Recitation 7
September 30, 2010

1. **Problem 2.35, page 130 in the text.** Verify the expected value rule

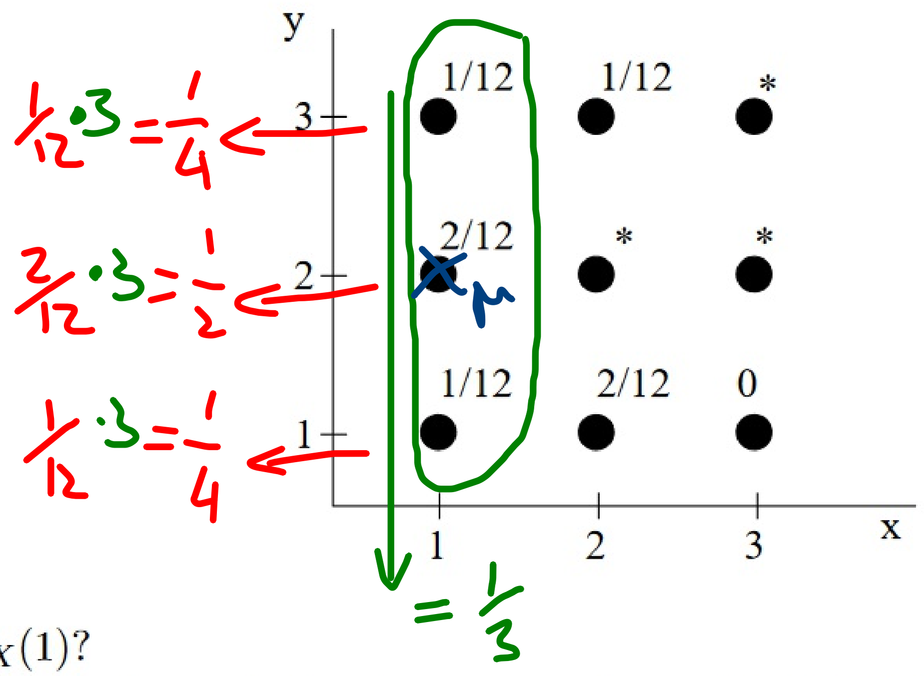
$$\mathbf{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y),$$

using the expected value rule for a function of a single random variable. Then, use the rule for the special case of a linear function, to verify the formula

$$\mathbf{E}[aX + bY] = a\mathbf{E}[X] + b\mathbf{E}[Y],$$

where a and b are given scalars.

2. Random variables X and Y can take any value in the set $\{1, 2, 3\}$. We are given the following information about their joint PMF, where the entries indicated by a * are left unspecified:



(a) What is $p_X(1)$?

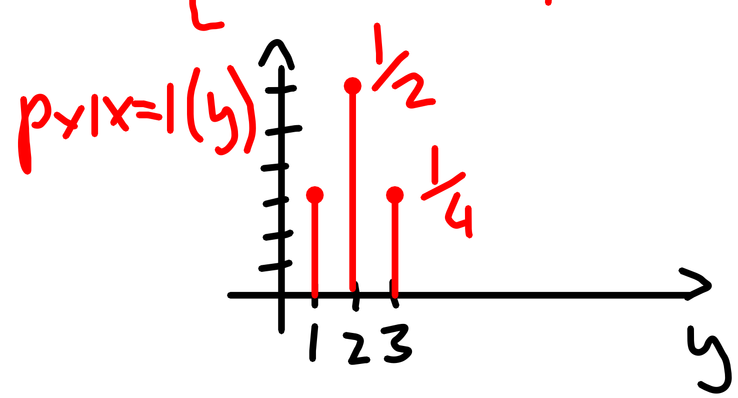
(b) Provide a clearly labeled sketch of the conditional PMF of Y given that $X = 1$.

(c) What is $\mathbf{E}[Y \mid X = 1]$?

(d) Is there a choice for the unspecified entries that would make X and Y independent?

a) $p_X(1) = \frac{4}{12} = \frac{1}{3}$

b) $p_{Y|X}(y|X=1) = \frac{p_{X,Y}(x,y)}{p_X(1)} = \left\{ (1; \tfrac{1}{12}\cdot 3), (2; \tfrac{2}{12}\cdot 3), (3, \tfrac{1}{12}\cdot 3) \right\}$

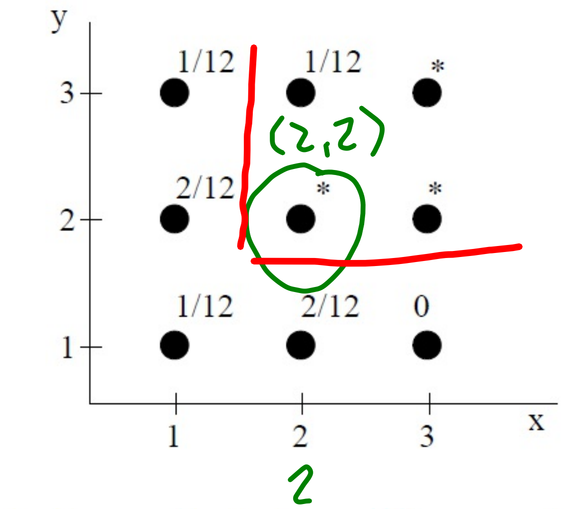


c) $E[Y|X=1] = \sum y \cdot p_{Y|X=1}(y) = 1\cdot\frac{1}{4} + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{4} = 2$

Let B be the event that $X \le 2$ and $Y \le 2$. We are told that conditioned on B, the random variables X and Y are independent.

(e) What is $p_{X,Y}(2,2)$?
(If there is not enough information to determine the answer, say so.)

(f) What is $p_{X,Y|B}(2,2 \mid B)$?
(If there is not enough information to determine the answer, say so.)



(e) Knowing that X and Y are conditionally independent given B, we must have

$$\frac{p_{X,Y}(1,1)}{p_{X,Y}(1,2)} = \frac{p_{X,Y}(2,1)}{p_{X,Y}(2,2)}$$

since the (X,Y) pairs in the equality are all in B. Thus

$$p_{X,Y}(2,2) = \frac{p_{X,Y}(1,2)p_{X,Y}(2,1)}{p_{X,Y}(1,1)} = \frac{(2/12)(2/12)}{1/12} = \frac{4}{12} = \frac{1}{3}.$$

e) I $\quad p_X(2) = \frac{1}{12} + \frac{2}{12} + p_{X,Y}(2,2)$

II $\quad 1 = \sum \frac{p_{X,Y}(x,y)}{p_X(2)}$

$$1 = \frac{1}{p_X(2)}\left(p_{X,Y}(2,1) + p_{X,Y}(2,2) + p_{X,Y}(2,3)\right)$$

$$1 = \frac{1}{p_X(2)}\left(\frac{2}{12} + p_{X,Y}(2,2) + \frac{1}{12}\right)$$

$$1 = \frac{3}{12 \cdot p_X(2)} + \frac{p_{X,Y}(2,2)}{p_X(2)} \quad \Big| \leftarrow \text{I}$$

$$= \frac{3}{12 \cdot \left(\frac{3}{12} + p_{X,Y}(2,2)\right)} + \frac{p_{X,Y}(2,2)}{\left(\frac{3}{12} + p_{X,Y}(2,2)\right)} \quad \Big| \cdot \left(\frac{3}{12} + p_{X,Y}(2,2)\right)$$

$$\cancel{\left(\frac{3}{12} + p_{X,Y}(2,2)\right)} = \frac{3}{12} \frac{\cancel{\left(\frac{3}{12} + p_{X,Y}(2,2)\right)}}{\cancel{\left(\frac{3}{12} + p_{X,Y}(2,2)\right)}} + \cancel{p_{X,Y}(2,2)}$$

$\Rightarrow$ not enough information

3. **Problem 2.33, page 128 in the text.** A coin that has probability of heads equal to p is tossed successively and independently until a head comes twice in a row or a tail comes twice in a row. Find the expected value of the number of tosses.

Tutorial 3

3. The joint PMF of the random variables X and Y is given by the following table:

$y = 3$	c	c	$2c$
$y = 2$	$2c$	0	$4c$
$y = 1$	$3c$	c	$6c$
	$x = 1$	$x = 2$	$x = 3$

(a) Find the value of the constant c.

(b) Find $p_Y(2)$.

(c) Consider the random variable $Z = YX^2$. Find $\mathbf{E}[Z \mid Y = 2]$.

(d) Conditioned on the event that $X \neq 2$, are X and Y independent? Give a one-line justification.

(e) Find the conditional variance of Y given that $X = 2$.