On the integrability of hybrid Hamiltonian systems

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Symplectic geometry

- Symplectic geometry is the natural framework for classical mechanics.
- Recall that a symplectic form ω on M is a 2-form such that $\mathrm{d}\omega = 0$ and $\mathrm{T}_x M \ni v \mapsto \omega_x(v, \cdot) \in \mathrm{T}_x^* M$ is an isomorphism of vector spaces.
- Given a function f on M, its its Hamiltonian vector field X_f is given by

$$\omega(X_f, \cdot) = \mathrm{d}f.$$

- The Poisson bracket $\{\cdot, \cdot\}$ is given by

$$\{f, g\} := \omega(X_f, X_g) = X_g(f) = -X_f(g)\,.$$

Theorem (Liouville–Arnold theorem)

Let $f_1, \ldots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0\ \forall i, j$) on a symplectic manifold (M^{2n}, ω). Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$.

1. *Any compact connected component of M_Λ is diffeomorphic to $\mathbb{T}^n$.*
2. *On a neighborhood of M_Λ there are coordinates (φ^i, J_i) such that*

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i,$$

and the Hamiltonian dynamics are given by

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \Omega^i(J_1, \ldots, J_n),$$
$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0.$$

Hybrid systems

Definition

A **hybrid system** is a 4-tuple $\mathscr{H} = (M, X, S, \Delta)$, formed by

1. a manifold M,
2. a vector field $X \in \mathfrak{X}(M)$,
3. a submanifold $S \subset M$ of codimension 1 or greater,
4. an embedding $\Delta \colon S \to M$.

The dynamics generated by $\mathscr{H}$ are the curves $c \colon I \subseteq \mathbb{R} \to M$ such that

$$\begin{aligned} \dot{c}(t) &= X(c(t))\,, && \text{if } c(t) \notin S\,, \\ c^+(t) &= \Delta(c^-(t))\,, && \text{if } c(t) \in S\,, \end{aligned}$$

where

$$c^\pm(t) = \lim_{\tau \to t^\pm} c(\tau)\,.$$

Hybrid Hamiltonian systems

Definition

A hybrid dynamical system (M, X, S, Δ) is said to be a **hybrid Hamiltonian system** and denoted by $\mathscr{H}_h$ if

1. $M \subseteq \mathrm{T}^*Q$ is a zero-codimensional submanifold of the cotangent bundle $\pi_Q \colon \mathrm{T}^*Q \to Q$ of a manifold Q,
2. S projects onto a codimension-one submanifold $\pi_Q(S)$ of Q,
3. $\pi_Q \circ \Delta = \pi_Q$,
4. $X = X_h$ is the Hamiltonian vector field of $h \in C^\infty(\mathrm{T}^*Q)$ w.r.t. the canonical symplectic form ω_Q, namely,

$$\omega_Q(X_h) = \mathrm{d}h\,.$$

Hybrid Hamiltonian systems

Physically,

- Q represents the space of positions,
- T^*Q the phase space,
- X_h the dynamics between the impacts,
- $\pi_Q(S)$ the hypersurface where impacts occur, and
- Δ the change of momenta on the impacts.

Hybrid Lie group action

Definition

A Lie group action $\Phi\colon G \times Q \to Q$ is called a **hybrid action for $\mathscr{H}_h$** if its cotangent lift $\Phi^{\mathrm{T}^*}\colon G \times \mathrm{T}^*Q \to \mathrm{T}^*Q$ satisfies the following conditions:

1. h is Φ^{T^*}-invariant, namely, $h \circ \Phi_g^{\mathrm{T}^*} = h$ for all $g \in G$,
2. the restriction $\Phi^{\mathrm{T}^*}\big|_{G\times S}$ is a Lie group action of G on S,
3. the impact map is equivariant w.r.t. this action, i.e.,

$$\Delta \circ \Phi_g^{\mathrm{T}^*}\Big|_S = \Phi_g^{\mathrm{T}^*} \circ \Delta\,, \quad \forall\, g \in G\,.$$

Hybrid momentum map

Definition

Let $\Phi\colon G \times Q \to Q$ be a hybrid action for $\mathscr{H}_h$. A momentum map $\mathbf{J}\colon \mathrm{T}^*Q \to \mathfrak{g}^*$ for the cotangent lift action Φ^{T^*} is called a **generalized hybrid momentum map** if, for each connected component $C \subseteq S$ and for each regular value μ_- of $\mathbf{J}$, there is another regular value μ_+ such that

$$\Delta(\mathbf{J}|_C{}^{-1}(\mu_-)) \subset \mathbf{J}^{-1}(\mu_+)\,.$$

In particular, if $\mu_- = \mu_+$ it is called a **hybrid momentum map**.
A **hybrid regular value** of $\mathbf{J}$ is a regular value of both $\mathbf{J}$ and $\mathbf{J}|_S$.

Hybrid momentum map

In other words, $\mathbf{J}$ is a generalized hybrid momentum map if, for every point in the connected component C of the switching surface S such that the momentum before the impact takes a value of μ_-, the momentum will take a value μ_+ after the impact; and it is a hybrid momentum map if its value does not change with the impacts.

Hybrid reduction

Proposition

If μ_- and μ_+ are regular values of $\mathbf{J}$ such that $\Delta\left(\mathbf{J}|_S{}^{-1}(\mu_-)\right) \subset \mathbf{J}^{-1}(\mu_+)$, then the isotropy subgroups in μ_- and μ_+ coincide, that is, $G_{\mu_-} = G_{\mu_+}$.

Hybrid reduction

Theorem (Colombo, de León, Eyrea Irazú, L. G., 2022)

Let $\Phi\colon G \times Q \to Q$ be a hybrid action on $\mathscr{H}_h$. Assume that G is connected and that $\Phi^{\mathrm{T}^*}\colon G \times \mathrm{T}^*Q \to \mathrm{T}^*Q$ is free and proper. Consider a sequence $\{\mu_i\}_{i\in I\subseteq\mathbb{N}}$ of hybrid regular values of $\mathbf{J}$, such that $\Delta\left(\mathbf{J}|_S{}^{-1}(\mu_i)\right) \subset \mathbf{J}^{-1}(\mu_{i+1})$. Let $G_{\mu_i} = G_{\mu_0}$ be the isotropy subgroup in μ_i under the co-adjoint action. Then, the reduction leads to a sequence of reduced hybrid forced Hamiltonian systems

$$\mathscr{H}_h^{\mu_i} = \left(\mathbf{J}^{-1}(\mu_i)/G_{\mu_0},\, X_{h_{\mu_i}}, \mathbf{J}|_S{}^{-1}(\mu_i)/G_{\mu_0}, (\Delta)_{\mu_i}\right).$$

Hybrid reduction

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbf{J}^{-1}(\mu_i) \arrow[d] & \mathbf{J}|_S{}^{-1}(\mu_i) \arrow[l, hook'] \arrow[r, "\Delta|_{\mathbf{J}^{-1}(\mu_i)}"] \arrow[d] & \mathbf{J}^{-1}(\mu_{i+1}) \arrow[d] & \cdots \arrow[l, hook'] \\
\cdots \arrow[r] & \frac{\mathbf{J}^{-1}(\mu_i)}{G_{\mu_0}} & \mathbf{J}|_S{}^{-1}(\mu_i)/G_{\mu_0} \arrow[l, hook'] \arrow[r, "(\Delta)_{\mu_i}"] & \frac{\mathbf{J}^{-1}(\mu_{i+1})}{G_{\mu_0}} & \cdots \arrow[l, hook']
\end{tikzcd}$$

Integrable hybrid Hamiltonian systems

- A particular case is when we have the Abelian Lie group action $\Phi \colon \mathbb{R}^n \times \mathrm{T}^*Q \to \mathrm{T}^*Q$ generated by the Hamiltonian flows of n functions $f_1, \ldots, f_n$ in involution.
- In that case, we can identify the momentum map with $F = (f_1, \ldots, f_n) \colon \mathrm{T}^*Q \to \mathbb{R}^n$.
- We may obtain action-angle coordinates for each time interval between impacts. The action-angle coordinates before and after the impact will be related by Δ.

Definition

Let (M, S, X, Δ) be a hybrid dynamical system. A function $f : M \to \mathbb{R}$ is called a **generalized hybrid constant of the motion** if

1. $Xf = 0$,
2. For each connected component $C \subseteq S$ and each $a \in \operatorname{Im} f$, there exists a $b \in \operatorname{Im} f$ such that

$$\Delta\left(f|_C{}^{-1}(a) \right) \subseteq f^{-1}(b) .$$

In particular, f is called a **hybrid constant of the motion** if, in addition, $b = a$ for each $a \in \operatorname{Im} f$.

Definition

Let Q be an n-dimensional manifold. A **completely integrable hybrid Hamiltonian system** is a 5-tuple $(\mathrm{T}^*Q, S, X_H, \Delta, F)$, formed by a hybrid Hamiltonian system $(\mathrm{T}^*Q, S, X_H, \Delta)$, together with a function $F = (f_1, \dots, f_n) \colon \mathrm{T}^*Q \to \mathbb{R}^n$ such that:

1. $\operatorname{rank} \mathrm{T}_x F = n \quad$ a.e.,
2. the functions $f_1, \dots, f_n$ are generalized hybrid constant of the motion
3. $\{f_i, f_j\} = X_{f_j}(f_i) = 0 \quad \forall\, i, j \in \{1, \dots, n\}$.

Theorem (L. G., Colombo, 2024)

Consider a completely integrable hybrid Hamiltonian system $(\mathrm{T}^*Q, S, X_H, \Delta)$, *with* $F = (f_1, \ldots, f_n)$, *where* $n = \dim Q$. *Let* M_Λ *be a regular level set of* F. *Then:*

1. *For each regular level set* M_Λ *and each connected component* $C \subseteq S$, *there exists a* $\Lambda' \in \mathbb{R}^n$ *such that* $\Delta(M_\Lambda \cap C) \subset M_{\Lambda'} = F^{-1}(\Lambda')$.
2. *On a neighbourhood* U_λ *of* M_Λ *there are coordinates* (φ^i, s_i) *s.t.*
    1. $\omega_Q = \mathrm{d}\varphi^i \wedge \mathrm{d}s_i$,
    2. *the action coordinates* s_i *are functions depending only on the integrals* $f_1, \ldots, f_n$,
    3. *the continuous part hybrid dynamics are given by*

$$\dot{\varphi}^i = \Omega^i(s_1, \ldots, s_n), \qquad \dot{s}_i = 0\,.$$

    4. *In these coordinates, for each connected component* $C \subseteq S$, *the impact map reads* $\Delta\colon (\varphi^i_-, s_i^-) \in M_\Lambda \cap C \mapsto (\varphi^i_+, s_i^+) \in M_{\Lambda'}$, *where* $s_1^+, \ldots, s_n^+$ *are functions depending only on* $s_1^-, \ldots, s_n^-$.

Rolling disk with a harmonic potential hitting fixed walls

- Consider a homogeneous circular disk of radius R and mass m moving in the plane.
- The configuration space is $Q = \mathbb{R}^2 \times \mathbb{S}^1$, with canonical coordinates (x, y, θ).
- The coordinates (x, y) represent then position of the center of the disk, while the coordinate θ represents the angle between a fixed reference point of the disk and the y-axis.

Rolling disk with a harmonic potential hitting fixed walls

- The Hamiltonian function $H\colon \mathrm{T}^*Q \to \mathbb{R}$ of the system is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2mk^2}p_\theta^2 + \frac{1}{2}\Omega^2(x^2 + y^2)\,,$$

where $(x, y, \theta, p_x, p_y, p_\theta)$ are the bundle coordinates in $\mathrm{T}^*(\mathbb{R}^2 \times \mathbb{S}^1)$.

Rolling disk with a harmonic potential hitting fixed walls

- Suppose that there are two rough walls situated at $y = 0$ and at $y = h > R$.
- Assume that the impact with a wall is such that the disk rolls without sliding and that the change of the velocity along the y-direction is characterized by an elastic constant e

Rolling disk with a harmonic potential hitting fixed walls

- Then, the switching surface is $S = C_1 \cup C_2$, where

$$C_1 = \left\{ \left(x, R, \theta, p_x, p_y, \frac{k^2}{R} p_x \right) \mid x, p_x, p_y \in \mathbb{R},\ \theta \in \mathbb{S}^1 \right\},$$

$$C_2 = \left\{ \left(x, h - R, \theta, p_x, p_y, \frac{k^2}{R} p_x \right) \mid x, p_x, p_y \in \mathbb{R},\ \theta \in \mathbb{S}^1 \right\},$$

and the impact map $\Delta \colon S \to \mathrm{T}^* Q$ is given by

$$\left(p_x^-, p_y^-, p_\theta^- \right) \mapsto \left(\frac{R^2 p_x^- + k^2 R p_\theta^-}{k^2 + R^2}, -e p_y^-, \frac{R p_x^- + k^2 p_\theta^-}{k^2 + R^2} \right)$$

Rolling disk with a harmonic potential hitting fixed walls

- For simplicity's sake, let us hereafter take $m = R = k = \Omega = 1$.
- The functions

$$f_1 = \frac{p_x^2 + x^2}{2}, \quad f_2 = \frac{p_y^2 + y^2}{2}, \quad f_3 = \frac{p_\theta^2}{2},$$

  are conserved quantities with respect to the Hamiltonian dynamics of H.
- Moreover, $\{f_i, f_j\} = 0$ and $\mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}f_3 \neq 0$ a.e.
- Let $F = (f_1, f_2, f_3) \colon \mathrm{T}^*(\mathbb{R}^2 \times \mathbb{S}) \to \mathbb{R}^3$.
- It is clear that, for $\Lambda \neq 0$, the level sets $F^{-1}(\Lambda)$ are diffeomorphic to $\mathbb{S} \times \mathbb{S} \times \mathbb{R}$.

Rolling disk with a harmonic potential hitting fixed walls

- In the intersection of their domains of definition, the functions

$$\phi^1 = \arctan\left(\frac{x}{p_x}\right), \quad \phi^2 = \arctan\left(\frac{y}{p_y}\right), \quad \phi^3 = \frac{\theta}{p_\theta}$$

  are coordinates on each level set $F^{-1}(\Lambda)$ for $\Lambda \neq 0$.
- Additionally, $\omega_Q = \mathrm{d}\phi^i \wedge \mathrm{d}f_i$.
- In these coordinates, the Hamiltonian function reads

$$H = f_1 + f_2 + f_3\,.$$

- Hence, its Hamiltonian vector field is simply

$$X_H = \frac{\partial}{\partial \phi^1} + \frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial \phi^3}\,.$$

Rolling disk with a harmonic potential hitting fixed walls

- In the action-angle coordinates (ϕ^i, f_i), the impact surface reads

$$S = \left\{ \left(\phi^i, f_i\right) \mid 2 f_2 \sin^2 \phi^2 = R^2 \text{ and } f_3 = \frac{2k^4 f_1 \cos^2 \phi^1}{R^2} \right\}$$
$$\cup \left\{ 2 f_2 \sin^2 \phi^2 = (h - R)^2 \text{ and } f_3 = \frac{2k^4 f_1 \cos^2 \phi^1}{R^2} \right\} .$$

Rolling disk with a harmonic potential hitting fixed walls

- The relations between the coordinates before, (ϕ_-^i, f_i^-), and after, (ϕ_+^i, f_i^+), are

$$\phi_+^1 = \phi_-^1\,, \qquad \phi_+^2 = -\arctan\left(\frac{\tan\phi_-^2}{e}\right)\,, \qquad \phi_+^3 = \phi_-^3\,,$$

$$f_1^+ = f_1^-\,, \qquad f_2^+ = e^2 f_2 + \tfrac{1-e^2}{2} a^2\,, \qquad f_3^+ = f_3^-\,,$$

where $a = R$ or $a = h - R$ depending on the wall where the impact takes place.

Merci pour votre attention!

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