Notes for Recitation 16

1 Combinatorial Proof

A ***combinatorial proof*** is an argument that establishes an algebraic fact by relying on counting principles. Many such proofs follow the same basic outline:

1. Define a set S.
2. Show that $|S| = n$ by counting one way.
3. Show that $|S| = m$ by counting another way.
4. Conclude that $n = m$.

Consider the following theorem:

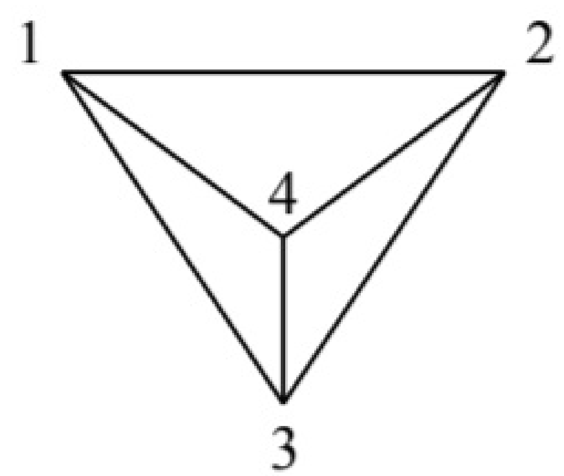
Theorem.

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

2 Triangles

Let $T = \{X_1, \ldots, X_t\}$ be a set whose elements X_i are themselves sets such that each X_i has size 3 and is $\subseteq \{1, 2, \ldots, n\}$. We call the elements of T "triangles". Suppose that for all "edges" $E \subseteq \{1, 2, \ldots, n\}$ with $|E| = 2$ there are exactly λ triangles $X \in T$ with $E \subseteq X$.

For example, if we might have the triangles depicted in the following diagram, which has $\lambda = 2$, $n = 4$, and $t = 4$:



In this example, each edge appears in exactly two of the following triangles:

$$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$$

Prove

$$\lambda \cdot \frac{n(n-1)}{2} = 3t$$

by counting the set

$$C = \{(E, X) : X \in T, E \subseteq X, |E| = 2\}$$

in two different ways.

3 Counting, counting, counting

Learning to count takes practice! Briefly justify your answers to the following questions. Not every problem can be solved with a cute formula; you may have to fall back on case analysis, explicit enumeration, or ad hoc methods. Do as many problems as you can and save the rest to study for Quiz II. You may leave factorials and binomial coefficients in your answers.

1. How many different arrangements are there of the letters in $BANANA$?

$$\frac{6!}{3!2!1!}$$

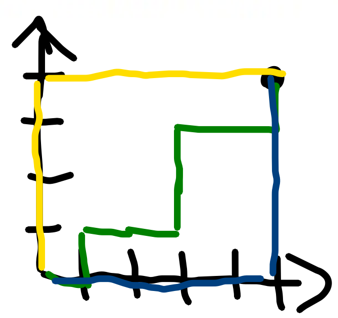
$6! \rightarrow$ if all letters were distinct

$3!2!1! \rightarrow$ divided by the permutations of the repetitive letters

$$\frac{(1+2+3)!}{1!2!3!}$$

2. How many different paths are there from point $(0, 0, 0)$ to point $(10, 20, 30)$ if every step increments one coordinate and leaves the other two unchanged?

Ex.: $(4,5)$



$|x| = 4$, $|z| = 5$

XXXX ~~ZZZZ~~Z
XZXZ XZX~~ZZ~~
XZZZ ZXZZXX
⋮

$$\frac{(|x|+|z|)!}{|x|!\,|z|!}$$

$$= \frac{(|x|+|y|+|z|)!}{|x|!\,|y|!\,|z|!}$$

3. Find the number of 5-card hands with exactly three aces.

$$\binom{4}{3} \cdot \binom{52-4}{2} = \binom{4}{3}\binom{48}{2}$$

4. Find the number of 5-card hands in which every suit appears at most twice.

Case 1: $\frac{13\cdot 12}{2!}\cdot\frac{13\cdot 12}{2!}\cdot 13\cdot 2\cdot\frac{4\cdot 3}{2!}$

Could choose 2 out of 4 suits

Case 2: $\frac{13\cdot 12}{2!}\cdot 13\cdot 13\cdot 13\cdot\binom{4}{1}$

Could choose all 4 suits

$$\binom{13}{2}\cdot 13^3\cdot 4+\underbrace{\binom{13}{2}^2\cdot 13\cdot\binom{4}{2}\cdot 2}_{\text{case 1}}$$

? $2\cdot 13$ remaining 2 suits have 26

5. There are 15 sidewalk squares in a row. Suppose that a ball is thrown down the row so that it bounces on 0, 1, 2, or 3 distinct sidewalk squares. How many different throws are possible? Two throws are considered to be equivalent if they bounce on the same squares in a different order.

$$\sum_{k}^{3}\frac{n!}{k!(n-k)!}=\sum_{k=0}^{3}\binom{n}{k}=\binom{15}{0}+\binom{15}{1}+\binom{15}{2}+\binom{15}{3}$$

52.728 949.104

6. In how many different ways can the numbers shown on a red die, a green die, and a blue die total up to 15? Assume that these are ordinary, 6-sided dice.

1	6	6	13
2	6	6	14
3	6	6	15
4	6	5	15
5	5	5	15
6	4	5	15

$\frac{3!}{2!}$ $3!$ $\frac{3!}{3!}$

$3+6+1$

$\frac{3!}{2!}+3!+\frac{3!}{3!}=$ 10 (in Solution) Why?

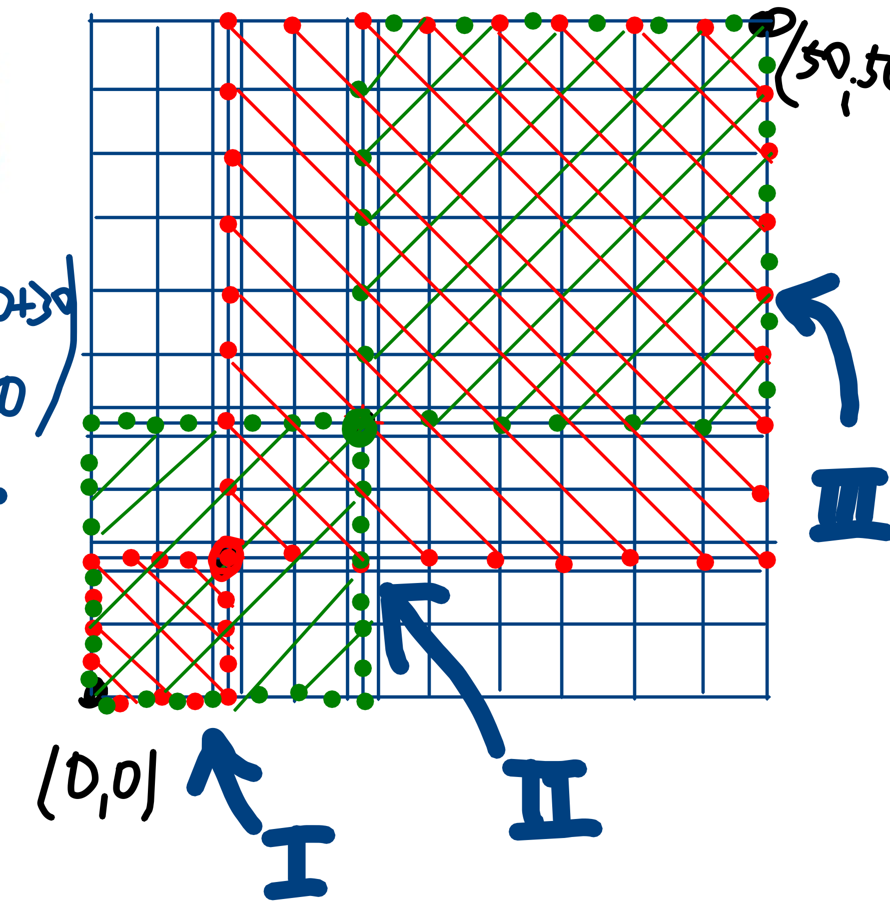
7. In how many ways can 20 indistinguishable pre-frosh be stored in four different crates if each crate must contain an even number of pre-frosh?

0 0 0 0 | 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 0 0

$$\frac{(10+3)!}{10!\,3!}=\binom{10}{3}$$

8. How many paths are there from point $(0,0)$ to $(50,50)$ if every step increments one coordinate and leaves the other unchanged and there are impassable boulders sitting at points $(10,10)$ and $(20,20)$?

$$\binom{50+50}{50} - \underbrace{\binom{10+10}{10}}_{\mathrm{I}}\underbrace{\binom{40+40}{40}}_{\mathrm{II}} - \underbrace{\binom{20+20}{20}}_{\mathrm{II}}\underbrace{\binom{30+30}{30}}_{\mathrm{III}} + \underbrace{\binom{10+10}{10}}_{\mathrm{I}}\cdot\underbrace{\binom{10+10}{10}}_{\mathrm{II}}\underbrace{\binom{30+30}{30}}_{\mathrm{III}}$$



Intersections between red and green areas

9. In how many ways can the 180 students in 6.042 be divided into 36 groups of 5?

$$\frac{180!}{5!\,(180-5)!} = \binom{180}{5}$$

10. In how many different ways can 10 indistinguishable balls be placed in four distinguishable boxes, such that every box gets 1, 2, 3, or 4 balls?

1 ball in every box first

then 6 balls left:

3, 3, 0, 0 $\quad \frac{4!}{2!2!}$

2, 2, 2, 0 $\quad \frac{4!}{3!1!}$

1, 1, 1, 3 $\quad \frac{4!}{3!1!}$

1, 1, 2, 2 $\quad \frac{4!}{2!2!}$

1, 2, 3, 0 $\quad 4!$

$= 6 + 4 + 4 + 6 + 24$

$= \underline{\underline{44}}$

11. In how many different ways can Blockbuster arrange 64 copies of *Cat in the Hat*, 96 copies of *Matrix Revolutions*, and 1 copy of *Amelie* on 5 shelves?

$$\frac{(64+96+1+4)!}{64! \cdot 96! \cdot 1! \cdot 4!}$$

4 There's more than one way...

In the beginning of today's recitation, we gave a combinatorial proof of the following theorem:

Theorem.

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

We can also prove this theorem using induction. Give such a proof.