The geometry of dissipation

Liouville–Arnol'd theorem for contact Hamiltonian systems

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Outline of the presentation

1. Main theorem
2. Symplectization
3. Proof
4. Example
5. Other notions of integrability

- Let $M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+ \colon f_\alpha(x) = r\Lambda_\alpha\}$.

Theorem (Colombo, de León, Lainz, L. G., 2023)

Let (M, η) be a $(2n+1)$-dimensional contact manifold. Suppose that $f_0, f_1, \ldots, f_n$ are functions in involution such that $(\mathrm{d}f_\alpha)$ has rank at least n. Then, $M_{\langle\Lambda\rangle_+}$ is invariant by the Hamiltonian flow of f_α and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$.

Moreover, there is a neighborhood U of $M_{\langle\Lambda\rangle_+}$ such that

1. *There exists coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on U such that the equations of motion are given by*
$$\dot{y}^\alpha = \Omega^\alpha(\tilde{A}_i)\,, \quad \dot{\tilde{A}}_i = 0\,, \quad \alpha \in \{0, \ldots, n\}\,, \, i \in \{1, \ldots, n\}\,.$$
2. *There exists a conformal change $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$, i.e. $\tilde{\eta} = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.*

Steps of the proof

1. Symplectize (M, η) and f_α, obtaining an exact symplectic manifold (M^Σ, θ) and homogeneous functions in involution f_α^Σ.
2. Prove a Liouville–Arnold theorem for exact symplectic manifolds with homogeneous functions in involution.
3. "Un-symplectize" the action-angle coordinates $(y_\Sigma^\alpha, A_\alpha^\Sigma)$ on M^Σ, yielding functions (y^α, A_Σ) on M.
4. Introduce action-angle coordinates $(y^\alpha, \tilde{A}_i)$ on M, where $\tilde{A}_i = -\frac{A_i}{A_0}$.

Exact symplectic manifolds: Liouville geometry

Definition

An **exact symplectic manifold** is a pair (M, θ), where M is a manifold and θ a one-form on N such that $\omega = -\mathrm{d}\theta$ is a symplectic form on M.

- The **Liouville vector field** Δ of (M, θ) is given by

$$\iota_\Delta \omega = -\theta.$$

- A tensor T is called **homogeneous of degree** n if $\mathcal{L}_\Delta T = nT$.

Symplectization of contact manifolds

Definition

Let (M, η) be a contact manifold and (M^Σ, θ) an exact symplectic manifold. A **symplectization** is a fibre bundle $\Sigma \colon M^\Sigma \to M$ such that

$$\sigma \Sigma^* \eta = \theta,$$

for a function σ on M^Σ called the **conformal factor**.

Symplectization of contact manifolds

$$\begin{array}{c}\text{Category of contact manifolds}\\ \updownarrow \\ \text{Category of exact symplectic manifolds}\end{array}$$

- Contact distribution $\ker \eta \longleftrightarrow$ symplectic potential θ
- Functions $\longleftrightarrow$ homogeneous functions of degree 1
- Hamiltonian vector fields $\longleftrightarrow$ Hamiltonian vector fields, homogeneous of degree 0

Symplectization of contact manifolds

Theorem

Given a symplectization $\Sigma\colon (M^\Sigma, \theta) \to (M, \eta)$ with conformal factor σ, there is a bijection between functions f on M and homogeneous functions of degree 1 f^Σ on M^Σ such that

$$\Sigma_* \left(X_{f^\Sigma} \right) = X_f \,.$$

This bijection is given by

$$f^\Sigma = \sigma \Sigma^* f \,.$$

Moreover, one has

$$\left\{ f^\Sigma, g^\Sigma \right\}_\theta = \{f, g\}_\eta^\Sigma \,.$$

Symplectization of contact manifolds

Example

$\Sigma = \pi_1 \colon (M \times \mathbb{R}^+, \theta = r\eta) \to (M, \eta)$ is a symplectization with conformal factor $\sigma = r$, for r the global coordinate on $\mathbb{R}^+$.

Liouville–Arnol'd theorem for exact symplectic manifolds

- We want to obtain action-angle coordinates $(\varphi^\alpha_\Sigma, J^\Sigma_\alpha)$ on (M^Σ, θ) in order to define functions $(\varphi^\alpha, J_\alpha)$ on (M, η)
- We need homogeneous objects on (M^Σ, θ) so that they have a correspondence with objects on (M, η).
- However, the classical Liouville–Arnold theorem does not take into account the homogeneity of θ and f^Σ_α.
- Moreover, we need to consider non-compact level sets of f^Σ_α.

Liouville–Arnol'd theorem for exact symplectic manifolds

Definition

A **homogeneous integrable system** is a triple (M, θ, F), where (M, θ) is an exact symplectic manifold and $F = (f_1, \ldots, f_n) \colon M \to \mathbb{R}^n$ is a map such that the functions $f_1, \ldots, f_n$ are in independent, in involution and homogeneous of degree 1 on M. The functions $f_1, \ldots, f_n$ are called **integrals**.

Liouville–Arnol'd theorem for exact symplectic manifolds

Consider a homogeneous integrable system (M, θ, F). Let U be an open neighbourhood of the level set $M_\Lambda = F^{-1}(\Lambda)$ (with $\Lambda \in \mathbb{R}^n$) such that:

1. $f_1, \ldots, f_n$ have no critical points in U,
2. the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ are complete,
3. the submersion $F\colon U \to \mathbb{R}^n$ is a trivial bundle over a domain $V \subseteq \mathbb{R}^n$.

Liouville–Arnol'd theorem for exact symplectic manifolds

Theorem (Colombo, de León, Lainz, L. G., 2023)

Let (M^{2n}, θ, F) be a homogeneous integrable system with $F = (f_1, \dots, f_n)$. Given $\Lambda \in \mathbb{R}^n$, suppose that $M_\Lambda = F^{-1}(\Lambda)$ is connected, and assume the statements from the previous slide. Then, $U \cong \mathbb{T}^k \times \mathbb{R}^{n-k} \times V$ and there is a chart $(\hat{U} \subseteq U; y^i, A_i)$ of M s.t.

1. $A_i = M_i^j f_j$, *where* M_i^j *are homogeneous functions of degree* 0 *depending only on* $f_1, \dots, f_n$,
2. $\theta = A_i \mathrm{d}y^i$,
3. $X_{f_i} = N_i^j \frac{\partial}{\partial y^j}$, *with* (N_i^j) *the inverse matrix of* (M_i^j).

Lemma

Let M be an n-dimensional manifold, and let $X_1, \ldots, X_n \in \mathfrak{X}(M)$ be linearly independent vector fields. If these vector fields are pairwise commutative and complete, then M is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $k \leq n$, where $\mathbb{T}^k$ denotes the k-dimensional torus.

Lemma

Let (M^{2n}, θ) *be an exact symplectic manifold and* $f_1, \dots, f_n \in \mathscr{C}^\infty(M)$ *s.t.*

1. $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \neq 0$,
2. $\{f_i, f_j\} = 0 \ \forall\, i, j$,
3. $X_{f_1}, \dots, X_{f_n}$ *are complete.*

Then, there exists n *functions* $g_i = M_i^j f_j \in \mathscr{C}^\infty(M)$ *which verify* ①, ②, ③, *and*

4. M_i^j *for* $i, j \in 1, \dots, n$ *are homogeneous functions of degree* 0 *and they depend only on* $f_1, \dots, f_n$,
5. $X_{g_1}, \dots, X g_k$ *are infinitesimal generators of* $\mathbb{S}^1$*-actions and their flows have period 1,*
6. $X_{g_{k+1}}, \dots, X_{g_n}$ *are infinitesimal generators of* $\mathbb{R}$*-actions.*

Lemma

Let $\pi \colon P \to M$ be a G-principal bundle over a connected and simply connected manifold. Suppose there exists a connection one-form A such that the horizontal distribution H is integrable. Then $\pi \colon M \to N$ is a trivial bundle and there exists a global section $\chi \colon N \to M$ such that $\chi^* A = 0$.

Proof of the theorem for exact symplectic manifolds

- We know that M_Λ is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$.
- W.l.o.g., assume that $X_{f_1}, \ldots, X_{f_k}$ are infinitesimal generators of $\mathbb{S}^1$-actions with period 1, and that $X_{g_{k+1}}, \ldots, X_{g_n}$ are infinitesimal generators of $\mathbb{R}$-actions.
- Let $\mathfrak{L} = \ker \theta$ and $\overline{U} = \{x \in U \mid f_i(x) \neq 0 \ \forall\, i \text{ and } \theta(x) \neq 0\}$.
- Since $F\colon U \to V$ is a trivial bundle, $U \cong V \times \mathbb{T}^k \times \mathbb{R}^{n-k}$ can be endowed with a Riemannian metric g, given by the product of flat metrics in $V \subseteq \mathbb{R}^n$, $\mathbb{T}^k$ and $\mathbb{R}^{n-k}$, which is flat and invariant by the Lie group action of $\mathbb{T}^k \times \mathbb{R}^{n-k}$.

Proof of the theorem for exact symplectic manifolds

- The distribution
$$\mathfrak{L}^\theta = \left(\mathfrak{L} \cap \langle X_{f_i} \rangle_{i=1}^n\right)^{\perp_g} \cap \mathfrak{L}$$
is
  1. invariant by the Lie group action of $\mathbb{T}^k \times \mathbb{R}^{n-k}$,
  2. contained in $\mathfrak{L}$,
  3. complementary to the vertical bundle:
$$\mathfrak{L}^\theta_x \oplus \langle X_{f_i}(x) \rangle_{i=1}^n = \mathrm{T}_x M\,, \quad \forall x \in \overline{U}\,.$$
- Moreover, $F\colon \overline{U} \to \overline{U}/(\mathbb{T}^k \times \mathbb{R}^{n-k})$ is a principal bundle and $\mathfrak{L}^\theta$ is a principal connection with connection one-form θ.
- The fact that $\theta \wedge \mathrm{d}\theta = 0$ implies that $\mathfrak{L}$ is integrable.
- Since it is the orthogonal complement of $\mathfrak{L}$ w.r.t. a flat metric, $\mathfrak{L}^\theta$ is integrable.

Proof of the theorem for exact symplectic manifolds

- Let $\hat{U} \subseteq \overline{U}$ be an open subset of $\overline{U}$ such that $\hat{V} = F(\hat{U})$ is simply connected.
- Then, there exists a global section χ of $F \colon \hat{U} \to \hat{V} \cong \hat{U}/(\mathbb{T}^k \times \mathbb{R}^{n-k})$ such that $\chi^*\theta = 0$.
- Let $\Phi \colon \mathbb{T}^k \times \mathbb{R}^{n-k} \times M \to M$ denote the action defined by the flows of X_{f_i}.
- For each point $x \in M_\Lambda = F^{-1}(\Lambda)$, the angle coordinates $(y^i(x))$ are determined by
$$\Phi(y^i(x), \chi(F(x))) = x\,.$$
- Notice that (y^i, f_i) are coordinates in $\hat{U}$ adapted to the foliation of M in M_Λ.

Proof of the theorem for exact symplectic manifolds

- In these coordinates,

$$\theta = A_i \mathrm{d}y^i + B^i \mathrm{d}f_j \,, \quad X_{f_i} = \frac{\partial}{\partial y^i} \,,$$

- Contracting θ with X_{f_i} yields $A_i = f_i$.
- Finally, notice that $\operatorname{Im} \chi = \cap_{i=1}^n (y^i)^{-1}(\mu_i)$. Hence,

$$0 = \chi^* \theta = B^i \mathrm{d}f_i \,.$$

- Since μ_i's are arbitrary values of y^i, the functions B^i are identically zero on all the manifold M and $\theta = f_i \mathrm{d}y^i$.

□

Construction of action-angle coordinates

In order to construct action-angle coordinates in a neighbourhood U of M_Λ, one has to carry out the following steps:

1. Fix a section χ of $F\colon U \to V$ such that $\chi^*\theta = 0$.
2. Compute the flows $\phi_t^{X_{f_i}}$ of the Hamiltonian vector fields X_{f_i}.
3. Let $\Phi\colon \mathbb{R}^n \times M \to M$ denote the action of $\mathbb{R}^n$ on M defined by the flows, namely,
$$\Phi(t_1, \ldots, t_n; x) = \phi_{t_1}^{X_{f_1}} \circ \cdots \circ \phi_{t_n}^{X_{f_n}}(x)\,.$$
4. It is well-known that the isotropy subgroup $G_{\chi(\Lambda)(\Lambda)} = \{g \in \mathbb{R}^n \mid \Phi(g, \chi(\Lambda)) = \chi(\Lambda)\}$, forms a lattice (that is, a $\mathbb{Z}$-submodule of $\mathbb{R}^n$). Pick a $\mathbb{Z}$-basis $\{e_1, \ldots, e_m\}$, where m is the rank of the isotropy subgroup.

Construction of action-angle coordinates

5. Complete it to a basis $\mathcal{B} = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ of $\mathbb{R}^n$.
6. Let (M_i^j) denote the matrix of change from the basis $\{X_{f_i}(\chi(\Lambda))\}$ of $\mathrm{T}_{\chi(\Lambda)} M_\Lambda \simeq \mathbb{R}^n$ to the basis $\{\partial/\partial y^i\}$. The action coordinates are the functions $A_i = M_i^j f_j$.
7. The angle coordinates (y^i) of a point $x \in M$ are the solutions of the equation

$$x = \Phi\left(y^i e_i; \chi \circ F(x)\right).$$

Definition

A **completely integrable contact system** is a triple (M, η, F), where (M, η) is a contact manifold and $F = (f_0, \ldots, f_n)\colon M \to \mathbb{R}^{n+1}$ is a map such that

1. $f_0, \ldots, f_n$ are in involution, i.e., $\{f_\alpha, f_\beta\} = 0\ \forall\, \alpha, \beta$,
2. $\operatorname{rank} \mathrm{T}F \geq n$ on a dense open subset $M_0 \subseteq M$.

The functions $f_0, \ldots, f_n$ are called **integrals**.

Proposition

Let (M, η, H) be a contact Hamiltonian system. Suppose that $\Sigma\colon M^\Sigma \to M$ is a symplectization such that $\theta = \sigma(\Sigma^*\eta)$ is the symplectic potential on M^Σ. Then, a function f on M is a dissipated quantity with respect to (M, η, H) if and only if f^Σ is a conserved quantity with respect to $(M^\Sigma, \theta, H^\Sigma)$.

Proposition

Let (M, η) be a contact manifold. Suppose that $\Sigma\colon M^\Sigma \to M$ is a symplectization such that $\theta = \sigma(\Sigma^*\eta)$ is the symplectic potential on M^Σ. Then, $(M^\Sigma, \theta, F^\Sigma)$, with $F^\Sigma = -\sigma(\Sigma^* F)$, is a homogeneous integrable system if and only if (M, η, F) is a completely integrable contact system.

Assumptions

1. Assume that the Hamiltonian vector fields $X_{f_0}, \ldots, X_{f_n}$ are complete.
2. Given $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $B \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be an open neighbourhood of Λ.
3. Let $\pi\colon U \to M_{\langle \Lambda \rangle_+}$ be a tubular neighbourhood of $M_{\langle \Lambda \rangle_+}$ such that $F|_U\colon U \to B$ is a trivial bundle over a domain $V \subseteq B$.

Theorem (Colombo, de León, Lainz, L. G., 2023)

Let (M, η, F) be a completely integrable contact system, where $F = (f_0, \dots, f_n)$. Consider the assumptions of the previous slide. Then:

1. $M_{\langle\Lambda\rangle_+}$ *is coisotropic, invariant by the Hamiltonian flow of* f_α*, and diffeomorphic to* $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ *for some* $k \leq n$.
2. *There exist coordinates* $(y^0, \dots, y^n, \tilde{A}_1, \dots, \tilde{A}_n)$ *on* U *such that the Hamiltonian vector fields of the functions* f_α *read*
$$X_{f_\alpha} = \overline{N}_\alpha^\beta X_{f_\beta}\,,$$
where $\overline{N}_\alpha^\beta$ *are functions depending only on* $\tilde{A}_1, \dots, \tilde{A}_n$.
3. *There exists a nowhere-vanishing function* $A_0 \in \mathscr{C}^\infty(U)$ *and a conformally equivalent contact form* $\tilde{\eta} = \eta / A_0$ *such that* $(y^i, \tilde{A}_i, y^0)$ *are Darboux coordinates for* $(M, \tilde{\eta})$*, namely,* $\tilde{\eta} = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.

Sketch of the proof

1. Consider the symplectization $\Sigma\colon M^\Sigma = M \times \mathbb{R}_+ \to M$, where $\Sigma = \pi_1$ is the canonical projection.
   - $\{f_\alpha, f_\beta\} = 0 \Rightarrow \ \{f_\alpha^\Sigma, f_\beta^\Sigma\} = 0$.
   - X_{f_α} complete $\Rightarrow \ X_{f_\alpha^\Sigma}$ complete.
   - $\operatorname{rank} \mathrm{d} f_\alpha \geq n \Rightarrow \ \operatorname{rank} \mathrm{d}(\underbrace{\sigma \Sigma^* f_\alpha}_{f_\alpha^\Sigma}) \geq n+1$.
   - $\Sigma\big((F^\Sigma)^{-1}(\Lambda)\big) = \left\{x \in M \mid \exists s \in \mathbb{R}^+\colon \ F(x) = \frac{\Lambda}{s}\right\} = M_{\langle\Lambda\rangle_+}$.
   - $X_{f_\alpha^\Sigma}$ are tangent to $(F^\Sigma)^{-1}(\Lambda) \Rightarrow X_{f_\alpha}$ are tangent to $M_{\langle\Lambda\rangle_+}$.
   - X_{f_α} commute and are tangent to $M_{\langle\Lambda\rangle_+} \Rightarrow M_{\langle\Lambda\rangle_+} \simeq \mathbb{T}^k \times \mathbb{R}^{n+1-k}$.
   - $F\colon U \to B$ is a trivial bundle $\Rightarrow F^\Sigma\colon \Sigma^{-1} U \to B$ is a trivial bundle.

   $\therefore$ We can apply the theorem for exact symplectic manifolds to obtain action-angle coordinates $(y^\alpha_\Sigma, A^\Sigma_\alpha)$ on $\Sigma^{-1}(U)$.

Sketch of the proof

2. In these coordinates,

$$\theta = A_\alpha^\Sigma \mathrm{d} y_\Sigma^\alpha \,, \qquad A_\alpha^\Sigma = M_\alpha^\beta f_\beta^\Sigma \,,$$

and

$$X_{f_\alpha^\Sigma} = N_\alpha^\beta \frac{\partial}{\partial y_\Sigma^\beta} \,, \quad (N_\beta^\alpha) = (M_\beta^\alpha)^{-1} \,.$$

Due to the homogeneity, there are functions $y^\alpha, A_\alpha, \overline{M}_\alpha^\beta$ and $\overline{N}_\alpha^\beta$ on M such that

$$A_\alpha^\Sigma = -\sigma \left(\Sigma^* A_\alpha \right) \,, \qquad y_\Sigma^\alpha = \Sigma^* y^\alpha \,,$$

$$M_\alpha^\beta = \Sigma^* \overline{M}_\alpha^\beta \,, \qquad N_\alpha^\beta = \Sigma^* \overline{N}_\alpha^\beta \,.$$

Sketch of the proof

3. Since $\sigma\left(\Sigma^*\eta\right) = \theta$, the contact form is given by
$$\eta = A_\alpha \mathrm{d}y^\alpha\,.$$
and
$$f_\alpha = \overline{M}_\alpha^\beta A_\beta\,, \quad X_{f_\alpha} = \overline{N}_\alpha^\beta \frac{\partial}{\partial y^\beta}\,,$$
4. Since $\Lambda \neq 0$, there is at least one nonvanishing f_α. Hence, there is at least one nonvanishing A_α. W.l.o.g. , assume that $A_0 \neq 0$. Then, $(y^i, \tilde{A}_i = -A_i/A_0, y^0)$ are Darboux coordinates for
$$\tilde{\eta} = \frac{1}{A_0}\eta = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i\,,$$

An example

- Let $M = \mathbb{R}^3 \setminus \{0\}$ with canonical coordinates (q, p, z), and $\eta = \mathrm{d}z - p\mathrm{d}q$.
- The functions $h = p$ and $f = z$ are in involution.
- Let $F = (h, f) \colon M \to \mathbb{R}^2$.
- $\operatorname{rank} \mathsf{T}F = 2$, and thus (M, η, F) is a completely integrable contact system.

An example

- Hypothesis of the theorem are satisfied:
  1. The Hamiltonian vector fields
  $$X_h = \frac{\partial}{\partial q}\,, \quad X_f = -p\frac{\partial}{\partial p} - z\frac{\partial}{\partial z}$$
  are complete,
  2. Since $F\colon (q, p, z) \mapsto (p, z)$ is the canonical projection, $F\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is a trivial bundle.

An example

- Consider the trivial symplectization $\Sigma\colon M^\Sigma = M \times \mathbb{R}_+ \to M$, with bundle coordinates (q, p, z, r) and conformal factor $\sigma = r$.
- Therefore, $\theta = r\mathrm{d}z - rp\mathrm{d}q$ is the symplectic potential on M^Σ, and the symplectizations of h and f are $h^\Sigma = -rp$ and $f^\Sigma = -rz$. Their Hamiltonian vector fields are

$$X_{h^\Sigma} = \frac{\partial}{\partial q}\,, \quad X_{f^\Sigma} = -p\frac{\partial}{\partial p} - z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r}\,.$$

- Consider a section $\chi\colon \mathbb{R}^2 \to M^\Sigma$ of $F^\Sigma = (h^\Sigma, f^\Sigma)$ such that $\chi^*\theta = 0$. For instance, one can choose $\chi(\Lambda_1, \Lambda_2) = \left(0, \frac{\Lambda_1}{\Lambda_2}, 1, \Lambda_2\right)$ in the points where $\Lambda_2 \neq 0$.

An example

- The Lie group action $\Phi\colon \mathbb{R}^2 \times M^\Sigma \to M^\Sigma$ defined by the flows of X_{h^Σ} and X_{f^Σ} is given by

$$\Phi(t, s; q, p, z, r) = \left(q + t, pe^{-s}, ze^{-s}, re^{s}\right) ,$$

whose isotropy subgroup is the trivial one.

- The angle coordinates (y^0_Σ, y^1_Σ) of a point $x \in M^\Sigma$ are determined by

$$\Phi\left(y^0_\Sigma, y^1_\Sigma, \chi(F(x))\right) = x \,.$$

- If the canonical coordinates of x are (q, p, z, r), then

$$y^0_\Sigma = q \,, \quad y^1_\Sigma = -\log z \,.$$

An example

- Since the isotropy subgroup is trivial, the action coordinates coincide with the functions in involution, namely,

$$A_0^\Sigma = h^\Sigma = -rp\,, \quad A_1^\Sigma = f^\Sigma = -rz\,.$$

- Projecting to M yields the functions

$$y^0 = q\,, \quad y^1 = -\log z\,, \quad A_0 = h = p\,, \quad A_1 = f = z\,.$$

An example

- The action coordinate is

$$\tilde{A} = -\frac{A_0}{A_1} = -\frac{p}{z}$$

In the coordinates $(y^0, y^1, \tilde{A})$ the Hamiltonian vector fields reads

$$X_h = \frac{\partial}{\partial y^0}, \quad X_f = \frac{\partial}{\partial y^1},$$

and there is a conformal contact form given by

$$\tilde{\eta} = -\frac{1}{A_1}\eta = \mathrm{d}y^1 - \tilde{A}\mathrm{d}y^0 .$$

An example

- Similarly,
$$\chi(\Lambda_1, \Lambda_2) = \left(\frac{\Lambda_2}{\Lambda_1}, 1, \frac{\Lambda_2}{\Lambda_1}, \Lambda_1 \right)$$
is a section of F^{Σ} in the points where $\Lambda_1 \neq 0$.
- Performing analogous computations as above one obtains the action-angle coordinates
$$\hat{y}^0 = q - \frac{z}{p}\,, \quad \hat{y}^1 = -\log p\,, \quad \hat{A} = -\frac{z}{p}\,,$$
such that
$$X_h = \frac{\partial}{\partial \hat{y}^0}\,, \quad X_f = \frac{\partial}{\partial \hat{y}^1}\,, \quad \hat{\eta} = -\frac{1}{p}\eta = \mathrm{d}\hat{y}^0 - \hat{A}\mathrm{d}\hat{y}^1\,.$$

Other notions of integrability

- Khesin and Tabachnikov, Liberman, Banyaga and Molino, Lerman, etc. have defined notions of contact complete integrability which are geometric but not dynamical, e.g. a certain foliation over a contact manifold.
- Boyer considers the so-called good Hamiltonians H, i.e., $R(H) = 0 \rightsquigarrow$ no dissipated quantities, "symplectic" dynamics.
- Miranda considered integrability of the Reeb dynamics when R is the generator of an S^1-action.
- We are interested in complete integrability of contact Hamiltonian dynamics.

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