Basic Epistemic Logic

A. V. Ravishankar Sarma

Indian Institute of Technology Kanpur

avrs@iitk.ac.in

October 26, 2020

Quotes

Knowledge and Wisdom

1. He that knows not, and knows not that he knows not is a fool. Shun him
2. He that knows not, and knows that he knows not is a pupil. Teach him.
3. He that knows, and knows not that he knows is asleep Wake him.
4. He that knows, and knows that he knows is a teacher. Follow him.

Quotes

We know what we know, we know that there are things we do not know, and we know that there are things we don't know we don't know

......Donald Rumsfeld (4 Sept 2002)

Example

We write that the agent knows that p as Kp, that she does not know that p and q as :$\neg K(p \wedge q)$, that she knows whether or not q as $Kq \vee K\neg q$, that she knows that she does not know that p implies q as $K\neg K(p \rightarrow q)$.

Introduction

1. Greek word *episteme* refers essentially just to knowledge, and the logic of belief may properly be called *doxastic*, from the Greek doxa
2. Epistemic logic as a branch of Modal Logic
3. Important application of Epistemic logic is in the area of Artificial Intelligence, modeling *common knowledge* as a phenomenon of something arising from the interaction of agents.
4. the role of *Agent* is central to Epistemic Logic: it is inconceivable to think of anything that is *known* or is *believed* without referring to one or more individuals who know or believe it.

Why Epistemic Reasoning?

1. Logic of knowledge and belief
2. Better representation of communication protocols.
3. Important in distributed computing.
4. Epistemic Game theory: knowing about what others know- as a strategy in games

Validities in Classical Logic vs Epistimic Logic

Knowing whether or not p

$p \lor \neg p$

Example

'Dean doesn't know whether Nixon knows that Dean knows that Nixon knows that McCord burgled O'Brien's office at Watergate'.
If we take Dean to be agent 1, Nixon to be agent 2, and p to be the statement McCord burgled O'Brien's office at Watergate, then this sentence can be expressed in the logic as:

Representation

we take Dean to be agent 1, Nixon to be agent 2, and p to be the statement 'McCord burgled O'Brien's office at Watergate', then this sentence can be expressed in the logic as
$\neg K1 \wedge \neg(K2\, Kl\, K2\, p) \wedge \neg K1\, (\neg K2\, K1\, k2\, p)$.

Epistemology and Epistemic Logic

Epistemology

Study of nature and scope of knowledge.
It is concerned with the following questions: What is knowledge? what can we know? What it means to say that some thing qualifies to be a knowledge claim?

Questions

Questions in Epistemic Logic

How do we formalize knowledge? How do we account for shared knowledge? How do we account for acquisition of knowledge through communication? How do we understand the knowledge required to perform certain actions?

Knowledge: Basic presupposition

JTB Account

A basic presupposition of any epistemic logic should be the JTB (Justified True Belief) conception of knowledge, which gives the following definition: S knows that p iff:

1. S believes that p
2. S is justified in believing that p
3. p is true.

Note: Although sensible, does not capture the nature of knowledge since all the three notions used are not yet clear and still subject of discussion

How do we come across knowledge claims?

1. Rationalists: Knowledge only comes from reason(ing): Plato, Descartes
2. Empiricists: Knowledge is derived from sense experience: Locke, Hume.
3. Immanuel Kant: Distinguishes analytic and synthetic knowledge argues that categories of analytical knowledge is derived purely from logical argument, where ad synthetic knowledge is not. Whether synthetic apriori is possible or not is debatable.

Language of Epistemic Logic

Definition (Basic Epistemic Language)

Let At be a set of atomic propositions, and Ag a set of agent symbols and Ot set of modal operators. The language $L(At, Op, Ag)$, the language for multi-agent epistemic logic, is generated by the following BNF:
$\phi ::= p|\neg\phi|(\phi\wedge\psi)|Ka\,\phi$.

The set Op depends on Ag. $Op = \{Ka \mid a \in Ag\}$. $L(At)$ are propositional formulas. $Ma\phi$ means Agent a does not know $\neg\phi$.

Three Requirements of Epistemic Logic

Modality

We want to be able to represent not only facts, but also the knowledge that each individual has about these facts, and knowledge about other individuals knowledge. Modal Operators will help us here $K_i\phi$

Non-monotonicity

Deducing things from what cannot be proved leads to non-monotonicy. Addition of new information leads to the withdraw of old conclusions.

Examples

Phase-1

i; guess two numbers, x and y. I tell you that x is greater than 1 and that y is greater than x. Their product is 12.
j: I dont know what x and y are?

Phase-2

i: Now I will give you the additional information that x is odd.
j: Now I know that x is 3.

What happened in the second phase of dialogue?

Non-monotonicty

Let Γ contains all the knowledge of agent i and formula K_iA cannot be proved from Γ. Then Agent does not know A.

John Macarthy

$\neg[\Gamma \vdash K_iA] \Downarrow \Gamma \vdash \neg(K_iA)$

Vagueness

Since there is a limit to our power of discriminaion, and our language is full of vague words whose meaning can be defined only by observation, it follows that we are a linguistic community only in the loose sense.

Language of Epistemic Logic

Definition (Basic Epistemic Language)

Let P be a set of atomic propositions, and A a set of agent symbols. The language $L_{\mathcal{K}}$, the language for multi-agent epistemic logic, is generated by the following form: $\phi ::= p|\neg\phi|(\phi \wedge \phi)|Ka\phi]$.

Primitive Symbols and formulas

We consider a set of agents (individuals) $I = \{1 \ldots\ldots n\}$ We assume that these individuals are not only interested in the objective reality but also in each others knowledge. Language is propositional logic$+$ modal operators $K_i\phi$. Let $P = \{p1, p2, p3 \ldots\ldots pn\}$ is a set of variables of the propositional calculus interpreted as primitive facts.

Primitive symbols and formulas

1. $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\} \cup \{K_i\phi \colon i \in I\}$ as the set of connectives.
2. k_i are modal operators; $K_i\phi$ means agent i knows ϕ.
3. $L_i\phi$ is equivalent to $\neg K_i \neg\phi$: Agent i thinks that ϕ is possible.

Well formed Formulas

1. If $pj \in P$ then $pj \in$ Well formed formula
2. If ϕ and ψ are well formed formulas, then $\neg\phi$ is a wff, and $\phi \wedge \psi$.
3. If $\phi \in$ wff and if $i \in I$ then $K_i\phi$ is a WFF.

Axioms of Knowledge (LK5)

1. All tautologies of Propositional Logic.
2. Consequential Closure: If the agent knows a collection of sentence Γ and ϕ is a logical consequence of Γ, then the agent also knows ϕ. Saying that an agent's knowledge is closed under logical consequence is equivalent to saying that the agent knows all logical axioms and that the agent's knowledge is closed under modus ponens. If the agent also knows ϕ. $K_i\phi \wedge K_i(\phi \rightarrow \psi) \rightarrow K_i\psi$.
3. Veridicality: What is known is true. $K_i\phi \rightarrow \phi$
4. Suppose that the agent knowsϕ; that is, he has been able to derive ϕ . Then he should be aware that he has derived ϕ; that is, he knows that he knows ϕ. We thus have the following axiom: $K_i\phi \rightarrow K_I K_I \phi$.
5. $L_i\phi \rightarrow K_i L_i \phi$.

1. Modus Ponens:$\{\phi, (\phi \rightarrow \psi)\} \vdash \psi$
2. R2: $\phi \downarrow K_i\phi$

Note

$A2$ together with $R2$ leads to Logical Omniscience. It forces a view that agents are ideal reasoners. If T is a theorem, then $K_i T$ is also a theorem. It is impossible to have $T \wedge \neg K_i T$.
All agents then know all valid formulas and also all its valid consequences. This does not seem to be a realistic model dealing with everyday reasoning. Even if wee know X is valid, wee may fail to know it, because of its computational intractability, or because we happen not to think of justification of X, or just because we are not interested in X.

Revision of LK5

1. If $\vdash X$ then $\vdash K_iX$. If $\phi \vdash X$ then $K_i\phi \vdash K_iX$. Now $X \wedge \neg K_iX$ is consistent. In the new system still preserves the necessitation rule. It states that necessitation rule is reasonable only for those formulas ϕ which are logically true, atleast true on the whole model. So, it is not necessary that $\phi \vdash X$ it follows that $\phi \vdash K_iX$.
2. Ψ is an axiom according to A1-A6, then so is $K_i\Psi$, for each i

Examples

Dialogue between Q and A

1. Q: Is this New Student Activity Center (NSAC)?
2. A: Yes

A is well informed, cooperative and responsible member of IITK community.Let ϕ be a factual and simple question. The presupposition for normal truthful answer is that A knows that ϕ: $K_A\phi$. Other presuppositions:

1. $\neg K_Q\phi \wedge \neg K_Q\neg\phi$: Q doesnt knowknow if ϕ.
2. $K_A\phi \vee \neg K_A\neg\phi$: Q thinks that A may know the answer.
3. $K_A\phi \wedge K_Q\phi$.
4. After two step communication: $K_Q K_A\phi \wedge K_A K_Q\phi$
5. $C(Q, A), \phi$

Basic Idea

1. Possible Worlds semantics: Beside the current state of affairs, there are other possible state of affairs for any given agent i: agents may be unable to distinguish the true world among all possible worlds.
2. An agent i is said to know a formula Ψ, if Ψ is true in all the worlds that are epistemically possible for him.
3. $k_i K_j \phi$ means agents i knows that k knows that ϕ.
4. Kripke Structure: $\{W, R, V\}$

Semantics

$V : At \rightarrow \{True, False\}$

The agent *a* considering possible a number of different situations that are consistent with the information that the agent has.

An agent *a*is said to know ϕ , if ϕ is true in all the situations that *a* considers possible,

Semantics of Epistemic Logic

We now define a relation $\models$ where $(M, s) \models \phi$ is read ϕ is true, or satisfied, in state s of structure$M$

1. $(M, s) \models \phi$ for a primitive proposition ϕ if $\Phi(s)(p) = true$
2. $(M, s) \models \neg\phi$ if $(M, s) \not\models \phi$.
3. $(M, s) \models \phi \wedge \psi$ iff $M, s \models \phi$ and $M, s \models \psi$.
4. $M, s \models Ka\phi$ iff $M, t \models \phi$ for all t such that $(s, t) \in R$.
5. $M, s \models La\phi$ iff there exists a t such that $(s, t) \in R$

Agent knows whether or not ϕ

pp: 11 [Handbook of Epistemic Logic.

Example

Kripke Structure

1. W a set of states (also called possible worlds).
2. v an assignment of truth values to the given propositions for each state $w \in W$. Also $V(w, pi) \in \{$True,False$\}$ for each state $w \in W$ and a primitive proposition $p_i \in P$
3. For each $i \in I$ R_I is an equivalent relation defined over W. R_i is called an accessibility relation of agent i.
4. $(w, u) \in R_i$ is read as u is accessible from w for agent i, or u is i accessible from w.
5. $(w, u) \in R_i$ holds iff agent i cannot distinguish the states of affairs w from the states of affairs u.
6. If w is actual state of the world, then agent i would consider u as possible state of the world.

Satisfaction Relation

ψ is true or satisfied in state u of model M:

The satisfaction relation $\models$ for Kripke model M is defined as follows:

1. $M, w \models p_i$ for primitive proposoitions $p_i \in P$ if $V(p, w)$ is true.
2. $M, w \models \neg\phi$ iff $M, w \not\models \phi$
3. $M, w \models \phi \wedge \psi$ iff $M, w \models \phi$ and $M, w \models \psi$
4. $M, w \models K_i\phi$ iff $M, u \models \phi$ for all u such that $(w, u) \in R_i$.
5. $M, w \models L_i\phi$ iff there exists a u such that $(w, u) \in R_i$ and $M, u \models \phi$.
6. $M, w \models \phi \vee \psi$ iff $M, w \models \phi$ or $M, w \models \psi$.

Note

Indistinguishability

We say that agent i knows fact ϕ in state w if ϕ is true at all u- accessible states, i.e., all states indistinguishable from w from agent i's point of view R_i is an equivalence relation(Reflexive, symmetric, transitive) ensures that everything known by i is true, and that i knows her own internal knowledge. $L_i\phi$ condition states that agent i think ϕ is possible in a state w, if there is atleast one state, say u, which us accessible from w, where ϕ is true.

Common Knowledge

A group has common knowledge C of a formula ϕ if:

1. Every one in the group knows that ϕ is true. (i.e., $K_i\phi$ for all $i \in I$) and
2. Everyone in the group knows that everyone knows that ϕ is true ($K_iK_j\phi$ for $i, j \in I$)
3. Everyone in the group knows that everyone knows that everyone knows that ϕ is true. ($K_iK_jK_i$ for all $i, j \in I$, and so on.

Every one knows ϕ

Every one knows ϕ can be represented as E_ϕ as follows: $K_1\phi \wedge K_2\phi \wedge \ldots \wedge K_n\phi$ for all $\{i, j\} \in I$. We express it using Common knowledge. We add the following axiom schemes to LK5 and call it CK5.

1. $C\phi \rightarrow K_i\phi$
2. $C\phi \rightarrow C\,C\phi$.

$M, w \models C\phi$ iff $M, w \models E^k\phi$ for each $k = 1 \ldots \ldots n$, where $E^I\phi = E\phi$ and $E^{i+1}\phi = E\,E^i\phi$.

1. $E_G\,\phi \leftrightarrow \bigcap_{i\in I} K_i\,\phi$.
2. $C_G\,\phi \leftrightarrow E_G\,(\phi \wedge\ C_G\phi)$.

Muddy Children Puzzle

Muddy Children Puzzle

Description

Let n children go to the park to play. When their mother comes to find them, she sees that k of them have mud on their foreheads. She then said, *atleast one of you has mud on your forehead*, and then asks, *Do you know if you have mud on your forehead?* The children respond simultaneously, **No.** It is proven inductively that they will constantly say, **No** for the first $k-1$ times and the children with muddy foreheads will answer, **Yes** at the *kth* time.

Assumptions

Everybody (Common knowledge) that the mother(father) is truthful, all the children can hear the mother, and can see the foreheads of the others.
Each can see the mud on others but not on his own forehead.
All the children are also assumed to be truthful and intelligent enough to make logical decisions.

Muddy Children puzzle: Case1

There is a proof that the first k - 1 times he asks the question, they will all say no but then the kth time the dirty children will answer Yes.

k=1

The proof is by induction on k.
For k = 1, the result is obvious: the dirty child sees that no one else is muddy, so he must be the muddy one

Case-2

k=2

So there are just two dirty children, *a* and *b*. Each answers no the first time, because of the mud on the other.

But, when *b* says no, *a* realizes that he must be muddy, for otherwise *b* would have known the mud was on his head and answered yes the first time. Thus, *a* answers yes the second time. But *b* goes through the same reasoning and answers yes.

Now there are three dirty children, a, b, c.
Child a argues as follows. Assume I dont have mud on my head. Then, by the $k = 2$ case, both b and c will answer yes the second time. When they dont, he realizes that the assumption was false, that he is muddy, and so will answer yes on the third question.
Similarly for b and c

Fathers Announcement

Let us denote the fact At least one child has a muddy forehead by m.
Notice that if $k > 1$, that is, more than one child has a muddy forehead, then every child can see at least one muddy forehead, and the children initially all know m.
Thus, it would seem, the father does not need to tell the children that m holds when $k > 1$.
But this is false!
In fact, had the father not announced m, the muddy children would never have been able to conclude that their foreheads are muddy.

Muddy-Children Puzzle

1. For $k = 1$, the child with muddy forehead sees that no one is muddy, since he knows that at least one child has a muddy forehead, he concludes that he must be the one, and says, **Yes**.
2. For $k = 2$, all the children answer, **No**, the first time, each of the muddy children sees just one muddy child, the first time he thought that the one he sees is the only one, and if it was like that, he would have said,**No** as in the case of just one muddy child, therefore, he concludes that he is muddy too.
3. In the general case, each muddy child sees $k - 1$ muddy children, he will keep saying, **No** and realise that he is muddy at the $(k - 1)$ th time.

Muddy children Puzzle: Two Children A, B

Let *a*, *b* are the two children
A: a has a muddy forehead; *B*: b has a muddy forehead. Knowledge operators: K_a and K_b

Muddy Children Puzzle

1. $A \vee B$: a is having muddy forehead or b is having muddy forehead.
2. $K_a(A \vee B)$; $K_b(A \vee B)$
3. $K_a K_b(A \vee B)$; $K_b K_a(A \vee B)$.
4. $K_a(K_b B \vee K_b \neg B)$
5. a knows that b doesnt know that his head is muddy: $K_a(\neg K_b B)$.
6. $K_a(X \to Y) \to K_a X \to K_a Y$
7. $K_a X \to X$; $X \Downarrow K_a X$.
8. b sees a: $K_a[K_b A \vee K_b \neg A]$.

Three Wisemen Puzzle

The king summons three of his wise men and pastes on their forehead a small black dot. The wise men are facing one another, so that they can all see the dot on each others forehead, but they do not know the color of the dot on their own. He tells them, On each of your foreheads, I have placed a dot which is either white or black. I have put a black dot on at least one of your foreheads. Do any of you know what the color on your forehead is?

Three Wisemen Puzzle

1. The wise man all answer in unison, "No, your majesty".
2. The king asks, "Do you now know what the color on your forehead is?"
3. They again answer in unison, "No, your majesty."
4. The king asks, "Do you now know what the color on your forehead is?"
5. All three wise men answer, "Your majesty, I have a black dot on my forehead." How do the wise men determine this?

Analysis of Three Wisemen Problem

After the first two questions, wise man A reasons as follows: Suppose that I have a white dot on my forehead. Then wise man B sees that I have a white dot and that C has a black dot. Now, in that case after the first question, wise man B could have reasoned to himself, Suppose that I (B) have a white dot on my forehead. Then C sees the white dot on A and on me. Then C could reason to himself. A's and B's dots are white but there is a black dot. The black dot must be mine.

Three Wisemen Contd

1. So C should have known the color of his dot before the king asked his first question, and should have answered the first question. But he did not answer it. This is a contradiction: therefore the supposition was wrong. I must have a black dot.
2. So B should have been known the color of his own dot as soon as he saw that C did not answer the first question, and B should have been able to answer the second question. But he did not answer it. This is a contradiction: therefore, the supposition was wrong. I must have a black dot.

Coordinated Attack

Two divisions of an army are camped on two hilltops overlooking a common valley. In the valley awaits the enemy. It is clear that if both divisions attack the enemy simultaneously, they will win the battle; whereas if only one division attacks, it will be defeated.

Coordinated Attack: Some Assumptions

The divisions do not initially have plans for launching an attack on the enemy, and the commanding general of the first division wishes to coordinate a simultaneous attack (at some time the next day).

Neither general will decide to attack unless he is sure that the other will attack with him. The generals can only communicate by means of a messenger.

Normally, it takes the messenger one hour to get from one encampment to the other. However, it is possible that he will get lost in the dark or, worse yet, be captured by the enemy. Fortunately, on this particular night, everything goes smoothly.

How long will it take them to coordinate an attack?

Will it ever End?

1. Actually, no!
2. For every (odd) round $2k+1$, left(A) would think that right(B) may have not received its message, and therefore won't attack.
3. That is, while left knows that—- right knows that left knows that k times
4. left(A) thinks that it's possible that right does not know that, and therefore that right won't attack, and therefore left should not attack.

Problem:

How to show that despite the fact that everything goes smoothly, no agreement can be reached and no general can decide to attack.

proof

1. Suppose General A sends a message to General B saying Let's attack at dawn, and the messenger delivers it an hour later.
2. General A does not immediately know whether the messenger succeeded in delivering the message. And because B would not attack at dawn if the messenger is captured and fails to deliver the message, A will not attack unless he knows that the message was successfully delivered.
3. Consequently, B sends the messenger back to A with an acknowledgment. Suppose the messenger delivers the acknowledgment to A an hour later.
4. Since B knows that A will not attack without knowing that B received the original message, he knows that A will not attack unless the acknowledgment is successfully delivered.

Coordinated attack: Failure to attain common Knowledge

1. Thus, B will not attack unless he knows that the acknowledgment has been successfully delivered. However, for B to know that the acknowledgment has been successfully delivered, A must send the messenger back with an acknowledgment to the acknowledgment.
2. Similar arguments can be used to show that no fixed finite number of acknowledgments, acknowledgments to acknowledgments, etc., suffices for the generals to attack.

Coordinated Attack

This endless sending of messages will never bring enough knowledge to both generals and so a coordinated attack cannot take place. This is called the coordinated attack problem.

Although common knowledge via communication is impossible, we can evolve with some theory to estimate probabilities of sufficient shared knowledge for a successful coordinated attack. Depending on the chosen strategy it seems sometimes important to ask for a message of receipt and sometimes this is not important at all.

PS: Online References

What does it all mean?

Even the simplest consensus problem cannot be solved over unreliable communication system.

Byzantine Generals Problem: The abstract problem

Each division of Byzantine army is directed by its own general. There are n Generals, some of which are traitors.
All armies are camped outside enemy castle, observing enemy.
They all communicate with each other by messengers.

Requirements:

1. G1: All loyal generals decide upon the same plan of action
2. G2: A small number of traitors cannot cause the loyal generals to adopt a bad plan. We do not have to identify the traitors.

Naive Solution

The general sends $v(i)$ to all other generals
To deal with two requirements:

1. All generals combine their information v(1), v(2), .., v(n) in the same way.
2. Consider Majority $(v(1), v(2), \dots v(n))$, ignore minority traitors.

Naive solution does not work

1. Traitors may send different values to different generals.
2. Loyal generals might get conflicting values from traitors

Requirement: Any two loyal generals must use the same value of v(i) to decide on same plan of action.

Simplified Version

Let there be one general and two lieutenants(one of them may be loyal and other may be a traitor)
A commanding general (commander) must send an order to his $n-1$ lieutenants.

Interactive consistency conditions:

1. IC1: All loyal lieutenants obey the same order.
2. IC2: If the commanding general is loyal, then every loyal lieutenant obeys the order he sends.
3. If General is loyal, $IC2 \Rightarrow IC1$
4. Each general sends his value v(i) by using the above solution, with other generals acting as lieutenants.

Requirements

1. A1. Every message that is sent is delivered correctly.
2. A2. The receiver of a message knows who sent it.
3. A3. The absence of a message can be detected.
4. A loyal general's signature cannot be forged, and any alteration of the contents of his signed messages can be detected.
5. Anyone can verify the authenticity of a general's signature.

3 generals: 1 commander and 2 lieutenants

1. Consider 3 generals, 1 traitor among them.
2. Two messages: Attack or Retreat
3. Two cases in the figure:A
4. L1 sees (A,R). Who is the traitor? C or L2?
5. Fig 1: L1 has to attack to satisfy IC2.
6. Fig 2: L1 attacks, L2 retreats. IC1 violated

No solutions with fewer than $3m + 1$ can cope up with m traitors.

It is shown that, using only oral messages, this problem is solvable if and only if more than two- thirds of the generals are loyal; so a single traitor can confound
two loyal generals.

www.cs.cornell.edu/courses/cs614/2004sp/papers/lsp82.pdf

Solution: (3m + 1) generals for m traitors

3 Generals cannot handle 1 traitor

1. If Commander is loyal, IC2 is always satisfied and IC1 follows from IC2
2. If Commander is traitor, then:
    1. Lieutenant 1 will attack
    2. Lieutenant 2 will retreat
3. IC1 is violated

Logical Omniscience

1. In Epistemic logics the notion of a world, or situation which is possible relative to an individual (knower) i.
2. The individual i knows a fact B iff B is true in all the worlds(Epistemic alternatives) which are possible for i.
3. It follows that if i knows A and also knows $A \rightarrow B$, then both A and $A \rightarrow B$ are true at all worlds possible for i and hence B is true at all the worlds possible for i so that i knows B.
4. In particular, i knows all B which are logically valid.
5. $K_i(A) \wedge K_i(A \rightarrow B) \rightarrow K_i(B)$

Logical Omniscience

Let ϕ, be formulas in $LK5$, and let Ka be an epistemic operator for $a \in A$. Let $\mathcal{K}$ be the set of all Kripke models, and $S5$ the set of Kripke models in which the accessibility relation is an equivalence. Then the following hold:

1. $\mathcal{K} \models Ka\phi \wedge Ka(\phi \to \psi) \to Ka\psi$ LO1
2. $\mathcal{K} \models \phi \Rightarrow \models Ka\phi$ LO2
3. $\mathcal{K} \models \phi \to \psi \Rightarrow \models Ka\,\phi \to Ka\,\psi$ LO3
4. $\mathcal{K} \models \phi \leftrightarrow \psi \Rightarrow \models Ka\phi \leftrightarrow Ka\psi$ LO4
5. $\mathcal{K} \models (Ka\phi \wedge Ka\psi) \to Ka(\phi \wedge \psi)$ LO5
6. $\mathcal{K} \models Ka\phi \to Ka(\phi \vee \psi)$ LO6
7. $S5 \models \neg(Ka\phi \wedge Ka\neg\phi)$ LO7

Explanation

1. LO1 says that knowledge is closed under consequences. LO2 expresses that agents know all (S5-)validities.
2. LO3-LO6 all assume that the agent is able to make logical deductions with respect to his knowledge, and, on top of that, LO7 ensures that his knowledge.

   is internally consistent.

   These properties reflect idealised notions of knowledge, that do not necessarily hold for human beings. For example, many people do not know all tautologies of propositional logic, so LO2 does not hold for them

Logical Omniscience

The fact that the above properties hold in all Kripke models is referred to as the problem of logical omniscience since they express that agents are omniscient, perfect logical reasoners

Problem

Logical omniscience requires an agent to know all logical consequences of its beliefs (that is, the set of beliefs held by the agent is closed under implication) and all valid sentences (including tautologies).

Epistemic Closure

If an agent knows a set of formulas G and if G logically implies a formula a then the agent will also know a (that is, if G and $G \rightarrow a$ are both true in every world then a must also be true in every world).
The problem with closure under implication is that it does not consider what an agent believes directly but what the world would be like if what it believed were true. Not actual knowledge but the Potential knowledge.

Unawareness

1. Why would an agent not know some fact ϕ? (i.e., why would $\neg Ki$ be true?)
   1. The agent may or may not believe ϕ, but has not ruled out all the $\neg\phi$ worlds
   2. The agent may believe ϕ and ruled-out the $\neg\phi$ worlds, but this was based on bad evidence, or was not justified, or the agent was epistemically lucky (eg., Gettier cases),...
   3. The agent has not yet entertained possibilities relevant to the truth of ϕ (the agent is unaware of).
2. Explicit vs Implicit Beliefs

Undesirable features

1. Knowing all axioms of set theory implies knowing all mathematics
2. Players of a chess game know the winning strategy of the game from the beginning.

Irrelevant Beliefs

Under the possible worlds interpretation, a valid sentence is one that is true in every world that the agent considers possible. This will also mean that the agent will need to know all tautologies in addition to its active beliefs.

Example

For instance, aside from what it(i) already knows, the agent should also know tautologies such as it is raining in Kanpur at this instant or it is not raining in Kanpur at this instant.

Problem

Possible worlds approach does not distinguish between an agent's active beliefs and tautologies that are irrelevant to the agent's beliefs.

Inconsistent Beliefs

An agent i cannot believe both a sentence a and its negation $\neg a$ without believing every sentence.

This follows from the fact that none of the possible worlds is consistent with $(a \wedge \neg a)$ [since $(a \wedge \neg a)$ is false in every world].

One of the possible solutions is to allow worlds that support both the truth and the falsity of a sentence and thus allowing inconsistent beliefs to be satisfiable.

Examples

Example:1

I point out to you that there is thick smoke coming out of a certain house and you conclude that the house is on fire. Here, if S stands for thick smoke and F stands or fire, then you already knew $S \to F$, and on learning that S , you also know F .
This situation, where knowledge is increased by the increase in evidence, is the only one that the $S5$ logic of knowledge can handle.

Example:2

Example:2

Nero Wolfe sits in his office and listens to a description of a scene of a murder, given by Archie Goodwin.
Afterwards, Wolfe knows who the murderer is, but Archie does not.
Wolfe is a fat and lazy man who almost never goes to the scene of crime, and has no evidence that Archie does not have.
How can he know more than Archie?

Example:3

I know the axioms of Peano arithmetic and also know that if Fermat's theorem is false, then this fact is deducible from the axioms.
Hence if F is false, then I know $\neg F$. If, on the other hand, F is true, clearly I cannot know $\neg F$, and hence, by the $S5$ axioms, I know that I do not know $\neg F$.
Combining this fact with my previous reasoning, I know F.
Hence either I know F or I know $\neg F$.
In other words, I know whether F is true. But of course, in fact I do not.