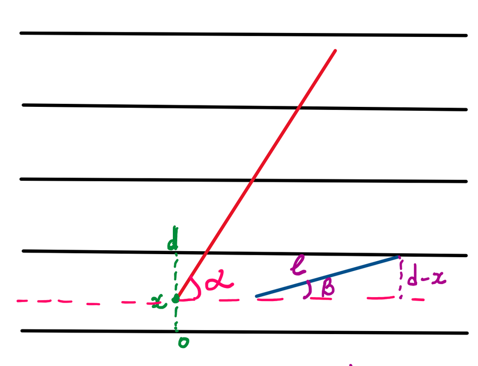
hw5

1



$\beta = \arcsin \frac{d-x}{\ell}$

$$g(x,y) = \left\lfloor \frac{x+\sin(y)\cdot \ell}{d} \right\rfloor$$ – кол-во пересеченных прямых

$x \in [0;d] \quad y\in[0;\pi] \quad f_{X,Y}(x,y) = \frac{1}{d\cdot\pi}$

$$\Pr[g(X,Y)=0] = \Pr\left[(x;y): \left\lfloor \frac{x+\sin(y)\ell}{d}\right\rfloor = 0\right] =$$

$$= \Pr[(x,y): x+\sin(y)\ell < d] = \Pr\left[(x,y): \sin(y) < \frac{d-x}{\ell}\right] \to \begin{cases} 1)\ \ell \geq d \\ 2)\ \ell < d\end{cases}$$

$$1)\ \int_0^d dx\, 2\cdot\int_0^{\arcsin\frac{d-x}{\ell}} dy \cdot \frac{1}{\pi d} = \frac{2}{\pi d}\int_0^d dx \int_0^{\arcsin\frac{d-x}{\ell}} dy = \frac{2}{\pi d}\int_0^d \arcsin\frac{d-x}{\ell}\,dx \ \textcircled{=}$$

$$* \int_a^b \arcsin\frac{d-x}{\ell}\,dx = \left| \begin{matrix} t = \frac{d-x}{\ell} \\ dt = -\frac{dx}{\ell}\end{matrix}\right| = -\ell\int_{\frac{d-a}{\ell}}^{\frac{d-b}{\ell}} \arcsin t\cdot dt = \ell \int_{\frac{d-b}{\ell}}^{\frac{d-a}{\ell}} \arcsin t\, dt =$$

$a, b \leq d$

$$* \int_{\frac{d-b}{\ell}}^{\frac{d-a}{\ell}} \arcsin t\, dt = \underbrace{\arcsin t\cdot t \Big|_{\frac{d-b}{\ell}}^{\frac{d-a}{\ell}}}_{\gamma} + \int_{\frac{d-b}{\ell}}^{\frac{d-a}{\ell}} \frac{-t\,dt}{\sqrt{1-t^2}} = \gamma + \sqrt{1-t^2}\Big|_{\frac{d-b}{\ell}}^{\frac{d-a}{\ell}}$$

$$\textcircled{=} \frac{2}{\pi d}\left(\arcsin x\cdot x\Big|_0^{\frac{d}{\ell}} + \sqrt{1-x^2}\Big|_0^{\frac{d}{\ell}}\right) = \frac{2}{\pi d}\left(\frac{d}{\ell}\arcsin\frac{d}{\ell} + \sqrt{1-\left(\frac{d}{\ell}\right)^2} - 1\right)$$

$$2)\ \int_{d-\ell}^d dx\, 2\cdot\int_0^{\arcsin\frac{d-x}{\ell}} dy\cdot\frac{1}{\pi d} = \frac{2}{\pi d}\int_{d-\ell}^d \arcsin\frac{d-x}{\ell}\,dx = \frac{2}{\pi d}\left(x\cdot\arcsin x\Big|_0^1 + \sqrt{1-x^2}\Big|_0^1\right) =$$
$$= \frac{2}{\pi d}\left(\frac{\pi}{2} - 1\right) = \frac{1}{d} - \frac{2}{\pi d}$$

$$\pi d\, E[g(X,Y)] = \int_0^\pi dy \int_0^d \left\lfloor\frac{x+\sin(y)\ell}{d}\right\rfloor dx = \int_0^\pi dy\left(\left\lfloor\frac{\sin y\,\ell}{d}\right\rfloor k(y) + \left(\left\lfloor\frac{\sin y\,\ell}{d}\right\rfloor + 1\right)(d-k(y))\right) \textcircled{=}$$

$$\exists\ \sin(y)\ell = t \Rightarrow k(y) + t = \left(\left\lfloor\frac{t}{d}\right\rfloor + 1\right)d$$
$$k(y) = \left\lfloor\frac{t}{d}\right\rfloor d + d - t$$

$$\textcircled{=} \int_0^\pi dy\left(\left\lfloor\frac{t}{d}\right\rfloor\left(\left\lfloor\frac{t}{d}\right\rfloor d + d - t\right) + \left(\left\lfloor\frac{t}{d}\right\rfloor + 1\right)\left(-\left\lfloor\frac{t}{d}\right\rfloor d + t\right)\right) = \left| b = \left\lfloor\frac{t}{d}\right\rfloor \right| =$$

$$= \int_0^\pi dy\left(\cancel{b^2 d} + \cancel{bd} - \cancel{bt} - \cancel{b^2 d} + \cancel{bt} - \cancel{bd} + t\right) = \ell\int_0^\pi \sin(y)\,dy = 2\ell \Rightarrow$$

$$\Rightarrow E[g(X,Y)] = \frac{2\ell}{\pi d}$$

2

X – непрерыв. с.в.
$\exists \alpha, \beta: \beta > \alpha \quad \Pr[\alpha \leq X \leq \beta]$
$g(x)$ – $\nearrow$ и диф-ма на $[\alpha;\beta]$

Д-ть: $Y = g(X)$, то $f_Y(y) = \begin{cases} \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}, & \text{если } y\in[g(\alpha), g(\beta)] \\ 0, & \text{иначе}\end{cases}$

$$f_Y(y) = \frac{\Pr(Y\in[y;y+\delta])}{\delta} = \frac{\Pr(X\in[g^{-1}(y); g^{-1}(y+\delta)])}{\delta} \textcircled{=}$$

$$g^{-1}(y+\delta) = g^{-1}(y) + (g^{-1})'(y)\delta + o(\delta)$$

$$\textcircled{=} \frac{\Pr(X\in[g^{-1}(y); g^{-1}(y) + \overbrace{(g^{-1})'(y)\delta + o(\delta)}^{\varepsilon}])}{\delta} = \frac{f_X(g^{-1}(y))\cdot(o(\delta) + (g^{-1})'(y)\delta)}{\delta} =$$

$$= f_X(g^{-1}(y))(g^{-1})'(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

$X \sim U(\alpha,\beta) \qquad g(x) = ax+b$

$$\Pr[c \leq ax+b \leq d] = \Pr\left[\frac{c-b}{a} \leq X \leq \frac{d-b}{a}\right] = \int_{\frac{c-b}{a}}^{\frac{d-b}{a}} \frac{1}{\beta-\alpha}\,dx = \left|\begin{matrix} t = ax+b \\ dt = a\,dx\end{matrix}\right| =$$

$$= \int_c^d \overbrace{\frac{1}{a(\beta-\alpha)}}^{f_Y(y)}\,dt$$

$$\frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))} = \frac{f_X\left(\frac{y-b}{a}\right)}{a} = \frac{1}{a(\beta-\alpha)} \ ✓$$

3

$$F_{X^2}(x) = \Pr(X^2 \leq x) = \Pr(X\leq\sqrt{x}) + \Pr(X \geq -\sqrt{x}) =$$
$$= \Pr(X\leq\sqrt{x}) + (1 - \Pr(X\leq -\sqrt{x})) = 1 + \int_{-\infty}^{\sqrt{x}} f_X(t)dt - \int_{-\infty}^{-\sqrt{x}} f_X(t)dt =$$
$$= 1 + \int_{-\sqrt{x}}^{\sqrt{x}} f_X(t)dt \Rightarrow F'_{X^2}(x) = f_{X^2}(x) = \frac{f_X(\sqrt{x}) + f_X(-\sqrt{x})}{2\sqrt{x}}$$

$$X\sim N(0,1) \qquad f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$$f_{X^2}(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-\frac{x}{2}} + e^{-\frac{x}{2}}}{2\sqrt{x}}\right) = \frac{e^{-\frac{x}{2}}}{\sqrt{2\pi x}}$$

4

$X\sim N(0,1) \qquad E[X\cdot\cos X] = ? \qquad E\left[\frac{X}{1+X^2}\right] = ?$

$f_X = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \qquad g(X) = X\cdot\cos X \qquad h(X) = \frac{X}{1+X^2}$

$$\sqrt{2\pi}\,E[g(X)] = \int_{-\infty}^{+\infty} \underbrace{x\cdot\cos x\cdot e^{-\frac{x^2}{2}}}_{u(x)} = 0$$

$u(-x) = -x\cdot\cos x\cdot e^{-\frac{x^2}{2}} = -u(x) \Rightarrow u$ – нечетная

$$\sqrt{2\pi}\,E[h(X)] = \int_{-\infty}^{+\infty} \underbrace{\frac{x}{1+x^2}e^{-\frac{x^2}{2}}}_{v(x)} = 0$$

$v(-x) = -\frac{x}{1+x^2}e^{-\frac{x^2}{2}} = -v(x) \Rightarrow v$ – нечетная

5

$X\sim N(0,1) \qquad E[\cos X] = ? \qquad Var[\cos X] = ?$

$f_X = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

$$\frac{1}{2}\cdot\sqrt{2\pi}\cdot E[\cos X] = \int_0^{+\infty}\cos x\cdot e^{-\frac{x^2}{2}}dx = \sqrt{\frac{\pi}{2e}} \Rightarrow E[\cos X] = \frac{1}{\sqrt{e}}$$

$$\frac{1}{2}\cdot\sqrt{2\pi}\,E[\cos^2 X] = \int_0^{+\infty}\cos^2 x\, e^{-\frac{x^2}{2}}dx = \frac{(1+e^2)\sqrt{\pi}}{2e^2\sqrt{2}} \Rightarrow E[\cos^2 X] = \frac{(1+e^2)\cancel{\sqrt{\pi}}\cdot\cancel{2}}{\cancel{2}\cdot e^2\cdot 2\cdot\cancel{\sqrt{\pi}}} = \frac{1+e^2}{2e^2} = \frac{1}{2} + \frac{1}{2e^2}$$

$$Var[\cos X] = \frac{1}{2} + \frac{1}{2e^2} - \frac{1}{e}$$

6

$$F(x,y) = \Pr(X\leq x \wedge Y\leq y)$$
$$f(z) = \Pr(X\leq z \wedge Y > z) = \Pr(X\leq z) - \Pr(X\leq z\wedge Y\leq z) =$$
$$= F(z,d) - F(z,z)$$

7

$$f_{X,Y}(x,y) = \begin{cases} x\cdot e^{-x(y+1)}, & x,y\geq 0 \\ 0, & \text{иначе}\end{cases}$$

$$f_X(x) = \int_0^{+\infty} x\cdot e^{-x(y+1)}dy = x\cdot e^{-x}\int_0^{+\infty} e^{-xy}\,dy = x\cdot e^{-x}\left(\frac{e^{-xy}}{-x}\Big|_0^{+\infty}\right) = e^{-x}$$

$$f_Y(y) = \int_0^{+\infty} x\cdot e^{-x(y+1)}dx = \frac{x\cdot e^{-x(y+1)}}{-(y+1)}\Big|_0^{+\infty} + \frac{1}{y+1}\int_0^{+\infty} e^{-x(y+1)}dy = \frac{1}{(y+1)^2}$$

$$f_{X|Y}(x|y) = x\cdot e^{-x(y+1)}(y+1)^2 \qquad f_{Y|X}(y|x) = x\cdot e^{-xy}$$