Homogeneous bi-Hamiltonian structures and integrable contact systems

Leonardo Colombo, Manuel de León, María Emma Eyrea Irazú,
and Asier López-Gordón

7th International Conference on Geometric Science of Information

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES

Contact geometry

Definition

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that the map

$$\begin{aligned} \flat_\eta \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

is an isomorphism of $\mathscr{C}^\infty(M)$-modules.

- There exists a unique vector field R on (M, η), called the **Reeb vector field**, given by $R = \flat_\eta^{-1}(\eta)$, or, equivalently,

$$\iota_R \mathrm{d}\eta = 0, \quad \iota_R \eta = 1.$$

Contact geometry

- The **Hamiltonian vector field** of $f \in \mathscr{C}^\infty(M)$ is given by

$$X_f = \flat_\eta^{-1}(\mathrm{d}f) - (R(f) + f)\, R\,,$$

- Around each point on M there exist **Darboux coordinates** (q^i, p_i, z) such that

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i\,,$$

$$R = \frac{\partial}{\partial z}\,,$$

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}\,.$$

Contact geometry

- The **Jacobi bracket** is given by
$$\{f, g\} = X_f(g) + gR(f).$$
- This bracket is bilinear and satisfies the Jacobi identity.
- However, unlike a Poisson bracket, it does not satisfy the Leibniz identity:
$$\{f, gh\} \neq \{f, g\}h + \{f, h\}g.$$

Contact Hamiltonian dynamics

Contact Hamiltonian vector fields allow modelling certain dissipative mechanical systems, as well as some thermodynamic systems.

In a Darboux chart, the integral curves $c(t) = (q^i(t), p_i(t), z(t))$ of X_h are determined by the **contact Hamilton equations**:

$$
\begin{aligned}
\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} &= \frac{\partial h}{\partial p_i} \circ c(t)\,, \\
\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -\frac{\partial h}{\partial q^i} \circ c(t) - p_i(t)\frac{\partial h}{\partial z} \circ c(t)\,, \\
\frac{\mathrm{d}z(t)}{\mathrm{d}t} &= p_i(t)\frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.
\end{aligned}
$$

Exact symplectic manifolds

Definition

An **exact symplectic manifold** is a pair (M, θ), where θ is a **symplectic potential** on M, i.e., $\omega = -\mathrm{d}\theta$ is a symplectic form on M. The **Liouville vector field** $\nabla \in \mathfrak{X}(M)$ is given by

$$\iota_{\nabla}\omega = -\theta\,.$$

A tensor field A on P is called k-homogeneous (for $k \in \mathbb{Z}$) if

$$\mathcal{L}_{\nabla}A = kA\,.$$

Trivial symplectisation

Definition

Let (M, η) be a co-oriented contact manifold. Then, the trivial bundle $\pi_1 : M^{\text{symp}} = M \times \mathbb{R}_+ \to M$, $\pi_1(x, r) = x$ can be endowed with the symplectic potential $\theta(x, r) = r\eta(x)$. The Liouville vector field reads $\nabla = r\partial_r$.
We will refer to $(M^{\text{symp}}, \theta)$ as the **trivial symplectisation** of (M, η).

Trivial symplectisation

Proposition

There is a one-to-one correspondence between functions $f(x)$ on M and 1-homogeneous functions $f^{\mathrm{symp}}(x,r) = -rf(x)$ on M^{symp} such that the symplectic $X_{f^{\mathrm{symp}}}$ and contact X_f Hamiltonian vector fields are related as follows:

$$\mathrm{T}\pi_1\left(X_{f^{\mathrm{symp}}}\right) = X_f\,.$$

Moreover, the Poisson $\{\cdot,\cdot\}_\theta$ and Jacobi $\{\cdot,\cdot\}_\eta$ brackets have the correspondence

$$\{f^{\mathrm{symp}}, g^{\mathrm{symp}}\}_\omega = \left(\{f,g\}_\eta\right)^{\mathrm{symp}}\,.$$

Definition

A **homogeneous Hamiltonian system** (M, θ, H) consists of a $2n$-dimensional exact symplectic manifold (M, θ) and a 1-homogeneous Hamiltonian function H. It is called a **homogeneous integrable system** if there exist n 1-homogeneous functions $f_1, \ldots, f_n$ such that

$$\{f_i, H\} = 0 = \{f_i, f_j\}, \quad 1 \leq i, j \leq n.$$

Definition

A **completely integrable contact system** is a co-oriented contact manifold (M, η) endowed with a Hamiltonian function $h \in \mathscr{C}^\infty(M)$ such that its trivial symplectisation $(M^{\mathrm{symp}}, \theta, h^{\mathrm{symp}})$ is a homogeneous integrable system.

Our Liouville–Arnol'd theorem permits constructing action-angle coordinates for completely integrable contact systems.

Compatible Poisson structures

Definition

Let M be a manifold. Two Poisson tensors are Λ and Λ_1 on M are said to be **compatible** if $\Lambda + \Lambda_1$ is also a Poisson tensor on M.

Definition

A vector field $X \in \mathfrak{X}(M)$ is called **bi-Hamiltonian** if it is a Hamiltonian vector field w.r.t. two compatible Poisson structures, namely,

$$X = \Lambda(\cdot, \mathrm{d}h) = \Lambda_1(\cdot, \mathrm{d}h_1)\,,$$

for two functions $h, h_1 \in \mathscr{C}^\infty(M)$.

Poisson – Nijehuis structures

- The linear map $\sharp_\Lambda \colon \mathrm{T}^*_x M \ni \alpha \mapsto \Lambda(\cdot, \alpha) \in \mathrm{T}_x M$ is an isomorphism iff Λ comes from a symplectic structure ω. In that case, $\flat_\omega \coloneqq \sharp_\Lambda^{-1}(v) = \iota_v \omega$.
- In that situation, we can define the $(1, 1)$-tensor field

$$N = \sharp_{\Lambda_1} \circ \sharp_\Lambda^{-1} = \sharp_{\Lambda_1} \circ \flat_\omega \,.$$

Poisson – Nijehuis structures

Theorem (Magri and Morosi, 1984)

Let (M, ω) *be a symplectic manifold and* Λ_1 *a bivector. If* Λ_1 *is a Poisson tensor compatible with* $\Lambda = \omega^{-1}$*, then the eigenvalues of the* $(1,1)$*-tensor field*

$$N = \sharp_{\Lambda_1} \circ \flat_\omega$$

are functions in involution w.r.t. both Poisson brackets.

Corollary

If a vector field $X \in \mathfrak{X}(M)$ *is bi-Hamiltonian w.r.t. to* ω *and* Λ_1 *(i.e.,* $X = \sharp_\omega \mathrm{d}h = \sharp_{\Lambda_1} \mathrm{d}h_1$*), then the eigenvalues of* N *form a family of conserved quantities in involution w.r.t. both Poisson brackets.*

Compatible Jacobi structures

- The theory of compatible Jacobi structures and Jacobi–Nijenhuis manifolds was developed by Iglesias, Monterde, Marrero, Nunes da Costa, Padrón and Petalidou in the 1990s and 2000s.
- Two Jacobi structures (Λ, E) and (Λ_1, E_1) on a manifold M are called compatible if $(\Lambda + \Lambda_1, E + E_1)$ is also a Jacobi structure on M.
- Given a Jacobi structure (Λ, E) on M, one can construct an associated Poisson structure $\tilde{\Lambda} = 1/r\Lambda + \partial_r \wedge E$ on $M \times \mathbb{R}_+$, which by construction is homogeneous of degree -1 w.r.t. $\nabla = r\partial_r$.
- Nunes da Costa (1998) showed that (Λ, E) and (Λ_1, E_1) are compatible Jacobi structures iff $\tilde{\Lambda}$ and $\tilde{\Lambda}_1$ are compatible Poisson structures.

Compatible Jacobi structures

- Unlike their Poisson counterpart, these results had not been applied for studying integrability of dynamics.

Remark

Magri's result cannot be generalised for completely integrable contact systems with compatible Jacobi structures.

Theorem (Fernandes, 1994)

Consider a $2n$-dimensional completely integrable Hamiltonian system (M, ω, H) with action-angle coordinates (s_i, φ^i) satisfying the following conditions:

(ND) *The Hessian matrix $\left(\frac{\partial^2 H}{\partial s_i \partial s_j}\right)$ of the Hamiltonian w.r.t. the action variables is non-degenerate in a dense subset of M.*

(BH) *The system is bi-Hamiltonian and the recursion operator N has n functionally independent real eigenvalues $\lambda_1, \ldots, \lambda_n$.*

Then, the Hamiltonian function can be written as

$$H(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} H_i(\lambda_i)\,,$$

where each H_i is a function that depends only on the corresponding λ_i.

Proposition

Let (M, θ, H) *be a homogeneous integrable Hamiltonian system satisfying the assumption (ND). Denote by* Λ *the Poisson structure defined by* $\omega = -\mathrm{d}\theta$*, and by* ∇ *the Liouville vector field corresponding to* θ*. If there is a Poisson structure* Λ_1 *on* M *compatible with* Λ*, it cannot be simultaneously* (-1)*-homogeneous (i.e.,* $\mathcal{L}_\nabla \Lambda_1 = -\Lambda_1$*) and satisfying (BH).*

Proof.

If N has n functionally independent eigenvalues, then $H = \sum_i H_i(\lambda_i)$. If Λ_1 is (-1)-homogeneous, then N is 0-homogeneous, so its eigenvalues are 0-homogeneous as well. Hence,

$$H = \nabla(H) = \sum_{i=1}^{n} H_i'(\lambda_i)\nabla(\lambda_i) = 0\,.$$

$\square$

Proposition

Let (M, θ, H) *be a homogeneous integrable Hamiltonian system satisfying the assumption (ND). Denote by* Λ *the Poisson structure defined by* $\omega = -\mathrm{d}\theta$*, and by* ∇ *the Liouville vector field corresponding to* θ*. If there is a Poisson structure* Λ_1 *on* M *compatible with* Λ*, it cannot be simultaneously* (-1)*-homogeneous (i.e.,* $\mathcal{L}_{\nabla}\Lambda_1 = -\Lambda_1$*) and satisfying (BH).*

Proof.

If N has n functionally independent eigenvalues, then $H = \sum_i H_i(\lambda_i)$. If Λ_1 is (-1)-homogeneous, then N is 0-homogeneous, so its eigenvalues are 0-homogeneous as well. Hence,

$$H = \nabla(H) = \sum_{i=1}^{n} H_i'(\lambda_i)\nabla(\lambda_i) = 0\,.$$

□

Corollary

Let (M, η, H) be a $(2n+1)$-dimensional integrable contact Hamiltonian system. If there is a second Jacobi structure (Λ_1, E_1) compatible with the Jacobi structure (Λ, E) defined by η, then the recursion operator $N = \sharp_{\tilde{\Lambda}_1} \circ \sharp_{\tilde{\Lambda}}^{-1}$ relating the associated Poisson structures on $M \times \mathbb{R}_+$ cannot have $(n+1)$ functionally independent real eigenvalues.

- Consequently, compatible Jacobi structures cannot be utilised to construct a set of independent functions in involution for a contact Hamiltonian system.
- Nevertheless, we can symplectise the contact Hamiltonian system and obtain a second Poisson structure compatible with the one defined by the exact symplectic structure.
- If N is 1-homogeneous and satisfies (BH), then its eigenvalues are n functionally independent and 1-homogeneous functions in involution, so they will project into n functions in involution w.r.t. the Jacobi bracket.

A toy example

- Let $M = \mathbb{R}^2$, and consider its cotangent bundle $\mathrm{T}^*M \simeq \mathbb{R}^4$ endowed with the canonical one-form $\theta_{\mathbb{R}^2}$.
- In bundle coordinates (x^1, x^2, p_1, p_2),

$$\theta_{\mathbb{R}^2} = p_1 \mathrm{d}x^1 + p_2 \mathrm{d}x^2 \rightsquigarrow \Lambda = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial p_2} \,.$$

- In this case, the Liouville vector field is $\nabla_M = p_i \partial_{p_i}$, the infinitesimal generator of homotheties on the fibers.
- A Poisson structure compatible with Λ is

$$\Lambda_1 = p_1 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial p_1} + p_2 x^2 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial p_2} \,.$$

A toy example

- The Nijenhuis tensor $N = \sharp_{\Lambda_1} \circ \sharp_{\Lambda}^{-1}$ reads

$$N = p_1 \left(\frac{\partial}{\partial x^1} \otimes \mathrm{d}x^1 + \frac{\partial}{\partial p_1} \otimes \mathrm{d}p_1 \right) + p_2 x^2 \left(\frac{\partial}{\partial x^2} \otimes \mathrm{d}x^2 + \frac{\partial}{\partial p_2} \otimes \mathrm{d}p_2 \right) ,$$

  with eigenvalues $\lambda_1 = p_1$ and $\lambda_2 = p_2 x^2$.

- The vector field

$$X = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} - p_2 \frac{\partial}{\partial p_2}$$

  is bi-Hamiltonian, it is the Hamiltonian vector field of $H = p_1 + p_2 x^2$ w.r.t. Λ, and the Hamiltonian vector field of $H_1 = \log(p_1 p_2 x^2)$ w.r.t. Λ_1. Moreover, λ_1 and λ_2 are first integrals of X.

A toy example bis

- Consider the contact Hamiltonian system $(M = \mathbb{R}^3, \eta, h)$, with η the canonical contact form, $\eta = \mathrm{d}z - p\mathrm{d}q$, and $h = p - z$.
- In bundle coordinates (q, p, z, r), the trivial symplectisation $(\mathbb{R}^4, \theta, H)$ of (M, η, h) reads

$$\theta = r\mathrm{d}z - rp\mathrm{d}q\,, \quad H = rz - rp\,,$$

  and Liouville vector field is $\nabla = r\partial_r$.
- This is the system from the previous example, as it becomes evident by performing the coordinate change

$$x^1 = q, \quad x^2 = z, \quad p_1 = -rp, \quad p_2 = r\,.$$

- Thus, we have the functions $\lambda_1 = p_1 = -rp$ and $\lambda_2 = p_2 x^2 = rz$, which are homogeneous of degree 1, in involution, and functionally independent on a dense subset.
- Projecting them to M, we obtain $\bar{\lambda}_1 = p$ and $\bar{\lambda}_2 = -z$, which are functionally independent and $\{\bar{\lambda}_1, \bar{\lambda}_2\} = \{\bar{\lambda}_1, h\} = \{\bar{\lambda}_2, h\} = 0$.

Main references

[1] V. I. Arnol'd. *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics. Springer-Verlag: New York, 1978.

[2] L. Colombo, M. de León, M. Lainz, and A. López-Gordón. "Liouville–Arnold Theorem for Homogeneous Symplectic and Contact Hamiltonian Systems". *Geom. Mech.*, **02**(03) (2025).

[3] K. Grabowska and J. Grabowski. "A Geometric Approach to Contact Hamiltonians and Contact Hamilton–Jacobi Theory". *J. Phys. A: Math. Theor.*, **55**(43) (2022).

[4] F. Magri, P. Casati, G. Falqui, and M. Pedroni. "Eight Lectures on Integrable Systems". In: *Integrability of Nonlinear Systems*. Ed. by Y. Kosmann-Schwarzbach, B. Grammaticos, and K. M. Tamizhmani. Lecture Notes in Physics. Springer: Berlin, Heidelberg, 1997.

[5] J. M. Nunes da Costa. "Compatible Jacobi Manifolds: Geometry and Reduction". *J. Phys. A: Math. Gen.*, **31**(3) (1998).

Merci pour votre attention!

- Feel free to contact me at alopez-gordon@impan.pl
- These slides are available at www.alopezgordon.xyz