Lecture 3

Basic Properties of Differentiable Functions

Proposition 1. Let $f:\mathcal{C}\to\mathcal{C}$ be defined in some neighbourhood of a and differentiable at a. Then, $\frac{\partial f}{\partial x}(a)$, $\frac{\partial f}{\partial y}(a)$ exist and satisfy

$$\frac{\partial f}{\partial x}(a)=-i\frac{\partial f}{\partial y}(a). \qquad (*)$$

Notes: 1. For a function $\varphi:\mathcal{C}\to\mathcal{R}$, the first order partial derivatives φ_x (also denoted as $\frac{\partial\varphi}{\partial x}$) and φ_y (also denoted as $\frac{\partial\varphi}{\partial y}$) at a point $\varsigma=(\alpha,\beta)$ are defined as , for $t\in\mathcal{R}$,

$$\varphi_x=\lim_{t\to 0}\frac{\varphi(\alpha+t,\beta)-\varphi(\alpha,\beta)}{t}\ ,\ \varphi_y=\lim_{t\to 0}\frac{\varphi(\alpha,\beta+t)-\varphi(\alpha,\beta)}{t}$$

2. For a function $f:\mathcal{C}\to\mathcal{C}$, given by $f=u+iv$, the first order partial derivatives f_x and f_y, at a point $a=\alpha+i\beta$, are defined as, for $t\in\mathcal{R}$,

$$f_x=\lim_{t\to 0}\frac{f(\alpha+t,\beta)-f(\alpha,\beta)}{t}=u_x+iv_x \qquad ,$$

$$f_y=\lim_{t\to 0}\frac{f(\alpha,\beta+t)-f(\alpha,\beta)}{t}=u_y+iv_y.$$

Proof of Proposition 1. Let $a = \alpha + i\beta$, $t \in \mathcal{R}$, $t \neq 0$. Then,

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t} = \lim_{t \to 0} \frac{f(\alpha + t, \beta) - f(\alpha, \beta)}{t} = \frac{\partial f}{\partial x}(a)$$

$$f'(a) = \lim_{t \to 0} \frac{f(a+it) - f(a)}{it} = \lim_{t \to 0} \frac{f(\alpha, \beta + t) - f(\alpha, \beta)}{it} = -i\frac{\partial f}{\partial y}(a).$$

The identity (*) follows by the above identities.

Proposition 2. Let $f = u + i\,v$ be differentiable at $a \in \mathcal{C}$. Then,

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a), \quad \frac{\partial v}{\partial x}(a) = -\frac{\partial u}{\partial y}(a).$$

Proof. By Prop. 1,

$$\frac{\partial f}{\partial x}(a) = -i\frac{\partial f}{\partial y}(a)$$

$$\Rightarrow \frac{\partial u}{\partial x}(a) + i\frac{\partial v}{\partial x}(a) = -i(\frac{\partial u}{\partial y}(a) + i\frac{\partial v}{\partial y}(a))$$

Equating real and imaginary parts of the above identity we get the result of Proposition 2.

Definition. Any one of the following equations

- $$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$
- $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are called ***Cauchy Riemann equations***.

Note that the above equations are equivalent.

Example. The function

$$f(z)=\begin{cases}\dfrac{(\overline{z})^2}{z} & if\ \ z\neq 0\\ 0 & if\ \ z=0\end{cases}$$

satisfies CR equations at 0, but is not diff. at 0.

<u>f is not diff. at 0</u>:

$$\lim_{\Delta z\to 0}\frac{f(\Delta z)}{\Delta z}=\lim_{\Delta z\to 0}\left(\frac{\overline{\Delta z}}{\Delta z}\right)^2=\lim_{\Delta z\to 0}\left(\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}\right)^2$$

$$\begin{aligned}&\to 1 && if\ \ \Delta y=0\\ &\to 1 && if\ \ \Delta x=0\\ &\to \left(\frac{1-i}{1+i}\right)^2 && if\ \ \Delta x=\Delta y\end{aligned}$$

CR Equations are satisfied at 0:

Note that

$$u(\Delta x,0)=\operatorname{Re} f(\Delta x,0)=\operatorname{Re}\Delta x=\Delta x$$
$$u(0,\Delta y)=\operatorname{Re} f(0,\Delta y)=\operatorname{Re}\frac{(i\Delta y)^2}{i\Delta y}=\operatorname{Re}(i\frac{\Delta y^2}{\Delta y})=0$$
$$v(\Delta x,0)=\operatorname{Im} f(\Delta x,0)=\operatorname{Im}\Delta x=0$$
$$v(0,\Delta y)=\operatorname{Im} f(0,\Delta y)=\operatorname{Im}(i\Delta y)=\Delta y.$$

Consequently,

$$\frac{u(\Delta x,0)-u(0,0)}{\Delta x}=\frac{\Delta x}{\Delta x}\to 1 \ as\ \Delta x\to 0$$
$$\frac{u(0,\Delta y)-u(0,0)}{\Delta y}=\frac{0}{\Delta y}\to 0 \ as\ \Delta y\to 0$$
$$\frac{v(\Delta x,0)-v(0,0)}{\Delta x}=\frac{0}{\Delta x}\to 0 \ as\ \Delta x\to 0$$
$$\frac{v(0,\Delta y)-v(0,0)}{\Delta y}=\frac{\Delta y}{\Delta y}\to 1 \ as\ \Delta y\to 0.$$

Therefore,

$$\frac{\partial u}{\partial x}(0)=\frac{\partial v}{\partial y}(0)=1;\ \frac{\partial v}{\partial x}(0)=-\frac{\partial u}{\partial y}(0)=0.$$

Exercise. Prove that for the following functions CR equations are satisfied at 0 but the functions are not differentiable at 0:

(i) $$f(z)=\begin{cases}\dfrac{z^5}{|z|^4} & if\ \ z\neq 0\\ 0 & if\ \ z=0\end{cases}$$

(ii) $$f(z)=\begin{cases}\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & if\ \ z\neq 0\\ 0 & if\ \ z=0\end{cases}$$

Definition. Any function $\varphi : \mathcal{C} \to \mathcal{R}$ having continuous partial derivatives up to second order and satisfying the equation $\nabla^2 \varphi \equiv \dfrac{\partial^2 \varphi}{\partial x^2} + \dfrac{\partial^2 \varphi}{\partial y^2} = 0$ (*) is called a ***Harmonic Function or Potential Function***. Equation (*) is called ***Laplace Equation*** and $\nabla^2 \varphi$ is called ***Laplacian*** of the function φ.

Harmonic Functions are widely used in the study of steady state temperatures, wave theory, two dimensional electrostatics, fluid flow, robotics etc.

Since real and imaginary parts $u \; and \; v$ of a complex differentiable function $f = u + i\,v$ satisfy CR-equations, it easily follows that $\nabla^2 u = 0, \nabla^2 v = 0$. Thus, real and imaginary parts of a complex differentiable function are Harmonic functions.

The converse of Prop.1, proved below, holds under the additional hypothesis that $\partial f / \partial x$, $\partial f / \partial y$ exist, are continuous and satisfying CR equations $\partial f / \partial x = -i \, \partial f / \partial y$.

Theorem. Let, for $f(z)$ defined in a domain G, $\partial f / \partial x$, $\partial f / \partial y$ exist, are continuous and CR Equation $\partial f / \partial x = -i \, \partial f / \partial y$ is satisfied at any point $z_0 = (x_0, y_0) \in G$. Then, $f'(z_0)$ exists and is given by $f'(z_0) = u_x(x_0, y_0) + i \, v_x(x_0, y_0)$.

Proof. Let,
$z_0 = x_0 + iy_0 \in G$ & $h = s + it$ *be such that* $|h| < \varepsilon$ *and* $z_0 + h \in G$.

Then,

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{u(x_0+s, y_0+t) - u(x_0, y_0)}{h} + i \frac{v(x_0+s, y_0+t) - v(x_0, y_0)}{h} \qquad (1)$$

Now,

$$\frac{u(x_0+s, y_0+t)-u(x_0, y_0)}{h}$$
$$=\frac{u(x_0+s, y_0+t)-u(x_0, y_0+t)}{h}+\frac{u(x_0, y_0+t)-u(x_0, y_0)}{h}$$
$$=\frac{\varphi(s,t)}{h}+\frac{s}{h}u_x(x_0, y_0)+\frac{t}{h}u_y(x_0, y_0) \quad . \qquad (2)$$

$(say,\ i.e.\ define\ \varphi(s,t)\ by\ this\ identity)$

Then,

$$\frac{\varphi(s,t)}{h}=\frac{s}{h}u_x(x_0+s_1, y_0+t)+\frac{t}{h}u_y(x_0, y_0+t_1)$$
$$-\frac{s}{h}u_x(x_0, y_0)-\frac{t}{h}u_y(x_0, y_0) \qquad \left(By\ MVT\ \&(2)\right)$$

$\to 0\ as\ h\to 0 \qquad (\because u_x\ \&\ u_y\ are\ continuous\ at\ z_0\ and\ \left|\frac{s}{h}\right|\le 1, \left|\frac{t}{h}\right|\le 1)$

Similarly,

$$\frac{v(x_0+s, y_0+t)-v(x_0, y_0)}{h}=\frac{\psi(s,t)}{h}+\frac{s}{h}v_x(x_0, y_0)+\frac{t}{h}v_y(x_0, y_0),$$

$where,\ \frac{\psi(s,t)}{h}\to 0$

$\therefore By$ (1),

$$\frac{f(z_0+h)-f(z_0)}{h}=\frac{s}{h}[u_x+iv_x]+\frac{t}{h}[u_y+iv_y]+\frac{\varphi+i\psi}{h}$$

$$=\frac{s}{h}[u_x+iv_x]+\frac{t}{h}[-v_x+iu_x]+\frac{\varphi+i\psi}{h} \qquad (since\ u_x=v_y\ \&\ u_y=-v_x)$$

$$=\frac{s}{h}[u_x+iv_x]+\frac{it}{h}[u_x+iv_x]+\frac{\varphi+i\psi}{h}$$

$$=u_x+iv_x+\frac{\varphi+i\psi}{h}$$

$$\to u_x+iv_x\ as\ h\to 0$$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{f(z_0+h)-f(z_0)}{h}$ exists & equals u_x+iv_x at z_0.

Cauchy Riemann Equations in Polar Coordinates

Let $f(z) = u(r,\theta) + i\,v(r,\theta)$. Then,

$$r\,u_r = v_\theta,\; r\,v_r = -u_\theta$$

are Cauchy Riemann equations in Polar coordinates.

The above equations can be easily obtained from CR equations in Cartesian coordinates as follows:

With $x = r\cos\theta,\; y = r\sin\theta,$

$$u_r = u_x x_r + u_y y_r = u_x(\cos\theta) + u_y(\sin\theta)$$

$$v_\theta = v_x x_\theta + v_y y_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

$$\therefore r\,u_r = v_\theta$$

by $CR\ Equations\ u_x = v_y,\, u_y = -v_x\ in\ cartesian\ coordinates$.

Similarly, $r\,v_r = -u_\theta$.

The following proposition readily follows from the above theorem:

Proposition 4. Let $f(z) = u(r,\theta) + i\,v(r,\theta)$. If $\dfrac{\partial f}{\partial r}, \dfrac{\partial f}{\partial \theta}$ exist, are continuous in some domain G, and satisfy the following Cauchy Riemann equations in polar coordinates at a point $z_0 = (r_0, \theta_0) \in G$,

$$\begin{aligned} r\,u_r(r_0,\theta_0) &= v_\theta(r_0,\theta_0) \\ r\,v_r(r_0,\theta_0) &= -u_\theta(r_0,\theta_0) \end{aligned}$$

then, $f'(z_0)$ exists and is given by

$$f'(z_0) = \frac{u_r(r_0,\theta_0) + i\; v_r(r_0,\theta_0)}{e^{i\theta_0}}.$$

Proof. Since, $r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$,

$$\begin{aligned} u_x = u_r r_x + u_\theta\,\theta_x &= u_r(\cos\theta) - u_\theta\,\frac{1}{r}(\sin\theta) \\ &= u_r(\cos\theta) + v_r(\sin\theta) \end{aligned}$$

$$\begin{aligned} v_x = v_r r_x + v_\theta\,\theta_x &= v_r(\cos\theta) - v_\theta\,\frac{1}{r}(\sin\theta) \\ &= v_r(\cos\theta) - u_r(\sin\theta) \end{aligned}$$

Therefore, using the above theorem,

$$\begin{aligned} \Rightarrow f'(z_0) &= u_x(x_0,y_0) + i\,v_x(x_0,y_0) \\ &= \frac{u_r(r_0,\theta_0) + i\; v_r(r_0,\theta_0)}{e^{i\theta_0}} \end{aligned}$$

Elementary Functions

Logarithmic Function. Define

$$\log z = \ln|z| + i \arg z$$

Since arg z takes multiple values for every z, log z also takes multiple values for every z and is therefore not a function.

To make it a well-defined function, the range of arg z has to be so restricted that it takes a unique value for every value of z in its domain of definition.

To this end, define

$$Log\, z = \ln|z| + i\, Arg\, z$$

where, $-\pi < Argz \leq \pi$. Arg z takes a unique value for every $z \neq 0$. Consequently, Log z is a well-defined function for every $z \neq 0$. *Log z* is called the Principal Branch of Logarithmic Function.

- Log z maps $\mathbb{C} - \{0\}$ onto the strip $\{w : -\pi < \operatorname{Im} z \leq \pi\}$.

- Log z is continuous at all points in its domain of definition except at the points on the negative real axis, since $Arg\, z$ is a continuous at all the points in its domain of definition, except the points on negative real axis.