The geometry of dissipation

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Geometric frameworks for dynamics

- By the Equivalence Principle, the laws of physics are the same for all observers, i.e., for all systems of coordinates.
- Hence, we should formulate the laws of physics in a coordinate-independent language, namely, the language of differential geometry.

Geometric frameworks for Hamiltonian dynamics

- As it is well-known, symplectic manifolds are the natural framework for Hamiltonian mechanics.
- Hamiltonian dynamics are conservative: the Hamiltonian flow preserves the symplectic form and the Hamiltonian function.

Geometric frameworks for dissipative dynamics

- The ubiquity of physical phenomena where the energy or the volume of the phase space are not preserved leads to the necessity of developing frameworks for non-conservative dynamics.
- In this dissertation, we consider three geometric frameworks for dissipative dynamics:
    - Hamiltonian (and Lagrangian) systems with external forces,
    - contact Hamiltonian (and Lagrangian) systems,
    - mechanical systems with impacts.

Contact geometry

Definition

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that the map

$$\begin{aligned} \flat_\eta \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

is an isomorphism of $\mathscr{C}^\infty(M)$-modules.

- There exists a unique vector field R on (M, η), called the **Reeb vector field**, given by $R = \flat_\eta^{-1}(\eta)$, or, equivalently,

$$\iota_R \mathrm{d}\eta = 0, \quad \iota_R \eta = 1.$$

Contact geometry

- The **Hamiltonian vector field** of $f \in \mathscr{C}^\infty(M)$ is given by

$$X_f = \flat_\eta^{-1}(\mathrm{d}f) - (R(f) + f)\, R\,,$$

- Around each point on M there exist **Darboux coordinates** (q^i, p_i, z) such that

$$\begin{aligned}
\eta &= \mathrm{d}z - p_i \mathrm{d}q^i,\\
R &= \frac{\partial}{\partial z},\\
X_f &= \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}.
\end{aligned}$$

Contact Hamiltonian systems

Definition

A **contact Hamiltonian system** is a triple (M, η, h) formed by a contact manifold (M, η) and a **Hamiltonian function** $h \in \mathscr{C}^\infty(M)$.

- The dynamics of (M, η, h) is determined by the integral curves of the Hamiltonian vector field X_h of h w.r.t. η.

Contact Hamiltonian systems

- In Darboux coordinates, these curves $c(t) = (q^i(t), p_i(t), z(t))$ are determined by the **contact Hamilton equations**:

$$
\begin{aligned}
\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} &= \frac{\partial h}{\partial p_i} \circ c(t)\,,\\
\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -\frac{\partial h}{\partial q^i} \circ c(t) + p_i(t)\frac{\partial h}{\partial z} \circ c(t)\,,\\
\frac{\mathrm{d}z(t)}{\mathrm{d}t} &= p_i(t)\frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.
\end{aligned}
$$

Contact Lagrangian systems

- The Darboux coordinate z can be regarded, from a variational point of view, as the action functional.
- Let L be an **action-dependent** Lagrangian function.
- Very loosely, the Herglotz functional $\mathcal{A}$ is like the usual action functional, but instead of being given by an integral is given by the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}[q(t)] = L\Big(q(t), \dot{q}(t), \mathcal{A}[q(t)]\Big)\,.$$

- One seeks for curves $q\colon I \subseteq \mathbb{R} \to Q$ that are extremals of $\mathcal{A}$.

Contact Lagrangian systems

- Given two fixed points $q_1, q_2 \in Q$ and an interval $[a, b]$, let

$$\Omega(q_1, q_2, [a, b]) = \left\{ c \in \mathscr{C}^2([a, b] \to Q) \mid c(a) = q_1,\ c(b) = q_2 \right\} .$$

- Consider the operator

$$\mathcal{Z} \colon \Omega(q_1, q_2, [a, b]) \to \mathscr{C}^2([a, b] \to \mathbb{R})$$

that assigns to each curve c the solution $\mathcal{Z}(c)$ of the following Cauchy problem:

$$\frac{\mathrm{d}\mathcal{Z}(c)(t)}{\mathrm{d}t} = L(c(t), \dot{c}(t), \mathcal{Z}(c)(t)) ,$$
$$\mathcal{Z}(c)(a) = z_a .$$

Contact Lagrangian systems

- The **Herglotz action functional** is the map

$$
\begin{aligned}
\mathcal{A}\colon \Omega(q_1, q_2, [a, b]) &\to \mathbb{R} \\
c &\mapsto \mathcal{Z}(c)(b)\,.
\end{aligned}
$$

- A curve $c \in \Omega(q_1, q_2, [a, b])$ is a critical point of $\mathcal{A}$ (i.e., $\mathrm{d}\mathcal{A}(c) = 0$) if and only if it satisfies the **Herglotz–Euler–Lagrange equations**:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^i}(c(t), \dot{c}(t), \mathcal{Z}(c)(t)) - \frac{\partial L}{\partial q^i}(c(t), \dot{c}(t), \mathcal{Z}(c)(t)) \\
&\quad - \frac{\partial L}{\partial v^i}(c(t), \dot{c}(t), \mathcal{Z}(c)(t))\frac{\partial L}{\partial z}(c(t), \dot{c}(t), \mathcal{Z}(c)(t)) = 0\,.
\end{aligned}
$$

Dissipated quantities

- In contact Hamiltonian dynamics dissipated quantities are akin to conserved quantities in symplectic dynamics.
- Energy (Hamiltonian function) is no longer conserved, but dissipated in a certain manner:

$$X_h(h) = -R(h)h\,.$$

Dissipated quantities

Example (linear dissipation)

Let $M = \mathbb{R}^3$ with canonical coordinates (q, p, z),

$$\eta = \mathrm{d}z - p\mathrm{d}q, \quad h = \frac{p^2}{2} + V(q) + \kappa z\,, \quad \kappa \in \mathbb{R}\,.$$

Then $X_h(h) = -\kappa h$, so

$$h \circ c(t) = e^{-\kappa t} h \circ c(0)\,,$$

along an integral curve c of X_h.

Dissipated quantities

Definition

Let (M, η, h) be a contact Hamiltonian system. A **dissipated quantity** is a solution $f \in \mathscr{C}^\infty(M)$ to the PDE

$$X_h(f) = -R(h)f\,.$$

Jacobi structure of a contact manifold

- The **Jacobi bracket** $\{\cdot,\cdot\}\colon \mathscr{C}^\infty(M)\times\mathscr{C}^\infty(M)\to\mathscr{C}^\infty(M)$ is given by

$$\{f,g\} = -\mathrm{d}\eta(\flat_\eta^{-1}\mathrm{d}f, \flat_\eta^{-1}\mathrm{d}g) - fR(g) + gR(f).$$

- It is a Lie bracket, namely, it is bilinear, skew-symmetric and satisfies the Jacobi identity.
- It satifsies the weak Leibniz identity:

$$\{f,gh\} = \{f,g\}h + \{f,h\}g - ghR(f).$$

Jacobi brackets and dissipated quantities

- The Jacobi bracket can also be expressed as follows:

$$\{f, g\} = X_f(g) + gR(f)\,.$$

Proposition

Let (M, η, h) be a contact Hamiltonian system and let $f \in \mathscr{C}^\infty(M)$. Then, f is a dissipated quantity (i.e., $X_h(f) = -R(h)f$) iff

$$\{f, h\} = 0\,.$$

Functions in involution

Definition

Let $\{\cdot,\cdot\}$ be a Jacobi bracket on M. A collection of functions $f_1,\ldots,f_k \in \mathscr{C}^\infty(M)$ are said to be **in involution** if

$$\{f_i, f_j\} = 0\,, \quad \forall\, i,j \in \{1,\ldots,k\}\,.$$

Remark

Unlike in the case of Poisson brackets, f_i and f_j being in involution does not imply that X_{f_i} is tangent to the level sets of f_j. Consequently, the submanifolds

$$M_\Lambda = \bigcap_{i=1}^{k} f_i^{-1}(\Lambda_i)\,, \quad \Lambda_i \in \mathbb{R}$$

are no longer invariant under the flows of $X_{f_i},\ldots,X_{f_k}$.

Liouville–Arnol'd theorem for contact Hamiltonian systems

- **Crucial idea:** replace the level sets (i.e., preimages of points) M_Λ by **preimages of rays**

$$M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+ \colon f_\alpha(x) = r\Lambda_\alpha \ \forall\, \alpha\}\,,$$

with $\alpha \in \{0, 1, \ldots, n\}$ and $\Lambda = (\Lambda_0, \Lambda_1, \ldots, \Lambda_n) \in \mathbb{R}^{n+1} \setminus \{0\}$.

Liouville–Arnol'd theorem for contact Hamiltonian systems

Theorem (Colombo, de León, Lainz, L. G., 2023)

Let (M, η) be a $(2n+1)$-dimensional contact manifold. Suppose that $f_0, f_1, \ldots, f_n$ are functions in involution such that $\operatorname{rank}\{\mathrm{d}f_\alpha\}_\alpha \geq n$. Then, $M_{\langle\Lambda\rangle_+}$ is invariant by the Hamiltonian flow of f_α and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$.

Moreover, there is a neighborhood U of $M_{\langle\Lambda\rangle_+}$ such that

1. *There exists coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on U in which the equations of motion read*

$$\dot{y}^\alpha = \Omega^\alpha(\tilde{A}_i)\,, \quad \dot{\tilde{A}}_i = 0\,, \quad \alpha \in \{0, \ldots, n\}\,, \, i \in \{1, \ldots, n\}\,.$$

2. *There exists a conformal change $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$, i.e. $\tilde{\eta} = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.*

Steps of the proof

1. Symplectize (M, η) and f_α, obtaining an exact symplectic manifold (M^Σ, θ) and homogeneous functions in involution f_α^Σ.
2. Prove a Liouville–Arnold theorem for exact symplectic manifolds with homogeneous functions in involution.
3. "Un-symplectize" the action-angle coordinates $(y_\Sigma^\alpha, A_\alpha^\Sigma)$ on M^Σ, yielding functions (y^α, A_Σ) on M.
4. Introduce action-angle coordinates $(y^\alpha, \tilde{A}_i)$ on M, where $\tilde{A}_i = -\frac{A_i}{A_0}$.

Example

- Let $M = \mathbb{R}^3 \setminus \{0\}$ with canonical coordinates (q, p, z), and $\eta = \mathrm{d}z - p\mathrm{d}q$.
- The functions $h = p$ and $f = z$ are in involution.
- We have a chart $\left(M \setminus \{z = 0\}; y^0, y^1, \tilde{A}\right)$, where

$$y^0 = q - q_0\,, \quad y^1 = \log z_0 - \log z \quad \tilde{A} = -\frac{p}{z}\,.$$

- In this chart,

$$X_h = \frac{\partial}{\partial y^0}\,, \quad X_f = \frac{\partial}{\partial y^1}$$

- It is a Darboux chart for the contact form

$$\tilde{\eta} = \frac{1}{A_0}\eta = \mathrm{d}y^0 - \tilde{A}\mathrm{d}y^1\,.$$

Dissipated quantities and stability

In a work in progress, we employ dissipated quantities to study the stability of contact Hamiltonian systems.

Proposition (de Lucas, L. G., Zawora)

Let (M, η, h) be a contact Hamiltonian system such that $X_h(x_0) = 0$. Suppose that $f_1, \ldots, f_k$ are dissipated quantities. If $(Rh)(x_0) > 0$ at an isolated point $x_0 \in \bigcap_{i=1}^{k} f_i^{-1}(0)$, then x_0 is an asymptotically stable equilibrium point.

Cocontact structures

- We would like to model dynamical systems which are dissipative and have an explicit time dependence.
- For instance, consider a friction forze that changes with time.
- Idea: adding explicit time dependence to contact dynamics.

Cocontact structures

Definition

A **cocontact manifold** is a triple (M, τ, η) such that:

1. M is a $(2n+2)$-dimensional manifold,
2. τ and η are 1-forms,
3. $\mathrm{d}\tau = 0$,
4. The map
$$\begin{aligned} \flat_{(\tau,\eta)} \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto (\iota_X \tau)\tau + \iota_X \mathrm{d}\eta + (\iota_X \eta)\,\eta \end{aligned}$$
is an isomorphism of $\mathscr{C}^\infty(M)$-modules.

Reeb and Hamiltonian vector fields

- **Reeb vector fields**: $R_t = \flat_{(\tau,\eta)}^{-1}(\tau),\ R_z = \flat_{(\tau,\eta)}^{-1}(\eta)$.
- **Hamiltonian vector field**:

$$X_f = \flat_{(\tau,\eta)}^{-1}(\mathrm{d}f) - \left(R_z(f) + f\right) R_z + \left(1 - R_t(f)\right) R_t \,.$$

- **Darboux coordinates** $\left(t, q^i, p_i, z\right)$:

$$\tau = \mathrm{d}t, \quad \eta = \mathrm{d}z - p_i \mathrm{d}q^i \,, \quad R_t = \frac{\partial}{\partial t}, \quad R_z = \frac{\partial}{\partial z} \,,$$
$$X_f = \frac{\partial}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z} \,.$$

Dissipated quantities

The definitions of cocontact Hamiltonian system and dissipated quantity are, *mutatis mutandis*, the same as in the (time-independent) contact case.

Definition

Let (M, τ, η, h) be a cocontact Hamiltonian system. A **dissipated quantity** is a function $f \colon M \to \mathbb{R}$ such that

$$X_h(f) = -R_z(h) f \,.$$

Noether's theorem for cocontact Hamiltonian systems

Theorem (Gaset, L. G., Rivas, 2023)

Consider the cocontact Hamiltonian system (M, τ, η, h). Let $Y \in \mathfrak{X}(M)$.

1. *If $\eta([Y, X_h]) = 0$ and $\tau(Y) = 0$, then $f = -\eta(Y)$ is a dissipated quantity.*
2. *Conversely, given a dissipated quantity f, the vector field $Y = X_f - R_t$ verifies $\eta([Y, X_h]) = 0$, $\tau(Y) = 0$ and $f = -\eta(Y)$.*

Definition

A **generalized infinitesimal dynamical symmetry** is a vector field $Y \in \mathfrak{X}(M)$ such that $\eta([Y, X_h]) = 0$ and $\tau(Y) = 0$.

Classification of infinitesimal symmetries

Generalized infinitesimal dynamical symmetries

$\tau(Y) = 0 \quad \eta([Y, X_h]) = 0$

Infinitesimal dynamical symmetries

$\tau(Y) = 0$

$[Y, X_h] = 0$

Infinitesimal conformal Hamiltonian symmetries

$\tau(Y) = 0 \quad \mathcal{L}_Y \eta = \rho\eta$

$Y(h) = \rho h$

Infinitesimal conformal cocontactomorphisms

$\tau(Y) = 0$

$\mathcal{L}_Y \eta = \rho\eta$

Infinitesimal strict Hamiltonian symmetries

$\tau(Y) = 0 \quad \mathcal{L}_Y \eta = 0$

$Y(h) = 0$

Cartan symmetries

$\tau(Y) = 0 \quad \mathcal{L}_Y \eta = \rho\eta + \mathrm{d}g$

$Y(h) = \rho h + g R_z(h)$

Example (The two-body problem with time-dependent friction)

- The phase space is $\mathbb{R} \times \mathrm{T}^*\mathbb{R}^6 \times \mathbb{R}$, with coords. $(t, \mathbf{q}^1, \mathbf{q}^2, \mathbf{v}^1, \mathbf{v}^2, z)$, where $\mathbf{q}^1 = (q^1_1, q^1_2, q^1_3)$ and $\mathbf{q}^2 = (q^2_1, q^2_2, q^2_3)$.
- The relative distance between the particles is $\mathbf{r} = \mathbf{q}^2 - \mathbf{q}^1$, whose Euclidean length will be denoted $r = |\mathbf{r}|$.
- The Hamiltonian function is

$$H = \frac{1}{2m_1}\mathbf{p}^1 \cdot \mathbf{p}^1 + \frac{1}{2m_2}\mathbf{p}^2 \cdot \mathbf{p}^2 + U(r) + \gamma(t) z\,,$$

  and the cocontact structure is given by the one-forms

$$\eta = \mathrm{d}z - \mathbf{p}^1 \cdot \mathrm{d}\mathbf{q}^1 - \mathbf{p}^2 \cdot \mathrm{d}\mathbf{q}^2\,, \quad \tau = \mathrm{d}t\,.$$

- The vector fields $Y_a = \frac{\partial}{\partial q^1_a} + \frac{\partial}{\partial q^2_a}\,, a \in \{1,2,3\}$ are infinitesimal strict Hamiltonian symmetries and the associated dissipated quantities are the components of $\mathbf{p}^1 + \mathbf{p}^2$.

Forced Hamiltonian systems

Given a manifold Q, let T^*Q be its cotangent bundle with canononical one-form θ_Q and canonical symplectic form $\omega_Q = -\mathrm{d}\theta_Q$.

Definition

A **forced Hamiltonian system** is a triple (Q, h, α) where Q is a manifold, $h \in \mathscr{C}^\infty(\mathrm{T}^*Q)$ is a function and $\alpha \in \Omega^1(\mathrm{T}^*Q)$ is a semibasic one-form (i.e., $\alpha(X) = 0$ for any vertical vector field X). The **forced Hamiltonian vector field** $X_{h,\alpha} \in \mathfrak{X}(\mathrm{T}^*Q)$ is given by

$$\iota_{X_{h,\alpha}}\omega_Q = \mathrm{d}h + \alpha\,.$$

Local expressions

Given local coordinates (q^i) on Q and the induced bundle coordinates (q^i, p_i) on T^*Q, we have that

$$
\begin{aligned}
\theta_Q &= p_i \mathrm{d}q^i \,, \\
\omega_Q &= \mathrm{d}q^i \wedge \mathrm{d}p_i \,, \\
\alpha &= \alpha_i(q, p)\mathrm{d}q^i \,, \\
X_{h,\alpha} &= \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial h}{\partial q^i} + \alpha_i\right)\frac{\partial}{\partial p_i} \,.
\end{aligned}
$$

Reduction *à la* Marsden–Weinstein

- Let G be a Lie group with Lie algebra $\mathfrak{g}$, and dual $\mathfrak{g}^*$.
- Consider a Lie group action $\Phi\colon G \times Q \to Q$ of G on a manifold Q and its cotangent lift $\Phi^{\mathsf{T}^*}\colon G \times \mathsf{T}^*Q \to \mathsf{T}^*Q$.
- Henceforth, assume that both of these actions are free and proper.

Reduction *à la* Marsden–Weinstein

- Let $\xi_{\mathrm{T}^*Q} \in \mathfrak{X}(\mathrm{T}^*Q)$ denote the infinitesimal generator of the action defined by $\xi \in \mathfrak{g}$, i.e.,

$$\xi_{\mathrm{T}^*Q}(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^{\mathrm{T}^*}_{\exp t\xi}(x)\,,$$

where $\exp\colon \mathfrak{g} \to G$ is the exponential map.

- Let $\mathbf{J}\colon \mathrm{T}^*Q \to \mathfrak{g}^*$ denote the natural momentum map, namely,

$$\mathbf{J}^{\xi}(x) := \langle \mathbf{J}(x), \xi \rangle = \iota_{\xi_{\mathrm{T}^*Q}} \theta_Q(x)\,,$$

Reduction *à la* Marsden–Weinstein

Proposition

Let (Q, h, α) be a forced Hamiltonian system such that h is Φ^{T^*}-invariant.

1. *For each $\xi \in \mathfrak{g}$, the function $\mathbf{J}^\xi$ is a conserved quantity iff*
$$\alpha(\xi_{\mathsf{T}^* Q}) = 0\,. \tag{1}$$
2. *If Eq. (1) holds, then α is ξ-invariant iff*
$$\iota_{\xi_{\mathsf{T}^* Q}} \mathrm{d}\alpha = 0\,.$$
3. *The subset*
$$\mathfrak{g}_\alpha = \left\{\xi \in \mathfrak{g} \mid \alpha(\xi_{\mathsf{T}^* Q}) = 0,\ \iota_{\xi_{\mathsf{T}^* Q}} \mathrm{d}\alpha = 0\right\}$$
is a Lie subalgebra of $\mathfrak{g}$.

Reduction *à la* Marsden–Weinstein

- Let G_α be the unique connected Lie subgroup of G whose Lie algebra is $\mathfrak{g}_\alpha$.
- Assume that G_α is a closed Lie subgroup of G.
- Let μ be a regular value of the natural momentum map $\mathbf{J}_\alpha \colon \mathrm{T}^*Q \to \mathfrak{g}_\alpha$.
- Denote by $G_{\alpha,\mu} \subseteq G_\alpha$ the isotropy subgroup of μ w.r.t. the coadjoint action.

Reduction *à la* Marsden–Weinstein

Theorem (de León, Lainz, L. G., 2021)

1. *$\mathbf{J}_\alpha^{-1}(\mu)$ is a submanifold of T^*Q and $X_{h,\alpha}$ is tangent to it.*
2. *$M_\mu := \mathbf{J}_\alpha^{-1}(\mu)/G_{\alpha,\mu}$ has a symplectic form ω_μ uniquely determined by*

$$\pi_\mu^*\omega_\mu = i_\mu^*\omega_Q\,,$$

where the maps $i_\mu\colon \mathbf{J}_\alpha^{-1}(\mu) \hookrightarrow \mathsf{T}^*Q$ and $\pi_\mu\colon \mathbf{J}_\alpha^{-1}(\mu) \to \mathbf{J}_\alpha^{-1}(\mu)/G_{\alpha,\mu}$ denote the inclusion and the projection, respectively.

3. *We have a reduced Hamiltonian function h_μ and force α_μ on M_μ given by*

$$h_\mu \circ \pi_\mu = h \circ i_\mu\,, \quad \pi_\mu^*\,\alpha_\mu = i_\mu^*\,\alpha\,.$$

4. *There is a vector field $X_{h_\mu,\,\alpha_\mu} \in \mathfrak{X}(M_\mu)$ such that*

$$\mathsf{T}\pi_\mu \circ X_{h,\,\alpha} \circ i_\mu = X_{h_\mu,\,\alpha_\mu} \circ \pi_\mu \quad \text{and} \quad \iota_{X_{h_\mu,\alpha_\mu}}\omega_\mu = \mathrm{d}h_\mu + \alpha_\mu\,.$$

Hybrid systems

Definition

A **hybrid system** is a 4-tuple $\mathscr{H} = (M, X, S, \Delta)$, formed by

1. a manifold M,
2. a vector field $X \in \mathfrak{X}(M)$,
3. a submanifold $S \subset M$ of codimension 1 or greater,
4. an embedding $\Delta \colon S \to M$.

The dynamics generated by $\mathscr{H}$ are the curves $c \colon I \subseteq \mathbb{R} \to M$ such that

$$\begin{aligned} \dot{c}(t) &= X(c(t))\,, && \text{if } c(t) \notin S\,,\\ c^+(t) &= \Delta(c^-(t))\,, && \text{if } c(t) \in S\,, \end{aligned}$$

where

$$c^\pm(t) = \lim_{\tau \to t^\pm} c(\tau)\,.$$

Hybrid Hamiltonian systems

Definition

A hybrid dynamical system (M, X, S, Δ) is said to be a **hybrid Hamiltonian system** and denoted by $\mathscr{H}_h$ if

1. $M \subseteq \mathrm{T}^*Q$ is a zero-codimensional submanifold of the cotangent bundle T^*Q of a manifold Q,
2. S projects onto a codimension-one submanifold $\pi_Q(S)$ of Q,
3. $\pi_Q \circ \Delta = \pi_Q$,
4. $X = X_h$ is the Hamiltonian vector field of $h \in \mathscr{C}^\infty(\mathrm{T}^*Q)$ w.r.t. ω_Q.

A **forced hybrid Hamiltonian system** is defined analogously by replacing X_h with $X_{h,\alpha}$.

Hybrid Hamiltonian systems

Physically,

- Q represents the space of positions,
- T^*Q the phase space,
- X_h the dynamics between the impacts,
- $\pi_Q(S)$ the hypersurface where impacts occur, and
- Δ the change of momenta on the impacts.

Example (The circular billiard)

- Consider a particle in the plane which moves freely inside the surface confined by the unit circle.
- The Hamiltonian is $H = \frac{p_x^2}{2} + \frac{p_y^2}{2}$.
- The switching surface is

$$S = \left\{ (x, y, p_x, p_y) \in \mathrm{T}^*\mathbb{R}^2 \mid x^2 + y^2 = 1 \text{ and } (p_x, p_y) \cdot (x, y) > 0 \right\} .$$

- The condition $(p_x, p_y) \cdot (x, y) > 0$ just means that, for an impact to occur, the momenta pointing to the wall must be positive.
- The impact map is $\Delta(x, y, p_x^-, p_y^-) \mapsto (x, y, p_x^+, p_y^+)$, where

$$\begin{aligned} p_x^+ &= p_x^- - 2(xp_x^- + yp_y^-)x \,, \\ p_y^+ &= p_y^- - 2(xp_x^- + yp_y^-)y \,. \end{aligned}$$

Hybrid Lie group action

Definition

A Lie group action $\Phi\colon G \times Q \to Q$ is called a **hybrid action for** $\mathscr{H}_h$ if its cotangent lift $\Phi^{\mathrm{T}^*}\colon G \times \mathrm{T}^*Q \to \mathrm{T}^*Q$ satisfies the following conditions:

1. h is Φ^{T^*}-invariant, namely, $h \circ \Phi_g^{\mathrm{T}^*} = h$ for all $g \in G$,
2. the restriction $\left.\Phi^{\mathrm{T}^*}\right|_{G\times S}$ is a Lie group action of G on S,
3. the impact map is equivariant w.r.t. this action, i.e.,

$$\Delta \circ \left.\Phi_g^{\mathrm{T}^*}\right|_S = \Phi_g^{\mathrm{T}^*} \circ \Delta\,, \quad \forall\, g \in G\,.$$

Hybrid momentum map

Definition

Let $\Phi\colon G \times Q \to Q$ be a hybrid action for $\mathscr{H}_h$. A momentum map $\mathbf{J}\colon \mathrm{T}^*Q \to \mathfrak{g}^*$ for the cotangent lift action Φ^{T^*} is called a **generalized hybrid momentum map** if, for each connected component $C \subseteq S$ and for each regular value μ_- of $\mathbf{J}$, there is another regular value μ_+ such that

$$\Delta(\mathbf{J}|_C{}^{-1}(\mu_-)) \subset \mathbf{J}^{-1}(\mu_+)\,.$$

In particular, if $\mu_- = \mu_+$ it is called a **hybrid momentum map**.
A **hybrid regular value** of $\mathbf{J}$ is a regular value of both $\mathbf{J}$ and $\mathbf{J}|_S$.

Hybrid momentum map

In other words, **J** is a generalized hybrid momentum map if, for every point in the connected component C of the switching surface S such that the momentum before the impact takes a value of μ_-, the momentum will take a value μ_+ after the impact; and it is a hybrid momentum map if its value does not change with the impacts.

Hybrid reduction

Proposition

If μ_- and μ_+ are regular values of $\mathbf{J}$ such that $\Delta\left(\mathbf{J}|_S{}^{-1}(\mu_-)\right) \subset \mathbf{J}^{-1}(\mu_+)$, then the isotropy subgroups in μ_- and μ_+ coincide, that is, $G_{\mu_-} = G_{\mu_+}$.

Hybrid reduction

Theorem (Colombo, de León, Eyrea Irazú, L. G., 2022)

Let $\Phi\colon G \times Q \to Q$ be a hybrid action on $\mathscr{H}_h$. Assume that G is connected and that $\Phi^{\mathsf{T}^*}\colon G \times \mathsf{T}^*Q \to \mathsf{T}^*Q$ is free and proper. Consider a sequence $\{\mu_i\}_{i\in I\subseteq\mathbb{N}}$ of hybrid regular values of $\mathbf{J}$, such that $\Delta\left(\mathbf{J}|_S{}^{-1}(\mu_i)\right) \subset \mathbf{J}^{-1}(\mu_{i+1})$. Let $G_{\mu_i} = G_{\mu_0}$ be the isotropy subgroup in μ_i under the co-adjoint action. Then, the reduction leads to a sequence of reduced hybrid forced Hamiltonian systems

$$\mathscr{H}_h^{\mu_i} = \left(\mathbf{J}^{-1}(\mu_i)/G_{\mu_0},\, X_{h_{\mu_i}}, \mathbf{J}|_S{}^{-1}(\mu_i)/G_{\mu_0}, (\Delta)_{\mu_i}\right).$$

Hybrid reduction

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathbf{J}^{-1}(\mu_i) \arrow[d] & \mathbf{J}|_S{}^{-1}(\mu_i) \arrow[l, hook'] \arrow[d] \arrow[r, "\Delta|_{\mathbf{J}^{-1}(\mu_i)}"] & \mathbf{J}^{-1}(\mu_{i+1}) \arrow[d] & \cdots \arrow[l, hook'] \\
\cdots \arrow[r] & \frac{\mathbf{J}^{-1}(\mu_i)}{G_{\mu_0}} & \mathbf{J}|_S{}^{-1}(\mu_i)/G_{\mu_0} \arrow[l, hook'] \arrow[r, "(\Delta)_{\mu_i}"] & \frac{\mathbf{J}^{-1}(\mu_{i+1})}{G_{\mu_0}} & \cdots \arrow[l, hook']
\end{tikzcd}
$$

Integrable hybrid Hamiltonian systems

- A particular case is when we have the Abelian Lie group action $\Phi\colon \mathbb{R}^n \times \mathsf{T}^*Q \to \mathsf{T}^*Q$ generated by the Hamiltonian flows of n functions $f_1, \ldots, f_n$ in involution.
- In that case, we can identify the momentum map with $F = (f_1, \ldots, f_n)\colon \mathsf{T}^*Q \to \mathbb{R}^n$.
- We may obtain action-angle coordinates for each time interval between impacts. The action-angle coordinates before and after the impact will be related by Δ.

Hamilton–Jacobi theory

- Let $\pi\colon E \to B$ is a vector bundle.
- Consider a dynamical system characterized by $X \in \mathfrak{X}(E)$.
- Idea: obtain a section $\gamma \in \Gamma(E)$ such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "X"] \arrow[d, "\pi"] & \mathsf{T}E \arrow[d, "\mathsf{T}\pi"'] \\
B \arrow[r, "X^\gamma"] \arrow[u, bend left, "\gamma"] & \mathsf{T}B \arrow[u, bend right, "\mathsf{T}\gamma"']
\end{tikzcd}$$

- If $c\colon I \subseteq \mathbb{R} \to B$ is an integral curve of X^γ, then $\gamma \circ c$ is an integral curve of X.

Hamilton–Jacobi theory

Under certain assumptions on X and γ, the diagram above is commutative iff a PDE known as the **Hamilton–Jacobi (HJ) equation** holds:

- If the bundle is $\pi_Q \colon \mathrm{T}^*Q \to Q$, the vector field is a forced Hamiltonian vector field $X = X_{h,\alpha}$, and γ is a closed one-form, the HJ equation is
$$\gamma^*(\mathrm{d}H + \alpha) = 0\,.$$

Hamilton–Jacobi theory

- In this dissertation we have also obtained two different HJ equations for a cocontact Hamiltonian vector field $X = X_h$ on $E = \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$.
- In that case, one can consider two possible bundles:

$$\begin{tikzcd}
 & \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \arrow[dl] \arrow[dr] & \\
\mathbb{R} \times Q \times \mathbb{R} & & \mathbb{R} \times Q
\end{tikzcd}$$

Hamilton–Jacobi theory

- We have also studied the HJ theory for hybrid systems.
- Essentially, in that case one has the usual HJ equation for the continuous dynamics between impacts.
- One has to impose a compatibility condition of the form

$$\operatorname{Im}(\Delta \circ \gamma_i) \subset \operatorname{Im} \gamma_{i+1} ,$$

where γ_i is the solution of the HJ equation between the i-th and $(i+1)$-th impacts.

Hamilton–Jacobi theory

- Discrete HJ equations can be obtained by replacing X and X^γ with their discrete flows.
- We have obtained a discrete HJ equation for forced discrete Hamiltonian systems.

Contact Lagrangian systems with impulsive constraints

- Constraints (both holonomic and nonholonomic) with discontinuities can lead to instantaneous changes on dynamical systems.
- Hence, this type of constraints, called **impulsive constraints**, can also be employed to model systems with impacts.
- For instance, one can think of a wall as a constraint.
- Impusive constraints have been deeply studied in classical mechanics and were given a geometric interpretation in the 1990s by Lacomba and Tulczyjew, Ibort *et al.*, and Cortés and Vinogradov.

Contact Lagrangian systems with impulsive constraints

- In this dissertation, we have extended the theory of impulsive constraints to contact Lagrangian systems.
- In addition, we have proven a **Carnot theorem** for contact Lagrangian systems subject to impulsive constraints, characterizing the changes of energy due to both the contact-type dissipation and the impulsive forces.

Nonsmooth Herglotz variational principle

- Let L be an **action-dependent** Lagrangian function.
- Recall that, roughly speaking, the Herglotz functional $\mathcal{A}$ is given by the ODE
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}[q(t)] = L\Big(q(t), \dot{q}(t), \mathcal{A}[q(t)]\Big)\,.$$
- One seeks for curves $q\colon I \subseteq \mathbb{R} \to Q$ that are extremals of $\mathcal{A}$.
- Usually, these curves are assumed to be at least $\mathscr{C}^2$.
- By considering curves that are $\mathscr{C}^0$ and piecewise $\mathscr{C}^2$ we can obtain a variational principle for systems with impacts. The impacts are precisely the points where the curve is not smooth.

Future research

- The Liouville–Arnol'd theorem is a first step in the study of completely integrable contact systems.
- Magri *et al.* studied the relation between bi-Hamiltonian structures, Poisson–Nijenhuis structures and integrable systems. It seems that Jacobi–Nijenhuis structures should have an analogous relation with integrable contact systems.
- We would like to find an algorithm for computing action-angle coordinates in an efficient manner. Perhaps, they are related with solutions of the HJ equation.
- It is pending to consider completely integrable contact systems with critical points, i.e., non-regular values of (f_α).

Future research

- Other structures employed in the study of classical integrable systems could be generalized to completely integrable contact systems: momentum polytopes, Haantjes tensors, etc.
- We intend to develop a Kolmogorov–Arnol'd–Moser (KAM) theory for contact Hamiltonian systems.
- We would like to study hybrid systems experiencing Zeno effect, i.e., the set of impacts is not discrete.
- It is pending to explore the applicability of our results concerning hybrid systems for mathematical billiard theory.

Publications derived from this thesis

[1] L. Colombo, M. de León, M. E. Eyrea Irazú, and A. López-Gordón. *Hamilton-Jacobi theory for nonholonomic and forced hybrid mechanical systems*. Accepted on *Geom. Mech.* arXiv: 2211.06252.

[2] L. Colombo, M. de León, M. Lainz, and A. López-Gordón. *Liouville-Arnold Theorem for Contact Hamiltonian Systems*. 2023. arXiv: 2302.12061.

[3] L. J. Colombo, M. de León, M. E. Eyrea Irazú, and A. López-Gordón. *Generalized Hybrid Momentum Maps and Reduction by Symmetries of Forced Mechanical Systems with Inelastic Collisions*. 2022. arXiv: 2112.02573.

[4] L. J. Colombo, M. de León, and A. López-Gordón. "Contact Lagrangian Systems Subject to Impulsive Constraints". *J. Phys. A: Math. Theor.*, **55**(42) (2022).

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[11] A. López-Gordón, L. Colombo, and M. de León. "Nonsmooth Herglotz Variational Principle". In: *2023 American Control Conference (ACC)*. 2023, pp. 3376–3381.

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Thanks for your kind attention!