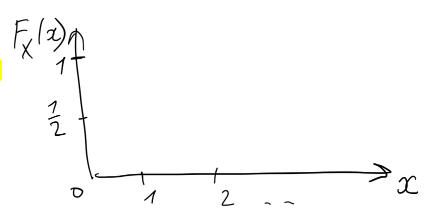
hw4

1



S_k – скачки F_X длина которых лежит в отрезке $[\frac{1}{k}; \frac{1}{k+1}]$
Следовательно, скачков S_k м.б. не больше $k+1$

Э M_{F_X} – мн-во всех точек разрыва

$M_{F_X} = \bigcup_{k=1}^{+\infty} S_k$ – не более чем счетно

2 X – с.в. $F_X \in C$ $F_X(x) = \Pr(X \le x)$

$Y = F_X(X)$. $F_Y = ?$

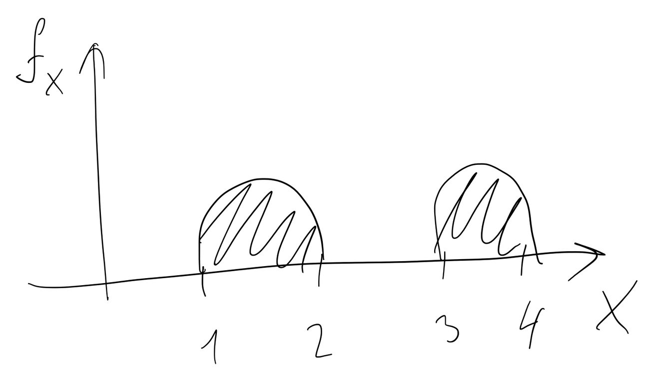
$$\Pr[a \le Y \le b] = \Pr[a \le F_X \le b] = \Pr\left[x : a \le \int_{-\infty}^{x} f_X(t)\,dt \le b\right]$$

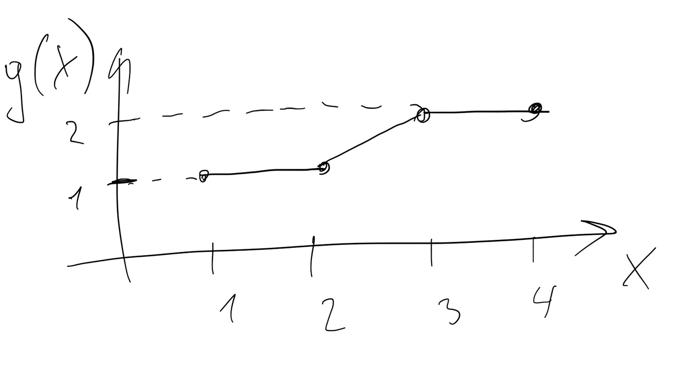
$$F_Y(y) = \Pr(Y \le y) = \Pr(F_X \le y) = \Pr\left(x^* : \int_{-\infty}^{x} f_X(t)\,dt \le y\right) \ominus$$

$$x^* \in (-\infty; a] : \int_{-\infty}^{a} f_X(t)\,dt = y$$

$$\ominus \int_{-\infty}^{a} f_X(t)\,dt, \text{ где } a : \int_{-\infty}^{a} f_X(t)\,dt = y$$

3





6 $X \sim N(0,1) \rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$Y = e^X$

$$\Pr[a \le Y \le b] = \Pr[a \le e^X \le b] = \Pr[\ln a \le x \le \ln b] =$$

$$= \int_{\ln a}^{\ln b} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \left| x = \ln y \right| = \int_a^b \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}\ln^2 y}}{y}\,dy$$

$$\Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}\ln^2 y}}{y}, \quad y \geq 0$$

$$\sqrt{2\pi}\,E[Y] = \int_{-\infty}^{+\infty} e^x e^{-\frac{x^2}{2}}\,dx = \int_{-\infty}^{+\infty} e^{\frac{2x - x^2}{2}}\,dx = \sqrt{e}\int_{-\infty}^{+\infty} e^{-\frac{(x-1)^2}{2}}\,d(x-1) = \sqrt{2\pi e}$$

$$\Rightarrow E[Y] = \sqrt{e}$$

$$\sqrt{2\pi}\,E[Y^2] = \int_{-\infty}^{+\infty} e^{2x} e^{-\frac{x^2}{2}} = e^2 \int_{-\infty}^{+\infty} e^{\frac{-4 + 4x - x^2}{2}}\,dx = e^2 \int_{-\infty}^{+\infty} e^{-\frac{(x-2)^2}{2}}\,d(x-2) = e^2\sqrt{2\pi}$$

$$Var[Y] = e^2 - e$$

7 $f_X(x) = \begin{cases} A x^\alpha e^{-\frac{x}{\beta}}, & \text{если } x \geq 0 \\ 0, & \text{иначе} \end{cases}$

* Будем считать, что $\beta > 0$

$$1 = \int_{-\infty}^{+\infty} A \cdot x^\alpha e^{-\frac{x}{\beta}}\,dx \Rightarrow \frac{1}{A} = \int_0^{+\infty} x^\alpha e^{-\frac{x}{\beta}}\,dx = \begin{vmatrix} t = \frac{x}{\beta} \\ dt = \frac{dx}{\beta} \end{vmatrix} =$$

$$= \int_0^{+\infty} (\beta t)^\alpha e^{-t}\,dt \cdot \beta = \beta^{\alpha+1} \int_0^{+\infty} t^\alpha e^{-t}\,dt = \beta^{\alpha+1}\Gamma(\alpha+1) \Rightarrow$$

$$\Rightarrow \boxed{A = \frac{1}{\beta^{\alpha+1}\Gamma(\alpha+1)}}$$

$$\frac{E[X]}{A} = \int_0^{+\infty} x^{\alpha+1} e^{-\frac{x}{\beta}}\,dx = \begin{vmatrix} t = \frac{x}{\beta} \\ dt = \frac{dx}{\beta} \end{vmatrix} = \int_0^{+\infty} (\beta \cdot t)^{\alpha+1} e^{-t}\,dt \cdot \beta =$$

$$= \beta^{\alpha+2} \int_0^{+\infty} t^{\alpha+1} e^{-t}\,dt = \beta^{\alpha+2}\Gamma(\alpha+2) \Rightarrow \boxed{E[X] = \beta \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}}$$

$$\frac{E[X^2]}{A} = \int_0^{+\infty} x^{\alpha+2} e^{-\frac{x}{\beta}}\,dx \Rightarrow E[X^2] = \beta^{\alpha+3}\Gamma(\alpha+3) \cdot A = \beta^2 \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)}$$

$$\boxed{Var[X] = \frac{\beta^2}{\Gamma(\alpha+1)}\left(\Gamma(\alpha+3) - \frac{\Gamma(\alpha+2)^2}{\Gamma(\alpha+1)}\right)}$$

8 1) $\lim\limits_{x \to +\infty} x \cdot \int_x^{+\infty} \frac{1}{z}\,dF(z) = 0$

$$\lim_{x \to +\infty} x \cdot \int_x^{+\infty} \frac{1}{z}\,dF(z) = \lim_{x \to +\infty} x \cdot \left(\frac{1}{z}F(z)\Big|_x^{+\infty} + \int_x^{+\infty} \frac{F(z)}{z^2}\,dz \right) =$$

$$= \lim_{x \to +\infty} x \cdot \left(-\frac{F(x)}{x} + \int_x^{+\infty} \frac{F(z)}{z^2}\,dz \right) = -1 + \lim_{x \to +\infty} \frac{\overset{\to 0}{\int_x^{+\infty} \frac{F(z)}{z^2}\,dz}}{\underset{\to 0}{\frac{1}{x}}} =$$

$$= -1 + \lim_{x \to +\infty} \frac{-F(x) \cdot \cancel{x^2}}{\cancel{x^2} \cdot (-1)} = -1 + 1 = 0$$

Лопиталь

2) $\lim\limits_{x \to 0+} x \cdot \int_x^{+\infty} \frac{1}{z}\,dF(z) = 0$

$$\lim_{x \to 0+} x \int_x^{+\infty} \frac{1}{z}\,dF(z) = \lim_{x \to 0+} x \cdot \left(-\frac{1}{x} \cdot F(x) + \int_x^{+\infty} \frac{F(z)}{z^2}\,dz \right) =$$

$$= -F(0) + \lim_{x \to 0+} \frac{\overset{\to +\infty}{\int_x^{+\infty} \frac{F(z)}{z^2}\,dz}}{\underset{\to +\infty}{\frac{1}{x}}} \ominus$$

Лопиталь

$$\nexists \lim_{x \to 0+} \int_x^{+\infty} \frac{F(z)}{z^2}\,dz = \lim_{x \to 0+} \left(\int_x^1 \frac{F(z)}{z^2}\,dz + \int_1^{+\infty} \frac{F(z)}{z^2}\,dz \right) =$$

$$= \underbrace{\lim_{x \to 0+} \int_x^1 \frac{F(z)}{z^2}\,dz}_{1} + \underbrace{\int_1^{+\infty} \frac{F(z)}{z^2}\,dz}_{2} \xrightarrow[x \to 0+]{} +\infty$$

$$2.\ \int_1^{+\infty} \frac{F(z)}{z^2}\,dz \le \int_1^{+\infty} \frac{1}{z^2}\,dz = \left(-\frac{1}{z}\right)\Big|_1^{+\infty} = 1$$

$$1.\ \lim_{x \to 0+} \int_x^1 \frac{F(z)}{z^2}\,dz \geq \lim_{x \to 0+} \int_x^1 \frac{F(0)}{z^2}\,dz = F(0) \cdot \lim_{x \to 0+} \int_x^1 \frac{1}{z^2}\,dz =$$

$$= F(0) \lim_{x \to 0+} \left(-1 + \frac{1}{x}\right) \xrightarrow[x \to +0]{} +\infty$$

$$\ominus -F(0) + \lim_{x \to 0+} \frac{-F(x)\,\cancel{x^2}}{\cancel{x^2}\,(-1)} = -F(0) + F(0) = 0$$