Nijenhuis–Jacobi structures and integrability of contact Hamiltonian systems

Asier López-Gordón
Work in progress with Leonardo J. Colombo, Manuel de León and María Emma Eyrea Irazú

Institute of Mathematical Sciences (ICMAT), CSIC, Madrid, Spain

XVI International ICMAT Summer School on Geometry, Dynamics and Field theory

Financially supported by Grants CEX2019-000904-S, PID2022-137909NB-C21 and RED2022-134301-T, funded by MCIN/AEI/10.13039/501100011033

Compatible Poisson structures

Definition

Let M be a manifold. Two Poisson tensors are Λ and Λ_1 on M are said to be **compatible** if $\Lambda + \Lambda_1$ is also a Poisson tensor on M.

Definition

A vector field $X \in \mathfrak{X}(M)$ is called **bi-Hamiltonian** if it is a Hamiltonian vector field w.r.t. two compatible Poisson structures, namely,

$$X = \Lambda(\mathrm{d}h, \cdot) = \Lambda_1(\mathrm{d}h_1, \cdot)\,,$$

for two functions $h, h_1 \in \mathscr{C}^\infty(M)$.

Compatible Poisson structures

Definition

Let M be a manifold. Two Poisson tensors are Λ and Λ_1 on M are said to be **compatible** if $\Lambda + \Lambda_1$ is also a Poisson tensor on M.

Definition

A vector field $X \in \mathfrak{X}(M)$ is called **bi-Hamiltonian** if it is a Hamiltonian vector field w.r.t. two compatible Poisson structures, namely,

$$X = \Lambda(\mathrm{d}h, \cdot) = \Lambda_1(\mathrm{d}h_1, \cdot)\,,$$

for two functions $h, h_1 \in \mathscr{C}^\infty(M)$.

Poisson – Nijehuis structures

- The linear map $\sharp_\Lambda \colon \mathrm{T}_x^* M \ni \alpha \mapsto \Lambda(\alpha, \cdot) \in \mathrm{T}_x M$ is an isomorphism iff Λ comes from a symplectic structure ω. In that case, $\flat_\omega := \sharp_\Lambda^{-1}(v) = \iota_v \omega$.
- In that situation, we can define the $(1,1)$-tensor field

$$N = \sharp_{\Lambda_1} \circ \sharp_\Lambda^{-1} .$$

Poisson – Nijehuis structures

Theorem (Magri and Morosi, 1984)

Let (M, ω) be a symplectic manifold and Λ_1 a bivector. Consider the $(1, 1)$-tensor field

$$N = \sharp_{\Lambda_1} \circ \sharp_{\omega}^{-1} .$$

If Λ_1 is a Poisson tensor compatible with Λ, then the Nijehuis torsion T_N of N vanishes. In that case, the eigenvalues of N are in involution w.r.t. both Poisson brackets.
The pair (Λ, N) is called a ***Poisson – Nijenhuis structure*** *on M.*

Poisson – Nijehuis structures

Corollary

If a vector field $X \in \mathfrak{X}(M)$ is bi-Hamiltonian w.r.t. to ω and Λ_1 (i.e., $X = \sharp_\omega \mathrm{d}h = \sharp_{\Lambda_1} \mathrm{d}h_1$), then the eigenvalues of N form a family of conserved quantities in involution w.r.t. both Poisson brackets.

Contact geometry

Definition

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that the map

$$\begin{aligned} \flat_\eta \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

is an isomorphism of $\mathscr{C}^\infty(M)$-modules.

- There exists a unique vector field R on (M, η), called the **Reeb vector field**, given by $R = \flat_\eta^{-1}(\eta)$, or, equivalently,

$$\iota_R \mathrm{d}\eta = 0, \quad \iota_R \eta = 1.$$

Contact geometry

- The **Hamiltonian vector field** of $f \in \mathscr{C}^\infty(M)$ is given by

$$X_f = \flat_\eta^{-1}(\mathrm{d}f) - (R(f) + f)\, R\,,$$

- Around each point on M there exist **Darboux coordinates** (q^i, p_i, z) such that

$$\begin{aligned}
\eta &= \mathrm{d}z - p_i \mathrm{d}q^i, \\
R &= \frac{\partial}{\partial z}, \\
X_f &= \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}.
\end{aligned}$$

Contact Hamiltonian systems

Definition

A **contact Hamiltonian system** is a triple (M, η, h) formed by a contact manifold (M, η) and a **Hamiltonian function** $h \in \mathscr{C}^\infty(M)$.

- The dynamics of (M, η, h) is determined by the integral curves of the Hamiltonian vector field X_h of h w.r.t. η.

Contact Hamiltonian systems

- In Darboux coordinates, these curves $c(t) = (q^i(t), p_i(t), z(t))$ are determined by the **contact Hamilton equations**:

$$
\begin{aligned}
\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} &= \frac{\partial h}{\partial p_i} \circ c(t)\,, \\
\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -\frac{\partial h}{\partial q^i} \circ c(t) + p_i(t) \frac{\partial h}{\partial z} \circ c(t)\,, \\
\frac{\mathrm{d}z(t)}{\mathrm{d}t} &= p_i(t) \frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.
\end{aligned}
$$

Jacobi manifolds

Definition

A **Jacobi structure** on a manifold M is a pair (Λ, E) where Λ is a bivector and E a vector field such that the composition rule $\{\cdot,\cdot\}$ on $\mathscr{C}^\infty(M)$ given by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f)\,,$$

is a Lie bracket, called the **Jacobi bracket**. The triple (M, Λ, E) is called a **Jacobi manifold**.

In particular, $\{\cdot,\cdot\}$ is a Poisson bracket iff $E \equiv 0$.

Jacobi structure of a contact manifold

- A contact manifold (M, η) is endowed with a Jacobi bracket determined by

$$\{f, g\} = -\mathrm{d}\eta(\flat_\eta^{-1}\mathrm{d}f, \flat_\eta^{-1}\mathrm{d}g) - fR(g) + gR(f).$$

- It can also be expressed as follows:

$$\{f, g\} = X_f(g) + gR(f)\,.$$

Jacobi brackets and dissipated quantities

Definition

Let (M, η, h) be a contact Hamiltonian system with Jacobi bracket $\{\cdot, \cdot\}$. A function $f \in \mathscr{C}^\infty(M)$ is called a **dissipated quantity** if

$$\{f, h\} = 0\,.$$

Completely integrable contact system

Definition

A **completely integrable contact system** is a triple (M, η, F), where (M, η) is a contact manifold and $F = (f_0, \ldots, f_n)\colon M \to \mathbb{R}^{n+1}$ is a map such that

1. $f_0, \ldots, f_n$ are in involution, i.e., $\{f_\alpha, f_\beta\} = 0 \ \forall\, \alpha, \beta$,
2. rank $\mathrm{T}F \geq n$ on a dense open subset $M_0 \subseteq M$.

The functions $f_0, \ldots, f_n$ are called **integrals**.

Liouville – Arnol'd theorem for contact systems

1. Given $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+ \colon f_\alpha(x) = r\Lambda_\alpha\}$.
2. Assume that the Hamiltonian vector fields $X_{f_0}, \ldots, X_{f_n}$ are complete.
3. Let $B \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be an open neighbourhood of Λ.
4. Let $\pi \colon U \to M_{\langle\Lambda\rangle_+}$ be a tubular neighbourhood of $M_{\langle\Lambda\rangle_+}$ such that $F|_U \colon U \to B$ is a trivial bundle over a domain $V \subseteq B$.

Liouville – Arnol'd theorem for contact systems

Theorem (Colombo, de León, Lainz, L.-G., 2023)

Let (M, η, F) be a completely integrable contact system, where $F = (f_0, \ldots, f_n)$. Consider the assumptions of the previous slide. Then:

1. *$M_{\langle\Lambda\rangle_+}$ is coisotropic, invariant by the Hamiltonian flow of f_α, and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ for some $k \leq n$.*
2. *There exists coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on U such that the equations of motion are given by*

$$\dot{y}^\alpha = \Omega^\alpha(\tilde{A}_1, \ldots, \tilde{A}_n)\,, \quad \dot{\tilde{A}}_i = 0\,.$$

3. *There exists a nowhere-vanishing function $A_0 \in \mathscr{C}^\infty(U)$ and a conformally equivalent contact form $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$, namely, $\tilde{\eta} = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.*

Our goal

- We would like to generalize Magri *et al.*'s constructions for integrable contact systems.
- That is, given a contact Hamiltonian system (M, η, h), we want to find a tensor N such that, if it satisfies certain compatibility conditions with (η, h), one can compute dissipated quantities in involution for it.

Compatible Jacobi structures

- Nunes da Costa (1998) introduced the notion of compatibility of Jacobi structures.

Definition

Two Jacobi structures (Λ, E) and (Λ_1, E_1) on a manifold M are said to be **compatible** if $(\Lambda + \Lambda_1, E + E_1)$ is also a Jacobi structure on M.

- She also proved several conditions which are equivalent to (Λ, E) and (Λ_1, E_1) being compatible.

Compatible Jacobi structures

- Nunes da Costa (1998) introduced the notion of compatibility of Jacobi structures.

Definition

Two Jacobi structures (Λ, E) and (Λ_1, E_1) on a manifold M are said to be **compatible** if $(\Lambda + \Lambda_1, E + E_1)$ is also a Jacobi structure on M.

- She also proved several conditions which are equivalent to (Λ, E) and (Λ_1, E_1) being compatible.

Jacobi – Nijenhuis structures

- A Jacobi –Nijenhuis structure (Λ, E, N) is a generalization of Nijenhuis – Poisson structures.
- These structures were introduced by Marrero, Monterde and Padrón (1999).
- Their relation with homogeneous Nijenhuis – Poisson structures was studied by Petalidou and Nunes da Costa (2001).

Jacobi – Nijenhuis structures

- The space $\mathfrak{X}(M) \times \mathscr{C}^\infty(M)$ can be endowed with the Lie bracket $[\cdot, \cdot]$ given by
$$\Big[(X, f), (Y, g)\Big] = \Big([X, Y], X(g) - Y(f)\Big) .$$
- *Mutatis mutandis*, the Nijenhuis torsion T_N of a linear operator $N \colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M) \times \mathscr{C}^\infty(M)$ is defined as usual:

$$\begin{aligned} T_N\Big((X, f), (Y, g)\Big) := \Big[N(X, f), N(Y, g)\Big] - N\Big[N(X, f), (Y, g)\Big] \\ - N\Big[(X, f), N(Y, g)\Big] + N^2\Big[(X, f), (Y, g)\Big] , \end{aligned}$$

Jacobi – Nijenhuis structures

- Given a Jacobi structure (Λ, E), we can define the $\mathscr{C}^\infty(M)$-modules homomorphism $\sharp_{(\Lambda,E)} \colon \Omega^1(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M) \times \mathscr{C}^\infty(M)$.
- It is an isomorphism iff (Λ, E) comes from a contact structure.

Jacobi – Nijenhuis structures

Definition

A **Jacobi – Nijenhuis structure** on a manifold M is a triple (Λ, E, N) where (Λ, E) is a Jacobi structure and $N \colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M) \times \mathscr{C}^\infty(M)$ is a $\mathscr{C}^\infty(M)$-linear map such that

$$
\begin{aligned}
&N \circ \sharp_{(\Lambda, E)} = \sharp_{(\Lambda, E)} \circ N^* \,, \\
&T_N \equiv 0 \,, \\
&\mathcal{C}\Big((\Lambda, E), N\Big) \equiv 0 \,.
\end{aligned}
$$

The 4-tuple (M, Λ, E, N) is calle a **Jacobi – Nijenhuis manifold**.

Jacobi – Nijenhuis structures

- In the previous slide, $\mathcal{C}$ denotes the **concomitant**. Its expression depends on N, (Λ, E) and a quite involved Lie bracket on $\Omega^1(M) \times \mathscr{C}^\infty(M)$.
- Let (Λ_1, E_1) be the Jacobi structure determined by

$$\Lambda_1(\beta, \alpha) = \left\langle \beta, N_1\big(\Lambda(\cdot, \alpha), 0\big) \right\rangle, \quad E_1 = N_1(E, 0)\,,$$

  where $N_1 \colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M)$ is the projection of N on the first component.
- If (Λ_1, E_1) is also coming from a contact structure, then

$$T_N \equiv 0 \Longleftrightarrow \mathcal{C}\Big((\Lambda, E), N\Big) \equiv 0\,.$$

The correspondence between Jacobi – Nijenhuis and homogeneous Nijenhuis – Poisson structures

Proposition (Petalidou and Nunes da Costa, 2001)

With any Jacobi – Nijenhuis manifold (M, Λ, E, N), *we can associate a homogeneous Nijenhuis – Poisson manifold, namely, a Nijenhuis – Poisson manifold* $(M \times \mathbb{R}, \tilde{\Lambda}, \tilde{N})$ *such that*

$$\mathcal{L}_{\frac{\partial}{\partial t}} \tilde{\Lambda} = -\tilde{\Lambda}\,, \quad \mathcal{L}_{\frac{\partial}{\partial t}} \tilde{N} = 0\,,$$

where t *denotes the canonical coordinate on the* $\mathbb{R}$ *component of* $M \times \mathbb{R}$.

Exact symplectic manifolds: Liouville geometry

Definition

An **exact symplectic manifold** is a pair (M, θ), where M is a manifold and θ a one-form on N such that $\omega = -\mathrm{d}\theta$ is a symplectic form on M.

- The **Liouville vector field** Δ of (M, θ) is given by

$$\iota_\Delta \omega = -\theta.$$

- A tensor T is called **homogeneous of degree** n if $\mathcal{L}_\Delta T = nT$.

Symplectization of contact manifolds

Definition

Let (M, η) be a contact manifold and (M^Σ, θ) an exact symplectic manifold. A **symplectization** is a fibre bundle $\Sigma \colon M^\Sigma \to M$ such that

$$\sigma \Sigma^* \eta = \theta,$$

for a function σ on M^Σ called the **conformal factor**.

Symplectization of contact manifolds

$$\text{Category of contact manifolds} \\ \updownarrow \\ \text{Category of exact symplectic manifolds}$$

- Contact distribution $\ker \eta \longleftrightarrow$ symplectic potential θ
- Functions $\longleftrightarrow$ homogeneous functions
- Hamiltonian vector fields $\longleftrightarrow$ Hamiltonian vector fields

Our result (under construction)

- Let (M, η) be a contact manifold with associated Jacobi structure (Λ, E).
- Suppose that there is another contact form η_1 on M with Jacobi structure (Λ_1, E_1).
- Let $N = \sharp_{(\Lambda_1, E_1)} \circ \sharp_{(\Lambda, E)}^{-1}$.
- (Λ, E) and (Λ_1, E_1) are compatible iff $T_N \equiv 0$.
- In that case, the eigenvalues of N are in involution w.r.t. the Jacobi brackets of both (Λ, E) and (Λ_1, E_1).

Our result (under construction)

- Let (Λ_1, E_1) be the Jacobi structure determined by

$$\Lambda_1(\beta, \alpha) = \left\langle \beta, N_1(\Lambda(\cdot, \alpha), 0) \right\rangle, \quad E_1 = N_1(E, 0)\,,$$

  where $N_1 \colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M)$ is the projection of N on the first component.

- Consider a contact Hamiltonian system (M, η, h) such that $X_h = Y_{h_1}$ is the Hamiltonian vector field of h w.r.t. η and the Hamiltonian vector field of h_1 w.r.t. h_1, namely,

$$X_h = Y_{h_1} = \Lambda(\cdot, \mathrm{d}h_1) + h_1 E_1\,.$$

- The spectrum of $\tilde{N}$ can be used to compute dissipated quantities in involution.

Main references

[1] L. Colombo, M. de León, M. Lainz, and A. L.-G. *Liouville-Arnold Theorem for Contact Hamiltonian Systems*. 2023. arXiv: 2302.12061.

[2] F. Magri, P. Casati, G. Falqui, and M. Pedroni. "Eight Lectures on Integrable Systems". In: *Integrability of Nonlinear Systems*. Lecture Notes in Physics. Springer: Berlin, Heidelberg, 1997.

[3] J. C. Marrero, J. Monterde, and E. Padron. "Jacobi–Nijenhuis Manifolds and Compatible Jacobi Structures". *C. R. Acad. Sci. Paris Sér. I Math.*, **329**(9) (1999).

[4] J. M. Nunes da Costa. "Compatible Jacobi Manifolds: Geometry and Reduction". *J. Phys. A: Math. Gen.*, **31**(3) (1998).

[5] F. Petalidou and J. Nunes Da Costa. "Local Structure of Jacobi–Nijenhuis Manifolds". *J. Geom. Phys.*, **45**(3-4) (2003).

Thanks for your kind attention!

✉ asier.lopez@icmat.es

🌐 www.alopezgordon.xyz