Nonsmooth Herglotz principle and contact Routh reduction

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Abstract.

1. Preliminaries: Contact Hamiltonian and Lagrangian systems

A *contact manifold* is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form. This 1-form η is called a *contact form.*

Given a contact manifold (M, η), there exists a unique vector field $\mathcal{R}$ on M such that

$$\iota_{\mathcal{R}} \mathrm{d}\eta = 0, \quad \iota_{\mathcal{R}} \eta = 1. \tag{1}$$

The vector field $\mathcal{R}$ is called the *Reeb vector field.*

In *Darboux coordinates* (q^i, p_i, z), the contact form is locally written

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i, \tag{2}$$

and the Reeb vector field is $\mathcal{R} = \frac{\partial}{\partial z}$. The contact structure η on M defines the so-called *musical isomorphisms*

$$\begin{aligned} \flat : TM &\to T^*M \\ v &\mapsto \iota_v \mathrm{d}\eta + \eta(v)\eta, \end{aligned} \tag{3}$$

and $\sharp = \flat^{-1}$.

Given a Hamiltonian function H on (M,η), we define the *(contact) Hamiltonian vector field* X_H by

$$\flat(X_H) = \mathrm{d}H - (\mathcal{R}(H) + H)\,\eta. \tag{4}$$

The triple (M,η,H) is called a *(contact) Hamiltonian system*. Eq. (4) is equivalent to

$$\eta(X_H) = -H, \tag{5a}$$

$$\mathcal{L}_{X_H}\eta = -\mathcal{R}(H)\eta. \tag{5b}$$

Additionally, the following identities hold:

$$X_H(H) = -\mathcal{R}(H)\ H, \tag{6a}$$

$$\iota_{X_H}\mathrm{d}\eta = \mathrm{d}H - \mathcal{R}(H)\eta. \tag{6b}$$

The *Jacobi bracket* $\{\cdot,\cdot\}$ of a contact manifold (M,η) is given by

$$\{f,g\} = X_f(g) + g\mathcal{R}(f) = -X_g(f) - f\mathcal{R}(g) = -\eta\left([X_f, X_g]\right) \tag{7}$$

for any pair of smooth functions f and g on M, where X_f and X_g are their associated Hamiltonian vector fields.

We say that a (local) diffeomorphism between two contact manifolds $F : (M,\eta) \to (N,\tau)$ is a (local) *contactomorphism* if $F^*\tau = \eta$. We say that F is a (local) *conformal contactomorphism* if $F^*\ker\tau = \ker\eta$ or, equivalently, $F^*\tau = \sigma\eta$, where $\sigma : M \to \mathbb{R}\backslash\{0\}$ is the conformal factor. In addition, we say that a vector field X on M is an *infinitesimal (conformal) contactomorphism* if its flow φ_t consists of (conformal) contactomorphisms. Equivalently, X is an infinitesimal conformal contactomorphism if $\mathcal{L}_X\eta = a_X\eta$ for some function $a_X \in \mathcal{C}^\infty(M)$. In particular, when $a_X = 0$, X is an infinitesimal contactomorphism. The function a_X is related to the conformal factors σ_t by

$$\sigma_t(x) = \exp\left(\int_0^t a_X\left(\varphi_\tau(x)\right)\mathrm{d}\tau\right). \tag{8}$$

Note that, by Eq. (5b), a Hamiltonian vector field is an infinitesimal conformal contactomorphism. Conversely, if X is an infinitesimal conformal contactomorphism, then X is the Hamiltonian vector field of $f = -\eta(X)$, and $a_X = -\mathcal{R}(f)$. Moreover X_f is an infinitesimal contactomorphism if and only if $\mathcal{R}(f) = 0$.

Integrating Eqs. (5b) and (6a), we have that

$$H \circ \varphi_t = \exp\left(\int_0^t -\mathcal{R}(H)(\varphi_\tau)\ \mathrm{d}\tau\right) H, \tag{9a}$$

$$\varphi_t^*\eta = \exp\left(\int_0^t -\mathcal{R}(H)(\varphi_\tau)\ \mathrm{d}\tau\right)\eta, \tag{9b}$$

where φ_t is the flow of X_H.

Let Q be an n-dimensional manifold. Consider a Lagrangian function $L : TQ \times \mathbb{R} \to \mathbb{R}$ and let us introduce the 1-form $\alpha_L = S^*(\mathrm{d}L)$, where S^* is the adjoint operator of the vertical endomorphism on TQ extended in the natural way to $TQ \times \mathbb{R}$. Let (q^i) be local coordinates on Q and let $(q^i, \dot{q}^i, z)$ be the induced coordinates on $TQ \times \mathbb{R}$. Locally, $S = \mathrm{d}q^i \otimes \frac{\partial}{\partial \dot{q}^i}$, so $\alpha_L = \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i$. Let η_L be a 1-form on $TQ \times \mathbb{R}$ given by $\eta_L = \mathrm{d}z - \alpha_L = \mathrm{d}z - \dfrac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i$.

One can show that η_L is a contact form if and only if L is *regular*, i.e., the Hessian matrix $(W_{ij}) = \left(\frac{\partial^2 L}{\partial \dot{q}^i \dot{q}^j}\right)$ is regular. Hereinafter, we shall assume that L is regular. The *energy* of the system is given by $E_L = \Delta(L) - L$, where $\Delta = \dot{q}^i \partial/\partial \dot{q}^i$ is the Liouville vector field on TQ naturally extended to $TQ \times \mathbb{R}$. Hence, $(TQ \times \mathbb{R}, \eta_L, E_L)$ is a contact Hamiltonian system. Its corresponding Reeb vector field $\mathcal{R}_L$ is locally

$$\mathcal{R}_L = \frac{\partial}{\partial z} - W^{ij}\frac{\partial^2 L}{\partial \dot{q}^i \partial z}\frac{\partial}{\partial \dot{q}^j}, \tag{10}$$

where $(W^{ij}) = (W_{ij})^{-1}$. The dynamics is given by the *Lagrangian vector field* Γ_L, defined by

$$\flat_L(\Gamma_L) = \mathrm{d}E_L - (E_L + \mathcal{R}_L(E_L))\,\eta_L, \tag{11}$$

where $\flat_L$ denotes the musical isomorphism defined by the contact form η_L. Eqs. (5) are now

$$\eta_L(\Gamma_L) = -E_L,\ \mathcal{L}_{\Gamma_L}\eta_L = -\mathcal{R}_L(E_L)\eta_L. \tag{12}$$

Moreover, Eqs. (9) are now written

$$E_L \circ \varphi_t = \exp\left(\int_0^t -\mathcal{R}_L(E_L)(\varphi_\tau)\mathrm{d}\tau\right) E_L, \tag{13a}$$

$$\varphi_t^*\eta_L = \exp\left(\int_0^t -\mathcal{R}_L(E_L)(\varphi_\tau)\mathrm{d}\tau\right) \eta_L, \tag{13b}$$

where φ_t is the flow of Γ_L.

Let us recall that a vector field Γ on $TQ \times \mathbb{R}$ is called a *SODE* (an acronym for *second order differential equation*) if $S(\Gamma) = \Delta$. Locally, a SODE is of the form

$$\Gamma = \dot{q}^i\frac{\partial}{\partial q^i} + \Gamma^i(q, \dot{q}, z)\frac{\partial}{\partial \dot{q}^i} + \Gamma^z(q, \dot{q}, z)\frac{\partial}{\partial z} \tag{14}$$

2. Routh reduction for contact Lagrangian systems

In the context of conservative Lagrangian systems, Routh's reduction procedure is a classical reduction technique which takes advantage of the conservation laws to define a reduced

Lagrangian function, so-called Routhian function, such that, when the conservation of momenta is taken into account, the solutions of the Euler-Lagrange equations for the Routhian are in correspondence with the solutions of the Euler-Lagrange equations for the original Lagrangian. Here we extend this procedure for contact Lagrangian systems.

Assume that the configuration space is of the form $Q = Q_1 \times Q_2$, and let us denote a point $(q^i) \in Q$ by $(q^i) = (q^1, q^j)$, where $q^1 \in Q_1$ and $(q^j) \in Q_2$, with $j = 2, \ldots, n$, $n = \dim Q$. Let $L(q^1, \dot{q}^1, q^j, \dot{q}^j, z)$ be a hyperregular Lagrangian on $TQ \times \mathbb{R}$. Suppose that q^1 is a cyclic coordinate, namely, $\partial L/\partial q^1 = 0$. Consider the momentum map

$$J_L(q^1, \dot{q}^1, q^j, \dot{q}^j, z) = \frac{\partial L}{\partial q^1}, \tag{15}$$

and fix a value of the momentum

$$\mu = \frac{\partial L}{\partial q^1}. \tag{16}$$

Since L is hyperregular, the last equation admits an inverse and allows us to write $\dot{q}^1$ as a function of (q^j), z and μ. Moreover, assume that L is natural, so that $\partial^2 L/\partial \dot{q}^i \partial z = 0$. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^1}\right) = \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}^1}, \tag{17}$$

so $\partial L/\partial \dot{q}^1$ is a dissipated quantity. Let us introduce the function

$$R^\mu\left(q^j, \dot{q}^j, z\right) = \left[L - \dot{q}^1 \mu \exp\left(\int_0^t \frac{\partial L}{\partial z}\left(q^i(\tau), z(\tau)\right)\, \mathrm{d}\tau\right)\right]\Bigg|_\mu, \tag{18}$$

which we will call the (*contact*) *Routhian function*. Here $|_\mu$ denotes that the relation $\mu = \partial L/\partial \dot{q}^1$ is used to replace $\dot{q}^1$ in terms of $(q^j, \dot{q}^j)$ and the parameter μ.

If we regard R^μ as a new contact Lagrangian system in the variables $(q^j, \dot{q}^j, z)$, then the solutions of the Herglotz equations for R^μ are in correspondence with those for L when one takes into account the relation $\mu = \frac{\partial L}{\partial \dot{q}^1}$. More precisely:

i) Any solution of the Herglotz equations for L with momentum $\mu = \frac{\partial L}{\partial \dot{q}^1}$ projects onto a solution of the Herglotz equations for R^μ, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R^\mu}{\partial \dot{q}^j}\right) - \frac{\partial R^\mu}{\partial q^j} - \frac{\partial R^\mu}{\partial \dot{q}^j}\frac{\partial R^\mu}{\partial z}. \tag{19}$$

These equations will be called the *Herglotz-Routh* equations.

ii) Conversely, any solution of the Herglotz-Routh equations for R^μ can be lifted to a solution of the Herglotz equations for L with $\partial L/\partial \dot{q}^1 = \mu\, \partial L/\partial z$.

3. Herglotz principle for nonsmooth Lagrangians

3.1. Herglotz variational principle for smooth Lagrangians

Consider a Lagrangian function $L : TQ \times \mathbb{R} \to \mathbb{R}$ and fix two points $q_1, q_2 \in Q$ and an interval $[a, b] \subset \mathbb{R}$. Let us denote by $\Omega(q_1, q_2, [a, b]) \subseteq (\mathcal{C}^\infty([a, b] \to Q)$ the space of smooth curves σ such that $\sigma(a) = q_1$ and $\sigma(b) = q_2$. This space has the structure of an infinite dimensional smooth manifold whose tangent space at σ is given by the set of vector fields over σ that vanish at the endpoints [1, Proposition 3.8.2], that is,

$$T_\sigma\Omega(q_1, q_2, [a, b]) = \{v_\sigma \in \mathcal{C}^\infty([a, b] \to TQ) \mid \tau_Q \circ v_\sigma = \sigma,\ v_\sigma(a) = 0,\ v_\sigma(b) = 0\} \tag{20}$$

The elements of $T_\sigma\Omega(q_1, q_2, [a, b])$ will be called *infinitesimal variations* of the curve σ. Let

$$\mathcal{Z} : \mathcal{C}^\infty([a, b] \to Q) \to \mathcal{C}^\infty([a, b] \to \mathbb{R}) \tag{21}$$

be the operator that assigns to each curve σ the function $\mathcal{Z}(\sigma)$ that solves the following ordinary differential equation (ODE):

$$\begin{aligned} &\frac{\mathrm{d}\mathcal{Z}(\sigma)(t)}{\mathrm{d}t} = L(\sigma(t), \dot\sigma(t), \mathcal{Z}(\sigma)(t)), \\ &\mathcal{Z}(\sigma)(a) = 0. \end{aligned} \tag{22}$$

Now we define the *action functional* $\mathcal{A}$ as the map which assigns to each curve the solution to the previous ODE evaluated at the endpoint, namely,

$$\begin{aligned} \mathcal{A} : \Omega(q_1, q_2, [a, b]) &\to \mathbb{R} \\ \sigma &\mapsto \mathcal{Z}(\sigma)(b) \end{aligned} \tag{23}$$

We will say that a path $\sigma \in \Omega(q_1, q_2, Q)$ satisfies the *Herglotz variational principle* if it is a critical point of $\mathcal{A}$, i.e.,

$$T_\sigma\mathcal{A} = 0. \tag{24}$$

These critical points are curves which satisfy the *Herglotz equations* [13]:

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q^i} + \frac{\partial L}{\partial \dot q^i}\frac{\partial L}{\partial z} = 0. \tag{25}$$

If the Lagrangian L is regular, one has the following equivalence between the Herglotz equations and the geometric dynamical equations (see Ref. [13]).

Theorem 1. *Let L be a regular contact Lagrangian system on $TQ \times \mathbb{R}$, and let Γ_L be the Hamiltonian vector field associated with the energy (given by Eq. (11)). Then*

i) Γ_L is a SODE on $TQ \times \mathbb{R}$,

ii) the integral curves of Γ_L are solutions of the Herglotz equations (25).

3.2. Herglotz principle for nonsmooth Lagrangians

Consider a configuration manifold Q and a submanifold with boundary $C \subset Q$, which represent the subset of admissible configurations. Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a regular Lagrangian. Let us introduce the *path space*

$$\mathcal{M} := \mathcal{T} \times \mathcal{Q}\left([0,1], \tau_i, \partial C, Q\right), \tag{26}$$

where

$$\begin{aligned} \mathcal{T} &:= \{c_t \in C^\infty([0,1], \mathbb{R}) \mid c_t' > 0 \text{ in } [0,1]\}\,, \\ \mathcal{Q}\left([0,1], \tau_i, \partial C, Q\right) &:= \big\{c_q : [0,1] \to Q \mid c_q \text{ is a } C^0, \text{ piecewise } C^2 \text{ curve}, \\ &\qquad c_q(\tau) \text{ has only one singularity at } \tau_i, c_q\left(\tau_i\right) \in \partial C\big\}\,. \end{aligned} \tag{27}$$

A path $c \in \mathcal{M}$ is a pair $c = (c_t, c_q)$. Given a path, the *associated curve* $q : [c_t(0), c_t(1)] \to Q$ is given by $q(t) = c_q \circ c_t^{-1}(t)$. Let $\mathcal{C}$ denote the set of all paths $q(t) \in Q$.

The moment of impact $\tau_i \in (0,1)$ is fixed in the τ-space, but can vary in the t-space according to $t_i = c_t(\tau_i)$. One can show that $\mathcal{T}$ and $\mathcal{Q}\left([0,1], \tau_i, \partial C, Q\right)$, and hence $\mathcal{M}$, are smooth manifolds [17]. Let $\widehat{\Omega}\left(q_1, q_2, [0,1]\right) \subset \mathcal{M}$ be the subset of curves such that $c_q(0) = q_1$ and $c_q(1) = q_2$. The tangent space at $c_q \in \mathcal{Q}$ is given by

$$T_{c_q}\mathcal{Q} = \big\{v : [0,1] \to TQ \mid v \text{ is a } C^0 \text{ piecewise } C^2 \text{ map}, v\left(\tau_i\right) \in T_{c_q(\tau_i)}\partial C\big\}\,. \tag{28}$$

Let

$$\widehat{\mathcal{Z}} : \widehat{\Omega}\left(q_1, q_2, [0,1]\right) \to \mathcal{T} \tag{29}$$

be the operator that assigns to each $c_q \in \mathcal{M}$ the solution of the following ODE:

$$\begin{aligned} \frac{\mathrm{d}\widehat{\mathcal{Z}}}{\mathrm{d}\tau} &= L\left(c_q(\tau), \frac{c_q'(\tau)}{c_q'(\tau)}, \widehat{\mathcal{Z}}(c_q, c_t)(\tau)\right) c_t'(\tau), \\ \widehat{\mathcal{Z}}(c_q, c_t)(0) &= \widehat{z}_0. \end{aligned} \tag{30}$$

Let

$$\begin{aligned} \widehat{\mathcal{A}} : \widehat{\Omega}\left(q_1, q_2, [0,1]\right) &\to \mathbb{R} \\ (c_q, c_t) &\mapsto \widehat{\mathcal{Z}}(c_q, c_t)(1). \end{aligned} \tag{31}$$

Theorem 2 (Nonsmooth contact variational principle)**.** *Let $L : TQ \times \mathbb{R}$ be a smooth and regular Lagrangian function. Let $c = (c_q, c_t)$ be a curve in $\widehat{\Omega}\left(q_1, q_2, [0,1]\right)$, and let $\chi(\tau) = \left(c_q(\tau), \frac{c_q'(\tau)}{c_t'(\tau)}, \widehat{\mathcal{Z}}(c)(\tau)\right) \subset TQ \times \mathbb{R}$. Then, c is a critical point of $\widehat{\mathcal{A}}$ if and only if*

$$\frac{\partial L}{\partial q^i}(\chi(\tau)) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}(\chi(\tau)) + \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau))\frac{\partial L}{\partial z}(\chi(\tau)) = 0, \tag{32a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_L(\chi(\tau)) = \frac{\partial L}{\partial z}(\chi(\tau))\ E_L(\chi(\tau)), \tag{32b}$$

for $\tau \in [0, \tau_i) \cup (\tau_i, 1]$, *and*

$$\frac{\partial L}{\partial \dot{q}^i}(\chi(\tau_i^-))v^i = \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau_i^+))v^i, \tag{33a}$$

$$E_L(\chi(\tau_i^-)) = E_L(\chi(\tau_i^+)) \tag{33b}$$

for any $v \in T_{c_q(\tau_i)}\partial C$.

Proof. Let $c = (c_q, c_t) \in \widehat{\Omega}\,(q_1, q_2, [0,1])$ be a curve. Consider a smoothly parametrized family of curves $c^\lambda = (c_q^\lambda, c_t^\lambda)$ in $\widehat{\Omega}\,(q_1, q_2, [0,1])$ such that $c^0 = c$ and

$$u = \left.\frac{\mathrm{d}c_q^\lambda}{\mathrm{d}\lambda}\right|_{\lambda=0}, \qquad \theta = \left.\frac{\mathrm{d}c_t^\lambda}{\mathrm{d}\lambda}\right|_{\lambda=0}. \tag{34}$$

Let $\varphi = T_c\widehat{\mathcal{Z}}(u,\theta)$, so that $T_c\widehat{\mathcal{A}}(u,\theta) = \varphi(1)$. Observe that $\varphi(0) = 0$, since $\widehat{\mathcal{Z}}(c^\lambda)(0) = \widehat{z}_0$ for every λ. Now,

$$\begin{aligned}
\varphi'(\tau) &= \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\lambda}\widehat{\mathcal{Z}}(c^\lambda(\tau))\right|_{\lambda=0} = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{\mathcal{Z}}(c^\lambda(\tau))\right|_{\lambda=0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[L\left(c_q^\lambda(\tau), \frac{c_q^{\lambda\prime}(\tau)}{c_q^{\lambda\prime}(\tau)}, \widehat{\mathcal{Z}}(c_q^\lambda, c_t^\lambda)(\tau)\right)c_t^{\lambda\prime}(\tau)\right]\right|_{\lambda=0} \\
&= \left[\frac{\partial L}{\partial q^i}(\chi(\tau))u^i(\tau) + \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau))\frac{1}{c_t'(\tau)}u'^i(\tau) - \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau))\frac{c_q'(\tau)}{c_t'(\tau)^2}\theta(\tau)\right. \\
&\quad \left. + \frac{\partial L}{\partial z}(\chi(\tau))\varphi(\tau)\right]c_t'(\tau) + L\left(c_q^\lambda(\tau), \frac{c_q^{\lambda\prime}(\tau)}{c_q^{\lambda\prime}(\tau)}, \widehat{\mathcal{Z}}(c_q^\lambda, c_t^\lambda)(\tau)\right)\theta'(\tau).
\end{aligned} \tag{35}$$

An integrating factor for this ODE is

$$\mu(\tau) = \exp\left(-\int_0^\tau \frac{\partial L}{\partial z}(\chi(s))\; c_t'(s)\mathrm{d}s\right), \tag{36}$$

so

$$\begin{aligned}
\varphi(\tau)\mu(\tau) &= \int_0^\tau \mu(s)c_t'(s)\left[\frac{\partial L}{\partial q^i}(\chi(s))u^i(s) + \frac{\partial L}{\partial \dot{q}^i}(\chi(s))\frac{1}{c_t'(s)}u'^i(s) - \frac{\partial L}{\partial \dot{q}^i}(\chi(s))\frac{c_q'(s)}{c_t'(s)^2}\theta'(s)\right]\,\mathrm{d}s \\
&\quad + \int_0^\tau \mu(s)L(\chi(s))\theta'(s)\,\mathrm{d}s \\
&= \int_0^\tau \mu(s)c_t'(s)u^i(s)\left[\frac{\partial L}{\partial q^i}(\chi(s)) + \frac{\partial L}{\partial \dot{q}^i}(\chi(s))\frac{1}{c_t'(s)}\right]\,\mathrm{d}s \\
&\quad + \int_0^\tau \mu(s)\theta'(s)\left[L(\chi(s)) - \frac{\partial L}{\partial \dot{q}^i}(\chi(s))\frac{c_q'(s)}{c_t'(s)}\right]\mathrm{d}s.
\end{aligned} \tag{37}$$

Integrating by parts and taking into account that $u(0) = u(1) = 0$ and $\theta(0) = \theta(1) = 0$, we obtain

$$\begin{aligned}
\varphi(1)\mu(1) = &\int_0^{\tau_i} \mu(s) c_t'(s) u^i(s) \left[\frac{\partial L}{\partial q^i}(\chi(s)) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q^i}(\chi(s)) + \frac{\partial L}{\partial \dot q^i}(\chi(s))\frac{\partial L}{\partial z}(\chi(s))\right] \mathrm{d}s \\
&- \int_0^{\tau_i} \mu(s)\theta(s)\frac{\mathrm{d}}{\mathrm{d}s}\left[L(\chi(s)) - \frac{\partial L}{\partial \dot q^i}(\chi(s))\frac{c_q'(s)}{c_t'(s)}\right] \mathrm{d}s \\
&+ \int_0^{\tau_i} \mu(s)\theta(s)\left[L(\chi(s)) - \frac{\partial L}{\partial \dot q^i}(\chi(s))\frac{c_q'(s)}{c_t'(s)}\right] \frac{\partial L}{\partial z}(\chi(s))\ c_t'(s)\mathrm{d}s \\
&+ \int_{\tau_i}^1 (\cdots)\ \mathrm{d}s \\
&+ \frac{\partial L}{\partial \dot q^i}(\chi(s)) u^i(s)\mu(s)\bigg|_{\tau_i^+}^{\tau_i^-} + \mu(s)\theta(s)\left[L(\chi(s)) - \frac{\partial L}{\partial \dot q^i}(\chi(s))\frac{c_q'(s)}{c_t'(s)}\right]\bigg|_{\tau_i^+}^{\tau_i^-}.
\end{aligned} \tag{38}$$

Since $\mu(\tau)$ is nonzero, $\varphi(0)$ vanishes for every (u, θ) (i.e., χ is a critical point of $\widehat{\mathcal{A}}$) if and only if

$$\frac{\partial L}{\partial q^i}(\chi(\tau)) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q^i}(\chi(\tau)) + \frac{\partial L}{\partial \dot q^i}(\chi(\tau))\frac{\partial L}{\partial z}(\chi(\tau)) = 0, \tag{39a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[L(\chi(\tau)) - \frac{\partial L}{\partial \dot q^i}(\chi(\tau))\frac{c_q'(\tau)}{c_t'(\tau)}\right] = \left[L(\chi(\tau)) - \frac{\partial L}{\partial \dot q^i}(\chi(\tau))\frac{c_q'(\tau)}{c_t'(\tau)}\right]\frac{\partial L}{\partial z}(\chi(\tau))\ c_t'(\tau), \tag{39b}$$

for $\tau \in [0, \tau_i) \cup (\tau_i, 1]$, and

$$\frac{\partial L}{\partial \dot q^i}(\chi(\tau_i^-))v^i = \frac{\partial L}{\partial \dot q^i}(\chi(\tau_i^+))v^i, \tag{40a}$$

$$L(\chi(\tau_i^-)) - \frac{\partial L}{\partial \dot q^i}(\chi(\tau_i^-))\frac{c_q'(\tau_i^-)}{c_t'(\tau_i^-)} = L(\chi(\tau_i^+)) - \frac{\partial L}{\partial \dot q^i}(\chi(\tau_i^+))\frac{c_q'(\tau_i^+)}{c_t'(\tau_i^+)}, \tag{40b}$$

for any $v \in T_{c_q(\tau_i)}\partial C$. The result follows from the chain rule and the definition of E_L. □

Remark 1. Equation (32b) is redundant. Indeed, we have that

$$\frac{\mathrm{d}E_L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q^i}\dot q^i - L\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i}\right)\dot q^i - \frac{\partial L}{\partial z}\dot z = \left(\frac{\partial L}{\partial \dot q^i}\dot q^i - L\right)\frac{\partial L}{\partial z} = E_L\frac{\partial L}{\partial z}, \tag{41}$$

along solutions of the Herglotz equations (32a).

Remark 2. Equations (13a) and (32b) are equivalent. As one can check straightforwardly,

$$\mathcal{R}_L(E_L) = -\frac{\partial L}{\partial z}, \tag{42}$$

so

$$\frac{\mathrm{d}E_L}{\mathrm{d}t} = -\mathcal{R}_L(E_L)E_L \tag{43}$$

along the solutions of the Herglotz equations (32a).

Example 1 (Billiard with dissipation)**.** Consider a particle moving in the plane confined to the surface defined by $x^2 + y^2 = 1$. The Lagrangian $L : T\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is given by

$$L(x, y, \dot{x}, \dot{y}, z) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \gamma z, \tag{44}$$

where γ is a real constant. Herglotz equations thus yield

$$\begin{aligned} \ddot{x} &= -\gamma\dot{x}, \\ \ddot{y} &= -\gamma\dot{y}, \\ \dot{z} &= L(x, y, \dot{x}, \dot{y}, z). \end{aligned} \tag{45}$$

Their solutions for the initial conditions $x(0) = x_0$, $y(0) = y_0$, $\dot{x}(0) = \dot{x}_0$, $\dot{y}(0) = \dot{y}_0$ and $E_L(0) = E_0$ are

$$\begin{aligned} x(t) &= x_0 + \frac{\dot{x}_0}{\gamma}\left(1 - e^{-t\gamma}\right), \\ y(t) &= y_0 + \frac{\dot{y}_0}{\gamma}\left(1 - e^{-t\gamma}\right), \\ z(t) &= -\frac{1}{\gamma}\left[\frac{1}{2}\left(\dot{x}_0^2 + \dot{y}_0^2\right)e^{-2t\gamma} + E_0 e^{-t\gamma}\right]. \end{aligned} \tag{46}$$

For simplicity's sake, let $(x_0, y_0) = (0, 0)$ and $(\dot{x}_0, \dot{y}_0) = (\gamma, \gamma)$. Then, the particle impacts with the wall at $t_1 = -1/\gamma \ \log(1 - 1/\sqrt{2})$. Conditions (33) are now

$$\begin{aligned} &\frac{1}{\sqrt{2}}\left(\dot{x}^- - \dot{y}^-\right) = \frac{1}{\sqrt{2}}\left(\dot{x}^+ - \dot{y}^+\right), \\ &\frac{1}{2}\left((\dot{x}^-)^2 + (\dot{y}^-)^2\right) + \gamma z_{|t_i^-} = \frac{1}{2}\left((\dot{x}^+)^2 + (\dot{y}^+)^2\right) + \gamma z_{|t_i^+}, \end{aligned} \tag{47}$$

where $\dot{x}^\pm = \dot{x}(t_i^\pm)$ and $\dot{y}^\pm = \dot{y}(t_i^\pm)$. Now, $z_{|t_i^+} = z_{|t_i^-}$, so $\dot{x}^+ = \dot{y}^+ = -\dot{x}^- = -\gamma$.

3.3. Hamiltonian counterpart

Given a Lagrangian function $L : TQ \times \mathbb{R} \to \mathbb{R}$, we can define the *Legendre transform*

$$\begin{aligned} \mathrm{Leg} : TQ \times \mathbb{R} &\to T^*Q \times \mathbb{R} \\ \left(q^i, \dot{q}^i, z\right) &\mapsto \left(q^i, \frac{\partial L}{\partial \dot{q}^i}, z\right). \end{aligned} \tag{48}$$

Hereinafter, we will assume that the Lagrangian L is *hypperregular*, i.e., the Legendre transform is a diffeomorphism. The Hamiltonian function H is then given by $H = E_L \circ \mathrm{Leg}$.

The Hamiltonian counterpart of Theorem 2 is as follows.

Proposition 3. *Let H be a regular Hamiltonian function on $T^*Q \times \mathbb{R}$. Let $\xi = (q^i, p_i, \mathcal{Z})$ be a continuous and piecewise C^2 curve on $T^*Q \times \mathbb{R}$ whose only singularities occur at t_i. Then,*

$$\begin{aligned} \frac{\mathrm{d}q^i}{\mathrm{d}t} &= \frac{\partial H}{\partial p_i}(\xi(t)), \\ \frac{\mathrm{d}p_i}{\mathrm{d}t} &= -\frac{\partial H}{\partial q^i}(\xi(t)) - p_i \frac{\partial H}{\partial z}(\xi(t)), \end{aligned} \tag{49}$$

for $t \neq t_i$, and

$$\begin{aligned} p_i(\xi(t_i^-))v^i &= p_i(\xi(t_i^+))v^i, \\ H(\xi(t_i^-)) &= H(\xi(t_i^+)). \end{aligned} \tag{50}$$

4. Contact Routh reduction for non-smooth contact systems

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