Hamilton–Jacobi theory for contact systems: autonomous and non-autonomous

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Abstract

In this paper we obtain two Hamilton–Jacobi equations for time dependent contact Hamiltonian systems. In these systems there is a dissipation parameter and the fact of obtaining two equations reflects whether we are looking for solutions that depend on this parameter or not. We also study the existence of complete solutions and the integrability problem. The theory is illustrated with two examples.

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Contents

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1 Introduction

Contact Hamiltonian systems have been widely used in the study of thermodynamics. Recently there has been a renewed interest in these systems, since, unlike symplectic Hamiltonian systems, they lead to dissipated rather than conserved quantities [25, 37, 38]. These systems are also relevant to describe mechanical systems with certain types of damping [23, 25, 37], quantum mechanics [9], circuit theory, control theory [22, 56, 60] and black holes [39], among many others. The underlying variational principle is the so-called Herglotz principle [24, 43], a generalization of the well-known Hamilton principle, which gives rise to action-dependent Lagrangian systems, which are becoming popular in theoretical physics [47–49].

Hamilton–Jacobi theory provides a remarkably powerful method to integrate the dynamics of many Hamiltonian systems. In particular, for a completely integrable system, if one knows a complete solution of the Hamilton–Jacobi problem, the dynamics of the system can be easily reduced to quadratures [40–42]. Geometrically, the Hamilton–Jacobi problem consists on finding a section γ of $\pi_Q\colon \mathrm{T}^*Q \to Q$ which transforms integral curves of a projected vector field X_H^γ on Q into integral curves of the dynamical vector field X_H on T^*Q [1, 6]. This idea can be naturally extended to other vector bundles. As a matter of fact, it has been applied in many other different contexts, such as nonholonomic systems [7, 13, 44, 57], singular Lagrangian systems [27, 28, 50], higher-order systems [12], field theories [4, 26, 31–33, 61, 62] or systems with external forces [19, 20]. A unifying Hamilton–Jacobi theory for almost-Poisson manifolds was developed in reference [29]. Hamilton–Jacobi theory has also been extended to Hamiltonian systems with non-canonical symplectic structures [55], non-Hamiltonian systems [58], locally conformally symplectic manifolds [35], Nambu–Poisson [14] and Nambu–Jacobi [15] manifolds, Lie algebroids [51] and implicit differential systems [34]. The applications of Hamilton–Jacobi theory include the relation between classical and quantum mechanics [3, 8, 54], information geometry [10, 11], control theory [59] and the study of phase transitions [46].

We have recently initiated the study of time-dependent contact Hamiltonian systems [16], and the underlying geometric structures, which we call co-contact, since they are a combination of cosymplectic (the setting for studying time-dependent Hamiltonian systems) and contact structures. Such structures thus consist of two 1-forms, τ and η, where τ is closed and $\tau \wedge \eta \wedge \mathrm{d}\eta^n$ is a volume form, in a $2n+2$-dimensional manifold. Obviously, the geometric model is $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ with canonical structures. In fact, in [16] we have been able to identify that such structures in turn define a Jacobi structure such that the characteristic foliation is formed by contact leaves.

The aim of the present paper is to develop a Hamilton–Jacobi theory for this type of Hamiltonian systems. We follow the line undertaken in previous papers, considering sections of the canonical fibrations $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to Q$ and $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to Q \times \mathbb{R}$ which allow us to project the Hamiltonian vector field to the base and by comparing the values on the section, we obtain the corresponding Hamilton–Jacobi equations. This study has led us to the symmetries and first integrals of the system as well as to the existence of complete solutions and the corresponding integrability problems.

The paper is structured as follows. Section 2 is devoted to review time-dependent contact Hamiltonian systems introducing the basic elements necessary to follow the rest of the paper. In Section 3 we study symmetries and dissipated quantities in cocontact systems. In Sections 4 and 5 we deal with the two main results of the paper: the action-independent approach and the action-dependent approach to the Hamilton–Jacobi equation, respectively. Finally, in Section 6 we discuss two examples that illustrate the theory previously developed.

From now on, all the manifolds and mappings are assumed to be smooth and second-countable. Sum over crossed repeated indices is understood.

2 Review on time-dependent contact systems

In this section we are going to review some fundamentals on cocontact geometry and time-dependent contact Hamiltonian systems introduced in [16].

2.1 Cocontact manifolds

Definition 2.1. *A* ***cocontact structure*** *on a $(2n+2)$-dimensional manifold M is a couple (τ,η), where $\tau,\eta\in\Omega^1(M)$ and $\mathrm{d}\tau=0$, such that $\tau\wedge\eta\wedge(\mathrm{d}\eta)^{\wedge n}$ is a volume form on M. In this case, (M,τ,η) is called a* ***cocontact manifold****.*

Given a cocontact manifold (M,τ,η), the distribution $\mathcal{H}=\ker\eta$ is called the **horizontal** or **contact distribution**. Notice that this distribution is non-integrable.

Example 2.2. Let (P,η_0) be a contact manifold and consider the product manifold $M=\mathbb{R}\times P$. Denoting by $\mathrm{d}t$ the pullback to M of the volume form in $\mathbb{R}$ and by η the pullback of η_0 to M, we have that $(\mathrm{d}t,\eta)$ is a cocontact structure on M.

Example 2.3. Let $(P,\tau,-\mathrm{d}\theta)$ be an exact cosymplectic manifold and consider the product manifold $M=P\times\mathbb{R}$. Denoting by z the coordinate in $\mathbb{R}$ we define the 1-form $\eta=\mathrm{d}z-\theta$. Then, (τ,η) is a cocontact structure on $M=P\times\mathbb{R}$.

Example 2.4 (Canonical cocontact manifold)**.** Let Q be an n-dimensional smooth manifold with local coordinates (q^i) and its cotangent bundle T^*Q with induced natural coordinates (q^i,p_i). Consider the product manifolds $\mathbb{R}\times\mathrm{T}^*Q$ with coordinates (t,q^i,p_i), $\mathrm{T}^*Q\times\mathbb{R}$ with coordinates (q^i,p_i,z) and $\mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R}$ with coordinates (t,q^i,p_i,z) and the canonical projections

$$
\begin{tikzcd}
 & \mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R} \arrow[dl, "\rho_1"'] \arrow[dd, "\pi"] \arrow[dr, "\rho_2"] & \\
\mathbb{R}\times\mathrm{T}^*Q \arrow[dr, "\pi_2"'] & & \mathrm{T}^*Q\times\mathbb{R} \arrow[dl, "\pi_1"] \\
 & \mathrm{T}^*Q &
\end{tikzcd}
$$

Let $\theta_0\in\Omega^1(\mathrm{T}^*Q)$ be the canonical 1-form of the cotangent bundle, which has local expression $\theta_0=p_i\mathrm{d}q^i$. Denoting by $\theta_2=\pi_2^*\theta$, we have that $(\mathrm{d}t,\theta_2)$ is a cosymplectic structure in $\mathbb{R}\times\mathrm{T}^*Q$. On the other hand, denoting by $\theta_1=\pi_1^*\theta_0$, we have that $\eta_1=\mathrm{d}z-\theta_1$ is a contact form in $\mathrm{T}^*Q\times\mathbb{R}$.

Finally, consider the 1-form $\theta=\rho_1^*\theta_2=\rho_2^*\theta_1=\pi^*\theta_0\in\Omega^1(\mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R})$ and let $\eta=\mathrm{d}z-\theta$. Then, $(\mathrm{d}t,\eta)$ is a cocontact structure in $\mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R}$. The local expression of the 1-form η is

$$\eta=\mathrm{d}z-p_i\mathrm{d}q^i\,.$$

Given a cocontact manifold (M,τ,η), we have the **flat isomorphism**.

$$\flat: v\in\mathrm{T}M\longmapsto(\iota_v\tau)\tau+\iota_v\mathrm{d}\eta+(\iota_v\eta)\,\eta\in\Omega^1(M)\,. \tag{1}$$

This isomorphism can be trivially extended to an isomorphism of $\mathscr{C}^\infty(M)$-modules $\flat:\mathfrak{X}(M)\to\Omega^1(M)$. The inverse of the flat morphism is denoted by $\sharp=\flat^{-1}\colon\Omega^1(M)\to\mathfrak{X}(M)$ and called the **sharp isomorphism**.

Moreover, we have the following results, whose proofs can be found in [16].

Proposition 2.5. *On every cocontact manifold (M,τ,η) there exist two vector fields R_t, R_z on M such that*

$$
\begin{aligned}
\iota_{R_t}\tau&=1\,, & \iota_{R_t}\eta&=0\,, & \iota_{R_t}\mathrm{d}\eta&=0\,,\\
\iota_{R_z}\tau&=0\,, & \iota_{R_z}\eta&=1\,, & \iota_{R_z}\mathrm{d}\eta&=0\,.
\end{aligned}
$$

or, equivalently, $R_t=\flat^{-1}(\tau)$ and $R_z=\flat^{-1}(\eta)$. These vector fields R_t and R_z are called ***time and contact Reeb vector fields*** *respectively.*

Theorem 2.6 (Cocontact Darboux theorem). *Given a cocontact manifold (M, τ, η), around every point $p \in M$ there exist local coordinates (t, q^i, p_i, z) such that*

$$\tau = \mathrm{d}t\,, \quad \eta = \mathrm{d}z - p_i \mathrm{d}q^i\,.$$

These coordinates are called ***canonical*** *or* ***Darboux*** *coordinates. In addition, in Darboux coordinates, the Reeb vector fields read*

$$R_t = \frac{\partial}{\partial t}\,, \quad R_z = \frac{\partial}{\partial z}\,.$$

Proposition 2.7. *Let (M, τ, η) be a cocontact manifold. Then, (M, Λ, E) is a Jacobi manifold, where*

$$\Lambda(\alpha, \beta) = -\mathrm{d}\eta(\sharp\alpha, \sharp\beta)\,, \quad E = -R_z\,.$$

The bivector Λ induces a $\mathscr{C}^\infty(M)$-module morphism $\hat{\Lambda}\colon \Omega^1(M) \to \mathfrak{X}(M)$ given by

$$\hat{\Lambda}(\alpha) = \Lambda(\alpha, \cdot) = \sharp\alpha - \alpha(R_z)R_z - \alpha(R_t)R_t\,. \tag{2}$$

It can be seen that $\ker \hat{\Lambda} = \langle \tau, \eta \rangle$. The morphism $\hat{\Lambda}$ is also denoted by $\sharp_\Lambda$ in the literature [18, 23].

Taking Darboux coordinates (t, q, p, z), the bivector Λ has local expression

$$\Lambda = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial z}\,,$$

and the Jacobi bracket reads

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} - \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial z}\right) - f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\,.$$

In particular, one has

$$\{q^i, q^j\} = \{p_i, p_j\} = 0\,, \qquad \{q^i, p_j\} = \delta^i_j\,, \qquad \{q^i, z\} = -q^i\,, \qquad \{p_i, z\} = -2p_i\,.$$

2.2 Cocontact Hamiltonian systems

Definition 2.8. *A* ***cocontact Hamiltonian system*** *is family (M, τ, η, H) where (τ, η) is a cocontact structure on M and $H : M \to \mathbb{R}$ is a Hamiltonian function. The* ***cocontact Hamilton equations*** *for a curve $\psi\colon I \subset \mathbb{R} \to M$ are*

$$\begin{cases} \iota_{\psi'}\mathrm{d}\eta = \mathrm{d}H - (\mathscr{L}_{R_s}H)\eta - (\mathscr{L}_{R_t}H)\tau\,, \\ \iota_{\psi'}\eta = -H\,, \\ \iota_{\psi'}\tau = 1\,, \end{cases} \tag{3}$$

where $\psi'\colon I \subset \mathbb{R} \to \mathrm{T}M$ is the canonical lift of ψ to the tangent bundle $\mathrm{T}M$. The ***cocontact Hamiltonian equations*** *for a vector field $X_H \in \mathfrak{X}(M)$ are:*

$$\begin{cases} \iota_{X_H}\mathrm{d}\eta = \mathrm{d}H - (\mathscr{L}_{R_s}H)\eta - (\mathscr{L}_{R_t}H)\tau\,, \\ \iota_{X_H}\eta = -H\,, \\ \iota_{X_H}\tau = 1\,, \end{cases} \tag{4}$$

or equivalently, $\flat(X_H) = \mathrm{d}H - (\mathscr{L}_{R_s}H + H)\,\eta + (1 - \mathscr{L}_{R_t}H)\,\tau$. The unique solution to these equations is called the ***cocontact Hamiltonian vector field****.*

Given a curve ψ with local expression $\psi(r) = (f(r), q^i(r), p_i(r), z(r))$, the third equation in (3) imposes that $f(r) = r + cnt$, thus we will denote $r \equiv t$, while the other equations read:

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i}\,, \\ \dot{p}_i = -\left(\dfrac{\partial H}{\partial q^i} + p_i \dfrac{\partial H}{\partial z}\right)\,, \\ \dot{z} = p_i \dfrac{\partial H}{\partial p_i} - H\,. \end{cases} \tag{5}$$

On the other hand, the local expression of the cocontact Hamiltonian vector field is

$$X_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}\,.$$

The integral curves of the cocontact Hamiltonian vector field satisfy the following variational principle [53], which is a Hamiltonian version of the Herglotz principle [43].

Theorem 2.9 (Hamiltonian formulation of the Herglotz principle)**.** *Given a Hamiltonian $H : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R}$, a curve $c = (\mathrm{Id}_\mathbb{R}, q, p, z) : [0, T] \to \mathrm{T}^*Q \times \mathbb{R}$ is an integral curve of X_H if and only if it is a critical point of the action map:*

$$\mathcal{A}(c) = \left(\int_0^T p(t)\dot{q}(t) - H(t, q(t), p(t), z(t))\right) \mathrm{d}t \tag{6}$$

among all curves satisfying $c(0) = c_0$, $c(T) = c_T$ and $\dot{z} = p(t)\dot{q}(t) - H(t, q(t), p(t), z(t))$.

3 Symmetries and dissipated quantities in cocontact systems

There are several notions of symmetries in contact mechanics depending on the structures they preserve [17, 36]. However, in the present paper we will restrict ourselves to what we call generalized dynamical symmetries (see [38] for other symmetries). In some cases we will restrict ourselves to the case of cocontact manifolds of the form $M = \mathbb{R} \times N$ where N is a contact manifold (see Example 2.2). In this case, the natural projection $\mathbb{R} \times N \to \mathbb{R}$ defines a global canonical coordinate t.

Definition 3.1. *Let (M, τ, η, H) be a cocontact Hamiltonian system and let X_H be its cocontact Hamiltonian vector field.*

- *If $M = \mathbb{R} \times N$ with N a contact manifold, a* ***generalized dynamical symmetry*** *is a diffeomorphism $\Phi\colon M \to M$ such that $\eta(\Phi_* X_H) = \eta(X_H)$ and $\Phi^* t = t$.*
- *An* ***infinitesimal generalized dynamical symmetry*** *is a vector field $Y \in \mathfrak{X}(M)$ such that $\eta([Y, X_H]) = 0$ and $i_Y \tau = 0$. In particular, if $M = \mathbb{R} \times N$ with N a contact manifold, the flow of Y is made of generalized dynamical symmetries.*

Definition 3.2. *Let (M, τ, η, H) be a cocontact Hamiltonian system. A* ***dissipated quantity*** *is a function $f \in \mathscr{C}^\infty(M)$ such that*

$$\mathscr{L}_{X_H} f = -R_z(H) f\,.$$

It is worth pointing out that, unlike in the contact case, the Hamiltonian function is not, in general, a dissipated quantity. Indeed, using that

$$\mathscr{L}_{X_H} H = -R_z(H)H + R_t(H)\,,$$

it is clear that H is a dissipated quantity if and only if it is time-independent, i.e. $R_t(H) = 0$. This resembles the cosymplectic case, where the Hamiltonian function is conserved if, and only if, it is time-independent (see [5]).

Proposition 3.3. *A function $f \in \mathscr{C}^\infty(M)$ is a dissipated quantity if and only if $\{H, f\} = R_t(f)$, where $\{\cdot,\cdot\}$ is the Jacobi bracket associated to the cocontact structure (τ, η).*

Proof. The Jacobi bracket of f and H is given by

$$\{f, H\} = \Lambda(\mathrm{d}f, \mathrm{d}H) + fE(H) - HE(f) = -\mathrm{d}\eta\,(\sharp\mathrm{d}f, \sharp\mathrm{d}H) - fR_z(H) + HR_z(f), \tag{7}$$

but

$$\sharp\mathrm{d}f = X_f + (R_z(f) + f)\,R_z - (1 - R_t(f))\,R_t, \tag{8}$$

so

$$\iota_{\sharp\mathrm{d}f}\mathrm{d}\eta = \iota_{X_f}\mathrm{d}\eta = \mathrm{d}f - R_z(f)\eta - R_t(f)\tau\,, \tag{9}$$

and thus

$$\mathrm{d}\eta(\sharp\mathrm{d}f, \sharp\mathrm{d}H) = X_H(f) + R_z(f)H - R_t(f)\,. \tag{10}$$

Hence,

$$\{H, f\} + R_t(f) = X_H(f) + R_z(H)f\,. \tag{11}$$

In particular, the right-hand side vanishes if and only if f is a dissipated quantity. □

Theorem 3.4 (Noether's theorem)**.** *Let Y be an infinitesimal generalized dynamical symmetry of the cocontact Hamiltonian system (M, τ, η, H). Then, $f = -\iota_Y\eta$ is a dissipated quantity of the system. Conversely, given a dissipated quantity $f \in \mathscr{C}^\infty(M)$, the vector field $Y = X_f - R_t$, where X_f is the Hamiltonian vector field associated to f, is an infinitesimal generalized dynamical symmetry and $f = -\iota_Y\eta$.*

Proof. Let $f = -\iota_Y\eta$, where Y is an infinitesimal generalized dynamical symmetry. Then,

$$\begin{aligned}\mathscr{L}_{X_H}f &= -\mathscr{L}_{X_H}\iota_Y\eta = -\iota_Y\mathscr{L}_{X_H}\eta - \iota_{[X_H,Y]}\eta = \iota_Y\left(R_z(H)\eta + R_t(H)\tau\right)\\ &= R_z(H)\iota_Y\eta = -R_z(H)f\,,\end{aligned}$$

and thus f is a dissipated quantity.

On the other hand, given a dissipated quantity f, let $Y = X_f - R_t$. Then, it is clear that $f = -\iota_Y\eta$. In addition, $\iota_Y\tau = 0$, and

$$\begin{aligned}\iota_{[X_H,Y]}\eta &= \mathscr{L}_{X_H}\iota_Y\eta - \iota_Y\mathscr{L}_{X_H}\eta = -\mathscr{L}_{X_H}f + \iota_Y\left(R_z(H)\eta + R_t(H)\tau\right)\\ &= R_z(H)f - R_z(H)\iota_Y\eta = 0\,.\end{aligned}$$

□

The symmetries presented yield dissipated quantities. However, we are also interested in finding conserved quantities. The latter are important due to their elation with complete solutions of the Hamilton–Jacobi problem (see Section 5).

Definition 3.5. *A **conserved quantity** of a cocontact Hamiltonian system (M, τ, η, H) is a function $g \in \mathscr{C}^\infty(M)$ such that*

$$\mathscr{L}_{X_H}g = 0\,.$$

Taking into account that every dissipated quantity changes with the same rate $R_z(H)$, we have the following result, whose proof is straightforward.

Proposition 3.6. *Consider a cocontact Hamiltonian system (M, τ, η, H).*

- *If f_1, f_2 are two dissipated quantities and $f_2 \neq 0$, then f_1/f_2 is a conserved quantity.*
- *If f is a dissipated quantity and g is a conserved quantity, then fg is a dissipated quantity.*

Note that if $H \neq 0$, given an infinitesimal generalized dynamical symmetry Y, we have a conserved quantity $g = -\dfrac{1}{H}\iota_Y\eta$.

4 Hamilton–Jacobi theory: the action-independent approach

Let $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$ be a cocontact Hamiltonian system. Consider a section γ of the bundle $\pi_Q^t : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q$, locally given by

$$\begin{aligned} \gamma : \mathbb{R} \times Q &\longrightarrow \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \\ (t, q^i) &\longmapsto \left(t, q^i, \gamma_i(t,q), S(t,q)\right) . \end{aligned} \tag{12}$$

Let us introduce the vector field X_H^γ on $\mathbb{R} \times Q$ given by

$$X_H^\gamma = \mathrm{T}\pi_Q^t \circ X_H \circ \gamma \,, \tag{13}$$

where X_H is the Hamiltonian vector field of $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$. Suppose that X_H^γ and X_H are γ-related, i.e.,

$$X_H \circ \gamma = \mathrm{T}\gamma \circ X_H^\gamma \,, \tag{14}$$

so that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \arrow[r, "X_H"] \arrow[d, "\pi_Q^t"] & \mathrm{T}(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}) \arrow[d, "\mathrm{T}\pi_Q^t"'] \\ \mathbb{R} \times Q \arrow[r, "X_H^\gamma"] \arrow[u, bend left, "\gamma"] & \mathrm{T}(\mathbb{R} \times Q) \arrow[u, bend right, "\mathrm{T}\gamma"'] \end{tikzcd}$$

Locally,

$$X_H \circ \gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + \gamma_i \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(\gamma_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z} \,, \tag{15}$$

and

$$\mathrm{T}\gamma \circ X_H^\gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + \left(\frac{\partial \gamma_i}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial \gamma_i}{\partial q^i}\right)\frac{\partial}{\partial p_i} + \left(\frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i}\frac{\partial H}{\partial p_i}\right)\frac{\partial}{\partial z} \,, \tag{16}$$

so Eq. (14) holds if and only if

$$-\left(\frac{\partial H}{\partial q^i} + \gamma_i \frac{\partial H}{\partial z}\right) = \frac{\partial \gamma_i}{\partial t} + \frac{\partial H}{\partial p_j}\frac{\partial \gamma_j}{\partial q^i} \,, \tag{17a}$$

$$\gamma_i \frac{\partial H}{\partial p_i} - H = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i}\frac{\partial H}{\partial p_i} \,. \tag{17b}$$

Definition 4.1. *Given a section $\alpha : \mathbb{R} \times Q \times \mathbb{R} \to \mathbb{R} \times \bigwedge^k(\mathrm{T}^*Q) \times \mathbb{R}$ and $t, z \in \mathbb{R}$, let*

$$\begin{aligned} \alpha_{(t,z)} : Q &\longrightarrow \bigwedge\nolimits^k(\mathrm{T}^*Q) \\ q &\longmapsto \mathrm{pr}_{\bigwedge^k(\mathrm{T}^*Q)}(\alpha(t,q,z)) \,, \end{aligned} \tag{18}$$

*where $\mathrm{pr}_{\bigwedge^k(\mathrm{T}^*Q)} : \mathbb{R} \times \bigwedge^k(\mathrm{T}^*Q) \times \mathbb{R} \to \bigwedge^k(\mathrm{T}^*Q)$ is the canonical projection. The **exterior derivative at fixed** z is the section of $\mathbb{R} \times \bigwedge^{k+1}(\mathrm{T}^*Q) \times \mathbb{R} \to Q \times \mathbb{R}$ given by*

$$\mathrm{d}_Q\alpha(t,q,z) = (t, \mathrm{d}\alpha_{(t,z)}(q), z) \,. \tag{19}$$

In local coordinates, for $f : \mathbb{R} \times Q \times \mathbb{R} \to \mathbb{R}$ and $\alpha : \mathbb{R} \times Q \times \mathbb{R} \to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$, the expressions are

$$\begin{aligned} \mathrm{d}_Q f &= \frac{\partial f}{\partial q^i}\mathrm{d}q^i, \\ \mathrm{d}_Q(\alpha_i \mathrm{d}q^i) &= \frac{\partial \alpha_j}{\partial q^i}\mathrm{d}q^i \wedge \mathrm{d}q^j \,. \end{aligned} \tag{20}$$

Definition 4.2. *Let $f \in \mathscr{C}^\infty(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R})$ The **fixed-time** 1**-jet** of f is given by*

$$j^1_t f(t,q) = (t, q, \mathrm{d}_Q f, f)\,.$$

Proposition 4.3. *Let γ be a section of $\pi^t_Q : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \times \mathbb{R} \to Q$. Then, $\operatorname{Im}\gamma(t,\cdot)$ for every $t \in \mathbb{R}$ is a Legendrian submanifold of $(\mathrm{T}^*Q \times \mathbb{R}, \eta)$ if and only if it is the image of the fixed-time 1-jet of a function, namely, $N = \operatorname{Im}\gamma$, where*

$$\gamma(t,q) = j^1_t f(t,q) = (t, q, \mathrm{d}_Q f, f)\,.$$

Proof. Let $t \in \mathbb{R}$ and let $\gamma : Q \times \mathbb{R} \to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ such that $\gamma(q,z) = \gamma(t,q,z) = (t, q, \alpha(t,q,z), f(t,q,z))$. Clearly, $\gamma^*\tau = 0$, hence $\operatorname{Im}\gamma$ is Legendrian if and only if $\gamma^*\eta = 0$. Thus,

$$\gamma^*\eta = f^*\mathrm{d}z - \alpha^*\theta_Q = \mathrm{d}_Q f - \alpha\,, \tag{21}$$

so $\gamma^*\eta_Q$ vanishes precisely when $\alpha = \mathrm{d}_Q f$. □

Now, suppose that $\operatorname{Im}\gamma$ is a Legendrian submanifold. By Proposition 4.3, we have that

$$\gamma_i = \frac{\partial S}{\partial q^i}\,, \tag{22}$$

so Eqs. (17) can be written as

$$-\left(\frac{\partial H}{\partial q^i} + \frac{\partial S}{\partial q^i}\frac{\partial H}{\partial z}\right) = \frac{\partial^2 S}{\partial t \partial q^i} + \frac{\partial H}{\partial p_j}\frac{\partial S}{\partial q^i \partial q^j}\,, \tag{23a}$$

$$\frac{\partial S}{\partial q^i}\frac{\partial H}{\partial p_i} - H = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q^i}\frac{\partial H}{\partial p_i}\,. \tag{23b}$$

Eq. (23a) implies that

$$\mathrm{d}_Q(H \circ j^1 S) + \mathrm{d}_Q\left(\mathscr{L}_{R_t} S\right) = 0\,, \tag{24}$$

while Eq. (23b) yields

$$H = -\frac{\partial S}{\partial t}\,, \tag{25}$$

that is,

$$H \circ j^1 S + \mathscr{L}_{\frac{\partial}{\partial t}} S = 0\,. \tag{26}$$

Clearly, Eq. (24) is implied by Eq. (26). We have thus proven the following.

Theorem 4.4 (Action-independent Hamilton–Jacobi Theorem)**.** *Let γ be a section of $\pi^t_Q : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q$ such that, for every $t \in \mathbb{R}$, $\operatorname{Im}\gamma(t,\cdot)$ is a Legendrian submanifold of $(\mathrm{T}^*Q \times \mathbb{R}, \eta)$. Then, X^γ_H and X_H are γ-related if and only if Eq. (26) holds. This equation will be called the **action-independent Hamilton–Jacobi** equation for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$. If $\gamma = j^1_t S$, the function S is called a **generating function** for H.*

4.1 The variational interpretation of the solution to Hamilton–Jacobi equation

Suppose that $q\colon [0,T] \to Q$ is a trajectory given by the cocontact Hamilton equations (5) for the Hamiltonian function $H\colon \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R}$. If $\gamma = j^1_t S$ is a solution to the Hamilton–Jacobi problem for H, the generating function S can be interpreted as the action of the lifted curve $j^1_t S \circ q$ up to a constant.

Theorem 4.5. *Suppose that $S \in \mathscr{C}^\infty(\mathbb{R} \times \mathrm{T}^*Q)$ is a generating function for H. Let $q\colon [0,T] \to Q$ be a curve such that $c\colon t \mapsto (t, q(t))$ is an integral curve of X^γ_H. Then,*

$$S(t, q(t)) = \mathcal{A}(j^1_t S \circ q)(t) + S_0 \tag{27}$$

for some $S_0 \in \mathbb{R}$, where $\mathcal{A}$ denotes the action map (6).

Proof. Assume that $S \in \mathscr{C}^\infty(\mathbb{R} \times \mathrm{T}^*Q)$ is a generating function for H. Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} S(t, q(t)) &= \frac{\partial S(t, q(t))}{\partial t} + \frac{\partial S(t, q(t))}{\partial q^i} \dot{q}^i \\
&= \frac{\partial S(t, q(t))}{\partial q^i} \dot{q}^i - H \circ j^1_t S \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{A}(j^1_t S \circ q)(t)\,,
\end{aligned} \tag{28}$$

where we have used the Hamilton–Jacobi equation (26) on the second step, and the definition of the action map (6) on the last step. Hence,

$$S(t, q(t)) = \mathcal{A}(j^1_t S \circ q)(t) + S_0$$

for some constant S_0. □

4.2 A new approach for the Hamilton–Jacobi problem in time-independent contact Hamiltonian systems

The analogous of Theorem 4.4 for autonomous contact Hamiltonian systems was developed in [30] (see also [21]):

Theorem 4.6 (Hamilton–Jacobi Theorem for autonomous systems)**.** *Let $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$ be a contact Hamiltonian system with contact Hamiltonian vector field X_H. Consider a section γ of $\pi_Q : \mathrm{T}^*Q \times \mathbb{R} \to Q$ such that $\operatorname{Im}\gamma$ is a Legendrian submanifold of $(\mathrm{T}^*Q \times \mathbb{R}, \eta)$. Then, X_H^γ and X_H are γ-related if and only if*

$$H \circ \gamma = 0. \tag{29}$$

The problem with this approach is that it cannot be used to completely integrate the system. Indeed, Eq. (29) implies that every integral curve of $X_H \circ \gamma$ is contained in $H^{-1}(0)$. This can be solved by regarding the contact Hamiltonian system $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$ as the cocontact Hamiltonian system $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \mathrm{d}t, \eta, H)$ such that $R_t(H) = 0$ and making use of Theorem 4.4. Suppose that S is of the form $S(t, q) = \alpha(q) + \beta(t)$. Then, Eq. (26) yields

$$H \circ j^1\alpha + \mathscr{L}_{\frac{\partial}{\partial t}} \beta = 0\,, \tag{30}$$

that is,

$$H\left(q^i, \frac{\partial \alpha}{\partial q^i}, z\right) + \dot{\beta}(t) = 0\,. \tag{31}$$

With a suitable choice of α and β, one can cover energy levels distinct from $H = 0$.

Example 4.7 (The free particle with linear dissipation)**.** Consider the contact Hamiltonian system $(\mathrm{T}^*\mathbb{R} \times \mathbb{R}, \eta, H)$ with Hamiltonian function

$$H(q, p, z) = \frac{p^2}{2} + \delta z\,. \tag{32}$$

Then, Eq. (31) becomes

$$\frac{\alpha'(q)^2}{2} + \dot{\beta}(t) + \delta(\alpha(q) + \beta(t)) = 0\,. \tag{33}$$

Suppose that $\dot{\beta}(t) + \delta\beta(t) = 0$. Then,

$$\alpha(q) = -\frac{\delta}{2} q^2\,, \quad \beta(t) = \lambda e^{-\delta t}\,, \tag{34}$$

for arbitrary $\lambda \in \mathbb{R}$. Thus, $S(t, q) = \lambda e^{-\delta t} - \frac{\delta}{2} q^2$

Now,
$$X_H^\gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q} = \frac{\partial}{\partial t} + p\frac{\partial}{\partial q} = \frac{\partial}{\partial t} - \delta q\frac{\partial}{\partial q}, \tag{35}$$
whose integral curves are of the form $\sigma(t) = (t, q_0 e^{-\delta t})$. Thus, the integral curves of $X_{H|\mathrm{Im}(\gamma)}$ are given by $\gamma\circ\sigma(t) = (t, q_0 e^{-\delta t}, -\delta q_0 e^{-\delta t}, -\frac{\delta}{2} q_0^2 e^{-2\delta t} + \lambda e^{-\delta t})$.

5 Hamilton–Jacobi theory: the action-dependent approach

In Section 4 we have introduced a Hamilton–Jacobi theory for time-dependent contact Hamiltonian systems. This approach was shown to be especially useful for dealing with time-independent contact Hamiltonian systems, where time is used as a free parameter. Nevertheless, this approach has a couple of drawbacks. First, one cannot define a notion of complete solution in the same fashion as it is done for symplectic Hamiltonian systems [6]. Additionally, time-independent solutions only cover the zero-energy level.

In order to solve these problems, in this section we propose an alternative approach, considering solutions of the Hamilton–Jacobi problem depending on the action variable z. Let us consider a section γ of the bundle $\pi_Q^{t,s} : \mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R} \to \mathbb{R}\times Q\times\mathbb{R}$, locally given by
$$\begin{aligned} \gamma : \mathbb{R}\times Q\times\mathbb{R} &\longrightarrow \mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R} \\ (t,q,z) &\longmapsto (t,q,\gamma_i(t,q,z),z)\ . \end{aligned} \tag{36}$$

As in the previous approach, assume that X_H^γ and X_H are γ-related, so that the following diagram commutes:
$$\begin{tikzcd} \mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R} \arrow[r, "X_H"] \arrow[d, shift left, "\pi_Q^{t,s}"] & \mathrm{T}(\mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R}) \arrow[d, shift right, "\mathrm{T}\pi_Q^{(t,s)}"'] \\ \mathbb{R}\times Q\times\mathbb{R} \arrow[r, "X_H^\gamma"] \arrow[u, bend left, "\gamma"] & \mathrm{T}(\mathbb{R}\times Q\times\mathbb{R}) \arrow[u, bend right, "\mathrm{T}\gamma"'] \end{tikzcd}$$

Locally,
$$X_H\circ\gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + \gamma_i\frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(\gamma_i\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}\,, \tag{37}$$
and
$$\mathrm{T}\gamma\circ X_H^\gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + \left(\frac{\partial\gamma_i}{\partial t} + \frac{\partial H}{\partial p_j}\frac{\partial\gamma_j}{\partial q^i} + \left(\gamma_j\frac{\partial H}{\partial p_j} - H\right)\frac{\partial\gamma_i}{\partial s}\right)\frac{\partial}{\partial p_i} + \left(\gamma_i\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}\,, \tag{38}$$
so X_H^γ and X_H are γ-related if and only if
$$-\left(\frac{\partial H}{\partial q^i} + \gamma_i\frac{\partial H}{\partial z}\right) = \frac{\partial\gamma_i}{\partial t} + \frac{\partial H}{\partial p_j}\frac{\partial\gamma_j}{\partial q^i} + \frac{\partial\gamma_i}{\partial z}\left(\gamma_j\frac{\partial H}{\partial p_j} - H\right)\,. \tag{39}$$

Note that $\mathrm{Im}\,\gamma$ is $(n+2)$-dimensional, so it no longer makes sense to require it to be Legendrian [16]. Demanding it to be coisotropic is not enough to obtain a satisfactory Hamilton–Jacobi equation. This motivates the following.

Let us recall that, given a Jacobi manifold (M,Λ,E) and a distribution $\mathcal{D}\subseteq \mathrm{T}M$, the orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$ is given by [18, 52]
$$\mathcal{D}_x^\perp = \hat\Lambda\left(\mathcal{D}_x^\circ\right)\,,$$
where $\mathcal{D}_x^\circ = \{\alpha_x\in\mathrm{T}_x^*M \mid \alpha_x(v) = 0,\ \forall\, v\in\mathcal{D}_x\}$ denotes the annihilator. In particular, a cocontact manifold is a Jacobi manifold (see Proposition 2.7) and its morphism $\hat\Lambda$ is given by Eq. (2).

Definition 5.1. *Let (M, τ, η) be a cocontact manifold and $i\colon N \hookrightarrow M$ a submanifold. The submanifold N is said to be* ***coisotropic*** *if $\mathrm{T}N^{\perp} \subseteq \mathrm{T}N$.*

The coisotropic condition can be written in local coordinates as follows.

Lemma 5.2. *Assume that an $(n+2)$-dimensional submanifold N of a $(2n+2)$-dimensional cocontact submanifold (M, τ, η) is locally the zero set of the constraint functions $\{\phi_a\}_{a=1}^{n}$. Then, N is coisotropic if and only if the following equation holds in Darboux coordinates:*

$$\left(\frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial z}\right) \frac{\partial \phi_b}{\partial p_i} - \frac{\partial \phi_a}{\partial p_i} \left(\frac{\partial \phi_b}{\partial q^i} + p_i \frac{\partial \phi_b}{\partial z}\right) = 0\,. \tag{40}$$

Proof. Assume that (M, τ, η) is a $(2n+2)$-dimensional cocontact manifold. Let $N \subseteq M$ be a k-dimensional submanifold given locally by the zero set of functions $\phi_a : U \to \mathbb{R}$, with $a \in \{1, \ldots, 2n+1-k\}$. We have that

$$\mathrm{T}N^{\perp} = \left\langle \{Z_a\}_{a=1}^{2n+1-k} \right\rangle ,$$

where

$$Z_a = \hat{\Lambda}(\mathrm{d}\phi_a) = \left(\frac{\partial \phi_a}{\partial q^i} + p_i \frac{\partial \phi_a}{\partial z}\right) \frac{\partial}{\partial p_i} - \frac{\partial \phi_a}{\partial p_i} \left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z}\right) .$$

Therefore, N is coisotropic if and only if $Z_a(\phi_b) = 0$ for all a, b, which in Darboux coordinates yields Eq. (40).

□

Proposition 5.3. *Let γ be a section of $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ over $\mathbb{R} \times Q \times \mathbb{R}$. Then $\operatorname{Im}\gamma$ is a coisotropic submanifold if and only if*

$$\frac{\partial \gamma_i}{\partial q^j} + \gamma_j \frac{\partial \gamma_i}{\partial z} = \frac{\partial \gamma_j}{\partial q^i} + \gamma_i \frac{\partial \gamma_j}{\partial z}. \tag{41}$$

Proof. Eq. (41) is obtained by applying the previous result to the submanifold $N = \operatorname{Im}\gamma$, which is locally defined by the constraints $\phi_i = p_i - \gamma_i$. □

Now suppose that $\operatorname{Im}\gamma$ is coisotropic in Eq. (39). Then, by means of Eq. (41) we obtain

$$\frac{\partial H}{\partial q^i} + \frac{\partial \gamma_i}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial \gamma_i}{\partial q^j} + \gamma_i \left(\frac{\partial \gamma_j}{\partial z} \frac{\partial H}{\partial p_j} + \frac{\partial H}{\partial z}\right) - H \frac{\partial \gamma_i}{\partial z} = 0\,, \tag{42}$$

or, globally,

$$\mathrm{d}_Q\left(H \circ \gamma\right) + \mathscr{L}_{\frac{\partial}{\partial z}}(H \circ \gamma)\gamma + \mathscr{L}_{R_t}\gamma = (H \circ \gamma)\mathscr{L}_{\frac{\partial}{\partial z}}\gamma\,. \tag{43}$$

Theorem 5.4 (Action-dependent Hamilton–Jacobi Theorem)**.** *Let γ be a section of $\pi_Q^{t,s} : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q \times \mathbb{R}$ such that $\operatorname{Im}\gamma$ is a coisotropic submanifold of $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta)$. Then, X_H^γ and X_H are γ-related if and only if Eq. (43) holds. This equation will be called the* ***action-dependent Hamilton–Jacobi*** *equation for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$.*

Definition 5.5. *Let $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$ be a cocontact Hamiltonian system. A* ***complete solution of the action-dependent Hamilton–Jacobi problem*** *for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$ is a local diffeomorphism $\Phi\colon \mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathrm{T}^*Q \times R$ such that, for each $\lambda \in \mathbb{R}^n$,*

$$\begin{aligned} \Phi_\lambda\colon \mathbb{R} \times Q \times \mathbb{R} &\longrightarrow \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \\ \left(t, q^i, z\right) &\longmapsto \Phi\left(t, q^i, \lambda, z\right) \end{aligned} \tag{44}$$

is a solution of the action-dependent Hamilton–Jacobi problem for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$.

Let $\alpha\colon \mathbb{R}\times Q\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$, and $\pi_i\colon\mathbb{R}^n\to\mathbb{R}$ denote the canonical projections. Let us define the functions $f_i=\pi_i\circ\alpha\circ\Phi^{-1}$ on $\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R}$, so that the following diagram commutes:

$$
\begin{tikzcd}
\mathbb{R}\times Q\times\mathbb{R}\times\mathbb{R}^n \arrow[r, shift left, "\Phi"] \arrow[d, "\alpha"] & \mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R} \arrow[l, shift left, "\Phi^{-1}"] \arrow[d, "f_i"] \\
\mathbb{R}^n \arrow[r, "\pi_i"] & \mathbb{R}
\end{tikzcd}
$$

Theorem 5.6. *Let $\Phi\colon \mathbb{R}\times Q\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R}$ be a complete solution of the Hamilton–Jacobi problem for $(\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R},\tau,\eta,H)$. Then,*

(i) *For each $i\in\{1,\dots,n\}$, the function $f_i=\pi_i\circ\alpha\circ\Phi^{-1}$ is a constant of the motion. However, these functions are not necessarily in involution, i.e., $\{f_i,f_j\}\neq 0$.*

(ii) *For each $i\in\{1,\dots,n\}$, the function $\hat{f}_i=gf_i$, where g is a dissipated quantity, is also a dissipated quantity. Moreover, if $R_t(H)=0$ and taking $g=H$, these functions are in involution, i.e., $\{\hat{f}_i,\hat{f}_j\}=0$.*

Proof. Observe that

$$
\operatorname{Im}\Phi_\lambda=\bigcap_{i=1}^n f_i^{-1}(\lambda_i)\,,
$$

where $\lambda=(\lambda,\dots,\lambda_n)\in\mathbb{R}^n$. In other words,

$$
\operatorname{Im}\Phi_\lambda=\{x\in\mathbb{R}\times \mathrm{T}^*Q\times\mathbb{R}\mid f_i(x)=\lambda_i, i=1,\dots,n\}\,.
$$

Therefore, since X_H is tangent to any of the submanifolds $\operatorname{Im}\Phi_\lambda$, we deduce that

$$
X_H(f_i)=0\,.
$$

Moreover, we can compute

$$
\{f_i,f_j\}=\Lambda(\mathrm{d}f_i,\mathrm{d}f_j)-f_iR_z(f_j)+f_jR_z(f_i)\,,
$$

but

$$
\Lambda(\mathrm{d}f_i,\mathrm{d}f_j)=\hat{\Lambda}(\mathrm{d}f_i)(f_j)=0\,,
$$

since $(\mathrm{T}\operatorname{Im}\Phi_\lambda)^\perp=\hat{\Lambda}((\mathrm{T}\operatorname{Im}\Phi_\lambda)^\circ)\subset \mathrm{T}\operatorname{Im}\Phi_\lambda$, so

$$
\{f_i,f_j\}=-f_iR_z(f_j)+f_jR_z(f_i)\,. \tag{45}
$$

We already know that the product of a conserved quantity and a dissipated quantity is a dissipated quantity 3.6. Let f_i and f_j be conserved quantities and take $g=H$. Then,

$$
\begin{aligned}
\{\hat{f}_i,\hat{f}_j\}&=\{Hf_i,Hf_j\}=f_j\{Hf_i,H\}+H\{Hf_i,f_j\}-f_jHR_z(Hf_i)\\
&=-f_jH\{H,f_i\}+f_if_jHR_z(H)-f_iH\{f_j,H\}-H^2\{f_j,f_i\}+f_iH^2R_z(f_j)-f_jHR_z(Hf_i)\\
&=0\,.
\end{aligned}
$$

□

From a complete solution of the Hamilton–Jacobi problem one can reconstruct the dynamics of the system. Given an integral curve σ of the vector field X_H^γ, $\Phi_\lambda\circ\sigma$ is an integral curve of X_H, thus recovering the dynamics of the original system.

5.1 Integrable contact Hamiltonian systems

Let $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$ be a contact Hamiltonian system. Let $\Phi\colon Q \times \mathbb{R}^n \times \mathbb{R} \to \mathrm{T}^*Q \times \mathbb{R}$ be a complete solution of the Hamilton–Jacobi problem for $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$. Then,

$$\mathcal{F} = \{\mathcal{F}_\lambda = \operatorname{Im}\Phi_\lambda \mid \lambda \in \mathbb{R}^n\} \subseteq \mathrm{T}^*Q \times \mathbb{R} \tag{46}$$

is a foliation in coisotropic submanifolds.

In the symplectic case the images of the complete solution for each choice of parameters λ form a Lagrangian foliation invariant under the Hamiltonian flow. This structure is called an integrable system. In analogy, we introduce the following definition:

Definition 5.7. *Let (M, η, H) be a contact Hamiltonian system and let $\mathcal{F}$ be a foliation consisting of $(n+1)$-dimensional coisotropic leaves invariant under the flow of the Hamiltonian vector field X_H. Then we call $(M, \eta, H, \mathcal{F})$ an* ***integrable system****.*

This definition can be compared to the ones given in [2, 45]:

- In [2], Boyer proposes a concept of completely integrable system in which he insists that the Hamiltonian vector field and the constants of motion are in involution. This is a particular case of our definition, in which both the Hamiltonian and the constants of the motion do not depend on z.

- In [45], Khesin and Tabachnikov call a foliation $\mathcal{F}^n$ **co-Legendrian** when it is transverse to $\mathcal{H}$ and $\mathrm{T}\mathcal{F} \cap \mathcal{H}$ is integrable. Then they define an integrable system as a particular of a co-Legendrian foliation with some extra regularity conditions. In the case that the dimension of the leaves is $n+1$, the following proposition shows that co-Legendrian foliations are particular cases of coisotropic foliations.

Proposition 5.8. *Let $i\colon N \hookrightarrow M$ be a submanifold of a $(2n+1)$-dimensional contact manifold (M, η). If N is an $(n+1)$-dimensional co-Legendrian submanifold, then it is also a coisotropic submanifold.*

Proof. Let us write $\mathrm{T}N = \mathcal{D}_\mathcal{H} + \mathcal{E}$, where $\mathcal{D}_\mathcal{H} = \mathrm{T}N \cap \mathcal{H}$. Then, $\mathrm{T}N^\perp = \mathcal{D}_\mathcal{H}^\perp \cap \mathcal{E}^\perp$. Obviously, η vanishes in $\mathrm{T}N \cap \mathcal{H}$. Moreover, since $\mathcal{D}_\mathcal{H}$ is integrable,

$$0 = \eta([v, w]) = \iota_{[v,w]}\eta = \mathscr{L}_v \iota_w \eta - \iota_w \mathscr{L}_v \eta = -\iota_w \iota_v \mathrm{d}\eta - \iota_w \mathrm{d}\iota_v \eta = -\iota_w \iota_v \mathrm{d}\eta\,, \tag{47}$$

for any $v, w \in \mathcal{D}_\mathcal{H}$, so $\mathrm{d}\eta_{|\mathcal{D}_\mathcal{H}} = 0$. Observe that $\hat{\Lambda}_{|\mathcal{H}} = \sharp_{|\mathcal{H}}$, and $\sharp_{|\mathcal{H}}\colon \mathcal{H} \to \langle R\rangle^\circ$, $\sharp_{|\mathcal{H}}^{-1}(v) = \iota_v \mathrm{d}\eta$ is an isomorphism. Since $\mathrm{d}\eta_{|\mathcal{D}_\mathcal{H}} = 0$, $\sharp_{|\mathcal{H}}^{-1}(\mathcal{D}_\mathcal{H}) \subseteq \mathcal{D}_\mathcal{H}^\circ$. Thus, $\mathcal{D}_\mathcal{H} \subseteq \hat{\Lambda}(\mathcal{D}_\mathcal{H}^\circ) = \mathcal{D}_\mathcal{H}^\perp$. By a dimension counting argument, we can see that both spaces are equal and, thus, $\mathcal{D}_\mathcal{H} = \mathcal{D}_\mathcal{H}^\perp$.

∎

We also note that a foliation $\tilde{\mathcal{F}}$ by Legendrian submanifolds can never be invariant by the Hamiltonian flow. Indeed, let $\tilde{F} \in \tilde{\mathcal{F}}$. The leaves of $\tilde{\mathcal{F}}$ are Lagrangian, thus $\mathrm{T}\tilde{F}_0 \subseteq \ker\eta$. Since $\eta(X_H) = -H$, X_H can only be tangent to the leaves at the zero set of H, hence its flow cannot leave invariant the whole foliation.

6 Examples

6.1 Freely falling particle with linear dissipation

Consider a particle of time-dependent mass $m(t)$ which is freely falling and subject to a dissipation linear in the velocity with proportionality constant γ. The Hamiltonian function $H : \mathbb{R} \times \mathrm{T}^*\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$H(t, q, p, z) = \frac{p^2}{2m(t)} + m(t)gq + \frac{\gamma}{m(t)}z\,, \tag{48}$$

where g is the gravity. The Hamiltonian vector field corresponding to this Hamiltonian function is

$$X = \frac{\partial}{\partial t} + \frac{p}{m(t)}\frac{\partial}{\partial q} - \left(m(t)g + \frac{\gamma}{m(t)}p\right)\frac{\partial}{\partial p} + \left(\frac{p^2}{2m(t)} - m(t)gq - \frac{\gamma}{m(t)}z\right)\frac{\partial}{\partial s}\,.$$

Its integral curves $(t(r), q(r), p(r), z(r))$ satisfy the system of differential equations

$$\dot t = 1\,, \qquad \dot q = \frac{p}{m(t)}\,, \qquad \dot p = -m(t)g - \frac{\gamma}{m(t)}p\,, \qquad \dot z = \frac{p^2}{2m(t)} - m(t)gq - \frac{\gamma}{m(t)}z\,.$$

Combining the second and third equations, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(m(t)\dot q) = -m(t)g - \gamma\dot q\,.$$

In order to solve the Hamilton–Jacobi problem, we look for a conserved quantity linearly independent from the Hamiltonian, i.e., a function f on $\mathrm{T}\mathbb{R}\times\mathbb{R}$ such that $X_H(f) = 0$. For the sake of simplicity, one can assume that f does not depend on q or z. Indeed, one can verify that

$$f(t,q,p,z) = e^{\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\left(p + g e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\int_1^t e^{-\int_1^u -\frac{\gamma}{m(s)}\mathrm{d}s} m(u)\mathrm{d}u\right)$$

is a conserved quantity. We can thus express the momentum p as a function of t and the real parameter λ, namely,

$$P(t,\lambda) = e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\left(\lambda - g e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\int_1^t e^{\int_1^u \frac{\gamma}{m(s)}\mathrm{d}s} m(u)\mathrm{d}u\right)\,,$$

and obtain a complete solution of the Hamilton–Jacobi problem for H:

$$\phi_\lambda : (t,q,z) \longmapsto \left(t, q, p \equiv e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\left(\lambda - g e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\int_1^t e^{\int_1^u \frac{\gamma}{m(s)}\mathrm{d}s} m(u)\mathrm{d}u\right), z\right)\,. \tag{49}$$

In this case, Eq. (41) holds trivially, so Φ_λ is coisotropic.

In addition, one can verify that

$$k(t,q,p,z) = p + g e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\int_1^t e^{\int_1^u \frac{\gamma}{m(s)}\mathrm{d}s} m(u)\mathrm{d}u = e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s} f(t,q,p,z)$$

is a dissipated quantity, that is, $\{k, H\} - \mathscr{L}_{R_t} k = 0$.

6.2 Damped forced harmonic oscillator

Consider the product manifold $\mathbb{R}\times\mathrm{T}Q\times\mathbb{R}$ with natural coordinates (t,q,p,z). The Hamiltonian function

$$H(t,q,p,s) = \frac{p^2}{2m} + \frac{k}{2}q^2 - qF(t) + \frac{\gamma}{m}s$$

describes a harmonic oscillator with elastic constant k, friction coefficient γ and subjected to an external time-dependent force $F(t)$ [16].

The Hamiltonian vector field is

$$X = \frac{\partial}{\partial t} + \frac{p}{m}\frac{\partial}{\partial q} + \left(-kq + F(t) - \frac{p}{m}\gamma\right)\frac{\partial}{\partial p} + \left(\frac{p^2}{2m} - \frac{k}{2}q^2 + qF(t) - \frac{\gamma}{m}s\right)\frac{\partial}{\partial s}\,,$$

and its integral curves $(t(r), q(r), p(r), s(r))$ satisfy

$$\dot t = 1\,, \qquad \dot q = \frac{p}{m}\,, \qquad \dot p = -kq + F(t) - \frac{p}{m}\gamma\,, \qquad \dot s = \frac{p^2}{2m} - \frac{k}{2}q^2 + qF(t) - \frac{\gamma}{m}s\,.$$

Combining the second and the third equations above, we obtain the second-order differential equation

$$m\ddot{q} + \gamma\dot{q} + kq = F(t)\,,$$

which corresponds to a damped forced harmonic oscillator.

One can check that the function

$$g(t,q,p,z) = e^{\frac{\gamma t}{2m}}\left(\frac{\sinh\left(\frac{\kappa t}{2m}\right)(2kmq+\gamma p)}{\kappa} + p\cosh\left(\frac{\kappa t}{2m}\right)\right) - \int_1^t \left(\cosh\left(\frac{\kappa s}{2m}\right) + \frac{\gamma\sinh\left(\frac{\kappa s}{2m}\right)}{\kappa}\right)\mathrm{d}s\,, \tag{50}$$

where $\kappa = \sqrt{\gamma^2 - 4km}$, is a conserved quantity. It is worth noting that, since $\sinh = x + \mathcal{O}(x^3)$ and $\cosh x = 1 + \mathcal{O}(x^2)$ near $x = 0$, $\sinh(ix) = i\sin x$ and $\cosh(ix) = \cos x$, the equation above is well-defined for any value of $\kappa \in \mathbb{C}$. Thus, we can write p in terms of t, q, z and a real parameter λ as

$$P(t,q,\lambda,z) = \frac{e^{-\frac{\gamma t}{2m}}\left(\kappa\int_1^t e^{\frac{s\gamma}{2m}}F(s)\left(\cosh\left(\frac{\kappa s}{2m}\right) + \frac{\gamma\sinh\left(\frac{\kappa s}{2m}\right)}{\kappa}\right)\mathrm{d}s - 2kmqe^{\frac{\gamma t}{2m}}\sinh\left(\frac{\kappa t}{2m}\right) + \kappa\lambda\right)}{\gamma\sinh\left(\frac{\kappa t}{2m}\right) + \kappa\cosh\left(\frac{\kappa t}{2m}\right)}\,,$$

and obtain a complete solution of the Hamilton–Jacobi problem:

$$\Phi_\lambda\colon (t,q,\lambda,z) \mapsto (t,q,p \equiv P(t,q,\lambda,z), z)\,. \tag{51}$$

Obviously Eq. (41) is satisfied, hence Φ_λ is coisotropic.

In addition,

$$\begin{aligned} f(t,q,p,z) = e^{-\frac{\gamma t}{m}}\Bigg[&e^{\frac{\gamma t}{2m}}\left(\frac{\sinh\left(\frac{\kappa t}{2m}\right)(2kmq+\gamma p)}{\kappa} + p\cosh\left(\frac{\kappa t}{2m}\right)\right) \\ &- \int_1^t e^{\frac{s\gamma}{2m}}F(s)\left(\cosh\left(\frac{\kappa s}{2m}\right) + \frac{\gamma\sinh\left(\frac{\kappa s}{2m}\right)}{\kappa}\right)\mathrm{d}s\Bigg]\,, \end{aligned}$$

is a dissipated quantity.

7 Conclusions and outlook

The main contributions of the present paper are the following ones:

- We have obtained two different Hamilton–Jacobi equations for time-dependent contact Hamiltonian systems: the so-called action independent and action-dependent approaches. In particular, the action-independent approach is useful for time-independent contact Hamiltonian systems, where the use of time as a free parameter allows to integrate the system at non-zero energy levels.
- The action-dependent approach permits to introduce a natural notion of complete solutions to the Hamilton–Jacobi problem. Each of these complete solutions is associated with a family of n independent dissipated quantities in involution (where n is the number of degrees of freedom of the system). Moreover, the image of a complete solution is a coisotropic submanifold.
- We introduce a new notion of integrable system in a contact manifold, taking into account the dynamics given by the Hamiltonian vector field, and extending the concept of complete solution. This allows us to study the dynamics outside the zero-energy level.

In future works, we will deepen in the study of infinitesimal symmetries [38], and in the reduction problem, in the Hamilton–Jacobi equations for the evolution vector field and its possible applications to thermodynamics as well as the extension to higher order systems. The study of the discrete Hamilton–Jacobi equations and applications to the construction of geometric integrators is also on the agenda.

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