Homogeneous bi-Hamiltonian structures and integrable contact systems

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Contact geometry

Definition

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that the map

$$\begin{aligned} \flat_\eta \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

is an isomorphism of $\mathscr{C}^\infty(M)$-modules.

- There exists a unique vector field R on (M, η), called the **Reeb vector field**, given by $R = \flat_\eta^{-1}(\eta)$, or, equivalently,

$$\iota_R \mathrm{d}\eta = 0, \quad \iota_R \eta = 1.$$

Contact geometry

- The **Hamiltonian vector field** of $f \in \mathscr{C}^\infty(M)$ is given by

$$X_f = \flat_\eta^{-1}(\mathrm{d}f) - (R(f) + f)\, R\,,$$

- Around each point on M there exist **Darboux coordinates** (q^i, p_i, z) such that

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i\,,$$

$$R = \frac{\partial}{\partial z}\,,$$

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}\,.$$

Contact geometry

- The **Jacobi bracket** is given by

$$\{f, g\} = X_f(g) + gR(f).$$

- This bracket is bilinear and satisfies the Jacobi identity.
- However, unlike a Poisson bracket, it does not satisfy the Leibniz identity:

$$\{f, gh\} \neq \{f, g\}h + \{f, h\}g.$$

Contact Hamiltonian dynamics

Contact Hamiltonian vector fields allow modelling certain dissipative mechanical systems, as well as some thermodynamic systems.

In a Darboux chart, the integral curves $c(t) = (q^i(t), p_i(t), z(t))$ of X_h are determined by the **contact Hamilton equations**:

$$\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} = \frac{\partial h}{\partial p_i} \circ c(t)\,,$$

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = -\frac{\partial h}{\partial q^i} \circ c(t) - p_i(t)\frac{\partial h}{\partial z} \circ c(t)\,,$$

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = p_i(t)\frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.$$

Exact symplectic manifolds

Definition

An **exact symplectic manifold** is a pair (M, θ), where θ is a **symplectic potential** on M, i.e., $\omega = -\mathrm{d}\theta$ is a symplectic form on M. The **Liouville vector field** $\nabla \in \mathfrak{X}(M)$ is given by

$$\iota_{\nabla}\omega = -\theta\,.$$

A tensor field A on P is called k-homogeneous (for $k \in \mathbb{Z}$) if

$$\mathcal{L}_{\nabla} A = kA\,.$$

Trivial symplectisation

Definition

Let (M, η) be a co-oriented contact manifold. Then, the trivial bundle $\pi_1 : M^{\mathrm{symp}} = M \times \mathbb{R}_+ \to M$, $\pi_1(x, r) = x$ can be endowed with the symplectic potential $\theta(x, r) = r\eta(x)$. The Liouville vector field reads $\nabla = r\partial_r$.
We will refer to $(M^{\mathrm{symp}}, \theta)$ as the **trivial symplectisation** of (M, η).

Trivial symplectisation

Proposition

There is a one-to-one correspondence between functions $f(x)$ on M and 1-homogeneous functions $f^{\mathrm{symp}}(x, r) = -rf(x)$ on M^{symp} such that the symplectic $X_{f^{\mathrm{symp}}}$ and contact X_f Hamiltonian vector fields are related as follows:

$$\mathrm{T}\pi_1\left(X_{f^{\mathrm{symp}}}\right) = X_f\,.$$

Moreover, the Poisson $\{\cdot, \cdot\}_\theta$ and Jacobi $\{\cdot, \cdot\}_\eta$ brackets have the correspondence

$$\{f^{\mathrm{symp}}, g^{\mathrm{symp}}\}_\omega = \left(\{f, g\}_\eta\right)^{\mathrm{symp}}\,.$$

Definition

A **homogeneous Hamiltonian system** (M, θ, H) consists of a $2n$-dimensional exact symplectic manifold (M, θ) and a 1-homogeneous Hamiltonian function H. It is called a **homogeneous integrable system** if there exist n 1-homogeneous functions $f_1, \ldots, f_n$ such that

$$\{f_i, H\} = 0 = \{f_i, f_j\}\,, \quad 1 \leq i, j \leq n\,.$$

Definition

A **completely integrable contact system** is a co-oriented contact manifold (M, η) endowed with a Hamiltonian function $h \in \mathscr{C}^\infty(M)$ such that its trivial symplectisation $(M^{\mathrm{symp}}, \theta, h^{\mathrm{symp}})$ is a homogeneous integrable system.

Our Liouville–Arnol'd theorem permits constructing action-angle coordinates for completely integrable contact systems.

Compatible Poisson structures

Definition

Let M be a manifold. Two Poisson tensors are Λ and Λ_1 on M are said to be **compatible** if $\Lambda + \Lambda_1$ is also a Poisson tensor on M.

Definition

A vector field $X \in \mathfrak{X}(M)$ is called **bi-Hamiltonian** if it is a Hamiltonian vector field w.r.t. two compatible Poisson structures, namely,

$$X = \Lambda(\cdot, \mathrm{d}h) = \Lambda_1(\cdot, \mathrm{d}h_1)\,,$$

for two functions $h, h_1 \in \mathscr{C}^\infty(M)$.

Poisson – Nijehuis structures

- The linear map $\sharp_\Lambda \colon \mathrm{T}^*_x M \ni \alpha \mapsto \Lambda(\cdot, \alpha) \in \mathrm{T}_x M$ is an isomorphism iff Λ comes from a symplectic structure ω. In that case, $\flat_\omega \coloneqq \sharp_\Lambda^{-1}(v) = \iota_v \omega$.
- In that situation, we can define the $(1,1)$-tensor field

$$N = \sharp_{\Lambda_1} \circ \sharp_\Lambda^{-1} = \sharp_{\Lambda_1} \circ \flat_\omega \,.$$

Poisson – Nijehuis structures

Theorem (Magri and Morosi, 1984)

Let (M, ω) *be a symplectic manifold and* Λ_1 *a bivector. If* Λ_1 *is a Poisson tensor compatible with* $\Lambda = \omega^{-1}$*, then the eigenvalues of the* $(1, 1)$*-tensor field*

$$N = \sharp_{\Lambda_1} \circ \flat_\omega$$

are functions in involution w.r.t. both Poisson brackets.

Corollary

If a vector field $X \in \mathfrak{X}(M)$ *is bi-Hamiltonian w.r.t. to* ω *and* Λ_1 *(i.e.,* $X = \sharp_\omega \mathrm{d}h = \sharp_{\Lambda_1} \mathrm{d}h_1$*), then the eigenvalues of* N *form a family of conserved quantities in involution w.r.t. both Poisson brackets.*

Compatible Jacobi structures

- The theory of compatible Jacobi structures and Jacobi–Nijenhuis manifolds was developed by Iglesias, Monterde, Marrero, Nunes da Costa, Padrón and Petalidou in the 1990s and 2000s.
- Two Jacobi structures (Λ, E) and (Λ_1, E_1) on a manifold M are called compatible if $(\Lambda + \Lambda_1, E + E_1)$ is also a Jacobi structure on M.
- Given a Jacobi structure (Λ, E) on M, one can construct an associated Poisson structure $\tilde{\Lambda} = 1/r\Lambda + \partial_r \wedge E$ on $M \times \mathbb{R}_+$, which by construction is homogeneous of degree –1 w.r.t. $\nabla = r\partial_r$.
- Nunes da Costa (1998) showed that (Λ, E) and (Λ_1, E_1) are compatible Jacobi structures iff $\tilde{\Lambda}$ and $\tilde{\Lambda}_1$ are compatible Poisson structures.

Theorem (Fernandes, 1994)

Consider a 2n-dimensional completely integrable Hamiltonian system (M, ω, H) *with action-angle coordinates* (s_i, φ^i) *satisfying the following conditions:*

(ND) *The Hessian matrix* $\left(\frac{\partial^2 H}{\partial s_i \partial s_j}\right)$ *of the Hamiltonian w.r.t. the action variables is non-degenerate in a dense subset of* M.

(BH) *The system is bi-Hamiltonian and the recursion operator* N *has* n *functionally independent real eigenvalues* $\lambda_1, \ldots, \lambda_n$.

Then, the Hamiltonian function can be written as

$$H(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} H_i(\lambda_i),$$

where each H_i *is a function that depends only on the corresponding* λ_i.

Proposition

Let (M, θ, H) be a homogeneous integrable Hamiltonian system satisfying the assumption (ND). Denote by Λ the Poisson structure defined by $\omega = -\mathrm{d}\theta$, and by ∇ the Liouville vector field corresponding to θ. If there is a Poisson structure Λ_1 on M compatible with Λ, it cannot be simultaneously (-1)-homogeneous (i.e., $\mathcal{L}_\nabla \Lambda_1 = -\Lambda_1$) and satisfying (BH).

Proof.

If N has n functionally independent eigenvalues, then $H = \sum_i H_i(\lambda_i)$. If Λ_1 is (-1)-homogeneous, then N is 0-homogeneous, so its eigenvalues are 0-homogeneous as well. Hence,

$$H = \nabla(H) = \sum_{i=1}^{n} H_i'(\lambda_i)\nabla(\lambda_i) = 0\,.$$

$\square$

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$\square$

Corollary

Let (M, η, H) *be a* $(2n+1)$*-dimensional integrable contact Hamiltonian system. If there is a second Jacobi structure* (Λ_1, E_1) *compatible with the Jacobi structure* (Λ, E) *defined by* η*, then the recursion operator* $N = \sharp_{\tilde{\Lambda}_1} \circ \sharp_{\tilde{\Lambda}}^{-1}$ *relating the associated Poisson structures on* $M \times \mathbb{R}_+$ *cannot have* $(n+1)$ *functionally independent real eigenvalues.*

- Consequently, compatible Jacobi structures cannot be utilised to construct a set of independent functions in involution for a contact Hamiltonian system.
- Nevertheless, we can symplectise the contact Hamiltonian system and obtain a second Poisson structure compatible with the one defined by the exact symplectic structure.
- If N is 1-homogeneous and satisfies (BH), then its eigenvalues are n functionally independent and 1-homogeneous functions in involution, so they will project into n functions in involution w.r.t. the Jacobi bracket.

A toy example

- Let $M = \mathbb{R}^2$, and consider its cotangent bundle $\mathrm{T}^*M \simeq \mathbb{R}^4$ endowed with the canonical one-form $\theta_{\mathbb{R}^2}$.
- In bundle coordinates (x^1, x^2, p_1, p_2),

$$\theta_{\mathbb{R}^2} = p_1 \mathrm{d}x^1 + p_2 \mathrm{d}x^2 \rightsquigarrow \Lambda = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial p_2} \,.$$

- In this case, the Liouville vector field is $\nabla_M = p_i \partial_{p_i}$, the infinitesimal generator of homotheties on the fibers.
- A Poisson structure compatible with Λ is

$$\Lambda_1 = p_1 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial p_1} + p_2 x^2 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial p_2} \,.$$

A toy example

- The Nijenhuis tensor $N = \sharp_{\Lambda_1} \circ \sharp_{\Lambda}^{-1}$ reads

$$N = p_1 \left(\frac{\partial}{\partial x^1} \otimes \mathrm{d}x^1 + \frac{\partial}{\partial p_1} \otimes \mathrm{d}p_1 \right) + p_2 x^2 \left(\frac{\partial}{\partial x^2} \otimes \mathrm{d}x^2 + \frac{\partial}{\partial p_2} \otimes \mathrm{d}p_2 \right) ,$$

  with eigenvalues $\lambda_1 = p_1$ and $\lambda_2 = p_2 x^2$.
- The vector field

$$X = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} - p_2 \frac{\partial}{\partial p_2}$$

  is bi-Hamiltonian, it is the Hamiltonian vector field of $H = p_1 + p_2 x^2$ w.r.t. Λ, and the Hamiltonian vector field of $H_1 = \log(p_1 p_2 x^2)$ w.r.t. Λ_1. Moreover, λ_1 and λ_2 are first integrals of X.

A toy example bis

- Consider the contact Hamiltonian system $(M = \mathbb{R}^3, \eta, h)$, with η the canonical contact form, $\eta = \mathrm{d}z - p\mathrm{d}q$, and $h = p - z$.
- In bundle coordinates (q, p, z, r), the trivial symplectisation $(\mathbb{R}^4, \theta, H)$ of (M, η, h) reads

$$\theta = r\mathrm{d}z - rp\mathrm{d}q\,, \quad H = rz - rp\,,$$

  and Liouville vector field is $\nabla = r\partial_r$.
- This is the system from the previous example, as it becomes evident by performing the coordinate change

$$x^1 = q, \quad x^2 = z, \quad p_1 = -rp, \quad p_2 = r\,.$$

- Thus, we have the functions $\lambda_1 = p_1 = -rp$ and $\lambda_2 = p_2 x^2 = rz$, which are homogeneous of degree 1, in involution, and functionally independent on a dense subset.
- Projecting them to M, we obtain $\bar{\lambda}_1 = p$ and $\bar{\lambda}_2 = -z$, which are functionally independent and $\{\bar{\lambda}_1, \bar{\lambda}_2\} = \{\bar{\lambda}_1, h\} = \{\bar{\lambda}_2, h\} = 0$.

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Merci pour votre attention!

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