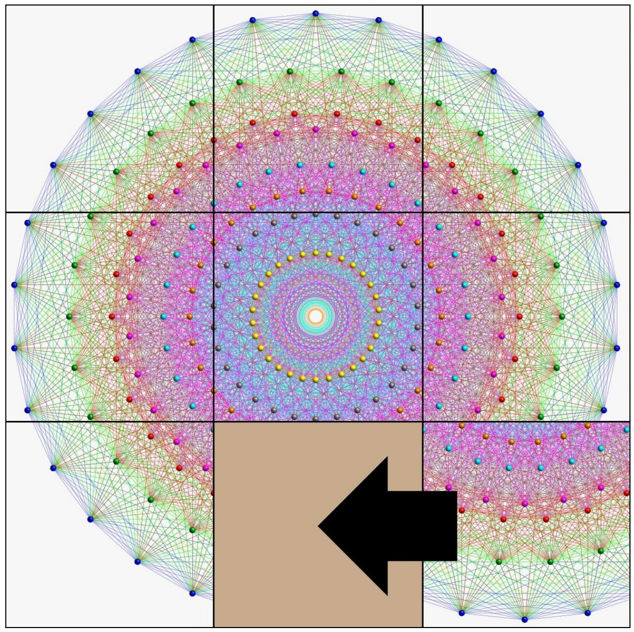
Reading Group(oid) Lecture Notes



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Lectures

Lecture 1 – O. Carballal (UCM)

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Abstract: We illustrate the need for groupoids through a series of examples, primarily following [**1, 2**], and discuss some of their recent applications in Mathematical Physics. We also outline potential directions for future topics of our Reading Groupoid.

Since F. Klein's *Erlangen program* and the pioneer work of S. Lie regarding the definition of geometric structures through their group of automorphisms, the notion of symmetry has been studied through the theory of groups and their actions.

Symmetry ≡ Groups & actions

Symmetry of homogeneous structures, such us homogeneous spaces, can be fully described using groups. This is no longer valid when analysing the symmetry of non-homogeneous structures.

The very first example of symmetry group is the *Euclidean group* $\mathrm{E}(n)$, consisting of rigid motions of $\mathbb{R}^n$. More concretely,

$$\mathrm{E}(n) := \{\phi : \mathbb{R}^n \to \mathbb{R}^n : \phi \text{ preserves distances}\}$$

Every $\phi \in \mathrm{E}(n)$ is univocally determined by a rotation $A \in \mathrm{O}(n)$ and a translation vector $b \in \mathbb{R}^n$ as $\phi(x) = Ax + b$ and, hence, $\mathrm{E}(n)$ is the semidirect product $\mathrm{E}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$. Let us now consider the action $\mathrm{E}(n) \curvearrowright \mathbb{R}^n$ given by $\phi \cdot x := \phi(x)$ for every $\phi \in \mathrm{E}(n)$ and $x \in \mathbb{R}^n$. Then, we define the symmetry group of a subset Ω as the subgroup G_Ω of $\mathrm{E}(n)$ formed by those rigid motions leaving Ω invariant; that is,

$$G_\Omega := \{\phi \in \mathrm{E}(n) : \phi(\Omega) = \Omega\}.$$

Let us now adopt the following *credo*:

G_Ω is large $\equiv$ Ω is very symmetric

Example 1 (Weinstein's tiling [**2**]). Let us study the symmetry group of the following tiling of $\mathbb{R}^2$ by 2×1-rectangles:

$$\Omega := (2\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \subset \mathbb{R}^2.$$

Within this context, every connected component of $\mathbb{R}^2 - \Omega$ is called a *tile*. Let Λ be the lattice determined by Ω, corresponding at the borders of the tiles,

$$\Lambda = (2\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z}) = 2\mathbb{Z} \times \mathbb{Z}.$$

Then, one easily sees that the symmetry group G_Ω of Ω consists of the following elements:

- translations by vectors $u \in \Lambda$, corresponding to the corner points of Λ;
- central reflections through the points of $\frac{1}{2}\Lambda = \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$, the latter corresponding to the centers of the tiles, and
- axial reflections along the vertical and horizontal lines trough the points of $\frac{1}{2}\Lambda$.

Since G_Ω is a big group, following the faith in our *credo*, Ω is very symmetric. Nevertheless, there are some problems:

- Ω and the lattice Λ have the same symmetry group $G_\Omega = G_\Lambda$, although Ω and Λ look quite different;
- G_Ω contains no local information about the action $G_\Omega \curvearrowright \mathbb{R}^2$, and
- the symmetry group of the finite tiling $\widetilde{\Omega} := \Omega \cap B$, where $B := [0, 2m] \times [0, n]$ is a 'bathroom floor', is isomorphic to the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (note that it is spanned by the horizontal and the vertical reflections through the midline of B, whose composition is the 180° rotation around the centre of B). In particular, we see that Ω is very symmetric but $\widetilde{\Omega}$ is not, even though they share the same patterns and independently of the number of tiles.

To address these issues, we introduce the *action groupoid* associated with $G_\Omega \curvearrowright \mathbb{R}^n$, defined as

$$\mathcal{G}_\Omega := \{(y, \phi, x) : \phi \in G_\Omega, x, y \in \mathbb{R}^n, y = \phi(x)\}, \tag{1}$$

and endowed with the partially defined multiplication

$$(z, \psi, y)(y, \phi, x) := (z, \psi \circ \phi, x). \tag{2}$$

In order to analyse the properties of this partially defined multiplication, we introduce the *source* $\mathrm{s} : \mathcal{G}_\Omega \to \mathbb{R}^n$ and *target* $\mathrm{t} : \mathcal{G}_\Omega \to \mathbb{R}^n$ maps as

$$\mathrm{s}(y, \phi, x) := x, \qquad \mathrm{t}(y, \phi, x) := y, \qquad (y, \phi, x) \in \mathcal{G}_\Omega.$$

One easily sees that (2) satisfies the following four properties:

(1) Composition. Given $g, h \in \mathcal{G}_\Omega$, gh is only defined when $\mathrm{s}(g) = \mathrm{t}(h)$. In that case, $\mathrm{s}(gh) = \mathrm{s}(h)$ and $\mathrm{t}(gh) = \mathrm{t}(g)$.
(2) Associativity. If $g, h, k \in \mathcal{G}_\Omega$ are such that $\mathrm{s}(g) = \mathrm{t}(h)$ and $\mathrm{s}(h) = \mathrm{t}(k)$, then $(gh)k = g(hk)$.
(3) Units. For all $x \in \mathbb{R}^n$, let us denote $1_x := (x, \mathrm{id}, x) \in \mathcal{G}_\Omega$. Then, $1_{\mathrm{t}(g)} g = g = g 1_{\mathrm{s}(g)}$ for all $g \in \mathcal{G}_\Omega$.
(4) Inverse. For all $g = (y, \phi, x) \in \mathcal{G}_\Omega$ there exists $g^{-1} := (x, \phi^{-1}, y)$ such that $gg^{-1} = 1_{\mathrm{t}(g)}$ and $g^{-1}g = 1_{\mathrm{s}(g)}$.

These are, exactly, the properties characterising a groupoid.

Definition 1 (Groupoid). A *groupoid* consists of two sets, $\mathcal{G}$ and M, together with maps

- $\mathrm{s}, \mathrm{t} : \mathcal{G} \to M$ (*source* and *target* projections);
- $m : \mathcal{G}^{(2)} := \{(g, h) : \mathrm{s}(g) = \mathrm{t}(h)\} \to \mathcal{G}, (g, h) \mapsto gh$ (*multiplication*);

- $u : M \to \mathcal{G}, x \mapsto 1_X$ (*unit*), and
- $i : \mathcal{G} \to \mathcal{G}, g \mapsto g^{-1}$ (*inverse*),

satisfying the following four properties:

(1) If $z \xleftarrow{g} y \xleftarrow{h} x$, then $z \xleftarrow{gh} x$;
(2) If $z \xleftarrow{g} y \xleftarrow{h} x \xleftarrow{k} v$, then $(gh)k = g(hk)$;
(3) There exists $x \xleftarrow{1_x} x$ such that for all $y \xleftarrow{g} x$, we have $1_y g = g = 1_x$, and
(4) If $y \xleftarrow{g} x$, there exists $x \xleftarrow{g^{-1}} y$ such that $gg^{-1} = 1_y$ and $g^{-1}g = 1_x$.

From now on, we will denote the groupoid as $\mathcal{G} \rightrightarrows M$, where the parallel arrows refer to the source and target maps.

Remark 2. (1) A groupoid is a small category where every arrow is invertible. Within this language, given a groupoid $\mathcal{G} \rightrightarrows M$, $\mathcal{G}$ is usually referred to as the set of *arrows* of the groupoid, M is the set of *objects* or *units* (also called the *base* of the groupoid), and $\mathcal{G}^{(2)}$ is the set of *composable arrows*.
(2) We write arrows from left to right to refer to the order of composition. Everything can be done analogously writing arrows from left to right by swapping the source and target maps s and t.
(3) Groupoids can be restricted to subsets of the set of units. That is, given a groupoid $\mathcal{G} \rightrightarrows M$, *its restriction* to $N \subset M$ is the groupoid

$$\mathcal{G}|_N := \mathrm{s}^{-1}(N) \cap \mathrm{t}^{-1}(N) \rightrightarrows N.$$

DEFINITION 3 (Orbit and isotropy group). Let $\mathcal{G} \rightrightarrows M$ be a groupoid and let $x \in M$ be a unit.

- The *orbit of* x is $O_x := \{y \in M : \text{there exists } g \in \mathcal{G}, y \xleftarrow{g} x\}$.
- The *isotropy group of* x is $\mathcal{G}_x := \mathrm{s}^{-1}(x) \cap \mathrm{t}^{-1}(x) = \{g \in \mathcal{G} : x \xleftarrow{g} x\}$.

EXAMPLE 2 (Weinstein's tiling continued [**2**]). Let us now go back to the tiling Ω of $\mathbb{R}^2$ introduced in Example 1. First of all, we consider the action groupoid $\mathcal{G}_\Omega \rightrightarrows \mathbb{R}^2$ associated with $G_\Omega \curvearrowright \mathbb{R}^2$, defined in (1); that is,

$$\mathcal{G}_\Omega := \{(y, \phi, x) : \phi \in G_\Omega, y \xleftarrow{\phi} x\} \rightrightarrows \mathbb{R}^2,$$

and let us now consider its restriction to the 'bathroom floor' $B = [0, 2m] \times [0, n]$, namely $\mathcal{G}_\Omega|_B \rightrightarrows B$. Then:

- $x, y \in B$ belong to the same orbit if they are similarly placed in their tiles, and
- the isotropy group of a point $x \in B$ is trivial, unless $x \in \frac{1}{2}\Lambda \cap B$, for which its isotropy is the Klein four group.

This means that $\mathcal{G}_\Omega|_B \rightrightarrows B$ detects local information.

Some applications to Mathematical Physics.

- Groupoid picture of Schwinger's quantum mechanics by F. M. Ciaglia, A. Ibort, G. Marmo and collaborators initiated in [**3**]. Recent work on the formulation of fields as functors between groupoids by A. Ibort, A. Mas and L. Schiavone [**4**].

- The material groupoid, a groupoidal approach to continuum mechanics, by M. de León, M. Epstein and V. Jiménez [**5**].
- Discrete Lagrangian and Hamiltonian mechanics on Lie groupoids by J. C. Marrero, D. Martín de Diego, E. Martínez and L. Colombo [**6, 7**].

Local Lie group actions. Let us consider the Lie group $\mathrm{PGL}(2,\mathbb{R})$ consisting of homographies of the projective line $\mathbb{RP}^1$. Explicitly, one has

$$\mathrm{PGL}(2,\mathbb{R}) = \mathrm{GL}(2,\mathbb{R})/\mathbb{R},$$

where $\mathbb{R}$ refers to the subgroup of $\mathrm{GL}(2,\mathbb{R})$ formed by non-zero diagonal scalar matrices. Clearly, $\mathrm{PGL}(2,\mathbb{R})$ acts on $\mathbb{RP}^1$ as

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot [x_0 : x_1] = [ax_0 + bx_1 : cx_0 + dx_1]. \tag{3}$$

Let us now analyse how $\mathrm{PLG}(2,\mathbb{R})$ acts on the affine line $\mathbb{R} \equiv \mathbb{RP}^1 - \{[1:0]\}$. Given an affine point $[x_0 : x_1] = [x : 1] \in \mathbb{R}$, where $x := x_0/x_1$, and $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \mathrm{PGL}(2,\mathbb{R})$, we have that

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot [x : 1] = \left[\frac{ax+b}{cx+d} : 1\right] \in \mathbb{R}, \tag{4}$$

provided that $cx + d \neq 0$. It is natural to consider the following question:

Does formula (4) define a Lie group action on $\mathbb{R}$?

Of course, the answer to this question is negative, since (4) is only defined on an open subset of $\mathrm{PGL}(2,\mathbb{R}) \times \mathbb{R}$. Indeed, it defines what is called a *local Lie group action* (see [**8**] for more details). This means, roughly speaking, that (4) only satisfies the defining properties of a Lie group action on a neighbourhood of the unit element of $\mathrm{PGL}(2,\mathbb{R})$.

Hopefully, this problem can be tackled successfully when one considers groupoids rather than groups. First of all, let us consider the so-called action groupoid associated with the Lie group action $\mathrm{PGL}(2,\mathbb{R}) \curvearrowright \mathbb{RP}^1$ defined in (3) as the groupoid $\mathcal{G} := \mathrm{PGL}(2,\mathbb{R}) \times \mathbb{RP}^1 \rightrightarrows \mathbb{RP}^1$ with source and target maps given by

$$\mathrm{s}\left([A], [x_0 : x_1]\right) = [x_0 : x_1], \qquad \mathrm{t}\left([A], [x_0 : x_1]\right) = [A] \cdot [x_0 : x_1],$$

and multiplication defined as

$$([B], [y_0 : y_1])([A], [x_0 : x_1]) := ([BA], [x_0, x_1]),$$

provided that $[A] \cdot [x_0 : x_1] = [y_0 : y_1]$. Then, the restriction of $\mathcal{G} \rightrightarrows \mathbb{RP}^1$ to the affine line $\mathbb{R}$ is the groupoid

$$\mathcal{G}|_{\mathbb{R}} = \mathrm{s}^{-1}(\mathbb{R}) \cap \mathrm{t}^{-1}(\mathbb{R}) = \left\{ \left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right], [x : 1] \right) \in \mathrm{PGL}(2,\mathbb{R}) \times \mathbb{R} : cx + d \neq 0 \right\} \rightrightarrows \mathbb{R}.$$

Within this context, the target map of $\mathcal{G}|_{\mathbb{R}} \rightrightarrows \mathbb{R}$ is just formula (4).

Singular spaces. Let $G \curvearrowright M$ be a smooth action of a Lie group G on a manifold M (see [**9**, Chapter 2] for more details on Lie group actions). Recall that the action is said to be

- *free* if $g \cdot x = x$ for some $x \in M$, then $g = e$, and
- *proper* if the map $\Phi : G \times M \ni (g, x) \mapsto (g \cdot x, x) \in M \times M$ is a proper map. That is, for every compact subset $K \subset M \times M$ we have that $\Phi^{-1}(K) \subset G \times M$ is compact. This is equivalent to the following: if (g_n) and (x_n) are sequences in G and M, respectively, such that (x_n) and $(g_n \cdot x_n)$ are convergent, then (g_n) contains a converging subsequence.

Provided that the action $G \curvearrowright M$ is free an proper, the quotient manifold M/G has a unique smooth structure such that the projection map $\pi : M \to M/G$ is a submersion. For instance, let us consider the action $G = \mathbb{Z}^k \curvearrowright M = \mathbb{R}^k$ given by translations:

$$(n_1, \ldots, n_k) \cdot (x^1, \ldots, x^k) := (x^1 + n_1, \ldots, x^k + n_k).$$

It can be easily seen that this action is free and proper. Hence, the quotient space $\mathbb{R}^k/\mathbb{Z}^k$, which is the k-dimensional torus $\mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$, possesses a unique smooth structure such that the projection $\pi : M \to M/G$ is a submersion.

What happens when the action $G \curvearrowright M$ is not free and proper?

If action $G \curvearrowright M$ is not free and proper, G/M is a "singular space"

EXAMPLE 3. Let us consider the actions $\mathrm{SO}(2) \curvearrowright \mathbb{R}^2$ and $\mathrm{SO}(3) \curvearrowright \mathbb{R}^3$ given by rotations. Both of these actions are proper, since the associated Lie groups are compact. Nevertheless, they are not free: the origin is a fixed point of every rotation. The "singular spaces" $\mathbb{R}^2/\mathrm{SO}(2)$ and $\mathbb{R}^3/\mathrm{SO}(3)$ are both homeomorphic to $[0, +\infty)$. Notwithstanding, they are not equivalent as singular spaces because their associated action groupoids $\mathrm{SO}(2) \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ and $\mathrm{SO}(3) \times \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ are not *Morita equivalent*. Roughly speaking, this can be justified as follows. The isotropy group of $x \in \mathbb{R}^2 - \{0\}$ is trivial, while the isotropy group of $x \in \mathbb{R}^3 - \{0\}$ is isomorphic to $\mathrm{SO}(2)$.

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