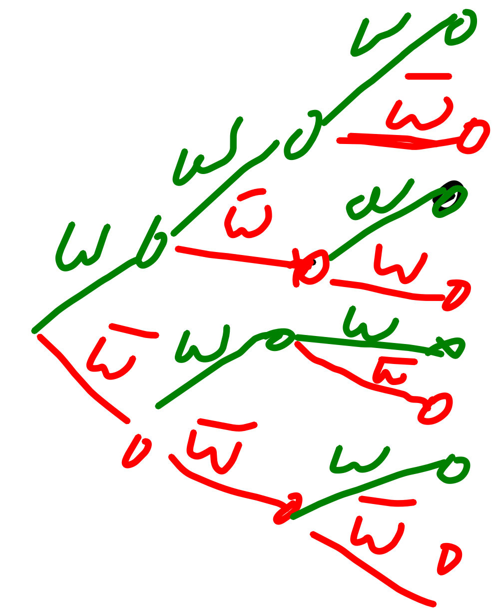
Problem 1. [points] Here's yet another fun 6.042 game! You pick a number between 1 and 6. Then you roll three fair, independent dice.

- If your number never comes up, then you lose a dollar.
- If your number comes up once, then you win a dollar.
- If your number comes up twice, then you win two dollars.
- If your number comes up three times, you win *four* dollars!



What is your expected payoff? Is playing this game likely to be profitable for you or not?

$p = \frac{1}{6}$ $\quad q = 1 - \frac{1}{6}$

Old way:

$$\Pr(L_i) = p^i q^{3-i} \binom{3}{i}$$

$$\boxed{Ex(L) = \sum_{i=0}^{3} i \cdot \Pr(L_i) = \sum_{i=0}^{3} i p^i q^{3-i} \binom{3}{i}}$$

$I_0 = \bar{w} + \bar{w} + \bar{w} \Rightarrow \Pr(I_0) = -3q$

$I_1 = w \Rightarrow P(I_1) = 1p$

$I_2 = w + w \Rightarrow P(I_2) = 2p$

$I_3 = w + w + w \Rightarrow \Pr(I_3) = 3p$

$I = I_0 + I_1 + I_2 + I_3$

$Ex(I) = -3q + 6p$

$= -3 \cdot \frac{5}{6} + 6 \cdot \frac{1}{6}$

$= -\frac{15}{6} + 1$

$= \frac{6-15}{6} = -\frac{9}{6} = -\frac{3}{2}$

Problem 2. [points] The number of squares that a piece advances in one turn of the game Monopoly is determined as follows:

- Roll two dice, take the sum of the numbers that come up, and advance that number of squares.
- If you roll *doubles* (that is, the same number comes up on both dice), then you roll a second time, take the sum, and advance that number of additional squares.
- If you roll doubles a second time, then you roll a third time, take the sum, and advance that number of additional squares.
- However, as a special case, if you roll doubles a third time, then you go to jail. Regard this as advancing zero squares overall for the turn.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$\frac{1}{6}$

$\frac{5}{6}$

(a) [pts] What is the expected sum of two dice, given that the same number comes up on both?

$$Ex(S_s) = \sum_{i=1}^{6} 2i \cdot Pr(S) = \sum_{i=1}^{6} 2i \cdot \cancel{\frac{1}{36}}\,\frac{1}{6} = (2+4+6+8+10+12)\cancel{\frac{1}{36}}\,\frac{1}{6}$$

$$= \frac{42}{\cancel{36}\,6} = \frac{7}{\cancel{6}\,1} = 7 \quad \boxed{= Ex(D_1+D_2 \mid E)}$$

(b) [pts] What is the expected sum of two dice, given that different numbers come up? (Use your previous answer and the Total Expectation Theorem.)

$$Ex(S_s) = \sum_{i} \sum_{j \neq i} (i+j)\frac{1}{36} = (2\cdot 3 + 2\cdot 4 + 4\cdot 5 + 4\cdot 6 + 6\cdot 7 + 4\cdot 8 + 4\cdot 9 + 2\cdot 10 + 2\cdot 11)\frac{1}{36}$$

$$Ex(D_1+D_2) = Ex(D_1+D_2 \mid E)\cdot Pr(E) + Ex(D_1+D_2 \mid \bar{E})\cdot Pr(\bar{E})$$

$$7 = 7 \cdot \frac{1}{6} + Ex(D_1+D_2 \mid \bar{E}) \cdot \frac{5}{6}$$

$$Ex(D_1+D_2 \mid \bar{E}) = \left(7 - \frac{7}{6}\right)\cdot\frac{6}{5} = \frac{35}{\cancel{6}}\cdot\frac{\cancel{6}}{5} = 7$$

(c) [pts] To simplify the analysis, suppose that we always roll the dice three times, but may ignore the second or third rolls if we didn't previously get doubles. Let the random variable X_i be the sum of the dice on the i-th roll, and let E_i be the event that the i-th roll is doubles. Write the expected number of squares a piece advances in these terms.

$$Ex(X) = E(X_1) + Ex(X_2) + Ex(X_3)$$

$$= E(X_1) + \overbrace{Ex(X_2 \mid E_1)\, Pr(E_1) + Ex(X_2 \mid \bar{E}_1)\, Pr(\bar{E}_1)} + Ex(X_3 \mid E_2) \cdot Pr(E_2) + Ex(X_3 \mid \bar{E}_2)\, Pr(\bar{E}_2)$$

$$= 7 + 7 \cdot \frac{1}{6} + 0 \cdot \frac{5}{6} + 7 \cdot \frac{1}{6} + 0 \cdot \frac{5}{6}$$

$$= 7 + \frac{7}{6} + \frac{7}{6}$$

$$= 7 + \frac{14}{6}$$

$$= 9\tfrac{1}{3}$$

$$= 9.33$$

& Consider $(X_1 + X_2 \mid E_1 + \bar{E}_2)$ etc

(d) [pts] What is the expected number of squares that a piece advances in Monopoly?

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$Pr(R) = \frac{1}{36}$$

$$S = S_1 + S_2 + S_3$$

$$Ex(S) = Ex(S_1) + Ex(S_2) + Ex(S_3)$$

$$= \sum_i \sum_j (i+j)\frac{1}{36} + \sum_i \sum_j (i+j)\left(\frac{1}{36}\right)^2 + \sum_i \sum_j (i+j)\left(\frac{1}{36}\right)^3$$