Integrability of contact Hamiltonian systems

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Outline of the presentation

Symplectic geometry

- Symplectic geometry is the natural framework for classical mechanics.
- Recall that a symplectic form ω on M is a 2-form such that $\mathrm{d}\omega = 0$ and $\mathrm{T}M \ni v \mapsto \iota_v \omega \in \mathrm{T}^* M$ is an isomorphism.
- Given a function f on M, its its Hamiltonian vector field X_f is given by

$$\iota_{X_f} \omega = \mathrm{d}f.$$

- The Poisson bracket $\{\cdot, \cdot\}$ is given by

$$\{f, g\} = \omega(X_f, X_g).$$

Theorem (Liouville–Arnold theorem)

Let $f_1, \ldots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0 \ \forall i, j$) on a symplectic manifold (M^{2n}, ω). Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$.

1. *Any compact connected component of M_Λ is diffeomorphic to $\mathbb{T}^n$.*
2. *On a neighborhood of M_Λ there are coordinates (φ^i, J_i) such that*

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i,$$

and the Hamiltonian dynamics are given by

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \Omega^i(J_1, \ldots, J_n),$$
$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0.$$

Contact geometry

Definition

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form.

- The contact form η defines an isomorphism

$$\begin{aligned} \flat \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

- There exists a unique vector field $\mathcal{R}$ on (M, η), called the **Reeb vector field**, such that $\flat(\mathcal{R}) = \eta$, that is,

$$\iota_{\mathcal{R}} \mathrm{d}\eta = 0, \quad \iota_{\mathcal{R}} \eta = 1.$$

Contact geometry

- The **Hamiltonian vector field** of $f \in C^\infty(M)$ is given by

$$\flat(X_f) = \mathrm{d}f - (\mathcal{R}(f) + f)\,\eta,$$

- Around each point on M there exist **Darboux coordinates** (q^i, p_i, z) such that

$$\begin{aligned}
\eta &= \mathrm{d}z - p_i \mathrm{d}q^i,\\
\mathcal{R} &= \frac{\partial}{\partial z},\\
X_f &= \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}.
\end{aligned}$$

Contact geometry

- The **Jacobi bracket** is given by

$$\{f, g\} = -\mathrm{d}\eta(\flat^{-1}\mathrm{d}f, \flat^{-1}\mathrm{d}g) - f\mathcal{R}(g) + g\mathcal{R}(f).$$

- This bracket is bilinear and satisfies the Jacobi identity.
- However, unlike a Poisson bracket, it does not satisfy the Leibnitz identity:

$$\{f, gh\} \neq \{f, g\}h + \{f, h\}g.$$

Dissipated quantities

- In contact Hamiltonian dynamics dissipated quantities are akin to conserved quantities in symplectic dynamics.
- Energy (Hamiltonian function) is no longer conserved, but dissipated in a certain manner:

$$X_H(H) = -\mathcal{R}(H)H\,.$$

Dissipated quantities

Example (linear dissipation)

Let

$$M = \mathbb{R}^3, \quad \eta = \mathrm{d}z - p\mathrm{d}q, \quad H = \frac{p^2}{2} + V(q) + \kappa z.$$

Then $X_H(H) = -\kappa H$, so

$$H \circ c(t) = e^{-\kappa t} H \circ c(0)\,,$$

along an integral curve c of X_H.

Dissipated quantities

Definition

An H-**dissipated quantity** is a solution $f \in C^\infty(M)$ to the PDE

$$X_H(f) = -\mathcal{R}(H) f \, .$$

- A function f is H-dissipated iff

$$\{f, H\} = 0.$$

- Noether's theorem: symmetries $\leftrightarrow$ dissipated quantities.

- Let $M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+ \colon f_\alpha(x) = r\Lambda_\alpha\}$.

Theorem (Colombo, de León, L., L.-G., 2023)

Let (M, η) be a $(2n+1)$-dimensional contact manifold. Suppose that $f_0, f_1, \ldots, f_n$ are functions in involution such that $(\mathrm{d}f_\alpha)$ has rank at least n. Then, $M_{\langle\Lambda\rangle_+}$ is invariant by the Hamiltonian flow of f_α and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$.

Moreover, there is a neighborhood U of $M_{\langle\Lambda\rangle_+}$ such that

1. *There exists coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on U such that the equations of motion are given by*

$$\dot{y}^\alpha = \Omega^\alpha(\tilde{A}_i)\,, \quad \dot{\tilde{A}}_i = 0\,, \quad \alpha \in \{0, \ldots, n\}\,, \, i \in \{1, \ldots, n\}\,.$$

2. *There exists a conformal change $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$, i.e. $\tilde{\eta} = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.*

Steps of the proof

1. Symplectize (M, η) and f_α, obtaining an exact symplectic manifold (M^Σ, θ) and homogeneous functions in involution f_α^Σ.
2. Prove a Liouville–Arnold theorem for exact symplectic manifolds with homogeneous functions in involution.
3. "Un-symplectize" the action-angle coordinates $(y_\Sigma^\alpha, A_\alpha^\Sigma)$ on M^Σ, yielding functions (y^α, A_Σ) on M.
4. Introduce action-angle coordinates $(y^\alpha, \tilde{A}_i)$ on M, where $\tilde{A}_i = -\frac{A_i}{A_0}$.

Exact symplectic manifolds: Liouville geometry

Definition

An **exact symplectic manifold** is a pair (M, θ), where M is a manifold and θ a one-form on N such that $\omega = -\mathrm{d}\theta$ is a symplectic form on M.

- The **Liouville vector field** Δ of (M, θ) is given by

$$\iota_\Delta \omega = -\theta.$$

- A tensor T is called **homogeneous of degree** n if $\mathcal{L}_\Delta T = nT$.

Symplectization of contact manifolds

Definition

Let (M, η) be a contact manifold and (M^Σ, θ) an exact symplectic manifold. A **symplectization** is a fibre bundle $\Sigma \colon M^\Sigma \to M$ such that

$$\sigma \Sigma^* \eta = \theta,$$

for a function σ on M^Σ called the **conformal factor**.

Symplectization of contact manifolds

Category of contact manifolds

$$\updownarrow$$

Category of exact symplectic manifolds

- Contact distribution $\ker \eta \longleftrightarrow$ symplectic potential θ
- Functions $\longleftrightarrow$ homogeneous functions of degree 1
- Hamiltonian vector fields $\longleftrightarrow$ Hamiltonian vector fields, homogeneous of degree 0

Symplectization of contact manifolds

Theorem

Given a symplectization $\Sigma\colon (M^\Sigma, \theta) \to (M, \eta)$ *with conformal factor* σ*, there is a bijection between functions* f *on* M *and homogeneous functions of degree 1* f^Σ *on* M^Σ *such that*

$$\Sigma_* (X_{f^\Sigma}) = X_f .$$

This bijection is given by

$$f^\Sigma = \sigma \Sigma^* f .$$

Moreover, one has

$$\left\{ f^\Sigma, g^\Sigma \right\}_\theta = \{f, g\}_\eta^\Sigma .$$

Symplectization of contact manifolds

Example

$\Sigma = \pi_1 \colon (M \times \mathbb{R}^+, \theta = r\eta) \to (M, \eta)$ is a symplectization with conformal factor $\sigma = r$, for r the global coordinate on $\mathbb{R}^+$.

Liouville–Arnold theorem for exact symplectic manifolds

- We want to obtain action-angle coordinates $(\varphi^\alpha_\Sigma, J^\Sigma_\alpha)$ on (M^Σ, θ) in order to define functions $(\varphi^\alpha, J_\alpha)$ on (M, η)
- We need homogeneous objects on (M^Σ, θ) so that they have a correspondence with objects on (M, η).
- However, the classical Liouville–Arnold theorem does not take into account the homogeneity of θ and f^Σ_α.
- Moreover, we need to consider non-compact level sets of f^Σ_α.

Liouville–Arnold theorem for exact symplectic manifolds

Theorem (Colombo, de León, Lainz, L.-G., 2023)

Let (M, θ) be an exact symplectic manifold. Suppose that the functions f_α, $\alpha = 1, \ldots, n$, on M are independent, in involution and homogeneous of degree 1. Let U be an open neighborhood of M_Λ such that:

1. *f_α have no critical points in U,*
2. *the Hamiltonian vector fields of X_{f_α} are complete,*
3. *the submersion $(f_\alpha)\colon U \to \mathbb{R}^n$ is a trivial bundle over $V \subseteq \mathbb{R}^n$.*

Then, $U \simeq \mathbb{R}^{n-m} \times \mathbb{T}^m \times V$, provided with action-angle coordinates (y^α, A_α) such that

$$\theta = A_\alpha \mathrm{d} y^\alpha, \qquad \frac{\mathrm{d} y^\alpha}{\mathrm{d} t} = \Omega^\alpha, \qquad \frac{\mathrm{d} A_\alpha}{\mathrm{d} t} = 0.$$

Sketch of proof

- Since X_{f_α} are n vector fields tangent to M_Λ, linearly independent and pairwise commutative, they generate the algebra $\mathbb{R}^n$ and $M_\Lambda \simeq \mathbb{R}^n/\mathbb{Z}^k$.
- Thus there are coordinates $y^\alpha = M_\alpha^\beta s^\beta$, where $X_{f_\alpha}(s^\beta) = \delta_\alpha^\beta$.
- The values of f_α define coordinates (J_α) on V.
- Since M_Λ is Lagrangian, $\theta = A_\alpha(J)\mathrm{d}y^\alpha + B^\alpha(y, J)\mathrm{d}J_\alpha$.
- Since f_α are homogeneous of degree 1, $\theta(X_{f_\alpha}) = f_\alpha$.
- By construction, $\Delta(y^\alpha) = 0$.
- With additional contractions with θ and ω, one concludes that $\theta = A_\alpha \mathrm{d}y^\alpha$, where $J_\beta = M_\beta^\alpha J_\alpha$.

Definition

A **completely integrable contact system** is a triple (M, η, F), where (M, η) is a contact manifold and $F = (f_0, \dots, f_n)\colon M \to \mathbb{R}^{n+1}$ is a map such that

1. $f_0, \dots, f_n$ are in involution, i.e., $\{f_\alpha, f_\beta\} = 0 \; \forall \, \alpha, \beta$,
2. $\operatorname{rank} \mathrm{T}F \geq n$ on a dense open subset $M_0 \subseteq M$.

The functions $f_0, \dots, f_n$ are called **integrals**.

Assumptions

1. Assume that the Hamiltonian vector fields $X_{f_0}, \ldots, X_{f_n}$ are complete.
2. Given $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $B \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be an open neighbourhood of Λ.
3. Let $\pi \colon U \to M_{\langle \Lambda \rangle_+}$ be a tubular neighbourhood of $M_{\langle \Lambda \rangle_+}$ such that $F|_U \colon U \to B$ is a trivial bundle over a domain $V \subseteq B$.

Theorem (Colombo, de León, L., L.-G., 2023)

Let (M, η, F) be a completely integrable contact system, where $F = (f_0, \dots, f_n)$. Consider the assumptions of the previous slide. Then:

1. *$M_{\langle\Lambda\rangle_+}$ is coisotropic, invariant by the Hamiltonian flow of f_α, and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ for some $k \leq n$.*
2. *There exists coordinates $(y^0, \dots, y^n, \tilde{A}_1, \dots, \tilde{A}_n)$ on U such that the equations of motion are given by*
$$\dot{y}^\alpha = \Omega^\alpha \quad \dot{\tilde{A}}_i = 0\,.$$
3. *There exists a nowhere-vanishing function $A_0 \in C^\infty(U)$ and a conformally equivalent contact form $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$, namely, $\tilde{\eta} = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.*

Sketch of the proof

1. Symplectize (M, η) and f_α, in order to apply the Liouville–Arnold theorem for exact symplectic manifolds
   - $\{f_\alpha, f_\beta\} = 0 \Rightarrow \{f_\alpha^\Sigma, f_\beta^\Sigma\} = 0$.
   - X_{f_α} complete $\Rightarrow X_{f_\alpha^\Sigma}$ complete.
   - $\operatorname{rank} \mathrm{d} f_\alpha \geq n \Rightarrow \operatorname{rank} \mathrm{d}(\underbrace{\sigma \Sigma^* f_\alpha}_{f_\alpha^\Sigma}) \geq n+1$.
   - $\Sigma\big((F^\Sigma)^{-1}(\Lambda)\big) = \big\{x \in M \mid \exists s \in \mathbb{R}^+ : \; F(x) = \frac{\Lambda}{s}\big\} = M_{\langle\Lambda\rangle_+}$.
   - X_{f_α} commute and are tangent to $M_{\langle\Lambda\rangle_+} \Rightarrow M_{\langle\Lambda\rangle_+} \simeq \mathbb{T}^k \times \mathbb{R}^{n+1-k}$.
2. "Un-symplectize" the action-angle coordinates $(y_\Sigma^\alpha, A_\alpha^\Sigma)$ on $\tilde{U}$, yielding functions (y^α, A_α) on U.
3. Introduce action-angle coordinates $(y^\alpha, \tilde{A}_i)$ on U
   - Since $\Lambda \neq 0$, $\exists A_\alpha \neq 0$. W.l.o.g., assume $A_0 \neq 0$.
   - Then $\left(y^\alpha, \tilde{A}_i = -\frac{A_i}{A_0}\right)$ are coordinates on U.

Sketch of the proof

- By construction, y^α are linear combinations of flows of X_{f_α}, namely,

$$X_{f_\alpha} = M^\alpha_\beta \frac{\partial}{\partial s^\beta}.$$

- Therefore, the dynamics are given by

$$\frac{\mathrm{d}y^\alpha}{\mathrm{d}t} = \Omega^\alpha, \qquad \frac{\mathrm{d}\tilde{A}_i}{\mathrm{d}t} = 0.$$

- $\theta^\Sigma = A^\Sigma_\alpha \mathrm{d}y^\alpha_\Sigma \rightsquigarrow \eta = A_\alpha \mathrm{d}y^\alpha$, so

$$\tilde{\eta} = \frac{1}{A_0}\eta = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i.$$

An example

- Let $M = \mathbb{R}^3 \setminus \{0\}$ with canonical coordinates (q, p, z), and $\eta = \mathrm{d}z - p\mathrm{d}q$.
- The functions $h = p$ and $f = z$ are in involution.
- Let $F = (h, f)\colon M \to \mathbb{R}^2$.
- $\operatorname{rank} \mathrm{T}F = 2$, and thus (M, η, F) is a completely integrable contact system.

An example

- Hypothesis of the theorem are satisfied:
  1. The Hamiltonian vector fields

$$X_h = \frac{\partial}{\partial q}\,, \quad X_f = -p\frac{\partial}{\partial p} - z\frac{\partial}{\partial z}$$

     are complete,
  2. Since $F\colon (q,p,z) \mapsto (p,z)$ is the canonical projection, $F\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is a trivial bundle.

An example

The invariant submanifolds are given by

$$M_{\langle \Lambda \rangle_+} = \{(q, p, z) \in M \mid \exists r \in \mathbb{R}_+ \colon p = r\Lambda_1 \, , z = r\Lambda_2\} \, ,$$

or, equivalently,

$$M_{\langle \Lambda \rangle_+} = \{(q, p, z) \in M \mid \exists r \in \mathbb{R}_+ \colon p = r \sin\varphi \, , z = r\cos\varphi\} \, ,$$

where $\langle \Lambda \rangle_+ = \langle (\sin\varphi, \cos\varphi) \rangle_+$ for some $\varphi \in [0, 2\pi)$.

An example

Consider the chart $(U; \xi, \zeta, \varphi)$, where

$$U = M \setminus \{z = 0\}\,, \quad \xi = q\,, \quad \zeta = z\,, \quad \varphi = \arctan\left(\frac{p}{z}\right) \pmod{2\pi}\,,$$

Suppose that $\langle \Lambda \rangle_+ \neq \langle (1,0) \rangle_+$. Then, one can write

$$M_{\langle \Lambda \rangle_+} = \left\{ (\xi, \zeta, \varphi) \mid \varphi = \arctan\left(\frac{\Lambda_1}{\Lambda_2}\right) \right\}\,.$$

An example

In the chart $(U; \xi, \zeta, \varphi)$, the Hamiltonian vector fields read

$$X_h = \frac{\partial}{\partial \xi}, \quad X_f = -\zeta \frac{\partial}{\partial \zeta}.$$

Therefore, the action $\Phi \colon \mathbb{R}^2 \to M$ defined by their flows is given by

$$\Phi(t, s; \xi, \zeta, \varphi) = \left(\xi + t, \zeta e^{-s}, \varphi\right).$$

An example

- Consider a reference point $x_0 \in M_{\langle \Lambda \rangle_+}$ with coordinates $(\xi_0, \zeta_0, \varphi_0)$.
- The angle coordinates (y^0, y^1) of a point $x \in M_{\langle \Lambda \rangle_+}$ with coordinates (ξ, ζ, φ_0) are given by

$$(\xi, \zeta, \varphi_0) = \Phi(y^0, y^1; \xi_0, \zeta_0, \varphi_0) = (\xi^0 + y^0, \zeta_0 e^{-y^1}, \varphi_0)\,,$$

that is

$$y^0 = \xi - \xi_0 = q - q_0\,, \quad y^1 = \log \zeta_0 - \log \zeta = \log z_0 - \log z\,.$$

An example

- We know that the contact form can be written as

$$\eta = A_0 \mathrm{d}y^0 + A_1 \mathrm{d}y^1 \,.$$

- Therefore,

$$A_0 = \eta\left(\frac{\partial}{\partial y^0}\right) = \eta\left(\frac{\partial}{\partial q}\right) = -p = -h \,,$$
$$A_1 = \eta\left(\frac{\partial}{\partial y^1}\right) = \eta\left(-z\frac{\partial}{\partial z}\right) = -z = -f \,.$$

- The action coordinate is

$$\tilde{A} = -\frac{A_1}{A_0} = -\frac{h}{f} = -\frac{p}{z} \,.$$

An example

- To sum up, we have a chart $\left(M \setminus \{z=0\}; y^0, y^1, \tilde{A}\right)$, where

$$y^0 = q - q_0\,, \quad y^1 = \log z_0 - \log z \quad \tilde{A} = -\frac{p}{z}\,.$$

- In this chart,

$$X_h = \frac{\partial}{\partial y^0}\,, \quad X_f = \frac{\partial}{\partial y^1}$$

- It is a Darboux chart for the contact form

$$\tilde{\eta} = \frac{1}{A_0}\eta = \mathrm{d}y^0 - \tilde{A}\mathrm{d}y^1\,.$$

- Notice that X_h is the Reeb vector field of $\tilde{\eta}$ and X_f is the Hamiltonian vector field of $\tilde{A}$ w.r.t. $\tilde{\eta}$.

Other notions of integrability

- Khesin and Tabachnikov, Liberman, Banyaga and Molino, Lerman, etc. have defined notions of contact complete integrability which are geometric but not dynamical, e.g. a certain foliation over a contact manifold.
- Boyer considers the so-called good Hamiltonians H, i.e., $\mathcal{R}(H) = 0 \rightsquigarrow$ no dissipated quantities, “symplectic” dynamics.
- Miranda considered integrability of the Reeb dynamics when $\mathcal{R}$ is the generator of an S^1-action.
- We are interested in complete integrability of contact Hamiltonian dynamics.

A **toric manifold** is a (compact) symplectic manifold (M, ω), a smooth effective Hamiltonian Lie group action of $\mathbb{T}^n$ and a moment map $\mathrm{J} = (\mathrm{J}_i)^i : M \to \mathfrak{g}^* \simeq \mathbb{R}^n$ that satisfies $\xi_M = X_{\langle \mathrm{J}, \xi \rangle}$ for all $\xi \in \mathfrak{g}$.
Note that, choosing a basis of the Lie algebra we obtain an integrable system $(\mathrm{J}_i)_i$.
Conversely, an integrable system provides the structure of a toric manifold. The flow of the Hamiltonian vector fields provides a Hamiltonian effective action of $\mathbb{T}^n$ and a the the action variables provide a moment map.
By the theorem of Atiyah, Guillemin and Sternberg, if M is compact, $\mathrm{J}(M)$ is a convex polytope.

The moment map and integrable systems

In the contact case, we may equivalently define a moment map $\hat{\mathrm{J}} = \langle (\mathrm{J}_\alpha)_\alpha \rangle_+ : M \to \mathbb{S}^n$, having coisotropic level sets.
Here $\mathbb{S}^n$ can be seen as $\mathbb{R}^n \setminus \{0\}$ modulo multiplication by positive integers.

$$\begin{array}{ccc} M^\Sigma & \xrightarrow{\mathrm{J}} & \mathbb{R}^{n+1} \setminus \{0\} \\ \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \pi_{\mathbb{S}}} \\ M & \xrightarrow{\hat{\mathrm{J}}} & \mathbb{S}^n \end{array}$$

The flow of the Hamiltonian vector fields also provides an effective action, which can be related using Σ to the symplectic one.

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Happy birthday, Manolo!