$E(X) = \sum P(X=k) \cdot k$

$E(X^2) = \sum P(X^2 = k^2) \cdot k^2 = \sum P(X=k) \cdot k^2$ square of an indicator is itself

$E(X+Y) = E(X) + E(Y)$

CS 70 Discrete Mathematics and Probability Theory
Summer 2018 Sinho Chewi and Vrettos Moulos

DIS 5C

1 Ball in Bins

You are throwing k balls into n bins. Let X_i be the number of balls thrown into bin i.

1. What is $\mathbb{E}[X_i]$?

$X_i = X_{i_1} + X_{i_2} + \cdots X_{i_k}$ if k^{th} ball falls into bin i.

$E(X_i) = E(X_{i_1} + X_{i_2} + X_{i_3} \cdots) = P(X_{i_1}) \cdot 1 + P(X_{i_2}) \cdot 1 \cdots = k \cdot P(X_{i_k}) = k \cdot \frac{1}{n}$

$X_{i_k} = \begin{cases} 1 & \text{if ball } k \text{ falls in bin } i \\ 0 & \text{else } (1-\frac{1}{n}) \end{cases}$

$P(X_{i_k} = 1) = \frac{1}{n}$

2. Compute $\mathbb{E}[X_i^2]$.

$E(X_i) = E((X_{i_1} + X_{i_2} + X_{i_3} \cdots)(X_{i_1} + X_{i_2} + \cdots))$ k^2 terms

$\sum E(X_{ij}^2)$ square of an indicator is itself $0^2 = 0$, $1^2 = 1$ $E(X_{ij}) = \frac{k}{n}$

$\underbrace{k^2 - k}_{k(k-1)}$ $E(X_{ij} \cdot X_{ik})$ k of event that both j and k lands to i $= \frac{1}{n^2}$ then

$\frac{k(k-1)}{n^2} + \frac{k}{n}$

3. What is the expected number of empty locations?

$E(X_{i_k}') = P(X_{i_k} = 0) = ((1-\frac{1}{n})^k) n$

4. What is the expected number of collisions?

\# balls − # occupied bins

$E(col) = k - (n - E(empty))$

2 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X = 0)$? → 3 non-Queen: $P(X=0) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} = \frac{\binom{48}{3}}{\binom{52}{3}}$

(b) What is $\mathbb{P}(X = 1)$? → 2 non Queen: $P(X=1) = \frac{\binom{48}{2}\binom{4}{1}}{\binom{52}{3}}$

(c) What is $\mathbb{P}(X = 2)$? → 1 non Queen $\frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}}$

(d) What is $\mathbb{P}(X = 3)$? → $\frac{\binom{4}{3}}{\binom{52}{3}}$

(e) Do the answers you computed in parts (a) through (d) add up to 1, as expected?

$\sum_{x \in \Omega} P(X=\omega) = 1$

(f) Compute $\mathbb{E}(X)$ from the definition of expectation.

$E(X) = \sum_{k=0}^{3} k \cdot P(X=k) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2\, P(X=2) + 3 \cdot P(X=3)$

(g) Suppose we define indicators X_i, $1 \leq i \leq 3$, where X_i is the indicator variable that equals 1 if the ith card is a queen and 0 otherwise. Compute $\mathbb{E}(X)$.

$E(X) = E(X_1 + X_2 + X_3)$
$= E(X_1) + E(X_2) + E(X_3)$
$\hookrightarrow P(X_1 = 1) \cdot 1 + P(X_1 = 0) \cdot 1$

$X_i = \begin{cases} 1 & \text{if } i\text{th card is } Q \\ 0 & \text{else} \end{cases}$

(h) Are the X_i indicators independent? Does this affect your solution to part (g)?

No they are not

No it doesn't b/c linearity of expectation

3 More Family Planning

(a) Suppose we have a random variable $N \sim Geom(1/3)$ representing the number of children of a randomly chosen family. Assume that within the family, children are equally likely to be boys and girls. Let B be the number of boys and G the number of girls in the family. What is the joint probability distribution of B, G?

(b) Given that we know there are 0 girls in the family, what is the most likely number of boys in the family?

(c) Now let X and Y be independent random variables representing the number of children in two independently, randomly chosen families. Suppose $X \sim Geom(p)$ and $Y \sim Geom(q)$. Using their joint distribution, find the probability that the number of children in the first family (X) is less than the number of children in the second family (Y). (You may use the convergence formula for a Geometric Series: $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ for $|r| < 1$)

(d) Show how you could obtain your answer from the previous part using an interpretation of the geometric distribution.