# Reduction of hybrid Hamiltonian systems with non-equivariant momentum maps

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# Hybrid systems

- A hybrid dynamical system is one which combines continuous and discrete transitions.
- The dynamics of such systems are continuous “most of the time”, except at some instants at which abrupt changes occur.
- This framework may be used to model mechanical systems with impacts.

# Hybrid systems

## Definition

A **hybrid system** is a 4-tuple $\mathscr{H} = (D, X, S, \Delta)$, formed by

1. a manifold $D$,
2. a vector field $X \in \mathfrak{X}(D)$,
3. a submanifold $S \subset D$ of codimension 1 or greater,
4. an embedding $\Delta \colon S \to D$.

The dynamics generated by $\mathscr{H}$ are the curves $c \colon I \subseteq \mathbb{R} \to D$ such that

$$\begin{aligned} \dot{c}(t) &= X(c(t))\,, && \text{if } c(t) \notin S\,, \\ c^+(t) &= \Delta(c^-(t))\,, && \text{if } c(t) \in S\,, \end{aligned}$$

where

$$c^\pm(t) = \lim_{\tau \to t^\pm} c(\tau)\,.$$

**Definition**

A simple hybrid system $\mathcal{H} = (D, X, S, \Delta)$ is said to be a **simple hybrid Hamiltonian system** if $X = X_H$ is the Hamiltonian vector field associated with a Hamiltonian system $(D, \omega, H)$.

# Lie group actions

- Consider a finite-dimensional Lie group $G$.
- Let $\mathfrak{g}$ be the Lie algebra of $G$, with dual $\mathfrak{g}^*$.
- Given a (left) Lie group action $\Phi \colon G \times D \to D$, for each $\xi \in \mathfrak{g}$, its associated **infinitesimal generator** on $D$ is the vector field $\xi_D \in \mathfrak{X}(D)$ given by

$$\xi_D(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(\exp(t\xi), x)\,, \forall\, x \in D\,.$$

# Lie group actions

- The **adjoint action** of $G$ on $\mathfrak{g}$ is given by

$$\mathrm{Ad}_g \xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g \exp(t\xi) g^{-1}, \quad \forall\, g \in G, \quad \forall\, \xi \in \mathfrak{g}.$$

- The **co-adjoint action** $\mathrm{Ad}^*_{g^{-1}} \mu$ of $g \in G$ on $\mu \in \mathfrak{g}^*$ is determined by

$$\langle \mathrm{Ad}^*_{g^{-1}} \mu, \xi \rangle = \langle \mu, \mathrm{Ad}_{g^{-1}} \xi \rangle, \quad \forall\, \xi \in \mathfrak{g}.$$

- A (left) Lie group action $\Phi\colon G \times D \to D$ on a symplectic manifold $(D, \omega)$ is called symplectic if

$$\Phi_g^* \omega = \omega \,, \quad \forall\, g \in G \,.$$

- If $\Phi$ is a symplectic action on $(D, \omega)$, a **momentum map** is a map $J\colon D \to \mathfrak{g}^*$ satisfying

$$\omega(\xi_D, \cdot) = \mathrm{d}\big(\langle J, \xi \rangle\big) \,, \forall\, \xi \in \mathfrak{g} \,.$$

## Definition

Let $\mathcal{H} = (D, X, S, \Delta)$ be a simple hybrid system. A Lie group action $\Phi : G \times D \to D$ of $G$ on $D$, is called a **hybrid action** if

1. $\Phi|_{G \times S}$ is a Lie group action of $G$ on $S$,
2. the impact map is equivariant with respect to this action, namely,

$$\Delta \circ \Phi_g\big|_S = \Phi_g \circ \Delta\,, \quad \forall g \in G\,.$$

## Definition

Suppose that $\mathcal{H} = (D, X_H, S, \Delta)$ is a hybrid Hamiltonian system with associated Hamiltonian system $(D, \omega, H)$, and assume that the action $\Phi$ is hybrid and symplectic. A momentum map $J$ is called a **generalized hybrid momentum map** for $\mathcal{H}$ if, for each regular value $\mu_-$ of $J$, and each connected component $C$ of $S$,

$$\Delta \left( J|_C{}^{-1}(\mu_-) \right) \subset J^{-1}(\mu_+),$$

for some regular value $\mu_+$.

The tuple $(D, X_H, S, \Delta, \omega, \Phi, J)$ will be called a **hybrid Hamiltonian $G$-space**. We will call $\mu \in \mathfrak{g}^*$ a **hybrid regular value** if it is a regular value of both $J$ and $J|_S$.

- Let $(D, X_H, S, \Delta, \omega, \Phi, J)$ be a hybrid Hamiltonian $G$-space.
- The **co-adjoint cocycle** associated with $J$ is the map $\sigma : G \to \mathfrak{g}^*$ determined by

$$\langle \sigma(g), \xi \rangle = J_\xi(\Phi_g(x)) - \mathrm{Ad}^*_{g^{-1}}(J_\xi(x)) , \quad \forall\, \xi \in \mathfrak{g} , \quad \forall\, x \in D .$$

- The **affine action** of $G$ on $\mathfrak{g}^*$ is given by

$$\Psi : (G, \mathfrak{g}^*) \ni (g, \mu) \mapsto \mathrm{Ad}^*_{g^{-1}} \mu + \sigma(g) \in \mathfrak{g}^* .$$

- By construction, the momentum map is equivalent w.r.t. this action, i.e.,

$$\Psi_g \circ J = J \circ \Phi_g\,, \quad \forall g \in G\,.$$

- Let $\tilde{G}_\mu$ denote the **isotropy subgroup** of $\mu \in \mathfrak{g}^*$ under the action $\Psi$, given by

$$\tilde{G}_\mu = \{g \in G : \Psi(g, \mu) = \mathrm{Ad}^*_{g^{-1}}\, \mu + \sigma(g) = \mu\}\,.$$

## Proposition

*Let $(D, X_H, S, \Delta, \omega, \Phi, J)$ be a hybrid Hamiltonian G-space. Assume that G is connected. If $\Delta$ is equivariant with respect to $\Phi$, and $\mu_-$, $\mu_+$ are regular values of J such that*

$$\Delta\left(J|_S{}^{-1}(\mu_-)\right) \subset J^{-1}(\mu_+)\,,$$

*then the isotropy subgroups at $\mu_-$ and at $\mu_+$ under the action $\Psi$ coincide, i.e., $\tilde{G}_{\mu_-} = \tilde{G}_{\mu_+}$.*

## Theorem

*Let $(D, X_H, S, \Delta, \omega, \Phi, J)$ be a hybrid Hamiltonian G-space. Assume that G is connected, and consider a discrete sequence $\Lambda = \{\mu_i\}$ of regular values of J such that $\Delta\left(J|_S{}^{-1}(\mu_i)\right) \subset J^{-1}(\mu_{i+1})$. Let $\tilde{G}_{\mu_i} = \tilde{G}_{\mu_0}$ be the isotropy subgroup in $\mu_i$ (for any $\mu_i$ in the sequence) under the affine action. Assume that $\Phi$ and $\Phi|_{\tilde{G}_\mu \times J^{-1}(\mu)}$ are free and proper actions. Then, for any $\mu_i \in \Lambda$, we have the reduced hybrid system*

$$(D_{\mu_i}, X_{H_{\mu_i}}, S_{\mu_i}, \Delta_{\mu_i}), \quad D_{\mu_i} := J^{-1}(\mu_i)/\tilde{G}_{\mu_i} .$$

The reduction scheme is summarized in the following commutative diagram:

$$
\begin{tikzcd}
\cdots \arrow[r] & J^{-1}(\mu_i) \arrow[d] & J|_S{}^{-1}(\mu_i) \arrow[l, hook'] \arrow[r, "\Delta|_{J^{-1}(\mu_i)}"] \arrow[d] & J^{-1}(\mu_{i+1}) \arrow[d] & \cdots \arrow[l, hook'] \\
\cdots \arrow[r] & \frac{J^{-1}(\mu_i)}{\tilde{G}_{\mu_0}} & S_{\mu_i} \arrow[l, hook'] \arrow[r, "\Delta_{\mu_i}"] & \frac{J^{-1}(\mu_{i+1})}{\tilde{G}_{\mu_0}} & \cdots \arrow[l, hook']
\end{tikzcd}
$$

# A toy example

- Let $Q = \mathbb{R}^2$, and consider $\mathrm{T}^*Q \simeq \mathbb{R}^4$ endowed with the canonical symplectic form $\omega_Q = \mathrm{d}q^i \wedge \mathrm{d}p_i$, where $(q^i, p_i)$ are bundle coordinates induced by the canonical coordinates $(q^i)$ of $Q$.
- Consider the Lie group action $\Phi\colon \mathbb{R}^2 \times \mathrm{T}^*Q \to \mathrm{T}^*Q$ of $G = \mathbb{R}^2$ on $\mathrm{T}^*Q$ given by

$$\Phi_{(a,b)}\left(q^1, q^2, p_1, p_2\right) = \left(q^1 + a, q^2 + a, p_1 + b, p_2 + b\right) .$$

- The associated infinitesimal generators are

$$\xi_1^{\mathrm{T}^*Q} = \partial_{q^1} + \partial_{q^2} , \quad \xi_2^{\mathrm{T}^*Q} = \partial_{p_1} + \partial_{p_2} .$$

- Note that $\Phi$ is a symplectic action.

# A toy example

- A momentum map $J\colon \mathrm{T}^*Q \to \mathfrak{g}^*$ for the action $\Phi$ is given by

$$J\left(q^1, q^2, p_1, p_2\right) = \left(p_1 + p_2, -q^1 - q^2\right) .$$

- Its co-adjoint cocycle is given by $\sigma(a, b) = (2b, -2a)$.
- The Hamiltonian function

$$H\left(q^1, q^2, p_1, p_2\right) = \frac{(p_1 - p_2)^2}{2} + V(q^1 - q^2) ,$$

where $V$ is a potential function depending only on $q^1 - q^2$, is $\Phi$-invariant.

# A toy example

- Consider the hybrid Hamiltonian system $\mathcal{H} = (D, X_H, S, \Delta)$, with $X_H$ the Hamiltonian vector field of $H$, and

$$S = \left\{ \left( q^1, q^2, p_1, p_2 \right) \mid q^1 - q^2 = c \,, \quad p_1 - p_2 < 0 \right\} \,,$$
$$\Delta \left( q^1, q^2, p_1, p_2 \right) = \left( q^1, q^2, p_1 - \frac{1+e}{2}(p_1 - p_2), p_2 + \frac{1+e}{2}(p_1 - p_2) \right) \,,$$

  where $c \in \mathbb{R}$ and $e \in [0, 1]$.
- The action $\Phi$ is a hybrid action for $\mathcal{H}$, and $J$ is a hybrid momentum map, i.e., $J \circ \Delta = J|_S$.
- The isotropy subgroup with respect to the affine action is trivial: $\tilde{G}_\mu = \{0\}$.

# A toy example

- Let $\mu = (\mu_1, \mu_2) \in \mathfrak{g}^*$ be a regular value of $J$, and consider the quotient manifold $D_\mu = J^{-1}(\mu)/\tilde{G}_\mu = J^{-1}(\mu)$, where

$$\begin{aligned} J^{-1}(\mu) &= \left\{(q^1, q^2, p_1, p_2) : J(q^1, q^2, p_1, p_2) = \mu\right\}, \\ &= \left\{(q^1, q^2, p_1, p_2) : (p_1 + p_2, -(q^1 + q^2)) = (\mu_1, \mu_2)\right\}. \end{aligned}$$

- We can use $(q^2\big|_{D_\mu}, p_2|_{D_\mu})$ as coordinates in $D_\mu$. With a slight abuse of notation, we will denote them simply by $(q^2, p_2)$.

# A toy example

- The reduced hybrid system is $\mathcal{H}_\mu = (D_\mu = J^{-1}(\mu), X_{H_\mu}, S_\mu, \Delta_\mu)$, where $X_{H_\mu}$ is the Hamiltonian vector field of

$$H_\mu(q^2, p_2) = \frac{(\mu_1 - 2p_2)^2}{2} + V(-\mu_2 - 2q^2)\,,$$

and

$$S_\mu = \left\{ \left(q^2, p_2, \mu_1, \mu_2\right) \mid -\mu_2 - 2q^2 = c\,, \quad \mu_1 - 2p_2 < 0 \right\}\,,$$

$$\Delta_\mu\left(q^2, p_2, \mu_1, \mu_2\right) = \left(-\mu_2 - q^2, q^2, (\mu_1 - p_2) - \frac{1+e}{2}(\mu_1 - 2p_2), \right.$$
$$\left. p_2 + \frac{1+e}{2}(\mu_1 - 2p_2)\right)\,.$$

# Conclusions and outlook

- We have obtained a reduction *à la* Marsden–Weinstein–Meyer for hybrid Hamiltonian systems.
- Our method does not require the momentum map to be equivariant, nor to be preserved by the impact map.
- We have illustrated the applicability of our theory with an academic example.
- Our result could be useful for developing a reduction by stages for hybrid Hamiltonian systems.

# Main references


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# Merci pour votre attention!

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- These slides are available at www.alopezgordon.xyz