For the problems below, recall the Law of Iterated Expectations and the Law of Total Variance:

$$\mathbf{E}[X] = \mathbf{E}\left[\mathbf{E}[X|Y]\right]$$

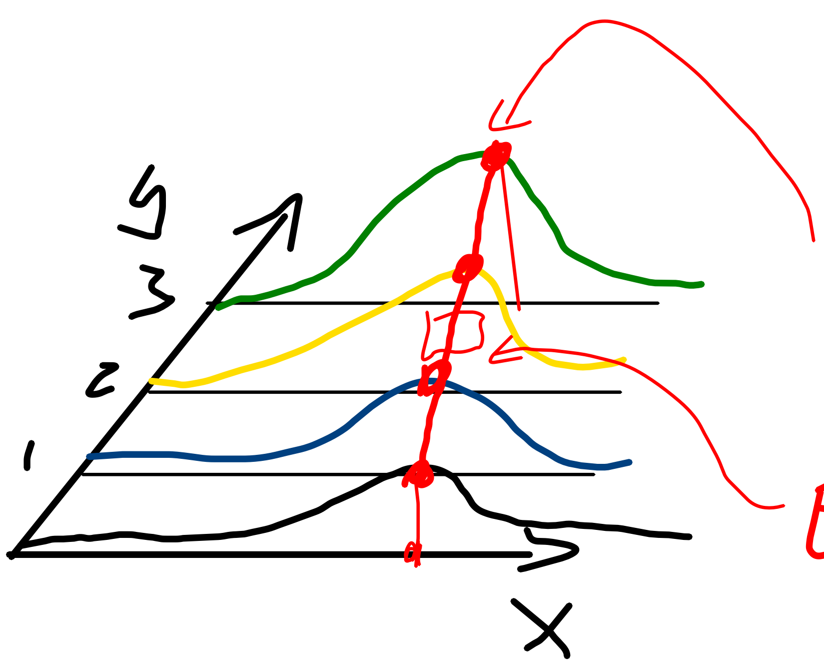
$$\text{var}(X) = \mathbf{E}\left[\text{var}(X|Y)\right] + \text{var}\left(\mathbf{E}[X|Y]\right).$$

1. Let X, Y, and Z be discrete random variables. Show the following generalizations of the law of iterated expectations.

   (a) $\mathbf{E}[Z] = \mathbf{E}\big[\mathbf{E}[Z \mid X, Y]\big]$.

   (b) $\mathbf{E}[Z \mid X] = \mathbf{E}\big[\mathbf{E}[Z \mid X, Y] \mid X\big]$.

   (c) $\mathbf{E}[Z] = \mathbf{E}\big[\mathbf{E}\big[\mathbf{E}[Z \mid X, Y] \mid X\big]\big]$.



$g(Y) = E[X|Y]$ new curve

$E[g(Y)] = E\big[E[X|Y]\big] = E[X]$

new point

expectation of the marginal

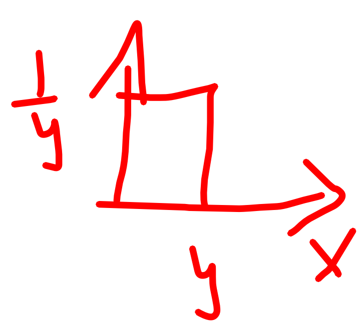
2. Example 4.17, page 223 in text.

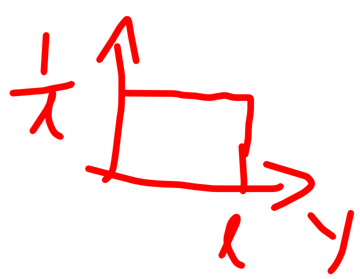
We start with a stick of length ℓ. We break it at a point which is chosen randomly and uniformly over its length, and keep the piece that contains the left end of the stick. We then repeat the same process on the piece that we were left with.

(a) What is the expected value of the length of the piece that we are left with after breaking twice?

(b) What is the variance of the length of the piece that we are left with after breaking twice?

a)



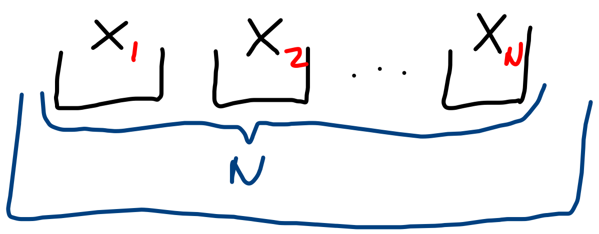


$$E[X] = E[E[X|Y]]$$
$$= E[Y/2] = \frac{1}{2}E[Y] = \frac{1}{2}\cdot\frac{1}{2}\cdot\ell$$

b)

$$\operatorname{var}(X) = E[\operatorname{var}(X|Y)] + \operatorname{var}(E[X|Y])$$
$$= E[E[(X-E[X])^2]] + \operatorname{var}(Y/2)$$
$$= E[E[(X-\tfrac{\ell}{4})^2]] + \left(\tfrac{1}{2}\right)^2 E[(Y-E[Y])^2]$$
$$= E[E[(X-\tfrac{\ell}{4})^2]] + \left(\tfrac{1}{2}\right)^2 E[(Y-\tfrac{\ell}{2})^2]$$

3. Widgets are stored in boxes, and then all boxes are assembled in a crate. Let X be the number of widgets in any particular box, and N be the number of boxes in a crate. Assume that X and N are independent integer-valued random variables, with expected value equal to 10, and variance equal to 16. Evaluate the expected value and variance of T, where T is the total number of widgets in a crate.



$T = \cancel{X \cdot N}$
$= X_1 + \dots + X_N$

$E[X] = 10$

$E[N] = 10$

$$E(T) = E[E[T|N]]$$
$$= E[E[X_1 + \dots + X_n | n]]$$
$$= E[n\,E(X)]$$
$$= \underbrace{\sum_n p_N(n) \cdot n}\, E(X)$$
$$= E[N] \cdot E[X]$$
$$= 10 \cdot 10 \quad = 100$$

Alternatively

$$E[T] = E[E(T|N)]$$
$$= E[N \cdot E(X)]$$
$$= E[N]\,E(X)$$
$$= 10 \cdot 10 = 100$$

T independent of N

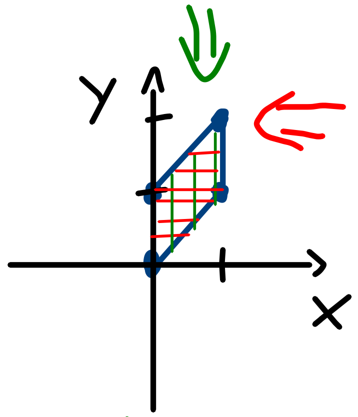
$$\mathrm{Var}(T) = E[\mathrm{Var}(T|N)] + \mathrm{Var}(E(T|N))$$
$$= E[N \cdot \mathrm{var}(X)] + \mathrm{Var}[N \cdot \overbrace{E[X]}^{\text{constant}}]$$
$$= E(N)\,\mathrm{var}(X) + (E[X])^2\,\mathrm{var}(N)$$

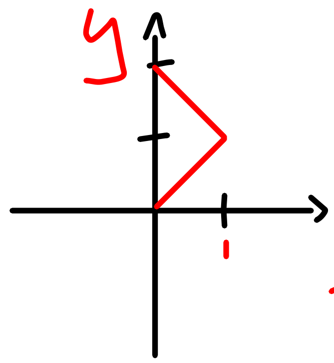
Tutorial 6
October 21/22, 2010

1. Let X be a discrete random variable with PMF p_X and let Y be a continuous random variable, independent from X, with PDF f_Y. Derive a formula for the PDF of the random variable $X+Y$.

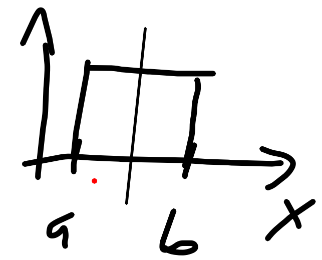
optional

2. The random variables X and Y are described by a joint PDF which is constant within the unit area quadrilateral with vertices $(0,0)$, $(0,1)$, $(1,2)$, and $(1,1)$. Use the law of total variance to find the variance of $X+Y$.



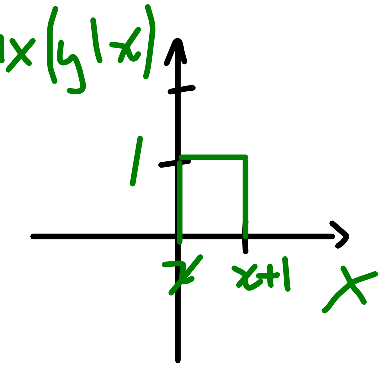


$f_{X|Y}(x|y)$



$$\operatorname{var}(x) = E[(X - E[X])^2] = E[X^2] - E[X]^2 =$$

$f_{Y|X}(y|x)$



$$f_{X,Y}(x,y) = 1$$

$$X|Y = X+Y$$

conditioning on X easier because cond. pdf

$$\operatorname{var}(\cancel{X}+Y) = E[\underbrace{\operatorname{var}(X+Y|X)}] + \operatorname{var}(\underbrace{E[X+Y|X]})$$

$$\boxed{\operatorname{var}(X+Y|X) = \operatorname{var}(Y|X) = \frac{((x+1)-x)^2}{12} = \frac{1}{12}}$$

$$\boxed{E[X+Y|X] = X + E[Y|X] = X + \frac{x+1+x}{2} = x + \frac{2x+1}{2} = 2x+1}$$

$$= E\left[\frac{1}{12}\right] + \operatorname{var}\left(2x+\frac{1}{2}\right) = \frac{1}{12} + 2^2\operatorname{var}(x) = \frac{1}{12} + \frac{4}{12} = \frac{5}{12}$$

3. (a) You roll a fair six-sided die, and then you flip a fair coin the number of times shown by the die. Find the expected value and the variance of the number of heads obtained.

(b) Repeat part (a) for the case where you roll two dice, instead of one.

a) $H = I_1 + \dots + I_D$ $\quad D: 1 \dots 6$ $\quad E(D) = \frac{1}{6}(1+2+3+4+5+6) = \frac{1}{6}(21) = \underline{\underline{3.5}}$

$$E[H] = E[H|D] = \underbrace{E[D]\,E[I_i]}_{\text{independence}} = 3.5 \cdot \frac{1}{2} = \underline{\underline{1.75}}$$

$$\begin{aligned} \operatorname{var}(I) &= \operatorname{var}(E[H|D]) + E[\operatorname{var}(H|D)] \\ &= \operatorname{var}(E[D]\,E[I_i]) + E[d \cdot \operatorname{var}(I)] \\ &= \operatorname{var}(1.75) + ? \\ &= \end{aligned}$$

Lecture 12 Bonus Video Problem

Problem 27.* We toss n times a biased coin whose probability of heads, denoted by q, is the value of a random variable Q with given mean μ and positive variance σ . Let X_i be a Bernoulli random variable that models the outcome of the ith toss (i.e., $X_i = 1$ if the ith toss is a head). We assume that $X_1, \ldots, X_n$ are conditionally independent, given $Q = q$. Let X be the number of heads obtained in the n tosses.

(a) Use the law of iterated expectations to find $\mathbf{E}[X_i]$ and $\mathbf{E}[X]$.

(b) Find $\text{cov}(X_i, X_j)$. Are $X_1, \ldots, X_n$ independent?

(c) Use the law of total variance to find $\text{var}(X)$. Verify your answer using the covariance result of part (b).