① Solve :-

$$u_{tt} - 4u_{xx} = 0 \,, \quad x > 0 \,, \quad t > 0$$

$$u(x,0) = |\sin x| \,, \quad x > 0$$

$$u_t(x,0) = 0 \,, \quad x \geq 0$$

$$u(0,t) = 0 \,, \quad t \geq 0.$$

Soln :- Extend u to $\tilde{u}$ s.t

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{if } x > 0, t > 0 \\ -u(-x,t) & \text{if } x < 0, t > 0. \end{cases}$$

Note :- Clearly this extension allows $\tilde{u}(0,t) = 0$ for $t \geq 0$.

and, $$\tilde{f}(x) = \tilde{u}(x,0) = \begin{cases} |\sin x| & , x > 0 \\ -|\sin x| & , x < 0 \end{cases}$$

$$\tilde{g}(x) = \tilde{u}_t(x,0) = 0 \,, \quad x \in \mathbb{R}.$$

By d'Alembert Formula

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{f}(x+2t) + \tilde{f}(x-2t)\right] + \frac{1}{2\cdot 2}\int_{x-2t}^{x+2t} 0 \, d\tau$$

$$= \begin{cases} \frac{1}{2}\left[|\sin(x+2t)| + |\sin(x-2t)|\right] & ; x > 2t \\ \frac{1}{2}\left[|\sin(x+2t)| - |\sin(2t-x)|\right] & ; x < 2t. \end{cases}$$

2. Solve :- $u_{xx} - u_{yy} = 1$

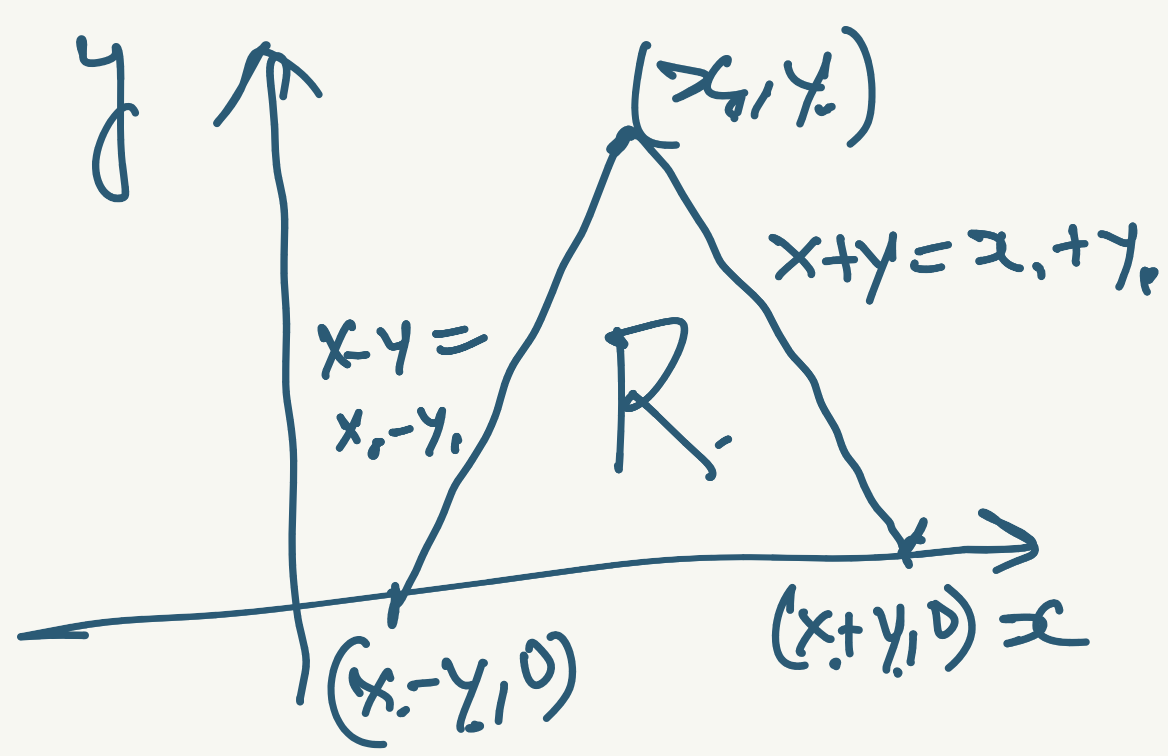
$u(x,0) = \sin x$

$u_y(x,0) = x.$

Soln:- Using Duhamel's formulae :-

$$u(x_0,y_0) = \frac{1}{2}\left[\sin(x_0+y_0) + \sin(x_0-y_0)\right] + \frac{1}{2}\int_{x_0-y_0}^{x_0+y_0} \tau \, d\tau - \frac{1}{2}\int_0^{y_0}\int_{y+x_0-y_0}^{-y+x_0+y_0} dx\,dy$$

(Formulae was deduced in the 1st wave equation lecture).



$$\boxed{u(x,y) = \frac{1}{2}\left[f(x+y)+f(x-y)\right] + \frac{1}{2}\int_{x-y}^{x+y} g(\tau)\,d\tau - \frac{1}{2}\iint_R h(x,y)\,dR. \quad \text{- Duhamel's Principle}}$$

$$= \frac{1}{2}\left[\sin(x_0+y_0) + \sin(x_0-y_0)\right] + x_0y_0 - \frac{y_0^2}{2}.$$

Alternatively:- Define:- $v(x,y) = u(x,y) - \frac{y^2}{2}$.

$\therefore\ v_{xx} = u_{xx}$ & $v_{yy} = u_{yy} - 1$

$\therefore\ v_{xx} - v_{yy} = u_{xx} - u_{yy} + 1 = 0$

Hence v satisfies the homogeneous wave eqn. Use D'Alembert now on v.

③ Solve :- $u_{tt} = c^2 u_{xx}$

$$u(x,t) = f(x) \text{ on } \{x+ct=0\}$$

$$u(x,t) = g(x) \text{ on } \{x-ct=0\}$$

where $f(0) = g(0)$.

The general solution of the wave eqn is

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

The characteristics are $x \pm ct = 0$.

Now, $u(x,t) = \phi(2x) + \psi(0) = f(x)$ on $x+ct=0$. —— (i)

$u(x,t) = \phi(0) + \psi(2x) = g(x)$ on $x-ct=0$. —— (ii)

Replacing x by $\frac{x+ct}{2}$ in (i) and x by $\frac{x-ct}{2}$ in (ii) one has

$$\phi(x+ct) = f\left(\frac{x+ct}{2}\right) - \psi(0)$$

$$\phi(x-ct) = g\left(\frac{x-ct}{2}\right) - \phi(0)$$

$\therefore$ The solution is given by

$$u(x,t) = f\left(\frac{x+ct}{2}\right) + g\left(\frac{x-ct}{2}\right) - f(0).$$

4. Solve:- $u_{tt} = c^2 u_{xx} + F(x)$; $0 < x < l$, $t > 0$

$u(x,0) = f(x)$; $0 \le x \le l$

$u_t(x,0) = g(x)$; $0 \le x \le l$

$u(0,t) = A$; $u(l,t) = B$, $t > 0$.

Soln:- Assume, $u(x,t) = v(x,t) + U(x)$

Substituting $u(x,t)$ we have;

$$v_{tt} = c^2(v_{xx} + U_{xx}) + F(x)$$

And if we choose U s.t $c^2 U_{xx} + F(x) = 0$ then v satisfies.

$$v_{tt} = c^2 v_{xx}$$

$$v(x,0) = u(x,0) - U(x) = f(x) - U(x)$$

$$v_t(x,0) = u_t(x,0) = g(x)$$

$$v(0,t) = 0 \; ; \; v(l,t) = 0$$

provided, $U(0) = A$ and $U(l) = B$.

$$\therefore U(x) = A + (B-A)\frac{x}{l} + \frac{x}{l}\int_0^l \left[\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi\right] d\eta - \int_0^x \left[\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi\right] d\eta$$

$\boxed{\text{The above is an ODE. Plz ask students to solve it themselves}}$

Use S.O.V for the wave equation. (No need to solve it, plz ask students to do it themselves).

⑤ <u>Solve :-</u>

$$u_{tt} = c^2 u_{xx} + h(x,t); \quad 0 < x < l, \quad t > 0$$

$$u(x,0) = f(x) \; ; \; 0 \le x < l$$

$$u_t(x,0) = g(x) \; ; \; 0 \le x < l$$

$$u(0,t) = 0 \; ; \; u(l,t) = 0 \; ; \; t \geq 0.$$

<u>Soln</u> :- We start by using the eigenfunctions one obtains from ~~the~~ solving the Homogeneous solution using S.O.V. which are given by $\sin\left(\frac{n\pi x}{l}\right)$.

Assume, $u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{l}\right)$ be the solution of the problem where $u_n(t)$ needs to be determined.

Also assume, $h(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{n\pi x}{l}\right)$

Thus by Fourier series,

$$h_n(t) = \frac{2}{l}\int_0^l h(x,t)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Assuming the convergence of (i) we have,

$$\sum_{n=1}^{\infty}\left[u_n''(t) + \lambda_n^2 u_n(t)\right]\sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} h_n(t)\sin\left(\frac{n\pi x}{l}\right)$$

where $\lambda_n = \frac{n\pi c}{l}$.

Multiplying both sides by $\sin\left(\frac{m\pi x}{l}\right)$ and integrating between $x=0$ & $x=l$ we have,

$$u_n''(t) + \lambda_n^2 u_n(t) = h_n(t)$$

$$\Rightarrow u_n(t) = a_n \cos \lambda_n t + b_n \sin \lambda_n t + \int_0^t \frac{1}{\lambda_n} h_n(\tau) \sin\left[\lambda_n (t-\tau)\right] d\tau .$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos \lambda_n t + b_n \sin \lambda_n t + \frac{1}{\lambda_n} \int_0^t h_n(\tau) \sin\left[\lambda_n (t-\tau)\right] d\tau \right\} \sin \frac{n\pi x}{l}$$

Now, $u(x,0) = f(x) = \sum_1 a_n \sin\left(\frac{n\pi x}{l}\right) \Rightarrow a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$||^{ly}$, $u_t(x,0) = g(x) = \sum_1 b_n \lambda_n \sin\left(\frac{n\pi x}{l}\right) \Rightarrow b_n = \frac{2}{l\lambda_n} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx .$

6. Let $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{-k\left(\frac{n\pi}{l}\right)^2 t}$ —— (i)

be the formal solution of the heat equation

$$u_t - k u_{xx} = 0 \;\; ; \;\; 0 < x < l,\ t > 0$$

$$u(0,t) = u(L,t) = 0 \;\; , \;\; t \geq 0$$

$$u(x,0) = f(x) \;\; ; \;\; 0 \leq x \leq l.$$

If the series, $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$ converges uniformly on $[0,L]$, then the series (i) converges uniformly and is the classical solution of the heat equation.

Soln:- Let $\epsilon > 0$, By Cauchy principle. $\exists\ N_\epsilon \le k \le l$ s.t

$$\left| \sum_{n=k}^{l} B_n \sin \frac{n\pi x}{l} \right| < \epsilon \quad \forall x \in [0,l]$$

Note that, $\displaystyle\sum_{n=k}^{l} B_n \sin\left(\frac{n\pi x}{l}\right) e^{-k\left(\frac{n\pi}{l}\right)^2 t}$ (Bcoz. finite sum).

is a classical solution of the heat equation.

hence by maximum principle.

$$\left| \sum_{n=k}^{l} B_n \sin \frac{n\pi x}{l}\, e^{-k\left(\frac{n\pi}{l}\right)^2 t} \right| < \epsilon \quad \forall (x,t) \in [0,l] \times [0,T]$$

$\therefore$ By Cauchy criterion, the series converges uniformly on $[0,l] \times [0,T]$ to a continuous function 'u'. Since u satisfies the initial & bdry data and also solves the heat eq$\underline{n}$ $\Rightarrow u$ is a classical solution.

7. Solve :- $u_t - u_{xx} = \sin t + x^2$ (find one non-trivial solution).

Soln :- Split $u = v + w$ such that

$$v_t - 4 v_{xx} = \sin t$$

$$w_t - 4 w_{xx} = x^2.$$

(Possible due to the principle of superposition).

Assuming, $v_{xx} = 0 \Rightarrow v_t = \sin t \Rightarrow v(x,t) = -\cos t.$

Again if $w_t = 0 \Rightarrow w(x,t) = -\dfrac{1}{4 \cdot 3 \cdot 4} x^4 = -\dfrac{x^4}{48}.$

$\therefore\ u(x,t) = -\cos t - \dfrac{x^4}{48}$ is a solution.