A friendly invitation to geometric mechanics

Basic Notions & Applied Topology Seminar

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A review on differential geometry

A manifold is essentially a topological space which locally looks like $\mathbb{R}^n$.

<u>Def.:</u> An n-dimensional topological manifold is a topological space M such that:

i) M is Hausdorff,

ii) M is second-countable,

iii) $\forall\, m \in M$, $\exists$ a neighbourhood $U \ni x$ and a homeomorphism $\varphi : U \longrightarrow \hat{U} \subseteq \mathbb{R}^n$, with $\hat{U}$ an open subset of $\mathbb{R}^n$.

The pair (U, φ) is called a *chart* on M. The component functions $(x^1, \dots, x^n)$ s.t. $\varphi(m) = (x^1(m), \dots, x^n(m))$ are called (local) *coordinates* on U.

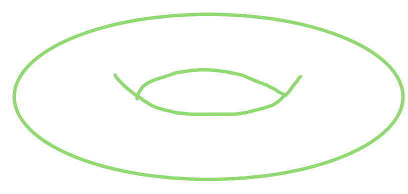
Examples of manifolds:

•) $\mathbb{R}^n$ with the global chart $(\mathbb{R}^n, Id_{\mathbb{R}^n})$.

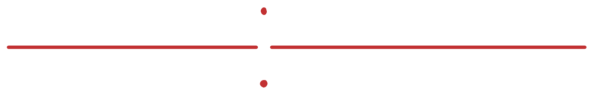
•) Any open subset $U \subseteq \mathbb{R}^n$ with the global chart $(U, Id_{\mathbb{R}^n}|_U)$.

•) The n-sphere $S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1 \}$

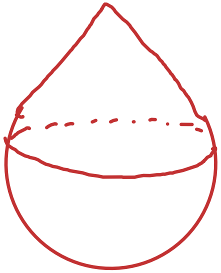
•) The n-torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$



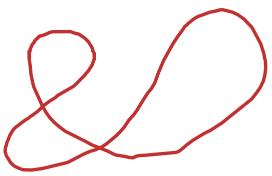
Examples of NOT manifolds

•) The line with two origins 

It is not Hausdorff.

•) The teardrop 

It is NOT locally Euclidean near the singularity.

•) Curves with self-intersections 

Topological properties of manifolds

- Let $M \neq \emptyset$ and $N \neq \emptyset$ be topological manifolds. They are homeomorphic iff $\dim(M) = \dim(N)$.
- Any topological manifold M
  - i) is locally path-connected,
  - ii) is connected iff it is path-connected,
  - iii) has countably many components, and each of them is a topological manifold,
  - iv) is locally compact,
  - v) is paracompact,
  - vi) has a countable fundamental group.

Differentiable Structures

- We would like to extend calculus to more general spaces.

- Using charts we can identify open subsets of a topological manifold with open subsets of $\mathbb{R}^n$.

<u>Def:</u> Let M be a topological manifold. Two charts (U,φ) and (V,ψ) on M are called <u>smoothly compatible</u> if

a) $U \cap V = \emptyset$, or

b) $U \cap V \neq \emptyset$ and

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subseteq \mathbb{R}^n \longrightarrow \psi(U \cap V) \subseteq \mathbb{R}^n$$

is C^∞, bijective, and its inverse is C^∞.

Def.: A smooth atlas is a collection of smoothly compatible charts covering M. A smooth manifold is a topological manifold equipped with a smooth atlas.

Proposition: Two smooth atlases A and $\tilde{A}$ on a topological manifold M define the same structure of smooth manifold on M iff $A \cup \tilde{A}$ is a smooth atlas. In other words, iff all the charts in A are compatible with all the charts on $\tilde{A}$.

Remark: In the rest of the talk

M = manifold = mfold. = smooth manifold.

<u>Def.</u>: Let M and N be manifolds. A map $F: M \to N$ is called <u>smooth</u> if $\forall x \in M$ $\exists$ charts (U, φ) of M with $U \ni x$ and (V, ψ) with $V \ni F(x)$ s.t.

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m \longrightarrow \psi(V) \subseteq \mathbb{R}^n$$

is C^∞. A <u>diffeomorphism</u> is a smooth invertible map whose inverse is also smooth.

The smooth maps $f: M \to \mathbb{R}$ called <u>smooth functions</u>. The set of all smooth functions on M is denoted by $C^\infty(M)$

<u>Remark</u>: All the maps mentioned hereinafter are smooth unless otherwise stated.

Remark: In differential geometry, we usually identify a map $F: M \to N$ with its representation in coordinates, i.e., we denote $\psi \circ F \circ \varphi^{-1}$ simply by F.

Def.: A Lie group G is a set equipped with the structures of (algebraic) group and smooth manifold, with the property that the multiplication $m: G \times G \to G$, $m(g,h) = g \cdot h$ and inverse $i: G \to G$, $i(g) = g^{-1}$ maps are smooth.

Example: matrix Lie groups. We can identify the set $M_{n \times n}$ of $n \times n$ real matrices with $\mathbb{R}^{n \times n}$. The general linear group is the open subset

$$GL(n, \mathbb{R}) = \{ A \in M_{n \times n} \mid \det A \neq 0 \}.$$

Proposition: $(\mathcal{C}^\infty(M), +, \cdot)$ is a ring, with

$$(f+g)(x) = f(x) + g(x) \qquad \forall x \in M$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$

In particular,

$$\mathbb{R} \equiv \{\text{constant functions}\} \subset \mathcal{C}^\infty(M).$$

Several objects on M will be $\mathcal{C}^\infty(M)$-linear and not only $\mathbb{R}$-linear

The intuitive idea of a tangent vector on a manifold is an arrow tangent to the manifold at a point, e.g the speed of wind at a point on the Earth's surface.

<u>Def.</u>: Let M be a manifold and $p \in M$. A <u>tangent vector</u> v at p can be defined in the two following equivalent ways:

- As a <u>derivation</u>: a linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying

$$v(fg) = f(p)\, vg + g(p)\, vf, \qquad \forall f, g \in C^\infty(M).$$

- As an <u>equivalence class of curves</u>: $v = [\gamma]_p$, where two curves $\gamma, \tilde{\gamma} : (-a, a) \subseteq \mathbb{R} \to M$ are equivalent if

$$\gamma(0) = \tilde{\gamma}(0) = p \qquad \& \qquad (f \circ \gamma)'(0) = (f \circ \tilde{\gamma})'(0) \quad \forall f \in C^\infty(M).$$

The set of all tangent vectors to M at p is denoted by T_pM and called the <u>tangent space of M at p.</u>

<u>Proposition:</u> T_pM is an n-dimensional real vector space, where $n = \dim M$.

<u>Def.:</u> The dual vector space to T_pM is called the <u>cotangent space to M at p</u> and denoted by T_p^*M.

The <u>differential</u> of $f \in C^\infty(M)$ at $p \in M$ is the unique covector $d_pf \in T_p^*M$ such that

$$d_pf \cdot v = vf \qquad \forall v \in T_pM.$$

Proposition: If $(U; x^1, \ldots, x^n)$ is a chart in M, then

$$\left(\frac{\partial}{\partial x^1}\bigg|_p, \ldots \frac{\partial}{\partial x^n}\bigg|_p \right)$$

is a basis of $T_p M$ for each $p \in U$, and

$$\left(d_p x^1, \ldots, d_p x^n \right)$$

is its dual basis.

Note that

$$v = \sum_i v^i \frac{\partial}{\partial x^i} \in T_p M \Rightarrow d_p f \cdot v = vf = \sum_i v^i \frac{\partial f}{\partial x^i}$$

and $d_p f$ is just the usual differential:

$$d_p f = \sum_i \frac{\partial f}{\partial x^i} dx^i.$$

Def.: The **tangent bundle** TM of a manifold M is

$$TM = \bigsqcup_{p\in M} T_pM$$

with the projection map $\tau_M : TM \to M$, $\tau_M(p,v) = p$.

Proposition: TM is a $2n$-dimensional manifold $(n = \dim M)$ and $\tau_M : TM \to M$ is a smooth map.

Sketch of the proof: If (U,φ) is a chart on M with $\varphi = (x^1, \dots, x^n)$ we can write $v = \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p$ $\forall v \in T_pM$, $\forall p \in U$. Hence, we can def. a chart $(\pi^{-1}(U), \tilde{\varphi})$ of TM by

$$\tilde{\varphi}(p,v) = \left(x^1(p), \dots, x^n(p), v^1, \dots, v^n\right).$$

Def.: A vector field on M is a map $X : M \longrightarrow TM$ such that $X(p) \in T_p M$ $\forall p \in M$, or equivalently,

$$\tau_M \circ X = Id_M .$$

The set of all vector fields on M is denoted by $\mathfrak{X}(M)$.

In coordinates,

$$X = \sum_i X^i \frac{\partial}{\partial x^i} , \qquad X^i \in C^\infty(M)$$

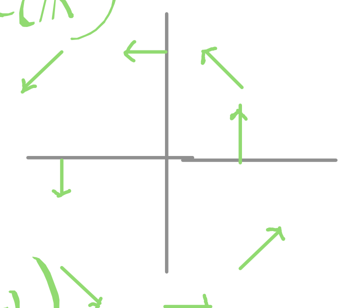
Def.: An <u>integral curve</u> of $X \in \mathfrak{X}(M)$ is a curve $\gamma : I \subseteq \mathbb{R} \to M$ satisfying

$$\gamma'(t) = X(\gamma(t)) \qquad \forall t \in I$$

In coordinates, $\gamma(t) = (x^1(t), \dots, x^n(t))$ is given by the system of first order ODEs

$$(x^i)'(t) = X^i(x^1(t), \dots, x^n(t)).$$

Example: The integral curves of $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$ satisfy



$$x'(t) = -y(t), \qquad y'(t) = x(t)$$

$$\rightsquigarrow \gamma(t) = \left(x_0\cos(t) - y_0\sin(t),\ y_0\cos(t) + x_0\sin(t)\right)$$

Def.: The cotangent bundle of M is $T^*M = \bigsqcup_{p\in M} T_p^*M$

with the projection $\pi_M : T^*M \longrightarrow M$, $\pi_M(p, \alpha) = p$.

Proposition: T^*M is a $2n$-dimensional mfold. as π_M is smooth.

Any system of coordinates $(x^1, \ldots, x^n)$ on M induces coords.

$(x^1, \ldots, x^n, p_1, \ldots, p_n)$ on T^*M s.t.

$$\pi_M(x^1, \ldots, x^n, p_1, \ldots, p_n) = (x^1, \ldots, x^n).$$

Fiber bundles → Locally like Cartesian products

* $\tau_M : TM \to M$ and $\pi_M : T^*M \to M$ are paradigmatic examples of vector bundles.

* A **fiber bundle** is a manifold E with a surjective smooth map $\pi : E \to M$ such that, for a sufficiently small neighbourhood U,

$$\pi^{-1}(U) \cong U \times F, \qquad F := \text{model fiber}$$

* If additionally $\pi^{-1}(x) \cong \mathbb{R}^n$ is a vector space $\forall x \in M$, we call it a **vector bundle**.

* Example: the Möbius strip can be described as a bundle $\pi : E \to S^1$ over the circle S^1.

Proposition: $\mathfrak{X}(M)$ is a $C^\infty(M)$-module. This means that

$$fX + gY \in \mathfrak{X}(M),$$

$\forall f, g \in C^\infty(M)$, $\forall X, Y \in \mathfrak{X}(M)$.

Def.: A K-form on M is an alternating $C^\infty(M)$-multilinear map

$$\omega : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{K\text{-times}} \longrightarrow C^\infty(M)$$

Alternating means that, $\forall\, X_1, \ldots, X_K \in \mathfrak{X}(M)$,

$$\omega(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_K) = -\,\omega(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_K).$$

The set of all K-forms on M is denoted by $\Omega^K(M)$.

We identify $\Omega^0(M) \equiv C^\infty(M)$.

Def: The *exterior* (or *wedge*) *product* $\wedge$ is a bilinear and associative product

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha \in \Omega^{k+\ell}(M), \qquad \forall \alpha \in \Omega^k(M), \ \forall \beta \in \Omega^\ell(M).$$

In coordinates, a 1-form α reads

$$\alpha = \sum_i \alpha_i \, dx^i$$

and a 2-form β reads

$$\beta = \sum_{i<j} \beta_{ij} \, dx^i \wedge dx^j$$

Remark: $\Omega^k(M) = \{0\} \quad \forall k > \dim(M).$

Theorem: $\exists!$ $\mathbb{R}$-linear operators $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ such that

i) $d^2 = 0$,

ii) $\forall f \in \mathcal{C}^\infty(M)$, df is the differential of f, i.e.

$$df \cdot X = Xf \qquad \forall X \in \mathfrak{X}(M).$$

iii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \quad \forall \alpha \in \Omega^k(M), \forall \beta \in \Omega^\ell(M).$

Def.: $d\alpha$ is called the <u>exterior derivative</u> of α.

In coordinates,

$$\alpha = \sum_i \alpha_i \, dx^i \implies d\alpha = \sum_{i,j} \frac{\partial \alpha_i}{\partial x^j} dx^j \wedge dx^i$$

$$\beta = \sum_{i,j} \beta_{ij} \, dx^i \wedge dx^j \implies d\beta = \sum_{i,j} \frac{\partial \beta_{ij}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j$$

Def.: A k-form α is called **closed** if $d\alpha = 0$.
It is called **exact** if $\exists$ $(k-1)$-form β s.t. $\alpha = d\beta$

Since $d^2 = 0$, exact $\Rightarrow$ closed

However, closed $\not\Rightarrow$ exact in general

Def : Set

$Z^p(M) = \{$closed p-forms on $M\} = \mathrm{Ker}\left(d : \Omega^p(M) \to \Omega^{p+1}(M)\right)$

$B^p(M) = \{$exact p-forms on $M\} = \mathrm{Im}\left(d : \Omega^{p-1}(M) \to \Omega^p(M)\right)$

They are real vector subspaces of $\Omega^p(M)$.

The p-th de Rham cohomology group of M is the quotient vector space

$$H^p_{dR}(M) = \frac{Z^p(M)}{B^p(M)}$$

By convention, we take $\Omega^p(M) = \{0\}$ for $p < 0$, so that $H^p_{dR}(M) = \{0\}$.

Theorem (de Rham): The de Rham cohomology of a smooth manifold is isomorphic to its singular cohomology, namely,

$$H^p_{dR}(M) \cong H^p(M, \mathbb{R})$$

closed k-form $\Rightarrow$ (globally) exact iff $H^k_{dR}(M) = \{0\}$.

Independently of the cohomology,

closed $\Rightarrow$ locally exact

<u>Poincaré Lemma:</u> For any closed k-form α and any point $x \in M$, there exists a sufficiently small neighbourhood U of x and a $(k-1)$-form $\beta \in \Omega^{k-1}(U)$ such that

$$\alpha\big|_U = d\beta$$

<u>Sketch of the proof:</u> U is diffeomorphic to $\hat{U} \subseteq \mathbb{R}^n$, and $\mathbb{R}^n$ has trivial cohomology.

Symplectic Geometry & Hamiltonian Mechanics

Def.: A symplectic form ω on M is a 2-form satisfying

i) $d\omega = 0$,

ii) $\omega_x(v,\cdot) = 0 \Rightarrow v = 0 \in T_x M \quad \forall x \in M$.

(M,ω) is called a symplectic manifold.

Example: In $\mathbb{R}^{2n}$ with Cartesian coordinates $(x^1,\dots,x^n,y^1,\dots,y^n)$,

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i .$$

Example: Recall that coordinates (x^i) on M induces coords. (x^i, p_i) on T^*M. The canonical one-form $\theta_M \in \Omega^1(T^*M)$ is given by

$$\theta_M = \sum_{i=1}^{n} p_i \, dx^i$$

Then,

$$\omega_M = -d\theta_M = -\sum_{i=1}^{n} dp_i \wedge dx^i = \sum_{i=1}^{n} dx^i \wedge dp_i$$

is symplectic. It is called the canonical symplectic form of T^*M.

Remark: The minus sign in the def. of ω_M is just a matter of convention.

<u>Theorem (Darboux):</u> Around each point of a symplectic manifold (M, ω) there are local coordinates (x^i, p_i) in which

$$\omega = \sum_{i=1}^{n} dx^i \wedge dp_i$$

These coords. are called Darboux (or canonical) coords.

In particular, every symplectic manifold is even-dimensional.

Topological restrictions

For a $2n$-dimensional manifold M to be symplectic, the following conditions are necessary:

* It must be orientable. The basis

$$\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}\right)$$ is positively oriented

* If M is compact, $H^2_{dR}(M) \neq \{0\}$.

Def.: Let (M, ω) be a symplectic manifold. The Hamiltonian vector field of $f \in C^\infty(M)$ is the unique $X_f \in \mathfrak{X}(M)$ such that

$$\omega(X_f, \cdot) = df.$$

In Darboux coords.,

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

Proposition: For any $f \in C^\infty(M)$, and for any integral curve $\gamma(t)$ of X_f,

$$\omega \circ \gamma(t) = \omega \circ \gamma(0) = \text{const.} \qquad \forall t.$$

<u>Def.:</u> A <u>Hamiltonian system</u> (M, ω, h) is a symplectic manifold (M, ω) with a fixed <u>Hamiltonian function</u> $h \in C^\infty(M)$.

Physically, h can be regarded as the total energy of the system.

The trajectories of (M, ω, h) are the integral curves $\gamma(t)$ of X_h. In Darboux coords., $\gamma(t) = (x^i(t), p_i(t))$ satisfies

$$(x^i)'(t) = \frac{\partial h}{\partial p_i}(\gamma(t)), \quad (p_i)'(t) = -\frac{\partial h}{\partial q^i}(\gamma(t))$$

These are the classical <u>Hamilton's equations</u>.

Example (Newton's second law of motion):

The trajectories $\gamma: I \subseteq \mathbb{R} \to \mathbb{R}^n$, $\gamma(t) = (x^i(t))$ of a physical system are given by the system of second-order ODEs

$$(x^i)''(t) = F^i(\gamma(t), \gamma'(t), \dots). \qquad (*)$$

Suppose that the force can be written as $F^i = \frac{\partial V}{\partial x^i}$ for some $V \in C^\infty(\mathbb{R}^n)$.

Consider the Hamiltonian system $(T^*\mathbb{R}^n \cong \mathbb{R}^{2n}, \omega_{\mathbb{R}^n}, h)$, where

$$h = \sum_{i=1}^{n} \frac{p_i^2}{2} + V(x^i).$$

In this case, Hamilton's eqs. read

$$(x^i)'(t) = \frac{\partial h}{\partial p_i} = p_i, \qquad (p_i)'(t) = -\frac{\partial h}{\partial x^i} = -\frac{\partial V}{\partial x^i},$$

which are equivalent to $(*)$.

Example: rigid bodies

* Consider a solid object (e.g. a bowling ball) whose deformations are negligible.

* To describe the position of such an object, e.g. the position of the center of mass ($\mathbb{R}^3$) and the angles of rotation of a fixed point in the border w.r.t. a reference point ($SO(3)$)

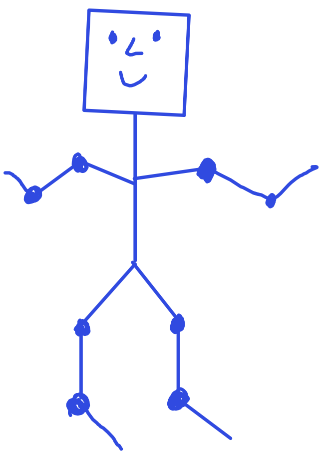
  Lie group of rotations on $\mathbb{R}^3$

* The configuration of such object will be determined by a point on $M = \mathbb{R}^3 \times SO(3)$. Its dynamics will be given by a Hamiltonian system (T^*M, ω_M, h).

Example: robots

To describe the motion of a humanoid robot, a drone, etc., we need to describe the rotations of its joints

$\rightsquigarrow$ products of Lie groups



Proposition: Let (M, ω, h) be a Hamiltonian system and $f \in C^\infty(M)$. The following statements are equivalent:

i) $f \circ \gamma(t)$ is constant along any trajectory $\gamma(t)$,

ii) $X_h(f) = 0$,

iii) $X_f(h) = 0$.

If these equivalent conditions are satisfied, f is called a constant of the motion.

Proof:

i) $\Leftrightarrow$ ii) For any integral curve $\gamma(t)$ of X_h,

$$(f \circ \gamma)'(t) = df(\gamma(t)) \cdot \gamma'(t) = df(\gamma(t)) \cdot X_h(\gamma(t)) = X_h(f)(\gamma(t))$$

ii) $\Leftrightarrow$ iii)

$$X_h(f) = df \cdot X_h = \Big(\omega(X_f, \cdot)\Big) \cdot X_h = \omega(X_f, X_h) = -\omega(X_h, X_f)$$

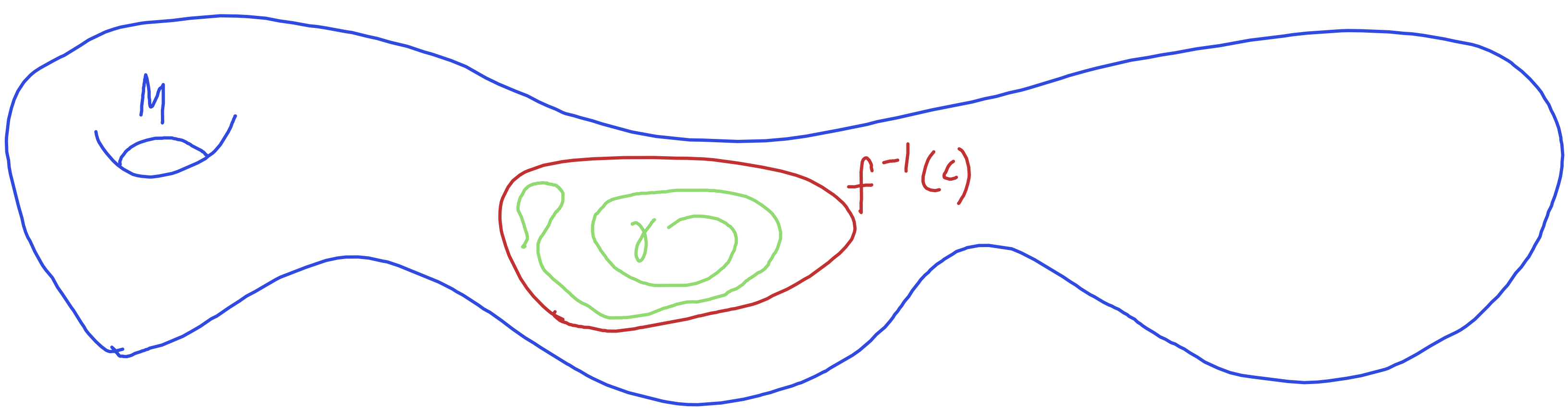
$$= -X_f(h).$$

Corollary: For any Hamiltonian system (M, ω, h), the Hamiltonian function is a conserved quantity

Proof: By the skew-symmetry of ω,

$$X_h h = \omega(X_h, X_h) = 0.$$

Remark: Note that $f \circ \gamma(t) = \text{const.}$ means that, for a fixed $\gamma(0)$, $\gamma(t)$ is contained in a level set $f^{-1}(c)$, $c \in \mathbb{R}$. In other words, trajectories are contained in level sets of conserved quantities



Def.: A $2n$-dimensional Hamiltonian system (M, ω, H) is called **completely integrable** if $\exists$ n constants of the motion $f_1, \dots, f_n$ such that

i) $df_1, \dots, df_n$ are linearly independent,

ii) $X_{f_i}(f_j) = 0 \quad \forall i, j \in \{1, \dots, n\}$.

Let $M_\Lambda = \bigcap_i f_i(\Lambda_i) = \{x \in M \mid f_1 = \Lambda_1, \dots, f_n = \Lambda_n\}, \quad \Lambda \in \mathbb{R}^n$

Theorem (Liouville-Arnol'd): Each compact and connected component of M_Λ is diffeomorphic to an n-torus T^n. The trajectories remain on their initial M_Λ. There are coords. (φ^i, s_i) in M such that (φ^i) are coords. in M_Λ, $\omega = \sum_i d\varphi^i \wedge ds_i$, and trajectories $\gamma(t) = (\varphi^i(t), s_i(t))$ are given by

$$\varphi^i(t) = \varphi^i(0) + \alpha^i t, \quad s_i(t) = s_i(0), \quad \alpha^i \in \mathbb{R} \text{ const.}$$

Reduction

* If a Lie group G acts "nicely" on a manifold M, then M/G is also a manifold. Moreover,

$$\dim M/G = \dim M - \dim G.$$

* Consider a Hamiltonian system (M, ω, h) such that the G-action preserves ω and h, i.e. G is the group of symmetries of (M, ω, h).

* If the action satisfies some additional conditions, it induces a reduced Hamiltonian system $(M/G, \omega_{red}, h_{red})$.

Geometric Numerical Integrators

* We have seen that Hamiltonian dynamics preserve different structures: the symplectic form, the energy, constants of the motion.
* To know these properties, we do not need to compute the explicit trajectories of the system.
* Geometric integrators are numerical methods that preserve geometric properties of the dynamical system.
* Classical numerical methods are designed to minimize the error in each time step, but do not care about the preservation of geometric structures or long-time behaviour.

More precisely, for a time step ε, and each integer n, we want a numerical approximation (q^n, p^n) to the exact solution $(q(n\cdot\varepsilon), p(n\cdot\varepsilon))$ of Hamilton's eqs. at time $n\cdot\varepsilon$.

An <u>integrator</u> is a smooth map $\Psi_{\varepsilon,H} : M \to M$ such that

$$(q^{n+1}, p^{n+1}) = \Psi_{\varepsilon,H}(q^n, p^n).$$

We will call it a <u>symplectic integrator</u> if

$$\omega = \sum_i dq_i^{n+1} \wedge dp_i^{n+1} = \sum_i dq_i^n \wedge dp_i^n.$$

Backward error interpretation

Theorem: For any positive integer, there exists a Hamiltonian function $H_{N,\varepsilon} = H + O(\varepsilon^N)$ such that the symplectic integrator

true Hamiltonian

$$\left(q^{n+1}, p^{n+1}\right) = \Psi_{\varepsilon,H}\left(q^n, p^n\right)$$

differs from the exact solutions $(q(t), p(t))$ of Hamilton's eqs. for $H_{N,\varepsilon}$ in order ε^{N+1}, namely,

$$q(n\cdot\varepsilon) = q^n + O(\varepsilon^{N+1}), \quad p(n\cdot\varepsilon) = p^n + O(\varepsilon^{N+1}).$$

Example: Symplectic Euler method

$$q^{n+1} = q^n + \varepsilon \frac{\partial h}{\partial p}(q^n, p^{n+1})$$

$$p^{n+1} = p^n - \varepsilon \frac{\partial h}{\partial q}(q^n, p^{n+1})$$

It is a symplectic integrator. Furthermore, it approximates the exact value of energy on the long time

The explicit Euler method

$$\frac{dy}{dt} = f(y(t)) \rightsquigarrow y^{n+1} = y^n + \varepsilon f(y^n)$$

is NOT symplectic

Example 2: The implicit midpoint rule

$$\frac{dy}{dt} = f(y(t)) \rightsquigarrow y^{n+1} - y^{n} = \varepsilon f\left(\frac{y_{n+1} + y_n}{2}\right)$$

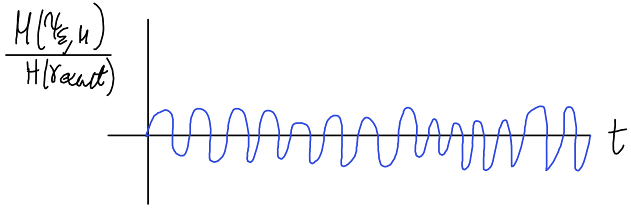
applied to Hamilton's eqs. is symplectic and approximates the exact energy on long times.

Near energy conservation

* In general, a numerical method cannot exactly preserve the Hamiltonian function and the symplectic form simultaneously, i.e.

$$dq^{n+1} \wedge dp^{n+1} = dq^n \wedge dp^n \quad \& \quad H(q^{n+1}, p^{n+1}) = H(q^n, p^n).$$

* However, some symplectic integrators exhibit, on the long-time, values of H oscillating around the correct constant value



* On the other hand, there are numerical integrators that exactly preserve H but not ω.

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Thank you for your attention!

Dziękuję za uwagę!

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