting plots for the four models

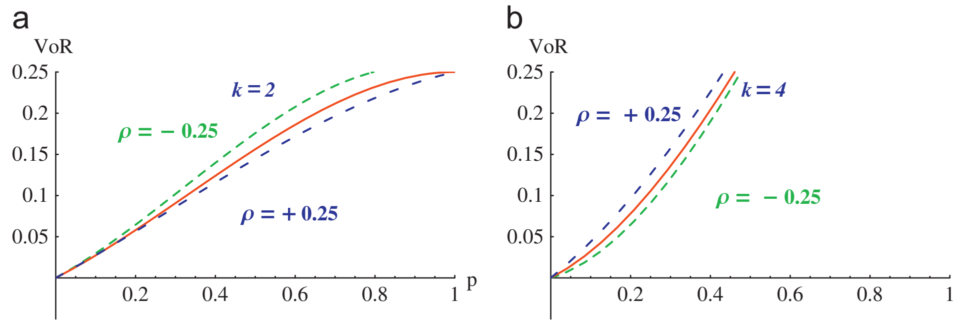

Fig. 11. VoR for the CPUF model as function of p and ρ for (a) $k=2$ and (b) $k=4$.

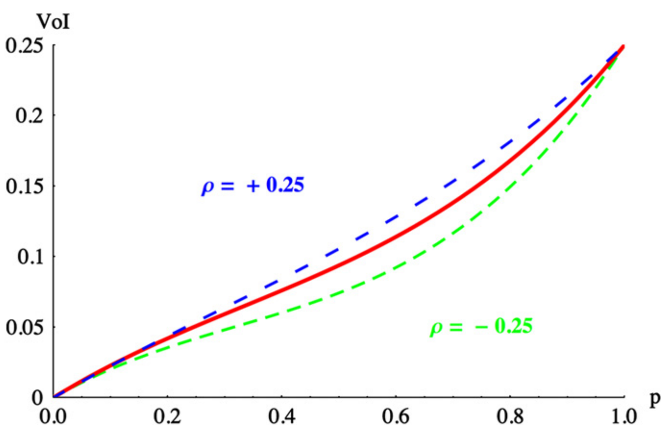

Fig. 12. VoI for the CPUF model as function of p and ρ.

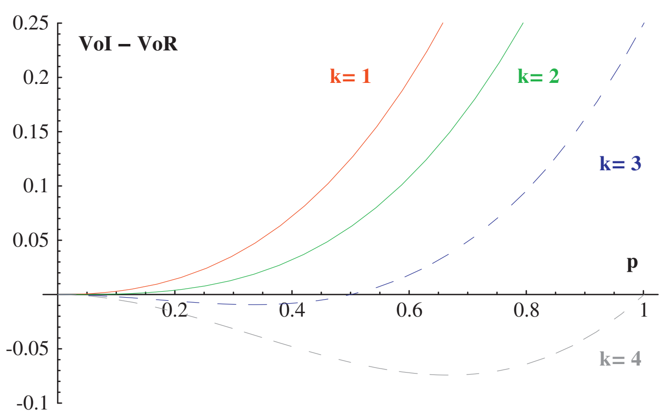

Fig. 13. Model CPUF: the difference between VoI and VoR terms as function of p and k.

are presented in Fig. 14 (as different values of ρ do not change the overall tendencies, we have kept it at 0 here to make the plots less messy). Here, the trajectories of motion are quite different, with the main principle being: the facility with the lower failure probability is kept closer to the center (i.e., the 1-median solution), while the more failure-prone facility stays on the periphery and plays a supporting role. For the MPUF-CI model (panel a on the figure) this results in the right facility following a straight-line trajectory to the center (as α increases this becomes the more dependable facility), while the left facility stays closer to the endpoint. While the centralization effect is still visible, it is counteracted by the separation induced by the increasingly different failure rate. For the MPUF-II model (panel b) we see the same tendency to keep the more reliable (right-most) facility closer to the center as α increases. Thus, while the centralization and co-location effects remain, they are weaker than before (co-location occurs at a higher value of α). For the CPUF models the trajectories are quite different than before. For the CI case (panel c), while the left facility essentially stays on the trajectory observed for the identical probability case (following the decentralization route), the facility on the right moves towards the center of the region. Essentially the same picture is observed for the CPUF-II model (panel d). In summary, when failure probabilities of different facilities are quite different, the more reliable facility is placed closer to the center and provides the main service, while the more failure-prone facility stays on the periphery and plays a supporting role.

We also comment briefly on the extension to the case when customer demand is not uniform. The analysis can certainly be extended (at least numerically) for specified spatial distributions of demand; for well-behaved distributions the extension appears to be rather straightforward. The details for two illustrative cases are presented in Appendix: the triangular distribution centered at 0.5 (i.e., customer demand is concentrated towards the center of the region) and inverse triangular distribution with peaks at 0 and 1 (i.e., customer demand is concentrated towards the endpoints). The numerical analysis of the results yields no new intuition: as expected, the initial locations (i.e., the 2-median and 2-center solutions) are affected by the demand distribution (closer to the center for the triangular case and closer to the endpoints for the inverse triangular case). However, as p and ρ are varied, exactly the same effects as observed for the uniform distribution earlier emerge for all four models. Of course, if one were to consider very general distributions (e.g., the ones with zero-density regions), more complex patterns might emerge.

6. Summary and future research

In this paper we studied the location problem with two facilities prone to failure on a unit segment, where customer demand is uniformly distributed over the segment. We considered median and center objectives, as well as customers having/not having advanced information about the operational status of the facilities, leading to four different models. For the most part we assumed identical failure probabilities for the facilities (in line with the assumptions made in most papers in the literature). Correlation of failure events was explicitly considered as well.

The relative simplicity of the setting allows us to derive most of the results in closed form, yielding transparent and managerially relevant insights. We observed that the objective function plays a strong role in determining optimal location patterns: median objective leads to facility centralization as the probability of failure increases, while the center objective leads to the opposite effect. Lack of advanced information about the state of the facilities

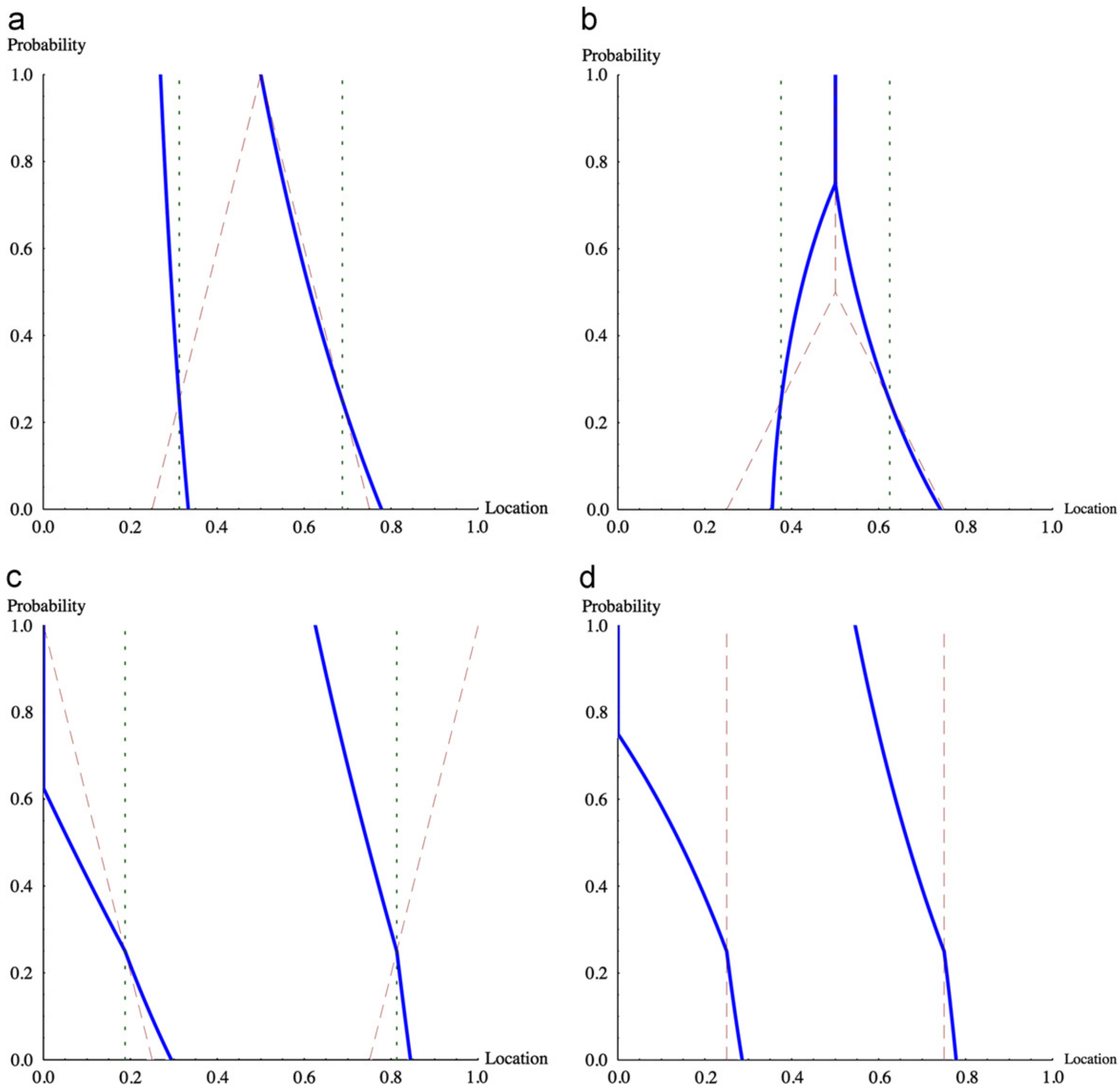

Fig. 14. The optimal locations for the four models when probability of failure for the right facility $\beta=0.25$ while the probability of failure of the left facility α is varied. Correlation $\rho=0$ in all cases. Dashed lines represent optimal location trajectories in the identical probability case, vertical dotted lines are the optimal 2-median locations.

accelerates the centralization effect in the median model (leading also to facility co-location), while the model with the center objective and no advanced information on the status of the facilities is quite unresponsive to the values of the probability of failure or the correlation of failures. Correlation of failures plays different role depending on the objective function. In the median models, positive correlation of failures reduces centralization and co-location effects, while the negative correlation of failures increases those effects. In the center model, positive correlation reduces the de-centralization effect, while the negative correlation increases this effect.

Most of the insights above were confirmed for models with 3–5 facilities via numerical experiments; the results for the models with median objectives and uncorrelated failure probabilities are also in line with those observed in numerical experiments for network models.

We also obtained the decomposition of optimal costs for each objective into the travel cost, reliability and informational components. It appears that the informational component tends to be the strongest in the median models, while the reliability component is the main driver in the center models.

Two model extensions were considered: allowing for non-identical failure probabilities and for non-uniform distributions of demand. While the demand distribution does not appear to have a strong affect on the main insights, the case of different failure probabilities yields a new principle: the more dependable facility should be kept closer to the center of the service region, while the more failure prone facility is kept closer to the region boundary. This principle, interacting with the centralization/decentralization effects described earlier leads to more complex location patterns. This research yields a number of directions for future research, including:

- Multi-facility problems on a line. The analysis of the general MPUF-CI model exists but for the other three models it is not available. The extension for the CPUF-CI case appears possible (though not direct); the analysis for the II appears very challenging.
- Analysis of network models allowing for different probability of failure. Our results indicate that the location patterns here may be quite different than in the identical probability of failure case.
- Study the problems on the plane. This appears to be non-trivial even when the distribution is uniform and the number of

facilities is low (we note that, in general, multi-facility planar models are quite challenging).

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Appendix

Proof of Proposition 1. Evaluating the integrals in (10) we find $Z^{\text{MPUF-CI}} = \frac{1}{4}(3x_1^2+3x_2^2-2x_1x_2-4x_2+2)+p(x_2-x_1)$.

From first order conditions

$$x_1 = \frac{2x_2+4p}{6}; \quad \text{and} \quad x_2 = \frac{2(x_2+2)-4p}{6}.$$

Solving for x_1,x_2 we obtain the unique solution

$$x_1 = \frac{1+2p}{4}, \quad x_2 = \frac{3-2p}{4}.$$

Note that we assumed $x_1 \le x_2$. Thus, optimal x_1 is the min $\{(1+2p)/4, 1/2\}$. and $x_2 = \max\{(3-2p)/4, 1/2\}$. It is easy to verify that this solution is a minimum. □

Proof of Lemma 1. First we observe that if $z \in [0,x_1]$, then $Z^{\text{CPUF-CI}}(z)$ is maximized at $z=0$, if $z \in [x_1,x_2]$, $Z^{\text{CPUF-CI}}_{x_1,x_2}(z)$ is maximized at $z=(x_1+x_2)/2$, and if $z \in [x_2,1]$, $Z^{\text{CPUF-CI}}_{x_1,x_2}(z)$ is maximized at $z=1$. This shows that $z \in \{0,(x_1+x_2)/2,1\}$.

To establish symmetry, first observe that a solution where $x_1,x_2 \le 1/2$ cannot possibly be optimal. In this case the maximum clearly occurs at $z=1$ and it is easy to show that for some $\epsilon > 0$, the solution $x_1, 1/2+\epsilon$ yields a lower objective function value. Thus, if $\{x_1, x_2\}$ is an optimal solution to $Z^{\text{CPUF-CI}}$ then $1/2 \in [x_1,x_2]$ (i.e., point $z=0$ is closer to x_1 and $z=1$ is closer to x_2).

Without loss of generality, assume that $x_1 \ge 1-x_2$ and let $\delta \ge 0$ be such that $x_2 = 1-x_1+2\delta$. Note that $(x_1+x_2)/2 = 1/2+\delta$. Consequently, evaluating the objective function at 0, $(x_1+x_2)/2$ and 1 we obtain

$$Z^{\text{CPUF-CI}} = \text{Max}\left\{ \begin{array}{l} (1-p)x_1+p_2(1-x_1+2\delta), \\ (1-p)(\frac{1}{2}-x_1)+p_2(\frac{1}{2}-x_1+2\delta), \\ (1-p)(x_1-2\delta)+p_2(1-x_1). \end{array} \right\}. \tag{25}$$

Note that, since $x_1,\delta \ge 0$,

$$(1-p)x_1+p_2(1-x_1+2\delta) \ge (1-p)(x_1-2\delta)+p_2(1-x_1),$$

indicating that only the first two terms in (25) need to be evaluated and therefore the customer location which maximizes the right-hand side of (11) must be $z \in \{0,\frac{1}{2}\}$.

Note that the first two terms are both increasing in δ. This implies that at an optimal solution $\delta = 0$ must hold, i.e., $x_1 = 1-x_2$. □

Optimal locations for non-identical failure probabilities. This material relates to Section 5. Here α,β represent the marginal failure probabilities of the left and right facilities, respectively, ρ is the correlation of failures and u is the covariance. Following the analysis in Section 2.3, the following formulas can be derived (x, y represent optimal locations for left and right facilities, respectively).

MPUF-CI model: optimal locations given by

$$x_{\text{MPUF-CI}} = \frac{u^2+(1-\alpha)(1-\beta)(1+(2+\alpha)\beta)-u(2-\beta+\alpha(1-2\beta))}{2((1-\alpha)(1-\beta)(2+\alpha+\beta)-u(2-\alpha-\beta))}$$

and

$$y_{\text{MPUF-CI}} = \frac{u^2-(1-\alpha)(1-\beta)(3+(2-\alpha)\beta)+u(2-3\beta-\alpha(1-2\beta))}{2((1-\alpha)(1-\beta)(2+\alpha+\beta)-u(2-\alpha-\beta))}.$$

MPUF-II model: note that optimal locations do not depend on ρ or u

$$x_{\text{MPUF-II}} = \frac{1+2(\beta^2-\beta(1+\alpha))}{2(2-(\alpha-\beta)^2)},$$

$$y_{\text{MPUF-II}} = \frac{3-2(\beta^2+\alpha(1-\beta))}{2(2-(\alpha-\beta)^2)}.$$

CPUF-CI model: here we assume that $\alpha \ge \beta$. Two cases must be considered. Let

$$x_{CPUF-CI} = \frac{1-(2-\beta)\alpha+u}{2(2-\alpha-\beta)}; \quad y_{CPUF-CI} = \frac{3-(2+\beta)\alpha+u}{2(2-\alpha-\beta)}.$$

If $x_{CPUF-CI} \ge 0$, then the solution is given above. Else, the optimal location for the left facility $x_{CPUF-CI} = 0$ and the location of the right facility is given by

$$y_{CPUF-CI} \in \left\{ \frac{\gamma}{\gamma+(\alpha-\beta)}, \frac{\gamma}{\gamma/2+(1-\beta)} \right\},$$

where $\gamma = 1-\alpha$ or $\gamma = \beta-u$, depending on which of these two possible values minimizes the value of the objective function.

CPUF-II model: Here the solution is hard to write down in closed form, as the formulas change depending on the parameter values. One can solve numerically for the intersection of the following set of planes:

$$\left\{ x(1-\alpha), 1-y(1-\beta)-x\beta, \tfrac{1}{2}(y-x)(1+2\alpha) \right\} \quad \text{if } \alpha \ge \beta;$$

and

$$\left\{ x(1-\alpha), 1-y(1-\beta)-x\beta, \tfrac{1}{2}(y-x)(1+2\beta) \right\} \quad \text{if } \alpha < \beta.$$

Optimal locations for triangular and inverse triangular distribution of consumer demand. For the triangular distribution, the following density function is assumed:

$$f(j) = 4j; \quad \text{if } j \in [0,1/2] \quad \text{and} \quad 4(1-j) \quad \text{if } j \in [1/2,1].$$

Using the same approach as before, the optimal locations for the MPUF-CI case are given by

$$x = \min\{(1/2)\sqrt{(1+p(1-\rho))/2}, 1/2\}; \quad y = 1-x.$$

For the MPUF-II case the optimal locations are

$$x = \min\{1/2\sqrt{(1+2p)/2}, 1/2\}; \quad y = 1-x.$$

It can be seen that the locations are closer to the center than for the uniform demand case. For the CPUF-CI and CPUF-II problems the optimal locations were obtained numerically.

For the inverted triangular distribution, we used the following density function:

$$f(j) = 2-4j \quad \text{if } j \in [0,1/2] \quad \text{and} \quad 2-4(1-j) \quad \text{if } j \in [1/2,1].$$

The optimal locations for the MPUF-CI case are

$$x = \min\{1/4(2-\sqrt{2(1-p(1-\rho))}, 1/2\}; \quad y = 1-x$$

with the corresponding formulas for the MPUF-II problems given by

$$x = \min\{1/4(2-\sqrt{2(1-p)}, 1/2\}; \quad y = 1-x.$$

Once again, the CPUF-CI and CPUF-II were solved numerically.

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