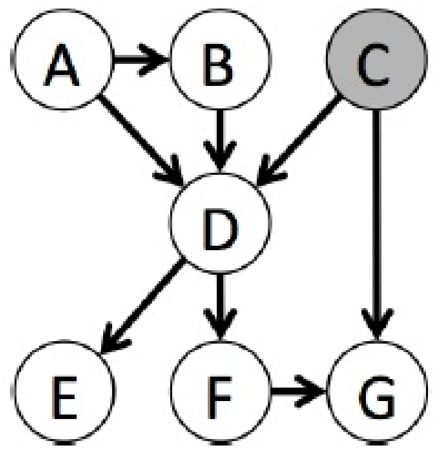
Q1. Variable Elimination

(a) For the Bayes' net below, we are given the query $P(A, E \mid +c)$. All variables have binary domains. Assume we run variable elimination to compute the answer to this query, with the following variable elimination ordering: B, D, G, F.



Complete the following description of the factors generated in this process:

After inserting evidence, we have the following factors to start out with:

$$P(A), P(B|A), P(+c), P(D|A,B,+c), P(E|D), P(F|D), P(G|+c,F)$$

When eliminating B we generate a new factor f_1 as follows:

$$f_1(A,+c,D) = \sum_b P(b|A)P(D|A,b,+c)$$

This leaves us with the factors:

$$P(A), P(+c), P(E|D), P(F|D), P(G|+c,F), f_1(A,+c,D)$$

When eliminating D we generate a new factor f_2 as follows:

$$f_2(A,+c,F,E) = \sum_d f_1(A,+c,d)\, P(F|d)\, P(E|d)$$

This leaves us with the factors:

$$P(A), P(+c), P(G|+c,F), f_2(A,+c,F,E)$$

When eliminating D we generate a new factor f_2 as follows:

$$f_2(A, +c, F, E) = \sum_d f_1(A, +c, d)\, P(F \mid d)\, P(E \mid d)$$

This leaves us with the factors:

$$P(A),\ P(+c),\ P(G \mid +c, F),\ f_2(A, +c, F, E)$$

When eliminating G we generate a new factor f_3 as follows:

$$f_3(+c, F) = \sum_g P(g \mid +c, F)$$

This leaves us with the factors:

$$P(A),\ P(+c),\ f_3(+c, F),\ f_2(A, +c, F, E)$$

When eliminating F we generate a new factor f_4 as follows:

$$f_4(A, +c, E) = \sum_f f_3(+c, f) \cdot f_2(A, +c, f, E)$$

This leaves us with the factors:

$$P(A),\ P(+c),\ f_4(A, +c, E)$$

(b) Write a formula to compute $P(A, E \mid +c)$ from the remaining factors.

$$P(A, E \mid +c) = \frac{f_4(A, E, +c)}{P(+c)}$$

(c) Among f_1, f_2, f_3, f_4, which is the largest factor generated, and how large is it? Assume all variables have binary domains and measure the size of each factor by the number of rows in the table that would represent the factor.

Self assessment If correct, write "**correct**" in the box. Otherwise, **write and explain** the correct answer.

(d) Find a variable elimination ordering for the same query, i.e., for $P(A, E \mid +c)$, for which the maximum size factor generated along the way is smallest. Hint: the maximum size factor generated in your solution should have only 2 variables, for a size of $2^2 = 4$ table. Fill in the variable elimination ordering and the factors generated into the table below.

Variable Eliminated	Factor Generated

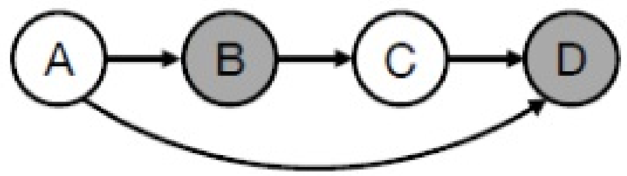
For example, in the naive ordering we used earlier, the first row in this table would have had the following two entries: B, $f_1(A, +c, D)$.

Self assessment If correct, write "**correct**" in the box. Otherwise, **write and explain** the correct answer.

Q2. Bayes Nets: Sampling

~~0.31~~	~~0.58~~	~~0.04~~	~~0.94~~	~~0.67~~	~~0.49~~	~~0.37~~	0.42	...

Consider the following Bayes Net, where we have observed that $B = +b$ and $D = +d$.



$P(A)$	
$+a$	0.5
$-a$	0.5

$P(B\|A)$		
$+a$	$+b$	0.8
$+a$	$-b$	0.2
$-a$	$+b$	0.4
$-a$	$-b$	0.6

$P(C\|B)$		
$+b$	$+c$	0.1
$+b$	$-c$	0.9
$-b$	$+c$	0.7
$-b$	$-c$	0.3

$P(D\|A,C)$			
$+a$	$+c$	$+d$	0.6
$+a$	$+c$	$-d$	0.4
$+a$	$-c$	$+d$	0.1
$+a$	$-c$	$-d$	0.9
$-a$	$+c$	$+d$	0.2
$-a$	$+c$	$-d$	0.8
$-a$	$-c$	$+d$	0.5
$-a$	$-c$	$-d$	0.5

$$P(+a,-c \mid +b,+d) = \frac{P(+a,-c,+b,+d)}{P(+b,+d)}$$

Rejection sampling:

$+a\ +b\ +c\ -d$ reject

$-a\ -b$ reject

$+a\ +b\ -c\ +d$ ✓

$$= \frac{\#(+a,-c,+b,+d)}{\#(+b,+d)} = \frac{1}{1} = 1$$

Likelihood weighting

$+a\ +b\ -c\ +d$ $\quad 0.8 \cdot 0.1 = 0.08$

$+a\ +b\ -c\ +d$ $\quad 0.08$

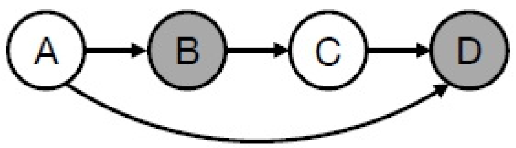
$-a\ +b\ -c\ +d$ $\quad 0.4 \cdot 0.5 = 0.02$

$+a\ +b\ -c\ +d$ $\quad 0.08$

$$= \frac{\sum W(+a,-c,+b,+d)}{\sum W(+b,+d)} = \frac{0.08 \cdot 3}{0.08 \cdot 3 + 0.02}$$

Q2. Bayes Nets: Sampling

Consider the following Bayes Net, where we have observed that $B = +b$ and $D = +d$.



$P(A)$	
$+a$	0.5
$-a$	0.5

$P(B \mid A)$		
$+a$	$+b$	0.8
$+a$	$-b$	0.2
$-a$	$+b$	0.4
$-a$	$-b$	0.6

$P(C \mid B)$		
$+b$	$+c$	0.1
$+b$	$-c$	0.9
$-b$	$+c$	0.7
$-b$	$-c$	0.3

$P(D \mid A, C)$			
$+a$	$+c$	$+d$	0.6
$+a$	$+c$	$-d$	0.4
$+a$	$-c$	$+d$	0.1
$+a$	$-c$	$-d$	0.9
$-a$	$+c$	$+d$	0.2
$-a$	$+c$	$-d$	0.8
$-a$	$-c$	$+d$	0.5
$-a$	$-c$	$-d$	0.5

(a) Consider doing Gibbs sampling for this example. Assume that we have initialized all variables to the values $+a, +b, +c, +d$. We then unassign the variable C, such that we have $A = +a$, $B = +b$, $C = ?$, $D = +d$. Calculate the probabilities for new values of C at this stage of the Gibbs sampling procedure.

$P(C = +c$ at the next step of Gibbs sampling$) = \dfrac{0.1 \cdot 0.6}{0.1 \cdot 0.6 + 0.9 \cdot 0.1} = \dfrac{0.06}{0.06 + 0.09} = \dfrac{0.06}{0.15} = \dfrac{6}{15}$

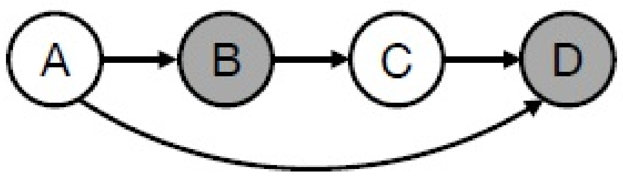
$P(C = -c$ at the next step of Gibbs sampling$) = \dfrac{0.9 \cdot 0.1}{0.1 \cdot 0.6 + 0.9 \cdot 0.1} = \dfrac{0.09}{0.06 + 0.09} = \dfrac{0.09}{0.15} = \dfrac{9}{15}$

$$P(C \mid +a, +b, +d) = \frac{P(C, +a, +b, +d)}{P(+a, +b, +d)} = \frac{P(C, +a, +b, +d)}{\sum_c P(c, +a, +b, +d)}$$

$$= \frac{\cancel{P(+a)} \cdot \cancel{P(+b \mid +a)} \cdot P(C \mid +b) \cdot P(+d \mid C, +a)}{\sum \cancel{P(+a)} \cdot \cancel{P(+b \mid +a)} \cdot P(c \mid +b) \cdot P(+d \mid c, +a)}$$

$$= \frac{P(C \mid +b) \cdot P(+d \mid C, +a)}{\sum_c P(c \mid +b) \cdot P(+d \mid c, +a)} \Rightarrow \frac{P(+c \mid +b) \cdot P(+d \mid +c, +a)}{\sum_c P(c \mid +b) \cdot P(+d \mid c, +a)}$$

Consider the following Bayes Net, where we have observed that $B = +b$ and $D = +d$.



$P(A)$	
$+a$	0.5
$-a$	0.5

$P(B\|A)$		
$+a$	$+b$	0.8
$+a$	$-b$	0.2
$-a$	$+b$	0.4
$-a$	$-b$	0.6

$P(C\|B)$		
$+b$	$+c$	0.1
$+b$	$-c$	0.9
$-b$	$+c$	0.7
$-b$	$-c$	0.3

$P(D\|A,C)$			
$+a$	$+c$	$+d$	0.6
$+a$	$+c$	$-d$	0.4
$+a$	$-c$	$+d$	0.1
$+a$	$-c$	$-d$	0.9
$-a$	$+c$	$+d$	0.2
$-a$	$+c$	$-d$	0.8
$-a$	$-c$	$+d$	0.5
$-a$	$-c$	$-d$	0.5

(b) Consider a sampling scheme that is a hybrid of rejection sampling and likelihood-weighted sampling. Under this scheme, we first perform rejection sampling for the variables A and B. We then take the sampled values for A and B and extend the sample to include values for variables C and D, using likelihood-weighted sampling.

(i) Below is a list of candidate samples. Mark the samples that would be rejected by the rejection sampling portion of the hybrid scheme.

- [x] $-a$ $-b$
- [ ] $+a$ $+b$
- [x] $+a$ $-b$
- [ ] $-a$ $+b$

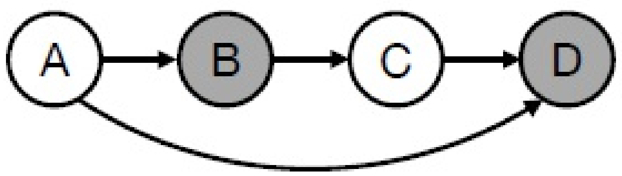
(ii) To decouple from part (i), you now receive a *new* set of samples shown below. Fill in the weights for these samples under our hybrid scheme.

				Weight
$-a$	$+b$	$-c$	$+d$	0.5
$+a$	$+b$	$-c$	$+d$	0.1
$+a$	$+b$	$-c$	$+d$	0.1
$-a$	$+b$	$+c$	$+d$	0.2
$+a$	$+b$	$+c$	$+d$	0.6

(iii) Use the weighted samples from part (ii) to calculate an estimate for $P(+a \mid +b, +d)$. $= \dfrac{\Sigma\, w(+a,+b,+d)}{\Sigma\, w(+b,+d)} = \dfrac{0.8}{1.5}$

The estimate of $P(+a \mid +b, +d)$ is 4/15

Consider the following Bayes Net, where we have observed that $B = +b$ and $D = +d$.



$P(A)$	
$+a$	0.5
$-a$	0.5

$P(B\|A)$		
$+a$	$+b$	0.8
$+a$	$-b$	0.2
$-a$	$+b$	0.4
$-a$	$-b$	0.6

$P(C\|B)$		
$+b$	$+c$	0.1
$+b$	$-c$	0.9
$-b$	$+c$	0.7
$-b$	$-c$	0.3

$P(D\|A,C)$			
$+a$	$+c$	$+d$	0.6
$+a$	$+c$	$-d$	0.4
$+a$	$-c$	$+d$	0.1
$+a$	$-c$	$-d$	0.9
$-a$	$+c$	$+d$	0.2
$-a$	$+c$	$-d$	0.8
$-a$	$-c$	$+d$	0.5
$-a$	$-c$	$-d$	0.5

(c) We now attempt to design an alternative hybrid sampling scheme that combines elements of likelihood-weighted and rejection sampling. For each proposed scheme, indicate whether it is valid, i.e. whether the weighted samples it produces correctly approximate the distribution $P(A, C| + b, +d)$.

(i) *First collect a likelihood-weighted sample for the variables A and B. Then switch to rejection sampling for the variables C and D. In case of rejection, the values of A and B and the sample weight are* ***thrown away****. Sampling then restarts from node* ***A****.*

◯ Valid ◯ Invalid

(ii) *First collect a likelihood-weighted sample for the variables A and B. Then switch to rejection sampling for the variables C and D. In case of rejection, the values of A and B and the sample weight are* ***retained****. Sampling then restarts from node* ***C****.*

◯ Valid ◯ Invalid