3. Problem 3.23, page 191 in the text.
Let the random variables X and Y have a joint PDF which is uniform over the triangle with vertices $(0,0)$, $(0,1)$, and $(1,0)$.

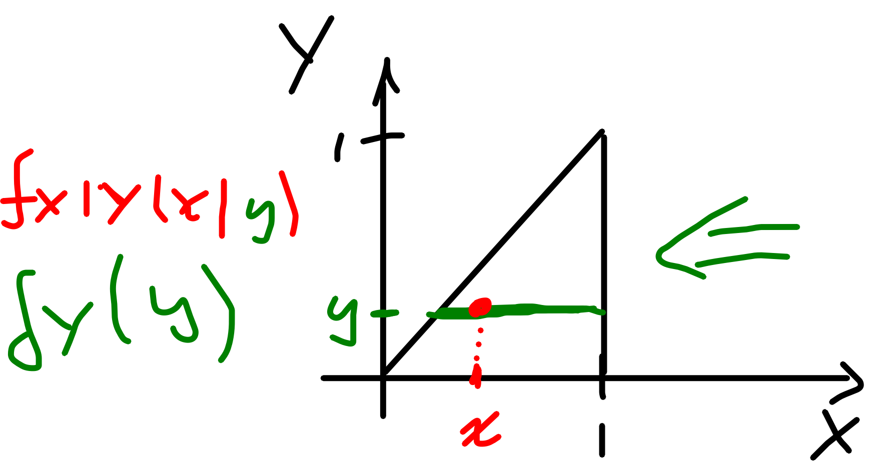
(a) Find the joint PDF of X and Y.

(b) Find the marginal PDF of Y.

(c) Find the conditional PDF of X given Y.

(d) Find $\mathbf{E}[X \mid Y = y]$, and use the total expectation theorem to find $\mathbf{E}[X]$ in terms of $\mathbf{E}[Y]$.

(e) Use the symmetry of the problem to find the value of $\mathbf{E}[X]$.

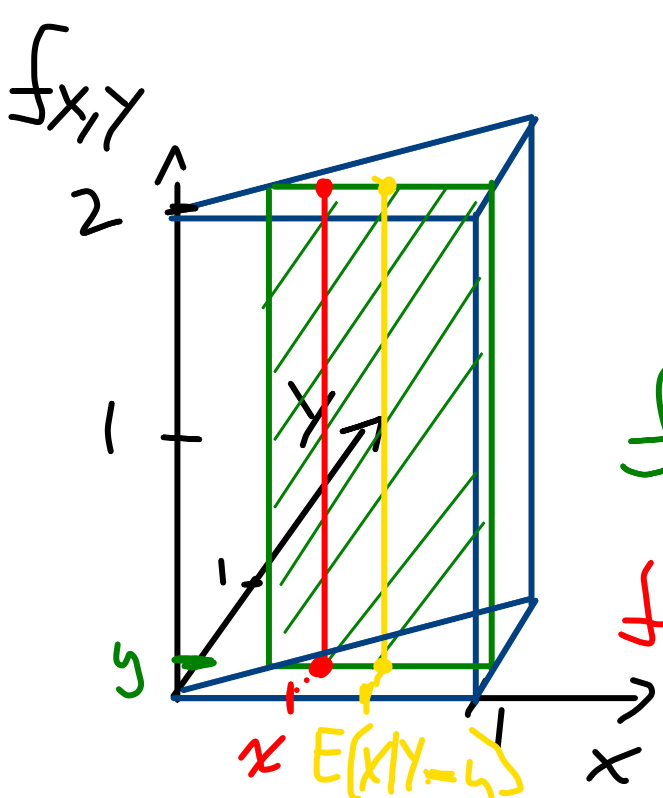


a) $f_{X,Y}(x,y) = \frac{1}{\text{Area}} = \frac{1}{1/2} = 2$

b) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{1-y} 2\,dx = 2\cdot x\Big|_0^{1-y} = 2\cdot(1-y)$

c) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}$

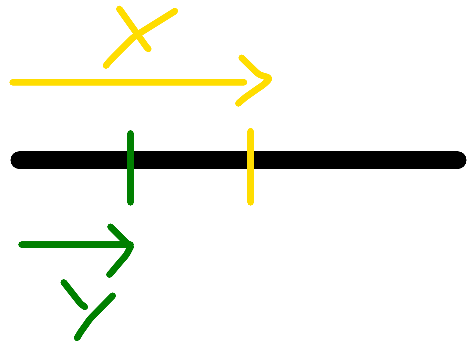
d) $E[X|Y=y] = \int x \cdot f_{X|Y}(x|y)\,dx$



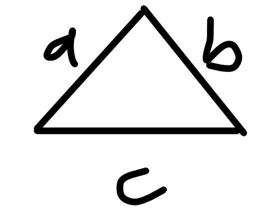
$f_Y(y) = \text{Vol}(\Delta y)$

$f_{X|Y}(x|y) = \frac{\text{Vol}(\Delta x, \Delta y)}{\text{Vol}(\Delta y)}$

4. We have a stick of unit length, and we break it into three pieces. We choose randomly and independently two points on the stick using a uniform PDF, and we break the stick at these points. What is the probability that the three pieces we are left with can form a triangle?



T : Triangle



$$P(T) = P\left(X \geq \tfrac{1}{2}\right) \cdot P\left(Y \mid X \leq X\right)$$

$$= P\left(X \geq \tfrac{1}{2}\right) \cdot P\left(Y \leq X\right)$$

independence

$$= \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{1}{4}$$

$$a+b > c$$

$$A+B > C \qquad C = 1-(A+B)$$

$$A+B > 1-A-B$$

$$2A+2B > 1$$

$$\underbrace{A}_{X}+\underbrace{B}_{Y} > \frac{1}{2}$$

$$X \geq \frac{1}{2}$$

$$Y \leq X \geq \frac{1}{2}$$

Prof. Joe Blitzstein (Department of Statistics, Harvard University)

1 Joint, Conditional, and Marginal Distributions

1. A random point (X, Y, Z) is chosen uniformly in the ball $B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$.

   (a) Find the joint PDF of X, Y, Z.

   (b) Find the joint PDF of X, Y.

   (c) Find an expression for the marginal PDF of X, as an integral.

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1 Joint, Conditional, and Marginal Distributions

2. Let X and Y be i.i.d. Unif$(0,1)$. Find the expected value and the standard deviation of the distance between X and Y. $\rightarrow$ LOTUS

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1 Joint, Conditional, and Marginal Distributions

4. A group of $n \geq 2$ people decide to play an exciting game of Rock-Paper-Scissors. As you may recall, Rock smashes Scissors, Scissors cuts Paper, and Paper covers Rock (despite Bart Simpson saying "Good old rock, nothing beats that!").

   → Multinomial

   Usually this game is played with 2 players, but it can be extended to more players as follows. If exactly 2 of the 3 choices appear when everyone reveals their choice, say $a, b \in \{Rock, Paper, Scissors\}$ where a beats b, the game is decisive: the players who chose a win, and the players who chose b lose. Otherwise, the game is indecisive and the players play again.

   For example, with 5 players, if one player picks Rock, two pick Scissors, and two pick Paper, the round is indecisive and they play again. But if 3 pick Rock and 2 pick Scissors, then the Rock players win and the Scissors players lose the game.