Applications of Modal Logic

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What are Conditional Statements?

Conditional Logic (Logic of Conditionals, is concerned with the logical and semantical properties of a certain class of sentences occurring in a natural language.

If. . . then

1. If it is a square, then it is a rectangle.
2. If you strike the match, it will light.
3. If it is raining, then we are taking Taxi.
4. If you had struck the match, it would have lit.
5. If I were feeling warm, I would remove my jacket.
6. When I find a good man, I will praise him.
7. You will need my number should you ever wish to call me.
8. No Hitler, no A-Bomb.

Role of conditionals in mathematical, practical and causal reasoning cannot be ignored

If P then Q

Antecedent and Consequent

P: antecedent, protasis
Q: consequent, apodosis

Analyzing Conditionals

1. Conditionals as truth-functional binary connectives: material conditional
2. Conditionals as non-truth-functional, but truth-conditional binary connectives: Stalnaker - Lewis (1968)
3. Conditionals as non truth conditional binary connectives: Edgington.

Indicative and subjunctive conditionals

Indicative

If Godse did not kill Gandhi ji, someone else did.

Subjunctive Conditional

If Godse had not killed Gandhiji, someone else would have

(1) is true, given what we know about Gandhijis death.
(1) is true, under the assumption that Godse did kill Gandhiji, only if one believes in conspiracy theories.
The fact that the antecedent is false(As Godse did shoot Gandhiji) does not by itself make the sentence true.
They have different truth conditions.

Indicative and Subjunctive Conditionals

Indicative conditionals

IND in antecedent, IND in consequent.

1. If Sita is rich, then she is happy.
2. If Sita becomes rich, she will be happy.

Subjunctive:

Subjunctive conditionals= SUBJ/PAST in antecedent, SUBJ/WOULD in consequent.

1. If Sita were/was rich, she would be happy.
2. If Sita had been rich, she would have been happy

Propositional Validity:

Tautology

A wff ϕ is a tautology or logical truth iff $V(\phi) = 1$ for all assignment of truth-value to the propositional atoms of ϕ. ($\models \phi$)

Logical Consequence

ϕ is a logical consequence of a set of formulae iff every assignment of truth value that makes all the formulae of Γ true makes ϕ true. ($\Gamma \models \phi$)

Validities that are Good

Validities of Material Conditional

1. $\phi \rightarrow \psi, \phi \models \psi$ (modus ponens)
2. $\phi \rightarrow \psi, \neg\psi \models \neg\phi$ (modus tollens)
3. $(\phi \vee \psi) \models \neg\phi \rightarrow \psi$ (Stalnakers direct argument"; aka disjunctive syllogism)
4. $\models (((\phi \wedge \psi) \rightarrow \gamma) \leftrightarrow (\phi \rightarrow (\psi \rightarrow \gamma)))$ (import export)
5. $\models [(\phi \vee \psi) \rightarrow \gamma] \leftrightarrow [(\phi \rightarrow \gamma) \wedge (\psi \rightarrow \gamma)]$ (simplification of disjunctive antecedents)

Bad Validities

1. $\neg\phi \models (\phi \rightarrow \psi)$ (falsity of the antecedent)
2. $\phi \models (\psi \rightarrow \phi))$ (truth of the consequent)
3. $(\phi \rightarrow \psi) \models (\neg\psi \rightarrow \neg\phi))$ (contraposition)
4. $(\phi \rightarrow \psi), (\psi \rightarrow \gamma) \models (\phi \rightarrow \gamma))$ (transitivity)
5. $(\phi \rightarrow \psi) \models ((\phi \wedge \gamma) \rightarrow \psi)$ (antecedent strengthening)
6. $\models \neg(\phi \rightarrow \psi) \leftrightarrow (\phi \wedge \neg\psi)$ (negation)

Undesirable validities;

Paradox of material Implication

The paradox of the truth of the antecedent:

1. a. X will teach his class at 12am. Therefore, if X dies at 9am, X will teach his class at 12am.
2. X missed the only train to Haridwar this morning and had to stay in Kanpur. So, if John was in Haridwar this morning, X missed the only train to Haridwar this morning and had to stay in Kanpur.

Contraposition, Strengthening,Transitivity

Contraposition

If Goethe had lived past 1832, he would not be alive today. Therefore (??) If Goethe was alive today, he would not have lived past 1832.

Strenghening:

If Ravi adds sugar in his coffee, he will find it better. Therefore (??) If Ravi adds sugar and salt in his coffee, he will find it better

Transitivity

If I quit my job, I won't be able to afford my rent of my apartment. If I win a million, I will quit my job. b.(??)If I win a million, I won't be able to afford my apartment.

Negation of Conditional

1. It is not true that if God exists, criminals will go to heaven.
2. (??) Hence God exists, and criminals wont go to heaven.

The expected understanding of negation

1. If God exists, criminals won't go to heaven.
2. $\neg$ (if p then q) = if p then $\neg q$.

Dorothy Edgington's Proof of the Existence of God

there are counter intuitive results to formulate if. . . then as $\neg p \vee q$.

Argument

If God does not exist($\neg G$), then it is not the case that if I $(P \rightarrow A)$).
pray (P), my prayers will be answered(A): $(\neg(G \rightarrow \neg A)$
I dont Pray $(\neg P)$.
Therefore, God Exist (G).

Contd. . .

Formal Analysis

1. $(\neg G \rightarrow \neg(P \rightarrow A))$
2. $(\neg G \rightarrow (P \wedge \neg A))$
3. $(G \vee (P \wedge \neg A))$
4. I don't pray: P=0
5. $G \vee (0 \wedge \neg A)$
6. $G \vee 0$
7. **therefore,**G (God Exist).

Call fort the revision of semantics of conditionals

1. The examples raise a problem for the pragmatics of conditionals, and do not call for a revision of the semantics. (Quine 1950 on indicative conditionals, Grice 1968, Lewis 1973).
2. The examples call for a revision of the semantics of conditionals (Quine 1950 on counterfactual conditionals, Stalnaker 1968, Lewis 1973)

Limits of Truth Functionality

Quine: 1950

Whatever the proper analysis of the contrafactual conditional may be, we may be sure in advance that it cannot be truth-functional; for, obviously ordinary usage demands that some contrafactual conditionals with false antecedents and false consequents be true and that other contrafactual conditionals with false antecedents and false consequents be false (Quine 1950)

Example:

1. If I weighed more than 150 kg, I would weigh more than 100 kg.
2. If I weighed more than 150 kg, I would weigh less than 25 kg.

Suppose I weigh 70 kg. Then the antecedent and consequent of both conditionals are presently false (put in present tense), yet the first is true, the second false.

Strict Implication(1932)

$\neg$, $\wedge$

X represents It is impossible that A is true. It is written as $\neg\Diamond X$ here.

Some Definitions

1. $A \multimap B = \neg A \vee_i B = \neg\Diamond(A \wedge \neg B)$
2. $(A = B) = (A \multimap B) \wedge (B \multimap A)$
3. $A \vee_i B =_{\text{Def}} \neg\Diamond(\neg A \wedge \neg B)$.
4. $A \multimap B =_{\text{Def}} \neg\Diamond(A \wedge \neg B)$.
5. $A = B =_{Def} \neg\Diamond(A \wedge \neg B) \wedge \neg\Diamond(B \wedge \neg A)$
6. $A \circ B$ represents intensional conjunction. $A \circ B = \neg\neg\Diamond(A \wedge B)$. A and B are consistent. It is possible that both A and B are true,
7. $A \circ B =_{\text{Def}} \neg(A \multimap \neg B)$

Consequences

Paradox of material implication

The strict conditional solves the paradoxes of material implication.
In particular: $\not\models (p \multimap (q \multimap p))$.
Why? Construct model for $\Diamond(p \wedge \Diamond(q \wedge \neg p))$.

Problems with Strict implication:

However, the strict conditional is still monotonic:

1. $\Box(p \rightarrow q) \models \Box(\neg q \rightarrow \neg p)$
2. $\Box(p \rightarrow q) \models \Box(p \wedge r \rightarrow q)$
3. $\Box(p \rightarrow q), \Box(q \rightarrow r) \models \Box(p \rightarrow r)$

Conclusion: must do better

Citeris Paribus conditionals:

1. If it does not rain tomorrow, we will go to the cricket
2. If it does not rain tomorrow, then other things being equal, we will go to the cricket
3. If A and *CA* then B, where *CA* is citeris paribus clause.
4. *CA* might include, some thing like, we are not invaded by Martians, A is flying saucers arrive from the Mars etc.
5. The conditional $A > B$ is true in a world if B is true at every world at which $A \wedge CA$ is true.
6. How to spell out this idea explicitly??

Context dependency:

If I overtake now, there will be an accident. You, on the other hand, are sitting in the passenger seat and cannot see the oncoming traffic.

Semantics of C

1. First, we extend our formal language with the connective $>$. Thus, if A and B are formulas of the extended language, so is $A > B$. Let the language be $\mathcal{L}$. Logic of Modal operators K_v.
2. $\{W, R, V\}$ for the normal modal operators. But, for conditionals interpretation changes a bit. $\{W, \{R_A : A \in \mathcal{L}\}, v\}$
3. R_A is a collection of binary relations on W.
4. Intuitively $w1 R_A w2$ means that A is true at $w2$, which is, ceteris paribus, the same as $w1$.
5. Truth conditions for $\Box A$, $\Diamond A$ is same as K_v.

Logic of C

1. $v_w(\Diamond A) = 1$ if, for some $w1 \in W$ such that $wRw1$, $v_w(A) = 1$; and 0 otherwise.
2. $v_w(\Box A) = 1$ if, for all $w1 \in W$ such that $wRw1$, $v_w(A) = 1$; and 0 otherwise
3. $v_w(A > B) = 1$ iff for all $w1$ such that $wR_A w1$, $v_w(B) = 1$
4. $[A]$ be the class of worlds where A is true, $\{w\colon v_w(A) = 1\}$.

Truth Conditions for $A > B$

$A > B$ is true at w iff $f_A(w) \subseteq [B]$.
Since no constraints are placed on the relations R_A, C is the analogue for conditional logics of the modal logic K.

Semantic Tableaux for C

1. $A > B, i, iR_A j \downarrow B, j$.
2. $\neg(A > B), i \downarrow iR_A j, (B, j)$

Not theorems in C

1. $A > B \not\models_C (A \wedge C) \rightarrow B$.
2. $A > B, B > C \not\models_C A > C$.
3. $A > B \not\models_c \neg B > \neg A$

Extensions of C

Conditions on R_A

There is nothing in the semantics, so far, that requires A to be true at $w1$ iff $wR_A w1$.

1. $f_A(w) \subseteq [A]$ Natural Condition.
2. If the world, w, is already such that A is true there, then, presumably, the worlds that are essentially the same as w, except that A is true there, must include w itself. If $w \in [A]$, then $w \in f_A(w)$

Tableaux for C+

1. $\neg(A > B), i \downarrow (ir_A j), A, j \neg B, j$
2. $\downarrow \neg A, j \vee A, i(ir_A i)$

Theorems in C+

$A, A > B \vdash_{C+} B$

$p > r \nvdash_{C+} p > (r \wedge q)$.

Similarity of Spheres

1. Worlds accessible to w via r_A should be thought of as the worlds most similar to w at which A is true.(Stalnaker, David Lewis)
2. Each world w comes with a system of spheres. All the words that fall within the sphere are are more similar to w than any world that falls outside the sphere.
3. Technically, for any world w, there is a set of subsets of W $\{S0^w, S1^w \dots Sn^w\}$ (for some n) such that $w \in S0 \subseteq S1 \subseteq S2 \dots Sn = W$

Further Constraints

1. $f_A(w) \subseteq [A]$ [Natural]
2. If $w \in [A]$ then $w \in f_A(w)$ the worlds that are essentially the same as w except that A is true there, , must include w itself.
3. If there are any worlds at which A is true then $f_A(w) \not\equiv \emptyset$, then $f_A(w) \not\equiv \emptyset$.
4. If $[A] \not\equiv \emptyset$ then $f_A(w) \not\equiv \emptyset$.
5. If $f_A(w) \subseteq [B]$ and $f_B(w) \subseteq [A]$ then $f_A(w) = f_B(w)$
6. If $f_A(w) \cap [B]\ \not\emptyset$, then $f_{A \cap B}(w) \subseteq f_A(w)$
7. If $x \in f_A(w)$ and $y \in f_A w$, then $x = y$.(C2)
8. If $w \in [A]$ and $w' \in f_A(w)$, then $w = w'$. (dropping c2 condition)

Assumptions

1. **Uniqueness Assumption** For any antecedent and evaluation world, there will be unique most similar antecedent world.
2. **Limit Assumption:** For an antecedent and evaluation world, there is always a set of most similar antecedent worlds.

Examples

1. ......If this one inch line were more than an inch long......it would be more than one inch.
2. If it will either rain tomorrow or it wont, then it will rain tomorrow. If it will either rain tomorrow or it wont, then it wont rain tomorrow.