Assign-6

① Deduce the Laplace Equation in polar coordinate.

Proof:- $\Delta u = u_{xx} + u_{yy} = 0$.

In polar coordinate $x = r\cos\theta$; $y = r\sin\theta$

$$\left\{\begin{array}{l} r_x = r/x \ ; \ r_y = y/r. \\ \theta_x = -y/r^2 \ ; \ \theta_y = \frac{x}{r^2}. \\ r_{xx} = y^2/r^3 \ ; \ r_{yy} = x^2/r^3. \\ \theta_{xx} = \frac{2xy}{r^4} \ ; \ \theta_{yy} = \frac{2xy}{r^4}. \end{array}\right.$$

Define, $v(r,\theta) := u(x,y)$

$\therefore$ $u_x = v_r r_x + v_\theta \theta_x$

and, $u_y = v_r r_y + v_\theta \theta_y$.

Also, $u_{xx} = (v_{rr} r_x + v_{r\theta}\theta_x) r_x + (v_{\theta r} r_x + v_{\theta\theta}\theta_x)\theta_x + v_r r_{xx} + v_\theta \theta_{xx}$

$= v_{rr} r_x^2 + 2 v_{r\theta}\theta_x r_x + v_{\theta\theta}\theta_x^2 + v_r r_{xx} + v_\theta \theta_{xx}$.

$$= \frac{x^2}{r^2} v_{rr} + \frac{y^2}{r^3} v_r + \left[-\frac{2xy}{r^3}\right] v_{r\theta} + \frac{2xy}{r^4} v_\theta + \frac{y^2}{r^4} v_{\theta\theta} \quad \text{—— (1)}$$

and, $u_{yy} = [v_{rr} r_y + v_{r\theta} \theta_y] r_y + v_r r_{yy} + [v_{\theta r} r_y + v_{\theta\theta} \theta_y] \theta_y + v_\theta \theta_{yy}$

$= v_{rr} r_y^2 + 2 v_{r\theta} \theta_y r_y + v_r r_{yy} + v_{\theta\theta} \theta_y^2 + \theta_{yy} v_\theta.$

$= \frac{y^2}{r^2} v_{rr} + \frac{x^2}{r^3} v_r + \frac{2xy}{r^3} v_{r\theta} - \frac{2xy}{r^4} v_\theta + \frac{x^2}{r^4} v_{\theta\theta}$ ← (2)

$\therefore$ Adding (1) & (2) we get,

$u_{xx} + u_{yy} = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0.$ ∎

(2) Solve :- $\Delta u = 0$ in $B(0;a) (\subseteq \mathbb{R}^2)$ (In polar coordinate)

$u(a,\theta) = h(\theta).$

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0$$

$$u(a,\theta) = h(\theta)$$

Using separation of variable,

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\Rightarrow R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$\Rightarrow \frac{\Theta''}{\Theta} = -\frac{r^2R - rR'}{R} = -\lambda \text{ (say)}$$

$\therefore$ $\Theta'' + \lambda\Theta = 0$; $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$

Eigenpairs are $(0,1)$; $(n^2, \sin n\theta)$; $(n^2, \cos n\theta) = (\lambda_n, \varphi_n)$

$n = 1, 2, 3, \ldots$

Also, $r^2 R'' + r R' - \lambda R = 0$

Substituting, $R(r) = r^\alpha$ one has,

$[\alpha(\alpha-1) + \alpha - \lambda] r^\alpha = 0 \Rightarrow \alpha = \pm\sqrt{\lambda}$

$\therefore$ For $\lambda = 0$ we get two linearly independent sol$\underline{n}$ 1 and $\ln r$.

$\therefore$ Using the principle of superposition;

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right].$$

(Note :- we have not considered the sol$\underline{n}$ involving $\ln r$ and $r^{-\alpha}$)

$$\therefore h(\theta) = u(r,\theta) = A_0 + \sum_{1}^{n} a^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

$\therefore$ A_0, A_n and B_n are the Fourier coefficient of the Fourier expansion of h.

$$\Rightarrow A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos(n\phi)\,d\phi$$

$$\& \; B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin(n\phi)\,d\phi$$

$$\text{and } A_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi.$$

$$\boxed{\begin{aligned} 1+2\sum\left(\frac{r}{a}\right)^n \cos(\theta-\phi)n &= 1+2\,\mathrm{Re}\sum\left(\frac{re^{i(\theta-\phi)}}{a}\right)^n \\ &= 1+2\,\mathrm{Re}\cdot\frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}} \\ &= \frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}. \end{aligned}}$$

$$\therefore u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi} h(\phi)\left[\cos(n\phi)\cos(n\theta)+\sin(n\phi)\sin(n\theta)\right]d\phi$$

$$\therefore u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\left\{1+2\sum_1^{\infty}\left(\frac{r}{a}\right)^n\cos(n(\theta-\phi))\right\}d\phi.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\left\{\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}\right\}d\phi.$$

(3) Show that the value of u at any point is just the average value of u on a circle centered on that point if $\Delta u = 0$.

Soln:- From (ii); $\Delta u = 0$

$$u(r,\theta) = h(\theta)$$

$$\Rightarrow u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2}\, h(\varphi)\, d\varphi$$

At, $r = 0$ $\quad u(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)\, d\varphi = \frac{1}{2\pi}\int_0^{2\pi} u(a,\varphi)\, d\varphi.$

$\therefore$ The value of u at any point is just the average value of u on a circle centered on that point.

(4) Let φ be such that $\varphi(x_0)\,\varphi(x_1) < 0$ for x_0 and x_1 on boundary of a smooth, bounded domain Ω. If $\Delta u = 0$ in Ω and $u|_{\partial\Omega} = \varphi$, show that u must change sign in Ω. (φ is continuous).

Proof:- Let any solution u of $\Delta u = 0$ in Ω

$$u|_{\partial\Omega} = \varphi$$

does not change sign.

WLOG, let $u(x) > 0$ in Ω.

Then by maximum principle, the maximum and minimum of u must be attend on the bdry.

and hence, $\min\limits_{\bar\Omega} u \overset{M.P}{=} \min\limits_{\partial\Omega} u$ But, $\min\limits_{\bar\Omega} u \geq 0$. ⊗

$\Rightarrow$ $\min\limits_{\partial\Omega} \varphi < 0$ is not possible -- a contradiction. ▨