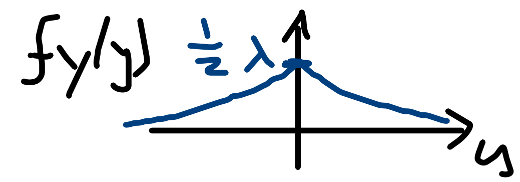
1. Let X be a discrete random variable that takes the values 1 with probability p and -1 with probability $1-p$. Let Y be a continuous random variable independent of X with the Laplacian (two-sided exponential) distribution

$$f_Y(y) = \tfrac{1}{2}\lambda e^{-\lambda|y|},$$

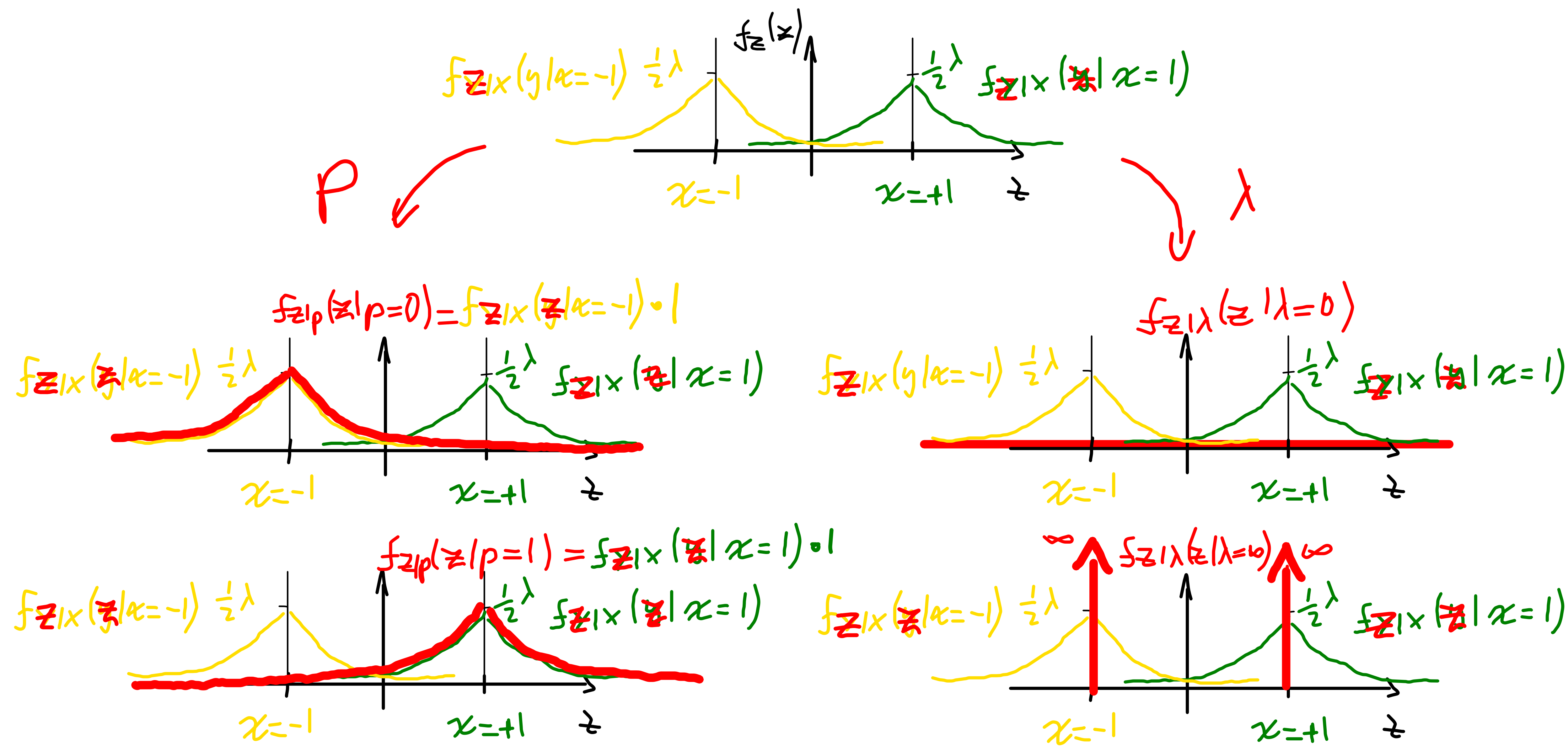


and let $Z = X + Y$. Find $\mathbf{P}(X = 1 \mid Z = z)$. Check that the expression obtained makes sense for $p \to 0^+$, $p \to 1^-$, $\lambda \to 0^+$, and $\lambda \to \infty$.

$$P(X=1 \mid Z=z) = \frac{p_X(x=1)\cdot f_{Z|X}(z|x=1)}{f_Z(z)} = \frac{p_X(x=1)\cdot f_{Z|X}(z|x=1)}{p_X(x=1)\cdot f_{Z|X}(z|x=1) + p_X(x=-1)\cdot f_{Z|X}(z|x=-1)}$$

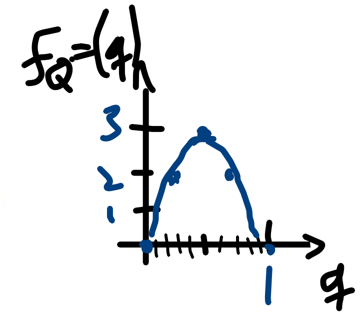
$$= \frac{p\cdot\frac{1}{2}\lambda e^{-\lambda|z-1|}}{p\cdot\frac{1}{2}\lambda e^{-\lambda|z-1|} + (1-p)\frac{1}{2}\lambda e^{-\lambda|z+1|}} = \;=\; \frac{pe^{-\lambda|z-1|}}{(1-p)e^{-\lambda|z+1|} + pe^{-\lambda|z-1|}}\cdot\frac{e^{\lambda|z-1|}}{e^{\lambda|z-1|}}$$

$$= \frac{p}{(1-p)e^{-\lambda(|z+1|-|z-1|)} + p}$$



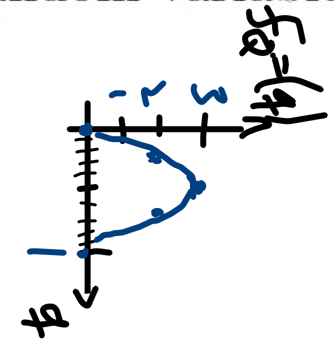
2. Let Q be a continuous random variable with PDF

$$f_Q(q) \;=\; \begin{cases} 6q(1-q), & \text{if } 0 \le q \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

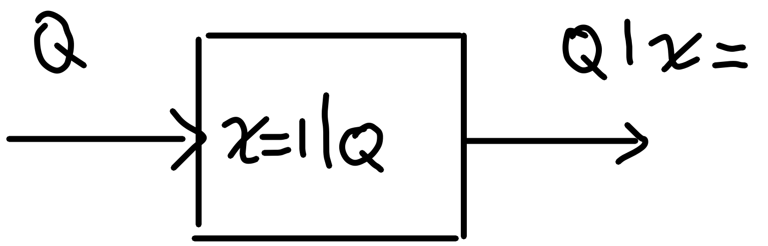


This Q represents the probability of success of a Bernoulli random variable X, i.e.,

$$\mathbf{P}(X = 1 \mid Q = q) \;=\; q.$$



Find $f_{Q|X}(q|x)$ for $x \in \{0, 1\}$ and all q.



$$f_{Q|X}(q|x=1) = \frac{\int_0^1 f_Q(q)\, p_{X|Q}(x=1|q)\, dq}{p_X(x)}$$

$$= \frac{\int_0^1 f_Q(q)\, p_{X|Q}(x=1|q)\, dq}{\int_0^1 f_Q(q)\, p_{X|Q}(x=1|q)\, dq + \int_0^1 f_Q(q)\, p_{X|Q}(x=0|q)\, dq}$$

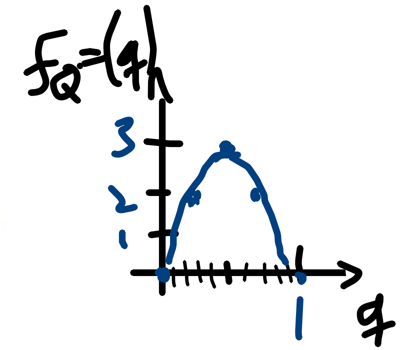
$$= \frac{\int_0^1 6q(1-q)\cdot q\, dq}{\int_0^1 6q(1-q)\cdot q\, dq + \int_0^1 6q(1-q)\cdot(1-q)q\, dq} = \frac{\int_0^1 6q^2(1-q)\, dq}{\int_0^1 6q^2(1-q)\, dq + \int_0^1 6q(1-q)^2\, dq}$$

$$= \frac{\int 6q^2 + 6q^3\, dq}{\int 6q^2 + 6q^3\, dq + \int_0^1 6\cancel{q}(q - 2q^2 + q^3)\, dq} = \frac{6\left(\frac{1}{2}q^3 + \frac{1}{3}q^4\right)}{6\left(\frac{1}{2}q^3 + \frac{1}{3}q^4\right) + 6(q}$$

$^2 - q^3 + \frac{1}{3}q^4)$

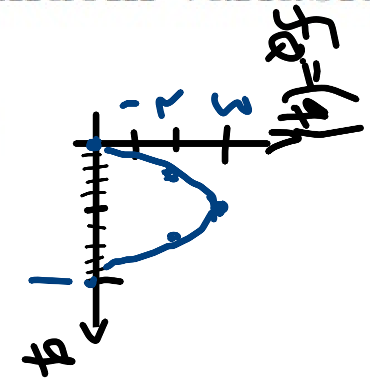
2. Let Q be a continuous random variable with PDF

$$f_Q(q) = \begin{cases} 6q(1-q), & \text{if } 0 \le q \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

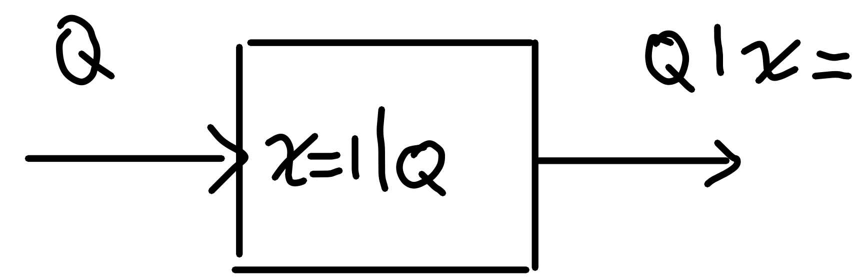


This Q represents the probability of success of a Bernoulli random variable X, i.e.,

$$\mathbf{P}(X = 1 \mid Q = q) = q.$$



Find $f_{Q|X}(q|x)$ for $x \in \{0, 1\}$ and all q.



$$f_{Q|X}(q|x=1) = \frac{\cancel{\int_0^1} f_Q(q)\, p_{X|Q}(x=1|q)\, \cancel{dq}}{p_X(x)}$$

$$= \frac{\cancel{\int_0^1} f_Q(q)\, p_{X|Q}(x=1|q)\, \cancel{dq}}{\int_0^1 f_Q(q)\, p_{X|Q}(x=1|q)\, dq + \cancel{\int_0^1 f_Q(q)\, p_{X|Q}(x=0|q)\, dq}}$$

$$= \frac{\cancel{\int_0^1} 6q(1-q)\cdot q\, \cancel{dq}}{\int_0^1 6q(1-q)\cdot q\, dq + \cancel{\int_0^1 6q(1-q)\cdot(1-q)\, dq}} = \frac{\cancel{\int_0^1} 6q^2(1-q)\, \cancel{dq}}{\int_0^1 6q^2(1-q)\, dq + \cancel{\int_0^1 6q(1-q)^2\, dq}}$$

$$= \frac{6q^2(1-q)}{\int 6q^2 + 6q^3\, dq + \cancel{\int_0^1 6q(q-2q^2+q^3)\, dq}} = \frac{\cancel{6}q^2(1-q)}{\cancel{6}(\frac{1}{2}q^3 + \frac{1}{3}q^4)\Big|_0^1 + \cancel{6(\frac{1}{2}q^2 - \frac{2}{3}q^3 + \frac{1}{4}q^4)}}$$

$$= \frac{\cancel{q^2}(1-q)}{\cancel{q^2}(\frac{1}{2}q + \frac{1}{3}q^2)\Big|_0^1} = \frac{1-q}{(\frac{1}{2}q + \frac{1}{3}q^2)\Big|_0^1} = \frac{1-q}{\frac{1}{2}+\frac{1}{3}} = \frac{1-q}{5/6} = \frac{6}{5}(1-q)$$

SECTION 8.1. Bayesian Inference and the Posterior Distribution

Problem 1. Artemisia moves to a new house and she is "fifty-percent sure" that the phone number is 2537267. To verify this. she uses the house phone to dial 2537267. she obtains a busy signal. and concludes that this is indeed the correct number. Assuming that the probability of a typical seven-digit phone number being busy at any given time is 1%. what is the probability that Artemisia's conclusion was correct?

Problem 2. Nefeli. a student in a probability class. takes a multiple-choice test with 10 questions and 3 choices per question. For each question. there are two equally likely possibilities, independent of other questions: either she knows the answer, in which case she answers the question correctly. or else she guesses the answer with probability of success 1/3.

(a) Given that Nefeli answered correctly the first question, what is the probability that she knew the answer to that question?

(b) Given that Nefeli answered correctly 6 out of the 10 questions, what is the posterior PMF of the number of questions of which she knew the answer?

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1 Joint, Conditional, and Marginal Distributions

3. Let U_1, U_2, U_3 be i.i.d. Unif$(0, 1)$, and let $L = \min(U_1, U_2, U_3)$, $M = \max(U_1, U_2, U_3)$.

   (a) Find the marginal CDF and marginal PDF of M, and the joint CDF and joint PDF of L, M.

   Hint: for the latter, start by considering $P(L \geq l, M \leq m)$.

   (b) Find the conditional PDF of M given L.

Stat 110 Strategic Practice 7, Fall 2011

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1 Joint, Conditional, and Marginal Distributions

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   (a) Find the marginal CDF and marginal PDF of M, and the joint CDF and joint PDF of L, M.

   Hint: for the latter, start by considering $P(L \geq l, M \leq m)$.

   (b) Find the conditional PDF of M given L.

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(independent version → Poisson)

1 Joint, Conditional, and Marginal Distributions

5. A chicken lays n eggs. Each egg independently does or doesn't hatch, with probability p of hatching. For each egg that hatches, the chick does or doesn't survive (independently of the other eggs), with probability s of survival. Let $N \sim \text{Bin}(n, p)$ be the number of eggs which hatch, X be the number of chicks which survive, and Y be the number of chicks which hatch but don't survive (so $X + Y = N$). Find the marginal PMF of X, and the joint PMF of X and Y. Are they independent?

→ Multinomial