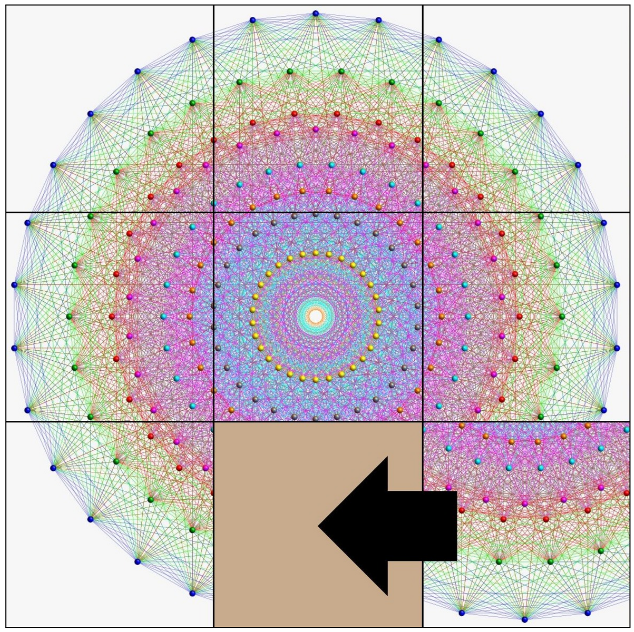
Reading Group(oid) Lecture Notes



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Lectures

Lecture 1 – O. Carballal (UCM)

Date: 7 October, 2025

Abstract: We illustrate the need for groupoids through a series of examples, primarily following [**2**, Lecture 1] and [**3**], and discuss some of their recent applications in Mathematical Physics. We also outline potential directions for future topics of our Reading Groupoid.

Since F. Klein's *Erlangen program* and the pioneer work of S. Lie regarding the definition of geometric structures through their group of automorphisms, the notion of symmetry has been studied through the theory of groups and their actions.

Symmetry ≡ Groups & actions

Symmetry of homogeneous structures, such us homogeneous spaces, can be fully described using groups. This is no longer valid when analysing the symmetry of non-homogeneous structures.

The very first example of symmetry group is the *Euclidean group* $\mathrm{E}(n)$, consisting of rigid motions of $\mathbb{R}^n$. More concretely,

$$\mathrm{E}(n) := \{\phi : \mathbb{R}^n \to \mathbb{R}^n : \phi \text{ preserves distances}\}$$

Every $\phi \in \mathrm{E}(n)$ is univocally determined by a rotation $A \in \mathrm{O}(n)$ and a translation vector $b \in \mathbb{R}^n$ as $\phi(x) = Ax + b$ and, hence, $\mathrm{E}(n)$ is the semidirect product $\mathrm{E}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$. Let us now consider the action $\mathrm{E}(n) \curvearrowright \mathbb{R}^n$ given by $\phi \cdot x := \phi(x)$ for every $\phi \in \mathrm{E}(n)$ and $x \in \mathbb{R}^n$. Then, we define the symmetry group of a subset Ω as the subgroup G_Ω of $\mathrm{E}(n)$ formed by those rigid motions leaving Ω invariant; that is,

$$G_\Omega := \{\phi \in \mathrm{E}(n) : \phi(\Omega) = \Omega\}\,.$$

Let us now adopt the following *credo*:

G_Ω is large ≡ Ω is very symmetric

Example 1 (Weinstein's tiling [**3**])**.** Let us study the symmetry group of the following tiling of $\mathbb{R}^2$ by 2×1-rectangles:

$$\Omega := (2\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \subset \mathbb{R}^2.$$

Within this context, every connected component of $\mathbb{R}^2 - \Omega$ is called a *tile*. Let Λ be the lattice determined by Ω, corresponding at the borders of the tiles,

$$\Lambda = (2\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z}) = 2\mathbb{Z} \times \mathbb{Z}.$$

Then, one easily sees that the symmetry group G_Ω of Ω consists of the following elements:

- translations by vectors $u \in \Lambda$, corresponding to the corner points of Λ;
- central reflections through the points of $\frac{1}{2}\Lambda = \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$, the latter corresponding to the centers of the tiles, and
- axial reflections along the vertical and horizontal lines trough the points of $\frac{1}{2}\Lambda$.

Since G_Ω is a big group, following the faith in our *credo*, Ω is very symmetric. Nevertheless, there are some problems:

- Ω and the lattice Λ have the same symmetry group $G_\Omega = G_\Lambda$, although Ω and Λ look quite different;
- G_Ω contains no local information about the action $G_\Omega \curvearrowright \mathbb{R}^2$, and
- the symmetry group of the finite tiling $\widetilde{\Omega} := \Omega \cap B$, where $B := [0, 2m] \times [0, n]$ is a 'bathroom floor', is isomorphic to the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (note that it is spanned by the horizontal and the vertical reflections through the midline of B, whose composition is the 180º rotation around the centre of B). In particular, we see that Ω is very symmetric but $\widetilde{\Omega}$ is not, even though they share the same patterns and independently of the number of tiles.

To address these issues, we introduce the *action groupoid* associated with $G_\Omega \curvearrowright \mathbb{R}^n$, defined as

$$\mathcal{G}_\Omega := \{(y, \phi, x) : \phi \in G_\Omega, x, y \in \mathbb{R}^n, y = \phi(x)\}, \tag{1}$$

and endowed with the partially defined multiplication

$$(z, \psi, y)(y, \phi, x) := (z, \psi \circ \phi, x). \tag{2}$$

In order to analyse the properties of this partially defined multiplication, we introduce the *source* $\mathrm{s} : \mathcal{G}_\Omega \to \mathbb{R}^n$ and *target* $\mathrm{t} : \mathcal{G}_\Omega \to \mathbb{R}^n$ maps as

$$\mathrm{s}(y, \phi, x) := x, \qquad \mathrm{t}(y, \phi, x) := y, \qquad (y, \phi, x) \in \mathcal{G}_\Omega.$$

One easily sees that (2) satisfies the following four properties:

(1) Composition. Given $g, h \in \mathcal{G}_\Omega$, gh is only defined when $\mathrm{s}(g) = \mathrm{t}(h)$. In that case, $\mathrm{s}(gh) = \mathrm{s}(h)$ and $\mathrm{t}(gh) = \mathrm{t}(g)$.
(2) Associativity. If $g, h, k \in \mathcal{G}_\Omega$ are such that $\mathrm{s}(g) = \mathrm{t}(h)$ and $\mathrm{s}(h) = \mathrm{t}(k)$, then $(gh)k = g(hk)$.
(3) Units. For all $x \in \mathbb{R}^n$, let us denote $1_x := (x, \mathrm{id}, x) \in \mathcal{G}_\Omega$. Then, $1_{\mathrm{t}(g)}g = g = g1_{\mathrm{s}(g)}$ for all $g \in \mathcal{G}_\Omega$.
(4) Inverse. For all $g = (y, \phi, x) \in \mathcal{G}_\Omega$ there exists $g^{-1} := (x, \phi^{-1}, y)$ such that $gg^{-1} = 1_{\mathrm{t}(g)}$ and $g^{-1}g = 1_{\mathrm{s}(g)}$.

These are, exactly, the properties characterising a groupoid.

Definition 1 (Groupoid)**.** A *groupoid* consists of two sets, $\mathcal{G}$ and M, together with maps

- $\mathrm{s}, \mathrm{t} : \mathcal{G} \to M$ (*source* and *target* projections);
- $m : \mathcal{G}^{(2)} := \{(g, h) : \mathrm{s}(g) = \mathrm{t}(h)\} \to \mathcal{G}, (g, h) \mapsto gh$ (*multiplication*);
- $u : M \to \mathcal{G}, x \mapsto 1_X$ (*unit*), and
- $i : \mathcal{G} \to \mathcal{G}, g \mapsto g^{-1}$ (*inverse*),

satisfying the following four properties:

(1) If $z \xleftarrow{g} y \xleftarrow{h} x$, then $z \xleftarrow{gh} x$;

(2) If $z \xleftarrow{g} y \xleftarrow{h} x \xleftarrow{k} v$, then $(gh)k = g(hk)$;
(3) There exists $x \xleftarrow{1_x} x$ such that for all $y \xleftarrow{g} x$, we have $1_y g = g = 1_x$, and
(4) If $y \xleftarrow{g} x$, there exists $x \xleftarrow{g^{-1}} y$ such that $gg^{-1} = 1_y$ and $g^{-1}g = 1_x$.

From now on, we will denote the groupoid as $\mathcal{G} \rightrightarrows M$, where the parallel arrows refer to the source and target maps.

Remark 2. (1) A groupoid is a small category where every arrow is invertible. Within this language, given a groupoid $\mathcal{G} \rightrightarrows M$, $\mathcal{G}$ is usually referred to as the set of *arrows* of the groupoid, M is the set of *objects* or *units* (also called the *base* of the groupoid), and $\mathcal{G}^{(2)}$ is the set of *composable arrows*.
(2) We write arrows from left to right to refer to the order of composition. Everything can be done analogously writing arrows from left to right by swapping the source and target maps s and t.
(3) Groupoids can be restricted to subsets of the set of units. That is, given a groupoid $\mathcal{G} \rightrightarrows M$, *its restriction* to $N \subset M$ is the groupoid

$$\mathcal{G}|_N := \mathrm{s}^{-1}(N) \cap \mathrm{t}^{-1}(N) \rightrightarrows N.$$

Definition 3 (Orbit and isotropy group)**.** Let $\mathcal{G} \rightrightarrows M$ be a groupoid and let $x \in M$ be a unit.

- The *orbit of* x is $O_x := \{y \in M : \text{there exists } g \in \mathcal{G}, y \xleftarrow{g} x\}$.
- The *isotropy group of* x is $\mathcal{G}_x := \mathrm{s}^{-1}(x) \cap \mathrm{t}^{-1}(x) = \{g \in \mathcal{G} : x \xleftarrow{g} x\}$.

Example 2 (Weinstein's tiling continued [**3**])**.** Let us now go back to the tiling Ω of $\mathbb{R}^2$ introduced in Example 1. First of all, we consider the action groupoid $\mathcal{G}_\Omega \rightrightarrows \mathbb{R}^2$ associated with $G_\Omega \curvearrowright \mathbb{R}^2$, defined in (1); that is,

$$\mathcal{G}_\Omega := \{(y, \phi, x) : \phi \in G_\Omega, y \xleftarrow{\phi} x\} \rightrightarrows \mathbb{R}^2,$$

and let us now consider its restriction to the 'bathroom floor' $B = [0, 2m] \times [0, n]$, namely $\mathcal{G}_\Omega|_B \rightrightarrows B$. Then:

- $x, y \in B$ belong to the same orbit if they are similarly placed in their tiles, and
- the isotropy group of a point $x \in B$ is trivial, unless $x \in \frac{1}{2}\Lambda \cap B$, for which its isotropy is the Klein four group.

This means that $\mathcal{G}_\Omega|_B \rightrightarrows B$ detects local information.

Some applications to Mathematical Physics.

- Groupoid picture of Schwinger's quantum mechanics by F. M. Ciaglia, A. Ibort, G. Marmo and collaborators initiated in [**4**]. Recent work on the formulation of fields as functors between groupoids by A. Ibort, A. Mas and L. Schiavone [**5**].
- The material groupoid, a groupoidal approach to continuum mechanics, by M. de León, M. Epstein and V. Jiménez [**6**].
- Discrete Lagrangian and Hamiltonian mechanics on Lie groupoids by J. C. Marrero, D. Martín de Diego, E. Martínez and L. Colombo [**7, 8**].

Local Lie group actions. Let us consider the Lie group $\mathrm{PGL}(2,\mathbb{R})$ consisting of homographies of the projective line $\mathbb{RP}^1$. Explicitly, one has

$$\mathrm{PGL}(2,\mathbb{R}) = \mathrm{GL}(2,\mathbb{R})/\mathbb{R},$$

where $\mathbb{R}$ refers to the subgroup of $\mathrm{GL}(2,\mathbb{R})$ formed by non-zero diagonal scalar matrices. Clearly, $\mathrm{PGL}(2,\mathbb{R})$ acts on $\mathbb{RP}^1$ as

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot [x_0 : x_1] = [ax_0 + bx_1 : cx_0 + dx_1]. \tag{3}$$

Let us now analyse how $\mathrm{PLG}(2,\mathbb{R})$ acts on the affine line $\mathbb{R} \equiv \mathbb{RP}^1 - \{[1:0]\}$. Given an affine point $[x_0 : x_1] = [x : 1] \in \mathbb{R}$, where $x := x_0/x_1$, and $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \mathrm{PGL}(2,\mathbb{R})$, we have that

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot [x : 1] = \left[\frac{ax+b}{cx+d} : 1\right] \in \mathbb{R}, \tag{4}$$

provided that $cx + d \neq 0$. It is natural to consider the following question:

$$\boxed{\textit{Does formula (4) define a Lie group action on } \mathbb{R}\textit{?}}$$

Of course, the answer to this question is negative, since (4) is only defined on an open subset of $\mathrm{PGL}(2,\mathbb{R}) \times \mathbb{R}$. Indeed, it defines what is called a *local Lie group action* (see [**9**] for more details). This means, roughly speaking, that (4) only satisfies the defining properties of a Lie group action on a neighbourhood of the unit element of $\mathrm{PGL}(2,\mathbb{R})$.

Hopefully, this problem can be tackled successfully when one considers groupoids rather than groups. First of all, let us consider the so-called action groupoid associated with the Lie group action $\mathrm{PGL}(2,\mathbb{R}) \curvearrowright \mathbb{RP}^1$ defined in (3) as the groupoid $\mathcal{G} := \mathrm{PGL}(2,\mathbb{R}) \times \mathbb{RP}^1 \rightrightarrows \mathbb{RP}^1$ with source and target maps given by

$$\mathrm{s}\left([A], [x_0 : x_1]\right) = [x_0 : x_1], \qquad \mathrm{t}\left([A], [x_0 : x_1]\right) = [A] \cdot [x_0 : x_1],$$

and multiplication defined as

$$([B], [y_0 : y_1])([A], [x_0 : x_1]) := ([BA], [x_0, x_1]),$$

provided that $[A] \cdot [x_0 : x_1] = [y_0 : y_1]$. Then, the restriction of $\mathcal{G} \rightrightarrows \mathbb{RP}^1$ to the affine line $\mathbb{R}$ is the groupoid

$$\mathcal{G}|_{\mathbb{R}} = \mathrm{s}^{-1}(\mathbb{R}) \cap \mathrm{t}^{-1}(\mathbb{R}) = \left\{ \left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right], [x : 1] \right) \in \mathrm{PGL}(2,\mathbb{R}) \times \mathbb{R} : cx + d \neq 0 \right\} \rightrightarrows \mathbb{R}.$$

Within this context, the target map of $\mathcal{G}|_{\mathbb{R}} \rightrightarrows \mathbb{R}$ is just formula (4).

Singular spaces. Let $G \curvearrowright M$ be a smooth action of a Lie group G on a manifold M (see [**10**, Chapter 2] for more details on Lie group actions). Recall that the action is said to be

- *free* if $g \cdot x = x$ for some $x \in M$, then $g = e$, and

- *proper* if the map $\Phi : G \times M \ni (g, x) \mapsto (g \cdot x, x) \in M \times M$ is a proper map. That is, for every compact subset $K \subset M \times M$ we have that $\Phi^{-1}(K) \subset G \times M$ is compact. This is equivalent to the following: if (g_n) and (x_n) are sequences in G and M, respectively, such that (x_n) and $(g_n \cdot x_n)$ are convergent, then (g_n) contains a converging subsequence.

Provided that the action $G \curvearrowright M$ is free an proper, the quotient manifold M/G has a unique smooth structure such that the projection map $\pi : M \to M/G$ is a submersion. For instance, let us consider the action $G = \mathbb{Z}^k \curvearrowright M = \mathbb{R}^k$ given by translations:

$$(n_1, \dots, n_k) \cdot (x^1, \dots, x^k) := (x^1 + n_1, \dots, x^k + n_k).$$

It can be easily seen that this action is free and proper. Hence, the quotient space $\mathbb{R}^k/\mathbb{Z}^k$, which is the k-dimensional torus $\mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$, possesses a unique smooth structure such that the projection $\pi : M \to M/G$ is a submersion.

What happens when the action $G \curvearrowright M$ is not free and proper?

If action $G \curvearrowright M$ is not free and proper, G/M is a "singular space"

Example 3. Let us consider the actions $\mathrm{SO}(2) \curvearrowright \mathbb{R}^2$ and $\mathrm{SO}(3) \curvearrowright \mathbb{R}^3$ given by rotations. Both of these actions are proper, since the associated Lie groups are compact. Nevertheless, they are not free: the origin is a fixed point of every rotation. The "singular spaces" $\mathbb{R}^2/\mathrm{SO}(2)$ and $\mathbb{R}^3/\mathrm{SO}(3)$ are both homeomorphic to $[0, +\infty)$. Notwithstanding, they are not equivalent as singular spaces because their associated action groupoids $\mathrm{SO}(2) \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ and $\mathrm{SO}(3) \times \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ are not *Morita equivalent*. Roughly speaking, this can be justified as follows. The isotropy group of $x \in \mathbb{R}^2 - \{0\}$ is trivial, while the isotropy group of $x \in \mathbb{R}^3 - \{0\}$ is isomorphic to $\mathrm{SO}(2)$.

Lecture 2 – Bartosz M. Zawora (KMMF - UW)

Date: 14 October, 2025

Abstract: During the lecture, I will recall the definition of a groupoid from the previous session and present the definition of a Lie groupoid. I will provide many useful and important examples and prove some fundamental properties. The lecture will follow Mackenzie's book "General Theory of Lie Groupoids and Lie Algebroids".

1. Lie groupoids: definition and examples

Definition 4. A groupoid consists of two sets $\mathcal{G}$ and M, with maps,
$s : \mathcal{G} \to M$ (source),
$t : \mathcal{G} \to M$ (target),
$u : M \to \mathcal{G} \mid x \mapsto 1_x$ (unit),
$i : \mathcal{G} \to \mathcal{G} \mid g \longmapsto g^{-1}$ (inverse),
$\kappa : \mathcal{G} * \mathcal{G} \to \mathcal{G} \mid (h, g) \longmapsto hg$ where $\mathcal{G} * \mathcal{G} := \{(h, g) \in \mathcal{G} \times \mathcal{G} \mid s(h) = t(g)\}$,
satisfying

(1) $s(hg) = s(g)$ and $t(hg) = t(h)$ for all $(h, g) \in \mathcal{G} * \mathcal{G}$,
(2) $j(hg) = (jh)g$ for every $j, h, y \in \mathcal{G}$ such that $s(j) = t(h)$ and $s(h) = t(g)$,
(3) $s(1_x) = t(1_x) = x$ for any $x \in M$,
(4) $g1_{s(g)} = g$ and $1_{t(y)}g = g$ for every $g \in \mathcal{G}$,
(5) For each $g \in \mathcal{G}$, there exists two-sided inverse g^{-1} such that $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g^{-1}g = 1_{s(g)}$, and $gg^{-1} = 1_{t(g)}$.

To simplify the notation, $\mathcal{G} \rightrightarrows M$ denotes a groupoid.

Proposition 5. Let $\mathcal{G} \rightrightarrows M$ and $g \in \mathcal{G}$ with $s(g) = x$ and $t(g) = y$.

(1) If $h \in \mathcal{G}$ with $s(h) = g$ and $hg = g$, then $h = 1_y$.
If $j \in \mathcal{G}$ with $t(j) = x$ and $gj = g$, then $j = 1_x$.
(2) If $h \in \mathcal{G}$ with $s(h) = y$ and $hg = 1_x$, then $h = g^{-1}$.
If $j \in \mathcal{G}$ with $t(j) = x$ and $gj = 1_y$, then $j = g^{-1}$.

Definition 6. Let $G \rightrightarrows M$. Then, $\mathcal{G}_x := s^{-1}(x)$ is called a s-fibre over $x \in M$, similarly $\mathcal{G}^y := t^{-1}(x)$ is a t-fibre over $y \in M$, and $\mathcal{G}_x^y := \mathcal{G}_x \cap \mathcal{G}^y = s^{-1}(x) \cap t^{-1}(y)$. Moreover, the set $\mathcal{G}_x \cap \mathcal{G}^y$ is called the vertex group. The set of identity elements 1_M is defined as $\{1_m \mid m \in M\}$.

Definition 7. A Lie groupoid $\mathcal{G} \rightrightarrows M$ is a groupoid $\mathcal{G} \rightrightarrows M$ with smooth manifold structures on $\mathcal{G}$ and M such that $s, t \colon \mathcal{G} \to M$ are surjective submersions, $u \colon M \to \mathcal{G}$ is smooth, and $\kappa \colon \mathcal{G} * \mathcal{G} \to \mathcal{G}$ is also smooth.

Remark 8. Since $s, t\colon \mathcal{G} \to M$ are submersions it follows that $\mathcal{G}_x$, $\mathcal{G}^y$, and $(s \times t)^{-1}(x, x')$ are closed embedded submanifolds for some $x, x' \in M$.

Definition 9. Let $\mathcal{G} \rightrightarrows M$ and $g \in \mathcal{G}$ with $s(g) = x$ and $t(g) = y$. Then, the left-translation corresponding to g is $L_g\colon h \in \mathcal{G}^x \mapsto gh \in \mathcal{G}^y$. Analogously, the right-translation corresponding to g is $R_g\colon h \in \mathcal{G}_y \mapsto hg \in \mathcal{G}_x$

Definition 10. A Lie groupoid is s-connected if each of its fibres is connected. Likewise, for any other property α, a Lie groupoid $\mathcal{G} \rightrightarrows M$ is s-α if each of its fibres has property α.

Proposition 11. Let $\mathcal{G} \rightrightarrows M$. The inverse map $i\colon \mathcal{G} \to \mathcal{G}$ is a diffeomorphism.

Proof. Note that the tangent bundle to $\mathcal{G} * \mathcal{G}$ is given by

$$\mathrm{T}(\mathcal{G} * \mathcal{G}) = \{(Y, X) \in \mathrm{T}\mathcal{G} \times \mathrm{T}\mathcal{G} \mid \mathrm{T}s(Y) = \mathrm{T}t(X)\}\,.$$

Suppose that $h, g \in \mathcal{G}$ are such that $s(h) = t(g)$, $Y \in \mathrm{T}_h(\mathcal{G}_{s(h)})$, and $X \in \mathrm{T}_g(\mathcal{G}^{t(g)})$. Then, by Leibniz rule, it follows

$$\mathrm{T}_{(h,g)}\kappa(Y, X) = \mathrm{T}_h R_g(Y) + \mathrm{T}_g L_h(X)\,. \tag{5}$$

Define $\theta\colon \mathcal{G} * \mathcal{G} \to \mathcal{G} \times_t \mathcal{G}$ as

$$\theta(h, g) = (h, hg)\,,$$

where $\mathcal{G} \times_t \mathcal{G} := \{(h, g) \in \mathcal{G} \times \mathcal{G} \mid t(h) = t(g)\}$. Then, one can simply verify that θ is bijective with $\theta^{-1}(i, j) = (i, i^{-1}j)$. To prove that θ is an immersion, suppose that $\mathrm{T}_{(h,g)}\theta(Y, X) = (0, 0)$. Moreover, define $\pi_1\colon (h, g) \in \mathcal{G} \times \mathcal{G} \mapsto g \in \mathcal{G}$ and $\pi_2\colon (h, g) \in \mathcal{G} \times \mathcal{G} \mapsto h \in \mathcal{G}$. Therefore, since $\pi_2 \circ \theta = \pi_2$, it follows that $Y = 0$. Consequently, by (5), one has $X = 0$ and θ is an immersion.

Furthermore, by the fact that s and t are submersions and $\dim \mathcal{G} * \mathcal{G} = \dim \mathcal{G} \times_t \mathcal{G}$, one obtains that θ is a diffeomorphism. For $\mu\colon h \in \mathcal{G} \mapsto (h, 1_{t(h)}\mathcal{G} \times_t \mathcal{G}$, one has

$$(\pi_1 \circ \theta^{-1} \circ \mu)(g) = (\pi_1 \circ \theta^{-1})(g, 1_{t(g)}) = \pi_1(g, g^{-1}) = g^{-1}\,.$$

Hence, $(\pi_1 \circ \theta^{-1} \circ \mu) = i$. Since $(\pi_1 \circ \theta^{-1} \circ \mu)$ is smooth, it follows that i is also smooth. In addition, since inversion is its own inverse, it is therefore a diffeomorphism. □

Remark 12. The unit map $u\colon M \to \mathcal{G}$ is smooth and is an immersion, by the fact that

$$t \circ u = \mathrm{id}_M\,, \qquad s \circ u = \mathrm{id}_M\,.$$

Consequently, 1_M is a closed embedded submanifold of $\mathcal{G}$.

Example 4 (Lie group)**.** A Lie group is a Lie groupoid with a unique unit, namely $G \rightrightarrows \{*\}$.

Example 5 (Manifold)**.** Any manifold M can be regarded as a Lie groupoid on itself, where $s = t = \mathrm{id}_M$.

Example 6 (A pair groupoid)**.** For any manifold M, one has a pair groupoid $M \times M \rightrightarrows M$ with arrows $(y, x) \in M \times M$, $s(y, x) = x$, $t(y, x) = y$, $u(x) = (x, x)$, $i(y, x) = (x, y)$, and

$$\kappa((z, y), (y, x)) = (z, x).$$

Note that $\mathcal{G}_x^x$ is trivial.

Example 7 (Trivial groupoid)**.** For a Lie group G and a manifold M one can construct a Lie groupoid $M \times G \times M \rightrightarrows M$ with $s(y, g, x) = x$, $t(y, g, x) = y$, $u(x) = (x, e, x)$, $i(y, g, x) = (x, g^{-1}, y)$, and

$$\kappa((z, h, y), (y, g, x)) = (y, hg, x)\,.$$

Example 8 (Submersion groupoid)**.** For a surjective submersion $\pi\colon M \to Q$ one has a submersion groupoid

$$M \times_Q M \rightrightarrows M\,,$$

where $M \times_Q M = \{(y,x) \in M \times M \mid \pi(y) = \pi(x)\}$ and is given by the restriction of a pair groupoid $M \times M \rightrightarrows M$ to $M \times_Q M$.

Example 9 (Action groupoid)**.** Let $G \times M \to M$ be a Lie group action on a manifold M. Then, $G \times M \rightrightarrows M$ is an action Lie groupoid with $s(g,x) = x$, $t(g,x) = gx$, $u(x) = (e,x)$, $i(g,x) = (g^{-1}, gx)$, and

$$\kappa((g_2,y),(g_1,x)) = (g_2 g_1, x)\,,$$

of $y = g_1 x$. Note that $\mathcal{G} \times \mathcal{M}_x^x$ is isomorphic to G_x. An action Lie groupoid is denoted as $G \lhd M$.

Example 10. ADD DRAWINGS

Example 11 (Fundamental groupoid)**.** Let M be a smooth manifold and let $\Pi(M)$ be the set of homotopy classes $[\gamma]$ of continuous path $\gamma\colon [0,1] \to M$, relative to fixed and points $\gamma(0)$ and $\gamma(1)$. Then, $\Pi(M) \rightrightarrows M$ is a groupoid with $s([\gamma]) = \gamma(0)$, $t([\gamma])\gamma(1)$, $u(x) = [\gamma_x]$, where γ_x is a constant path at $x \in M$, and

$$\kappa([\delta],[\gamma]) = [\delta\gamma]\,,$$

where $\delta\gamma$ is the standard concatenation of γ and δ, namely

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Moreover, $i([\gamma]) = [\overleftarrow{\gamma}]$, where $\overleftarrow{\gamma}$ is the reverse path, namely $\overleftarrow{\gamma}(t) = \gamma(1-t)$.

Example 12. Let (E,q,M) be a vector bundle. Let $\Phi(E)$ denote the set of all vector space isomorphism $\xi\colon E_x \to E_y$ for some $x, y \in M$. Then, $\Phi(E) \rightrightarrows M$ is a frame groupoid with $s(\xi) = x$, $t(\xi) = y$, $u(x) = \mathrm{id}_{E_x}$, the inverse of ξ is just an inverse map of ξ, and multiplication κ is the composition of maps.

Example 13 (Jet groupoid)**.** For a local diffeomorphism $\varphi\colon U \to V$ between open sets of a manifold M, and given $x \in U$, let $j^1_x\varphi$ be the one-jet of φ at $x \in U$. Then, $J^1(M,M)$, the set of all such one-jets, has a natural groupoid structure $J^1(M,M) \rightrightarrows M$ with $s(j^1_x\varphi) = x$, $t(j^1_x\varphi) = \varphi(x)$, $\kappa(j^1_{\varphi(x)}\psi, j^1_x\varphi) = j^1_x(\psi\circ\varphi)$, and the inverse is given by $i(j^1_x\varphi) = j^1_{\varphi(x)}\varphi^{-1}$.

Example 14 (Gauge groupoid)**.** Let (P,M,G,π) be a principal bundle and let $(u_2,u_1,g) \in P \times P \times G \mapsto (u_2 g, u_1 g) \in P \times P$. Let $[(u_2,u_1)]$ denote the equivalence class in $(P\times P)/G$. Then,

$$\frac{P \times P}{G} \rightrightarrows M$$

is a Lie groupoid with $s([(u_2,u_1)]) = \pi(u_1)$, $t([(u_2,u_1)]) = \pi(u_2)$, $u(x) = [(v,v)]$ for $v \in \pi^{-1}(x)$, $i([(u_2,u_1)]) = [(u_1,u_2)]$, and

$$\kappa(([(u_3,u_2)]),([(u_2,u_1)])) = [(u_3,u_1)]\,.$$

Example 15. Let G be a Lie group with a Lie algebra $\mathfrak{g}$. Then, $\mathrm{T}^*G \rightrightarrows \mathfrak{g}^*$ is a Lie groupoid with $s(\theta) = \theta\circ \mathrm{T}_e L_g$, $t(\theta) = \theta \circ \mathrm{T}_e R_g$ for given $\theta \in \mathrm{T}^*_g G$. Additionally, for $\varphi \in \mathrm{T}^*_h G$, one has

$$\kappa(\varphi,\theta) = \varphi \circ \mathrm{T}_{hg} R_{g^{-1}} = \theta \circ \mathrm{T}_{hg} L_{h^{-1}}$$

and $i(\mu) = \mu \in \mathrm{T}^*_e G$ with $i(\theta) = \theta \circ \mathrm{T}_e L_g \circ \mathrm{T}_{g^{-1}} R_g \in \mathrm{T}^*_{g^{-1}} G$.

Example 16. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Applying tangent functor to each maps yields a Lie groupoid $\mathrm{T}\mathcal{G} \rightrightarrows \mathrm{T}M$.

2. Morphisms and subgroupoids

Definition 13. Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M'$ be groupoids. A morphism of groupoids $(\mathcal{G} \rightrightarrows M) \to \mathcal{G}' \rightrightarrows M'$ is a pair of maps (F, f) such that the following diagrams commute

$$\begin{tikzcd} \mathcal{G} \arrow[r, "F"] \arrow[d, "s"] & \mathcal{G}' \arrow[d, "s'"] \\ M \arrow[r, "f"] & M' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G} \arrow[r, "f"] \arrow[d, "t"] & \mathcal{G}' \arrow[d, "t'"] \\ M \arrow[r, "f"] & M' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G} \times \mathcal{G} \arrow[r, "F \times F"] \arrow[d, "\kappa"] & \mathcal{G}' \times \mathcal{G} \arrow[d, "\kappa'"] \\ \mathcal{G} \arrow[r, "F"] & \mathcal{G}' \end{tikzcd}$$

. If $M = M'$ and $f = \mathrm{id}_M$, then F is called a base-preserving morphism, or morphism over M. If $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M'$ are Lie groupoids, then (F, f) is a Lie groupoid morphism if F and f are smooth.

Remark 14. The smoothness of F already implies the smoothness of f. Moreover, The above definition implies that $F(h)F(g)$ is well defined whenever hg is.

Proposition 15. Let (F, f) be a groupoid morphism. Then,

(1) $F(1_x) = 1_{f(x)}$ for every $x \in M$,
(2) $F(g^{-1}) = F(g)^{-1}$ for every $g \in \mathcal{G}$.

Definition 16. A morphism (F, f) is an isomorphism of Lie groupoids if F and hence f are diffeomorphisms.

Example 17. Let $\mathcal{G} \rightrightarrows M$. Then, $\chi := (t, s)\colon \mathcal{G} \times \mathcal{G} \to M \times M \mid g \mapsto (t(g), s(g))$ is a morphism over M from $\mathcal{G} \rightrightarrows M$ and $M \times M \rightrightarrows M$. The map χ is called the anchor of $\mathcal{G}$.

Example 18. Let $G \lhd \mathfrak{g}^*$, where the action is given by $\mathrm{Ad}^*_{g^{-1}}$. Then, the left-trivialisation of

$$\lambda\colon (g, \nu) \in G \times \mathfrak{g}^* \longmapsto \nu \circ \mathrm{T}_g L_{g^{-1}}\,,$$

is an isomorphism over $\mathfrak{g}^*$ of $G \lhd \mathfrak{g}^*$ and $\mathrm{T}^*G \rightrightarrows \mathfrak{g}^*$, see Example 15.

Example 19. For any $\mathcal{G} \rightrightarrows M$, the tangent bundle projections $p_{\mathcal{G}}\colon \mathrm{T}\mathcal{G} \to \mathcal{G}$ and $p_M\colon \mathrm{T}M \to M$ give rise to a Lie groupoid morphism from $\mathrm{T}\mathcal{G} \rightrightarrows \mathrm{T}M$ to $\mathcal{G} \rightrightarrows M$.

Definition 17. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A Lie subgroupoid of $\mathcal{G} \rightrightarrows M$ is a Lie groupoid $\mathcal{G}' \rightrightarrows M'$ with injective immersions $\iota\colon \mathcal{G}' \to \mathcal{G}$ and $\iota_\circ\colon M' \to M$ such that $(\iota, \iota_\circ)$ is a Lie groupoid morphism. A Lie subgroupoid $\mathcal{G}' \rightrightarrows M'$ of $\mathcal{G} \rightrightarrows M$ is embedded if $\mathcal{G}'$ and M' are embedded sumbanifolds of $\mathcal{G}$ and M, respectively. A Lie subgroupoid $\mathcal{G}' \rightrightarrows M'$ of $\mathcal{G} \rightrightarrows M$ is wide if $M = M'$ and $\iota_\circ = \mathrm{id}_M$.

Example 20. If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid and $N \subset M$ is an open submanifold, then $\mathcal{G}_N^N := s^{-1}(N) \cap t^{-1}(N)$ is clearly a Lie subgroupoid.

Example 21. If $\mathcal{G} \rightrightarrows M$ is a groupoid, the inner (set) subgroupoid is defined as $\mathcal{IG} := \{g \in \mathcal{G}\colon s(g) = t(g)\}$. In general, if $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, this does not define a Lie subgroupoid.

Lecture 3– Rubén Izquierdo-López (ICMAT)

Date: 21 October, 2025

Abstract: In this lecture, I will present locally trivial Lie groupoids, and prove their ”correspondence” with gauge groupoids. The lecture essentially follows section 1.3 from Mackenzie's book “General Theory of Lie groupoids and Lie algebroids”

3. Locally trivial Lie groupoids

Before introducing the definition of locally trivial Lie groupoid, let us recall the notion of transitive groupoid.

Definition 18. Let $\mathcal{G} \rightrightarrows M$ be a groupoid (not necessarily Lie groupoid). Then, it is called **transitive** if the map $(t, s)\colon \mathcal{G} \to M \times M$ is surjective.

Hence, a groupoid $\mathcal{G} \rightrightarrows M$ is transitive if for every pair of points $x, y \in M$ there is an element $g \in \mathcal{G}$ such that $s(g) = x$ and $t(g) = y$:

$$\bullet^{y} \xleftarrow{\;g\;} \bullet^{x}$$

Notice that if a **set** groupoid $\mathcal{G} \rightrightarrows M$ is transitive, then it is trivial. Indeed, fix $x \in M$ and choose a right inverse of the restriction of the target map to the s-fiber of x:

$$\alpha\colon M \to \mathcal{G}_x\,, \quad t \circ \alpha = \mathrm{id}_M\ .$$

Then, we can build the following isomorphism of *set* groupoids:

$$\Phi\colon M \times \mathcal{G}_x^x \times M \to \mathcal{G}\,, \quad \Phi(z, g, y) := s(z) \cdot g \cdot s(y)^{-1}\,. \tag{6}$$

$$\begin{tikzcd}
\bullet^{z} & & \bullet^{y} \arrow[ll, dashed, bend right, "s(z)\cdot g\cdot s(y)^{-1}"'] \\
& \bullet^{x} \arrow[ul, bend left, "s(z)"] \arrow[ur, bend right, "s(y)"'] \arrow[loop above, "g"] &
\end{tikzcd}$$

What we are going to study now is how this picture translates into the theory of Lie groupoids. The adapted notion of transitivity to the smooth category is the following:

Definition 19. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. It is called **locally trivial** if the map $(t, s)\colon \mathcal{G} \to M \times M$ defines a surjective submersion.

Our first objective is to show that the isomorphism defined in Eq.(6) is well defined *locally*. In order to achieve this, we need to prove the following characterizations of Definition 19:

Proposition 20. Let $\mathcal{G} \rightrightarrows M$ denote a Lie groupoid and $x \in M$ be a fixed point. Then, the following statements are equivalent:

(1) The Lie groupoid $\mathcal{G} \rightrightarrows M$ is locally trivial.
(2) The restriction of the target map $t\colon \mathcal{G}_x \to M$ defines a surjective submersion.
(3) The map
$$\delta\colon \mathcal{G}_x \times \mathcal{G}_x \to \mathcal{G}\,, \quad \delta(g,h) := h \cdot g^{-1}\,,$$
is a surjective submersion.

Proof. Let us prove the equivalence of the first and last pair of conditions:

- $\boxed{(1) \iff (2)}$ (2) clearly follows from (1). Now, to prove the converse, notice that we may write $(t,s)\circ\delta = (t,t)$. Since, by hypothesis, the map on the left is a surjective submersion, it follows from the chain rule that (t,s) must be a surjective submersion as well, so that the Lie groupoid is locally trivial.
- $\boxed{(2) \iff (3)}$ Assume (2) and define $\mathcal{G}_x \times_M \mathcal{G} = \{(\eta,\xi) \in \mathcal{G}_x \times \mathcal{G} : t(\eta) = t(\xi)\}$, together with the following map:
$$\Psi\colon \mathcal{G}_x \times \mathcal{G}_x \to \mathcal{G}_x \times_M \mathcal{G}\,, \quad \Psi(h,g) := (h, gh^{-1})\,.$$
Using the fact that $t\colon \mathcal{G}_x \to M$ defines a surjective submersion, we have that $\mathcal{G}_x \times_M \mathcal{G}$ is a smooth manifold, and that Ψ defines a diffeomorphism. Hence, to show that δ is a surjective submersion, we only need to show that $\psi^{-1}\circ\delta$ is. A quick computation shows that $\psi^{-1}\circ\delta$ is the projection onto the second factor, which is clearly a surjective submersion. For the converse, it is enough to write $t\circ\delta = t\circ\pi$, where $\pi\colon \mathcal{G}_x \times_M \mathcal{G} \to \mathcal{G}_x$ denotes the projection onto the second factor. Now, the map on the left is clearly a surjective submersion, so $t\colon \mathcal{G}_x \to M$ is a surjective submersion as well.

□

Now we can show that a locally trivial Lie groupoid $\mathcal{G} \rightrightarrows M$ is, in fact, locally trivial. Indeed, choose $x \in M$ and let $\{U_i\}$ be an open cover of M, together with sections of $t\colon \mathcal{G}_x \to M$
$$s_i\colon U_i \to \mathcal{G}_x\,.$$
Now, if we define maps $\Phi\colon U_i \times \mathcal{G}_x^x \times U_i \to \mathcal{G}_{U_i}^{U_i}$ as in Eq.(6) (where the space in the right denotes the restriction groupoid), we have an isomorphism of Lie groupoids, so it is, in fact, locally trivial. The maps s_i are called **decomposing sections** and the collection $\{s_i\}$ is called **decomposing atlas**.

Let us give two examples of locally trivial Lie groupoids:

Example 22. Let $E \to M$ be a vector bundle. Then, the frame groupoid $\Phi(E)$, defined as the space of linear isomorphisms $E_x \to E_y$ is locally trivial.

Example 23. The gauge groupoid of a principal bundle is locally trivial.

In fact, we claim that every locally trivial groupoid is the gauge groupoid of a principal bundle. Indeed, by our previous discussion, we know that for fixed $x \in M$, the map
$$t\colon \mathcal{G}_x \to M$$

is a fiber bundle with standard fiber $\mathcal{G}_x^x$ (this can be seen taking a decomposing atlas). Furthermore, $\mathcal{G}_x^x$ acts on the right by multiplication:

$$\Phi\colon \mathcal{G}_x \times \mathcal{G}_x^x \to \mathcal{G}_x\,, \quad \Phi(h,g) := h \cdot g\,.$$

This action is free and defines a principal $\mathcal{G}_x^x$-bundle structure on $t\colon \mathcal{G}_x \to M$. We may wonder how does this construction depend on the choice of point $x \in M$. The relation is given as follows:

Proposition 21. Let $\mathcal{G} \rightrightarrows M$ be a locally trivial Lie groupoid and let $g \in \mathcal{G}$. Denote $x = s(g)$ and $y = t(g)$. Then, we have the following diffeomorphism

$$\Psi\colon \mathcal{G}_y \to \mathcal{G}_x\,, \quad \Phi(h) := h \cdot g$$

which, together with the "inner" Lie group isomorphism

$$\varphi\colon \mathcal{G}_y^y \to \mathcal{G}_x^x\,, \quad \varphi(k) = g^{-1} \cdot k \cdot g$$

defines an isomorphism of principal bundles over the identity on M.

Proof. Clearly, Ψ defines a diffeomorphism, and φ defines an isomorphism of Lie groups. To check that these two maps define an isomorphism of principal bundles, we need to show that Ψ is equivariant with respect to the action. Indeed, let $h \in \mathcal{G}_y$ and $k \in \mathcal{G}_y^y$. Then,

$$\Psi(h \cdot k) = h \cdot k \cdot g = (h \cdot g) \cdot (g^{-1} \cdot k \cdot g) = \Psi(h) \cdot \varphi(k)\,,$$

which completes the proof. $\square$

Summarizing, to each principal bundle, we can associated its gauge groupoid, which is locally trivial; and to each locally trivial Lie groupoid we may associate a principal bundle (after making a choice $x \in M$). These operations are inverses of each other. However, the isomorphisms depend upon choices, and they cannot be chosen naturally. Nevertheless, the isomorphisms depend functorially:

Proposition 22. Let M be a manifold, $\pi\colon P \to M$ be a principal bundle over M, and $\mathcal{G} \rightrightarrows M$ be a locally trivial Lie groupoid.

(1) Given $u_0 \in P$ and denoting $x_0 := \pi(u_0)$, the map

$$P \to ((P \times P)/G)_{x_0}\,, \quad u \mapsto [(u, u_0)]\,,$$

is a diffeomorphism, the map

$$G \to ((P \times P)/G)_{x_0}^{x_0}\,, \quad g \mapsto [(u_0 \cdot g, u_0)]\,,$$

is an isomorphism of Lie groups, and together they define an isomorphism of principal bundles over M. Furthermore, given a morphism of principal bundles $f\colon P \to P'$, where P' is a principal bundle over M', denoting $u_0' := f(u_0)$ and $x_0' := \pi'(u_0')$, we have a commutative diagram:

$$\begin{tikzcd} ((P \times P)/G)_{x_0} \arrow[r] & ((P' \times P')/G')_{x_0'} \\ P \arrow[u] \arrow[r] & P' \arrow[u] \end{tikzcd},$$

where the arrows are defined in the obvious manner.

(2) Given $x \in M$, the map
$$(\mathcal{G}_x \times \mathcal{G}_x)/\mathcal{G}_x^x \to \mathcal{G}\,, \quad [(g,h)] \mapsto gh^{-1}\,,$$
defines an isomorphism of Lie groupoids. Furthermore, given a morphism of Lie groupoids $\varphi\colon \mathcal{G} \to \mathcal{G}'$, where $\mathcal{G}'$ is a locally trivial Lie groupoid over M', if we define $x' := \varphi(x)$, we get a commutative diagram
$$\begin{tikzcd}
(\mathcal{G}_x \times \mathcal{G}_x)/\mathcal{G}_x^x \arrow[r] \arrow[d] & ({\mathcal{G}'}_{x'} \times {\mathcal{G}'}_{x'})/\mathcal{G}_{x'}^{x'} \arrow[d] \\
\mathcal{G} \arrow[r] & \mathcal{G}'
\end{tikzcd},$$
where the arrows are defined in the obvious manner.

Proof. The proof follows from a straight-forward computation. □

An important consequence of Proposition 22 is the following:

Corollary 23. Let $\mathcal{G} \rightrightarrows M$ denote a locally trivial Lie groupoid and $\mathcal{G}' \rightrightarrows M'$ denote a Lie groupoid (not necessarily locally trivial). Let $\varphi\colon \mathcal{G} \to \mathcal{G}'$ be a set morphism of groupoids. Then,

(1) Fix $x \in M$. If $\varphi\colon \mathcal{G}_x \to \mathcal{G}_{\varphi(x)}$ is smooth, so is $\varphi\colon \mathcal{G} \to \mathcal{G}'$.
(2) If φ is smooth in a neighborhood of the identity, φ is smooth.

Proof. (1) follows straight-forwardly using Proposition 22 (2). (2) follows by checking (1) for a particular $x \in M$, using right translations. □

As we mentioned already, the correspondence should not be thought of as an equivalence, given that it is not natural. However, we may use constructions from the world of principal bundles and carry them over to the world of locally trivial Lie groupoids. The most important of these constructions being the associated bundles. Given a principal G-bundle $\pi\colon P \to M$, and a left action of G on a manifold F, we can build a fiber bundle $P[F] \to M$ with standard fiber F by taking the quotient $P[F] := (P \times F)/G$, where $(p,f) \sim (p\cdot g, g^{-1}\cdot f)$. When G preserves a particular structure on F (group, vector space, hermitian...) the induced bundle $P[F]$ inherits this structure fiber wise. For instance if G acts on itself by conjugation, $P[G]$ is a Lie group bundle. When P is the principal bundle obtained from a locally trivial Lie groupoid $\mathcal{G}_x \to M$, the canonical Lie group bundle may be thought of as the inner groupoid:

Proposition 24. Let $\mathcal{G} \rightrightarrows M$ be a locally trivial Lie groupoid, $x \in M$, and $\mathcal{G}_x[\mathcal{G}_x^x]$ denote the canonical Lie group bundle. Then, the map
$$\mathcal{G}_x[\mathcal{G}_x^x] \to \mathcal{G}\,, \quad [(g,h)] \to g\cdot h\cdot g^{-1}$$
takes values in the inner subgroupoid $\mathcal{IG} \subset \mathcal{G}$.

Proof. The proof follows from a straight-forward computation. □

Lecture 4 – Asier López-Gordón (IMPAN)

Date: 4 November, 2025

4. Bisections

The main reference for this lecture are [**1**, Section 1.4] and [**11, 12**].

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with source $s\colon \mathcal{G} \to M$, target $t\colon \mathcal{G} \to M$, unit $u : M \to \mathcal{G} \mid x \mapsto 1_x$ and inverse $i : \mathcal{G} \to \mathcal{G} \mid g \longmapsto g^{-1}$.

On a group $G \ni g$, left-translations $L_g\colon h \mapsto gh$ and inner automorphisms $I_g\colon h \mapsto ghg^{-1}$ each form a group isomorphic to G itself, while right-tranlations $R_g\colon h \mapsto hg$ constitute a group anti-isomorphic to G.

On a groupoid, on the other hand, left-tranlations and right-tranlations only are defined on fibers of the target and source projections, respectively:

$$
\begin{aligned}
L_g &\colon \mathcal{G}^x \ni h \mapsto gh \in \mathcal{G}^y\,,\\
R_g &\colon \mathcal{G}_y \ni h \mapsto hg \in \mathcal{G}_x\,,
\end{aligned}
$$

where $g \in \mathcal{G}$ with $s(g) = x$ and $t(g) = y$. In order to extend these concepts to maps of the whole groupoid, we introduce the notion of bisections.

Definition 25. A *bisection* is a smooth section $\sigma\colon M \to \mathcal{G}$ of s (i.e., $s \circ \sigma = \mathrm{Id}_M$) such that $t \circ \sigma\colon M \to M$ is a diffeomorphism. The set of bisections of $\mathcal{G}$ is denoted by $\mathsf{Bis}(\mathcal{G})$.

In other words, a bisection is a smooth map $\sigma\colon M \to \mathcal{G}$ making the following diagram commutative:

$$
\begin{tikzcd}
& \mathcal{G} \arrow[dl, "t"'] \arrow[dr, "s"] & \\
M \arrow[rr, "t\circ\sigma", "\sim"'] & & M \arrow[ul, bend right, "\sigma"']
\end{tikzcd}
$$

where all the maps are smooth.

Proposition 26. The set of bisections $\mathsf{Bis}(\mathcal{G})$ is canonically endowed with a group structure, with multiplication

$$
\begin{gathered}
\mathsf{Bis}(\mathcal{G}) \times \mathsf{Bis}(\mathcal{G}) \ni (\sigma, \tau) \mapsto \sigma \star \tau \in \mathsf{Bis}(\mathcal{G})\\
\sigma \star \tau(x) \coloneqq \sigma\Big((t \circ \tau)(x)\Big)\tau(x)\,, \quad \forall\, x \in M\,.
\end{gathered}
\tag{7}
$$

The neutral element is just the unit map of the groupoid, while the inverse element σ^{-1} of σ is determined by the inverse map of the groupoid, namely,

$$\sigma^{-1}(x) \coloneqq i\Big(\sigma\Big((t\circ\sigma)^{-1}(x)\Big)\Big) = \Big(\sigma\Big((t\circ\sigma)^{-1}(x)\Big)\Big)^{-1}.$$

$$\begin{tikzcd}
z = t\circ\sigma(y) \arrow[r, bend right, "\sigma^{-1}(z)"'] & y = t\circ\tau(x) \arrow[l, "\sigma(y)"'] \arrow[r, bend right, "\tau^{-1}(y)"'] & x = s\circ\tau(x) \arrow[l, "\tau(x)"'] \arrow[ll, bend right, "\sigma\star\tau(x)"']
\end{tikzcd}$$

Proof. For any $\sigma, \tau \in \mathsf{Bis}(\mathcal{G})$, we have that $s(\sigma\star\tau) = s(\tau) = \mathrm{Id}_M$ and $t(\sigma\star\tau) = t(\sigma)$, which is a diffeomorphism. Thus, $\sigma\star\tau$ is also a bisection. The fact that $u\colon M \ni x \mapsto 1_x \in \mathcal{G}$ is the identity element is obvious. Finally, for any $x \in M$, we can write

$$\sigma^{-1}\Big((t\circ\sigma)(x)\Big) = \Big(\sigma(x)\Big)^{-1},$$

and thereupon

$$\sigma^{-1}\star\sigma(x) = \Big(\sigma(x)\Big)^{-1}\sigma(x) = 1_x\,.$$

Moreover,

$$\sigma\star\sigma^{-1}(x) = \sigma\Big((t\circ\sigma^{-1})(x)\Big)\sigma^{-1}(x) = \sigma\Big((t\circ\sigma^{-1})(x)\Big)\Big(\sigma\Big((t\circ\sigma)^{-1}(x)\Big)\Big)^{-1}$$

but

$$t\circ\sigma^{-1}(x) = s\circ\Big(\sigma\Big((t\circ\sigma)^{-1}(x)\Big)\Big) = (t\circ\sigma)^{-1}(x)\,,$$

and thus

$$\sigma\star\sigma^{-1}(x) = 1_x\,.$$

□

Remark 27. The map

$$\mathsf{Bis}(\mathcal{G}) \ni \sigma \mapsto t\circ\sigma \in \mathrm{Diff}(M)$$

is a group homomorphism. Furthermore, any morphism of Lie groupoids over M

$$f\colon\ (\mathcal{G} \rightrightarrows M) \to (\mathcal{H} \rightrightarrows M)$$

induces a morphism

$$\begin{aligned}\mathsf{Bis}(f)\colon \mathsf{Bis}(\mathcal{G}) &\to \mathsf{Bis}(\mathcal{H})\\ \sigma &\mapsto f\circ\sigma\,.\end{aligned}$$

Hence, we can regard Bis as a functor from the category of Lie groupoids over M to the category of groups:

$$\mathsf{Bis}\colon \mathsf{LieGroupoids}_M \to \mathsf{Groups}\,.$$

In particular, isomorphic Lie groupoids have isomorphic groups of bisections.

Remark 28. In principle, $\mathsf{Bis}(\mathcal{G})$ is only a group. Under certain assumptions on $\mathcal{G}$ and M, it is possible to endow it with a structure of Lie group (see [**11**] and references therein).

Example 24 (Lie group)**.** Given a Lie group G, we have the Lie groupoid $G \rightrightarrows \{*\}$. Then, $\mathsf{Bis}(G) \cong G$.

Example 25 (Manifold)**.** Given a manifold M, we can define the Lie groupoid $M \rightrightarrows M$ with $s = t = \mathrm{Id}_M$, whose group of bisections is $\mathsf{Bis}(M) = \{\mathrm{Id}_M\}$.

Example 26 (Pair groupoid)**.** For $M \times M \to M$, any bisection is of the form

$$\sigma \colon M \times M \to M\,, \quad \sigma(x) = \Big(\varphi(x), x\Big)\,,$$

with $\varphi \in \mathrm{Diff}(M)$. Hence,

$$\mathsf{Bis}(M \times M) \cong \mathrm{Diff}(M)\,.$$

Example 27 (Trivial groupoid)**.** Consider the Lie groupoid $M \times G \times M \rightrightarrows M$, for a Lie group G and a manifold M, with $s(y, g, x) = x$, $t(y, g, x) = y$, $u(x) = (x, e, x)$, $i(y, g, x) = (x, g^{-1}, y)$, and

$$\kappa((z, h, y), (y, g, x)) = (y, hg, x)\,.$$

Any bisection $\sigma \colon M \to M \times G \times M$ can be written as

$$\sigma(x) = \Big(\varphi(x), \theta(x), x\Big)\,,$$

where $\varphi \in \mathrm{Diff}(M)$ and $\theta \colon M \to G$ is a smooth map. The multiplication is given by

$$\sigma_1 \star \sigma_2(x) = \Big(\varphi_1 \circ \varphi_2(x), \Big(\theta_1 \circ \varphi_2(x)\Big)\theta_2(x), x\Big)\,, \quad \forall\, x \in M\,,$$

for any pair of bisections $\sigma_i = (\varphi_i, \theta_i, \mathrm{Id}_M) \in \mathsf{Bis}(M \times G \times M)$, $i = 1, 2$. The inverse is given by

$$\sigma^{-1}(x) = \Big(\varphi^{-1}(x), \theta^{-1}\Big(\varphi^{-1}(x)\Big), x\Big)\,, \quad \forall\, x \in M\,.$$

Example 28 (Action groupoid)**.** Consider an action Lie groupoid $G \lessdot M$:

$$\begin{tikzcd}
z = hgx \arrow[r, bend right, "(h^{-1},z)"'] & y = gx \arrow[l, "(h,gx)"'] \arrow[r, bend right, "(g^{-1},y)"'] & x \arrow[l, "(g,x)"'] \arrow[ll, bend right, "(hg,x)"']
\end{tikzcd}$$

A bisection of G is a smooth map

$$\sigma \colon M \to G \times M\,, \quad \sigma(x) = \Big(\rho(x), x\Big)$$

such that $M \ni x \mapsto t \circ \sigma(x) = \rho(x) \cdot x \in M$ is a diffeomorphism.

The Lie group action of G on M is transitive (i.e., for any $x, y \in M$, there exists a $g \in G$ such that $y = gx$) if and only if for each $\varphi \in \mathrm{Diff}(M)$ there exists a bisection σ such that

$$\varphi(x) = t \circ \sigma(x) = \rho(x) \cdot x\,, \quad \forall x \in M\,.$$

Example 29 (Frame groupoid)**.** Let (E, q, M) be a vector bundle. Let us recall that the set $\Phi(E)$ of all vector space isomorphisms

$$\xi_x^y \colon E_x \to E_y\,, \quad x, y \in M\,,$$

can be equipped with the structure of a Lie groupoid, the so-called frame groupoid $\Phi(E) \rightrightarrows M$.

$$\begin{tikzcd}
z = t(\chi^z_y) \arrow[r, bend right, "(\chi^z_y)^{-1}"'] & y = t(\xi^y_x) \arrow[l, "\chi^z_y"'] \arrow[r, bend right, "(\xi^y_x)^{-1}"'] & x = s(\xi^y_x) \arrow[l, "\xi^y_x"'] \arrow[ll, bend right, "\chi^z_y \cdot \xi^y_x"']
\end{tikzcd}$$

There is a one-to-one correspondence between bisections of $\Phi(E)$ and VB automorphisms of the VB (E, q, M). Indeed, if $\sigma \in \mathsf{Bis}(\Phi(E))$, we can define $\overline{\sigma}\colon E \to E$ by

$$\overline{\sigma}\colon E_x \ni v_x \mapsto \sigma(x)(v_x) = \xi^y_x(v_x) \in E_y\,.$$

Then,

$$\overline{\sigma \star \tau}(v_x) = \sigma \star \tau(x)(v_x) = \xi^z_y \cdot \chi^y_x(v_x) = \overline{\sigma} \circ \overline{\tau}(v_x)\,, \quad \forall v_x \in E_x\,, \quad \forall x \in M\,, \tag{8}$$

that is, the multiplication of bisections corresponds to the composition of automorphisms. The identity bisection (i.e., the unit map of the groupoid) corresponds to the identity automorphism:

$$\overline{u} = \mathrm{Id}_E\ .$$

Therefore,

$$\overline{\sigma^{-1}} = (\overline{\sigma})^{-1}\ .$$

Conversely, given a VB automorphism

$$\begin{tikzcd}
E \arrow[r, "\varphi", "\sim"'] \arrow[d, "q"] & E \arrow[d, "q"] \\
M \arrow[r, "\varphi_0", "\sim"'] & M
\end{tikzcd}$$

we can define a bisection σ by

$$\sigma(x) = \varphi_x\colon E_x \to E_{\varphi(x)}\,, \quad \forall \in M\,.$$

The local triviality of the VB guarantees the smoothness of σ.

Each VB automorphism $\overline{\sigma}\colon E \to E$, and thus each bisection $\sigma \in \mathsf{Bis}(\Phi(E))$, induces a map of sections of E:

$$\tilde{\sigma}\colon \Gamma(E) \to \Gamma(E)\,,$$

given by

$$\tilde{\sigma}(\mu)\Big(t \circ \sigma(x)\Big) = \overline{\sigma}_x\Big(\mu(x)\Big) = \sigma(x)\mu(x)\,, \quad \forall \mu \in \Gamma(E)\,, \quad \forall x \in M\,,$$

that is,

$$\tilde{\sigma}(\mu)(x) = \sigma\Big((t \circ \sigma)^{-1}(x)\Big)\mu\Big((t \circ \sigma)^{-1}(x)\Big)\,.$$

It also induces a map of smooth functions

$$\begin{aligned}\hat{\sigma}\colon \mathscr{C}^\infty(M) &\to \mathscr{C}^\infty(M)\\ f &\mapsto f \circ (t \circ \sigma)^{-1}\,.\end{aligned}$$

The map of sections inherits the $\mathbb{R}$-linerity from the associated VB automorphism. In particular,

$$\tilde{\sigma}(\mu_1 + \mu_2) = \tilde{\sigma}(\mu_1) + \tilde{\sigma}(\mu_2)\,, \quad \forall\, \mu_1, \mu_2 \in \Gamma(E)\,.$$

By equation (8), the multiplication of bisections corresponds to the composition of the associated maps of sections:

$$\widetilde{\sigma \star \mu} = \tilde{\sigma} \circ \tilde{\mu} .$$

Moreover,

$$\begin{aligned}\tilde{\sigma}(f\mu)(x) &= \sigma\Big((t\circ\sigma)^{-1}(x)\Big) f\Big((t\circ\sigma)^{-1}(x)\Big)\mu\Big((t\circ\sigma)^{-1}(x)\Big)\\ &= \hat{\sigma}(f)(x)\,\tilde{\sigma}(\mu)(x)\,, \quad \forall\,\mu\in\Gamma(E)\,, \quad \forall f\in\mathscr{C}^\infty(M) \quad \forall x\in M\,,\end{aligned}$$

i.e.

$$\tilde{\sigma}(f\mu) = \hat{\sigma}(f)\,\tilde{\sigma}(\mu)\,.$$

In particular, the map $\tilde{\sigma}\colon \Gamma(E)\to\Gamma(E)$ is not an automorphism of the $\mathscr{C}^\infty(M)$-module $\Gamma(E)$.

Notation 29. The maps $\overline{\sigma}, \hat{\sigma}$ and $\tilde{\sigma}$ are all denoted by $\overline{\sigma}$ in Mackenzie's book.

Example 30 (Gauge groupoid)**.** In the previous lecture, we saw an isomorphism of principal bundles over M relating (P, M, G, π) and the gauge groupoid (also known as Atiyah groupoid) $\frac{P\times P}{G} \rightrightarrows M$ (see Proposition 22). In particular, given $u_0\in P$ with $x_0 \coloneqq \pi(u_0)$, the map

$$P \ni u \mapsto \Big[(u,u_0)\Big] \in \frac{P\times P}{G}\bigg|_{x_0}$$

is a diffeomorphism. Let $\varphi\colon P\to P$ be a principal bundle automorphism, such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\varphi_0} & M \end{array}$$

is commutative, and

$$\varphi(ug) = \varphi(u)g\,.$$

These type of principal bundle automorphisms are called *gauge transformations*. Composing the gauge transformation φ with the principal bundle isomorphism above, we can define an inner automorphism

$$\begin{aligned}\frac{P\times P}{G} &\to \frac{P\times P}{G}\\ [v,u] &\mapsto \Big[\varphi(v),\varphi(u)\Big]\,.\end{aligned}$$

Its associated bisection $\sigma\in\mathsf{Bis}((P\times P)/G)$ is given by

$$\begin{aligned}\sigma\colon M &\to \frac{P\times P}{G}\\ x=\pi(u) &\mapsto \Big[\varphi(u),u\Big]\,.\end{aligned}$$

Moreover, a locally trivial Lie groupoid $\mathcal{G}\rightrightarrows M$ and the gauge groupoid of its vertex bundle at any $x_0\in M$ are isomorphic Lie groupoids over M

$$\frac{\mathcal{G}_{x_0}\times\mathcal{G}_{x_0}}{\mathcal{G}_{x_0}^{x_0}} \xrightarrow[\sim]{} \mathcal{G}\,, \quad [(g,h)]\mapsto gh^{-1}\,,$$

Therefore, their corresponding groups of bisections are isomorphic:

$$\mathsf{Bis}\,(\mathcal{G}) \cong \mathsf{Bis}\left(\frac{\mathcal{G}_{x_0} \times \mathcal{G}_{x_0}}{\mathcal{G}_{x_0}^{x_0}}\right) .$$

Hence, each bisection $\sigma \in \mathsf{Bis}(\mathcal{G})$ can be written as

$$\sigma \colon x = t(g) \mapsto \varphi(g) g^{-1} , \quad \forall g \in \mathcal{G}_{x_0} ,$$

where $\varphi \colon \mathcal{G}_{x_0} \to \mathcal{G}_{x_0}$ is an automorphism of the principal bundle $(\mathcal{G}_{x_0}, M, \mathcal{G}_{x_0}^{x_0}, t)$.

Definition 30. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A bisection $\sigma \in \mathsf{Bis}(\mathcal{G})$ defines three maps $L_\sigma, R_\sigma, I_\sigma \colon \mathcal{G} \to \mathcal{G}$ as follows:

- The *left-translation*
$$L_\sigma(g) = \sigma\Big(t(g)\Big) \cdot g \,. \tag{9}$$
- The *right-translation*
$$R_\sigma(g) = g \cdot \sigma\Big((t \circ \sigma)^{-1}\Big(s(g)\Big)\Big) \,. \tag{10}$$
- The *inner automorphism*
$$I_\sigma(g) = \sigma\Big(t(g)\Big) \cdot g \cdot \Big(\sigma\Big(s(g)\Big)\Big)^{-1} \,. \tag{11}$$

The following properties can be proven straightforwardly:

$$\begin{aligned} L_{\sigma\star\tau} &= L_\sigma \circ L_\tau \,, \\ R_{\sigma\star\tau} &= R_\tau \circ R_\sigma \,, \\ I_{\sigma\star\tau} &= I_\sigma \circ I_\tau \,, \\ I_\sigma &= L_\sigma \circ R_{\sigma^{-1}} = R_{\sigma^{-1}} \circ L_\sigma \,, \end{aligned}$$

for any $\sigma, \tau \in \mathsf{Bis}(\mathcal{G})$. Therefore, the sets of left-translations, right-translations, and inner automorphisms each form a group under composition; the maps $\sigma \mapsto L_\sigma$ and $\sigma \mapsto I_\sigma$ are group isomorphisms, while $\sigma \mapsto R_\sigma$ is an anti-isomorphism.

Definition 31. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and let $U \subseteq M$ be an open subset. A *local bisection* is a smooth map $\sigma \colon U \to \mathcal{G}$ such that $s \circ \sigma = \mathrm{Id}_U$ and $t \circ \sigma U \to t \circ \sigma(U)$ is a diffeomorphism. The set of local bisections of $\mathcal{G}$ on U is denoted by $\mathsf{Bis}_U(\mathcal{G})$.

Each local bisection $\sigma \in \mathsf{Bis}_U(\mathcal{G})$ defines:

- The *local left-translation* $L_\sigma \colon \mathcal{G}^U \to \mathcal{G}^V$, given by (9).
- The *local right-translation* $R_\sigma \colon \mathcal{G}_V \to \mathcal{G}_U$, given by (10).
- The *local inner automorphism* $I_\sigma \colon \mathcal{G}_U^U \to \mathcal{G}_V^V$, given by (11).

Proposition 32. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. For any $g \in \mathcal{G}$, there exists a sufficiently small open subset $U \subseteq M$ and a local bisection $\sigma \in \mathsf{Bis}_U(\mathcal{G})$ such that $g = \sigma(U)$.

PROOF. Let C be a complementary vector subspace to both $\ker \mathsf{T}_g s$ and $\ker \mathsf{T}_g t$ in $\mathsf{T}_g\mathcal{G}$, namely,

$$\mathsf{T}_g\mathcal{G} = \ker \mathsf{T}_g s \oplus C = \ker \mathsf{T}_g t \oplus C\,.$$

Since the source map is a surjective submersion, it has local sections. On a sufficiently small neighbourhood $\tilde{U} \subseteq M$ of $s(g)$, we can choose (e.g. working in local coordinates) a smooth map $\sigma\colon \tilde{U} \to \mathcal{G}$ such that $s \circ \sigma = \mathrm{Id}_{\tilde{U}}$ and

$$\mathsf{T}_{s(g)}\sigma \left(\mathsf{T}_{s(g)}M\right) = C\,.$$

Thus,

$$\mathsf{T}_{s(g)}(t \circ \sigma)\colon V \to \mathsf{T}_{s(g)}(t \circ \sigma)(V)$$

is an isomorphism. Consequently, there exists a sufficiently small neighbourhood $U \subseteq \tilde{U}$ such that $t \circ \sigma|_U$ is a diffeomorphism onto its image. $\square$

Lecture 5 – Juan Manuel López Medel (ICMAT)

Date: 18 November

Abstract: In this session we first record basic facts about local bisections and give an explicit formula for multiplication in the tangent groupoid. We continue with Section 1.5. Components and Transitivity. We show that the identity–component subgroupoid C is generated by symmetric neighbourhoods of the unit section, is open, and yields connected reductions of principal bundles by choosing components of the total space. Finally, orbits are shown to be submanifolds and transitive Lie groupoids are locally trivial.

We first start with two easy corollaries from the last proposition of the previous lecture:

Corollary 33. Let $G \rightrightarrows M$ be a Lie groupoid. For each $x \in M$, $t_x : G_x \to M$ is of constant rank.

Proof. Take $g, h \in G_x$. Then $j = gh^{-1}$ is defined and so there is a local bisection $\sigma \in \mathrm{Bis}_U G$ with $t(h) \in U$ and $\sigma(t(h)) = j$.
Now $L_\sigma : G_x^U \to G_x^V$, where $V = (t \circ \sigma)(U)$, maps h to g and $t_x \circ L_\sigma = (t \circ \sigma) \circ t_x$. Hence the ranks of t_x at g and h are equal. □

Corollary 34. Let G be a Lie groupoid on M. Then for all $x, y \in M$, G_x^y is a closed embedded submanifold of G_x, of G^y and of G. In particular, each vertex group G_x^x is a Lie group.

Proof. Since $G_x^y = t_x^{-1}(y)$ is the preimage of a point under a map of constant rank, it is a closed embedded submanifold of G_x and hence of G. A similar argument applies for $G_x^y \subseteq G^y$. It follows that $G_x^x \times G_x^x$ is a closed embedded submanifold of $G * G$ and so the restriction of the multiplication in G is smooth. □

The set of all L_σ for $\sigma \in \mathrm{Bis}_U G$ and $U \subseteq M$ open is not a pseudogroup on G since it is not closed under restriction. The following result, which will be needed in Chapter 3, shows that this is unimportant.

Proposition 35. Let G be a Lie groupoid on M. Let $\varphi : \mathcal{U} \to \mathcal{V}$ be a diffeomorphism from $\mathcal{U} \subseteq G$ open to $\mathcal{V} \subseteq G$ open, and let $f : U \to V$ be a diffeomorphism from $U = t(\mathcal{U}) \subseteq M$ to $V = t(\mathcal{V}) \subseteq M$, such that $s \circ \varphi = f \circ t$ and $t \circ \varphi = f \circ t$ and $\varphi(gh) = \varphi(g)h$ whenever $(g, h) \in G * G$, $g \in \mathcal{U}$ and $gh \in \mathcal{U}$. Then φ is the restriction to $\mathcal{U}$ of a unique local left–translation $L_\sigma : G^U \to G^V$ where $\sigma \in \mathrm{Bis}_U G$.

Proof. For $x \in U$ choose $g \in \mathcal{U}^x$ and define $\sigma(x) = \varphi(g)g^{-1}$; clearly $\sigma(x)$ is well defined. Since the restriction $t : \mathcal{U} \to U$ is a submersion, σ is smooth and is therefore a local bisection on U with $t \circ \sigma = f$.
That $L_\sigma(g) = \varphi(g)$ for $g \in \mathcal{U}$ is clear, as is the uniqueness. □

Thus any local diffeomorphism $\mathcal{U} \to \mathcal{V}$ which commutes with the $R_h : G_{t(h)} \to G_{s(h)}$ in the sense of last proposition is the restriction of a local left-translation $L_\sigma : G^{t(\mathcal{U})} \to G^{t(\mathcal{V})}$.

In any case, for a general Lie groupoid G we will often treat the set of local bisections as if it were a pseudogroup on M with law of composition $\star$: if $\sigma \in \mathrm{Bis}_U G$ with $(t \circ \sigma)(U) = V$ and $\tau \in \mathrm{Bis}_{V'} G$ with $(t \circ \tau)(V') = W$, then $\tau \star \sigma$ is the local bisection in $(t \circ \sigma)^{-1}(V') \cap V$ defined by

$$(\tau \star \sigma)(x) = \tau((t \circ \sigma)(x))\sigma(x),$$

providing $(t \circ \sigma)^{-1}(V') \cap V$ is not void. We will abusively refer to the set of all local bisections, together with this composition, as the *pseudogroup of local bisections of* G, and will denote it by $\mathrm{Bis}^{\mathrm{loc}}(G)$.
We will often need to consider the effect of base-preserving morphisms on bisections.

Definition 36. Let $\varphi : G \to G'$ be a morphism of Lie groupoids over M. Then the maps $\mathrm{Bis}G \to \mathrm{Bis}G'$, $\sigma \mapsto \varphi \circ \sigma$, and $\mathrm{Bis}^{\mathrm{loc}}G \to \mathrm{Bis}^{\mathrm{loc}}G'$, $\sigma \mapsto \varphi \circ \sigma$, are both denoted by $\tilde{\varphi}$ and called *induced morphisms* (of groups, and abusively pseudogroups, respectively).

The following important formula for the multiplication in a tangent groupoid makes essential use of the concept of bisection.

Theorem 37. Let $G \rightrightarrows M$ be a Lie groupoid, and denote the multiplication by κ. Let $X \in T_g(G)$ and $Y \in T_h(G)$ have $T(s)(X) = T(t)(Y) = w$. Then

$$X \bullet Y = T(\kappa)(X, Y) = T(L_\sigma)(Y) + T(R_\tau)(X) - T(L_\sigma)T(R_\tau)(T(1)(w)),$$

where σ, τ are any (local) bisections of G for which $\sigma(s(g)) = g$ and $\tau(s(h)) = h$.

Proof. First suppose that $\sigma = 1_m$. Then $X - T(1)T(s)(X)$ is defined and is annulled by $T(s)$. So $(X - T(1)T(s)(X)) \bullet 0_h$ is defined and we can write

$$X \bullet Y = (X - T(1)T(s)(X)) \bullet 0_h + T(1)T(t)(Y) \bullet Y.$$

Using (5), and remembering that $T(1)T(t)(Y)$ is an identity in TG, we have

$$X \bullet Y = T(R_h)(X - T(1)T(s)(X)) + Y. \tag{12}$$

Now consider the general case. We have $L_\sigma(1_{s(g)}) = g$; write $Z = T(L_\sigma^{-1})(X)$. Since L_σ commutes with s, we can apply (12) to get

$$Z \bullet Y = T(R_g)(Z - T(1)T(s)(Z)) + Y.$$

R_g is the restriction to $G_{th} \to G_{s(h)}$ of R_τ. Also, $X \bullet Y = T(L_\sigma)(Z) \bullet Y = T(L_\sigma)(Z \bullet Y)$, since $\kappa \circ (L_\sigma \times \mathrm{id}) = L_\sigma \circ \kappa$.
We obtain the equation of the proposition. □

This gives a complete description of the multiplicative structure of TG. In particular, if $X \in T_g G$ and σ is a (local) bisection with $\sigma(s(g)) = g$, then

$$T(\ell)(X) = T(R_\sigma^{-1})T(1)T(s)(X) + T(L_\sigma^{-1})T(1)T(t)(X) - T(R_\sigma^{-1})T(L_\sigma^{-1})(X).$$

5. Components and transitivity

We first extend to groupoids two elementary facts about topological groups:

- the component of the identity is a subgroup and
- that subgroup is generated by any neighbourhood of the identity.

Proposition 38. Let $G \rightrightarrows M$ be a Lie groupoid. For each $x \in M$ let C_x denote the connectedness component of 1_x in G_x. Then $C = C(G) = \cup_{x\in M} C_x$ is a wide Lie subgroupoid of G, called the identity-component subgroupoid of G.

Proof. By definition C contains each $1_x,\ x \in M$, so it is certainly wide. Take $g \in C_x^y$ and $h \in C_y^z$ and consider $hg = R_g(h) \in C_x^z$. Because $R_g : G_y \to G_z$ is a diffeomorphism, it maps components to components; since $g = R_g(1_y) \in R_g(C_y)$ we have $C_x \cap R_g(C_y) \neq \emptyset$ and therefore $C_x = R_g(C_y)$. Hence $hg \in C_x$. So C is closed under multiplication. Taking $g \in C_x^y$ again, we have $1_y \in R_{g^{-1}}(C_x) \cap C_y$ so $R_{g^{-1}}(C_x) = C_y$ and hence $g^{-1} = 1_x g^{-1} \in C_y$, which proves that C is closed under inversion.
Let N be a tubular neighbourhood for 1_M, and regard N as a vector bundle over M with its zero section identified with $1\colon M \to G$. Then $N \subseteq C$. Now the C_x, $x \in M$, are leaves of the foliation defined by $T^sG = \ker T(s)$ and so C is the union of the integral manifolds of T^sG which meet the open set N. Hence C is open and is therefore a Lie subgroupoid of G. $\square$

It is implicit in this proof that the t-fibres C^y are the identity components of the t-fibres G^y of G, and that, for $g \in G_x^y$, the component of G_x containing g is $C_y g$ and the component of G^y containing g is gC_x. Clearly the various components of any one s–fibre need not be diffeomorphic.
If M is connected, then $C = (\cap_x C_x) \cap 1_M$ is connected, since each $C_x \cap 1_M$ is nonvoid. Conversely, if C is connected then $M = t(C)$ is connected.

Proposition 39. Let G be a Lie groupoid on M. Then the identity–component subgroupoid $C = C(G)$ is open in G.

Proof. Let $\varphi\colon \mathbb{R}^p \times \mathbb{R}^q \to \mathcal{U} \subseteq G$ be a distinguished chart for the foliation induced by T^sG, where $\mathcal{U} \cap 1_M \neq \varnothing$ and $\varphi(\{0\} \times \mathbb{R}^q) = \mathcal{U} \cap 1_M$. Then clearly $\mathcal{U} \subseteq C$. Taking the union of such $\mathcal{U}$ we obtain an open neighbourhood of 1_M in G which is contained in C. Now C is the union of those leaves of the foliation which intersect the open neighbourhood and so is itself open. $\square$

Proposition 40. Let Ω be a locally trivial Lie groupoid on a connected base M. Then $C = C(\Omega)$ is also locally trivial.

Proof. Let $\{\sigma_i : U_i \to \Omega_{m_i}\}$ be a section–atlas for Ω. We can assume the U_i are connected, so each $\sigma_i(U_i)$ lies in a single component C_i of Ω_{m_i}. Choose any $m_i \in U_i$ and define $\tau_i : U_i \to C_{t(\sigma_i(m_i))}$ by $\tau_i(x) = \sigma_i(x)\sigma_i(m_i)^{-1}$.
Next consider any $t_x : \Omega_x \to M$. This is a surjective submersion and therefore open; hence $t_x(C_x)$ is open in M. Now the $t_x(C_x)$, as x ranges through M, are either disjoint or equal, and therefore are also closed. Since M is connected, we must have $t_x(C_x) = M$ for all $x \in M$. For each i we can therefore choose a $\xi_i \in C_{m_i}$ with $t\xi_i = t\sigma_i(m_i)$. Then $\{\nu_i : U_i \to C_m\}$ defined by $\nu_i(x) = \tau_i(x)\xi_i$ gives a section-atlas for C. $\square$

Of course it is true in general that if a Lie groupoid has a subgroupoid which is locally trivial, then the ambient groupoid itself must be locally trivial. The construction of the identity component subgroupoid thus has a principal bundle analogue.
Consider a principal bundle $P(M, G, \pi)$ on a connected base. Choose any component P_o of P. Define

$$H = \{g \in G \mid P_o g = P_o\}.$$

It is clear that H is an open subgroup of G. By an argument similar to last proposition, $\pi_o = \pi|_{P_o} : P_o \to M$ is surjective; it remains a surjective submersion since P_o is open in P. Further, $P_o \times H \to P_o$ remains a free action, and its orbits are equal to the fibres of π_o, by an argument similar to that in Proposition 38.

Definition 41. The principal bundle $P_o(M, H, \pi_o)$ is a *connected reduction* of $P(M, G)$.

Example 31. Let P be the space $\mathbb{R} \times \mathbb{Z}$ and let G be the discrete space $\mathbb{Z} \times \mathbb{Z}$ with the semidirect product group structure

$$(m_1, n_1)(m_2, n_2) = (m_1 + m_2,\ (-1)^{m_2} n_1 + n_2).$$

Let G act on P to the right by

$$(x, p)(m, n) = (x + m, (-1)^m p + n)$$

and let $\pi : P \to M = \mathbb{S}^1$ be $\pi(x, p) = e^{2\pi i x}$. It is easy to verify that $P(M, G, \pi)$ is a principle bundle.
The component of P through $(0, 0)$ is $P_o = \mathbb{R} \times \{0\}$. Evidently, $H = \mathbb{Z} \times \{0\}$. The bundle $P_o(M, H)$ can be identified with the covering $\mathbb{R}(\mathbb{S}^1, \mathbb{Z})$.
Note that $P(M, G)$ is the pullback of the universal cover $\mathbb{R}^2(K, G)$ of the Klein bottle K along the map $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \to K = \mathbb{R}^2/G$ induced by $\mathbb{R} \to \mathbb{R}^2,\ x \mapsto (x, 0)$.

This example illustrates that H need not be normal in G.

Proposition 42. Let G be a Lie groupoid on M. Let $\mathcal{U}$ be a symmetric set (that is, $\mathcal{U} \supseteq 1_M$ and $\mathcal{U}^{-1} = \mathcal{U}$) such that each $\mathcal{U}_x$ is open in G_x. Then the subgroupoid H generated by $\mathcal{U}$ has H_x open in G_x for all $x \in M$.

Proof. We prove this by induction.
Since $\mathcal{U}$ is symmetric, H is merely the set of all possible products of elements from $\mathcal{U}$. Choosing $x \in M$, the set of all n-fold products $g_n \cdots g_1$ from $\mathcal{U}$ with $sg_1 = x$ is the union of all $\mathcal{U}_{t(j)} \subseteq G_x$ where j is an $(n-1)$-fold product from $\mathcal{U}$. Since $R_j : G_{t(j)} \to G_x$ is a diffeomorphism, the set of all n–fold products from $\mathcal{U}$ which lie in G_x is open in G_x. Hence H_x is open in G_x. □

Definition 43. A set $\mathcal{U}$ satisfying the conditions of the previous proposition, will be called a symmetric s-neighbourhood of 1_M (or, of the base) in G.

Proposition 44. Let G be a Lie groupoid on M, and let H be a wide Lie subgroupoid of G. If each H_x is open in G_x, $x \in M$, then each H_x is also closed in G_x, $x \in M$.

Proof. The complement $G_x \setminus H_x$ is the union of all $H_{t(g)} g$ as g ranges over $G_x \setminus H_x$. Since $H_{t(g)}$ is open in $G_{t(g)}$ it follows that $H_{t(g)} g$ is open in G_x. □

The following result is now immediate.

Proposition 45. Let G be a Lie groupoid on M, and let $\mathcal{U}$ be a symmetric s-neighbourhood of 1_M in G. Then $\mathcal{U}$ generates the identity-component subgroupoid C of G.

* * * * *

As we know, a groupoid $G \rightrightarrows M$ is transitive if and only if the anchor is surjective. At the other extreme, the following case is often encountered.

Definition 46. A groupoid $G \rightrightarrows M$ is *totally intransitive* if the image of $(t, s) : G \to M \times M$ is the base subgroupoid Δ_M of $M \times M$.

For any $G \rightrightarrows M$, the image of the anchor $(t, s) : G \to M \times M$ is an equivalence relation on M. The equivalence class of $x \in M$, denoted $\mathcal{O}_x = \theta_x(G)$, is the *transitivity component* of G through x, or the *transitivity orbit* of G through x. Note that $\mathcal{O}_x = t_x(G_x) = s^x(G^x)$. Except where confusion is possible, we will generally just write *orbit*.

Example 32. The transitivity components of a fundamental groupoid $\Pi(M)$ are the connectedness components of M, and the transitivity components of an action groupoid $G < M$ are the orbits of the action. For a surjective submersion $q : M \to Q$, the transitivity components of $R(q) \subseteq M \times M$ are the fibres of q.
A Lie Groupoid Bibundle considered as a Lie groupoid is totally intransitive. Groups are both transitive and totally intransitive.

For a set groupoid $G \rightrightarrows M$, each restriction $G^{\mathcal{O}}$ to an orbit $\mathcal{O}$ is a transitive subgroupoid of G, and G can be regarded as the disjoint union of the transitive subgroupoids $G^{\mathcal{O}}$, as $\mathcal{O}$ runs through the orbits. In the case of Lie groupoids, it is clear that the smooth structure cannot be obtained so simply. A good example to keep in mind throughout this section is the cotangent groupoid example 15.

Theorem 47. Let G be a Lie groupoid on M. Then for each $x \in M$, the orbit $\mathcal{O}_x = t_x(G_x)$ is a submanifold of M.

Proof. Denote G_x by P and G_x^x by W. Then, for the same reason as in Corollary 34, the restriction of the groupoid multiplication to $P \times W \to P$ is a smooth action of a Lie group on a manifold. It is easily seen to be proper: if $K, L \subset P$ are compact then $\{ w \in W \mid Kw \cap L \neq \varnothing \}$ is the image under $P \times_{t_x} P \to W$, $(h, g) \mapsto h^{-1}g$, of the closed subset $K \times_{t_x} L = (K \times L) \cap (P \times_{t_x} P)$ of the compact set $K \times L$ and is therefore compact. Since the action is also free, it follows that $\{(gw, g) \mid g \in P,\ w \in W\}$ is a closed embedded submanifold of $P \times P$ and so there is a quotient manifold structure on P/W.
Define $i : P/W \to M$ by $i(gW) = t_x(g)$. Then i is smooth and injective. Since $P \to P/W$ is a submersion, $\mathrm{rk}_g W(i) = \mathrm{rk}_g(t_x)$ for all $g \in P$, and so i is of constant rank with image $\mathcal{O}_x$. Now an injection of constant rank is an immersion. $\square$

In particular, $G_x(\mathcal{O}_x, G_x^x, t_x)$ is a principal bundle. It is clear that, as sets,

$$\frac{G_x \times G_x}{G_x^x} = G_{\mathcal{O}_x}^{\mathcal{O}_x}.$$

Since t_x is of constant rank it follows, as in Proposition 20, that the division map $\delta_x : G_x \times G_x \to G$ is also of constant rank. Hence $\frac{G_x \times G_x}{G_x^x} \to G$ is an injection of constant rank, and therefore an immersion. Thus $G_{\mathcal{O}_x}^{\mathcal{O}_x}$ is a submanifold of G, and we have the following result:

Theorem 48. Let G be a Lie groupoid on M and let $\mathcal{O}$ be an orbit of M. Then there is a manifold structure on $G^{\mathcal{O}}$ with respect to which it is a submanifold of G and a Lie groupoid on $\mathcal{O}$. Further, $G^{\mathcal{O}}$ is locally trivial.

Corollary 49. A transitive Lie groupoid is locally trivial.

For ease of reference we state explicitly the following results, proved above.

Corollary 50. Let G be a Lie groupoid on M, and let $x \in M$. Then $\delta_x : G_x \times G_x \to G$ is of constant rank. Further, G is locally trivial if and only if δ_x is a surjective submersion.

Example 33. Let $G \times M \to M$ be a smooth action of a Lie group on a manifold and let $G < M$ be the action groupoid. Applying Corollary 33 shows that each evaluation map is of constant rank, applying Theorem 47 that the orbits are submanifolds of M, and transitive Lie groupoids are locally trivial, that if G acts transitively, then M is equivariantly diffeomorphic to a homogeneous space. The fact that transitive Lie groupoids are locally trivial, also includes the existence of local cross-sections for closed subgroups of Lie groups.

To avoid misunderstanding, we will refer to 'the orbit foliation of the base' only when the foliation is known to be regular; in general we refer to 'the orbit partition of the base'. We know that any Lie groupoid may be restricted to any open subset of the base, and that a Lie groupoid may be restricted to any of its orbits. The following general result concerning restrictions is easily proved.

Proposition 51. Let $G \rightrightarrows M$ be a Lie groupoid and let $\iota : S \to M$ be a submanifold such that $\iota \times \iota : S \times S \to M \times M$ and $(t, s) : G \to M \times M$ are transversal. Then G_S^S with its submanifold structure is a Lie subgroupoid of G.

A submanifold satisfying the condition is said to be transversal to G.

Lecture 6 – Paula Alba San Miguel (UNED)

Date: 25^{th} November 2025

Abstract: In this section we discuss Lie groupoid actions on smooth manifolds, the morphisms between such actions, and their intrinsic definition. We also apply results from previous sections to the study of actions.

6. Actions

We are interested in defining the action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a manifold M', so it generalizes the action of a Lie group on a manifold. The lecture follows section 1.6 from Mackenzie's book "General Theory of Lie groupoids and Lie algebroids".

In order to make use of the groupoid structure of $\mathcal{G} \rightrightarrows M$ is reasonable to expect that not every element of $\mathcal{G}$ will be able to act on any point of M'. Also, if $(h, g) \in \mathcal{G} * \mathcal{G}$, we will ask h to be able to act on gm' for every m' g can act on, and that $(hg)m' = h(gm')$.

Definition 52 (Action of a Lie groupoid)**.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $f : M' \to M$ be a smooth map. Let $G * M'$ be the pullback manifold:

$$G * M' = \{(g, m') \in G \times M' | s(g) = f(m')\}$$

An *action of* $\mathcal{G}$ on $f : M' \to M$ is a smooth map $\Phi : G * M' \to M'$ that we will note $\Phi(g, m') = gm'$ such that:

(1) $f(gm') = t(g)$, $\forall (g, m') \in G * M'$.
(2) $h(gm') = (hg)m'$, $\forall (h, g) \in \mathcal{G} * \mathcal{G}$ and $m' \in M'$ such that $(g, m'), (h, m') \in G * M'$,
(3) $1_{f(m')}m' = m'$ $\forall m' \in M'$.

For $m' \in M'$, the subset $\mathcal{G}[m'] = \{gm' | g \in \mathcal{G}_{f(m')}\}$ is the orbit of m'.

Remark 53. In case f is a surjective submersion, its fibres, that we will denote M'_m, are closed embedded submanifolds of M' and, for each $g \in \mathcal{G}^n_m$, the map $\Phi|_{\{g\} \times M'_{s(g)}}$ is a diffeomorphism from $\{g\} \times M'_{s(g)}$ to $\{g\} \times M'_{t(g)}$.

If that is not the case, the fibres aren't necessarily manifold, so the action need not be smooth when restricted to the fibers.

Remark 54. Each isotropy group $\mathcal{G}^x_x$ acts on M'_x, and the action is a Lie group action when it is smooth when restricted to the fibres.

Example 34 (Trivial action)**.** For every groupoid $\mathcal{G} \rightrightarrows M$, we can define its action on the trivial bundle $M \times F$ as follows:

$$\mathcal{G} * (M \times F) = \bigsqcup_{m \in s(\mathcal{G})} \mathcal{G}_m \times (\{m\} \times F) \to M \times F$$
$$(g, s(g), a) \mapsto (t(g), a)$$

Example 35 (Gauge groupoid acting on its principal bundle)**.** Given a principal bundle $P(M, G, \pi)$, the groupoid $\frac{P \times P}{G} \rightrightarrows M$ acts on $\pi : P \to M$ as follows:

$$\frac{P \times P}{G} * P \to P$$
$$([p_1, p_2], p) \mapsto p_1$$

For $s([p_1, p_2]) = \pi(p_2) = p$, and then, there exists $g \in G$ such that $[p_1, p_2] = [p_1 g, p_2 g]$ and $p_2 g = p$.

Example 36 (Gauge groupoid acting on an associated fibre bundle)**.** Given a principal bundle $P(M, G, \pi)$, let $\frac{P \times P}{G} \rightrightarrows M$ be its Lie groupoid. Given $\phi : G \times F \to F$ a left action of G on a manifold F, we consider the asociated fiber bundle $\mathtt{p} : \frac{(P \times F)}{G} \to M$. Then, $\frac{P \times P}{G} \rightrightarrows M$ can act on $\frac{(P \times F)}{G}$ through the smooth map $\mathtt{p}$:

$$\frac{P \times P}{G} * \frac{P \times F}{G} \to \frac{P \times F}{G}$$
$$([p_1, p_2], \langle p_2, a \rangle) \mapsto \langle p_1, a \rangle$$

Notice that $s([p_1, p_2]) = \pi(p_2) = \mathtt{p}(\langle p_2, a \rangle)$ and $t([p_1, p_2]) = \pi(p_1) = \mathtt{p}(\langle p_1, a \rangle)$

In order to gain a better understanding of grouopid actions, we will study those maps which, while transforming the manifold on which the action takes place, remain invariant under the action.

Definition 55 (Equivariant map)**.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and let $\mathcal{G} * M^1 \to M^1$ and $\mathcal{G} * M^2 \to M^2$ be actions of $\mathcal{G}$ on $f_i : M^i \to M$, for $i = 1, 2$. Then, a smooth map $\psi : M^1 \to M^2$ such that $f_2 \circ \psi = f_1$ is said to be $\mathcal{G}$ *equivariant* if $\psi(gm') = g\psi(m')$ for all $(g, m') \in \mathcal{G} * M^1$.

$$\begin{tikzcd} M^1 \arrow[rr, "\psi"] \arrow[dr, "f_1"'] & & M^2 \arrow[dl, "f_2"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G} * M^1 \arrow[r, "id \times \psi"] \arrow[d, "\Phi^1"'] & \mathcal{G} * M^2 \arrow[d, "\Phi^2"] \\ M^1 \arrow[r, "\psi"'] & M^2 \end{tikzcd}$$

It is also interesting to study groupoid morphisms that transform the action of one groupoid to the action of the other.

Definition 56 (Equivariant groupoid morphism)**.** If $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ are Lie groupoids and $\mathtt{p} : M' \to M$ and $\mathtt{q} : N' \to N$ are smooth maps, let $\mathcal{G} * M' \to M'$ and $\mathcal{H} * N' \to N'$ be actions on $\mathtt{p}$ and $\mathtt{q}$. Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a morphism of Lie groupoids over $f : M \to N$ and let $\psi : M' \to N'$ be a smooth map such that $\mathtt{q} \circ \psi = f \circ \mathtt{p}$. Then, ψ is said to be φ *equivariant* if $\psi(gm') = \varphi(g)\psi(m')$ for all $(g, m') \in \mathcal{G} * M'$.

$$\begin{tikzcd} M' \arrow[r, "\psi"] \arrow[d, "\mathtt{p}"'] & N' \arrow[d, "\mathtt{q}"] \\ M \arrow[r, "f"] & N \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G} * M' \arrow[r, "\varphi\times\psi"] \arrow[d, "\Phi_{\mathcal{G}}"'] & \mathcal{H} * N' \arrow[d, "\Phi_{\mathcal{H}}"] \\ M' \arrow[r, "\psi"'] & N' \end{tikzcd}$$

On lecture 3, we proved that, if $\mathcal{G} \rightrightarrows M$ is locally trivial, it is isomorphic to $\frac{P\times P}{G} \rightrightarrows M$ with $P = \mathcal{G}_x$ and $G = \mathcal{G}_x^x$ with $x \in M$. Let us call that isomorphism φ.

Theorem 57. Let $\mathcal{G} \rightrightarrows M$ be a locally trivial groupoid, and let $\mathcal{G} * M' \to M'$ be an action of $\mathcal{G}$ on a surjective submersion $\mathtt{q} : M' \to M$. Then, for any $m \in M$ if we call $P = \mathcal{G}_m$, $G = \mathcal{G}_m^m$ and $F = M'_m$, the fibre bundle $(M', \mathtt{q}, M)$ is diffeomorphic to $(\frac{P\times F}{G}, \mathtt{p}, M)$ through a diffeomorphism of fibrations that is equivariant with respect to the isomorphism of groupoids $\varphi : \frac{P\times P}{G} = \frac{\mathcal{G}_m \times \mathcal{G}_m}{\mathcal{G}_m^m} \to \mathcal{G}$ seen in Lecture 3.

That is, all actions of locally trivial groups are of the type of example 36.

Proof. The isomorphism φ is defined as $\varphi([g,h]) = gh^{-1}$, so the map:

$$\psi : \frac{P \times F}{G} \to M'$$
$$\langle g, a\rangle \mapsto ga$$

is equivariant with respect to φ, since $\mathtt{q} \circ \psi(\langle gh, h^{-1}a\rangle) = \mathtt{q}(ga) = t_x(g) = \mathtt{p}(\langle gh, h^{-1}a\rangle)$ and $\psi([g,h]\cdot\langle h,a\rangle) = \psi(\langle g,a\rangle) = ga = (gh^{-1})ha = \varphi([g,h])\psi(\langle h,a\rangle)$.

It only remains to proof that ψ is a diffeomorphism. $\square$

Proposition 58. Let $P(M, \pi, G)$ be a principal bundle and let B and B' be two associated fibre bundles corresponding to actions $G \times F \to F$ and $G \times F' \to F'$ of G on manifolds F and F'.

(1) If $f : F \to F'$ is a G-equivariant map, then $\tilde{f} : B \to B' : \langle u, a\rangle \mapsto \langle u, f(a)\rangle$ is a well defined morphism of fibre bundles over M, and is $\frac{P\times P}{M}$-equivariant.
(2) If $\varphi : B \to B'$ is a $\frac{P\times P}{M}$-equivariant morphism of fibre bundles over M, then $\varphi = \tilde{f}$ for some G-equivariant map f.

Proof. (1) is easy to verify

For (2), if φ is a $\frac{P\times P}{M}$-equivariant morphism of fibre bundles, $\varphi([p_1,p_2]\langle p_2,a\rangle) = [p_1,p_2]\varphi(\langle p_2,a\rangle)$. This implies that $\varphi(\langle p,a\rangle) = \langle p, f(a)\rangle$. Using (1), we have proven the result. $\square$

Example 37 (Inner automorphism action)**.** Given a locally trivial Lie groupoid $\mathcal{G} \rightrightarrows M$, the inner automorphism action is the map:

$$\mathcal{G} * \mathcal{IG} \to \mathcal{IG}$$
$$(g,h) \mapsto ghg^{-1}$$

If $\mathcal{G}$ corresponds to a principal bundle $P(m, G, \pi)$, Theorem 6 shows that $\mathcal{IG}$ is equivariantly isomorphic as a Lie group bundle to the inner group bundle $\frac{P\times G}{G}$

Given a Lie group action $G \times M \to M$, we constructed the action groupoid $G \times M \rightrightarrows M$. Now we are going to construct the action groupoid of an action $\mathcal{G} * M' \to M'$ of the Lie groupoid $\mathcal{G} \rightrightarrows M$ on $f : M' \to M$.

It will be $\mathcal{G} * M' \rightrightarrows M'$ with $s'(g, m') = m'$, $t'(g, m') = gm'$, inclusion $i(m') = (1_{f(m')}, m')$, multiplication $\kappa((h, n'), (g, m')) = (hg, m')$ and inversion $(g, m')^{-1} = (g^{-1}, gm')$.

Definition 59 (Action groupoid)**.** The groupoid we constructed is noted by $\mathcal{G}\measuredangle f$ or $\mathcal{G}\measuredangle M'$ and is the action groupoid associated to the action of $\mathcal{G}$ on $f : M' \to M$

Note that we can defined a canonical morphism beetwen $\mathcal{G} \rightrightarrows M$ and $\mathcal{G} * M' \rightrightarrows M'$:

$$\begin{tikzcd} \mathcal{G} * M' \arrow[r, "\pi_1"] \arrow[d] & \mathcal{G} \arrow[d] \\ M' \arrow[r, "f"] & M' \end{tikzcd}$$

Example 38 (Action groupoid of a Lie groupoid acting on itself)**.** Any groupoid $\mathcal{G} \rightrightarrows M$ can act on itself through the target map as follows:

$$\mathcal{G} * \mathcal{G} \to \mathcal{G}$$
$$(g, h) \mapsto gh$$

And, considering its action groupoid, its anchor defines an isomorphism of Lie groupoids with $\mathcal{G} \times_s \mathcal{G} = \{(g, h) | s(g) = s(h)\}$ the subgroupoid of the pair groupoid.

$$\begin{tikzcd} \mathcal{G} * \mathcal{G} \arrow[rr, "(s',t')"] \arrow[dr, "t',s'"] & & \mathcal{G} \times_s \mathcal{G} \arrow[dl, "\bar{t},\bar{s}"'] \\ & \mathcal{G} & \end{tikzcd}$$

Note that $(\bar{t}, \bar{s})(t', s')(g, h) = (\bar{t}, \bar{s})(gh, h) = (t'(g, h), s'(g, h))$.

In particular, for any Lie group $G \rightrightarrows \{x\}$, the action groupoid $G\measuredangle G$ is isomorphic to the pair groupoid $G \times G \rightrightarrows G$.

Actions groupoids can be characterized intrinsically as follows.

Definition 60 (Action morphism)**.** Let $\varphi : \mathcal{G}' \to \mathcal{G}$ be a morphism of Lie groupoids over $f : M' \to M$, and let φ' be:

$$\varphi' : \mathcal{G}' \to f'\mathcal{G} = \mathcal{G} * M'$$
$$g' \mapsto (\varphi(g'), s'(g'))$$

Note that, since (φ, f) is a groupoid morphism, $s(\varphi(g') = f(s'(g'))$.

φ is said to be an action morphism if φ' is a diffeomorphism.

Remark 61. Given an action $\mathcal{G} * M' \to M'$, the canonical Lie groupoid morphism beetwen $\mathcal{G} * M' \rightrightarrows M'$ and $\mathcal{G} \rightrightarrows M$ is an action morphism.

Since we are calling φ an action groupoid, are there an action associated to it? And, is φ the canonical morphism of it?

Theorem 62. *Let $\varphi : \mathcal{G}' \to \mathcal{G}$ be an action groupoid over a map $f : M' \to M$. Then $t' \circ (\varphi')^{-1} : f'\mathcal{G} \to M'$ is an action of $\mathcal{G}$ on $f : M' \to M$.*

Proof. Since φ' is a diffeomorphism, it maps $\mathcal{G}'_{m'}$ into $\mathcal{G}_{f(m')} \times \{m'\}$, and $\varphi|_{\mathcal{G}'_{m'}} : \mathcal{G}'_{m'} \to \mathcal{G}_{f(m')}$ is a bijection, and $\varphi^{-1}(g) = (\varphi')^{-1}(g, m')$, so:

$$f(gm') = (f \circ t' \circ (\varphi')^{-1})(g, m') = (t \circ \varphi \circ (\varphi')^{-1})(g, m') = t(g)$$

Let us see that $h(gm') = (hg)m'$. If $n' = t' \circ (\varphi')^{-1}(g, m')$, $f(n') = t(g) = s(h)$ and $\varphi^{-1}(h) = (\varphi')^{-1}(h, n')$. Since $s'((\varphi')^{-1}(h, n')) = n' = t'((\varphi')^{-1}(g, m'))$, so:

$$\Big((\varphi')^{-1}(h, n'), (\varphi')^{-1}(g, m')\Big) \in \mathcal{G} * \mathcal{G}$$

And its multiplication is in $\mathcal{G}'_{m'}$, so it is mapped by φ to hg, because it is bijective. $\square$

It is easy to prove that this action and the canonical morphism associated to it are mutual inverses.

Proposition 63.
(1) Let $\varphi : \mathcal{G}' \to \mathcal{G}'$ be an action morphism, and let $\mathcal{G} * M' \to M'$ be the associated action. Then, $\varphi' : \mathcal{G}' \to \mathcal{G} \measuredangle M'$ is an isomorphism of Lie groupoids over M', and $\pi_1 \circ \varphi' = \varphi$.
(2) Let $\mathcal{G} * M' \to M'$ be an action of a Lie groupoid on a smooth map $f : M' \to M$. Then, the action induced by the action morphism $\varphi' : f'\mathcal{G} \to \mathcal{G}$ is the original action.

Lecture 7 – [Tomasz Sobczak (KMMF-UW)]

Date: 2 December

Abstract: TBA.

Lecture 8 – [Juan Manuel López Medel (ICMAT)]

Date: 9 December

Abstract: In the eight lecture we will mainly talk about linear actions, and the relation with representation. We study a proposition that gives us a lot of examples as Riemannian frame groupoid, complex frame groupoid... Lastly, we show how are the linear actions on the fundamental groupoid

7. Linear actions and frame groupoids

Definition 64. Let (E, q, M) be a v.b. and let $\Omega * E \to E$ be an action of a locally trivial Lie groupoid Ω on q. The action is *linear* if for each $\xi \in \Omega$, the diffeomorphism $E_{s(\xi)} \to E_{t(\xi)}$ is a linear isomorphism.

Definition 65. Let Ω be a locally trivial Lie groupoid on M and let (E, q, M) be a v.b. Then a *representation* of Ω on (E, q, M) is a morphism $\rho : \Omega \to \Phi(E)$ of Lie groupoids over M.

Proposition 66. Let $\Omega \rightrightarrows M$ be a l.t.L.g. and let (E, q, M) be a v.b. If $\Omega * E \to E$ is a linear action of Ω on E then the associated map $\Omega \to \Phi(E)$, $\xi \mapsto (u \mapsto \xi u)$ is a representation; if $\rho : \Omega \to \Phi(E)$ is a representation then $(\xi, u) \mapsto \rho(\xi)(u)$ is a linear action.

Proof. One direction follows from nothing that $\Phi(E) * E \to E$ is a linear action. For the converse, suppose we have a linear action of $\Omega * E \to E$. It has to be proved that the set map $\Omega \to \Phi(E)$ is smooth. Take a chart $\psi : U \times V \to E_U$ for E and use it to write the map locally as $\Omega_U * (U \times V) \to U \times V$. It then follows easily that the associated map $\Omega_U \to U \times GL(V) \cong \Phi(E)_U$ is smooth. □

Now we study invariant sections.

Definition 67. Let $\Omega * M' \to M'$ be an action of a l.t.L.g. Ω on a smooth surjection (M', q, M). Then a section $\mu \in \Gamma M'$ is Ω-invariant if $\xi\mu(s(\xi)) = \mu(t(\xi))$, for all $\xi \in \Omega$. The set of Ω-invariant sections of M' is denoted $(\Gamma M')^{\Omega}$.

If $\Omega * E \to E$ is a linear action on a vector bundle, then $(\Gamma E)^{\Omega}$ is an $\mathbb{R}$-vector space with respect to pointwise operations, but not usually a module over the ring of smooth functions on M. A general fibre bundle doesn't have to admit any (global) sections. In particular, a vector bundle equipped with a linear action, $(\Gamma E)^{\Omega}$ may consist of the zero section alone.

Proposition 68. Let $\Omega \rightrightarrows M$ be a locally trivial Lie groupoid and let $\Omega * E \to E$ be a linear action of Ω on a vector bundle (E, q, M). Choose $m \in M$, and write V for E_m and G for Ω_m^m. Then the evaluation map

$$(\Gamma E)^{\Omega} \to V^G, \ \mu \mapsto \mu(m)$$

is an isomorphism of $\mathbb{R}$-vector spaces.

PROOF. The map is clearly injective. Given $v \in V^G$, define μ by $\mu(x) = \xi v$ where ξ is any element of Ω_m^x. Clearly $\mu(x)$ is well defined; μ is smooth because $T_m : \Omega_m \to M$ is a surjective submersion. $\square$

Example 39. Consider the principle bundle $SO(2)(S^1, \mathbb{Z}_2, p)$ where $\mathbb{Z}_2$ is embedded in $SO(2)$ as $\{1, -1\}$ and p is $z \mapsto z^2$. Let Ω be the gauge groupoid and let E be the vector bundle $SO(2) \times \mathbb{R}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $\mathbb{R}$ by multiplication; this is the Möbius band. Then $(\Gamma E)^\Omega \cong \mathbb{R}^{\mathbb{Z}_2}$ is the zero space.

Proposition 69. Let $\Omega * E \to E$ be a linear action of a locally trivial Lie groupoid Ω on a v.b. E. For each $x \in M$, define $E^{\mathcal{I}\Omega}|_x$ to be

$$E_x^{\Omega_x^x} = \{u \in E_x;\ \lambda u = u,\ \forall \lambda \in \Omega_x^x\}.$$

Then $E^{\mathcal{I}\Omega}$ is a subvector bundle of E, and there is a natural trivialization

$$M \times V^G \to E^{\mathcal{I}\Omega}.$$

PROOF. Let $\{\omega_i : U_i \to \Omega_m\}$ be a section-atlas for Ω, and write $V = E_m,\ G = \Omega_m^m$. Define $\psi_i : U_i \times V^G \to E^{\mathcal{I}\Omega}|_{U_i}$ by $(x, v) \mapsto \omega_i(x)(v)$. Then $\psi_{i,x}$ maps V^G isomorphically onto $E^{\mathcal{I}\Omega}|_x$.
To define the trivialization, note that for $x \in M$ and any two $\xi, \xi' \in \Omega_m^x$, the maps $V^G \to E^{\mathcal{I}\Omega}|_x,\ v \mapsto \xi v$ and $v \mapsto \xi' v$ are identical. $\square$

Proposition 70. Let (E, q, M) be a vector bundle and $n \geq 1$.

(i) The action $\Phi(E) * \mathrm{Hom}^n(E; M \times \mathbb{R}) \to \mathrm{Hom}^n(E; M \times \mathbb{R})$ defined by $\xi\varphi = \varphi \circ (\xi^{-1})^n$ is smooth.
(ii) The action $\Phi(E) * \mathrm{Hom}^n(E; E) \to \mathrm{Hom}^n(E; E)$ defined by $\xi\varphi = \xi \circ \varphi \circ (\xi^{-1})^n$ is smooth.
(iii) Let (E', p', M) be another v.b. on the same base. Then the action

$$(\Phi(E) \times_{M \times M} \Phi(E')) * \mathrm{Hom}(E; E') \to \mathrm{Hom}(E; E')$$

defined by $(\xi, \xi')\varphi = \xi' \circ \varphi \circ \xi^{-1}$ is smooth.

In (iii) the product is the product of locally trivial Lie groupoids over a fixed base in the sense of the following result, which is easily proved.

Proposition 71. Let Ω and Ω' be l.t.L.g. over the same base M. Let $\Omega \times_{M \times M} \Omega'$ be the pullback manifold of the two anchors $(t, s) : \Omega \to M \times M$ and $(t', s') : \Omega' \to M \times M$. Then, with the componentwise algebraic structure, $\Omega \times_{M \times M} \Omega'$ is a locally trivial Lie groupoid on M and the two projections $\Omega \times_{M \times M} \Omega' \to \Omega$ and $\Omega \times_{M \times M} \Omega' \to \Omega'$ are surjective submersions.

The Lie groupoid structure on $\Omega \times_{M \times M} \Omega'$ can be defined for weaker local triviality conditions, but the full properties of a product in the categorical sense over the fixed base M will be lost.

Remark 72. $\mathrm{Hom}^n(E; E')$ denotes the vector bundle on M whose fibre over $x \in M$ is the space of n-multilinear maps $E_x \times \cdots \times E_x \to E'_x$ and whose bundle structure is induced from the bundle structure of E and E' in the usual way. The actions (i) and (ii) restrict to the subbundles $\mathrm{Alt}^n(E; E')$ and $\mathrm{Sym}^n(E; E')$ of alternating and symmetric multilinear maps; further, $\mathrm{Hom}(E; E')$ in (iii) could be replaced by $\mathrm{Hom}^n(E; E')$ and the obvious action.

Lastly, there are analogous actions of $\Phi(E)$ on the tensor bundles $\otimes^r_s E$.
Before we start with the examples, we recall a past Theorem:

Theorem 73. Let Ω be a l.t.L.g. on M and let $\Omega * M' \to M'$ be a smooth action of Ω on the fibre bundle (M', q, M). Let $\mu \in \Gamma M'$ be an Ω-deformable section. Then the stabilizer groupoid $\Omega\{\mu\}$ of Ω at μ is a closed embedded reduction of Ω.

Example 40. Let (E, q, M) be a vector bundle, and let $\langle , \rangle$ be a Riemannian structure in E, regarded as a section of $\mathrm{Hom}^2(E; M \times \mathbb{R})$. Then $\langle , \rangle$ is $\Phi(E)$-deformable with respect to the action of (70)(i), since any two vector spaces of the same dimension with any positive-definite inner products, are isometric. Denote the stabilizer groupoid of $\langle , \rangle$ by $\Phi_{\mathcal{O}}(E)$. By Theorem (73), it is a locally trivial Lie groupoid on M, the *Riemannian frame groupoid* or *orthonormal frame groupoid* of $(E, \langle , \rangle)$. A section $\sigma : U \to \Phi_{\mathcal{O}}(E)_m$ of $\Phi_{\mathcal{O}}(E)$ is a moving frame for E, and the local triviality of $\Phi_{\mathcal{O}}(E)$ is equivalent to the existence of moving frames in E.

Example 41. A *complex structure* in a vector bundle (E, q, M) is an endomorphism $J : E \to E$ such that $J^2 = -id$. Such a J has constant rank and so, regarded as a $(1, 1)$ tensor field, J is $\Phi(E)$-deformable with respect to the action of (70)(ii). Denote the stabilizer groupoid of J by $\Phi_{\mathbb{C}}(E)$. By Theorem (73) it is a locally trivial Lie groupoid on M, the *complex frame groupoid* of (E, J).
If E further has a Hermitian metric $\langle , \rangle$ for the given J, then the argument of the previous example applies, and there is a l.t.L.g. $\Phi_U(E) \leq \Phi_{\mathbb{C}}(E)$ of Hermitian frames. $\Phi_U(E)$ is the *Hermitian frame groupoid* or *unitary frame groupoid* of $(E, J, \langle , \rangle)$.

Example 42. Similarly, if Δ is a determinant function in a vector bundle (E, q, M), regarded as a never-zero section of $\mathrm{Alt}^r(E; M \times \mathbb{R})$ where $r = \mathrm{rank} E$, then Δ is $\Phi(E)$-deformable and the groupoid of orientation-preserving isomorphisms between the fibres of E is a closed embedded reduction of $\Phi(E)$, denoted $\Phi^+(E)$.

Example 43. Let (L, q, M) be a v.b. and let $[,]$ be a section of $\mathrm{Alt}^2(L; L)$ such that each

$$[,]_x : L_x \times L_x \to L_x$$

is a Lie algebra bracket. We call such a section a *field of Lie algebra brackets* in L.

A field of Lie algebra brackets need not be $\Phi(L)$-deformable. For example, let $\mathfrak{g}$ be a non-abelian Lie algebra with bracket $[,]$ and in $L = \mathbb{R} \times \mathfrak{g}$ define $[\ ,\ ]_t = t[\ ,\ ]$. However Theorem (73) implies that if $[\ ,\ ]$ is a field of Lie algebra brackets in a vector bundle L and if the fibres of L are pairwise isomorphic as Lie algebras, then L admits an atlas of charts which fibrewise are Lie algebra isomorphisms. In this case, $(L, [\ ,\ ])$ is a Lie algebra bundle and we denote the stabilizer groupoid by $\Phi_{\mathrm{Aut}}(L)$.

Example 44. Let μ be a section of a vector bundle E on a connected base M. Then, by a similar argument, E has an atlas of charts $U \times V \to E_U$ such that the local representatives $U \to V$ of μ are constant, if and only if μ is either never zero or always zero. This result is well known in the case of tangent vector fields.

A similar treatment may be applied to any tensor structure on a vector bundle.
It is worth noting that in the previous two examples, the condition of pairwise isomorphism, or of being never zero or always zero, need hold only on each component of the base separately. A similar comment applies to the next example.

Example 45. Let $(E^\nu, q_\nu, M),\ \nu = 1,2$, be vector bundles on base M, and let $\varphi : E^1 \to E^2$ be a morphism, considered as a section of $\mathrm{Hom}(E^1; E^2)$. Then φ is $\Phi(E^1) \times_{M\times M} \Phi(E^2)$-deformable if and only if it is of constant rank. Now Theorem (73) shows that if this is the case, there are atlases $\{\psi_i^\nu : U_i \times V^\nu \to E^\nu_{U_i}\},\ \nu = 1,2$, and a linear map $F : V^1 \to V^2$ such that each $\varphi : E^1_{U_i} \to E^2_{U_i}$ is represented by $(x,v) \to (x, f(v))$.

This is a vector bundle version of the standard characterization of a subimmersion (though in order to apply to general subimmersions $M \to N$ it must be extended to varying base manifolds). This result will be useful in the abstract theory of transitive Lie algebroids.

Actions of Lie groupoids on bundles where the fibres have algebraic structures of other types can handled in a similar way. We briefly consider the case of actions on Lie group bundles.

Consider an LGB (K, q, M). An action of a locally trivial Lie groupoid $\Omega \rightrightarrows M$ on K is said to be an *action by Lie group isomorhisms* if each isomorphism $\xi : K_x \to K_y,\ \xi \in \Omega$, is a Lie group isomorphism.
Let $\Phi(K)$ denote the groupoid of all Lie group isomorphisms between the fibres of K. Using the standard Lie group structure on the automorphism group of the fibre type of K, $\Phi(K)$ has a l.t.L.g. structure, and actions of a Lie groupoid Ω on K by Lie group isomorphisms can be identified with Lie groupoid morphisms $\Omega \to \Phi(K)$.

Definition 74. Let Ω be a locally trivial Lie groupoid on M and let (K, q, M) be an LGB. An *extension* of Ω by K is a sequence

$$K \overset{\iota}{\rightarrowtail} \Psi \overset{\pi}{\twoheadrightarrow} \Omega$$

in which Ψ is a locally trivial Lie groupoid on M, ι and π are groupoid morphisms over M, ι is an embedding, π is a surjective submersion, and $\mathrm{im}(\iota) = \mathrm{ker}(\pi)$.

It is easy to see that the condition that Ψ be locally trivial is superfluous.

Example 46. Let $K \overset{\iota}{\rightarrowtail} \Psi \overset{\pi}{\twoheadrightarrow} \Omega$ be an extension as in the definition with K an abelian LGB. For $\xi \in \Omega_x^y,\ x, y \in M$, choose $\xi' \in \Psi_x^y$ with $\pi(\xi') = \xi$ and define $\rho(\xi) : K_x \to K_y$ as $\lambda \mapsto \xi'\lambda\xi'^{-1}$, the restriction of $I_{\xi'}$. It is clear that $\rho(\xi)$ is well defined. Now $I : \Psi \to \Phi(K)$ is smooth and π is a surjective submersion so $\rho : \Omega \to \Phi(K)$ is smooth. This ρ is the representation associated to the extension $K \rightarrowtail \Psi \twoheadrightarrow \Omega$.

Example 47. We consider the fundamental Lie Groupoid:

$$\Pi_1(M) := \{[\gamma]; \gamma : [0,1] \to M\}.$$

The linear action $\rho([\gamma]) : E_x \to E_y$ must be a linear isomorphism, depends only on the path, and verify:

- $\rho([\gamma_2 * \gamma_1]) = \rho([\gamma_2]) \circ \rho([\gamma_1])$,
- $\rho([c_x]) = id_{E_x}$ for a constant path c_x.

We need E to have a flat connection, an horizontal parralel transport,

$$\tau_\gamma : E_{\gamma(0)} \to E_{\gamma(1)},$$

which depends only on the class of γ.
So $\rho([\gamma]) := \tau_\gamma$ defines a linear action on $\Pi_1(M)$. The linear action of $\Pi_1(M)$ over the fibre bundle $E \to M$ is given by

$$[\gamma] : x \to y \mapsto \rho([\gamma]) : E_x \to E_y.$$

Lecture 9 – O. Carballal (UCM)

Date: 16 December, 2025

Abstract: In this talk, I will present several constructions in the category of Lie groupoids, including pullbacks, fibered products, and homotopy pullbacks. Then, some notions of equivalence between Lie groupoids will be discussed, focusing in particular on Morita equivalence, which plays a central rôle in the modern approach to differentiable stacks, foliation theory, and geometric quantisation. Several of the constructions and equivalence notions will be illustrated through examples coming from the theory of foliations.

Let us start discussing the problematic of arbitrary pull-backs in the category of smooth manifolds.

Example 48. Let $f, g : \mathbb{R} \to \mathbb{R}^2$ be the smooth maps defined by $f(t) := (t, 0)$ and $g(t) := (t, t^3)$. We clearly have that

Lecture 10 – Asier López-Gordón (IMPAN)

Date: 13 January, 2026

Abstract: In this talk, I will present the notion of Lie algebroid and illustrate it with several examples.

The main reference for this lecture is [**2**].

Let me start by recalling how the Lie algebra of a Lie group is defined. Let G be a Lie group, and let $\mathfrak{X}(G)$ denote the set of vector fields on G. The subset of left-invariant vector fields on G is given by

$$\mathfrak{X}_L(G) \coloneqq \{X \in \mathfrak{X}(G) \mid (L_g)_* X = X\,, \quad \forall\, g \in G\}\ .$$

Note that

$$(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y]\ , \quad \forall\, g \in G\,, \quad \forall X, Y \in \mathfrak{X}(G)\,.$$

Thus, $X_L(G)$ is a Lie subalgebra of $X(G)$ with respect to the Lie bracket $[\cdot, \cdot]$ of vector fields on G, namely,

$$X, Y \in \mathfrak{X}_L(G) \Longrightarrow [X, Y] \in \mathfrak{X}_L(G)\,.$$

Moreover, as a real vector space, $\mathfrak{X}_L(G)$ is isomorphic to the tangent space $\mathsf{T}_e G$ at the identity. More precisely, we can define the isomorphism

$$\begin{aligned} \Psi \colon \mathsf{T}_e G \ni v &\mapsto \overleftarrow{v} \in \mathfrak{X}_L(G) \\ \overleftarrow{v_g} &\coloneqq \mathsf{T}_e L_g(v)\,, \end{aligned}$$

with inverse

$$\Psi^{-1}(\overleftarrow{v}) = \overleftarrow{v_e}\,.$$

Indeed, for every $g, h \in G$, we have that

$$\left((L_g)_* \overleftarrow{v}\right)_h = \mathsf{T}_{g^{-1}\cdot h} L_g(\overleftarrow{v_{g^{-1}\cdot h}}) = \mathsf{T}_{g^{-1}\cdot h} L_g \circ \mathsf{T}_e L_{g^{-1}\cdot h}(v) = \mathsf{T}_e L_h(v) = \overleftarrow{v_h}\,.$$

Definition 75. The Lie algebra of a Lie group G is the vector space $\mathfrak{g} \coloneqq \mathrm{Lie}(G)$ endowed with the Lie bracket

$$\begin{aligned} [\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ [v, w] &\coloneqq [\overleftarrow{v}, \overleftarrow{w}]\big|_e = [\Psi(v), \Psi(w)]\big|_e\,. \end{aligned}$$

We would now like to extend this notion to Lie groupoids.

Definition 76. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A *left-invariant vector field* on $\mathcal{G}$ is a vector field $X \in \mathfrak{X}(\mathcal{G})$ such that:

(1) X is tangent to the t-fibers, i.e. $\mathsf{T}_g t(X_g) = 0$ for all $g \in \mathcal{G}$,
(2) $\mathsf{T}_h L_g(X_h) = X_{gh}$ for every $(g, h) \in \mathcal{G}^{(2)}$.

We shall denote the set of left-invariant vector fields by $\mathfrak{X}_L(\mathcal{G})$.

Lemma 77. The set of left-invariant vector fields is closed under the Lie bracket of vector fields, i.e., $\mathfrak{X}_L(\mathcal{G})$ is a Lie subalgebra of $(\mathfrak{X}(\mathcal{G}), [\cdot, \cdot])$.

PROOF. A vector field $X \in \mathfrak{X}(\mathcal{G})$ is tangent to the t-fibers if and only if

$$X(f \circ t) = 0\,, \quad \forall f \in \mathscr{C}^\infty(\mathcal{G})\,.$$

Hence, for each pair of vector fields $X, Y \in \mathfrak{X}_L(\mathcal{G})$ we have that

$$[X,Y](f \circ t) = X \circ Y \circ f \circ t - Y \circ X \circ f \circ t = 0\,, \quad \forall f \in \mathscr{C}^\infty(\mathcal{G})\,,$$

which implies that $[X,Y] \in \mathfrak{X}_L(\mathcal{G})$.

For each fixed $g \in \mathcal{G}$, with $s(g) = x$ and $t(g) = y$, the left translation

$$L_g \colon t^{-1}(x) \to t^{-1}(y)$$

is a diffeomorphism. For each $X, Y \in \mathfrak{X}_L(\mathcal{G})$, we have that

$$(L_g)_* X|_{t^{-1}(x)} = X|_{t^{-1}(y)}\,, \quad (L_g)_* Y|_{t^{-1}(x)} = Y|_{t^{-1}(y)}\,,$$

and thereupon

$$\begin{aligned}(L_g)_*[X,Y]|_{t^{-1}(x)} &= (L_g)_*\left[X|_{t^{-1}(x)}, Y|_{t^{-1}(x)}\right] = \left[(L_g)_* X|_{t^{-1}(x)}, (L_g)_* Y|_{t^{-1}(x)}\right] \\ &= \left[X|_{t^{-1}(y)}, Y|_{t^{-1}(y)}\right] = [X,Y]|_{t^{-1}(y)}\,.\end{aligned}$$

Hence,

$$\mathsf{T}_h L_g [X,Y]_h = [X,Y]gh\,, \quad \forall g, h \in \mathcal{G}^{(2)}\,.$$

□

We define the following vector bundle over M:

$$\mathrm{A}(\mathcal{G}) \coloneqq \ker \mathsf{T}t|_{u(M)} = u^*(\ker \mathsf{T}t)\,,$$

and denote its space of sections by $\Gamma(\mathrm{A}(\mathcal{G}))$.

Lemma 78. As vector spaces, $\Gamma(\mathrm{A}(\mathcal{G}))$ and $\mathfrak{X}_L(\mathcal{G})$ are isomorphic.

PROOF. The isomorphism $\varphi\colon \Gamma(\mathrm{A}(\mathcal{G})) \to \mathfrak{X}_L(\mathcal{G})$ is given by

$$\varphi(\alpha) = \overleftarrow{\alpha}\,, \quad \overleftarrow{\alpha_g} \coloneqq \mathsf{T}_e L_g(\alpha_{s(g)})\,, \quad \forall\, g \in \mathcal{G}\,,$$

and its inverse reads

$$\varphi^{-1}(X) = X|_{u(M)}\,.$$

Clearly, $\alpha \in \ker \mathsf{T}t|_{u(M)}$ is equivalent to $\overleftarrow{\alpha}$ being tangent to the t-fibers. Moreover,

$$\mathsf{T}_h L_g(\overleftarrow{\alpha_h}) = T_h L_g \circ \mathsf{T}_e L_h(\alpha_{s(h)}) = \mathsf{T}_e L_{gh}(\alpha_{s(h)}) = \mathsf{T}_e L_{gh}(\alpha_{s(gh)}) = \overleftarrow{\alpha_{gh}}\,.$$

□

The *anchor* is the vector bundle map $\rho\colon\ \mathrm{A}(\mathcal{G}) \to \mathsf{T}M$ given by

$$\rho(\alpha) = \mathsf{T}_{1_x} s(\alpha)\,, \quad \forall\, \alpha \in \mathrm{A}(\mathcal{G})_x = \ker(\mathsf{T}_{1_x} t)\,, \quad \forall\, x \in M\,,$$

Lemma 79. The bracket of sections $\Gamma(\mathrm{A}(\mathcal{G}))$ satisfies the Leibniz identity

$$[\alpha, f\beta] = f[\alpha,\beta] = \rho(\alpha)(f)\beta\,, \quad \forall\, \alpha, \beta \in \Gamma(\mathrm{A}(\mathcal{G}))\,, \quad \forall\, f \in \mathscr{C}^\infty(M)\,.$$

PROOF. For each $\alpha \in \mathrm{A}(\mathcal{G})_x$, we have that

$$s_*(\overleftarrow{\alpha}) = s_*\big(\varphi^{-1}(\alpha)\big) = \mathsf{T}_{1_x} s(\alpha) = \rho(\alpha)\,,$$

i.e. $\overleftarrow{\alpha}$ and $\rho(\alpha)$ are s-related. In addition, for each $f \in \mathscr{C}^\infty(M)$ and for all $g \in \mathcal{G}$

$$\overleftarrow{f\alpha}_g = \varphi(f\alpha)_g = \mathsf{T}_e L_g\big(f(x)\,\alpha(x)\big) = f(x)\,\mathsf{T}_e L_g\big(\alpha(x)\big) = f(x)\overleftarrow{\alpha}_g = \big((f \circ s)\overleftarrow{\alpha}\big)(g)\,,$$

where $x = s(g)$, i.e.,

$$\overleftarrow{f\alpha} = s^*(f)\,\overleftarrow{\alpha}\,.$$

Hence,

$$
\begin{aligned}
\overleftarrow{[\alpha, f\beta]} &\coloneqq \left[\overleftarrow{\alpha}, \overleftarrow{f\beta}\right] = \left[\overleftarrow{\alpha}, s^* f\, \overleftarrow{\beta}\right] = s^*(f)\left[\overleftarrow{\alpha}, \overleftarrow{\beta}\right] + \overleftarrow{\alpha}\big(s^*(f)\big)\overleftarrow{\beta} \\
&= \overleftarrow{f[\alpha,\beta]} + \overleftarrow{\rho(\alpha)(f)\beta} = \overleftarrow{f[\alpha,\beta] + \rho(\alpha)(f)\beta}\,,
\end{aligned}
$$

where we have used the standard Leibniz rule for the Lie bracket of vector fields on the third step. Consequently,

$$
[\alpha, f\beta] = \varphi\big(\overleftarrow{[\alpha, f\beta]}\big) = \varphi\big(\overleftarrow{f[\alpha,\beta] + \rho(\alpha)(f)\beta}\big) = f[\alpha,\beta] + \rho(\alpha)(f)\beta\,.
$$

□

Definition 80. A *Lie algebroid* is a vector bundle $A \to M$ endowed with:

- a Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$,
- a vector bundle morphism

  $$
  \begin{tikzcd}
  A \arrow[rr, "\rho"] \arrow[dr] & & \mathsf{T}M \arrow[dl] \\
  & M &
  \end{tikzcd}
  $$

  satisfying the Leibniz identity:

  $$
  [\alpha, f\beta]_A = f[\alpha,\beta]_A + \rho(\alpha)(f)\beta\,, \quad \forall\, \alpha,\beta \in \Gamma(A)\,, \quad \forall\, f \in \mathscr{C}^\infty(M)\,.
  $$

The map ρ is called the *anchor* of A.

Given a Lie groupoid $\mathcal{G} \to M$, we call $\mathrm{A}(\mathcal{G})$ the *Lie algebroid of* $\mathcal{G}$. A Lie algebroid A is called integrable if $A \simeq \mathrm{A}(\mathcal{G})$ for some Lie group.

Proposition 81. The map $\rho\colon \Gamma(A) \to \mathfrak{X}(M)$ induced by the anchor preserves the brackets, namely,

$$
\rho\big([\alpha,\beta]_A\big) = [\rho(\alpha), \rho(\beta)]\,, \quad \forall\, \alpha,\beta \in \Gamma(A)\,.
$$

Proof. Let $\alpha,\beta,\gamma \in \Gamma(A)$ and $f \in C^\infty(M)$. Applying the Leibniz rule twice, we have that

$$
\begin{aligned}
[\alpha, [\beta, f\gamma]_A]_A &= [\alpha, f[\beta,\gamma]_A + (\rho(\beta)f)\gamma]_A \\
&= [\alpha, f[\beta,\gamma]_A]_A + [\alpha, (\rho(\beta)f)\gamma]_A \\
&= f[\alpha,[\beta,\gamma]_A]_A + (\rho(\alpha)f)[\beta,\gamma]_A + (\rho(\beta)f)[\alpha,\gamma]_A + (\rho(\alpha)\rho(\beta)f)\gamma\,,
\end{aligned}
$$

and analogously

$$
[\beta, [\alpha, f\gamma]_A]_A = f[\beta,[\alpha,\gamma]_A]_A + (\rho(\beta)f)[\alpha,\gamma]_A + (\rho(\alpha)f)[\beta,\gamma]_A + (\rho(\beta)\rho(\alpha)f)\gamma
$$

By the Jacobi identity,

$$
\begin{aligned}
[[\alpha,\beta]_A, f\gamma]_A &= [\alpha,[\beta,f\gamma]_A]_A - [\beta,[\alpha,f\gamma]_A]_A \\
&= f\left([\alpha,[\beta,\gamma]_A]_A - [\beta,[\alpha,\gamma]_A]_A\right) + \left(\rho(\alpha)\rho(\beta)f - \rho(\beta)\rho(\alpha)f\right)\gamma \\
&= f[[\alpha,\beta]_A,\gamma]_A + ([\rho(\alpha),\rho(\beta)]_{\mathfrak{X}(M)}f)\gamma
\end{aligned}
\tag{13}
$$

On the other hand, by the Leibniz rule we have

$$
[[\alpha,\beta]_A, f\gamma]_A = f[[\alpha,\beta]_A,\gamma]_A + (\rho([\alpha,\beta]_A)f)\gamma \tag{14}
$$

Combining equations (13) and (14) yields

$$
([\rho(\alpha),\rho(\beta)]_{\mathfrak{X}(M)}f)\gamma = (\rho([\alpha,\beta]_A)f)\gamma\,, \quad \forall\, \alpha,\beta,\gamma \in \Gamma(A)\,, \quad \forall f \in \mathscr{C}^\infty(M)\,,
$$

from where the seeked identity follows. □

Example 49 (Tangent bundle)**.** For any manifold M, we can regard its tangent bundle $A = \mathsf{T}M \to M$ as an algebroid, with $\rho \equiv \mathrm{Id}_{\mathsf{T}M}$ and $[\cdot,\cdot]_A$ the usual Lie bracket on $\mathfrak{X}(M)$. As a matter of fact, it is an integrable Lei algebroid, because $\mathsf{T}M = \mathrm{A}(M \times M)$ for the groupoid $M \times M \rightrightarrows M$.

Example 50 (Involutive distribution)**.** A vector sub-bundle $D \subseteq \mathsf{T}M$ such that $[X, Y] \in \Gamma(D)$ for all $X, Y \in \Gamma(D)$ is a Lie algebroid with $\rho = \iota \colon A \hookrightarrow \mathsf{T}M$ the natural inclusion. In fact, by the Frobenius theorem, it is an integrable Lie algebroid.

Example 51 (Lie algebra)**.** A Lie algebra $\mathfrak{g}$ can be regarded as a trivial Lie algebroid $A = \mathfrak{g} \to \{\star\}$.

Definition 82. Let $A \to M$ be a Lie algebroid with anchor ρ. The *isotropy Lie algebra* of A at $x \in M$ is the vector subspace
$$\mathfrak{g}_x \coloneqq \ker \rho_x \subseteq A_x$$
equipped with the Lie bracket
$$[\alpha_x, \beta_x]_{\mathfrak{g}} \coloneqq [\alpha, \beta]_A(x)\,.$$

If $A = \mathrm{A}(\mathcal{G})$ for a Lie groupoid $\mathcal{G} \rightrightarrows M$, then
$$\mathfrak{g}_x = \ker \mathsf{T}_{1_x} s \cap \ker \mathsf{T}_{1_x} t\,,$$
which is the Lie algebra of the vertex group (a.k.a. the isotropy group) $\mathcal{G}_x^x = s^{-1}(x) \cap t^{-1}(x)$.

For a general Lie algebroid $A \to M$, we can define a bundle of Lie algebras
$$\mathfrak{g}(A) = \ker \rho = \bigcup_{x \in M} \ker \rho_x\,,$$
which in general is **not** smooth. We call A a *regular Lie algebroid* if ρ has constant rank, so $\mathfrak{g}(A)$ is a smooth bundle. If A is regular, we have a short exact sequence of Lie algebroids:
$$0 \longrightarrow \ker \rho \longrightarrow A \longrightarrow \operatorname{Im} \rho \longrightarrow 0\,.$$
Moreover, $\operatorname{Im} \rho \subseteq \mathsf{T}M$ is an integrable distribution, and its associated integral foliation $\mathcal{F}$ is called the *orbit foliation* of the regular algebroid. It is also possible to define a (singular) orbit foliation for non-regular Lie algebroids:
$$\mathcal{F}_A = \{\mathcal{O}_i\}\,, \quad \mathsf{T}_x \mathcal{O}_i = \operatorname{Im} \rho_x\,, \quad \forall\, x \in \mathcal{O}_i\,.$$
A Lie algebroid is called *transitive* if $\operatorname{Im} \rho = \mathsf{T}M$.

Definition 83. Let $A_1 \to M$ and $A_2 \to M$ be two Lie algebroids over the same base. A *morphism of Lie algebroids* is a vector bundle morphism $\Phi \colon A_1 \to A_2$ such that:
$$\Phi\big([\alpha, \beta]_{A_1}\big) = \big[\Phi(\alpha), \Phi(\beta)\big]_{A_2}\,, \quad \forall\, \alpha, \beta \in \Gamma(A_1)\,,$$
and
$$\rho_2 \circ \Phi = \rho_1\,.$$

Example 52 (Atiyah algebroid)**.** Let $\pi \colon P \to M$ be a G-principal bundle. Its Atiyah algebroid is the vector bundle $A \coloneqq \mathsf{T}P/G \to P/G$, with the anchor $\rho \colon A \to \mathsf{T}M$ induced by $\mathsf{T}\pi \colon \mathsf{T}P \to \mathsf{T}M$:
$$\begin{tikzcd}
\mathsf{T}P \arrow[loop left, "G"] \arrow[r] \arrow[d, "\mathsf{T}\pi"'] & A = \mathsf{T}P/G \arrow[dl, "\rho"] \\
\mathsf{T}M &
\end{tikzcd}\,.$$
The Lie bracket $[\cdot, \cdot]_A$ is induced by the restriction of the Lie bracket of $\mathfrak{X}(P)$ to the set $\mathfrak{X}_G(P)$ of G-invariant vector fields, namely,
$$\mathfrak{X}_G(P) = \{X \in \mathfrak{X}(P) \mid g_* X = X\,, \quad \forall g \in G\}\,.$$
One can easily see that A is the Lie algebroid of the gauge groupoid (a.k.a. Atiyah groupoid) $\mathcal{G} = (P \times P)/G \rightrightarrows M$.

If $A \simeq \mathrm{A}(\mathcal{G})$ is an integrable and transitive Lie algebroid, then the anchor
$$\rho \colon \mathrm{A}(\mathcal{G}) = \ker \mathsf{T}t|_{u(M)} \ni \alpha \mapsto \mathsf{T}_{1_x} s(\alpha) \in \mathsf{T}M$$

is surjective, and thus the restriction of the target map $t\colon \mathcal{G}_x \to M$ is a surjective submersion. By Proposition 20, this in turn implies that $\mathcal{G} \rightrightarrows M$ is locally trivial. By Proposition 22, $\mathcal{G}$ is isomorphic to the gauge groupoid $(\mathcal{G}_x \times \mathcal{G}_x)/\mathcal{G}_x^x$. In conclusion, an integrable transitive Lie algebroid $A \to M$ is isomorphic to the Atiyah algebroid of some principal bundle $P \to M$.

Example 53 (Action algebroid)**.** Let $a\colon \mathfrak{g} \to \mathfrak{X}(M)$ be an infinitesimal action of a Lie algebra $\mathfrak{g}$ on a manifold G. We define the associated action algebroid as the vector bundle $A = M \times \mathfrak{g} \to M$, with the anchor

$$\rho\colon M \times \mathfrak{g} \ni (x, \xi) \mapsto a(\xi)_x \in \mathsf{T}_x M\,,$$

and the Lie bracket on $\Gamma(A) \simeq \mathscr{C}^\infty(M, \mathfrak{g})$

$$[f, g]_A(x) = [f(x), g(x)]_{\mathfrak{g}} + \left(\mathcal{L}_{a(f(x))} g\right)(x) - \left(\mathcal{L}_{a(g(x))} f\right)(x)\,.$$

As one could expect, given a Lie group action of G on M, the action groupoid has as associated Lie algebroid the action algebroid of $\mathfrak{g} = \mathrm{Lie}(G)$.

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