Darboux theorem for homogeneous contact forms

Summer school on Geometry, Dynamics & Field Theory 2025

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There are several situations in geometry and physics in which a ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{R}$, ...) grading appears:

* The algebra of differential forms with the wedge product.
* The spin of particles.
* Intensive/extensive variables in thermodynamics
* Symplectisation/Poissonisation of contact/Jacobi mfolds.
* Supermanifolds
* Higher tangent bundles

Why homogeneity?

<u>Theorem (Euler):</u> Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function. The following statements are equivalent:

i) f is k-homogeneous ($k \in \mathbb{Z}$), namely

$$f(tx^1, \dots, tx^n) = t^k f(x^1, \dots, x^n) \qquad \forall t \in \mathbb{R} \setminus \{0\}.$$

ii) f is a solution of the PDE

$$k \cdot f = \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}.$$

In other words, homogeneous functions are eigenfunctions of

$$X = \sum_{i=1}^{n} x^i \, \partial_{x^i} . \qquad (*)$$

In particular, linear = 1-homogeneous

We can extend this idea to manifolds by considering a vector field X that is locally of the form $(*)$ in some coords.

Def.: A vector field ∇ on a manifold M is called a weight vector field if in a neighbourhood of every point of M there are local coordinates (x^a) such that

$$\nabla = \sum_{a=1}^{n} w_a \, x^a \, \partial_{x^a} \, ,$$

where $w_a =: \deg(x^a) \in \mathbb{R}$ is called the weight of x^a.

Such coordinates are called homogeneous coordinates.

The pair (M, ∇) is called a homogeneity manifold.

Def.: Let (M, ∇) be a homogeneity manifold.

A tensor field A on M is called w-homogeneous ($w \in \mathbb{R}$) if

$$\mathcal{L}_{\nabla} A = w \cdot A.$$

Examples of homogeneity manifolds

* A vector bundle $\pi : E \to M$ and the Euler vector field ∇_E (the generator of homotheties on the fibers).
  In bundle coords., $\pi : (x^i, y^a) \mapsto (x^i)$,

$$\nabla_E = \sum_a y^a \partial_{y^a} .$$

* The second-order tangent bundle

$$\tau : T^2M \ni (x^i, \dot{x}^i, \ddot{x}^i) \longmapsto (x^i) \in M$$

with $\deg(x^i) = 0$, $\deg(\dot{x}^i) = 1$, $\deg(\ddot{x}^i) = 2$.

* An exact symplectic manifold $(M, \omega = d\theta)$ with a Liouville vector field ∇, i.e.

$$\mathcal{L}_{\nabla} \omega = \omega .$$

* Weight vector fields with non-integer weights appear in BH thermodynamics

↳ F. Belgiorno, "Quasi-homogeneous thermodynamics and black holes", J. Math. Phys. 44, 1089 (2003)

Let (M, ∇) be a homogeneity mfold.

There are two different situations on an open subset $U \subseteq M$

* $\nabla\big|_U \neq 0$,

* $\exists\ x_0 \in U$ s.t. $\nabla(x_0) = 0$.

Remark: Any nowhere-vanishing vector field $X \in \mathfrak{X}(M)$ is a weight vector field. However, its weights are not canonical.

Indeed, since X is nowhere zero, $\exists$ local coords. (x^a) such that $X = \partial_{x^1}$. For any $\{w_1, \dots, w_n\} \subset \mathbb{R}$ with $w_1 \neq 0$, we can def. a new system of coords.

$$y^1 = e^{w_1 x^1}, \qquad y^i = e^{w_i x^1} x^i, \quad 2 \leq i \leq n$$

So that

$$X = \sum_{a=1}^{n} w_a y^a \partial_{y^a}, \quad \text{i.e.} \qquad \deg(y^a) = w_a.$$

On the other hand, in a neighbourhood of any point at which a weight vector field vanishes, its weights are canonical.

Proposition (Grabowska & Grabowski, 2024): $\nabla \in \mathfrak{X}(M)$ is a weight vector field on M iff $T_{x_0}X$ is diagonal $\forall x_0 \in M$ s.t. $\nabla(x_0) = 0$.

Let (x^a) be a system of homog. coords. around x_0, i.e.

$$\nabla = \sum_a w_a x^a \partial_{x^a}, \quad \text{with} \quad \Gamma := \{w_1, \dots, w_n\} \subset \mathbb{R}.$$

Then, any other system of homog. coords. around x_0 has weights in Γ.

Homogeneous Poincaré Lemma (Grabowska & Grabowski, '24):

Let (M, ∇) be a homogeneity mfold. Let $\omega \in \Omega^k(M)$ be a λ-homogeneous k-form ($k>0$). In a nbh. of $x_0 \in M$, $\exists$ λ-homog. $(k-1)$-form α s.t. $d\alpha = \omega$ if one of the following conditions holds:

i) $\nabla(x_0) = 0$,

ii) $\nabla(x_0) \neq 0$ and $k>1$,

iii) $\nabla(x_0) \neq 0$, $k=1$ and $\omega \neq 0$.

In the cases i) and ii), it is possible to additionally choose an α s.t. $\alpha(x_0) = 0$.

Darboux theorem for homogeneous symplectic forms (G&G '24)

Let (M, ∇) be a homogeneity mfold., and let ω be a λ-homog. symplectic form on M. Then, around every $x_0 \in M$ s.t. $\nabla(x_0)=0$, there is a system of homog. coords. (q^i, p_i) such that

$$\omega = \sum_i dp_i \wedge dq^i, \qquad \nabla = \sum_i \left(w_{q^i} q^i \partial_{q^i} + w_{p_i} p_i \partial_{p_i} \right).$$

Idea of the proof:

1) (Graded) linear algebra $\rightsquigarrow$ $\exists$ graded basis (e_a) of $T_{x_0}M$ s.t. $\omega(x_0) = \sum_i e^*_{i+n} \wedge e^*_i$.

2) Choose (homogeneous) coords. $(\bar{q}^i, \bar{p}_i)$ s.t. $d\bar{q}^i(x_0) = e^*_i$, $d\bar{p}_i(x_0) = e^*_{i+n}$.

3) Def. $\omega_0 := d\bar{p}_i \wedge d\bar{q}^i$, so that $\omega_0(x_0) = \omega(x_0)$, and $\omega_t = (1-t)\,\omega_0 + t\omega_1$ $t \in [0,1]$, so that $\omega = \omega_1$

4) Moser's trick: obtain a smooth isotopy Φ_t s.t. $\Phi_t^* \omega_t = \omega_0$ and $(\Phi_t)_* \nabla = \nabla$. $\rightsquigarrow$

$$\omega = \omega_1 = (\Phi_1^{-1})^* \omega_0 = \sum_i d\big(\underbrace{\bar{p}_i \circ \Phi_t^{-1}}_{p_i}\big) \wedge d\big(\underbrace{\bar{q}^i \circ \Phi_t^{-1}}_{q^i}\big).$$

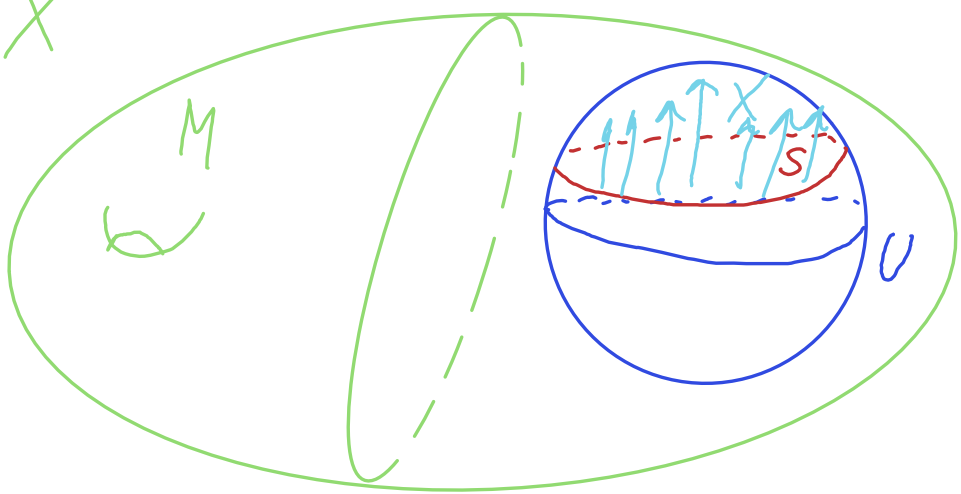
Homogeneous straightening lemma (Grabowski & LG):

Let (M, ∇) be a homogeneity mfold., and let $X \in \mathfrak{X}(M)$ be a $(-\lambda)$-homogeneous vector field. Assume that $\nabla(x_0) = 0$ and $X(x_0) \neq 0$ at $x_0 \in M$. Then, in a neighbourhood of x_0, there is a chart of homog. coords. $(U; z, y^i)$ such that

$$X = \partial_z, \qquad \nabla = \lambda z \partial_z + \sum_i w_i y^i \partial_{y^i}.$$

Sketch of the proof: Let $(U; x^a)$ be a chart of homog. coords. around x_0, i.e., $\nabla = \sum_a w_a x^a \partial_{x^a}$. Since $X(x_0) \neq 0$, not all $X(x^a)$ can vanish. W.l.o.g., assume that $X(x^1) \neq 0$ on U.

The hypersurface $S = \{x^1 = 0\} \subset U$ is a homogeneous submanifold (i.e., $\nabla|_S$ is tangent to S) and it is transverse to X



(S, ∇_S) is a homog. mfold. $\rightsquigarrow$ $\exists$ coords. (y^i) s.t.

$$\nabla = \sum_i w_i y^i \partial_{y^i}.$$

As in the proof of the standard straightening lemma, these coords. induce coords (z, y^i) in a neighbourhood of x_0 in M s.t. $X = \partial_z$.

These coords. are homog. Indeed,

$$[X, \nabla] = \lambda X \Rightarrow TF^X_{-t} \circ \nabla \circ F^X_t = \nabla + \lambda t$$

In particular,

$$\nabla(z, y^i) = \nabla(0, y^i) + \lambda z\, X(0, y^i) = \nabla_S(y^i) + \lambda z\, X(0, y^i)$$

$$= \sum_i w_i y^i \partial_{y^i} + \lambda z \partial_z \qquad \square$$

<u>Def.:</u> A <u>contact distribution</u> is a corank-one distribution $C \subset TM$ which is maximally non-integrable, that is, the skew-symmetric bilinear map

$$\nu : C \times_M C \longrightarrow TM/C, \quad \nu(X,Y) = \gamma([X,Y]),$$

with $\gamma : TM \to TM/C$ the natural projection is non-integrable.

Locally, $C = \ker \eta$, where η is a (local) one-form such that $\eta \wedge (d\eta)^n$ is nowhere zero $(\dim M = 2n+1)$.

Def.: A (global) one-form η on a mfold M^{2n+1} s.t. $\eta \wedge (d\eta)^n$ is a volume form is called a contact form.

The Reeb vector field $R \in \mathfrak{X}(M)$ is uniquely determined by

$$R \in \ker d\eta \quad \& \quad \eta(R).$$

Remark: A contact form is never unique. Indeed, $\ker \eta = \ker (f\eta)$ $\forall$ nowhere-vanishing $f \in C^\infty(M)$.

Darboux theorem for homogeneous contact forms (Grabowski, LG)

Let (M, ∇) be a homogeneity mfold., and let η be a λ-homog. contact form on M. Then, in a neighbourhood of each point $x_0 \in M$ s.t. $\nabla(x_0) = 0$, there exists a system of homog. coords. (q^i, p_i, z) s.t.

$$\eta = dz + \sum_i p_i \, dq^i,$$

$$\nabla = \sum_i \left(w_{q^i} q^i \partial_{q^i} + w_{p_i} p_i \partial_{p_i} \right) + \lambda z \partial_z .$$

Sketch of the proof:

1) The Reeb v.f. R is nowhere-vanishing and (-1)-homogeneous. Hence, $\exists$ coords. $(\bar{z}, y^a)$ around x_0 s.t. $R = \partial_{\bar{z}}$.

Then, $\mathcal{L}_{\partial_{\bar{z}}} \eta = 1$ and $\mathcal{L}_{\partial_{\bar{z}}} d\eta = 0 \implies$

$$\eta = d\bar{z} + \sum_a A_a(y)\, dy^a.$$

2) Consider the hypersurface $S = \{\bar{z} = 0\}$. It is a homogeneous submanifold (i.e. $\nabla_S = \nabla|_S$ is tangent to S) and $\omega = d\eta|_S$ is a λ-homog. symplectic form. By the Darboux theorem for homog. symp. forms, $\exists$ coords. (q^i, p_i) around $x_0 \in S$ s.t. $\omega = \sum_i dp_i \wedge dq^i$.

3) Note that $d\eta$ does not depend on $\bar{z}$. Thus, locally,

$$d\eta = \sum_i dp_i \wedge dq^i .$$

Therefore, $\alpha := \sum_a A_a dy^a - \sum_i p_i dq^i$ is a closed λ-homog. one-form.

4) By the Homog. Poincaré lemma, $\alpha = df$ with f a λ-homog. function s.t. $f(x_0) = 0$.

Finally,

$$\eta = d\bar{z} + \sum_a A_a dy^a = d\bar{z} + \sum_i p_i dq^i + df = dz + \sum_i p_i dq^i,$$

$z = \bar{z} + f$.

<u>Def.:</u> Let (M, ∇) be a homog. mfold. A (co)distribution $D \subset TM$ (resp. $D \subset T^*M$) is called <u>homogeneous</u> if the (co)tangent lift $d_T \nabla$ (resp. $d_{T^*} \nabla$) is tangent to D.

<u>Conjecture:</u> A homogeneous (co)distribution is locally generated by homogeneous vector fields (resp. one-forms).

We know this is true if ∇ is $\mathbb{N}$-graded and complete.

* "Conjecture" is my pretentious way of saying "work in progress".

Note D is endowed with a double homogeneity structure

$$\nabla \text{ and } \nabla_{TM}\big|_D, \qquad [\nabla, \nabla_{TM}] = 0 \text{ (compatible)}$$

with $\nabla_{TM} = \sum_i v^i \partial_{v^i}$ the Euler vector field of TM.

If ∇ is $\mathbb{N}$-graded and complete, then D can be covered by an atlas of bi-homogeneous coords. (i.e., fibered coords. w.r.t. $D \longrightarrow M$ and homog. w.r.t. ∇)
[Grabowski & Rotkiewicz, 2011]

In the associated local trivialisation, these coords. provide homog. vector fields (one-forms) generating D.

Homogeneous Frobenius theorem (Grabowski & LG):

Let (M, ∇) be a homog. mfold, and let D be an involutive distribution of rank k which is locally generated by homog. vector fields. Around every $x_0 \in M$ s.t. $\nabla(x_0) = 0$ $\exists$ homog. chart $(U; x^1, \dots, x^n)$ such that

$$D|_U = \langle \partial_{x^1}, \dots, \partial_{x^k} \rangle$$

and the slices

$$N = \{ x^{k+1} = \text{const.}, \dots, x^n = \text{const} \} \subset U$$

are integral submanifolds.

__Def.:__ A __presymplectic form__ ω on M is a closed 2-form of constant rank. Its __characteristic distribution__ is given by

$$C_\omega = \ker \omega.$$

__Theorem (Darboux):__ Around every point of M, there are local coords. (q^i, p_i, z^a) s.t.

$$\omega = \sum_i dp_i \wedge dq^i \qquad (*)$$

Problem: If (M, ∇) is a homog. mfold. and ω is homog., can we find homog. coords. (q^i, p_i, z^c) in which ω has the form $(*)$?

If our conjecture is true, the answer is YES.

Def.: A one-form ω on a mfold. M^m is said to have

* odd class $2s+1 \leqslant m$ at $x \in M$ if

$$\omega \wedge (d\omega)^s (x) \neq 0 \quad \& \quad (d\omega)^{s+1}(x) = 0.$$

* even class $2s+2 \leqslant m$ at x if

$$\omega \wedge (d\omega)^s (x) \neq 0 \;\; \& \;\; (d\omega)^{s+1}(x) \neq 0 \quad \& \quad \omega \wedge (d\omega)^{s+1}(x) = 0.$$

Theorem (Darboux): In a sufficiently small neighbourhood of x where ω has constant class, there are coords. (q^i, p_i, z^a) s.t.

$$\omega = dz^0 + \sum_{i=1}^{s} p_i\, dq^i \;\; (\text{odd}) \quad (**) \qquad \Bigg| \qquad \omega = \sum_{i=1}^{s+1} p_i\, dq^i \;\; (\text{even}) \quad (***)$$

<u>Problem:</u> If (M,∇) is a homog. mfold. and ω is homog., can we find homog. coords. (q^i, p_i, z^a) in which ω has the form $(**)$ or $(***)$?

Future work

* Extending our results to supermanifolds.

* Bi-homogeneity: ∇_1, ∇_2 s.t. $[\nabla_1, \nabla_2] = 0$.

* Homogeneous multisymplectic forms

* Applications to Pfaffian systems / exterior differential systems

  ↳ studying differential eqs. as ideals generated by differential forms

References

J. Grabowski and M. Rotkiewicz, "Graded bundles and homogeneity structures", J. Geom. Phys. 62 (2012)

K. Grabowska, J. Grabowski and Z. Ravanpak, "VB-structures and generalizations", Ann. Global Anal. Geom. 62, 1 (2022)

K. Grabowska and J. Grabowski, "Homogeneity supermanifolds and homogeneous Darboux theorem", 2024, arXiv: 2411.00537

Thank you for your attention!

Feel free to contact me at alopez-gordon@impan.pl

These slides are available at alopezgordon.xyz