1 Head Count

Consider flipping a fair coin twice.

(a) What is the sample space Ω generated from these flips?

TT , TH , HT , HH } sample points have no prob assigned yet.

(b) Define a random variable X to be the number of heads. What is the distribution of X?

each sample point maps to a single RV

$X=0$: TT, TH; $X=2$: HH, TH → $X=1$

$X = \#$ heads $= 0, 1, 2$

$P(X=0) = P(TT) = \frac{1}{4}$

$X \begin{cases} 0 & \text{wp } 0.25 \\ 1 & \text{wp } 0.5 \\ 2 & \text{wp } 0.25 \end{cases}$

(c) Define a random variable Y to be 1 if $\omega = (H,T)$ and 0 otherwise. What is the distribution of Y?

$Y = \begin{cases} 1 & \text{if HT} \\ 0 & \text{else} \end{cases}$ $P(H,T) = \frac{1}{4}$

$Y = \begin{cases} 1 & \text{wp } 0.25 \\ 0 & \text{wp } 0.75 \end{cases}$

(d) Define a third random variable $Z = X + Y$. What is the distribution of Z?

$Z = \begin{cases} 0 \to \text{if } X=0, Y=0 \\ \cancel{1 \to \text{if } X=1, Y=0} \text{ or } X=0, Y=1 \\ 2 \to \text{if } X=1, Y=1 \text{ or } X=2, Y=0 \\ 3 \to \text{if } X=2, Y=1 \end{cases}$

$P(Z=0) = P(X=0 \cap Y=0)$
$= P(X=0) \cdot P(Y=0 \mid X=0)$
$\frac{1}{4} \cdot 1 = 1/4$

2 Maybe Lossy Maybe Not

$P(Z=2) = P(X=1 \cap Y=1) + P(P(X=2) \cap Y=0)$
$P(X=1) \cdot P(Y=1 \mid X=1)$

Let us say that Alice would like to send a message to Bob, over some channel. Alice has a message of length 4 and sends 5 packets.

(a) Packets are dropped with probability p. What is probability that Bob can successfully reconstruct Alice's message?

1

need poly. degree → $4-1=3$ → need 4 points to construct poly.

Goal: Probability that at most 1 is lost. (0 or 1)

$\binom{5}{1}(1-p)^4 \cdot p + \binom{5}{0}(1-p)^5 \cdot p^0$

$\boxed{\text{Binomial } \binom{n}{k}(1-p)^{n-k} p^k}$

(b) Again, packets can be dropped with probability p. The channel may additionally corrupt 1 packet. Alice realizes this and sends 3 additional packets. What is the probability that Bob receives enough packets to successfully reconstruct Alice's message?

(1 corruption → 2 extra needed)

Alice now sends $5 + 3\text{extra} = 8$
original message $= 4$ packets

$\underbrace{n}_{4} + \underbrace{2(1)}_{\text{one corruption}} = 6$

Bob needs at least 6 packages (the rest can be dropped it is ok)

→ Probability that 0, 1 or 2 packages are lost.

$\hookrightarrow \sum_{k=0}^{2} \binom{8}{k}(1-p)^{8-k} \cdot p^k$

(c) Again, packets can be dropped with probability p. This time, packets may be corrupted with probability q. Consider the original scenario where Alice sends 5 packets for a message of length 4. What is probability that Bob can successfully reconstruct Alice's message?

Alice sends 5, we need 4 real ones (if 1 drop + no corruption).
o o o o o   we need 5 with 1 corruption (no drop)

$$\binom{5}{0}(1-p)^5 + \underbrace{\binom{5}{1}(1-p)^4 p}_{\text{one drop, no corruption}} + \underbrace{\binom{5}{1}\cdot(1-p)^4 p\,(1-q)^4 q}_{\text{one corruption and it is dropped}}$$

no drop, no corruption

3 Telebears

Lydia has just started her CalCentral enrollment appointment. She needs to register for a marine science class and CS 70. There are no waitlists, and she can attempt to enroll once per day in either class or both. The CalCentral enrollment system is strange and picky, so the probability of enrolling in the marine science class is μ and the probability of enrolling in CS 70 is κ. These events are independent. Let M be the number of days it takes to enroll in the marine science class, and C be the number of days it takes to enroll in CS 70.

(a) What distribution do M and C follow? Are M and C independent?

$P(\text{marine enroll}) = \mu$
$P(\text{CS70 enroll}) = K$ independent

Yes. Geometric →

calendar

can't	can't	can't
can't	enrolled	

(b) For some integer $k \geq 1$, what is $\mathbb{P}[C \geq k]$?

takes at least k days : $P(x=k)+$

$\Rightarrow (1-K)^{k-1}$

$\boxed{\text{Geom } (1-p)^{k-1} p}$ (2)

$P(X \geq k)$

if occurs happen on k^{th} day

(c) For some integer $k \geq 1$, what is the probability that she is enrolled in both classes before day k?

$$P(M < k)\cdot P(C < k) = (1 - P(M \geq k)) + (1 - P(C \geq k))$$

$$(1-(1-K)^{k-1}) + (1-(1-\mu)^{k-1})$$

4 Fishy Computations

Use the Poisson distribution to answer these questions:

$\boxed{(3)\ \text{Poisson} \quad P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}}$

(a) Suppose that on average, a fisherman catches 20 salmon per week. What is the probability that he will catch exactly 7 salmon this week?

$X \sim \text{Poiss}(20)$ $\quad P(X=7) = \frac{20^7 \cdot e^{-20}}{7!}$

(b) Suppose that on average, you go to Fisherman's Wharf twice a year. What is the probability that you will go at most once in 2018?

$X \sim \text{Poisson}(2)$

$$P(X=1) + P(X=0) = P(X \leq 1)$$

(c) Suppose that in March, on average, there are 5.7 boats that sail in Laguna Beach per day. What is the probability there will be *at least* 3 boats sailing throughout the *next two days* in Laguna?

$X_1 \sim \text{Poisson}(5.7)$ day 1

$X_2 \sim \text{Poisson}(5.7)$ day 2

$Y \sim \text{Poisson}(11.4)$ days

$$P(Y \geq 3) = 1 - P(Y < 3)$$