Homogeneous symplectic manifolds and integrable contact systems

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Symplectic geometry

- It is well-known that a symplectic manifold (M, ω) is the natural geometric framework for a Hamiltonian system.
- The Hamiltonian vector field X_h of a function $h \in \mathscr{C}^\infty(M)$ is given by $\omega(X_h, \cdot) = 0$.
- In a neighbourhood of each point in M there are canonical (or Darboux) coordinates (q^i, p_i) in which

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i\,, \quad X_h = \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i}\,.$$

Liouville–Arnol'd theorem

Theorem (Liouville–Arnol'd)

Let $f_1, \ldots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0\ \forall i, j$) on a symplectic manifold (M^{2n}, ω). Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$ be a regular level set.

1. *Any compact connected component of M_Λ is diffeomorphic to $\mathbb{T}^n$.*
2. *On a neighbourhood of M_Λ there are coordinates (φ^i, J_i) such that*

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i\,,$$

and $f_i = f_i(J_1, \ldots, J_n)$, so the Hamiltonian vector fields read

$$X_{f_i} = \frac{\partial f_i}{\partial J_j} \frac{\partial}{\partial \varphi^j}\,.$$

Liouville–Arnol'd theorem

Corollary

Let (M^{2n}, ω, h) be a Hamiltonian system. Suppose that $f_1, \ldots, f_n$ are independent conserved quantities (i.e. $X_h(f_i) = 0\ \forall\ i$) in involution. Then, on a neighbourhood of M_Λ there are Darboux coordinates (φ^i, J_i) such that $h = h(J_1, \ldots, J_n)$, so the Hamiltonian dynamics are given by

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \frac{\partial h}{\partial J_i}\frac{\partial}{\partial \varphi^i}\,,$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0\,.$$

Definition

The tuple $(M, \omega, (f_1, \ldots, f_n))$ is called a **(completely) integrable system**. Sometimes, we will refer to a Hamiltonian system (M^{2n}, ω, h) that has n independent first integrals in involution as a **(completely) integrable Hamiltonian system**.

Definition

The coordinates (φ^i) are called **angle coordinates** (or angle variables), and the coordinates (J_i) are called **action coordinates** (or action variables).

Remark

The Liouville–Arnol'd theorem was extended to non-compact invariant submanifolds by Fiorani, Giachetta and Sardanashvily (2002). One has to assume that the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ are complete, which holds automatically in the compact case.

Example (The n-dimensional harmonic oscillator)

- Consider $\mathbb{R}^{2n}$, with canonical coordinates (x_i, p_i), $i \in \{1, \ldots, n\}$, equipped with the symplectic form ω and the Hamiltonian function h,

$$\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}p_i \,, \quad h = \sum_{i=1}^{n} \left(\frac{p_i^2}{2} + \frac{x_i^2}{2} \right)$$

- The functions $f_i = \frac{p_i^2}{2} + \frac{x_i^2}{2}$ are independent and involution, and one can write $h = \sum_{i=1}^{n} f_i$.
- Angle coordinates are $\varphi^i = \arctan\left(\frac{x_i}{p_i}\right)$ and action coordinates are f_i.
- Hamilton's equations read

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = 1 \,, \qquad \frac{\mathrm{d}f_i}{\mathrm{d}t} = 0 \,.$$

Remark

The explicit computation of action-angle coordinates for a detailed physical model can be challenging and potentially worthy of publication.

MR4664599 - Poisson structure and action-angle variables for the Hirota equation
Zhang, Yu; Tian, Shou-Fu
Z. Angew. Math. Phys. **74** (2023), no. 6, Paper No. 236, 18 pp.

MR4644726 - Action-angle formalism for extreme mass ratio inspirals in Kerr spacetime
Kerachian, Morteza; Polcar, Lukáš; Skoupý, Viktor; Efthymiopoulos, Christos; Lukes-Gerakopoulos, Georgios
Phys. Rev. D **108** (2023), no. 4, Paper No. 044004, 22 pp.

MR4626427 - On the Poisson structure and action-angle variables for the complex modified Korteweg–de Vries equation
Yin, Zhe-Yong; Tian, Shou-Fu
J. Geom. Phys. **192** (2023), Paper No. 104952, 19 pp.
(Reviewer: Leandro, Eduardo S. G.)

MR4736518 - On inverse scattering approach and action-angle variables to the Harry-Dym equation
Yin, Zhe-Yong; Tian, Shou-Fu
J. Math. Phys. **65** (2024), no. 4, Paper No. 043506, 22 pp.

MR4698015 - Action-angle variables and conservation laws expressed in terms of scattering data for an integrable hierarchy associated with the Zakharov-Ito system
Wu, Zhi-Jia; Tian, Shou-Fu; Yin, Zhe-Yong
Phys. D **460** (2024), Paper No. 134062, 8 pp.

MR4689321 - On the Poisson structure and action-angle variables for the Fokas-Lenells equation
Gao, Yun-Zhi; Tian, Shou-Fu; Fan, Hai-Ning
J. Geom. Phys. **197** (2024), Paper No. 105099, 17 pp.

A crash course on contact geometry

Maximally non-integrable distributions

Definition

We will say that a distribution $D \subset \mathrm{T}M$ on a manifold M is **maximally non-integrable** if the bilinear map

$$\nu_D \colon D \times_M D \ni (X, Y) \mapsto \gamma\Big([X, Y]\Big) \in \mathrm{T}M/D$$

is non-degenerate. Here $[\cdot, \cdot]$ denotes the Lie bracket of vector fields with image in D, and $\gamma \colon \mathrm{T}M \to \mathrm{T}M/D$ is the canonical projection.

Contact distributions

Definition

Let M be a $(2n+1)$-dimensional manifold. A **contact distribution** C on M is a maximally non-integrable distribution of corank 1. The pair (M, C) is called a **contact manifold**.

Distributions as kernels of 1-forms

- Note that a distribution D of corank 1 on M can be locally written as the kernel of a (local) 1-form α on M.
- It is easy to see that D is integrable iff

  $$\alpha \wedge \mathrm{d}\alpha = 0$$

  for any local 1-form α such that $D = \ker \alpha$.
- On the contrary, D is maximally non-integrable iff

  $$\alpha \wedge \mathrm{d}\alpha^n = \alpha \wedge \underbrace{\mathrm{d}\alpha \wedge \cdots \wedge \mathrm{d}\alpha}_{n \text{ times}} \neq 0$$

  for any local 1-form α such that $D = \ker \alpha$.

Contact forms

Definition

Let (M, C) be a contact manifold such that C can be globally written as the kernel of a global 1-form η on M. Then, C is said to be a **co-orientable** contact distribution, η is called a **contact form**, and the pair (M, η) is called a **co-oriented contact manifold**.

Contact forms

Remark (Not existence and not uniqueness of contact forms)

- Not all contact manifolds are co-orientable. Nevertheless, there always exists a co-orientable double covering space.
- A co-orientable contact distribution C does not fix the contact form η, but rather the equivalence class

$$\eta \sim \tilde{\eta} \iff \ker \eta = \ker \tilde{\eta} \iff \exists\, f \colon M \to \mathbb{R} \setminus \{0\} \text{ such that } \tilde{\eta} = f\eta\,.$$

Contact forms

Remark

Several authors refer to co-oriented contact manifolds as contact manifolds. The term “contact structure” is used to refer either to the contact distribution or to the contact form, so I will not use it in order to avoid ambiguity.

Example (Odd-dimensional Euclidean space)

$\eta = \mathrm{d}z - \sum_{i=1}^{n} y^i \mathrm{d}x^i$, in $\mathbb{R}^{2n+1}$ with canonical coordinates (x^i, y^i, z).

Example (Trivial bundle over the cotangent bundle)

The cotangent bundle T^*Q of Q is endowed with the tautological 1-form θ_Q. The trivial bundle $\pi_1 : \mathrm{T}^*Q \times \mathbb{R} \to \mathrm{T}^*Q$ can be equipped with the contact form $\eta_Q = \mathrm{d}r - \pi^*\theta_Q$, with r the canonical coordinate of $\mathbb{R}$. If (q^i) are coordinates in Q which induce bundle coordinates (q^i, p_i) in T^*Q and (q^i, p_i, r) in $\mathrm{T}^*Q \times \mathbb{R}$, we have

$$\theta_Q = p_i \mathrm{d}q^i \,, \quad \eta_Q = \mathrm{d}r - p_i \mathrm{d}q^i \,.$$

Example (Projective space)

Let $M = \mathbb{R}^n \times \mathbb{RP}^{n-1}$. Consider the open subsets

$$U_k = \{(x, [y]) \in M \mid y^k \neq 0\} ,$$

where $x = (x^1, \ldots, x^n), y = (y^1, \ldots, y^k, \ldots, y^n) \in \mathbb{R}^n$. We have the local contact forms

$$\eta_k = \mathrm{d}x^k - \sum_{i \neq k} \frac{y_i}{y_k} \mathrm{d}x^i \in \Omega^1(U_k) .$$

If a global contact form η on M existed, then $\eta \wedge \mathrm{d}\eta^n$ would define an orientation. Hence, M is not co-orientable if $n - 1$ is even.

Example (Projective cotangent bundle $\mathbb{P}(\mathrm{T}^*N)$)

This space is the set of equivalence classes $[(x, \alpha)]$ of points of T^*N with the equivalence relation

$$(x, \alpha) \sim (y, \beta) \quad \text{iff} \quad x = y \quad \text{and} \quad \exists \lambda \in \mathbb{R} \setminus \{0\} \quad \text{s.t.} \quad \alpha = \lambda \beta\,.$$

Similarly to $\mathbb{R}^n \times \mathbb{RP}^{n-1}$, it can be equipped with a contact distribution which will not be co-orientable if N is odd-dimensional.

The Reeb vector field

Definition

Let (M, η) be a co-oriented contact manifold. The **Reeb vector field** of (M, η) is the unique vector field $\mathcal{R} \in \mathfrak{X}(M)$ such that

$$\mathcal{R} \in \ker \mathrm{d}\eta\,, \qquad \eta(\mathcal{R}) = 1\,.$$

The tangent bundle $\mathrm{T}M$ of a co-oriented contact manifold (M, η) can be decomposed as the Whitney sum

$$\mathrm{T}M = \ker \eta \oplus \ker \mathrm{d}\eta = C \oplus \langle \mathcal{R} \rangle \,.$$

Note that the complement of the contact distribution $C = \ker \eta$ depends on the choice of contact form, or, equivalently, on the choice of the Reeb vector field.

Proposition

Let η be a 1-form on a manifold M. The map

$$\flat_\eta \colon \mathfrak{X}(M) \to \Omega^1(M)\,, \quad \flat_\eta(X) = \eta(X)\eta + \iota_X \mathrm{d}\eta$$

is a $\mathscr{C}^\infty(M)$-module isomorphism iff η is a contact form.

Note that the Reeb vector field can be equivalently defined as $\mathcal{R} = \flat_\eta^{-1}(\eta)$.

Darboux coordinates

Theorem

Let (M, η) *be a* $(2n+1)$*-dimensional co-oriented contact manifold. Around each point* $x \in M$ *there exist local coordinates* (q^i, p_i, z), $i \in \{1 \ldots, n\}$ *such that the contact form reads*

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i \,.$$

Consequently, the Reeb vector field is written as

$$\mathcal{R} = \frac{\partial}{\partial z} \,.$$

These coordinates are called ***canonical*** *or* ***Darboux*** *coordinates.*

Jacobi structures

- Consider a manifold M endowed with a bivector field $\Lambda \in \mathrm{Sec}(\bigwedge^2 \mathrm{T}M)$ and a vector field $E \in \mathfrak{X}(M)$.
- Define the bracket $\{\cdot, \cdot\} \colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f)\,.$$

- Lichnerowicz (1977) showed that it is a Lie bracket iff

$$[\Lambda, E] = 0\,, \quad [\Lambda, \Lambda] = 2E \wedge \Lambda\,,$$

  where $[\cdot, \cdot]$ denotes the Schouten–Nijenhuis bracket.
- In that case, (Λ, E) is called a **Jacobi structure** on M, $\{\cdot, \cdot\}$ is called a Jacobi bracket, and (M, Λ, E) is called a Jacobi manifold.

Jacobi structures

Remark

A Poisson structure Λ is a Jacobi structure with $E \equiv 0$.

Jacobi structures

- A Jacobi structure (Λ, E) defines a $\mathscr{C}^\infty(M)$-module morphism

$$\sharp_\Lambda \colon \Omega^1(M) \to \mathfrak{X}(M)\,, \qquad \sharp_\Lambda(\alpha) = \Lambda(\alpha, \cdot)\,.$$

- This defines a so-called orthogonal complement $D^{\perp_\Lambda} = \sharp_\Lambda(D^\circ)$, for a distribution D with annihilator D°.
- A submanifold N of M is called **coisotropic** if $\mathrm{T}N^{\perp_\Lambda} \subseteq \mathrm{T}N$.

Jacobi structures

- Two Jacobi structures (Λ, E) and $(\tilde{\Lambda}, \tilde{E})$ on M are **conformally equivalent** if there exists a nowhere-vanishing function f on M such that
$$\tilde{\Lambda} = f\Lambda\,, \quad \tilde{E} = \sharp_\Lambda \mathrm{d}f + fE\,.$$

Remark

The orthogonal complement coincides for conformally equivalent Jacobi structures, namely, $D^{\perp_\Lambda} = D^{\perp_{\tilde{\Lambda}}}$ for any distribution D.

Jacobi structures

Definition

Let (M, Λ, E) be a Jacobi manifold with Jacobi bracket $\{\cdot, \cdot\}$. A collection of functions $f_1, \ldots, f_k \in \mathscr{C}^\infty(M)$ will be said to be **in involution** if

$$\{f_i, f_j\} = 0\,, \forall\, i, j \in \{1, \ldots, k\}\,.$$

Jacobi structures

- For each function $f \in \mathscr{C}^\infty(M)$, we can define a vector field

$$X_f = \sharp_\Lambda(\mathrm{d}f) + fE\,,$$

  or, equivalently,

$$X_f(g) = \{f, g\} + gE(f)\,, \quad \forall g \in \mathscr{C}^\infty(M)\,.$$

- Following the nomenclature of Dazord, Lichnerowicz, Marle, *et al.*, we will refer to X_f as the **Hamiltonian vector field of f**.
- However, X_f does not satisfy the properties of a usual Hamiltonian vector field (w.r.t. a symplectic or Poisson structure). In particular,

$$\{f, g\} = 0 \not\Leftrightarrow X_f(g) = 0\,.$$

Jacobi structure defined by a contact form

- A co-oriented contact manifold (M^{2n+1}, η) is endowed with a Jacobi structure (Λ, E) given by

$$\Lambda(\alpha, \beta) = -\mathrm{d}\eta\left(\flat_\eta^{-1}(\alpha), \flat_\eta^{-1}(\beta)\right), \quad E = -\mathcal{R},$$

  where $\mathcal{R}$ is the Reeb vector field.
- Any contact form $\tilde{\eta}$ defining the same contact distribution, i.e., $\ker \tilde{\eta} = \ker \eta$, defines a conformally equivalent Jacobi structure.

Contact Hamiltonian vector field

- Let (M, η) be a co-oriented contact manifold. The Hamiltonian vector field of $f \in \mathscr{C}^\infty(M)$ is uniquely determined by

$$\eta(X_f) = -f \,, \quad \mathcal{L}_{X_f}\eta = -\mathcal{R}(f)\eta \,.$$

- In Darboux coordinates

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z} \,.$$

Contact Hamiltonian vector field

Remarks

- The Reeb vector field is the Hamiltonian vector field of $f \equiv -1$.
- Every Hamiltonian vector field is an infinitesimal contactomorphism (i.e., its flow preserves the contact distribution $C = \ker \eta$). Conversely, if $Y \in \mathfrak{X}(M)$ is an infinitesimal contactomorphism, then it is the Hamiltonian vector field of $f = -\eta(Y)$.
- Knowing $C = \ker \eta$ and X_f does not fix η nor f. As a matter of fact, X_f is the Hamiltonian vector field of $g = f/a$ with respect to $\tilde{\eta} = a\eta$, for any non-vanishing $a \in \mathscr{C}^\infty(M)$.

Contact Hamiltonian systems

Definition

A **contact Hamiltonian system** (M, η, h) is a co-oriented contact manifold (M, η) with a fixed **Hamiltonian function** $h \in \mathscr{C}^\infty(M)$.

- The dynamics of (M, η, h) is determined by the integral curves of the Hamiltonian vector field X_h of h w.r.t. η.

Contact Hamiltonian systems

- In Darboux coordinates, these curves $c(t) = (q^i(t), p_i(t), z(t))$ are determined by the **contact Hamilton equations**:

$$
\begin{aligned}
\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} &= \frac{\partial h}{\partial p_i} \circ c(t)\,,\\
\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -\frac{\partial h}{\partial q^i} \circ c(t) - p_i(t)\frac{\partial h}{\partial z} \circ c(t)\,,\\
\frac{\mathrm{d}z(t)}{\mathrm{d}t} &= p_i(t)\frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.
\end{aligned}
$$

Example (The harmonic oscillator with linear damping)

Consider the solution $x \colon \mathbb{R} \to \mathbb{R}$ of the second-order ordinary differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) = -x(t) - \kappa \frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,$$

where $\kappa \in \mathbb{R}$. Defining $p = \mathrm{d}x/\mathrm{d}t$, we can reduce it to the system of first-order ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = p(t)\,, \quad \frac{\mathrm{d}p}{\mathrm{d}t}(t) = -x(t) - \kappa p(t)\,.$$

We can obtain this system as the two first contact Hamilton equations from the contact Hamilton system $(\mathbb{R}^3, \eta, h)$, where $\eta = \mathrm{d}z - p\mathrm{d}x$ and

$$h = \frac{p^2}{2} + \frac{x^2}{2} + \kappa z\,.$$

Example (The parachute equation)

- Consider a particle of mass m falling in a fluid under the constant gravitational acceleration g.
- The friction of the fluid is a drag force, namely, of the form $m\gamma\dot{x}^2$, with γ a positive constant.
- The equation of motion (2nd Newton's law)

$$\ddot{x} = \gamma\dot{x}^2 - g$$

can be obtained from the contact Hamilton equations of the contact Hamiltonian system $(\mathbb{R}^3, \eta = \mathrm{d}z - p\mathrm{d}x, h)$, with

$$h = \frac{gm^2\left(e^{2\gamma x} - 1\right)}{2m\gamma} + \frac{(p - 2\gamma z)^2}{2m}\,.$$

Exact symplectic manifolds and homogeneous Liouville–Arnol'd theorem

Exact symplectic manifolds

Definition

An **exact symplectic manifold** is a pair (M, θ), where θ is a **symplectic potential** on M, i.e., $\omega = -\mathrm{d}\theta$ is a symplectic form on M. The **Liouville vector field** $\nabla \in \mathfrak{X}(M)$ is given by

$$\iota_{\nabla}\omega = -\theta\,.$$

A tensor field A on P is called k-homogeneous (for $k \in \mathbb{Z}$) if

$$\mathcal{L}_{\nabla}A = kA\,.$$

Exact symplectic manifolds

Proposition

Let (M, θ) *be an exact symplectic manifold. Given a vector field* $Y \in \mathfrak{X}(M)$*, the following statements are equivalent:*

1. Y *is an infinitesimal homogeneous symplectomorphism, i.e.,* $\mathcal{L}_Y \theta = 0$;
2. Y *is an infinitesimal symplectomorphism (i.e.,* $\mathcal{L}_Y \mathrm{d}\theta = 0$*) and commutes with the Liouville vector field* ∇,
3. Y *is the Hamiltonian vector field of* $f = \theta(Y)$ *and* f *is a homogeneous function of degree* 1.

Homogeneous integrable system

Definition

A **homogeneous integrable system** consists of an exact symplectic manifold (M^{2n}, θ) and a map $F = (f_1, \ldots, f_n) \colon M \to \mathbb{R}^n$ such that the functions $f_1, \ldots, f_n$ are independent, in involution and homogeneous of degree 1 (w.r.t. the Liouville vector field ∇ of θ) on a dense open subset $M_0 \subseteq M$. We will denote it by (M, θ, F).

For simplicity's sake, in this talk I will assume that $M_0 = M$.

Proposition

Let (M, θ, F) be a homogeneous integrable system. Then, for each $\Lambda \in \mathbb{R}^n$, the level set $M_\Lambda = F^{-1}(\Lambda)$ is a Lagrangian submanifold, and

$$\varphi_t^{\nabla}(M_\Lambda) = M_{t\Lambda} = F^{-1}(t\Lambda)\,,$$

where φ_t^{∇} denotes the flow of the Liouville vector field ∇.

Some remarks

- Around each point of an exact symplectic manifold (M, θ), there is a system of canonical coordinates (q^i, p_i) where

$$\theta = p_i \mathrm{d}q^i\,, \quad \nabla = p_i \frac{\partial}{\partial p_i}\,.$$

- Note that coordinates may be canonical for $\omega = -\mathrm{d}\theta$ but not for θ. For instance, in the coordinates $\tilde{q}^i = q^i$, $\tilde{p}_i = p_i + e^{q_i}$ we have

$$\theta = \sum_i (\tilde{p}_i - e^{\tilde{q}^i})\mathrm{d}\tilde{q}^i\,, \quad \omega = \mathrm{d}\tilde{q}^i \wedge \mathrm{d}\tilde{p}_i\,, \quad \nabla = \left(\tilde{p}_i - e^{\tilde{q}^i}\right) \frac{\partial}{\partial \tilde{p}_i}\,.$$

- In particular, the Liouville–Arnol'd theorem provides coordinates which are canonical for ω, but not necessarily for θ or ∇.

Homogeneous Liouville – Arnol'd theorem

Theorem (Colombo, de León, Lainz, L. G., 2023)

Let (M, θ, F) be a homogeneous integrable system with $F = (f_1, \ldots, f_n)$. Given $\Lambda \in \mathbb{R}^n$, suppose that $M_\Lambda = F^{-1}(\Lambda)$ is connected. Assume that, in a neighbourhood U of M_Λ, the Hamiltonian vector fields X_{f_i} are complete, $\operatorname{rank} \mathrm{T}F|_U = n$ and $F|_U \colon U \to F(U) =: V$ is a trivial bundle. Then, $U \simeq \mathbb{T}^k \times \mathbb{R}^{n-k} \times V$ and there is a chart $(\hat{U} \subseteq U; y^i, A_i)$ of M s.t.

1. $A_i = M_i^j f_j$, *where* M_i^j *are homogeneous functions of degree* 0 *depending only on* $f_1, \ldots, f_n$,
2. $\theta = A_i \mathrm{d}y^i$,
3. $X_{f_i} = N_i^j \frac{\partial}{\partial y^j}$, *with* (N_i^j) *the inverse matrix of* (M_i^j).

Lemma

Let M be an n-dimensional manifold, and let $X_1, \ldots, X_n \in \mathfrak{X}(M)$ be linearly independent vector fields. If these vector fields are pairwise commutative and complete, then M is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $k \leq n$.

Lemma

Let (M^{2n}, θ, F) *be a homogeneous integrable system, with* $F = (f_1, \ldots, f_n)$*. Assume that the Hamiltonian vector fields* X_{f_i} *are complete. Then, there exists n functions* $g_i = M_i^j f_j \in \mathscr{C}^\infty(M)$ *such that*

1. $\Big(M, \theta, (g_1, \ldots, g_n)\Big)$ *is also a homogeneous integrable system,*
2. $X_{g_1}, \ldots, X_{g_k}$ *are infinitesimal generators of* $\mathbb{S}^1$*-actions and their flows have period 1,*
3. $X_{g_{k+1}}, \ldots, X_{g_n}$ *are infinitesimal generators of* $\mathbb{R}$*-actions,*
4. M_i^j *for* $i, j \in 1, \ldots, n$ *are homogeneous functions of degree* 0*, and they depend only on* $f_1, \ldots, f_n$*.*

Lemma

Let $\pi \colon P \to M$ be a G-principal bundle over a connected and simply connected manifold. Suppose there exists a connection one-form A such that the horizontal distribution H is integrable. Then $\pi \colon P \to M$ is a trivial bundle and there exists a global section $\chi \colon M \to P$ such that $\chi^* A = 0$.

Proof of the theorem

- W.l.o.g., assume that $X_{f_1}, \ldots, X_{f_k}$ are infinitesimal generators of $\mathbb{S}^1$-actions with period 1, and that $X_{g_{k+1}}, \ldots, X_{g_n}$ are infinitesimal generators of $\mathbb{R}$-actions. Restrict V so that it is simply connected.
- We know that $M_\Lambda \simeq \mathbb{T}^k \times \mathbb{R}^{n-k}$, so we have the trivial $(\mathbb{T}^k \times \mathbb{R}^{n-k})$-principal bundle $F \colon U \simeq V \times \mathbb{T}^k \times \mathbb{R}^{n-k} \to V \subseteq \mathbb{R}^n$.
- We can endow U with a flat, and thus $(\mathbb{T}^k \times \mathbb{R}^{n-k})$-invariant, Riemannian metric g, and construct an integrable horizontal distribution
$$\mathrm{H} = \left(\ker \theta \cap \langle X_{f_i} \rangle_{i=1}^{n} \right)^{\perp_g} \cap \ker \theta\,,$$
with connection one-form θ.
- Then, there exists a global section χ of the principal bundle such that $\chi^* \theta = 0$.

Proof of the theorem

- For each point $x \in M_\Lambda = F^{-1}(\Lambda)$, the angle coordinates $(y^i(x))$ are determined by
$$\Phi\left(y^i(x), \chi(F(x))\right) = x\,,$$
where $\Phi \colon \mathbb{T}^k \times \mathbb{R}^{n-k} \times M \to M$ denotes the action defined by the flows of X_{f_i}. Thus, $X_{f_i} = \partial_{y^i}$.
- In coordinates (f_i, y^i),
$$\chi(f_i) = \left(f_i, 0\right)\,, \quad \theta = A_i(f_j, y^j)\mathrm{d}y^i + B^i(f_j, y^j)\mathrm{d}f_i\,.$$
- Contracting θ with X_{f_i} yields $A_i = f_i$. Moreover,
$$0 = \mathcal{L}_{X_{f_j}}\theta = \mathcal{L}_{\partial_{y^j}}\left(f_i\mathrm{d}y^i + B^i\mathrm{d}f_i\right) = \frac{\partial B^i}{\partial y^j}\mathrm{d}f_i \Longrightarrow \theta = f_i\mathrm{d}y^i + B^i(f_j)\mathrm{d}f_i\,.$$
- Since $\chi^*\theta = 0$, we conclude that $\theta = f_i\mathrm{d}y^i$.

Q.E.D.

Liouville–Arnol’d theorem for contact Hamiltonian systems

Trivial symplectization of a co-oriented contact manifold

Definition

Let (M, η) be a co-oriented contact manifold. Then, the trivial bundle $\pi_1 : M^{\text{symp}} = M \times \mathbb{R}_+ \to M$, $\pi_1(x, r) = x$ can be endowed with the symplectic potential $\theta(x, r) = r\eta(x)$. The Liouville vector field reads $\nabla = r\partial_r$.

We will refer to $(M^{\text{symp}}, \theta)$ as the **trivial symplectization** of (M, η).

Remark

I will present a more general setting at the end of the talk.

Trivial symplectization of a co-oriented contact manifold

Proposition

There is a one-to-one correspondence between functions $f(x)$ *on* M *and 1-homogeneous functions* $f^{\mathrm{symp}}(x, r) = -r f(x)$ *on* M^{symp} *such that the symplectic* $X_{f^{\mathrm{symp}}}$ *and contact* X_f *Hamiltonian vector fields are related as follows:*

$$\mathrm{T}\pi_1\left(X_{f^{\mathrm{symp}}}\right) = X_f\,.$$

Moreover, the Poisson $\{\cdot,\cdot\}_\theta$ *and Jacobi* $\{\cdot,\cdot\}$ *brackets have the correspondence*

$$\{f^{\mathrm{symp}}, g^{\mathrm{symp}}\}_\omega = \left(\{f, g\}_\eta\right)^{\mathrm{symp}}\,.$$

Definition

A **completely integrable contact system** is a triple (M, η, F), where (M^{2n+1}, η) is a co-oriented contact manifold and $F = (f_0, \ldots, f_n)\colon M \to \mathbb{R}^{n+1}$ is a map such that

1. $f_0, \ldots, f_n$ are in involution, i.e., $\{f_\alpha, f_\beta\} = 0 \ \forall\, \alpha, \beta \in \{0, \ldots, n\}$,
2. $\operatorname{rank} \mathrm{T}F \geq n$ on a dense open subset $M_0 \subseteq M$.

Proposition

Let (M, η) be a co-oriented contact manifold and $F \colon M \to \mathbb{R}^{n+1}$ a smooth map. Consider the trivial symplectization, i.e., $M^{\mathrm{symp}} = M \times \mathbb{R}_+$ endowed with the symplectic potential $\theta(x, r) = r\eta(x)$, and the map $F^{\mathrm{symp}}(x, r) = -rF(x)$. Then, $(M^{\mathrm{symp}}, \theta, F^{\mathrm{symp}})$ is a homogeneous integrable system iff (M, η, F) is a completely integrable contact system.

Some notation

- For each $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $\langle\Lambda\rangle_+$ denote the ray generated by Λ, namely,
$$\langle\Lambda\rangle_+ := \left\{x \in \mathbb{R}^{n+1} \mid \exists \in \mathbb{R}_+ \colon x = r\Lambda\right\} .$$
- Consider the preimages $M_{\langle\Lambda\rangle_+}$ of those rays by a map $F\colon M \to \mathbb{R}^{n+1}$, namely,
$$M_{\langle\Lambda\rangle_+} := F^{-1}\left(\langle\Lambda\rangle_+\right) .$$

Theorem (Colombo, de León, Lainz, L. G., 2023)

Let $\left(M, \eta, F\right)$ be a completely integrable contact system, where $F = (f_0, \ldots, f_n)$. Suppose that the contact Hamiltonian vector fields X_{f_i} are complete. Given $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, assume that U is a neighbourhood of $M_{\langle\Lambda\rangle_+}$ s.t. $F|_U \colon U \to B$ is a trivial bundle. Then:

1. *$M_{\langle\Lambda\rangle_+}$ is coisotropic, invariant by the Hamiltonian flow of f_α, and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ for some $k \leq n$.*
2. *There exist coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on U such that the Hamiltonian vector fields of the functions f_α read*
$$X_{f_\alpha} = \overline{N}_\alpha^\beta X_{f_\beta}\,,$$
where $\overline{N}_\alpha^\beta$ are functions depending only on $\tilde{A}_1, \ldots, \tilde{A}_n$.
3. *There exists a nowhere-vanishing function $A_0 \in \mathscr{C}^\infty(U)$ and a conformally equivalent contact form $\tilde{\eta} = \eta/A_0 = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.*

Sketch of the proof

1. Translate the problem to the exact symplectic manifold $(M^{\mathrm{symp}} = M \times \mathbb{R}_+, \theta = r\eta)$.
   - $\{f_\alpha, f_\beta\} = 0 \Rightarrow \ \{f_\alpha^{\mathrm{symp}}, f_\beta^{\mathrm{symp}}\} = 0.$
   - X_{f_α} complete $\Rightarrow \ X_{f_\alpha^{\mathrm{symp}}}$ complete.
   - $\operatorname{rank} \mathrm{d}f_\alpha \geq n \Rightarrow \ \operatorname{rank} \mathrm{d}(\underbrace{r\pi_1^* f_\alpha}_{f_\alpha^{\mathrm{symp}}}) \geq n+1.$
   - $\pi_1\left((F^{\mathrm{symp}})^{-1}(\Lambda)\right) = \left\{x \in M \mid \exists s \in \mathbb{R}^+ : \ F(x) = \frac{\Lambda}{s}\right\} = M_{\langle \Lambda \rangle_+}.$
   - $X_{f_\alpha^{\mathrm{symp}}}$ are tangent to $(F^{\mathrm{symp}})^{-1}(\Lambda) \Rightarrow X_{f_\alpha}$ are tangent to $M_{\langle \Lambda \rangle_+}$.
   - X_{f_α} commute and are tangent to $M_{\langle \Lambda \rangle_+} \Rightarrow M_{\langle \Lambda \rangle_+} \simeq \mathbb{T}^k \times \mathbb{R}^{n+1-k}$.
   - $F \colon U \to B$ is a trivial bundle $\Rightarrow F^{\mathrm{symp}} \colon \pi_1^{-1} U \to B$ is a trivial bundle.

   $\therefore$ We can apply the theorem for exact symplectic manifolds to obtain action-angle coordinates $(y^\alpha_{\mathrm{symp}}, A_\alpha^{\mathrm{symp}})$ on $\pi_1^{-1}(U)$.

Sketch of the proof

2. In these coordinates,

$$\theta = A_\alpha^{\text{symp}} \mathrm{d} y_{\text{symp}}^\alpha\,, \qquad A_\alpha^{\text{symp}} = M_\alpha^\beta f_\beta^{\text{symp}}\,,$$

and

$$X_{f_\alpha^{\text{symp}}} = N_\alpha^\beta \frac{\partial}{\partial y_{\text{symp}}^\beta}\,, \quad (N_\beta^\alpha) = (M_\beta^\alpha)^{-1}\,.$$

Due to the homogeneity, there are functions $y^\alpha, A_\alpha, \overline{M}_\alpha^\beta$ and $\overline{N}_\alpha^\beta$ on M such that

$$A_\alpha^{\text{symp}} = -r\left(\pi_1^* A_\alpha\right)\,, \qquad y_{\text{symp}}^\alpha = \pi_1^* y^\alpha\,,$$
$$M_\alpha^\beta = \pi_1^* \overline{M}_\alpha^\beta\,, \qquad N_\alpha^\beta = \pi_1^* \overline{N}_\alpha^\beta\,.$$

Sketch of the proof

3. Since $r\left(\pi_1^*\eta\right) = \theta$, the contact form is given by

$$\eta = A_\alpha \mathrm{d}y^\alpha\,.$$

   and

$$f_\alpha = \overline{M}_\alpha^\beta A_\beta\,,\quad X_{f_\alpha} = \overline{N}_\alpha^\beta \frac{\partial}{\partial y^\beta}\,,$$

4. Since $\Lambda \neq 0$, there is at least one nonvanishing f_α. Hence, there is at least one nonvanishing A_α. W.l.o.g. , assume that $A_0 \neq 0$. Then, $(y^i, \tilde{A}_i = -A_i/A_0, y^0)$ are Darboux coordinates for

$$\tilde{\eta} = \frac{1}{A_0}\eta = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i\,,$$

Construction of action-angle coordinates

In order to construct action-angle coordinates in a neighbourhood U of M_Λ, one has to carry out the following steps:

1. Fix a section χ of $F\colon U \to V$ such that $\chi^*\theta = 0$.
2. Compute the flows $\varphi_t^{X_{f_i}}$ of the Hamiltonian vector fields X_{f_i}.
3. Let $\Phi\colon \mathbb{R}^n \times M \to M$ denote the action of $\mathbb{R}^n$ on M defined by the flows, namely,

$$\Phi(t_1, \ldots, t_n; x) = \varphi_{t_1}^{X_{f_1}} \circ \cdots \circ \varphi_{t_n}^{X_{f_n}}(x)\,.$$

4. It is well-known that the isotropy subgroup $G_{\chi(\Lambda)(\Lambda)} = \{g \in \mathbb{R}^n \mid \Phi(g, \chi(\Lambda)) = \chi(\Lambda)\}$, forms a lattice (that is, a $\mathbb{Z}$-submodule of $\mathbb{R}^n$). Pick a $\mathbb{Z}$-basis $\{e_1, \ldots, e_m\}$, where m is the rank of the isotropy subgroup.
5. Complete it to a basis $\mathcal{B} = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ of $\mathbb{R}^n$.

Construction of action-angle coordinates

6. Let (M^j_i) denote the matrix of change from the basis $\{X_{f_i}(\chi(\Lambda))\}$ of $\mathrm{T}_{\chi(\Lambda)}M_\Lambda \simeq \mathbb{R}^n$ to the basis $\{e_i\}$. The action coordinates are the functions $A_i = M^j_i f_j$.
7. The angle coordinates (y^i) of a point $x \in M$ are the solutions of the equation
$$x = \Phi\big(y^i e_i; \chi \circ F(x)\big) .$$

An example

- Let $M = \mathbb{R}^3 \setminus \{0\}$ with canonical coordinates (q, p, z), and $\eta = \mathrm{d}z - p\mathrm{d}q$.
- The functions $h = p$ and $f = z$ are in involution.
- Let $F = (h, f)\colon M \to \mathbb{R}^2$.
- $\operatorname{rank} \mathrm{T}F = 2$, and thus (M, η, F) is a completely integrable contact system.

An example

- Hypothesis of the theorem are satisfied:
  1. The Hamiltonian vector fields
  $$X_h = \frac{\partial}{\partial q}, \quad X_f = -p\frac{\partial}{\partial p} - z\frac{\partial}{\partial z}$$
  are complete,
  2. Since $F\colon (q, p, z) \mapsto (p, z)$ is the canonical projection, $F\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is a trivial bundle.

An example

- Therefore, $\theta = r\mathrm{d}z - rp\mathrm{d}q$ is the symplectic potential on $M^{\mathrm{symp}} = M \times \mathbb{R}_+$, and the symplectizations of h and f are $h^{\mathrm{symp}} = -rp$ and $f^{\mathrm{symp}} = -rz$. Their Hamiltonian vector fields are

$$X_{h^{\mathrm{symp}}} = \frac{\partial}{\partial q}, \quad X_{f^{\mathrm{symp}}} = -p\frac{\partial}{\partial p} - z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r}.$$

- Consider a section $\chi\colon \mathbb{R}^2 \to M^{\mathrm{symp}}$ of $F^{\mathrm{symp}} = (h^{\mathrm{symp}}, f^{\mathrm{symp}})$ such that $\chi^*\theta = 0$. For instance, one can choose $\chi(\Lambda_1, \Lambda_2) = \left(0, \frac{\Lambda_1}{\Lambda_2}, 1, \Lambda_2\right)$ in the points where $\Lambda_2 \neq 0$.
- The Lie group action $\Phi\colon \mathbb{R}^2 \times M^{\mathrm{symp}} \to M^{\mathrm{symp}}$ defined by the flows of $X_{h^{\mathrm{symp}}}$ and $X_{f^{\mathrm{symp}}}$ is given by

$$\Phi(t, s; q, p, z, r) = \left(q + t, pe^{-s}, ze^{-s}, re^{s}\right),$$

whose isotropy subgroup is the trivial one.

An example

- The angle coordinates $(y^0_{\mathrm{symp}}, y^1_{\mathrm{symp}})$ of a point $x \in M^{\mathrm{symp}}$ are determined by

$$\Phi\left(y^0_{\mathrm{symp}}, y^1_{\mathrm{symp}}, \chi(F(x))\right) = x\,.$$

- If the canonical coordinates of x are (q, p, z, r), then

$$y^0_{\mathrm{symp}} = q\,, \quad y^1_{\mathrm{symp}} = -\log z\,.$$

- Since the isotropy subgroup is trivial, the action coordinates coincide with the functions in involution, namely,

$$A_0^{\mathrm{symp}} = h^{\mathrm{symp}} = -rp\,, \quad A_1^{\mathrm{symp}} = f^{\mathrm{symp}} = -rz\,.$$

- Projecting to M yields the functions

$$y^0 = q\,, \quad y^1 = -\log z\,, \quad A_0 = h = p\,, \quad A_1 = f = z\,.$$

An example

- The action coordinate is

$$\tilde{A} = -\frac{A_0}{A_1} = -\frac{p}{z}$$

In the coordinates $(y^0, y^1, \tilde{A})$ the Hamiltonian vector fields reads

$$X_h = \frac{\partial}{\partial y^0}\,, \quad X_f = \frac{\partial}{\partial y^1}\,,$$

and there is a conformal contact form given by

$$\tilde{\eta} = -\frac{1}{A_1}\eta = \mathrm{d}y^1 - \tilde{A}\mathrm{d}y^0\,.$$

An example

- Similarly,

$$\chi(\Lambda_1,\Lambda_2) = \left(\frac{\Lambda_2}{\Lambda_1}, 1, \frac{\Lambda_2}{\Lambda_1}, \Lambda_1\right)$$

  is a section of F^{symp} in the points where $\Lambda_1 \neq 0$.

- Performing analogous computations as above one obtains the action-angle coordinates

$$\hat{y}^0 = q - \frac{z}{p}, \quad \hat{y}^1 = -\log p, \quad \hat{A} = -\frac{z}{p},$$

  such that

$$X_h = \frac{\partial}{\partial \hat{y}^0}, \quad X_f = \frac{\partial}{\partial \hat{y}^1}, \quad \hat{\eta} = -\frac{1}{p}\eta = \mathrm{d}\hat{y}^0 - \hat{A}\mathrm{d}\hat{y}^1 .$$

Generalisation to not co-oriented contact manifolds

$\mathbb{R}^\times$-principal bundles

- Consider the multiplicative group of non-zero real numbers $\mathrm{GL}(1,\mathbb{R}) = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.
- Let $\pi\colon P \to M$ be an $\mathbb{R}^\times$-principal bundle, and denote the $\mathbb{R}^\times$-action by Φ, and the Euler vector field by ∇.
- In a local trivialization $\pi^{-1}(U) \simeq U \times \mathbb{R}^\times$ of P, they read

$$\pi(x,s) = x\,, \quad h_t(x,s) = (x,ts)\,, \quad \nabla = s\frac{\partial}{\partial s}\,.$$

Homogeneous symplectic forms

Definition

Let $\pi\colon P \to M$ be an $\mathbb{R}^\times$-principal bundle with Euler vector field ∇. A tensor field A on P is called k-homogeneous (for $k \in \mathbb{Z}$) if

$$\mathcal{L}_\nabla A = kA\,.$$

Definition

A **symplectic $\mathbb{R}^\times$-principal bundle** is an $\mathbb{R}^\times$-principal bundle $\pi\colon P \to M$ endowed with a 1-homogeneous symplectic form ω on P. We will denote it by $(P, \pi, M, \nabla, \omega)$

Contact manifolds and symplectic $\mathbb{R}^\times$-principal bundles

Theorem (Grabowski, 2013)

There is a canonical one-to-one correspondence between contact distributions $C \subset \mathrm{T}M$ on M and symplectic $\mathbb{R}^\times$-principal bundles $\pi\colon P \to M$ over M.
More precisely, the symplectic $\mathbb{R}^\times$-principal bundle associated with C is $(C^\circ)^\times = C^\circ \setminus 0_{\mathrm{T}^*M} \subset \mathrm{T}^*M$ (i.e., the annihilator of C with the zero section removed), whose symplectic form is the restriction to $(C^\circ)^\times$ of the canonical symplectic form ω_M on T^*Q. It is called the ***symplectic cover*** *of (M, C).*

Remark

Every symplectic $\mathbb{R}^\times$-principal bundle $(P, \pi, M, \nabla, \omega)$ is an exact symplectic manifold. Indeed, the 1-form $\theta = -\iota_\nabla \omega$ is a symplectic potential for ω.
Conversely, an exact symplectic manifold (M, θ) is a symplectic $\mathbb{R}^\times$-principal bundle if the Liouville vector field ∇ is complete.

Contact Hamiltonian vector fields

Theorem (Grabowska and Grabowski, 2022)

*Let $(P, \pi, M, \nabla, \omega)$ be the symplectic cover of (M, C). Then, the Hamiltonian vector field X_h of a 1-homogeneous function $h \in \mathscr{C}^\infty(P)$ is π-projectable. The vector field $X_h^c := \mathrm{T}\pi(X_h) \in \mathfrak{X}(M)$ is called the **contact Hamiltonian vector field** of h.*

Proposition

Let $(P^{2n}, \pi, M, \nabla, \omega)$ *be the symplectic cover of the contact manifold* (M, C)*, and let* $F = (f_1, \ldots, f_n)\colon P \to \mathbb{R}^n$ *a map such that* $(M, \theta = -\iota_\nabla \omega, F)$ *is a homogeneous integrable system. Then:*

1. $\pi\left(F^{-1}(\Lambda)\right)$ *is coisotropic, invariant by the flows of* $X^c_{f_1}, \ldots, X^c_{f_n}$*, and diffeomorphic to* $\mathbb{T}^k \times \mathbb{R}^{n-k}$ *for some* $k \leq n$.
2. *There exist coordinates* $(y^1, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_{n-1})$ *such that*

$$X^c_{f_\alpha} = \overline{N}_\alpha^\beta \frac{\partial}{\partial y^\beta},$$

   where $\overline{N}_\alpha^\beta$ *are functions depending only on* $\tilde{A}_1, \ldots, \tilde{A}_{n-1}$.

Intermezzo: other notions of contact integrability

Intermezzo: other notions of contact integrability

- Khesin and Tabachnikov, Liberman, Banyaga and Molino, Lerman, etc. have defined notions of contact complete integrability which are geometric but not dynamical, e.g. a certain foliation over a contact manifold.
- Miranda (2005, 2014) considered integrability of the Reeb dynamics when $\mathcal{R}$ is the generator of an $\mathbb{S}^1$-action.
- Boyer (2011) calls a contact Hamiltonian system (M^{2n+1}, η, h) completely integrable if there exist $n+1$ independent functions in involution $f_0 \equiv h, f_1, \ldots, f_n$ such that $X_h(f_1) = \cdots = X_h(f_n) = 0$. This implies that $\mathcal{R}(h) = 0$, what he calls a "good Hamiltonian". Then, the two first contact Hamilton equations are the classical Hamilton equations $\rightsquigarrow$ conservative dynamics:

$$\mathcal{L}_{X_h}\eta = 0\,, \quad X_h(h) = 0\,.$$

Intermezzo: other notions of contact integrability

- B. Jovanović and V. Jovanović (2012, 2015) considered noncommutative integrability for the flows of contact Hamiltonian vector fields, assuming the functions in involution to be Reeb-invariant.
- Recently (a month before this seminar), B. Jovanović submitted a preprint in which he studies the non-commutative integrability of contact systems on a contact manifold (M, C) using the Jacobi structure on the space of sections of a contact line bundle L. In this new work, he no longer assumes the contact Hamiltonian to be Reeb-invariant.

Theorem (B. Jovanović, 2025)

Consider a "contact Hamiltonian system" $(M, C, h \in \mathrm{Sec}(L))$ *with symmetries* $s_0 = h, \ldots, s_p \in \mathrm{Sec}(L)$ *s.t.*

$$\{s_i, s_a\} = 0, \quad i = 0, \ldots, r, \quad a = 0, \ldots, p, \quad p + r = 2n,$$

and assume that $X_{s_0}, \ldots, X_{s_r}$ *are complete. Let* $\pi \colon M \setminus M_0 \to \mathbb{RP}^p$, $\pi(x) = \left[s_0(x), \ldots, s_n(x)\right]$ *be the associated momentum map and let* $M_{reg} \subseteq M$ *be an open subset in which* $\operatorname{rank} \mathrm{T}\pi = p$. *Then,*

$$\ker \mathrm{T}\pi_x = \mathrm{span}\{X_0(x), \ldots, X_r(x)\}\,, \quad \forall\, x \in M_{reg}.$$

A connected component $M^0_{\mathbf{c}}$ *of* $M_{\mathbf{c}} = \pi^{-1}(\mathbf{c}) \cap M_{reg}$ *is diffeomorphic to* $\mathbb{T}^l \times \mathbb{R}^{r+1-l}$. *There exist coordinates* (φ_μ, x_k) *of* $M^0_{\mathbf{c}}$ *in which the contact dynamics read*

$$\dot{\varphi}_\mu = \omega_\mu = const\,, \quad \dot{x}_k = a_k = const\,.$$

Theorem (B. Jovanović, 2025)

Furthermore, the contact symmetries $\operatorname{span}\{X_0, \dots, X_r\}$ *are also tangent to the zero locus* $M_{\mathbf{0}} = \{x \in M \mid s_0(x) = \cdots = s_p(x) = 0\}$. *Let* $M_{\mathbf{0},reg}$ *be an open subset of* $M_{\mathbf{0}}$ *such that each point has a neighborhood* U *with local sections* $s_{0U}, \dots, s_{pU}$ *that are independent in a chart* (U, α_U)*:*

$$M_{\mathbf{0},reg} \cap U = \{x \in U \mid s_{0U}(x) = 0, \dots, s_{pU}(x) = 0,\ ds_{0U} \wedge \cdots \wedge ds_{pU}|_x \neq 0\}.$$

Then,

$$\dim \ker T\pi_x = r\,, \quad \forall\, x \in M_{\mathbf{0},reg}$$

and a connected component $M_{\mathbf{0}}^0$ *of* $M_{\mathbf{0},reg}$ *is diffeomorphic to* $\mathbb{T}^l \times \mathbb{R}^{r-l}$ *with linearized dynamics.*

Bi-Hamiltonian systems

Bi-Hamiltonian systems

Problem

Given a Hamiltonian system (M^{2n}, ω, h), we would like to find n independent conserved quantities in involution $f_1, \ldots, f_n$, in order to construct action-angle coordinates (φ^i, J_i).

Magri *et al.* developed a method for constructing such conserved quantities by computing the eigenvalues of a $(1,1)$-tensor field N verifying certain compatibility conditions.

Compatible Poisson structures

Definition

Let M be a manifold. Two Poisson tensors are Λ and Λ_1 on M are said to be **compatible** if $\Lambda + \Lambda_1$ is also a Poisson tensor on M.

Definition

A vector field $X \in \mathfrak{X}(M)$ is called **bi-Hamiltonian** if it is a Hamiltonian vector field w.r.t. two compatible Poisson structures, namely,

$$X = \Lambda(\mathrm{d}h, \cdot) = \Lambda_1(\mathrm{d}h_1, \cdot)\,,$$

for two functions $h, h_1 \in \mathscr{C}^\infty(M)$.

Poisson – Nijehuis structures

- The linear map $\sharp_\Lambda \colon \mathrm{T}^*_x M \ni \alpha \mapsto \Lambda(\alpha, \cdot) \in \mathrm{T}_x M$ is an isomorphism iff Λ comes from a symplectic structure ω. In that case, $\flat_\omega := \sharp_\Lambda^{-1}(v) = \iota_v \omega$.
- In that situation, we can define the $(1,1)$-tensor field

$$N = \sharp_{\Lambda_1} \circ \sharp_\Lambda^{-1} = \sharp_{\Lambda_1} \circ \flat_\omega \,.$$

Poisson – Nijehuis structures

Theorem (Magri and Morosi, 1984)

Let (M, ω) *be a symplectic manifold and* Λ_1 *a bivector. Consider the* $(1,1)$*-tensor field*

$$N = \sharp_{\Lambda_1} \circ \flat_\omega \,.$$

If Λ_1 *is a Poisson tensor compatible with* Λ*, then the Nijehuis torsion* T_N *of* N *vanishes. In that case, the eigenvalues of* N *are in involution w.r.t. both Poisson brackets.*
The pair (Λ, N) *is called a* ***Poisson – Nijenhuis structure*** *on* M*.*

The Nijehuis torsion of N is the vector valued 2-form T_N on M given by

$$T_N(X, Y) := N^2[X, Y] - N[NX, Y] - N[X, NY] + [NX, NY] \,, \quad \forall\, X, Y \in \mathfrak{X}(M) \,.$$

Poisson – Nijehuis structures

Corollary

If a vector field $X \in \mathfrak{X}(M)$ is bi-Hamiltonian w.r.t. to ω and Λ_1 (i.e., $X = \sharp_\omega \mathrm{d}h = \sharp_{\Lambda_1} \mathrm{d}h_1$), then the eigenvalues of N form a family of conserved quantities in involution w.r.t. both Poisson brackets.

Compatible Jacobi structures

- The theory of compatible Jacobi structures and Jacobi–Nijenhuis manifolds was developed by Iglesias, Monterde, Marrero, Nunes da Costa, Padrón and Petalidou in the 1990s and 2000s.
- Two Jacobi structures (Λ, E) and (Λ_1, E_1) on a manifold M are called compatible if $(\Lambda + \Lambda_1, E + E_1)$ is also a Jacobi structure on M.
- Given a Jacobi structure (Λ, E) on M, one can construct an associated Poisson structure $\tilde{\Lambda} = 1/r\Lambda + \partial_r \wedge E$ on $M \times \mathbb{R}_+$, which by construction is homogeneous of degree –1 with respect to $\nabla = r\partial_r$.
- Nunes da Costa (1998) showed that (Λ, E) and (Λ_1, E_1) are compatible Jacobi structures iff $\tilde{\Lambda}$ and $\tilde{\Lambda}_1$ are compatible Poisson structures.

Theorem (Fernandes, 1994)

Consider a $2n$-dimensional completely integrable Hamiltonian system (M, ω, H) with action-angle coordinates (s_i, φ^i) satisfying the following conditions:

(ND) *The Hessian matrix $\left(\frac{\partial^2 H}{\partial s_i \partial s_j}\right)$ of the Hamiltonian with respect to the action variables is non-degenerate in a dense subset of M.*

(BH) *The system is bi-Hamiltonian and the recursion operator N has n functionally independent real eigenvalues $\lambda_1, \ldots, \lambda_n$.*

Then, the Hamiltonian function can be written as

$$H(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} H_i(\lambda_i),$$

where each H_i is a function that depends only on the corresponding λ_i.

Proposition

Let (M, θ, H) be a homogeneous integrable Hamiltonian system satisfying the assumption (ND). Denote by Λ the Poisson structure defined by $\omega = -\mathrm{d}\theta$, and by ∇ the Liouville vector field corresponding to θ. If there is a Poisson structure Λ_1 on M compatible with Λ, it cannot be simultaneously (-1)-homogeneous (i.e., $\mathfrak{L}_{\nabla}\Lambda_1 = -\Lambda_1$) and satisfying (BH).

Proof.

If N has n functionally independent eigenvalues, then $H = \sum_i H_i(\lambda_i)$. If Λ_1 is (-1)-homogeneous, then N is 0-homogeneous, so its eigenvalues are 0-homogeneous as well. Hence,

$$H = \nabla(H) = \sum_{i=1}^{n} H_i'(\lambda_i)\nabla(\lambda_i) = 0\,.$$

□

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$$H = \nabla(H) = \sum_{i=1}^{n} H_i'(\lambda_i) \nabla(\lambda_i) = 0\,.$$

$\square$

Corollary

Let (M, η, H) *be a* $(2n+1)$*-dimensional integrable contact Hamiltonian system. If there is a second Jacobi structure* (Λ_1, E_1) *compatible with the Jacobi structure* (Λ, E) *defined by* η*, then the recursion operator* $N = \sharp_{\tilde{\Lambda}_1} \circ \sharp_{\tilde{\Lambda}}^{-1}$ *relating the associated Poisson structures on* $M \times \mathbb{R}_+$ *cannot have* $(n+1)$ *functionally independent real eigenvalues.*

- Consequently, compatible Jacobi structures cannot be utilised to construct a set of independent functions in involution for a contact Hamiltonian system.
- Nevertheless, we can symplectise the contact Hamiltonian system and obtain a second Poisson structure compatible with the one defined by the exact symplectic structure.
- If N is 1-homogeneous and satisfies (BH), then its eigenvalues are n functionally independent and 1-homogeneous functions in involution, so they will project into n functions in involution with respect to the Jacobi bracket.

A toy example

- Let $M = \mathbb{R}^2$, and consider its cotangent bundle $\mathrm{T}^*M \simeq \mathbb{R}^4$ endowed with the canonical one-form $\theta_{\mathbb{R}^2}$.
- In bundle coordinates (x^i, p_i), it reads $\theta_M = p_i \mathrm{d}x^i$. It defines the symplectic form $\omega_M = -\mathrm{d}\theta_M = \mathrm{d}x^i \wedge \mathrm{d}p_i$, and the Poisson structure

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial p_2} .$$

- In this case, the Liouville vector field is $\nabla_M = p_i \partial_{p_i}$, the infinitesimal generator of homotheties on the fibers.
- A Poisson structure compatible with Λ is

$$\Lambda_1 = p_1 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial p_1} + p_2 x^2 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial p_2} .$$

A toy example

- The Nijenhuis tensor $N = \sharp_{\Lambda_1} \circ \sharp_{\Lambda}^{-1}$ reads

$$N = p_1 \left(\frac{\partial}{\partial x^1} \otimes \mathrm{d}x^1 + \frac{\partial}{\partial p_1} \otimes \mathrm{d}p_1 \right) + p_2 x^2 \left(\frac{\partial}{\partial x^2} \otimes \mathrm{d}x^2 + \frac{\partial}{\partial p_2} \otimes \mathrm{d}p_2 \right) .$$

- The eigenvalues of N are $\lambda_1 = p_1$ and $\lambda_2 = p_2 x^2$, which are homogeneous of degree 1, in involution with respect to both Λ and Λ_1, and functionally independent on the dense subset $U = \mathrm{T}^*M \setminus \left(\{p_2 = 0\} \cap \{x^2 = 0\} \right)$.
- The vector field

$$X = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} - p_2 \frac{\partial}{\partial p_2}$$

is bi-Hamiltonian. Indeed, it is the Hamiltonian vector field of $H = p_1 + p_2 x^2$ with respect to Λ, and the Hamiltonian vector field of $H_1 = \log(p_1 p_2 x^2)$ with respect to Λ_1. Moreover, λ_1 and λ_2 are first integrals of X.

A toy example

- In the coordinates (φ^i, λ_i),

$$\theta = \sum_{i=1}^{2} \lambda_i \mathrm{d}\varphi^i\,, \quad \Lambda = \sum_{i=1}^{2} \frac{\partial}{\partial \varphi^i} \wedge \frac{\partial}{\partial \lambda^i}\,, \quad \Lambda_1 = \sum_{i=1}^{2} \lambda_i \frac{\partial}{\partial \varphi^i} \wedge \frac{\partial}{\partial \lambda^i}\,,$$

$$X = \partial_{\varphi^1} + \partial_{\varphi^2}\,, \quad H = \lambda_1 + \lambda_2\,.$$

A toy example bis

- Consider the contact Hamiltonian system $(M = \mathbb{R}^3, \eta, h)$, with η the canonical contact form, $\eta = \mathrm{d}z - p\mathrm{d}q$, and $h = p - z$.
- In bundle coordinates (q, p, z, r), the trivial symplectisation $(\mathbb{R}^4, \theta, H)$ of (M, η, h) reads

$$\theta = r\mathrm{d}z - rp\mathrm{d}q\,, \quad H = rz - rp\,,$$

  and Liouville vector field is $\nabla = r\partial_r$.
- This is the system from the previous example, as it becomes evident by performing the coordinate change

$$x^1 = q, \quad x^2 = z, \quad p_1 = -rp, \quad p_2 = r\,.$$

- Thus, we have the functions $\lambda_1 = p_1 = -rp$ and $\lambda_2 = p_2 x^2 = rz$, which are homogeneous of degree 1, in involution, and functionally independent on a dense subset.

A toy example bis

- Projecting them to M, we obtain $\bar{\lambda}_1 = p$ and $\bar{\lambda}_2 = -z$, which are functionally independent and $\{\bar{\lambda}_1, \bar{\lambda}_2\} = \{\bar{\lambda}_1, h\} = \{\bar{\lambda}_2, h\} = 0$.
- Moreover, the angle coordinates $\varphi^1 = x^1 = q$ and $\varphi^2 = \log x^2 = \log z$ are 0-homogeneous, so they project into M. With a slight abuse of notation, we will also denote by φ^1 and φ^2 to the corresponding functions on M.
- Let $\bar{\lambda} = -\bar{\lambda}_1/\bar{\lambda}_2 = p/z$. In the chart $(U = M \setminus \{z = 0\}; \varphi^1, \varphi^2, \bar{\lambda})$, the contact Hamiltonian vector field reads $X_h = \partial_{\varphi^1} + \partial_{\varphi^2}$.
- Moreover, $\bar{\eta} = \mathrm{d}\varphi^2 - \bar{\lambda}\mathrm{d}\varphi^1$ is a contact form on U conformal to η (i.e., $\ker \bar{\eta} = \ker \eta$), and X_h is the Hamiltonian vector field of $\bar{h} = \bar{\lambda} - 1$ with respect to $\bar{\eta}$.

Conclusions

Future research

- Interesting examples
- A method for computing action-angle coordinates $\rightsquigarrow$ Hamilton–Jacobi equation?
- Delzant's theorem: classifying Hamiltonian actions by the image of the associated moment map, which is a polytope

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Dziękuję za uwagę!
Vă mulțumesc pentru atenție!

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