Partial Differential Equations - MSO 203B

Assignment 3 - Classification of 2nd Order PDE and Laplace equation

Tutorial Problems

1. Classify the following PDE as per their ellipticity, parabolicity or hyperbolicity.
   - $u_{xx} \pm (\operatorname{sech}^4 x) u_{yy} = 0$.
   - $u_{xx} + (2 \operatorname{cosec} y) u_{xy} + (\operatorname{cosec}^2 y) u_{yy} = 0$.
   - $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$.
2. Show that the behaviour of a PDE based on ellipticity, parabolicity or hyperbolicity depends on the domain.
3. Reduce $y^2 u_{xx} - x^2 u_{yy} = 0$ into its canonical form.
4. Reduce the PDE $u_{xx} + x^2 u_{yy} = 0$ into its canonical form.
5. Reduce $u_{xx} - 4u_{xy} + 4u_{yy} = \exp y$ into its canonical form and solve it.
6. Deduce the Laplace equation on a disk using polar coordinate.
7. Solve the Laplace equation on a disk of radius R, using polar coordinate subject to the condition $u(R, \theta) = h(\theta)$ with h being a continuous function.
8. Show that the value of u at any point is just the average value of u on the circle centered at that point provided u solves the Laplace Equation.
9. Let ϕ be a continuous function which changes sign on $\partial\Omega$. If $\Delta u = 0$ and $u = \phi$ on $\partial\Omega$, show that u must change sign in Ω provided Ω is bounded and smooth domain.

Practice Problems

1. Reduce all the equations in problem 1 to its canonical form.
2. Solve the problem $u_{tt} - u_{xx} = 0$.
3. Show that Laplacian is rotation invariant.
4. Solve the problem $-\Delta u = \lambda u$ in a square.
5. Solve the problem $-\Delta u = 1$ in a rectangle in $\mathbf{R}^2$.

END