Nijenhuis–Jacobi structures and integrability of contact Hamiltonian systems

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Liouville–Arnol'd theorem

Theorem (Liouville–Arnol'd)

Let $f_1, \ldots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0\ \forall i, j$) on a symplectic manifold (M^{2n}, ω). Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$ be a regular level set.

1. *Any compact connected component of M_Λ is diffeomorphic to $\mathbb{T}^n$.*
2. *On a neighborhood of M_Λ there are coordinates (φ^i, J_i) such that*

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i \,,$$

and $f_i = f_i(J_1, \ldots, J_n)$, so the Hamiltonian vector fields read

$$X_{f_i} = \frac{\partial f_i}{\partial J_j} \frac{\partial}{\partial \varphi^j} \,.$$

Liouville–Arnol'd theorem

Corollary

Let (M^{2n}, ω, h) *be a Hamiltonian system. Suppose that* $f_1, \ldots, f_n$ *are independent conserved quantities (i.e.* $X_h(f_i) = 0 \ \forall\, i$*) in involution. Then, on a neighborhood of* M_Λ *there are Darboux coordinates* (φ^i, J_i) *such that* $H = H(J_1, \ldots, J_n)$*, so the Hamiltonian dynamics are given by*

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \frac{\partial H}{\partial J_i} \frac{\partial}{\partial \varphi^i}\,,$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0\,.$$

Problem

Given a Hamiltonian system (M^{2n}, ω, h), we would like to find n independent conserved quantities in involution $f_1, \ldots, f_n$, in order to construct action-angle coordinates (φ^i, J_i).

Magri *et al.* developed a method for constructing such conserved quantities by computing the eigenvalues of a $(1,1)$-tensor field N verifying certain compatibility conditions.

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Magri *et al.* developed a method for constructing such conserved quantities by computing the eigenvalues of a $(1,1)$-tensor field N verifying certain compatibility conditions.

Compatible Poisson structures

Definition

Let M be a manifold. Two Poisson tensors are Λ and Λ_1 on M are said to be **compatible** if $\Lambda + \Lambda_1$ is also a Poisson tensor on M.

Definition

A vector field $X \in \mathfrak{X}(M)$ is called **bi-Hamiltonian** if it is a Hamiltonian vector field w.r.t. two compatible Poisson structures, namely,

$$X = \Lambda(\mathrm{d}h, \cdot) = \Lambda_1(\mathrm{d}h_1, \cdot)\,,$$

for two functions $h, h_1 \in \mathscr{C}^\infty(M)$.

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Poisson – Nijehuis structures

- The linear map $\sharp_\Lambda \colon \mathrm{T}^*_x M \ni \alpha \mapsto \Lambda(\alpha, \cdot) \in \mathrm{T}_x M$ is an isomorphism iff Λ comes from a symplectic structure ω. In that case, $\sharp_\omega := \sharp_\Lambda^{-1}(v) = \iota_v \omega$.
- In that situation, we can define the $(1,1)$-tensor field

$$N = \sharp_{\Lambda_1} \circ \sharp_\Lambda^{-1} .$$

Poisson – Nijehuis structures

Theorem (Magri and Morosi, 1984)

Let (M, ω) be a symplectic manifold and Λ_1 a bivector. Consider the $(1, 1)$-tensor field

$$N = \sharp_{\Lambda_1} \circ \sharp_{\omega}^{-1} .$$

If Λ_1 is a Poisson tensor compatible with Λ, then the Nijehuis torsion T_N of N vanishes. In that case, the eigenvalues of N are in involution w.r.t. both Poisson brackets.
The pair (Λ, N) is called a ***Poisson – Nijenhuis structure*** *on M.*

Poisson – Nijehuis structures

Corollary

If a vector field $X \in \mathfrak{X}(M)$ is bi-Hamiltonian w.r.t. to ω and Λ_1 (i.e., $X = \sharp_\omega \mathrm{d}h = \sharp_{\Lambda_1} \mathrm{d}h_1$), then the eigenvalues of N form a family of conserved quantities in involution w.r.t. both Poisson brackets.

Contact geometry

Definition

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that the map

$$\begin{aligned} \flat_\eta \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

is an isomorphism of $\mathscr{C}^\infty(M)$-modules.

- There exists a unique vector field R on (M, η), called the **Reeb vector field**, given by $R = \flat_\eta^{-1}(\eta)$, or, equivalently,

$$\iota_R \mathrm{d}\eta = 0, \quad \iota_R \eta = 1.$$

Contact geometry

- The **Hamiltonian vector field** of $f \in \mathscr{C}^\infty(M)$ is given by

$$X_f = \flat_\eta^{-1}(\mathrm{d}f) - (R(f) + f)\, R\,,$$

- Around each point on M there exist **Darboux coordinates** (q^i, p_i, z) such that

$$\begin{aligned}
\eta &= \mathrm{d}z - p_i \mathrm{d}q^i, \\
R &= \frac{\partial}{\partial z}, \\
X_f &= \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}.
\end{aligned}$$

Contact Hamiltonian systems

Definition

A **contact Hamiltonian system** is a triple (M, η, h) formed by a contact manifold (M, η) and a **Hamiltonian function** $h \in \mathscr{C}^\infty(M)$.

- The dynamics of (M, η, h) is determined by the integral curves of the Hamiltonian vector field X_h of h w.r.t. η.

Contact Hamiltonian systems

- In Darboux coordinates, these curves $c(t) = (q^i(t), p_i(t), z(t))$ are determined by the **contact Hamilton equations**:

$$
\begin{aligned}
\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} &= \frac{\partial h}{\partial p_i} \circ c(t)\,, \\
\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -\frac{\partial h}{\partial q^i} \circ c(t) + p_i(t)\frac{\partial h}{\partial z} \circ c(t)\,, \\
\frac{\mathrm{d}z(t)}{\mathrm{d}t} &= p_i(t)\frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.
\end{aligned}
$$

Jacobi manifolds

Definition

A **Jacobi structure** on a manifold M is a pair (Λ, E) where Λ is a bivector and E a vector field such that the composition rule $\{\cdot,\cdot\}$ on $\mathscr{C}^\infty(M)$ given by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f)\,,$$

is a Lie bracket, called the **Jacobi bracket**. The triple (M, Λ, E) is called a **Jacobi manifold**.

In particular, $\{\cdot,\cdot\}$ is a Poisson bracket iff $E \equiv 0$.

Jacobi structure of a contact manifold

- A contact manifold (M, η) is endowed with a Jacobi bracket determined by

$$\{f, g\} = -\mathrm{d}\eta(\flat_\eta^{-1}\mathrm{d}f, \flat_\eta^{-1}\mathrm{d}g) - fR(g) + gR(f).$$

- It can also be expressed as follows:

$$\{f, g\} = X_f(g) + gR(f)\,.$$

Jacobi brackets and dissipated quantities

Definition

Let (M, η, h) be a contact Hamiltonian system with Jacobi bracket $\{\cdot, \cdot\}$. A function $f \in \mathscr{C}^\infty(M)$ is called a **dissipated quantity** if

$$\{f, h\} = 0\,.$$

Completely integrable contact system

Definition

A **completely integrable contact system** is a triple (M, η, F), where (M, η) is a contact manifold and $F = (f_0, \dots, f_n)\colon M \to \mathbb{R}^{n+1}$ is a map such that

1. $f_0, \dots, f_n$ are in involution, i.e., $\{f_\alpha, f_\beta\} = 0 \ \forall\, \alpha, \beta$,
2. $\operatorname{rank} \mathrm{T}F \geq n$ on a dense open subset $M_0 \subseteq M$.

The functions $f_0, \dots, f_n$ are called **integrals**.

Liouville – Arnol'd theorem for contact systems

1. Given $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+ \colon f_\alpha(x) = r\Lambda_\alpha\}$.
2. Assume that the Hamiltonian vector fields $X_{f_0}, \ldots, X_{f_n}$ are complete.
3. Let $B \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be an open neighbourhood of Λ.
4. Let $\pi \colon U \to M_{\langle\Lambda\rangle_+}$ be a tubular neighbourhood of $M_{\langle\Lambda\rangle_+}$ such that $F|_U \colon U \to B$ is a trivial bundle over a domain $V \subseteq B$.

Liouville – Arnol'd theorem for contact systems

Theorem (Colombo, de León, Lainz, L.-G., 2023)

Let (M, η, F) be a completely integrable contact system, where $F = (f_0, \ldots, f_n)$. Consider the assumptions of the previous slide. Then:

1. *$M_{\langle\Lambda\rangle_+}$ is coisotropic, invariant by the Hamiltonian flow of f_α, and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ for some $k \leq n$.*
2. *There exists coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on U such that the equations of motion are given by*

$$\dot{y}^\alpha = \Omega^\alpha(\tilde{A}_1, \ldots, \tilde{A}_n)\,, \quad \dot{\tilde{A}}_i = 0\,.$$

3. *There exists a nowhere-vanishing function $A_0 \in \mathscr{C}^\infty(U)$ and a conformally equivalent contact form $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$, namely, $\tilde{\eta} = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.*

Our goal

- We would like to generalize Magri *et al.*'s constructions for integrable contact systems.
- That is, given a contact Hamiltonian system (M, η, h), we want to find a tensor N such that, if it satisfies certain compatibility conditions with (η, h), one can compute dissipated quantities in involution for it.

Compatible Jacobi structures

- Nunes da Costa (1998) introduced the notion of compatibility of Jacobi structures.

Definition

Two Jacobi structures (Λ, E) and (Λ_1, E_1) on a manifold M are said to be **compatible** if $(\Lambda + \Lambda_1, E + E_1)$ is also a Jacobi structure on M.

- She also proved several conditions which are equivalent to (Λ, E) and (Λ_1, E_1) being compatible.

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Jacobi – Nijenhuis structures

- A Jacobi–Nijenhuis structure (Λ, E, N) is a generalization of Nijenhuis – Poisson structures.
- These structures were introduced by Marrero, Monterde and Padrón (1999).
- Their relation with homogeneous Nijenhuis – Poisson structures was studied by Petalidou and Nunes da Costa (2001).

Jacobi – Nijenhuis structures

- The space $\mathfrak{X}(M) \times \mathscr{C}^\infty(M)$ can be endowed with the Lie bracket $[\cdot, \cdot]$ given by
$$\Big[(X, f), (Y, g)\Big] = \Big([X, Y], X(g) - Y(f)\Big) .$$
- *Mutatis mutandis*, the Nijenhuis torsion T_N of a linear operator $N \colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M) \times \mathscr{C}^\infty(M)$ is defined as usual:
$$\begin{aligned} T_N\Big((X, f), (Y, g)\Big) \coloneqq \Big[N(X, f), N(Y, g)\Big] - N\Big[N(X, f), (Y, g)\Big] \\ - N\Big[(X, f), N(Y, g)\Big] + N^2\Big[(X, f), (Y, g)\Big] , \end{aligned}$$

Jacobi – Nijenhuis structures

- Given a Jacobi structure (Λ, E), we can define the $\mathscr{C}^\infty(M)$-modules homomorphism $\sharp_{(\Lambda,E)} \colon \Omega^1(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M) \times \mathscr{C}^\infty(M)$.
- It is an isomorphism iff (Λ, E) comes from a contact structure.

Jacobi – Nijenhuis structures

Definition

A **Jacobi – Nijenhuis structure** on a manifold M is a triple (Λ, E, N) where (Λ, E) is a Jacobi structure and $N\colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M) \times \mathscr{C}^\infty(M)$ is a $\mathscr{C}^\infty(M)$-linear map such that

$$
\begin{aligned}
&N \circ \sharp_{(\Lambda,E)} = \sharp_{(\Lambda,E)} \circ N^* \,,\\
&T_N \equiv 0 \,,\\
&\mathcal{C}\Big((\Lambda, E), N\Big) \equiv 0 \,.
\end{aligned}
$$

The 4-tuple (M, Λ, E, N) is calle a **Jacobi – Nijenhuis manifold**.

Jacobi – Nijenhuis structures

- In the previous slide, $\mathcal{C}$ denotes the **concomitant**. Its expression depends on N, (Λ, E) and a quite involved Lie bracket on $\Omega^1(M) \times \mathscr{C}^\infty(M)$.
- Let (Λ_1, E_1) be the Jacobi structure determined by

$$\Lambda_1(\beta, \alpha) = \Big\langle \beta, N_1\big(\Lambda(\cdot, \alpha), 0\big) \Big\rangle , \quad E_1 = N_1(E, 0) ,$$

  where $N_1 \colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M)$ is the projection of N on the first component.
- If (Λ_1, E_1) is also coming from a contact structure, then

$$T_N \equiv 0 \Longleftrightarrow \mathcal{C}\Big((\Lambda, E), N\Big) \equiv 0 .$$

The correspondence between Jacobi – Nijenhuis and homogeneous Nijenhuis – Poisson structures

Proposition (Petalidou and Nunes da Costa, 2001)

With any Jacobi – Nijenhuis manifold (M, Λ, E, N)*, we can associate a homogeneous Nijenhuis – Poisson manifold, namely, a Nijenhuis – Poisson manifold* $(M \times \mathbb{R}, \tilde{\Lambda}, \tilde{N})$ *such that*

$$\mathcal{L}_{\frac{\partial}{\partial t}} \tilde{\Lambda} = -\tilde{\Lambda}\,, \quad \mathcal{L}_{\frac{\partial}{\partial t}} \tilde{N} = 0\,,$$

where t *denotes the canonical coordinate on the* $\mathbb{R}$ *component of* $M \times \mathbb{R}$*.*

Exact symplectic manifolds: Liouville geometry

Definition

An **exact symplectic manifold** is a pair (M, θ), where M is a manifold and θ a one-form on N such that $\omega = -\mathrm{d}\theta$ is a symplectic form on M.

- The **Liouville vector field** Δ of (M, θ) is given by

$$\iota_\Delta \omega = -\theta.$$

- A tensor T is called **homogeneous of degree** n if $\mathcal{L}_\Delta T = nT$.

Symplectization of contact manifolds

Definition

Let (M, η) be a contact manifold and (M^Σ, θ) an exact symplectic manifold. A **symplectization** is a fibre bundle $\Sigma \colon M^\Sigma \to M$ such that

$$\sigma \Sigma^* \eta = \theta,$$

for a function σ on M^Σ called the **conformal factor**.

Symplectization of contact manifolds

$$\begin{gathered}\text{Category of contact manifolds}\\ \updownarrow \\ \text{Category of exact symplectic manifolds}\end{gathered}$$

- Contact distribution $\ker \eta \longleftrightarrow$ symplectic potential θ
- Functions $\longleftrightarrow$ homogeneous functions
- Hamiltonian vector fields $\longleftrightarrow$ Hamiltonian vector fields

Our result (under construction)

- Let (M, η) be a contact manifold with associated Jacobi structure (Λ, E).
- Suppose that there is another contact form η_1 on M with Jacobi structure (Λ_1, E_1).
- Let $N = \sharp_{(\Lambda_1, E_1)} \circ \sharp_{(\Lambda, E)}^{-1}$.
- (Λ, E) and (Λ_1, E_1) are compatible iff $T_N \equiv 0$.
- In that case, the eigenvalues of N are in involution w.r.t. the Jacobi brackets of both (Λ, E) and (Λ_1, E_1).

Our result (under construction)

- Let (Λ_1, E_1) be the Jacobi structure determined by

$$\Lambda_1(\beta, \alpha) = \left\langle \beta, N_1(\Lambda(\cdot, \alpha), 0) \right\rangle, \quad E_1 = N_1(E, 0),$$

  where $N_1 \colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M)$ is the projection of N on the first component.

- Consider a contact Hamiltonian system (M, η, h) such that $X_h = Y_{h_1}$ is the Hamiltonian vector field of h w.r.t. η and the Hamiltonian vector field of h_1 w.r.t. h_1, namely,

$$X_h = Y_{h_1} = \Lambda(\cdot, \mathrm{d}h_1) + h_1 E_1 .$$

- The spectrum of $\tilde{N}$ can be used to compute dissipated quantities in involution.

Main references

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Thanks for your kind attention!

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