Alternative Explanations for a Counting Problem

Question: You are hired at a company with twelve employees (including yourself), each of whom drive a car to work, and twelve parking spaces arranged in a row. All orderings of the twelve cars are equally likely. Suppose that, you park in a space that is not at one of the ends of the row. As you leave your office, you know that exactly five of your colleagues have left work before you. Assuming that you remember nothing about where these colleagues had parked, what is the probability that you will find both spaces on either side of your car unoccupied?

Method	**Using combinations** (locations for empty slots)	**Using ordering** (all possible sequences)	**Using Probability** (individual probabilities)
Why	We don't really care about the order of the cars what left, or whose car left, but we only care about which parking spaces are empty. The empty parking spaces are undistinguishable.	This is a bit more tedious but also works. In this case you can first assume a full parking space where all 12 cars are parked, and then out of those 11, 5 of them leave, and count the number of different ways 5 cars could leave.	Consider all the locations you could park your car, and the probability that both cars next to yours ends up leaving.
Define A and B	**A** = #ways picking 2 out of 5 empty spots next to your car **B**=#ways picking 5 empty slots	**B** = #car sequences where any 5 car can leave **A** = #sequences where both cars next to yours leave	No A, B here since we are not counting now but directly thinking probabilistically
Calculate A and B	B = $\binom{11}{5}$ A = $\binom{9}{3}$	B = $10 \cdot 11! \cdot \binom{11}{5}$ A = $\binom{11}{5}\binom{5}{2} \cdot 2! \cdot 10!$	No A, B
Intuition & Explanation	We know your car is still in the parking lot taking one slot, so we only consider other 11 cars. **B:** any 5 out of those 11 slots can be emptied when that coworker leaves so 11 choose 5 will give the number of ways to pick any 5 slots to be emptied. **A:** fixing the slot of your car leaves 11 more slots. You also want to fix the two slots next to your car with two empty slots since this is precisely the scenario we want to consider, which leaves 9 more slots whose faiths we didn't determine. Since the wanted scenario requires 2 empty spaces to be next to your car, the remaining 3 out of the 5 empty spaces has to be among the 9 slots we didn't fix under the scenario. There are 9 choose 3 ways that the remaining empty slots can be picked	**B:** You could have parked your car in 10 places (the question says you didn't park on either ends). Once you decide on a location for your car, your 11 coworkers can park in 11! different ways around your car. And finally, after everybody is parked, 5 cars out of the 11 coworker cars will leave. **A:** not only 5 out of 11 coworker cars are picked to leave but also now we want the case where out of the 5 cars to leave, 2 of them are on the two sides of your car. We start by picking which 2 out of the 5 will be next to your car: 5 choose 2. Then we also need to consider those two cars to switch places, it can be car X and Y on your left and right respectively, or Y and X. Since we care about specific sequences of parked cars, we need to consider the number of ways to place the 2 cars we picked around your car, which is 2! = 2 ways. Lastly, we need to count the different sequences made by the remaining cars: considering we made sure the 2 cars next to yours are among the 5 that will leave, we can think of your car and the 2 cars around yours as a chunk that will stay together. So we count that chunk of 3 cars as one since they will occupy one location together. That leaves 9 more cars that was initially parked in the parking lot. So, now we are ordering 10 things in total: 9 other cars + the chunk we want. There are 10! ways to order these together.	You didn't park on slot 1 or slot 12, which remains 10 slots in between that you can park your car at. The probability of you parking in any specific slot if uniformly distributed so the probability of your car being in slot 2 (or any of the 10 slots) is 1/10. We know 5 cars will leave among the 11 coworker cars. So, the probability that the car on one side of your car leaving is 5/11. Then, assuming that car has already left, there are 4 more cars out of the remaining 10 coworker cars that could also leave. So the probability that the car on the other side of your car leaving is 4/10. Since each of these individual situations (i.e you parking on slot 2, and both sides leaving; you parking on slot 3, and both sides leaving; ... ; you parking on slot 11 and both sides leaving) are disjoint events, we can sum them up to find the total probability.
Final Expression	$\frac{\binom{9}{3}}{\binom{11}{5}} = \frac{2}{11}$	$\frac{\binom{11}{5}\binom{5}{2}\cdot 2!\cdot 10!}{10\cdot 11!\cdot \binom{11}{5}} = \frac{2}{11}$	$\frac{1}{10}\left(\frac{5}{11}\cdot\frac{4}{10}\right)+\ldots+\frac{1}{10}\left(\frac{5}{11}\cdot\frac{4}{10}\right)$ $= 10\cdot\frac{1}{10}\left(\frac{5}{11}\cdot\frac{4}{10}\right) = \left(\frac{5}{11}\cdot\frac{4}{10}\right) = \frac{2}{11}$