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Manuel de León, Manuel Lainz, and Asier López-Gordón

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Manuel de León,[a] Manuel Lainz, and Asier López-Gordón[b]

AFFILIATIONS

Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) Calle Nicolás Cabrera, 13-15, Campus Cantoblanco, UAM, 28049 Madrid, Spain

[a]**Also at:** Real Academia de Ciencias Exactas, Físicas y Naturales, Calle Valverde, 22, 28004 Madrid, Spain.
[b]**Author to whom correspondence should be addressed:** asier.lopez.gordon@csic.es

ABSTRACT

This paper is devoted to the study of mechanical systems subjected to external forces in the framework of symplectic geometry. We obtain Noether's theorem for Lagrangian systems with external forces, among other results regarding symmetries and conserved quantities. We particularize our results for the so-called Rayleigh dissipation, i.e., external forces that are derived from a dissipation function, and illustrate them with some examples. Moreover, we present a theory for the reduction in Lagrangian systems subjected to external forces, which are invariant under the action of a Lie group.

I. INTRODUCTION

In this paper, we study the geometry and symmetries of Hamiltonian and Lagrangian systems with external forces, focusing on the so-called systems with Rayleigh dissipation. Mechanical systems with external forces are common in engineering[1–3] but can also arise in a more sophisticated manner, for instance, after a process of reduction in a nonholonomic system with symmetries.[3–5] As is well-known (see Refs. 6 and 7), external forces can be regarded as semibasic one-forms on the tangent or cotangent bundle. Our approach is based on the symplectic structure obtained from a regular Lagrangian in the Lagrangian formulation and the geometry of the tangent bundle. There are other ways to treat with symmetries, for instance, a variational approach as in Ref. 8.

The main result in the presence of symmetries is the celebrated Noether theorem. See Ref. 9 for the original paper by Noether (see also Refs. 10 and 11). In our case, despite the existence of a non-conservative external force, we are able to extend the Noether theorem and to also obtain new conserved quantities. Our approach is just an appropriate modification of the well-known results for conservative mechanical systems (i.e., with no external forces).[7,12–27] Hence, we first define point-base symmetries (that is, those provided by vector fields on the configuration manifold Q) and then symmetries on the tangent bundle.

There are other approaches that can be found in the previous literature and have some relation with ours. For instance, Cantrijn[1] considered Lagrangian systems that depend explicitly on time and defined a two-form on $\mathbb{R} \times TQ$ that depends on the Poincaré–Cartan two-form of the Lagrangian and the semibasic one-form representing the external force. Alternatively, van der Schaft[28,29] considered a framework stemming from system theory in which an "observation" manifold appears together with the usual state space and obtained Noether's theorem for Hamiltonian system in this frame. Other approaches using variational tools can be found in Ref. 8. However, in our approach, no additional structure or objects are introduced besides the proper external force.

This paper is organized as follows. In Secs. II and IV, we review Hamiltonian and Lagrangian systems with external forces, respectively. In Sec. III, we cover the relation between fiber bundle morphisms and semibasic one-forms. In Sec. V, we present some (as far as we know) original results concerning symmetries and constants of the motion for mechanical systems with external forces. In Sec. VI, we study the symmetries and constants of the motion in the Hamiltonian framework. We relate these symmetries with the ones obtained for Lagrangian systems in Sec. V. In Sec. VII, we particularize the results of Secs. V and VI for the Rayleigh dissipation. Classically,[30–32] only external forces that are linear on the velocities are regarded as examples of Rayleigh dissipation. However, following Lurie[33] and Minguzzi,[34] we consider a wider family of external forces as Rayleigh dissipation, namely, forces that are derived from a dissipation function (which is not necessarily quadratic on the velocities). Finally, in Sec. VIII, we present a scheme for reduction in Lagrangian systems subjected to external forces, which are invariant under the action of a Lie group.

II. HAMILTONIAN SYSTEMS SUBJECT TO EXTERNAL FORCES

An external force is geometrically interpreted as a semibasic one-form on T^*Q. Let us recall[6,7,35] that a one-form γ on T^*Q is called *semibasic* if

$$\gamma(Z) = 0$$

for all vertical vector fields Z.

Remark 1. This definition can be extended to any fiber bundle $\pi : E \to M$. Indeed, a one-form γ on E is called semibasic if

$$\gamma(Z) = 0$$

for all vertical vector fields Z on E. If (x^i, y^a) are fibered (bundle) coordinates, then the vertical vector fields are locally generated by $\{\partial/\partial y^a\}$. Hence, γ is a semibasic one-form if it is locally written as

$$\gamma = \gamma_i(x, y)\mathrm{d}x^i.$$

A Hamiltonian system with external forces is given by a Hamiltonian function $H : T^*Q \to \mathbb{R}$ and a semibasic one-form γ on T^*Q. Let $\omega_Q = -\mathrm{d}\alpha_Q$ be the canonical symplectic form of T^*Q. Locally, these objects can be written as

$$\begin{aligned}
\alpha_Q &= p_i \mathrm{d}q^i,\\
\omega_Q &= \mathrm{d}q^i \wedge \mathrm{d}p_i,\\
\gamma &= \gamma_i(q, p)\mathrm{d}q^i,\\
H &= H(q, p),
\end{aligned}$$

where (q^i, p_i) are bundle coordinates in T^*Q.

The dynamics of the system is given by the vector field $X_{H,\gamma}$, defined by

$$\iota_{X_{H,\gamma}}\omega_Q = \mathrm{d}H + \gamma.$$

If X_H is the Hamiltonian vector field for H, that is,

$$\iota_{X_H}\omega_Q = \mathrm{d}H, \tag{1}$$

and Z_γ is the vector field defined by

$$\iota_{Z_\gamma}\omega_Q = \gamma,$$

then we have

$$X_{H,\gamma} = X_H + Z_\gamma.$$

Locally, the above equations can be written as

$$\begin{aligned}
X_H &= \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i},\\
\gamma &= \gamma_i \mathrm{d}q^i,\\
Z_\gamma &= -\gamma_i \frac{\partial}{\partial p_i},\\
X_{H,\gamma} &= \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + \gamma_i\right)\frac{\partial}{\partial p_i}.
\end{aligned} \tag{2}$$

Then, a curve $[q^i(t), p_i(t)]$ in T^*Q is an integral curve of $X_{H,\gamma}$ if and only if it satisfies the forced motion equations,

$$\begin{aligned}
\frac{\mathrm{d}q^i}{\mathrm{d}t} &= \frac{\partial H}{\partial p_i},\\
\frac{\mathrm{d}p_i}{\mathrm{d}t} &= -\left(\frac{\partial H}{\partial q^i} + \gamma_i\right).
\end{aligned}$$

III. SEMIBASIC FORMS AND FIBERED MORPHIMS

Given a semibasic one-form γ on TQ, one can define the following morphism of fiber bundles:[6,7]

$$D_\gamma : TQ \to T^*Q,$$
$$\langle D_\gamma(v_q), w_q\rangle = \gamma(v_q)(u_{w_q})$$

for every $v_q, w_q \in T_qQ,\ u_{w_q} \in T_{w_q}(TQ)$, with $T\tau_Q(u_{w_q}) = w_q$. In local coordinates, if

$$\gamma = \gamma_i(q,\dot{q})\mathrm{d}q^i,$$

then

$$D_\gamma(q^i,\dot{q}^i) = \big(q^i, \gamma_i(q^i,\dot{q}^i)\big).$$

Here, $(q^i,\dot{q}^i)$ are bundle coordinates in TQ.

Conversely, given a morphism of fiber bundles

$$\begin{tikzcd} D : TQ \arrow[rr] \arrow[dr, "\tau_q"'] & & T^*Q \arrow[dl, "\pi_Q"] \\ & Q & \end{tikzcd},$$

we define a semibasic one-form γ on TQ by

$$\gamma_D(v_q)(u_{v_q}) = \big\langle D(v_q), T\tau_Q(u_{v_q})\big\rangle,$$

where $v_q \in T_qQ,\ u_{v_q} \in T_{v_q}(TQ)$.

If locally D is given by

$$D(q^i,\dot{q}^i) = (q^i, D_i(q,\dot{q})),$$

then

$$\gamma_D = D_i(q,\dot{q})\mathrm{d}q^i.$$

Therefore, there exists a one-to-one correspondence between semibasic one-forms and fibered morphisms from TQ to T^*Q.

IV. LAGRANGIAN SYSTEMS WITH EXTERNAL FORCES

We shall now consider a Lagrangian system with a Lagrangian function L subjected to external forces. An external force is given by a semibasic one-form β on TQ. In bundle coordinates, we have

$$\beta = \beta_i(q,\dot{q})\mathrm{d}q^i.$$

If $L : TQ \to \mathbb{R}$, then $\omega_L = -\mathrm{d}\alpha_L$ is the *Poincaré–Cartan two-form*, where $\alpha_L = S^*(\mathrm{d}L)$. Here, S is the vertical endomorphism of TQ, which in local coordinates is given by

$$S = \mathrm{d}q^i \otimes \frac{\partial}{\partial \dot{q}^i}.$$

Hence,

$$\omega_L = \mathrm{d}q^i \wedge \mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^i}\right).$$

Then, the dynamics is given by the vector field $\xi_{L,\beta}$ via the following equation:

$$\iota_{\xi_{L,\beta}}\omega_L = \mathrm{d}E_L + \beta, \tag{3}$$

where $E_L = \Delta(L) - L$ is the energy of the system and Δ is the *Liouville vector field*,

$$\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}.$$

Here, we are assuming that L is *regular*, that is, the Hessian matrix

$$(W_{ij}) = \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \tag{4}$$

is invertible. It can be easily proven that ω_L is symplectic if and only if L is regular.[7] Let ξ_β be the vector field given by

$$\iota_{\xi_\beta}\omega_L = \beta$$

and ξ_L be the vector field given by

$$\iota_{\xi_L}\omega_L = \mathrm{d}E_L, \tag{5}$$

then

$$\xi_{L,\beta} = \xi_L + \xi_\beta.$$

We have

$$\xi_\beta = -\beta_i W^{ij}\frac{\partial}{\partial \dot{q}^j},$$

where (W^{ij}) is the inverse matrix of (W_{ij}). Then, $\xi_{L,\beta}$ is a *second order differential equation (SODE)*, meaning that

$$S(\xi_{L,\beta}) = S(\xi_L) = \Delta. \tag{6}$$

We know that

$$\xi_L = \dot{q}^i\frac{\partial}{\partial q^i} + \xi^i\frac{\partial}{\partial \dot{q}^i},$$

where

$$\xi^i\frac{\partial p_j}{\partial \dot{q}^i} + \dot{q}^i\frac{\partial p_j}{\partial q^i} - \frac{\partial L}{\partial q^j} = 0. \tag{7}$$

Then,

$$\xi_{L,\beta} = \dot{q}^i\frac{\partial}{\partial q^i} + \left(\xi^i - \beta_j W^{ij}\right)\frac{\partial}{\partial \dot{q}^i}.$$

Hence, a solution of $\xi_{L,\beta}$, $[q^i(t)]$, satisfies

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \dot{q}^i,$$
$$\frac{\mathrm{d}\dot{q}^i}{\mathrm{d}t} = \xi^i - \beta_j W^{ji}.$$

Therefore, from Eq. (7), we get

$$\ddot{q}^i\frac{\partial p_j}{\partial \dot{q}^i} + \dot{q}^i\frac{\partial p_j}{\partial q^i} - \frac{\partial L}{\partial q^j} + \beta_k W^{ki}\frac{\partial p_j}{\partial \dot{q}^i} = 0.$$

Since $p_j = \partial L/\partial \dot{q}^j$, the term $\partial p_j/\partial \dot{q}^i$ is equal to W_{ji}, and thus, we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = -\beta_i.$$

If we construct the Legendre transform[7]

$$\begin{tikzcd} TQ \arrow[rr, "\mathrm{Leg}"] \arrow[dr, "\tau_q"'] & & T^*Q \arrow[dl, "\pi_Q"] \\ & Q & \end{tikzcd}$$

(and let L be *hyperregular*, that is, Leg is a diffeomorphism), then we can define the external force γ on T^*Q by

$$\mathrm{Leg}^*\gamma = \beta.$$

Thus, $\xi_{L,\beta}$ and $X_{H,\gamma}$ are Leg-related, that is, Leg takes $\xi_{L,\beta}$ onto $X_{H,\gamma}$, where H is defined by

$$H \circ \mathrm{Leg} = E_L. \tag{8}$$

Definition 1. In what follows, we will refer to the pair (L, β) for a forced Lagrangian system given by a Lagrangian L and a semibasic one-form β. The corresponding vector field $\xi_{L,\beta}$, given by Eq. (3), will be called *forced Euler–Lagrange vector field.*

Remark 2. Take

$$\alpha_L = S^*(\mathrm{d}L) = p_i \mathrm{d}q^i,$$

where $p_i = \partial L/\partial \dot{q}^i$; then α_L is a semibasic one-form on TQ, and the corresponding fibered map is just the Legendre transform,

$$\mathrm{Leg} : TQ \to T^*Q.$$

V. SYMMETRIES AND CONSTANTS OF THE MOTION IN THE LAGRANGIAN DESCRIPTION

Let $f : TQ \to \mathbb{R}$ be an arbitrary function and $\tau_Q : TQ \to Q$ be the projection. Then, the *vertical lift*[36,37] of f is a function $f^v : TQ \to Q$ given by

$$f^v = f \circ \tau_Q.$$

Any one-form ω in Q can be naturally regarded as a function on TQ, which we shall denote by $\iota\omega$. If X is a vector field on Q, its *vertical lift* is the unique vector field X^v on TQ such that

$$X^v(\iota\omega) = (\alpha(X))^v$$

for every one-form α on Q. The *complete lift* of a function f on Q is the function f^c on TQ given by

$$f^c = \iota(\mathrm{d}f).$$

The *complete lift* of a vector field X on Q is the vector field X^c on TQ such that

$$X^c(f^c) = (X(f))^c$$

for every function f on Q. If X generates locally a one-parameter group of transformations on Q, then X^c generates the induced transformations on TQ.[7] Locally, if X is given by

$$X = X^i \frac{\partial}{\partial q^i},$$

then its *vertical lift* is

$$X^v = X^i \frac{\partial}{\partial \dot{q}^i}$$

and its *complete lift* is

$$X^c = X^i \frac{\partial}{\partial q^i} + \dot{q}^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}.$$

Let (L, β) be a Lagrangian system with a Lagrangian function L and external force β; denote by $\xi_{L,\beta}$ the corresponding forced Euler–Lagrange vector field.

Definition 2. A function f on TQ is called a *constant of the motion* (or a *conserved quantity*) if $\xi_{L,\beta}(f) = 0$.

Suppose that, for a certain coordinate q^i, $\partial L/\partial q^i = \beta_i$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = 0,$$

and $p_i = \partial L/\partial \dot{q}^i$ is a constant of the motion. This motivates the following theorem.

Theorem 1 (Noether's theorem for dissipative systems). *Let X be a vector field on Q. Then, $X^c(L) = \beta(X^c)$ if and only if $X^v(L)$ is a constant of the motion.*

Proof. By Eq. (3), we can write

$$\begin{aligned}(\mathrm{d}E_L + \beta)(X^c) &= \left(\iota_{\xi_{L,\beta}}\omega_L\right)(X^c) = -\mathrm{d}\alpha_L(\xi_{L,\beta}, X^c)\\ &= -\xi_{L,\beta}(\alpha_L(X^c)) + X^c(\alpha_L(\xi_{L,\beta})) + \alpha_L([\xi_{L,\beta}, X^c]).\end{aligned}$$

Now, since $\xi_{L,\beta}$ is a SODE, we have

$$\alpha_L(\xi_{L,\beta}) = \iota_{\xi_{L,\beta}}(S^* \mathrm{d}L) = (S\xi_{L,\beta})L = \Delta L.$$

It is easy to see that $SX^c = X^v$. Moreover, $[\xi_{L,\beta}, X^c]$ is a vertical vector field, and thus, $S[\xi_{L,\beta}, X^c] = 0$. Then,

$$(\mathrm{d}E_L + \beta)(X^c) = -\xi_{L,\beta}(X^v L) + X^c(\Delta L).$$

On the other hand, we can write

$$\mathrm{d}E_L(X^c) = X^c(E_L) = X^c(\Delta L) - X^c(L).$$

Combining these last two equations, one deduces

$$\xi_{L,\beta}(X^v L) = X^c(L) - \beta(X^c).$$

In particular, the right-hand side vanishes if and only if the left-hand side does. □

Definition 3. Consider the forced Lagrangian system (L, β). Then, the following holds:

(i) A *symmetry of the forced Lagrangian system* is a vector field X on Q such that $X^c(L) = \beta(X^c)$.
(ii) A *Lie symmetry* is a vector field X on Q such that $[X^c, \xi_{L,\beta}] = 0$.
(iii) A *Noether symmetry* is a vector field X on Q such that $X^c(E_L) + \beta(X^c) = 0$ and $\mathcal{L}_{X^c}\alpha_L$ is exact.

Proposition 2. If X is a vector field on Q such that

$$\mathrm{d}(\mathcal{L}_{X^c}\alpha_L) = 0,$$

then X is a Lie symmetry if and only if

$$\mathcal{L}_{X^c}\beta = -\mathrm{d}(X^c(E_L)).$$

Proof. Indeed,

$$\begin{aligned}\iota_{[X^c,\xi_{L,\beta}]}\omega_L &= \mathcal{L}_{X^c}(\iota_{\xi_{L,\beta}}\omega_L) - \iota_{\xi_{L,\beta}}(\mathcal{L}_{X^c}\omega_L)\\ &= \mathcal{L}_{X^c}(\mathrm{d}E_L + \beta) + \iota_{\xi_{L,\beta}}\mathrm{d}(\mathcal{L}_{X^c}\alpha_L)\\ &= \mathrm{d}(X^c(E_L)) + \mathcal{L}_{X^c}\beta.\end{aligned}$$

Since ω_L is non-degenerate, then $[X^c, \xi_{L,\beta}]$ vanishes if and only if $\iota_{[X^c,\xi_{L,\beta}]}\omega_L$ does (Fig. 1). □

Proposition 3. A Noether symmetry is a Lie symmetry if and only if

$$\iota_{X^c}\mathrm{d}\beta = 0.$$

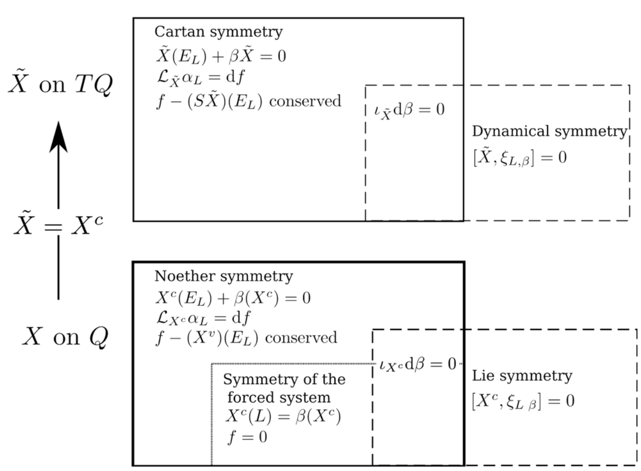

FIG. 1. Types of symmetries on Q and on TQ. The complete lifts of Lie symmetries and Noether symmetries correspond to dynamical symmetries and Cartan symmetries, respectively. The symmetries of the forced Lagrangian system are the subset of Noether symmetries for which $\mathcal{L}_{X^c}\alpha_L = 0$, so f is a constant that can be taken as 0 without loss of generality. The Noether symmetries that satisfy $\iota_{X^c}\mathrm{d}\beta = 0$ are also Lie symmetries and analogous with the intersection between Cartan and dynamical symmetries.

Proof. Since $\mathcal{L}_{X^c}\alpha_L$ is exact, it can be written as $\mathcal{L}_{X^c}\alpha_L = \mathrm{d}f$ for some function $f : TQ \to \mathbb{R}$. Obviously, $\mathrm{d}(\mathcal{L}_{X^c}\alpha_L) = \mathrm{d}(\mathrm{d}f) = 0$. In addition,

$$\begin{aligned}\mathcal{L}_{X^c}\beta &= \iota_{X^c}(\mathrm{d}\beta) + \mathrm{d}(\iota_{X^c}\beta) = \iota_{X^c}(\mathrm{d}\beta) + \mathrm{d}(\beta(X^c))\\ &= \iota_{X^c}(\mathrm{d}\beta) - \mathrm{d}(X^c(E_L)).\end{aligned}$$

By Proposition 2, the result holds. □

Proposition 4. Let X be a vector field on Q such that

$$\mathcal{L}_{X^c}\alpha_L = \mathrm{d}f.$$

Then, X is a Noether symmetry if and only if $f - X^v(L)$ is a conserved quantity.

Proof. Indeed,

$$\begin{aligned}\mathrm{d}f &= \mathcal{L}_{X^c}\alpha_L = \iota_{X^c}(\mathrm{d}\alpha_L) + \mathrm{d}(\iota_{X^c}\alpha_L) = \iota_{X^c}(\mathrm{d}\alpha_L) + \mathrm{d}(\iota_{X^c}S^*\mathrm{d}L)\\ &= \iota_{X^c}(\mathrm{d}\alpha_L) + \mathrm{d}(\iota_{SX^c}\mathrm{d}L) = \iota_{X^c}(\mathrm{d}\alpha_L) + \mathrm{d}(X^vL),\end{aligned}$$

so

$$\iota_{\xi_{L,\beta}}\iota_{X^c}(\mathrm{d}\alpha_L) = \iota_{\xi_{L,\beta}}(\mathrm{d}(f - X^vL)) = \xi_{L,\beta}(f - X^v(L)),$$

but

$$\iota_{\xi_{L,\beta}}\iota_{X^c}\mathrm{d}\alpha_L = \iota_{X^c}\iota_{\xi_{L,\beta}}\omega_L = \iota_{X^c}(\mathrm{d}E_L + \beta) = X^c(E_L) + \beta(X^c),$$

and the result holds. □

Observe that this last proposition is a generalization of Theorem 1. In other words, every symmetry of the forced Lagrangian system is a Noether symmetry. In fact, if f is a constant function, clearly, X^vL is a conserved quantity. Moreover,

$$\mathcal{L}_{X^c}\alpha_L = 0,$$

so

$$\begin{aligned}0 &= (\mathcal{L}_{X^c}\alpha_L)(\xi_{L,\beta}) = X^c\big(\alpha_L(\xi_{L,\beta})\big) - \alpha_L([X^c,\xi_{L,\beta}])\\ &= X^c(\Delta L) - S[X^c,\xi_{L,\beta}]L = X^c(\Delta L),\end{aligned}$$

and thus,

$$0 = X^c(E_L) + \beta(X^c) = X^c(\Delta L) - X^c(L) + \beta(X^c) = -X^c(L) + \beta(X^c).$$

Remark 3. A Noether symmetry is a symmetry of the forced Lagrangian system if and only if $\mathcal{L}_{X^c}\alpha_L = 0$.

We have just discussed infinitesimal symmetries on Q, the so-called point-like symmetries.[19] We shall now cover symmetries that are not necessarily point-like, that is, vector fields on TQ.

Definition 4. A *dynamical symmetry* of $\xi_{L,\beta}$ is a vector field $\tilde{X}$ on TQ such that $[\tilde{X},\xi_{L,\beta}] = 0$. A *Cartan symmetry* is a vector field $\tilde{X}$ on TQ such that $\tilde{X}(E_L) + \beta(\tilde{X}) = 0$ and $\mathcal{L}_{\tilde{X}}\alpha_L = \mathrm{d}f$.

Remark 4. Let X be a vector field on Q. Then, the following holds:

(i) X is a Lie symmetry if and only if X^c is a dynamical symmetry.
(ii) X is a Noether symmetry if and only if X^c is a Cartan symmetry.

Proposition 5. If $\tilde{X}$ is a vector field on TQ such that

$$\mathrm{d}(\mathcal{L}_{\tilde{X}}\alpha_L) = 0,$$

then $\tilde{X}$ is a dynamical symmetry if and only if

$$\mathrm{d}(\tilde{X}(E_L)) = -\mathcal{L}_{\tilde{X}}\beta.$$

Proposition 6. A Cartan symmetry is a dynamical symmetry if and only if

$$\iota_{\tilde{X}}\mathrm{d}\beta = 0.$$

Proposition 7. Let $\tilde{X}$ be a vector field on TQ such that

$$\mathcal{L}_{\tilde{X}}\alpha_L = \mathrm{d}f.$$

Then, $\tilde{X}$ is a Cartan symmetry if and only if $f - (S\tilde{X})L$ is a constant of the motion.

The proofs are completely analogous to those for point-like symmetries. Note that Theorem 1 cannot be generalized for symmetries on TQ since $[\xi_{L,\beta},\tilde{X}]$ is not a vertical vector field for a general $\tilde{X}$ on TQ.

VI. SYMMETRIES AND CONSTANTS OF THE MOTION IN THE HAMILTONIAN DESCRIPTION

Let α and $\hat{X}$ be a one-form and a vector field on T^*Q, respectively. We say that α is a *first integral* of $\hat{X}$ if $\alpha(\hat{X}) = 0$. Similarly, a function F on T^*Q is called a *first integral* of $\hat{X}$ if $\mathrm{d}F(\hat{X}) = \hat{X}(F) = 0$.

Let (H, γ) be a Hamiltonian system with Hamiltonian function H and external force γ. Let $X_{H,\gamma}$ be the corresponding Hamiltonian vector field. A first integral of $X_{H,\gamma}$ is called a *constant of the motion* or a *conserved quantity*.

Let F and G be two functions on T^*Q. Let X_F and X_G be their corresponding Hamiltonian vector fields, namely, $\iota_{X_F}\omega_Q = \mathrm{d}F$ and $\iota_{X_G}\omega_Q = \mathrm{d}G$. The *Poisson bracket* of F and G is given by

$$\{F, G\} = \omega_Q(X_F, X_G).$$

Let α and β be one-forms on T^*Q, with X_α and X_β being their corresponding Hamiltonian vector fields. Then, their Poisson bracket is defined as

$$\{\alpha, \beta\} = -\iota_{[X_\alpha, X_\beta]}\omega_Q.$$

Clearly,

$$\begin{aligned} X_{H,\gamma}(F) &= \iota_{X_{H,\gamma}}\mathrm{d}F = \iota_{X_{H,\gamma}}(\iota_{X_F}\omega_Q) = -\iota_{X_F}(\iota_{X_{H,\gamma}}\omega_Q) \\ &= -\omega_Q(X_H, X_F) - \gamma(X_F) = \{F, H\} - \gamma(X_F), \end{aligned}$$

and hence, F is a constant of the motion if and only if

$$\{F, H\} = \gamma(X_F).$$

Proposition 8. If $\hat{X}$ is a vector field on T^*Q such that $\mathcal{L}_{\hat{X}}\alpha_Q$ is closed, then $\hat{X}$ commutes with $X_{H,\gamma}$ if and only if

$$\mathrm{d}(\hat{X}(H)) = -\mathcal{L}_{\hat{X}}\gamma.$$

Proposition 9. Let $\hat{X}$ be a vector field on T^*Q such that

$$\mathcal{L}_{\hat{X}}\alpha_Q = \mathrm{d}f.$$

Then, $\hat{X}(H) + \gamma(\hat{X}) = 0$ if and only if $f - \alpha_Q(\hat{X})$ is a constant of the motion. Additionally, $\hat{X}$ commutes with $X_{H,\gamma}$ if and only if

$$\iota_{\hat{X}}\mathrm{d}\gamma = 0.$$

Now, suppose that (L, β) is a Lagrangian system such that $H \circ \mathrm{Leg} = E_L$ and $\mathrm{Leg}^*\gamma = \beta$. Let $\tilde{X}$ be a vector field on TQ and $\hat{X}$ be the Leg-related vector field on T^*Q. Then, the following holds:

(i) $\hat{X}$ commutes with $X_{H,\gamma}$ if and only if $\tilde{X}$ is a dynamical symmetry of (L, β).
(ii) $\mathcal{L}_{\hat{X}}\alpha_Q = \mathrm{d}f$ if and only if $\mathcal{L}_{\tilde{X}}\alpha_L = \mathrm{d}g$, where $g = f \circ \mathrm{Leg}$.
(iii) Suppose that $\mathcal{L}_{\hat{X}}\alpha_Q = \mathrm{d}f$. Then, the following assertions are equivalent.
 (a) $\hat{X}(H) + \gamma(\hat{X}) = 0$.
 (b) $f - \alpha_Q(\hat{X})$ is a conserved quantity.
 (c) $\tilde{X}(E_L) + \beta(\tilde{X}) = 0$.
 (d) $f \circ \mathrm{Leg} - \alpha_L(\tilde{X})$ is a conserved quantity.

VII. RAYLEIGH DISSIPATION

A. Rayleigh dissipation function and one-form

Rayleigh[30] considered the hypothesis that there is a non-conservative force linear on the velocities. This external force can be described as a semibasic one-form on TQ as follows:

$$R = R_{ij}(q)\dot{q}^i \mathrm{d}q^j, \tag{9}$$

where R_{ij} is symmetric. Of course, R can be described as a bilinear form on TQ,

$$\begin{aligned} R &: TQ \times TQ \to \mathbb{R}, \\ R(q^i, \dot{q}_1^i, \dot{q}_2^j) &= R_{ij}\dot{q}_1^i \dot{q}_2^j, \end{aligned}$$

or, in other words, a symmetric $(0, 2)$-tensor R on Q. Since R is a $(0, 2)$-tensor on Q, it defines a linear mapping

$$\tilde{R} : TQ \to T^*Q$$

by

$$\tilde{R}(v_q) = \iota_{v_q} R,$$

that is,

$$\tilde{R}(v_q)(w_q) = R(v_q, w_q).$$

Therefore,

$$\tilde{R}(q^i, \dot{q}^i) = (q^i, R_{ij}(q)\dot{q}^i\dot{q}^j),$$

so $\tilde{R}$ defines a semibasic one-form $\bar{R}$ on TQ given by

$$\bar{R} = R_{ij}(q)\dot{q}^i \mathrm{d}q^j. \tag{10}$$

In the literature,[31,32] the Rayleigh dissipation function is defined as

$$\mathcal{R}(q, \dot{q}) = \frac{1}{2} R_{ij}(q)\dot{q}^i\dot{q}^j, \tag{11}$$

so we can write

$$\bar{R} = \frac{\partial \mathcal{R}}{\partial \dot{q}^i}\mathrm{d}q^i = S^*(\mathrm{d}\mathcal{R}).$$

Note that external forces of the form $S^*(\mathrm{d}\mathcal{F})$, for some function $\mathcal{F}$, are quite more general than the form (9) originally proposed by Rayleigh.[30] That is, we do not need to require $\mathcal{R}$ to be of the form (11). In fact, more general dissipation functions are studied in Refs. 33 and 34. This function $\mathcal{R}$ can be physically interpreted as a potential that depends on the velocities from which the external force is derived.

Remark 5. $\mathcal{R}$ and $\tilde{\mathcal{R}} = \mathcal{R} + f$ define the same one-form $\bar{R}$ for $f : Q \to \mathbb{R}$ arbitrary.

We shall now consider a Lagrangian system with a hyperregular Lagrangian function L, which is subject to an external force linear on the velocities. Suppose that the force can be described through the Rayleigh dissipation function $\mathcal{R}$. Then, the equations of motion of the system are the integral curves of the vector field $\xi_{L,\bar{R}}$, given by

$$\iota_{\xi_{L,\bar{R}}}\omega_L = \mathrm{d}E_L + \bar{R}.$$

Let $\xi_{\bar{R}}$ be the vector field given by

$$\iota_{\xi_{\bar{R}}}\omega_L = \bar{R}.$$

Then,

$$\xi_{L,\bar{R}} = \xi_L + \xi_{\bar{R}},$$

where ξ_L is the vector field given by Eq. (5). We have

$$\xi_{\bar{R}} = -R_{ik}\dot{q}^k W^{ij}\frac{\partial}{\partial \dot{q}^j},$$

where W^{ij} is the inverse of the Hessian matrix of the Lagrangian (4). Then, $\xi_{L,\bar{R}}$ is a SODE in the sense of Eq. (6), and the equations of motion of the system are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = -R_{ij}(q)\dot{q}^j = -\frac{\partial \mathcal{R}}{\partial \dot{q}^i}.$$

It is easy to see that

$$\xi_{L,\beta}(E_L) + \Delta(\mathcal{R}) = 0,$$

where Δ is the Liouville vector field. In particular, if $\mathcal{R}$ is of the form (11), then $\Delta(\mathcal{R}) = 2\mathcal{R}$.

We can also consider the Hamiltonian formalism for the Rayleigh dissipation. Indeed, since we have assumed L to be hyperregular, we can always define the external force $\hat{R}$ on T^*Q by

$$\bar{R} = \mathrm{Leg}^*\hat{R}$$

and consider the Hamiltonian function given by Eq. (8). Locally, $\hat{R}$ can be written as

$$\hat{R} = R_{ij}(q)p_i \mathrm{d}q^j.$$

Then, the equations of motion of the system are the integral curves of the vector field $X_{H,\hat{R}}$, given by

$$\iota_{X_{H,\hat{R}}}\omega_Q = \mathrm{d}H + \hat{R}.$$

If $Z_{\hat{R}}$ is the vector field defined by

$$\iota_{Z_{\hat{R}}}\omega_Q = \hat{R},$$

then we have

$$X_{H,\hat{R}} = X_H + Z_{\hat{R}},$$

where X_H is the Hamiltonian vector field given by Eq. (1). In canonical coordinates, X_H and $\hat{R}$ are given by Eqs. (2) and (10), respectively, and we have

$$Z_{\hat{R}} = -R_{ij}(q)p_i\frac{\partial}{\partial p_j},$$

$$X_{H,\hat{R}} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + R_{ij}(q)p_j\right)\frac{\partial}{\partial p_i},$$

where we have made use of the fact that R_{ij} is symmetric. Thus, the equations of motion are given by

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i},$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\left(\frac{\partial H}{\partial q^i} + R_{ij}(q)p_j\right).$$

As we have shown in Sec. III, given the semibasic one-form $\bar{R}$, we can define the following morphism of fibered bundles:

$$\begin{tikzcd} D_{\bar{R}} : TQ \arrow[rr] \arrow[dr, "\tau_q"'] & & T^*Q \arrow[dl, "\pi_Q"] \\ & Q & \end{tikzcd},$$

$$\langle D_{\bar{R}}(v_q), w_q\rangle = \bar{R}(v_q)(u_{w_q})$$

for every $v_q, w_q \in T_qQ$, $u_{w_q} \in T_{w_q}(TQ)$, with $T\tau_Q(u_{w_q}) = w_q$. In local coordinates, we have

$$D_{\bar{R}}(q^i, \dot{q}^i) = \left(q^i, R_{ij}(q)\dot{q}^j\right).$$

B. Constants of the motion for Rayleigh dissipation

We shall now consider the case in which the external force is derived from a dissipation function $\mathcal{R}$ (not necessarily quadratic in the velocities).

Lemma 10. Consider a semibasic one-form $\bar{R}$ on TQ given by

$$\bar{R} = S^*(\mathrm{d}\mathcal{R})$$

for some function $\mathcal{R} : TQ \to \mathbb{R}$, where S^* is the adjoint of the vertical endomorphism. Then, for each vector field X on Q,

$$\bar{R}(X^c) = X^v(\mathcal{R}),$$
$$\mathcal{L}_{X^c}\bar{R} = S^*(\mathrm{d}(X^c(\mathscr{R}))). \tag{12}$$

Similarly, for each vector field $\tilde{X}$ on TQ,

$$\bar{R}(\tilde{X}) = (S\tilde{X})(\mathcal{R}).$$

Proof. Indeed,

$$\bar{R}(\tilde{X}) = \iota_{\tilde{X}}\bar{R} = \iota_{\tilde{X}}(S^*\mathrm{d}\mathcal{R}) = \iota_{S\tilde{X}}\mathrm{d}\mathcal{R} = (S\tilde{X})(\mathcal{R}).$$

In particular, $SX^c = X^v$. Equation (12) can be shown by direct computation in bundle coordinates. □

Proposition 11. Let X be a vector field on Q. Then, $X^c(L) = X^v(\mathcal{R})$ if and only if $X^v(L)$ is a constant of the motion.

Example 1 (Fluid resistance). Consider a body of mass m moving through a fluid that fully encloses it. For the sake of simplicity, suppose that the motion takes place along one dimension. Then, the drag force[38,39] is given by

$$\bar{R} = \frac{1}{2}\rho\, CA\dot{q}^2\mathrm{d}q,$$

where C is a dimensionless constant depending on the body shape, ρ is the mass density of the fluid, and A is the area of the projection of the object on a plane perpendicular to the direction of motion. For the sake of simplicity, suppose that the density is uniform, and then, $k = CA\rho/2$ is constant. The dissipation function is thus given by

$$\mathcal{R} = \frac{k}{3}\dot{q}^3.$$

If the body is not subject to forces besides the drag, its Lagrangian is $L = m\dot{q}^2/2$. Consider the vector field $X = e^{k/mq}\partial/\partial q$. We can verify that $X^c(L) = X^v(\mathcal{R})$, so $X^v(L) = me^{k/mq}\dot{q}$ is a constant of the motion. In particular, when $k \to 0$, we recover the conservation of momentum.

Proposition 12. If $\mathcal{L}_{X^c}\alpha_L$ is closed, then X is a Lie symmetry of $(L, \bar{R})$ if and only if

$$\mathrm{d}(X^c(E_L)) = -S^*(\mathrm{d}(X^c\mathscr{R})).$$

Proposition 13. If $\mathcal{L}_{X^c}\alpha_L = \mathrm{d}f$ for some function $f : TQ \to \mathbb{R}$, then the following statements are equivalent:

(i) *X is a Noether symmetry.*
(ii) *$X^c(E_L) + X^v(\mathcal{R}) = 0$.*
(iii) *$f - X^v(L)$ is a constant of the motion.*

Moreover, a Noether symmetry is a Lie symmetry if and only if $\iota_{X^c}\mathrm{d}\bar{R} = 0$.

Let $\tilde{X}$ be a vector field on TQ. If $\mathcal{L}_{\tilde{X}}\alpha_L$ is closed, then $\tilde{X}$ is a dynamical symmetry if and only if

$$\mathrm{d}(\tilde{X}(E_L) + (S\tilde{X})(\mathcal{R})) = -\iota_{\tilde{X}}\mathrm{d}\bar{R}.$$

Proposition 14. If $\mathcal{L}_{\tilde{X}}\alpha_L = \mathrm{d}f$, then the following statements are equivalent:

(i) *$\tilde{X}$ is a Cartan symmetry.*
(ii) *$\tilde{X}(E_L) + (S\tilde{X})(\mathcal{R}) = 0$.*
(iii) *$f - (S\tilde{X})(L)$ is a conserved quantity.*

We shall now cover some examples proposed in Ref. 34 and obtain their constants of motion.

Example 2 (A rotating disk). Let us consider a disk of mass m and radius r placed on a horizontal surface. Let φ be the angle of rotation of the disk with respect to a reference axis. The Lagrangian of the disk is $L = T = mr^2\dot{\varphi}^2/4$, and its Rayleigh dissipation function is $\mathcal{R} = \mu mgr\dot{\varphi}/2$. The Poincaré–Cartan one-form is $\alpha_L = mr^2\dot{\varphi}/2\,\mathrm{d}\varphi$. The external force is $\bar{R} = \mu mgr/2\,\mathrm{d}\varphi$.

Consider the vector field $\tilde{X} = r\dot{\varphi}\partial/\partial\varphi + \mu g\partial/\partial\dot{\varphi}$. Clearly, $\tilde{X}(E_L) = \tilde{X}(L) = (S\tilde{X})(\mathcal{R})$. We have that

$$\mathcal{L}_{\tilde{X}}\alpha_L = \frac{\mu mgr^2}{2}\mathrm{d}\varphi + \frac{mr^3}{2}\dot{\varphi}\mathrm{d}\dot{\varphi} = \mathrm{d}f,$$

where

$$f = \frac{\mu mgr^2}{2}\varphi + \frac{mr^3}{4}\dot{\varphi}^2$$

modulo a constant and $(S\tilde{X})(L) = mr^3\dot{\varphi}^2/2$, so

$$f - (S\tilde{X})(L) = \frac{\mu mgr^2}{2}\varphi - \frac{mr^3}{4}\dot{\varphi}^2$$

is a constant of the motion. Since $\bar{R}$ is closed, $\iota_{\tilde{X}}\mathrm{d}\bar{R} = 0$ is trivially satisfied, so $\tilde{X}$ is a dynamical symmetry and a Cartan symmetry.

However, since $\bar{R}$ is closed, it is not strictly an external force. In fact, the Lagrangian

$$\tilde{L} = L + \frac{\mu mgr}{2}\varphi$$

leads to the same equations of motion as L with the external force $\bar{R}$.

Example 3 (The rotating stone polisher). Consider a system formed by two concentric rings of the same mass m and radius r, which are placed over a rough surface and rotate in opposite directions. Let (x, y) be the position of the center and θ be the orientation of the machine. Let ω be the angular velocity of the rings. The Rayleigh dissipation function is given by

$$\mathcal{R} = 2\mu mgr\omega + \frac{\mu mg}{2r\omega}(\dot{x}^2 + \dot{y}^2),$$

and the Lagrangian is $L = T = m(\dot{x}^2 + \dot{y}^2 + r^2\dot{\theta}^2 + r^2\omega^2)$. The Poincaré–Cartan one-form is $\alpha_L = 2m(\dot{x}\mathrm{d}x + \dot{y}\mathrm{d}y + r^2\dot{\theta}\mathrm{d}\theta)$, and the external force is

$$\bar{R} = \frac{\mu mg}{r\omega}(\dot{x}\mathrm{d}x + \dot{y}\mathrm{d}y).$$

Let $\tilde{X}^{(1)} = 2r\omega\partial/\partial x + \mu g\partial/\partial\dot{x}$ and $\tilde{X}^{(2)} = 2r\omega\partial/\partial y + \mu g\partial/\partial\dot{y}$. We can check that $\tilde{X}^{(i)}(E_L) = \tilde{X}^{(i)}(L) = (S\tilde{X}^{(i)})(\mathcal{R})$ (for $i = 1, 2$). We have that $\mathcal{L}_{\tilde{X}^{(i)}}\alpha_L = \mathrm{d}f_i$ for $f_1 = 2\mu mgx$ and $f_2 = 2\mu mgy$, along with $(S\tilde{X}^{(1)})(L) = 4mr\dot{x}$ and $(S\tilde{X}^{(2)})(L) = 4mr\dot{y}$, so $2mr\omega\dot{x} - \mu mgx$ and $2mr\omega\dot{y} - \mu mgy$ are constants of the motion.

VIII. MOMENTUM MAP AND REDUCTION

It is well-known that if a d-dimensional symmetry group is acting over a physical system, then the number of independent degrees of freedom is reduced by d. In other words, Q is reduced by d dimensions, so TQ and T^*Q are reduced by $2d$ dimensions. Therefore, $2d$ variables can be eliminated from the equations of motion. This fact can be exploited in a systematic way by means of the procedure known as *reduction*, which is due to Marsden and Weinstein.[35,40]

Let G be a Lie group acting on Q, and consider the lifted action to TQ using tangent prolongation, that is, if $\Phi_g : Q \to Q$ is the diffeomorphism given by $\Phi_g(q) = gq$ for each $g \in G$ and $q \in Q$, then the lifted action is defined by

$$T\Phi_g : TQ \to TQ.$$

In what follows, we shall assume every group action considered to be free and proper. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ be its dual. Let $L : TQ \to \mathbb{R}$ be a Lagrangian function subjected to an external force β. Suppose that the G-action leaves L invariant, and hence, α_L and ω_L are invariant. Then, the natural momentum map,[35]

$$J : TQ \to \mathfrak{g}^*,$$
$$J(v_q)(\xi) = \alpha_L(v_q)\big(\xi_Q^c(v_q)\big),$$

is equivariant and Hamiltonian. For each $\xi \in \mathfrak{g}$ and $v_\in TQ$, $J\xi : TQ \to \mathbb{R}$ is the function given by

$$J\xi(v_q) = \langle J(v_q), \xi\rangle.$$

Lemma 15. Let $\xi \in \mathfrak{g}$. Then, the following holds:

(i) *$J\xi$ is a conserved quantity for $\xi_{L,\beta}$ if and only if*

$$\beta(\xi_Q^c) = 0. \tag{13}$$

(ii) *If the previous equation holds, then ξ leaves β invariant if and only if*

$$\iota_{\xi_Q^c} \mathrm{d}\beta = 0. \tag{14}$$

In addition, the vector subspace of $\mathfrak{g}$ given by

$$\mathfrak{g}_\beta = \left\{\xi \in \mathfrak{g} \mid \beta(\xi_Q^c) = 0,\ \iota_{\xi_Q^c}\mathrm{d}\beta = 0\right\}$$

is a Lie subalgebra of $\mathfrak{g}$.

Proof. (i) We have that

$$J\xi = \alpha_L(\xi_Q^c) = \iota_{\xi_Q^c}\alpha_L,$$

so

$$\mathrm{d}(J\xi) = \mathrm{d}(\iota_{\xi_Q^c}\alpha_L) = \mathcal{L}_{\xi_Q^c}\alpha_L - \iota_{\xi_Q^c}\mathrm{d}\alpha_L = \iota_{\xi_Q^c}\omega_L.$$

Contracting this equation with $\xi_{L,\beta}$, one gets

$$\iota_{\xi_{L,\beta}}(\mathrm{d}(J\xi)) = \xi_{L,\beta}(J\xi),$$

on the left-hand side, and

$$\iota_{\xi_{L,\beta}}\iota_{\xi_Q^c}\omega_L = -\iota_{\xi_Q^c}\iota_{\xi_{L,\beta}}\omega_L = -\iota_{\xi_Q^c}(\mathrm{d}E_L + \beta) = -\xi_Q^c(E_L) - \beta(\xi_Q^c),$$

on the right-hand side.

Thus, $J\xi$ is a conserved quantity for $\xi_{L,\beta}$ if and only if

$$\xi_Q^c(E_L) + \beta(\xi_Q^c) = 0. \tag{15}$$

Now, observe that

$$\begin{aligned}\xi_Q^c(E_L) &= \xi_Q^c(\Delta(L)) - \xi_Q^c(L) = \xi_Q^c(\Delta(L)) = \mathcal{L}_{\xi_Q^c}(\Delta(L)) = \mathcal{L}_{\xi_Q^c}(\iota_\Delta \mathrm{d}L)\\ &= \iota_{[\xi_Q^c,\Delta]}\mathrm{d}L + \iota_\Delta(\mathcal{L}_{\xi_Q^c}\mathrm{d}L) = \iota_{[\xi_Q^c,\Delta]}\mathrm{d}L = [\xi_Q^c, \Delta](L)\end{aligned}$$

since $\xi_Q^c(L) = 0$ by the G-invariance of L, but $[\xi_Q^c, \Delta] = 0$, and thus,

$$\xi_Q^c(E_L) = 0$$

for each $\xi \in \mathfrak{g}$, that is, E_L is G-invariant. By Eq. (15), $J\xi$ is a conserved quantity for $\xi_{L,\beta}$ if and only if

$$\beta(\xi_Q^c) = 0. \tag{16}$$

(iii) For each $\xi \in \mathfrak{g}_\beta$, we have that

$$\mathcal{L}_{\xi_Q^c}\beta = \mathrm{d}(\iota_{\xi_Q^c}\beta) + \iota_{\xi_Q^c}\mathrm{d}\beta = \mathrm{d}(\beta(\xi_Q^c)) + \iota_{\xi_Q^c}\mathrm{d}\beta.$$

If Eq. (16) holds, then β is $\mathfrak{g}_\beta$-invariant (i.e., $\mathcal{L}_{\xi_Q^c}\beta = 0$) if and only if

$$\iota_{\xi_Q^c}\mathrm{d}\beta = 0.$$

For $\mathfrak{g}_\beta$ being a Lie subalgebra, it is necessary and sufficient that $[\xi,\eta] \in \mathfrak{g}_\beta$ for each $\xi,\eta \in \mathfrak{g}_\beta$. Since $\xi \in \mathfrak{g} \mapsto \xi_Q \in \mathfrak{X}(Q)$ is a Lie algebra antihomomorphism,[41] this is equivalent to

$$\beta\big([\xi_Q,\eta_Q]^c\big) = 0,$$
$$\iota_{[\xi_Q,\eta_Q]^c}\mathrm{d}\beta = 0,$$

but $[\xi_Q,\eta_Q]^c = [\xi_Q^c,\eta_Q^c]$ since the complete lift is a morphism between Lie algebras. Then,

$$\begin{aligned}\beta\big([\xi_Q,\eta_Q]^c\big) &= \beta\big([\xi_Q^c,\eta_Q^c]\big) = \iota_{[\xi_Q^c,\eta_Q^c]}\beta = \mathcal{L}_{\xi_Q^c}\iota_{\eta_Q^c}\beta - \iota_{\eta_Q^c}\mathcal{L}_{\xi_Q^c}\beta\\ &= \xi_Q^c(\beta(\eta_Q^c)) - \eta_Q^c(\beta(\xi_Q^c)) - \iota_{\eta_Q^c}(\iota_{\xi_Q^c}\mathrm{d}\beta) = 0,\end{aligned}$$

by Eqs. (14) and (13). Similarly,

$$\begin{aligned}\iota_{[\xi_Q,\eta_Q]^c}\mathrm{d}\beta &= \iota_{[\xi_Q^c,\eta_Q^c]}\mathrm{d}\beta = \mathcal{L}_{\xi_Q^c}\iota_{\eta_Q^c}\mathrm{d}\beta - \iota_{\eta_Q^c}\mathcal{L}_{\xi_Q^c}\mathrm{d}\beta\\ &= \mathcal{L}_{\xi_Q^c}\iota_{\eta_Q^c}\mathrm{d}\beta - \iota_{\eta_Q^c}\mathrm{d}\mathcal{L}_{\xi_Q^c}\beta = 0.\end{aligned}$$

□

It is worth mentioning that our Lemma 15 (i) was previously obtained by Marsden and West[42] (Theorem 3.1.1), albeit from a variational approach.

Corollary 16. *For each $\xi \in \mathfrak{g}_\beta$, ξ_Q^c is a Noether symmetry and it is a symmetry of the forced Lagrangian system.*

Proof. Since α_L is invariant,

$$\mathcal{L}_{\xi_Q^c}\alpha_L = 0$$

for each $\xi \in \mathfrak{g}$. In combination with Eq. (15), this implies that ξ_Q^c is a Noether symmetry. By Remark 3, it is also a symmetry of the forced Lagrangian system. □

Theorem 17. *Let $G_\beta \subset G$ be the Lie subgroup generated by $\mathfrak{g}_\beta$ and $J_\beta : TQ \to \mathfrak{g}_\beta^*$ be the reduced momentum map. Let $\mu \in \mathfrak{g}_\beta^*$ be a regular value of J_β and $(G_\beta)_\mu$ be the isotropy group in μ. Then, the following holds:*

(i) *$J_\beta^{-1}(\mu)$ is a submanifold of TQ, and $\xi_{L,\beta}$ is tangent to it.*

(ii) *The quotient space $M_\mu := J_\beta^{-1}(\mu)/(G_\beta)_\mu$ is endowed with an induced symplectic structure ω_μ, namely,*

$$\pi_\mu^*\omega_\mu = \iota_\mu^*\omega_L,$$

where $\pi_\mu : J_\beta^{-1}(\mu) \to M_\mu$ and $\iota_\mu : J_\beta^{-1}(\mu) \hookrightarrow TQ$ denote the projection and the inclusion, respectively.

(iii) *L induces a function $L_\mu : M_\mu \to \mathbb{R}$ defined by*

$$L_\mu \circ \pi_\mu = L \circ \iota_\mu.$$

Moreover, we can introduce a function $E_{L_\mu} : M_\mu \to \mathbb{R}$, given by $E_{L_\mu} = \Delta_\mu(L_\mu) - L_\mu$, which satisfies

$$E_{L_\mu} \circ \pi_\mu = E_L \circ \iota_\mu. \tag{17}$$

(iv) *β induces a reduced semibasic one-form β_μ on M_μ, given by*

$$\pi_\mu^*\beta_\mu = \iota_\mu^*\beta.$$

Proof. For a proof of the first three assertions, see Refs. 35, 41, and 43. Observe that

$$\begin{aligned}\Delta(L)\circ\iota_\mu &= \iota_\mu^*\Delta(L) = \iota_\mu^*(\iota_\Delta \mathrm{d}L) = \iota_{\iota_{\mu*}\Delta}(\iota_\mu^*\mathrm{d}L) = (\iota_{\mu*}\Delta)(\iota_\mu^* L)\\ &= (\iota_{\mu*}\Delta)(L\circ\iota_\mu) = (\iota_{\mu*}\Delta)(L_\mu\circ\pi_\mu) = (\pi_{\mu*}\Delta_\mu)(L_\mu\circ\pi_\mu)\\ &= \pi_\mu^*(\Delta_\mu(L_\mu)) = \Delta_\mu(L_\mu)\circ\pi_\mu,\end{aligned}$$

where Δ is the Liouville vector field on TQ and Δ_μ is a π_μ-related vector field on M_μ, namely,

$$\pi_{\mu*}\Delta_\mu = \iota_{\mu*}\Delta.$$

Then, we can introduce a function $E_{L_\mu} : M_\mu \to \mathbb{R}$, given by $E_{L_\mu} = \Delta_\mu(L_\mu) - L_\mu$, which satisfies Eq. (17). Since β is $(G_\beta)_\mu$-invariant, it induces a reduced semibasic one-form β_μ on M_μ. □

Corollary 18. The vector field ξ_{L_μ,β_μ}, defined by

$$\iota_{\xi_{L_\mu,\beta_\mu}}\omega_\mu = \mathrm{d}E_{L_\mu} + \beta_\mu,$$

determines the dynamics on M_μ. It is π_μ-related to $\xi_{L,\beta}$.

Remark 6. In the Rayleigh dissipation case $\beta = S^*(\mathrm{d}\mathcal{R})$, according to Lemma 10, one can equivalently define $\mathfrak{g}_\beta \equiv \mathfrak{g}_\mathcal{R}$ as

$$\mathfrak{g}_\mathcal{R} = \{\xi \in \mathfrak{g} \mid \xi_Q^v(\mathcal{R}) = 0, S^*\mathrm{d}\xi_Q^c(\mathcal{R}) = 0\},$$

where the condition $S^*\mathrm{d}\xi_Q^c(\mathcal{R}) = 0$ means that $\xi_Q^c(\mathcal{R})$ is basic: it does not depend on $\dot{q}^i$. If additionally $\mathcal{R}$ is $\mathfrak{g}_\mathcal{R}$-invariant, i.e., $\xi_Q^c(\mathcal{R}) = 0$ for each $\xi \in \mathfrak{g}_\mathcal{R}$, then it induces a dissipation function $\mathcal{R}_\mu : M_\mu \to \mathbb{R}$ given by

$$\mathcal{R}_\mu \circ \pi_\mu = \mathcal{R} \circ \iota_\mu.$$

Remark 7 (Reconstruction of the dynamics). Knowing the integral curves on M_μ, we want to obtain the integral curves on $J^{-1}(\mu)$. Let $c(t)$ and $[c(t)]$ be the integral curves of $\xi_{L,\beta}$ and ξ_{L_μ,β_μ}, respectively, with $c(0) = p_0$. Let $d(t) \in J^{-1}(\mu)$ be a smooth curve such that $d(0) = p_0$ and $[d(t)] = [c(t)]$. We can write

$$c(t) = \Phi_{g(t)}(d(t)) \tag{18}$$

for $g(t) \in (G_\beta)_\mu$. Then, we have to find $g(t)$ in order to express $c(t)$ in terms of $[c(t)]$. Now,[35]

$$\begin{aligned}\xi_{L,\beta}(c(t)) &= c'(t)\\ &= T\Phi_{g(t)}(d(t))(d'(t)) + T\Phi_{g(t)}(d(t))(TL_{g(t)^{-1}}(g'(t)))_Q^c(d(t)),\end{aligned}$$

and using the Φ_g-invariance of $\xi_{L,\beta}$, one gets

$$\xi_{L,\beta}(d(t)) = d'(t) + (TL_{g(t)^{-1}}(g'(t)))_Q^c(d(t)).$$

In order to solve this equation, we first solve the algebraic problem,

$$\xi_Q^c(d(t)) = \xi_{L,\beta}(d(t)) - d'(t)$$

for $\xi(t) \in \mathfrak{g}_\beta$, and then, solve

$$g'(t) = TL_{g(t)}\xi(t)$$

for $g(t)$. The integral curve sought is given by Eq. (18).

Example 4 (Angular momentum). Consider $Q = \mathbb{R}^n\backslash\{0\}$, a Lagrangian function L on TQ that is spherically symmetric, say, $L(q,\dot{q}) = L(\|q\|,\|\dot{q}\|)$. Consider the Lie group $G = \mathrm{SO}(n) = \{O \in \mathbb{R}^{n\times n} \mid O^tO = \mathrm{Id}, \det(O) = 1\}$ acting by rotations on Q. The action can be lifted to TQ via the tangent lift. Explicitly, for $O \in \mathrm{SO}(3)$, we let

$$\begin{aligned} g_O : Q &\to Q,\\ (q) &\mapsto (O\cdot q),\\ Tg_O : TQ &\to TQ,\\ (q,\dot{q}) &\mapsto (O\cdot q, O\cdot\dot{q}).\end{aligned}$$

The group $\mathrm{SO}(n)$ acts freely and properly. The Lie algebra of the group is given by $\mathfrak{g} = \mathfrak{so}(n) = \{o \in \mathbb{R}^{n\times n} \mid o^t + o = 0\}$. In the case $n = 3$, this algebra $\mathfrak{g}$ will be identified with the algebra of three dimensional vectors with the cross product by taking

$$\begin{pmatrix} 0 & -\xi_3 & \xi_2\\ \xi_3 & 0 & -\xi_1\\ -\xi_2 & \xi_1 & 0\end{pmatrix} \mapsto \begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}.$$

The infinitesimal generator of $\xi \in \mathfrak{g}$ is given by

$$\begin{aligned}\xi_Q(q) &= (\xi\times q),\\ \xi_Q^c(q,\dot{q}) &= (\xi\times q, \xi\times\dot{q}),\\ \xi_Q^v(q,\dot{q}) &= (0, \xi\times\dot{q}).\end{aligned}$$

One can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by using the inner product on $\mathbb{R}^3$. The moment map is then given by[35] (Example 4.2.15)

$$J(q,\dot{q}) = q\times\dot{q}.$$

Identifying $\mathfrak{g}^* \simeq \mathbb{R}^3$, one sees that the coadjoint actions of G are the usual one (by rotations). Let $\mu \in \mathfrak{g}^*$, $\mu \neq 0$; then, the isotropy group $G_\mu \simeq S^1$ of μ under the coadjoint action, which is the rotations around the axis μ.

We look for Rayleigh potentials $\mathcal{R}$ such that $\mathfrak{g}_{\mathcal{R}} = \mathfrak{g}$. The condition that $\xi^v_Q(\mathcal{R}) = 0$ implies that $\mathcal{R}$ is spherically symmetric on the velocities. Then, the condition that $\xi^c_Q(q, \dot{q})$ is semi-basic means that the terms that are not spherically symmetric on the positions cannot involve the velocities. That is,

$$\mathcal{R} = A(q) + B(\|q\|, \|\dot{q}\|).$$

Without loss of generality, one can take $\mu = (0, 0, \mu_0)$. Hence, if $(q, v) \in J^{-1}(\mu)$, both q and $\dot{q}$ lie on the xy-plane. Moreover, they must satisfy the equation $\dot{q}^1 p_2 - \dot{q}^1 q^2 = \mu_0$. We can finally apply Theorem 17 to our system and find out that (M_μ, L_μ), which is a Hamiltonian system over a two-dimensional manifold[35] (Example 4.3.4).

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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