INDIAN INSTITUTE OF TECHNOLOGY, KANPUR
DEPARTMENT OF MATHEMATICS & STATISTICS

MSO 202/A COMPLEX ANALYSIS
ASSIGNMENT 6

Q1. Determine the location and kind of singularities in the finite complex plane and at infinity of the following functions.
(i) $\tan^2 \pi z$, (ii) $\cot(z^2)$, (iii) $1/(\cos z - \sin z)$.

Sol: (i) Since $\cos \pi z = 0$ for $z_n = (2n+1)/2$, $n \in \mathbb{Z}$, z_n are the poles of order 2 of $\tan^2 \pi z$. Since $z_n \to \infty$, we have non-isolated singularity at infinity of $\tan^2 \pi z$.
(ii) Since $\sin z^2 = 0$ for $z_n^2 = n\pi$, $n \in \mathbb{Z}$. For $n = 0, 1, 2, \ldots$, $z_n = \pm\sqrt{n\pi}$ and for $n = -1, -2, -3, \cdots$, $z_n = \pm i\sqrt{-n\pi}$. These are simple poles of $\cot(z^2)$. Again we have sequeces of isolated singular points going to infinity, we have a non-isolated singularity at infinity of $\cot(z^2)$.
(iii) We have $\cos z = \sin z$ if and only if $e^{iz} + e^{-iz} = i(e^{iz} - e^{-iz})$. Hence

$$e^{2iz} = -\frac{1+i}{1-i} = \frac{e^{i\pi/4}}{e^{-i\pi/4}} = e^{i\pi/2+2n\pi i}, \quad n \in \mathbb{Z}.$$

Hence $z_n = (\pi/4)+n\pi$, $n \in \mathbb{Z}$, are the simple poles of $f(z) = 1/(\cos z - \sin z)$ since $(\cos z - \sin z)' \neq 0$ at z_n. Again, we have $z_n \to \infty$, we have non-isolated singularity at infinity.

Q2.

Q3. Evaluate:
(i) $\int_0^\pi \frac{d\theta}{2+\cos\theta}$, (ii) $\int_0^{2\pi} \frac{1+4\cos\theta}{17-8\cos\theta}\, d\theta.$ (iii) $\int_{-\infty}^\infty \frac{dx}{x^4+16}$.

Q4. Find the Cauchy principal value:

$$\int_{-\infty}^\infty \frac{x^2}{x^4-1}\, dx.$$

Q5 Integrating e^{-z^2} around the boundary of the rectangle with the vertices $-a$, a, $a+ib$ and $-a+ib$, $a > 0$, then letting $a \to \infty$ and using the fact that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

show that

$$\int_0^\infty e^{-x^2} \cos 2bx \; dx = \frac{\sqrt{\pi}}{2} e^{-b^2}.$$