Darboux theorem for homogeneous contact forms

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Asier López-Gordón, joint work with Janusz Grabowski
Intitute of Mathematics, Polish Academy of Sciences

There are several situations in geometry and physics in which a $(\mathbb{N}, \mathbb{Z}, \mathbb{Z}_2, \mathbb{R}, \dots)$ grading appears:

* The algebra of differential forms with the wedge product.
* The spin of particles.
* Intensive/extensive variables in thermodynamics
* Symplectisation / Poissonisation of contact/Jacobi mfolds.
* Supermanifolds
* Higher tangent bundles

Why homogeneity?

Theorem (Euler): Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function. The following statements are equivalent:

i) f is k-homogeneous ($k \in \mathbb{Z}$), namely

$$f(t x^1, \dots, t x^n) = t^k f(x^1, \dots, x^n) \qquad \forall t \in \mathbb{R} \setminus \{0\}.$$

ii) f is a solution of the PDE

$$k \cdot f = \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}.$$

In other words, homogeneous functions are eigenfunctions of

$$X = \sum_{i=1}^{n} x^i \, \partial_{x^i} . \qquad (*)$$

In particular, linear = 1-homogeneous

We can extend this idea to manifolds by considering a vector field X that is locally of the form $(*)$ in some coords.

Def.: A vector field ∇ on a manifold M is called a *weight vector field* if in a neighbourhood of every point of M there are local coordinates (x^a) such that

$$\nabla = \sum_{a=1}^{n} w_a \, x^a \, \partial_{x^a} ,$$

where $w_a =: \deg(x^a) \in \mathbb{R}$ is called the weight of x^a.

Such coordinates are called *homogeneous coordinates*.

The pair (M, ∇) is called a *homogeneity manifold*.

<u>Def.:</u> Let (M, ∇) be a homogeneity manifold.

A tensor field A on M is called w-homogeneous $(w \in \mathbb{R})$ if

$$\mathcal{L}_{\nabla} A = w \cdot A .$$

Examples of homogeneity manifolds

* A vector bundle $\pi : E \longrightarrow M$ and the Euler vector field ∇_E (the generator of homotheties on the fibers).

In bundle coords., $\pi : (x^i, y^a) \longmapsto (x^i)$,

$$\nabla_E = \sum_a y^a \partial_{y^a} .$$

* The second-order tangent bundle

$$\tau : T^2M \ni (x^i, \dot{x}^i, \ddot{x}^i) \longmapsto (x^i) \in M$$

with $\deg(x^i) = 0$, $\deg(\dot{x}^i) = 1$, $\deg(\ddot{x}^i) = 2$.

* An exact symplectic manifold $(M, \omega = d\theta)$ with a Liouville vector field ∇, i.e.

$$\mathcal{L}_{\nabla} \omega = \omega.$$

* Weight vector fields with non-integer weights appear in BH thermodynamics

↳ F. Belgiorno, "Quasi-homogeneous thermodynamics and black holes", J. Math. Phys. 44, 1089 (2003)

Let (M, ∇) be a homogeneity mfold.

There are two different situations on an open subset $U \subseteq M$

* $\nabla\big|_U \neq 0$,

* $\exists\ x_0 \in U$ s.t. $\nabla(x_0) = 0$.

Remark: Any nowhere-vanishing vector field $X \in \mathfrak{X}(M)$ is a weight vector field. However, its weights are not canonical.

Indeed, since X is nowhere zero, $\exists$ local coords. (x^a) such that $X = \partial_{x^1}$. For any $\{w_1, \dots, w_n\} \subset \mathbb{R}$ with $w_1 \neq 0$, we can def. a new system of coords.

$$y^1 = e^{w_1 x^1}, \qquad y^i = e^{w_i x^1} x^i, \quad 2 \leq i \leq n$$

so that

$$X = \sum_{a=1}^{n} w_a y^a \partial_{y^a}, \quad \text{i.e.} \qquad \deg(y^a) = w_a.$$

On the other hand, in a neighbourhood of any point at which a weight vector field vanishes, its weights are canonical.

Proposition (Grabowska & Grabowski, 2024): $\nabla \in \mathfrak{X}(M)$ is a weight vector field on M iff $T_{x_0} X$ is diagonal $\forall x_0 \in M$ s.t. $\nabla(x_0) = 0$.

Let (x^a) be a system of homog. coords. around x_0, i.e.

$$\nabla = \sum_a w_a x^a \partial_{x^a}, \quad \text{with} \quad \Gamma := \{w_1, \dots, w_n\} \subset \mathbb{R}.$$

Then, any other system of homog. coords. around x_0 has weights in Γ.

Homogeneous Poincaré Lemma (Grabowska & Grabowski, '24):

Let (M, ∇) be a homogeneity mfold. Let $\omega \in \Omega^k(M)$ be a λ-homogeneous k-form $(k>0)$. In a nbh. of $x_0 \in M$, $\exists$ λ-homog. $(k-1)$-form α s.t. $d\alpha = \omega$ if one of the following conditions holds:

i) $\nabla(x_0) = 0$,

ii) $\nabla(x_0) \neq 0$ and $k>1$,

iii) $\nabla(x_0) \neq 0$, $k=1$ and $\omega \neq 0$.

In the cases i) and ii), it is possible to additionally choose an α s.t. $\alpha(x_0) = 0$.

Darboux theorem for homogeneous symplectic forms (GJ & G '24)

Let (M, ∇) be a homogeneity mfold., and let ω be a λ-homog. symplectic form on M. Then, around every $x_0 \in M$ s.t. $\nabla(x_0)=0$, there is a system of homog. coords. (q^i, p_i) such that

$$\omega = \sum_i dp_i \wedge dq^i, \qquad \nabla = \sum_i \left(w_{q^i}\, q^i \partial_{q^i} + w_{p_i}\, p_i \partial_{p_i} \right).$$

Idea of the proof:

1) (Graded) linear algebra $\rightsquigarrow$ $\exists$ graded basis (e_a) of $T_{x_0}M$ s.t. $\omega(x_0) = \sum_i e^*_{i+n} \wedge e^*_i$.

2) Choose (homogeneous) coords. $(\bar{q}^i, \bar{p}_i)$ s.t.
$d\bar{q}^i(x_0) = e^*_i$, $d\bar{p}_i(x_0) = e^*_{i+n}$.

3) Def. $\omega_0 := d\bar{p}_i \wedge d\bar{q}^i$, so that $\omega_0(x_0) = \omega(x_0)$,
and $\omega_t = (1-t)\,\omega_0 + t\omega_1$ $t \in [0,1]$, so that $\omega = \omega_1$

4) Moser's trick: obtain a smooth isotopy Φ_t s.t.
$\Phi_t^* \omega_t = \omega_0$ and $(\Phi_t)_* \nabla = \nabla$. $\rightsquigarrow$

$$\omega = \omega_1 = (\Phi_1^{-1})^* \omega_0 = \sum_i d\underbrace{(\bar{p}_i \circ \Phi_t^{-1})}_{p_i} \wedge d\underbrace{(\bar{q}^i \circ \Phi_t^{-1})}_{q^i}.$$

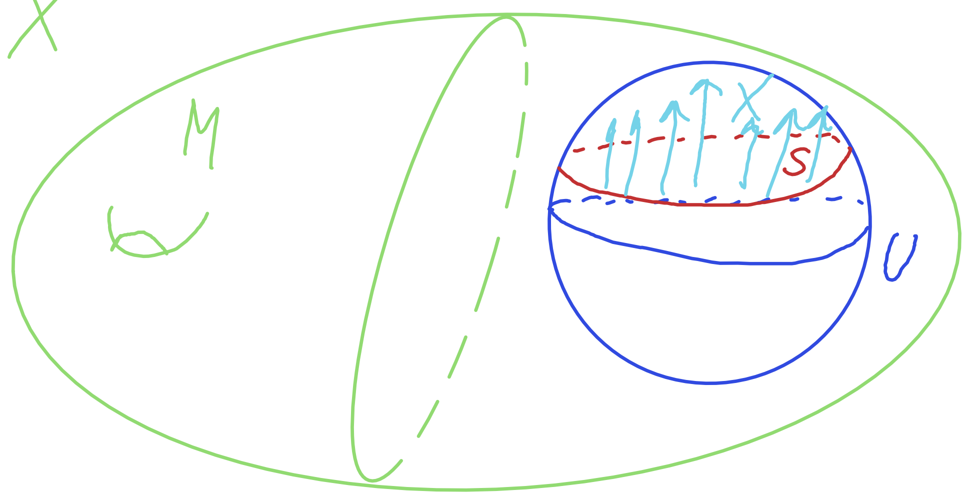
Homogeneous straightening lemma (Grabowski & LG):

Let (M, ∇) be a homogeneity mfold., and let $X \in \mathfrak{X}(M)$ be a $(-\lambda)$-homogeneous vector field. Assume that $\nabla(x_0) = 0$ and $X(x_0) \neq 0$ at $x_0 \in M$. Then, in a neighbourhood of x_0, there is a chart of homog. coords. $(U; z, y^i)$ such that

$$X = \partial_z, \qquad \nabla = \lambda z \partial_z + \sum_i w_i y^i \partial_{y^i}.$$

Sketch of the proof: Let $(U; x^a)$ be a chart of homog. coords. around x_0, i.e., $\nabla = \sum_a w_a x^a \partial_{x^a}$. Since $X(x_0) \neq 0$, not all $X(x^a)$ can vanish. W.l.o.g., assume that $X(x^1) \neq 0$ on U.

The hypersurface $S = \{x^1 = 0\} \subset U$ is a homogeneous submanifold (i.e., $\nabla|_S$ is tangent to S) and it is transverse to X



(S, ∇_S) is a homog. mfold. $\rightsquigarrow$ $\exists$ coords. (y^i) s.t.

$\nabla = \sum_i w_i y^i \partial_{y^i}$.

As in the proof of the standard straightening lemma, these coords. induce coords (z, y^i) in a neighbourhood of x_0 in M s.t. $X = \partial_z$.

These coords. are homog. Indeed,

$$[X, \nabla] = \lambda X \Rightarrow TF^X_{-t} \circ \nabla \circ F^X_t = \nabla + \lambda t$$

In particular,

$$\nabla(z, y^i) = \nabla(0, y^i) + \lambda z X(0, y^i) = \nabla_S(y^i) + \lambda z X(0, y^i)$$

$$= \sum_i w_i y^i \partial_{y^i} + \lambda z \partial_z$$

$\square$

Def.: A contact distribution is a corank-one distribution $C \subset TM$ which is maximally non-integrable, that is, the skew-symmetric bilinear map

$$\nu : C \times_M C \longrightarrow TM/C, \quad \nu(X,Y) = \gamma([X,Y]),$$

with $\gamma: TM \to TM/C$ the natural projection is non-integrable.

Locally, $C = \ker \eta$, where η is a (local) one-form such that $\eta \wedge (d\eta)^n$ is nowhere zero $(\dim M = 2n+1)$.

Def.: A (global) one-form η on a mfold. M^{2n+1} s.t. $\eta \wedge (d\eta)^n$ is a volume form is called a contact form.

The Reeb vector field $R \in \mathfrak{X}(M)$ is uniquely determined by

$$R \in \ker d\eta \quad \& \quad \eta(R) = 1.$$

Remark: A contact form is never unique. Indeed, $\ker \eta = \ker(f\eta)$ $\forall$ nowhere-vanishing $f \in C^\infty(M)$.

Darboux theorem for homogeneous contact forms (Grabowski, LG)

Let (M, ∇) be a homogeneity mfold., and let η be a λ-homog. contact form on M. Then, in a neighbourhood of each point $x_0 \in M$ s.t. $\nabla(x_0) = 0$, there exists a system of homog. coords. (q^i, p_i, z) s.t.

$$\eta = dz + \sum_i p_i \, dq^i,$$

$$\nabla = \sum_i \left(w_{q^i} q^i \partial_{q^i} + w_{p_i} p_i \partial_{p_i} \right) + \lambda z \partial_z.$$

Sketch of the proof:

1) The Reeb V.f. R is nonwhere-vanishing and (-1)-homogeneous. Hence, $\exists$ coords. $(\bar{z}, y^a)$ around x_0 s.t. $R = \partial_{\bar{z}}$.

Then, $L_{\partial_{\bar{z}}} \eta = 1$ and $L_{\partial_{\bar{z}}} d\eta = 0 \Longrightarrow$

$$\eta = d\bar{z} + \sum_a A_a(y)\, dy^a.$$

2) Consider the hypersurface $S = \{\bar{z} = 0\}$. It is a homogeneous submanifold (i.e. $\nabla_S = \nabla|_S$ is tangent to S) and $\omega = d\eta|_S$ is a λ-homog. symplectic form. By the Darboux theorem for homog. symp. forms, $\exists$ coords. (q^i, p_i) around $x_0 \in S$ s.t. $\omega = \sum_i dp_i \wedge dq^i$.

3) Note that $d\eta$ does not depend on $\bar{z}$. Thus, locally,

$$d\eta = \sum_i dp_i \wedge dq^i .$$

Therefore, $\alpha := \sum_a A_a dy^a - \sum_i p_i dq^i$ is a closed λ-homog. one-form.

4) By the Homog. Poincaré lemma, $\alpha = df$ with f a λ-homog. function s.t. $f(x_0) = 0$.

Finally,

$$\eta = d\bar{z} + \sum_a A_a dy^a = d\bar{z} + \sum_i p_i dq^i + df = dz + \sum_i p_i dq^i,$$

$z = \bar{z} + f$.

Def.: Let (M, ∇) be a homog. mfold. A (co)distribution $D \subset TM$ (resp. $D \subset T^*M$) is called homogeneous if the (co)tangent lift $d_T \nabla$ (resp. $d_{T^*}\nabla$) is tangent to D.

Conjecture: A homogeneous (co)distribution is locally generated by homogeneous vector fields (resp. one-forms).

We know this is true if ∇ is $\mathbb{N}$-graded and complete.

* "Conjecture" is my pretentious way of saying "work in progress".

Note D is endowed with a double homogeneity structure

$$\nabla \text{ and } \nabla_{TM}\Big|_D, \qquad [\nabla, \nabla_{TM}] = 0 \text{ (compatible)}$$

with $\nabla_{TM} = \sum_i v^i \partial_{v^i}$ the Euler vector field of TM.

If ∇ is $\mathbb{N}$-graded and complete, then D can be covered by an atlas of bi-homogeneous coords. (i.e., fibered coords. w.r.t. $D \longrightarrow M$ and homog. w.r.t. ∇)
[Grabowski & Rotkiewicz, 2011]

In the associated local trivialisation, these coords. provide homog. vector fields (one-forms) generating D.

Homogeneous Frobenius theorem (Grabowski & LG):

Let (M,∇) be a homog. mfold, and let D be an involutive distribution of rank k which is locally generated by homog. vector fields. Around every $x_0 \in M$ s.t. $\nabla(x_0)=0$ $\exists$ homog. chart $(U; x^1, \dots, x^n)$ such that

$$D\big|_U = \langle \partial_{x^1}, \dots, \partial_{x^k} \rangle$$

and the slices

$$N = \{ x^{k+1} = \text{const.}, \dots, x^n = \text{const} \} \subset U$$

are integral submanifolds.

Def.: A presymplectic form ω on M is a closed 2-form of constant rank. Its characteristic distribution is given by

$$C_\omega = \ker \omega.$$

Theorem (Darboux): Around every point of M, there are local coords. (q^i, p_i, z^a) s.t.

$$\omega = \sum_i dp_i \wedge dq^i \qquad (*)$$

Problem: If (M, ∇) is a homog. mfold. and ω is homog., can we find homog. coords. (q^i, p_i, z^c) in which ω has the form $(*)$?

If our conjecture is true, the answer is YES.

Def.: A one-form ω on a mfold. M^m is said to have

* odd class $2s+1 \leq m$ at $x \in M$ if

$$\omega \wedge (d\omega)^s(x) \neq 0 \quad \& \quad (d\omega)^{s+1}(x) = 0.$$

* even class $2s+2 \leq m$ at x if

$$\omega \wedge (d\omega)^s(x) \neq 0 \quad \& \quad (d\omega)^{s+1}(x) \neq 0 \quad \& \quad \omega \wedge (d\omega)^{s+1}(x) = 0.$$

Theorem (Darboux): In a sufficiently small neighbourhood of x where ω has constant class, there are coords. (q^i, p_i, z^a) s.t.

$$\omega = dz^0 + \sum_{i=1}^{s} p_i\, dq^i \ \text{(odd)} \quad (**) \qquad \Big| \qquad \omega = \sum_{i=1}^{s+1} p_i\, dq^i \ \text{(even)} \quad (***)$$

Problem: If (M, ∇) is a homog. mfold. and ω is homog., can we find homog. coords. (q^i, p_i, z^a) in which ω has the form (**) or (***)?

Future work

* Extending our results to supermanifolds.

* Bi-homogeneity: ∇_1, ∇_2 s.t. $[\nabla_1, \nabla_2] = 0$.

* Homogeneous multisymplectic forms

* Applications to Pfaffian systems / exterior differential systems

  ↳ studying differential eqs. as ideals generated by differential forms

References

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Thank you for your attention!

Feel free to contact me at alopez-gordon@impan.pl

These slides are available at alopezgordon.xyz