Logic of Vagueness

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Outline

1. A brief History of Many-valued Logic: Systems of Many Valued Logic
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3. Sorites Paradox: Possible Solutions
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Historical Sketch of Many-Valued Logic

Rosser and Turquette 1952

Ever since there was first a clear enunciation of the principle Every proposition is either true or false, there have been those who questioned it.

With the development of an axiomatic treatment of logic, it has become possible to construct systems of logic in which this principle is not valid.

One way to obtain a usable set of axioms for scuh a purpose is to replace this principle by an alternative one such as Every statement is true or false or tertium.

Rosser and Turquette, 1952] Rosser, B. and Turquette, A. (1952). Many-valued Logics. North-Holland.

Brief Historical Sketch

1. Debate on Truth Values started already in Antiquity
2. Modern Prehistory (around 1900): Peirce, (MacColl, Vasiliev)
3. Modern Pioneers (from 1918 onwards):
4. Lukasiewicz, Kleene, Bochvar, Post,
5. Further important developments: Zadeh and Fuzzy Logic

So far ...

Let us consider wffs $L1$, $L2$.

1. Class of wffs of $L1$ are included in $L2$. Class of theorems/valid inferences of $L1$ are properly included in the class of theorems and valid wfss of $L2$. In this sense, $L2$ is an extensions of $L1$ [Classical Logic]
2. The class of wffs of $L1$ and class of wffs of $L2$ coincide, but the class of theorems/valid inferences of $L1$ differ from the class of theorems and valid inferences of $L2$. In this sense, $L1$ and $L2$ are deviations from each other.

Principles of bivalence and excluded middle

Principle of Bivalence

A wff is either true or false (it takes only two values). every declarative sentence expressing a proposition has exactly one truth value, either true or false.
A logic satisfying this principle is called a two-valued logic or bivalent logic.

Law of excluded Middle

$p \vee \neg p$ as theorem of logical system.

Note

The difference between the principle and the law is important because there are logics which validate the law but which do not validate the principle.For example, the three-valued Logic of Paradox (LP) validates the law of excluded middle, but not the law of non-contradiction, $\neg(P \wedge \neg P)$, and its intended semantics is not bivalent.

Many-valued Logic

Many-valued logic extends the scope of classical logic by considering a set of truth values larger from the usual set $\{T, F\}$.In Many valued logic, we deal with:

1. Vague predicates: Rich, poor, tall, bald ...etc.
2. Future contingent sentences: *Tomorrow it will rain*, or *Reimann Hypothesis will be proved after the next 100 years etc.*
3. Liars Sentences: *this sentence is false*
4. The sentence Ravi is tall is neither true nor false.,

Origin of 3-valued Logic: Luckasewicz, 1930

I can assume without contradiction that my presence in Warsaw at a certain moment in time next year, e.g., at noon on 21st December, is not settled at the present moment either positively or negatively.

It is therefore possible but not necessary that I shall be present in Warsaw at the stated time.

On this presupposition the statement I shall be present in Warsaw at noon on 21st December next year is neither true nor false at the present moment. For if it were true at the present moment my future presence in Warsaw would have to be necessary, which contradicts the presupposition, and if it were false at the present moment, my future presence in Warsaw would have to be impossible, which again contradicts the presupposition.

Luckasewicz

1. The above statement is neither true not false and must have a third value 1/2
2. The third value: possible.
3. He suggested that statements about the future that are contingent or dependent on circumstances should not be regarded as true or false but only as indeterminate, or *I*.
4. Hence the law of the excluded middle does not apply to such statements and our argument for fatalism does not go through.

Argument of Fatalism

Suppose that there is to be an election tomorrow and that your favored candidate is Narendra Modi. By the law $A \vee A$, either Narendra Modi will win the election tomorrow or Narendra Modi will not win the election tomorrow.
This disjunction is logically true, so it is true now, and so one of its disjuncts must be true now.

Argument for Fatalism

Suppose it is the first.
Then Narendra Modi will win, and if he will win, there is no need for you to vote for him. Suppose it is the second that is true.
Then Narendra Modi wont win, and if he wont win, there is no point in your voting for him.
Either way your vote is wasted.

Aristotle: De Interpretatione 9

If a thing is white now, it was true before to say that it would be white, so that of anything that has taken place, it was always true to say it is or it will be. But if it was always true to say that a thing is or will be, it is not possible that it should not be or not come to be, and when a thing cannot not come to be, it is impossible that it should not come to be, and when it is impossible that it should not come to be, it must come to be. All then, that is about to be must of necessity take place. It results from this that nothing is uncertain or fortuitous, for if it were fortuitous, it would not be necessary

Aristotle Sea battle argument

Suppose that a sea battle will not be fought tomorrow. Then it was also true yesterday (and the week before, and last year) that it will not be fought, since any true statement about what will be the case was also true in the past.
But all past truths are now necessary truths; therefore it is now necessarily true that the battle will not be fought, and thus the statement that it will be fought is necessarily false. Therefore it is not possible that the battle will be fought. In general, if something will not be the case, it is not possible for it to be the case. This conflicts with the idea of our own free will: that we have the power to determine the course of events in the future, which seems impossible if what happens, or does not happen, is necessarily going to happen, or not happen.

Aristotle

1. Bivalence demands that either of these contradictories is true: (a) There will be a sea-battle tomorrow, (b) There will not be a sea-battle tomorrow.
2. But if (a) is true now, tomorrow's battle is necessary, and if (b) is true, the battle is impossible.
3. There is no room to say that it might happen.
4. Hence, for Aristotle, 'it is not necessary that of every affirmation and opposite negation one should be true and the other false

Fatalism

Fatalism

We are powerless to do anything other than what we actually do. Everything that happens, happens out of necessity.

Logical Fatalism

Diodorous and the necessity of past:4c BCE

1. What is true of past is necessary.
2. Suppose that there is a sea battle on 1/1/2100.
3. Then, it was true on 2012 that, there would be sea battle on 1/1/2100.
4. Then it was true of the past that there would be sea battle on 1/1/2100.
5. So it is necessary that there will be a sea battle on 1/1/2100

Note: If there is a sea battle on 1/1/2100, it is necessary that there will be a sea battle on 1/1/2100 (it is impossible that there should not be one)

Some Motivations for Many-valued Logic

1. Indeterminism:
2. Failure of Law of excluded middle.
3. Sorites paradox: Modus ponens
4. Liars Paradox:

Language

1. Let C be the class of connectives of classical propositional logic $\{\wedge, \neg, \vee, \rightarrow, \leftrightarrow\}$.
2. The classical propositional calculus can be thought of as defined by the structure $\{V, D, \{f_c \colon c \in C\}\}$.
3. V is the set of truth values $\{1, 0\}$.
4. D is the set of designated values $\{1\}$; these are the values that are preserved in valid inferences.
5. For every connective, c, f_c is the truth function it denotes.
6. Thus, $f_\neg$ is a one place function such that $f_\neg(0) = 1$ and $f_\neg(1) = 0$; $f_\wedge$ is a two-place function such that $f_{\wedge(x,y)} = 1$ if $x = y = 1$, and $f_{\wedge(x,y)} = 0$, otherwise.

CL: Classical Logic; MVL: Many Valued Logic; FL: Fuzzy Logic

Interpretation and Validity in CL

1

Interpretation

An interpretation, v, is a map from the propositional parameters to V set of designated values.
An interpretation is extended to a map from all formulas into V by applying the appropriate truth functions recursively.

1. $(\neg(p \vee q)) = f_\neg((p \vee q)) = f_\neg(f_\vee((p),(q)))$.
2. So if $(p) = 1$ and $(q) = 0$, $((\neg(p \vee q)) = f_\neg(f_\vee(1,0)) = f_\neg(1) = 0.)$

Semantic Validity

An inference is semantically valid just if there is no interpretation that assigns all the premises a value in D, but assigns the conclusion a value not in D.

[1] Graham Priest, 7.2.3, pp.121

Generalized Structure for MVL

[2]

1. Given some propositional language with connectives C (maybe the same as those of the classical propositional calculus, maybe different), a logic is defined by a structure $\{V, D, \{f_c : c \in C\}\}$.
2. V is the set of truth values: it may have any number of members (≥ 1).
3. D is a subset of V, and is the set of designated values.
4. For every connective, c, f_c is the corresponding truth function.
5. If c is an n-place connective, f_c is an n-place function with inputs and outputs in V.

[2] Graham Priest: 7.2.4

Interpretation in MVL[3]

1. An interpretation for the language is a map, , from propositional parameters into V.
2. This is extended to a map from all formulas of the language to V by applying the appropriate truth functions recursively.
3. if c is an n-place connective, $(c(A1 \ldots\ldots An)) = f_c((A1), \ldots (An))$.
4. $\Sigma \models A$ iff there is no interpretation, , such that for all $B \in \Sigma$, $(B) \in D$, but $(A) \notin D$.
5. A is a logical truth iff $\phi \models A$, i.e., iff for every interpretation $(A) \in D$.

If V is finite, the logics said to be finitely many valued. If V has n members, it is said to be an n-valued logic.

[3] Graham Priest: 7.2.5

Validity in Two valued and Many-Valued Logic

Deductive Validity

An argument is deductively valid if and only if its conclusion is true when- ever its premises are all true.

Deductive Validity and Designated Values

There is no need to revise this, except to generalize it to designated values.
An argument is deductively valid if and only if its conclusion is designated whenever its premises are all designated.

Tautology

A Tautology is a formula which takes the designated values 1 under any valuation v.

Invalidities in L3

1. $\nvDash_{L3} \; p \vee \neg p$
2. $\nvDash_{L3} \; \neg [p \wedge \neg\neg p]$
3. $\nvDash_{L3} \; p \leftrightarrow \neg p$

Russell's Paradox

Consider the set of all sets that are not members of themselves: Z.
Then ask, `Is this set a member of itself?`
If it is a member of itself, then it shouldn't be, and vice versa.

Formal Representation

$Z = \{x : \; x \notin x\}$
$Z \notin Z \;\; \leftrightarrow \;\; Z \notin Z.$

Barber's Paradox

Analogy

The set of all sets in the Russell paradox is like the barber in a small, remote town who shaves all the men who don't shave themselves.
"Does the barber shave himself?
He shaves himself if and only if he doesn't shave himself.

Validty in Many-Valued Logic

many-valued logic has given a notion of validity as specified. The specification means that we designate a certain value, if the premises all take the designated value, so must the conclusion

Some Three valued Logics for Vagueness

1. The Swedish logician Soren Halldén in1949,
2. Stephan Körner in 1955

Hallden Three Valued Logic

1. Hallden assigned a proposition nonsensical or meaningless, by which he means only that it is neither true nor false. A proposition is meaningless if it is neither true nor false.
2. Jack is a bald philosopher.' is a typical case of neither true nor false, thus 'Jack is a bald philosopher.' is meaningless.
3. If the is component meaningless' (N), the complex proposition is 'meaningless' too.
4. introduces a new one-place operator '+' to mean that '+p' is 'p is meaningful',
5. if 'p is meaningless' is true, '+p' is false.
6. Thus $\neg(+p)$ will mean that p is 'meaningless
7. 'meaninglessness' could be a designated value too, here.

Halden Validities and Invalidities

1. $\vdash_{H3}\ p \vee \neg p.$
2. $\nvdash_{H3}\ ((p \vee \neg p) \rightarrow +p) \wedge (p \vee \neg p \quad \rightarrow +p$

Limitations of Hallden's Logic

1. Some intuitive inferences re invalid.
2. B. Russell is a bald philospher. Therefore B. Russell is a philosopher.
3. $+(p \to q)$ takes the value zero.

Korners Three Valued Logic

1. He introduced a logic of inexact concept and applied it into the philosophy of science.
2. The inexactness stems from borderline cases for concepts defined by examples
3. Such a concept F divides objects into positive candidates, which must be ‘elected’ as positive instances of F, negative candidates, which must he elected as negative instances, and neutral candidates, which may be elected as positive or as negative instances, depending on a free choice.
4. There is a three-valued pre-election logic and a two-valued post-election logic.

Sorites Paradox

1. The sorites paradox is one of the most resilient paradoxes in philosophy.
2. A great deal of work in recent times has been devoted to dealing with this paradox and the phenomenon of vagueness which gives rise to it.
3. But this work has largely been conducted in the absence of a good definition of vagueness

What is Vagueness?

1. Common Understanding: A predicate is vague iff it permits borderline cases
2. Example: predicate such as red is vague because there are cases of borderline red: reddish-orange objects etc.
3. Borderline Case: a borderline case is one that neither falls under the predicate in question nor does it not fall under the predicate.
4. Otavio Bueno, Mark Colyvan: a definition of vagueness should not prejudice the question of how best to deal with vagueness, and both gappy and glutty approaches are serious contenders here. The standard definition, rules out a paraconsistent glutty approach to vagueness right from the start.
5. Most philosophers are rather sympathetic to approaches to the Sorites paradox that involve truth-value gaps

http://www.colyvan.com/papers/jwiv.pdf

Sorites Paradox: Origin

1. 3 BC: Eubulides of Megara, contemporary of Aristotle.
2. Is one many? is two many? and so on.
3. Sorites: a heaper, or accumulator, one who adds things.Soros, means heap, and hence it is called Heaps paradox.
4. Paradox: Apparent true premises, apparently valid, but has false or absurd conclusion.

The Paradox of Heap

1. P1: Zero grains are not heap
2. P2: The addition of a single grain cannot make the difference between what is not a heap and what is heap.
3. C: Nothing is a heap.

Paradox of the bald person

1. A hairy person is not bald.
2. Major Premise: The loss of single hair cannot turn hairy man bald.
3. C: Nobody is bald.

Sorites Paradox: Example

1. Someone who is 7 feet in height is tall.
2. If someone who is 7 feet in height is tall, then someone 6'11.9" in height is tall.
3. If someone who is 6'11.9" in height is tall, then someone 6'11.8" in height is tall. ......
4. C: Someone who is 3' in height is tall.

Sorites Premise

For any height h, if someone's height is h and he is tall, then someone whose height is $h - 0.1$ is also tall.

This is a universal claim about all heights.

Paradox

This is a paradox, since it looks like each of the premises is true, but the conclusion is clearly false. Nonetheless, the reasoning certainly appears to be valid.

Another Instance of Sorites Paradox

1. 10,000 grains of sand is a heap of sand.
2. 10,000 grains of sand is a heap of sand, then 9999 grains of sand is a heap of sand.
3. 9999 grains of sand is a heap of sand, then 9998 grains of sand is a heap of sand.
4. ……
5. C. 1 grain of sand is a heap of sand.

Sorites Premise: For any number n, if n grains of sand is a heap, then $n - 1$ grains of sand is a heap.

Instance:2

1. 1. A man with 1 hair on his head is bald.
2. If a man with 1 hair on his head is bald, a man with 2 hairs on his head is bald.
3. If a man with 2 hairs on his head is bald, a man with 3 hairs on his head is bald.
4. ....
5. C. A man with 100,000 hairs on his head is bald.

Sorites Premise:For any number n, if someone with n hairs on his head is bald, then someone with $n+1$ hairs on their head is bald.

Resolution of Paradox

Response to the sorites paradox will fall into one of three cate- gories:

1. Rejecting the initial premise.
2. Rejecting one of the other premises, and/or the sorites premise.
3. Rejecting the validity of the argument.

The problem is that none of these looks initially promising.

Quotation

When the prophet, a complacent fat man, Arrived at the mountain top He cried: Woe to my knowledge! I intended to see good white lands And bad black lands. But the scene is gray.

……Stephen Crane

Fuzzy sets: Introduction

1. Zadeh introduced the term fuzzy logic in his seminal work "Fuzzy sets," which described the mathematics of fuzzy set theory (1965).
2. The idea of fuzzy sets and fuzzy logic were not accepted well within academic circles because some of the underlying mathematics had not yet been explored.
3. It was Lukasiewicz who first proposed a systematic alternative to the bivalued logic of Aristotle. The third value Lukasiewicz proposed can be best translated as possible, and he assigned it a numeric value between True and False.
4. Later he explored four-valued logic and five-valued logic, and then he declared that, in principle, there was nothing to prevent the derivation of infinite-valued logic

Basic Ideas of Fuzzy Logic

1. FL provides the opportunity for modeling conditions that are inherently imprecisely defined.
2. Fuzzy techniques in the form of approximate reasoning provide decision support and expert systems with powerful reasoning capabilities.
3. The permissiveness of fuzziness in the human thought process suggests that much of the logic behind thought processing is not traditional two valued logic or even multivalued logic, but logic with fuzzy truths, fuzzy connectiveness, and fuzzy rules of inference.
4. To be Updated ......