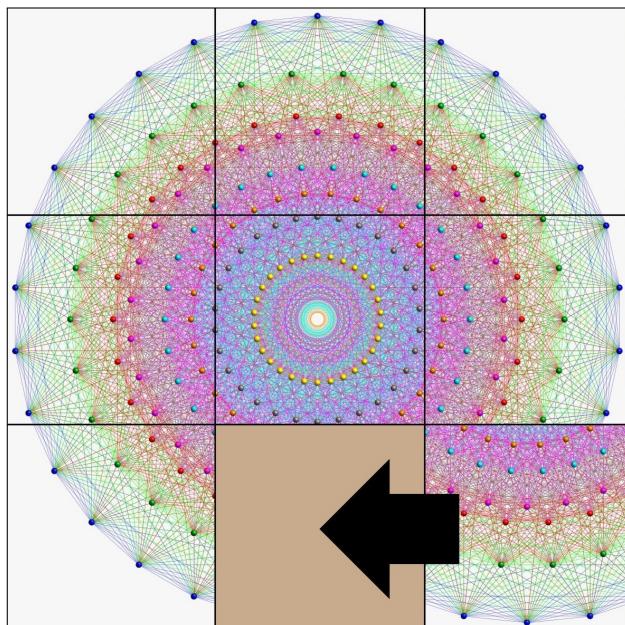
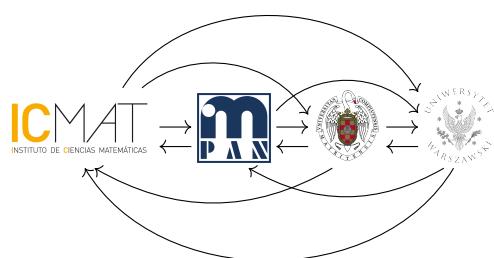


Reading Group(oid) Lecture Notes



Organisers/editors:
Oscar Carballal (UCM, Madrid)
Rubén Izquierdo-López (ICMAT, Madrid)
Asier López-Gordón (IMPAN, Warsaw)
Juan Manuel López Medel (ICMAT, Madrid)
Bartosz Maciej Zawora (KMMF-UW, Warsaw)



Contents

Lectures	3
Lecture 1 – O. Carballal (UCM)	5
Lecture 2 – Bartosz M. Zawora (KMMF - UW)	11
1. Lie groupoids: definition and examples	11
2. Morphisms and subgroupoids	14
Lecture 3 – Rubén Izquierdo-López (ICMAT)	15
3. Locally trivial Lie groupoids	15
Lecture 4 – Asier López-Gordón (IMPAN)	19
4. Bisections	19
Bibliography	27

Lectures

Lecture 1 – O. Carballal (UCM)

Date: 7 October, 2025

Abstract: We illustrate the need for groupoids through a series of examples, primarily following [2, Lecture 1] and [3], and discuss some of their recent applications in Mathematical Physics. We also outline potential directions for future topics of our Reading Groupoid.

Since F. Klein's *Erlangen program* and the pioneer work of S. Lie regarding the definition of geometric structures through their group of automorphisms, the notion of symmetry has been studied through the theory of groups and their actions.

$$\text{Symmetry} \equiv \text{Groups } \mathcal{E} \text{ actions}$$

Symmetry of homogeneous structures, such as homogeneous spaces, can be fully described using groups. This is no longer valid when analysing the symmetry of non-homogeneous structures.

The very first example of symmetry group is the *Euclidean group* $E(n)$, consisting of rigid motions of \mathbb{R}^n . More concretely,

$$E(n) := \{\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n : \phi \text{ preserves distances}\}$$

Every $\phi \in E(n)$ is univocally determined by a rotation $A \in O(n)$ and a translation vector $b \in \mathbb{R}^n$ as $\phi(x) = Ax + b$ and, hence, $E(n)$ is the semidirect product $E(n) = O(n) \ltimes \mathbb{R}^n$. Let us now consider the action $E(n) \curvearrowright \mathbb{R}^n$ given by $\phi \cdot x := \phi(x)$ for every $\phi \in E(n)$ and $x \in \mathbb{R}^n$. Then, we define the symmetry group of a subset Ω as the subgroup G_Ω of $E(n)$ formed by those rigid motions leaving Ω invariant; that is,

$$G_\Omega := \{\phi \in E(n) : \phi(\Omega) = \Omega\}.$$

Let us now adopt the following *credo*:

$$G_\Omega \text{ is large} \equiv \Omega \text{ is very symmetric}$$

Example 1 (Weinstein's tiling [3]). Let us study the symmetry group of the following tiling of \mathbb{R}^2 by 2×1 -rectangles:

$$\Omega := (2\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \subset \mathbb{R}^2.$$

Within this context, every connected component of $\mathbb{R}^2 - \Omega$ is called a *tile*. Let Λ be the lattice determined by Ω , corresponding at the borders of the tiles,

$$\Lambda = (2\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z}) = 2\mathbb{Z} \times \mathbb{Z}.$$

Then, one easily sees that the symmetry group G_Ω of Ω consists of the following elements:

- **translations** by vectors $u \in \Lambda$, corresponding to the corner points of Λ ;
- **central reflections** through the points of $\frac{1}{2}\Lambda = \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$, the latter corresponding to the centers of the tiles, and
- **axial reflections** along the vertical and horizontal lines through the points of $\frac{1}{2}\Lambda$.

Since G_Ω is a big group, following the faith in our *credo*, Ω is very symmetric. Nevertheless, there are some problems:

- Ω and the lattice Λ have the same symmetry group $G_\Omega = G_\Lambda$, although Ω and Λ look quite different;
- G_Ω contains no local information about the action $G_\Omega \curvearrowright \mathbb{R}^n$, and
- the symmetry group of the finite tiling $\tilde{\Omega} := \Omega \cap B$, where $B := [0, 2m] \times [0, n]$ is a ‘bathroom floor’, is isomorphic to the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (note that it is spanned by the horizontal and the vertical reflections through the midline of B , whose composition is the 180° rotation around the centre of B). In particular, we see that Ω is very symmetric but $\tilde{\Omega}$ is not, even though they share the same patterns and independently of the number of tiles.

To address these issues, we introduce the *action groupoid* associated with $G_\Omega \curvearrowright \mathbb{R}^n$, defined as

$$\mathcal{G}_\Omega := \{(y, \phi, x) : \phi \in G_\Omega, x, y \in \mathbb{R}^n, y = \phi(x)\}, \quad (1)$$

and endowed with the **partially defined multiplication**

$$(z, \psi, y)(y, \phi, x) := (z, \psi \circ \phi, x). \quad (2)$$

In order to analyse the properties of this partially defined multiplication, we introduce the *source* $s : \mathcal{G}_\Omega \rightarrow \mathbb{R}^n$ and *target* $t : \mathcal{G}_\Omega \rightarrow \mathbb{R}^n$ maps as

$$s(y, \phi, x) := x, \quad t(y, \phi, x) := y, \quad (y, \phi, x) \in \mathcal{G}_\Omega.$$

One easily sees that (2) satisfies the following four properties:

- (1) **COMPOSITION.** Given $g, h \in \mathcal{G}_\Omega$, gh is only defined when $s(g) = t(h)$. In that case, $s(gh) = s(h)$ and $t(gh) = t(g)$.
- (2) **ASSOCIATIVITY.** If $g, h, k \in \mathcal{G}_\Omega$ are such that $s(g) = t(h)$ and $s(h) = t(k)$, then $(gh)k = g(hk)$.
- (3) **UNITS.** For all $x \in \mathbb{R}^n$, let us denote $1_x := (x, \text{id}, x) \in \mathcal{G}_\Omega$. Then, $1_{t(g)}g = g = g1_{s(g)}$ for all $g \in \mathcal{G}_\Omega$.
- (4) **INVERSE.** For all $g = (y, \phi, x) \in \mathcal{G}_\Omega$ there exists $g^{-1} := (x, \phi^{-1}, y)$ such that $gg^{-1} = 1_{t(g)}$ and $g^{-1}g = 1_{s(g)}$.

These are, exactly, the properties characterising a groupoid.

Definition 1 (Groupoid). A *groupoid* consists of two sets, \mathcal{G} and M , together with maps

- $s, t : \mathcal{G} \rightarrow M$ (*source* and *target* projections);
- $m : \mathcal{G}^{(2)} := \{(g, h) : s(g) = t(h)\} \rightarrow \mathcal{G}, (g, h) \mapsto gh$ (*multiplication*);
- $u : M \rightarrow \mathcal{G}, x \mapsto 1_X$ (*unit*), and
- $i : \mathcal{G} \rightarrow \mathcal{G}, g \mapsto g^{-1}$ (*inverse*),

satisfying the following four properties:

- (1) If $z \xleftarrow{g} y \xleftarrow{h} x$, then $z \xleftarrow{gh} x$;

- (2) If $z \xleftarrow{g} y \xleftarrow{h} x \xleftarrow{k} v$, then $(gh)k = g(hk)$;
- (3) There exists $x \xleftarrow{1_x} x$ such that for all $y \xleftarrow{g} x$, we have $1_y g = g = 1_x$, and
- (4) If $y \xleftarrow{g} x$, there exists $x \xleftarrow{g^{-1}} y$ such that $gg^{-1} = 1_y$ and $g^{-1}g = 1_x$.

From now on, we will denote the groupoid as $\mathcal{G} \rightrightarrows M$, where the parallel arrows refer to the source and target maps.

Remark 2. (1) A groupoid is a small category where every arrow is invertible. Within this language, given a groupoid $\mathcal{G} \rightrightarrows M$, \mathcal{G} is usually referred to as the set of *arrows* of the groupoid, M is the set of *objects* or *units* (also called the *base* of the groupoid), and $\mathcal{G}^{(2)}$ is the set of *composable arrows*.

- (2) We write arrows from left to right to refer to the order of composition. Everything can be done analogously writing arrows from left to right by swapping the source and target maps s and t .
- (3) Groupoids can be restricted to subsets of the set of units. That is, given a groupoid $\mathcal{G} \rightrightarrows M$, its *restriction* to $N \subset M$ is the groupoid

$$\mathcal{G}|_N := s^{-1}(N) \cap t^{-1}(N) \rightrightarrows N.$$

Definition 3 (Orbit and isotropy group). Let $\mathcal{G} \rightrightarrows M$ be a groupoid and let $x \in M$ be a unit.

- The *orbit* of x is $O_x := \{y \in M : \text{there exists } g \in \mathcal{G}, y \xleftarrow{g} x\}$.
- The *isotropy group* of x is $\mathcal{G}_x := s^{-1}(x) \cap t^{-1}(x) = \{g \in \mathcal{G} : x \xleftarrow{g} x\}$.

Example 2 (Weinstein's tiling continued [3]). Let us now go back to the tiling Ω of \mathbb{R}^2 introduced in Example 1. First of all, we consider the action groupoid $\mathcal{G}_\Omega \rightrightarrows \mathbb{R}^2$ associated with $G_\Omega \curvearrowright \mathbb{R}^2$, defined in (1); that is,

$$\mathcal{G}_\Omega := \{(y, \phi, x) : \phi \in G_\Omega, y \xleftarrow{\phi} x\} \rightrightarrows \mathbb{R}^2,$$

and let us now consider its restriction to the ‘bathroom floor’ $B = [0, 2m] \times [0, n]$, namely $\mathcal{G}_\Omega|_B \rightrightarrows B$. Then:

- $x, y \in B$ belong to the same orbit if they are similarly placed in their tiles, and
- the isotropy group of a point $x \in B$ is trivial, unless $x \in \frac{1}{2}\Lambda \cap B$, for which its isotropy is the Klein four group.

This means that $\mathcal{G}_\Omega|_B \rightrightarrows B$ detects local information.

Some applications to Mathematical Physics.

- Groupoid picture of Schwinger’s quantum mechanics by F. M. Ciaglia, A. Ibort, G. Marmo and collaborators initiated in [4]. Recent work on the formulation of fields as functors between groupoids by A. Ibort, A. Mas and L. Schiavone [5].
- The material groupoid, a groupoidal approach to continuum mechanics, by M. de León, M. Epstein and V. Jiménez [6].
- Discrete Lagrangian and Hamiltonian mechanics on Lie groupoids by J. C. Marrero, D. Martín de Diego, E. Martínez and L. Colombo [7, 8].

Local Lie group actions. Let us consider the Lie group $\mathrm{PGL}(2, \mathbb{R})$ consisting of homographies of the projective line \mathbb{RP}^1 . Explicitly, one has

$$\mathrm{PGL}(2, \mathbb{R}) = \mathrm{GL}(2, \mathbb{R})/\mathbb{R},$$

where \mathbb{R} refers to the subgroup of $\mathrm{GL}(2, \mathbb{R})$ formed by non-zero diagonal scalar matrices. Clearly, $\mathrm{PGL}(2, \mathbb{R})$ acts on \mathbb{RP}^1 as

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \cdot [x_0 : x_1] = [ax_0 + bx_1 : cx_0 + dx_1]. \quad (3)$$

Let us now analyse how $\mathrm{PGL}(2, \mathbb{R})$ acts on the affine line $\mathbb{R} \equiv \mathbb{RP}^1 - \{[1 : 0]\}$. Given an affine point $[x_0 : x_1] = [x : 1] \in \mathbb{R}$, where $x := x_0/x_1$, and $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \in \mathrm{PGL}(2, \mathbb{R})$, we have that

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \cdot [x : 1] = \left[\frac{ax + b}{cx + d} : 1 \right] \in \mathbb{R}, \quad (4)$$

provided that $cx + d \neq 0$. It is natural to consider the following question:

Does formula (4) define a Lie group action on \mathbb{R} ?

Of course, the answer to this question is negative, since (4) is only defined on an open subset of $\mathrm{PGL}(2, \mathbb{R}) \times \mathbb{R}$. Indeed, it defines what is called a *local Lie group action* (see [9] for more details). This means, roughly speaking, that (4) only satisfies the defining properties of a Lie group action on a neighbourhood of the unit element of $\mathrm{PGL}(2, \mathbb{R})$.

Hopefully, this problem can be tackled successfully when one considers groupoids rather than groups. First of all, let us consider the so-called action groupoid associated with the Lie group action $\mathrm{PGL}(2, \mathbb{R}) \curvearrowright \mathbb{RP}^1$ defined in (3) as the groupoid $\mathcal{G} := \mathrm{PGL}(2, \mathbb{R}) \times \mathbb{RP}^1 \rightrightarrows \mathbb{RP}^1$ with source and target maps given by

$$s([A], [x_0 : x_1]) = [x_0 : x_1], \quad t([A], [x_0 : x_1]) = [A] \cdot [x_0 : x_1],$$

and multiplication defined as

$$([B], [y_0 : y_1])([A], [x_0 : x_1]) := ([BA], [x_0 : x_1]),$$

provided that $[A] \cdot [x_0 : x_1] = [y_0 : y_1]$. Then, the restriction of $\mathcal{G} \rightrightarrows \mathbb{RP}^1$ to the affine line \mathbb{R} is the groupoid

$$\mathcal{G}|_{\mathbb{R}} = s^{-1}(\mathbb{R}) \cap t^{-1}(\mathbb{R}) = \left\{ \left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right], [x : 1] \right) \in \mathrm{PGL}(2, \mathbb{R}) \times \mathbb{R} : cx + d \neq 0 \right\} \rightrightarrows \mathbb{R}.$$

Within this context, the target map of $\mathcal{G}|_{\mathbb{R}} \rightrightarrows \mathbb{R}$ is just formula (4).

Singular spaces. Let $G \curvearrowright M$ be a smooth action of a Lie group G on a manifold M (see [10, Chapter 2] for more details on Lie group actions). Recall that the action is said to be

- *free* if $g \cdot x = x$ for some $x \in M$, then $g = e$, and

- *proper* if the map $\Phi : G \times M \ni (g, x) \mapsto (g \cdot x, x) \in M \times M$ is a proper map. That is, for every compact subset $K \subset M \times M$ we have that $\Phi^{-1}(K) \subset G \times M$ is compact. This is equivalent to the following: if (g_n) and (x_n) are sequences in G and M , respectively, such that (x_n) and $(g_n \cdot x_n)$ are convergent, then (g_n) contains a converging subsequence.

Provided that the action $G \curvearrowright M$ is free and proper, the quotient manifold M/G has a unique smooth structure such that the projection map $\pi : M \rightarrow M/G$ is a submersion. For instance, let us consider the action $G = \mathbb{Z}^k \curvearrowright M = \mathbb{R}^k$ given by translations:

$$(n_1, \dots, n_k) \cdot (x^1, \dots, x^k) := (x^1 + n_1, \dots, x^k + n_k).$$

It can be easily seen that this action is free and proper. Hence, the quotient space $\mathbb{R}^k/\mathbb{Z}^k$, which is the k -dimensional torus $\mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$, possesses a unique smooth structure such that the projection $\pi : M \rightarrow M/G$ is a submersion.

What happens when the action $G \curvearrowright M$ is not free and proper?

If action $G \curvearrowright M$ is not free and proper, G/M is a “singular space”

Example 3. Let us consider the actions $\mathrm{SO}(2) \curvearrowright \mathbb{R}^2$ and $\mathrm{SO}(3) \curvearrowright \mathbb{R}^3$ given by rotations. Both of these actions are proper, since the associated Lie groups are compact. Nevertheless, they are not free: the origin is a fixed point of every rotation. The “singular spaces” $\mathbb{R}^2/\mathrm{SO}(2)$ and $\mathbb{R}^3/\mathrm{SO}(3)$ are both homeomorphic to $[0, +\infty)$. Notwithstanding, they are not equivalent as singular spaces because their associated action groupoids $\mathrm{SO}(2) \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ and $\mathrm{SO}(3) \times \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ are not *Morita equivalent*. Roughly speaking, this can be justified as follows. The isotropy group of $x \in \mathbb{R}^2 - \{0\}$ is trivial, while the isotropy group of $x \in \mathbb{R}^3 - \{0\}$ is isomorphic to $\mathrm{SO}(2)$.

Lecture 2 – Bartosz M. Zawora (KMMF - UW)

Date: 14 October, 2025

Abstract: During the lecture, I will recall the definition of a groupoid from the previous session and present the definition of a Lie groupoid. I will provide many useful and important examples and prove some fundamental properties. The lecture will follow Mackenzie's book "General Theory of Lie Groupoids and Lie Algebroids".

1. Lie groupoids: definition and examples

Definition 4. A groupoid consists of two sets \mathcal{G} and M , with maps,

$s : \mathcal{G} \rightarrow M$ (source),

$t : \mathcal{G} \rightarrow M$ (target),

$u : M \rightarrow \mathcal{G} \mid x \mapsto 1_x$ (unit),

$i : \mathcal{G} \rightarrow \mathcal{G} \mid g \mapsto g^{-1}$ (inverse),

$\kappa : \mathcal{G} * \mathcal{G} \rightarrow \mathcal{G} \mid (h, g) \mapsto hg$ where $\mathcal{G} * \mathcal{G} := \{(h, g) \in \mathcal{G} \times \mathcal{G} \mid s(h) = t(g)\}$,

satisfying

- (1) $s(hg) = s(g)$ and $t(hg) = t(h)$ for all $(h, g) \in \mathcal{G} * \mathcal{G}$,
- (2) $j(hg) = (jh)g$ for every $j, h, g \in \mathcal{G}$ such that $s(j) = t(h)$ and $s(h) = t(g)$,
- (3) $s(1_x) = t(1_x) = x$ for any $x \in M$,
- (4) $g1_{s(g)} = g$ and $1_{t(y)}g = g$ for every $g \in \mathcal{G}$,
- (5) For each $g \in \mathcal{G}$, there exists two-sided inverse g^{-1} such that $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g^{-1}g = 1_{s(g)}$, and $gg^{-1} = 1_{t(g)}$.

To simplify the notation, $\mathcal{G} \rightrightarrows M$ denotes a groupoid.

Proposition 5. Let $\mathcal{G} \rightrightarrows M$ and $g \in \mathcal{G}$ with $s(g) = x$ and $t(g) = y$.

- (1) If $h \in \mathcal{G}$ with $s(h) = g$ and $hg = g$, then $h = 1_y$.
If $j \in \mathcal{G}$ with $t(j) = x$ and $gj = g$, then $j = 1_x$.
- (2) If $h \in \mathcal{G}$ with $s(h) = y$ and $hg = 1_x$, then $h = g^{-1}$.
If $j \in \mathcal{G}$ with $t(j) = x$ and $gj = 1_y$, then $j = g^{-1}$.

Definition 6. Let $\mathcal{G} \rightrightarrows M$. Then, $\mathcal{G}_x := s^{-1}(x)$ is called a s -fibre over $x \in M$, similarly $\mathcal{G}^y := t^{-1}(y)$ is a t -fibre over $y \in M$, and $\mathcal{G}_x^y := \mathcal{G}_x \cap \mathcal{G}^y = s^{-1}(x) \cap t^{-1}(y)$. Moreover, the set $\mathcal{G}_x \cap \mathcal{G}^y$ is called the vertex group. The set of identity elements 1_M is defined as $\{1_m \mid m \in M\}$.

Definition 7. A Lie groupoid $\mathcal{G} \rightrightarrows M$ is a groupoid $\mathcal{G} \rightrightarrows M$ with smooth manifold structures on \mathcal{G} and M such that $s, t : \mathcal{G} \rightarrow M$ are surjective submersions, $u : M \rightarrow \mathcal{G}$ is smooth, and $\kappa : \mathcal{G} * \mathcal{G} \rightarrow \mathcal{G}$ is also smooth.

Remark 8. Since $s, t: \mathcal{G} \rightarrow M$ are submersions it follows that \mathcal{G}_x , \mathcal{G}^y , and $(s \times t)^{-1}(x, x')$ are closed embedded submanifolds for some $x, x' \in M$.

Definition 9. Let $\mathcal{G} \rightrightarrows M$ and $g \in \mathcal{G}$ with $s(g) = x$ and $t(g) = y$. Then, the left-translation corresponding to g is $L_g: h \in \mathcal{G}^x \mapsto gh \in \mathcal{G}^y$. Analogously, the right-translation corresponding to g is $R_g: h \in \mathcal{G}_y \mapsto hg \in \mathcal{G}_x$

Definition 10. A Lie groupoid is s -connected if each of its fibres is connected. Likewise, for any other property α , a Lie groupoid $\mathcal{G} \rightrightarrows M$ is $s\text{-}\alpha$ if each of its fibres has property α .

Proposition 11. Let $\mathcal{G} \rightrightarrows M$. The inverse map $i: \mathcal{G} \rightarrow \mathcal{G}$ is a diffeomorphism.

PROOF. Note that the tangent bundle to $\mathcal{G} * \mathcal{G}$ is given by

$$T(\mathcal{G} * \mathcal{G}) = \{(Y, X) \in T\mathcal{G} \times T\mathcal{G} \mid Ts(Y) = Tt(X)\}.$$

Suppose that $h, g \in \mathcal{G}$ are such that $s(h) = t(g)$, $Y \in T_h(\mathcal{G}_{s(h)})$, and $X \in T_g(\mathcal{G}^{t(g)})$. Then, by Leibniz rule, it follows

$$T_{(h,g)}\kappa(Y, X) = T_hR_g(Y) + T_gL_h(X). \quad (5)$$

Define $\theta: \mathcal{G} * \mathcal{G} \rightarrow \mathcal{G} \times_t \mathcal{G}$ as

$$\theta(h, g) = (h, hg),$$

where $\mathcal{G} \times_t \mathcal{G} := \{(h, g) \in \mathcal{G} \times \mathcal{G} \mid t(h) = t(g)\}$. Then, one can simply verify that θ is bijective with $\theta^{-1}(i, j) = (i, i^{-1}j)$. To prove that θ is an immersion, suppose that $T_{(h,g)}\theta(Y, X) = (0, 0)$. Moreover, define $\pi_1: (h, g) \in \mathcal{G} \times \mathcal{G} \mapsto g \in \mathcal{G}$ and $\pi_2: (h, g) \in \mathcal{G} \times \mathcal{G} \mapsto h \in \mathcal{G}$. Therefore, since $\pi_2 \circ \theta = \pi_2$, it follows that $Y = 0$. Consequently, by (5), one has $X = 0$ and θ is an immersion.

Furthermore, by the fact that s and t are submersions and $\dim \mathcal{G} * \mathcal{G} = \dim \mathcal{G} \times_t \mathcal{G}$, one obtains that θ is a diffeomorphism. For $\mu: h \in \mathcal{G} \mapsto (h, 1_{t(h)})\mathcal{G} \times_t \mathcal{G}$, one has

$$(\pi_1 \circ \theta^{-1} \circ \mu)(g) = (\pi_1 \circ \theta^{-1})(g, 1_{t(g)}) = \pi_1(g, g^{-1}) = g^{-1}.$$

Hence, $(\pi_1 \circ \theta^{-1} \circ \mu) = i$. Since $(\pi_1 \circ \theta^{-1} \circ \mu)$ is smooth, it follows that i is also smooth. In addition, since inversion is its own inverse, it is therefore a diffeomorphism. \square

Remark 12. The unit map $u: M \rightarrow \mathcal{G}$ is smooth and is an immersion, by the fact that

$$t \circ u = \text{id}_M, \quad s \circ u = \text{id}_M.$$

Consequently, 1_M is a closed embedded submanifold of \mathcal{G} .

Example 4 (Lie group). A Lie group is a Lie groupoid with a unique unit, namely $G \rightrightarrows \{\ast\}$.

Example 5 (Manifold). Any manifold M can be regarded as a Lie groupoid on itself, where $s = t = \text{id}_M$.

Example 6 (A pair groupoid). For any manifold M , one has a pair groupoid $M \times M \rightrightarrows M$ with arrows $(y, x) \in M \times M$, $s(y, x) = x$, $t(y, x) = y$, $u(x) = (x, x)$, $i(y, x) = (x, y)$, and

$$\kappa((z, y), (y, x)) = (z, x).$$

Note that \mathcal{G}_x^x is trivial.

Example 7 (Trivial groupoid). For a Lie group G and a manifold M one can construct a Lie groupoid $M \times G \times M \rightrightarrows M$ with $s(y, g, x) = x$, $t(y, g, x) = y$, $u(x) = (x, e, x)$, $i(y, g, x) = (x, g^{-1}, y)$, and

$$\kappa((z, h, y), (y, g, x)) = (y, hg, x).$$

Example 8 (Submersion groupoid). For a surjective submersion $\pi: M \rightarrow Q$ one has a submersion groupoid

$$M \times_Q M \rightrightarrows M,$$

where $M \times_Q M = \{(y, x) \in M \times M \mid \pi(y) = \pi(x)\}$ and is given by the restriction of a pair groupoid $M \times M \rightrightarrows M$ to $M \times_Q M$.

Example 9 (Action groupoid). Let $G \times M \rightarrow M$ be a Lie group action on a manifold M . Then, $G \times M \rightrightarrows M$ is an action Lie groupoid with $s(g, x) = x$, $t(g, x) = gx$, $u(x) = (e, x)$, $i(g, x) = (g^{-1}, gx)$, and

$$\kappa((g_2, y), (g_1, x)) = (g_2 g_1, x),$$

of $y = g_1 x$. Note that $\mathcal{G} \times \mathcal{M}_x^x$ is isomorphic to G_x . An action Lie groupoid is denoted as $G \triangleleft M$.

Example 10. ADD DRAWINGS

Example 11 (Fundamental groupoid). Let M be a smooth manifold and let $\Pi(M)$ be the set of homotopy classes $[\gamma]$ of continuous path $\gamma: [0, 1] \rightarrow M$, relative to fixed and points $\gamma(0)$ and $\gamma(1)$. Then, $\Pi(M) \rightrightarrows M$ is a groupoid with $s([\gamma]) = \gamma(0)$, $t([\gamma])\gamma(1)$, $u(x) = [\gamma_x]$, where γ_x is a constant path at $x \in M$, and

$$\kappa([\delta], [\gamma]) = [\delta\gamma],$$

where $\delta\gamma$ is the standard concatenation of γ and δ , namely

Add

Moreover, $i([\gamma]) = [\overleftarrow{\gamma}]$, where $\overleftarrow{\gamma}$ is the reverse path, namely $\overleftarrow{\gamma}(t) = \gamma(1 - t)$.

Example 12. Let (E, q, M) be a vector bundle. Let $\Phi(E)$ denote the set of all vector space isomorphism $\xi: E_x \rightarrow E_y$ for some $x, y \in M$. Then, $\Phi(E) \rightrightarrows M$ is a frame groupoid with $s(\xi) = x$, $t(\xi) = y$, $u(x) = \text{id}_{E_x}$, the inverse of ξ is just an inverse map of ξ , and multiplication κ is the composition of maps.

Example 13 (Jet groupoid). For a local diffeomorphism $\varphi: U \rightarrow V$ between open sets of a manifold M , and given $x \in U$, let $j_x^1\varphi$ be the one-jet of φ at $x \in U$. Then, $J^1(M, M)$, the set of all such one-jets, has a natural groupoid structure $J^1(M, M) \rightrightarrows M$ with $s(j_x^1\varphi) = x$, $t(j_x^1\varphi) = \varphi(x)$, $\kappa(j_{\varphi(x)}^1\psi, j_x^1\varphi) = j_x^1(\psi \circ \varphi)$, and the inverse is given by $i(j_x^1\varphi) = j_{\varphi(x)}^1\varphi^{-1}$.

Example 14 (Gauge groupoid). Let (P, M, G, π) be a principal bundle and let $(u_2, u_1, g) \in P \times P \times G \mapsto (u_2g, u_1g) \in P \times P$. Let $[(u_2, u_1)]$ denote the equivalence class in $(P \times P)/G$. Then,

$$\frac{P \times P}{G} \rightrightarrows M$$

is a Lie groupoid with $s([(u_2, u_1)]) = \pi(u_1)$, $t([(u_2, u_1)]) = \pi(u_2)$, $u(x) = [(v, v)]$ for $v \in \pi^{-1}(x)$, $i([(u_2, u_1)]) = [(u_1, u_2)]$, and

$$\kappa([(u_3, u_2)], [(u_2, u_1)]) = [(u_3, u_1)].$$

Example 15. Let G be a Lie group with a Lie algebra \mathfrak{g} . Then, $T^*G \rightrightarrows \mathfrak{g}^*$ is a Lie groupoid with $s(\theta) = \theta \circ T_e L_g$, $t(\theta) = \theta \circ T_e R_g$ for given $\theta \in T_g^*G$. Additionally, for $\varphi \in T_h^*G$, one has

$$\kappa(\varphi, \theta) = \varphi \circ T_{hg}R_{g^{-1}} = \theta \circ T_{hg}L_{h^{-1}}$$

and $i(\mu) = \mu \in T_e^*G$ with $i(\theta) = \theta \circ T_e L_g \circ T_{g^{-1}}R_g \in T_{g^{-1}}^*G$.

Example 16. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Applying tangent functor to each maps yields a Lie groupoid $T\mathcal{G} \rightrightarrows TM$.

2. Morphisms and subgroupoids

Definition 13. Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M'$ be groupoids. A morphism of groupoids $(\mathcal{G} \rightrightarrows M) \rightarrow (\mathcal{G}' \rightrightarrows M')$ is a pair of maps (F, f) such that the following diagrams commute

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{F} & \mathcal{G}' \\ \downarrow s & & \downarrow s' \\ M & \xrightarrow{f} & M' \end{array} \quad \begin{array}{ccc} \mathcal{G} & \xrightarrow{f} & \mathcal{G}' \\ \downarrow t & & \downarrow t' \\ M & \xrightarrow{f} & M' \end{array} \quad \begin{array}{ccc} \mathcal{G} \times \mathcal{G} & \xrightarrow{F \times F} & \mathcal{G}' \times \mathcal{G}' \\ \downarrow \kappa & & \downarrow \kappa' \\ \mathcal{G} & \xrightarrow{F} & \mathcal{G}' \end{array} . \text{ If } M = M'$$

and $f = \text{id}_M$, then F is called a base-preserving morphism, or morphism over M . If $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M'$ are Lie groupoids, then (F, f) is a Lie groupoid morphism if F and f are smooth.

Remark 14. The smoothness of F already implies the smoothness of f . Moreover, The above definition implies that $F(h)F(g)$ is well defined whenever hg is.

Proposition 15. Let (F, f) be a groupoid morphism. Then,

- (1) $F(1_x) = 1_{f(x)}$ for every $x \in M$,
- (2) $F(g^{-1}) = F(g)^{-1}$ for every $g \in \mathcal{G}$.

Definition 16. A morphism (F, f) is an isomorphism of Lie groupoids if F and hence f are diffeomorphisms.

Example 17. Let $\mathcal{G} \rightrightarrows M$. Then, $\chi := (t, s): \mathcal{G} \times \mathcal{G} \rightarrow M \times M \mid g \mapsto (t(g), s(g))$ is a morphism over M from $\mathcal{G} \rightrightarrows M$ and $M \times M \rightrightarrows M$. The map χ is called the anchor of \mathcal{G} .

Example 18. Let $G \triangleleft \mathfrak{g}^*$, where the action is given by $\text{Ad}_{g^{-1}}^*$. Then, the left-trivialisation of

$$\lambda: (g, \nu) \in G \times \mathfrak{g}^* \longmapsto \nu \circ T_g L_{g^{-1}},$$

is an isomorphism over \mathfrak{g}^* of $G \triangleleft \mathfrak{g}^*$ and $T^*G \rightrightarrows \mathfrak{g}^*$, see Example 15.

Example 19. For any $\mathcal{G} \rightrightarrows M$, the tangent bundle projections $p_{\mathcal{G}}: T\mathcal{G} \rightarrow \mathcal{G}$ and $p_M: TM \rightarrow M$ give rise to a Lie groupoid morphism from $T\mathcal{G} \rightrightarrows TM$ to $\mathcal{G} \rightrightarrows M$.

Definition 17. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A Lie subgroupoid of $\mathcal{G} \rightrightarrows M$ is a Lie groupoid $\mathcal{G}' \rightrightarrows M'$ with injective immersions $\iota: \mathcal{G}' \rightarrow \mathcal{G}$ and $\iota_o: M' \rightarrow M$ such that (ι, ι_o) is a Lie groupoid morphism. A Lie subgroupoid $\mathcal{G}' \rightrightarrows M'$ of $\mathcal{G} \rightrightarrows M$ is embedded if \mathcal{G}' and M' are embedded submanifolds of \mathcal{G} and M , respectively. A Lie subgroupoid $\mathcal{G}' \rightrightarrows M'$ of $\mathcal{G} \rightrightarrows M$ is wide if $M = M'$ and $\iota_o = \text{id}_M$.

Example 20. If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid and $N \subset M$ is an open submanifold, then $\mathcal{G}_N^N := s^{-1}(N) \cap t^{-1}(N)$ is clearly a Lie subgroupoid.

Example 21. If $\mathcal{G} \rightrightarrows M$ is a groupoid, the inner (set) subgroupoid is defined as $\mathcal{IG} := \{g \in \mathcal{G}: s(g) = t(g)\}$. In general, if $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, this does not define a Lie subgroupoid.

Lecture 3 – Rubén Izquierdo-López (ICMAT)

Date: 21 October, 2025

Abstract: In this lecture, I will present locally trivial Lie groupoids, and prove their "correspondence" with gauge groupoids. The lecture essentially follows section 1.3 from Mackenzie's book "General Theory of Lie groupoids and Lie algebroids"

3. Locally trivial Lie groupoids

Before introducing the definition of locally trivial Lie groupoid, let us recall the notion of transitive groupoid.

Definition 18. Let $\mathcal{G} \rightrightarrows M$ be a groupoid (not necessarily Lie groupoid). Then, it is called **transitive** if the map $(t, s): \mathcal{G} \rightarrow M \times M$ is surjective.

Hence, a groupoid $\mathcal{G} \rightrightarrows M$ is transitive if for every pair of points $x, y \in M$ there is an element $g \in \mathcal{G}$ such that $s(g) = x$ and $t(g) = y$:

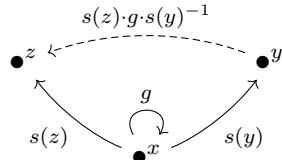
$$\bullet^y \xleftarrow{g} \bullet^x$$

Notice that if a **set** groupoid $\mathcal{G} \rightrightarrows M$ is transitive, then it is trivial. Indeed, fix $x \in M$ and choose a right inverse of the restriction of the target map to the s -fiber of x :

$$\alpha: M \rightarrow \mathcal{G}_x, \quad t \circ \alpha = \text{id}_M.$$

Then, we can build the following isomorphism of **set** groupoids:

$$\Phi: M \times \mathcal{G}_x^x \times M \rightarrow \mathcal{G}, \quad \Phi(z, g, y) := s(z) \cdot g \cdot s(y)^{-1}. \quad (6)$$



What we are going to study now is how this picture translates into the theory of Lie groupoids. The adapted notion of transitivity to the smooth category is the following:

Definition 19. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. It is called **locally trivial** if the map $(t, s): \mathcal{G} \rightarrow M \times M$ defines a surjective submersion.

Our first objective is to show that the isomorphism defined in Eq.(6) is well defined *locally*. In order to achieve this, we need to prove the following characterizations of Definition 19:

Proposition 20. Let $\mathcal{G} \rightrightarrows M$ denote a Lie groupoid and $x \in M$ be a fixed point. Then, the following statements are equivalent:

- (1) The Lie groupoid $\mathcal{G} \rightrightarrows M$ is locally trivial.
- (2) The restriction of the target map $t: \mathcal{G}_x \rightarrow M$ defines a surjective submersion.
- (3) The map

$$\delta: \mathcal{G}_x \times \mathcal{G}_x \rightarrow \mathcal{G}, \quad \delta(g, h) := h \cdot g^{-1},$$

is a surjective submersion.

PROOF. Let us prove the equivalence of the first and last pair of conditions:

- $\boxed{(1) \iff (2)}$ (2) clearly follows from (1). Now, to prove the converse, notice that we may write $(t, s) \circ \delta = (t, t)$. Since, by hypothesis, the map on the left is a surjective submersion, it follows from the chain rule that (t, s) must be a surjective submersion as well, so that the Lie groupoid is locally trivial.
- $\boxed{(2) \iff (3)}$ Assume (2) and define $\mathcal{G}_x \times_M \mathcal{G} = \{(\eta, \xi) \in \mathcal{G}_x \times \mathcal{G} : t(\eta) = t(\xi)\}$, together with the following map:

$$\Psi: \mathcal{G}_x \times \mathcal{G}_x \rightarrow \mathcal{G}_x \times_M \mathcal{G}, \quad \Psi(h, g) := (h, gh^{-1}).$$

Using the fact that $t: \mathcal{G}_x \rightarrow M$ defines a surjective submersion, we have that $\mathcal{G}_x \times_M \mathcal{G}$ is a smooth manifold, and that Ψ defines a diffeomorphism. Hence, to show that δ is a surjective submersion, we only need to show that $\psi^{-1} \circ \delta$ is. A quick computation shows that $\psi^{-1} \circ \delta$ is the projection onto the second factor, which is clearly a surjective submersion. For the converse, it is enough to write $t \circ \delta = t \circ \pi$, where $\pi: \mathcal{G}_x \times_M \mathcal{G} \rightarrow \mathcal{G}_x$ denotes the projection onto the second factor. Now, the map on the left is clearly a surjective submersion, so $t: \mathcal{G}_x \rightarrow M$ is a surjective submersion as well.

□

Now we can show that a locally trivial Lie groupoid $\mathcal{G} \rightrightarrows M$ is, in fact, locally trivial. Indeed, choose $x \in M$ and let $\{U_i\}$ be an open cover of M , together with sections of $t: \mathcal{G}_x \rightarrow M$

$$s_i: U_i \rightarrow \mathcal{G}_x.$$

Now, if we define maps $\Phi: U_i \times \mathcal{G}_x^x \times U_i \rightarrow \mathcal{G}_{U_i}^{U_i}$ as in Eq.(6) (where the space in the right denotes the restriction groupoid), we have an isomorphism of Lie groupoids, so it is, in fact, locally trivial. The maps s_i are called **decomposing sections** and the collection $\{s_i\}$ is called **decomposing atlas**.

Let us give two examples of locally trivial Lie groupoids:

Example 22. Let $E \rightarrow M$ be a vector bundle. Then, the frame groupoid $\Phi(E)$, defined as the space of linear isomorphisms $E_x \rightarrow E_y$ is locally trivial.

Example 23. The gauge groupoid of a principal bundle is locally trivial.

In fact, we claim that every locally trivial groupoid is the gauge groupoid of a principal bundle. Indeed, by our previous discussion, we know that for fixed $x \in M$, the map

$$t: \mathcal{G}_x \rightarrow M$$

is a fiber bundle with standard fiber \mathcal{G}_x^x (this can be seen taking a decomposing atlas). Furthermore, \mathcal{G}_x^x acts on the right by multiplication:

$$\Phi: \mathcal{G}_x \times \mathcal{G}_x^x \rightarrow \mathcal{G}_x, \quad \Phi(h, g) := h \cdot g.$$

This action is free and defines a principal \mathcal{G}_x^x -bundle structure on $t: \mathcal{G}_x \rightarrow M$. We may wonder how does this construction depend on the choice of point $x \in M$. The relation is given as follows:

Proposition 21. Let $\mathcal{G} \rightrightarrows M$ be a locally trivial Lie groupoid and let $g \in \mathcal{G}$. Denote $x = s(g)$ and $y = t(g)$. Then, we have the following diffeomorphism

$$\Psi: \mathcal{G}_y \rightarrow \mathcal{G}_x, \quad \Psi(h) := h \cdot g$$

which, together with the “inner” Lie group isomorphism

$$\varphi: \mathcal{G}_y^y \rightarrow \mathcal{G}_x^x, \quad \varphi(k) = g^{-1} \cdot k \cdot g$$

defines an isomorphism of principal bundles over the identity on M .

PROOF. Clearly, Ψ defines a diffeomorphism, and φ defines an isomorphism of Lie groups. To check that these two maps define an isomorphism of principal bundles, we need to show that Ψ is equivariant with respect to the action. Indeed, let $h \in \mathcal{G}_y$ and $k \in \mathcal{G}_y^y$. Then,

$$\Psi(h \cdot k) = h \cdot k \cdot g = (h \cdot g) \cdot (g^{-1} \cdot k \cdot g) = \Psi(h) \cdot \varphi(k),$$

which completes the proof. \square

Summarizing, to each principal bundle, we can associate its gauge groupoid, which is locally trivial; and to each locally trivial Lie groupoid we may associate a principal bundle (after making a choice $x \in M$). These operations are inverses of each other. However, the isomorphisms depend upon choices, and they cannot be chosen naturally. Nevertheless, the isomorphisms depend functorially:

Proposition 22. Let M be a manifold, $\pi: P \rightarrow M$ be a principal bundle over M , and $\mathcal{G} \rightrightarrows M$ be a locally trivial Lie groupoid.

(1) Given $u_0 \in P$ and denoting $x_0 := \pi(u_0)$, the map

$$P \rightarrow ((P \times P)/G)_{x_0}, \quad u \mapsto [(u, u_0)],$$

is a diffeomorphism, the map

$$G \rightarrow ((P \times P)/G)_{x_0}^{x_0}, \quad g \mapsto [(u_0 \cdot g, u_0)],$$

is an isomorphism of Lie groups, and together they define an isomorphism of principal bundles over M . Furthermore, given a morphism of principal bundles $f: P \rightarrow P'$, where P' is a principal bundle over M' , denoting $u'_0 := f(u_0)$ and $x'_0 := \pi'(u'_0)$, we have a commutative diagram:

$$\begin{array}{ccc} ((P \times P)/G)_{x_0} & \longrightarrow & ((P' \times P')/G')_{x'_0} \\ \uparrow & & \uparrow \\ P & \xrightarrow{\quad} & P' \end{array},$$

where the arrows are defined in the obvious manner.

(2) Given $x \in M$, the map

$$(\mathcal{G}_x \times \mathcal{G}_x)/\mathcal{G}_x^x \rightarrow \mathcal{G}, \quad [(g, h)] \mapsto gh^{-1},$$

defines an isomorphism of Lie groupoids. Furthermore, given a morphism of Lie groupoids $\varphi: \mathcal{G} \rightarrow \mathcal{G}'$, where \mathcal{G}' is a locally trivial Lie groupoid over M' , if we define $x' := \varphi(x)$, we get a commutative diagram

$$\begin{array}{ccc} (\mathcal{G}_x \times \mathcal{G}_x)/\mathcal{G}_x^x & \longrightarrow & (\mathcal{G}'_{x'} \times \mathcal{G}'_{x'})/\mathcal{G}'_{x'}^{x'} \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{\hspace{2cm}} & \mathcal{G}' \end{array},$$

where the arrows are defined in the obvious manner.

PROOF. The proof follows from a straight-forward computation. \square

An important consequence of Proposition 22 is the following:

Corollary 23. Let $\mathcal{G} \rightrightarrows M$ denote a locally trivial Lie groupoid and $\mathcal{G}' \rightrightarrows M'$ denote a Lie groupoid (not necessarily locally trivial). Let $\varphi: \mathcal{G} \rightarrow \mathcal{G}'$ be a set morphism of groupoids. Then,

- (1) Fix $x \in M$. If $\varphi: \mathcal{G}_x \rightarrow \mathcal{G}_{\varphi(x)}$ is smooth, so is $\varphi: \mathcal{G} \rightarrow \mathcal{G}'$.
- (2) If φ is smooth in a neighborhood of the identity, φ is smooth.

PROOF. (1) follows straight-forwardly using Proposition 22 (2). (2) follows by checking (1) for a particular $x \in M$, using right translations. \square

As we mentioned already, the correspondence should not be thought of as an equivalence, given that it is not natural. However, we may use constructions from the world of principal bundles and carry them over to the world of locally trivial Lie groupoids. The most important of these constructions being the associated bundles. Given a principal G -bundle $\pi: P \rightarrow M$, and a left action of G on a manifold F , we can build a fiber bundle $P[F] \rightarrow M$ with standard fiber F by taking the quotient $P[F] := (P \times F)/G$, where $(p, f) \sim (p \cdot g, g^{-1} \cdot f)$. When G preserves a particular structure on F (group, vector space, hermitian...) the induced bundle $P[F]$ inherits this structure fiber wise. For instance if G acts on itself by conjugation, $P[G]$ is a Lie group bundle. When P is the principal bundle obtained from a locally trivial Lie groupoid $\mathcal{G}_x \rightrightarrows M$, the canonical Lie group bundle may be thought of as the inner groupoid:

Proposition 24. Let $\mathcal{G} \rightrightarrows M$ be a locally trivial Lie groupoid, $x \in M$, and $\mathcal{G}_x[\mathcal{G}_x^x]$ denote the canonical Lie group bundle. Then, the map

$$\mathcal{G}_x[\mathcal{G}_x^x] \rightarrow \mathcal{G}, \quad [(g, h)] \mapsto g \cdot h \cdot g^{-1}$$

takes values in the inner subgroupoid $\mathcal{IG} \subset \mathcal{G}$.

PROOF. The proof follows from a straight-forward computation. \square

Lecture 4 – Asier López-Gordón (IMPAN)

Date: 4 November, 2025

4. Bisections

The main reference for this lecture are [1, Section 1.4] and [11, 12].

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with source $s: \mathcal{G} \rightarrow M$, target $t: \mathcal{G} \rightarrow M$, unit $u: M \rightarrow \mathcal{G} \mid x \mapsto 1_x$ and inverse $i: \mathcal{G} \rightarrow \mathcal{G} \mid g \mapsto g^{-1}$.

On a group $G \ni g$, left-translations $L_g: h \mapsto gh$ and inner automorphisms $I_g: h \mapsto ghg^{-1}$ each form a group isomorphic to G itself, while right-translations $R_g: h \mapsto hg$ constitute a group anti-isomorphic to G .

On a groupoid, on the other hand, left-translations and right-translations only are defined on fibers of the target and source projections, respectively:

$$\begin{aligned} L_g: \mathcal{G}^x &\ni h \mapsto gh \in \mathcal{G}^y, \\ R_g: \mathcal{G}_y &\ni h \mapsto hg \in \mathcal{G}_x, \end{aligned}$$

where $g \in \mathcal{G}$ with $s(g) = x$ and $t(g) = y$. In order to extend these concepts to maps of the whole groupoid, we introduce the notion of bisections.

Definition 25. A *bisection* is a smooth section $\sigma: M \rightarrow \mathcal{G}$ of s (i.e., $s \circ \sigma = \text{Id}_M$) such that $t \circ \sigma: M \rightarrow M$ is a diffeomorphism. The set of bisections of \mathcal{G} is denoted by $\text{Bis}(\mathcal{G})$.

In other words, a bisection is a smooth map $\sigma: M \rightarrow \mathcal{G}$ making the following diagram commutative:

$$\begin{array}{ccccc} & & \mathcal{G} & & \\ & \swarrow t & \downarrow \sigma & \searrow s & \\ M & \xrightarrow[t \circ \sigma]{\sim} & M & & \end{array}$$

where all the maps are smooth.

Proposition 26. The set of bisections $\text{Bis}(\mathcal{G})$ is canonically endowed with a group structure, with multiplication

$$\begin{aligned} \text{Bis}(\mathcal{G}) \times \text{Bis}(\mathcal{G}) &\ni (\sigma, \tau) \mapsto \sigma * \tau \in \text{Bis}(\mathcal{G}) \\ \sigma * \tau(x) &:= \sigma((t \circ \tau)(x))\tau(x), \quad \forall x \in M. \end{aligned} \tag{7}$$

The neutral element is just the unit map of the groupoid, while the inverse element σ^{-1} of σ is determined by the inverse map of the groupoid, namely,

$$\sigma^{-1}(x) := i\left(\sigma((t \circ \sigma)^{-1}(x))\right) = \left(\sigma((t \circ \sigma)^{-1}(x))\right)^{-1}.$$

$$\begin{array}{ccccc} & & \sigma \star \tau(x) & & \\ & \swarrow & & \searrow & \\ z = t \circ \sigma(y) & \xleftarrow{\sigma(y)} & y = t \circ \tau(x) & \xleftarrow{\tau(x)} & x = s \circ \tau(x) \\ \curvearrowleft & & \curvearrowright & & \curvearrowright \\ \sigma^{-1}(z) & & \tau^{-1}(y) & & \end{array}$$

PROOF. For any $\sigma, \tau \in \text{Bis}(\mathcal{G})$, we have that $s(\sigma \star \tau) = s(\tau) = \text{Id}_M$ and $t(\sigma \star \tau) = t(\sigma)$, which is a diffeomorphism. Thus, $\sigma \star \tau$ is also a bisection. The fact that $u: M \ni x \mapsto 1_x \in \mathcal{G}$ is the identity element is obvious. Finally, for any $x \in M$, we can write

$$\sigma^{-1}((t \circ \sigma)(x)) = (\sigma(x))^{-1},$$

and thereupon

$$\sigma^{-1} \star \sigma(x) = (\sigma(x))^{-1} \sigma(x) = 1_x.$$

Moreover,

$$\sigma \star \sigma^{-1}(x) = \sigma((t \circ \sigma^{-1})(x)) \sigma^{-1}(x) = \sigma((t \circ \sigma^{-1})(x)) \left(\sigma((t \circ \sigma)^{-1}(x)) \right)^{-1}$$

but

$$t \circ \sigma^{-1}(x) = s \circ \left(\sigma((t \circ \sigma)^{-1}(x)) \right) = (t \circ \sigma)^{-1}(x),$$

and thus

$$\sigma \star \sigma^{-1}(x) = 1_x.$$

□

Remark 27. The map

$$\text{Bis}(\mathcal{G}) \ni \sigma \mapsto t \circ \sigma \in \text{Diff}(M)$$

is a group homomorphism. Furthermore, any morphism of Lie groupoids over M

$$f: (\mathcal{G} \rightrightarrows M) \rightarrow (\mathcal{H} \rightrightarrows M)$$

induces a morphism

$$\begin{aligned} \text{Bis}(f): \text{Bis}(\mathcal{G}) &\rightarrow \text{Bis}(\mathcal{H}) \\ \sigma &\mapsto f \circ \sigma. \end{aligned}$$

Hence, we can regard Bis as a functor from the category of Lie groupoids over M to the category of groups:

$$\text{Bis}: \text{LieGroupoids}_M \rightarrow \text{Groups}.$$

In particular, isomorphic Lie groupoids have isomorphic groups of bisections.

Remark 28. In principle, $\text{Bis}(\mathcal{G})$ is only a group. Under certain assumptions on \mathcal{G} and M , it is possible to endow it with a structure of Lie group (see [11] and references therein).

Example 24 (Lie group). Given a Lie group G , we have the Lie groupoid $G \rightrightarrows \{\ast\}$. Then, $\text{Bis}(G) \cong G$.

Example 25 (Manifold). Given a manifold M , we can define the Lie groupoid $M \rightrightarrows M$ with $s = t = \text{Id}_M$, whose group of bisections is $\text{Bis}(M) = \{\text{Id}_M\}$.

Example 26 (Pair groupoid). For $M \times M \rightarrow M$, any bisection is of the form

$$\sigma: M \times M \rightarrow M, \quad \sigma(x) = (\varphi(x), x),$$

with $\varphi \in \text{Diff}(M)$. Hence,

$$\text{Bis}(M \times M) \cong \text{Diff}(M).$$

Example 27 (Trivial groupoid). Consider the Lie groupoid $M \times G \times M \rightrightarrows M$, for a Lie group G and a manifold M , with $s(y, g, x) = x$, $t(y, g, x) = y$, $u(x) = (x, e, x)$, $i(y, g, x) = (x, g^{-1}, y)$, and

$$\kappa((z, h, y), (y, g, x)) = (y, hg, x).$$

Any bisection $\sigma: M \rightarrow M \times G \times M$ can be written as

$$\sigma(x) = (\varphi(x), \theta(x), x),$$

where $\varphi \in \text{Diff}(M)$ and $\theta: M \rightarrow G$ is a smooth map. The multiplication is given by

$$\sigma_1 \star \sigma_2(x) = (\varphi_1 \circ \varphi_2(x), (\theta_1 \circ \varphi_2(x))\theta_2(x), x), \quad \forall x \in M,$$

for any pair of bisections $\sigma_i = (\varphi_i, \theta_i, \text{Id}_M) \in \text{Bis}(M \times G \times M)$, $i = 1, 2$. The inverse is given by

$$\sigma^{-1}(x) = (\varphi^{-1}(x), \theta^{-1}(\varphi^{-1}(x)), x), \quad \forall x \in M.$$

Example 28 (Action groupoid). Consider an action Lie groupoid $G \triangleleft M$:

$$\begin{array}{ccccc} & & (hg, x) & & \\ & \swarrow^{(h, gx)} & & \searrow_{(g^{-1}, y)} & \\ z = hgx & & y = gx & & x \\ & \searrow_{(h^{-1}, z)} & \swarrow^{(g, x)} & & \\ & & & & \end{array}$$

A bisection of G is a smooth map

$$\sigma: M \rightarrow G \times M, \quad \sigma(x) = (\rho(x), x)$$

such that $M \ni x \mapsto t \circ \sigma(x) = \rho(x) \cdot x \in M$ is a diffeomorphism.

The Lie group action of G on M is transitive (i.e., for any $x, y \in M$, there exists a $g \in G$ such that $y = gx$) if and only if for each $\varphi \in \text{Diff}(M)$ there exists a bisection σ such that

$$\varphi(x) = t \circ \sigma(x) = \rho(x) \cdot x, \quad \forall x \in M.$$

Example 29 (Frame groupoid). Let (E, q, M) be a vector bundle. Let us recall that the set $\Phi(E)$ of all vector space isomorphisms

$$\xi_x^y: E_x \rightarrow E_y, \quad x, y \in M,$$

can be equipped with the structure of a Lie groupoid, the so-called frame groupoid $\Phi(E) \rightrightarrows M$.

$$\begin{array}{ccccc}
& & \chi_y^z \cdot \xi_x^y & & \\
& \swarrow & & \searrow & \\
z = t(\chi_y^z) & \xleftarrow{\chi_y^z} & y = t(\xi_x^y) & \xleftarrow{\xi_x^y} & x = s(\xi_x^y) \\
& \searrow & & \swarrow & \\
& (\chi_y^z)^{-1} & & (\xi_x^y)^{-1} &
\end{array}$$

There is a one-to-one correspondence between bisections of $\Phi(E)$ and VB automorphisms of the VB (E, q, M) . Indeed, if $\sigma \in \text{Bis}(\Phi(E))$, we can define $\bar{\sigma}: E \rightarrow E$ by

$$\bar{\sigma}: E_x \ni v_x \mapsto \sigma(x)(v_x) = \xi_x^y(v_x) \in E_y.$$

Then,

$$\overline{\sigma \star \tau}(v_x) = \sigma \star \tau(x)(v_x) = \xi_x^z \cdot \chi_x^y(v_x) = \bar{\sigma} \circ \bar{\tau}(v_x), \quad \forall v_x \in E_x, \quad \forall x \in M, \quad (8)$$

that is, the multiplication of bisections corresponds to the composition of automorphisms. The identity bisection (i.e., the unit map of the groupoid) corresponds to the identity automorphism:

$$\bar{u} = \text{Id}_E.$$

Therefore,

$$\overline{\sigma^{-1}} = (\bar{\sigma})^{-1}.$$

Conversely, given a VB automorphism

$$\begin{array}{ccc}
E & \xrightarrow[\sim]{\varphi} & E \\
\downarrow q & & \downarrow q \\
M & \xrightarrow[\sim]{\varphi_0} & M
\end{array}$$

we can define a bisection σ by

$$\sigma(x) = \varphi_x: E_x \rightarrow E_{\varphi(x)}, \quad \forall x \in M.$$

The local triviality of the VB guarantees the smoothness of σ .

Each VB automorphism $\bar{\sigma}: E \rightarrow E$, and thus each bisection $\sigma \in \text{Bis}(\Phi(E))$, induces a map of sections of E :

$$\tilde{\sigma}: \Gamma(E) \rightarrow \Gamma(E),$$

given by

$$\tilde{\sigma}(\mu)(t \circ \sigma(x)) = \bar{\sigma}_x(\mu(x)) = \sigma(x)\mu(x), \quad \forall \mu \in \Gamma(E), \quad \forall x \in M,$$

that is,

$$\tilde{\sigma}(\mu)(x) = \sigma((t \circ \sigma)^{-1}(x))\mu((t \circ \sigma)^{-1}(x)).$$

It also induces a map of smooth functions

$$\begin{aligned}
\hat{\sigma}: \mathcal{C}^\infty(M) &\rightarrow \mathcal{C}^\infty(M) \\
f &\mapsto f \circ (t \circ \sigma)^{-1}.
\end{aligned}$$

The map of sections inherits the \mathbb{R} -linearity from the associated VB automorphism. In particular,

$$\tilde{\sigma}(\mu_1 + \mu_2) = \tilde{\sigma}(\mu_1) + \tilde{\sigma}(\mu_2), \quad \forall \mu_1, \mu_2 \in \Gamma(E).$$

By equation (8), the multiplication of bisections corresponds to the composition of the associated maps of sections:

$$\widetilde{\sigma} \star \widetilde{\mu} = \widetilde{\sigma} \circ \widetilde{\mu}.$$

Moreover,

$$\begin{aligned}\widetilde{\sigma}(f\mu)(x) &= \sigma((t \circ \sigma)^{-1}(x)) f((t \circ \sigma)^{-1}(x)) \mu((t \circ \sigma)^{-1}(x)) \\ &= \hat{\sigma}(f)(x) \widetilde{\sigma}(\mu)(x), \quad \forall \mu \in \Gamma(E), \quad \forall f \in \mathcal{C}^\infty(M) \quad \forall x \in M,\end{aligned}$$

i.e.

$$\widetilde{\sigma}(f\mu) = \hat{\sigma}(f) \widetilde{\sigma}(\mu).$$

In particular, the map $\widetilde{\sigma}: \Gamma(E) \rightarrow \Gamma(E)$ is not an automorphism of the $\mathcal{C}^\infty(M)$ -module $\Gamma(E)$.

Notation 29. The maps $\bar{\sigma}, \hat{\sigma}$ and $\widetilde{\sigma}$ are all denoted by $\bar{\sigma}$ in Mackenzie's book.

Example 30 (Gauge groupoid). In the previous lecture, we saw an isomorphism of principal bundles over M relating (P, M, G, π) and the gauge groupoid (also known as Atiyah groupoid) $\frac{P \times P}{G} \rightrightarrows M$ (see Proposition 22). In particular, given $u_0 \in P$ with $x_0 := \pi(u_0)$, the map

$$P \ni u \mapsto [(u, u_0)] \in \left. \frac{P \times P}{G} \right|_{x_0}$$

is a diffeomorphism. Let $\varphi: P \rightarrow P$ be a principal bundle automorphism, such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{\varphi_0} & M \end{array}$$

is commutative, and

$$\varphi(ug) = \varphi(u)g.$$

These type of principal bundle automorphisms are called *gauge transformations*. Composing the gauge transformation φ with the principal bundle isomorphism above, we can define an inner automorphism

$$\begin{aligned}\frac{P \times P}{G} &\rightarrow \frac{P \times P}{G} \\ [v, u] &\mapsto [\varphi(v), \varphi(u)].\end{aligned}$$

Its associated bisection $\sigma \in \text{Bis}((P \times P)/G)$ is given by

$$\begin{aligned}\sigma: M &\rightarrow \frac{P \times P}{G} \\ x = \pi(u) &\mapsto [\varphi(u), u].\end{aligned}$$

Moreover, a locally trivial Lie groupoid $\mathcal{G} \rightrightarrows M$ and the gauge groupoid of its vertex bundle at any $x_0 \in M$ are isomorphic Lie groupoids over M

$$\frac{\mathcal{G}_{x_0} \times \mathcal{G}_{x_0}}{\mathcal{G}_{x_0}^{x_0}} \xrightarrow{\sim} \mathcal{G}, \quad [(g, h)] \mapsto gh^{-1},$$

Therefore, their corresponding groups of bisections are isomorphic:

$$\text{Bis}(\mathcal{G}) \cong \text{Bis}\left(\frac{\mathcal{G}_{x_0} \times \mathcal{G}_{x_0}}{\mathcal{G}_{x_0}^{x_0}}\right).$$

Hence, each bisection $\sigma \in \text{Bis}(\mathcal{G})$ can be written as

$$\sigma: x = t(g) \mapsto \varphi(g)g^{-1}, \quad \forall g \in \mathcal{G}_{x_0},$$

where $\varphi: \mathcal{G}_{x_0} \rightarrow \mathcal{G}_{x_0}$ is an automorphism of the principal bundle $(\mathcal{G}_{x_0}, M, \mathcal{G}_{x_0}^{x_0}, t)$.

Definition 30. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A bisection $\sigma \in \text{Bis}(\mathcal{G})$ defines three maps $L_\sigma, R_\sigma, I_\sigma: \mathcal{G} \rightarrow \mathcal{G}$ as follows:

- The *left-translation*

$$L_\sigma(g) = \sigma(t(g)) \cdot g. \quad (9)$$

- The *right-translation*

$$R_\sigma(g) = g \cdot \sigma((t \circ \sigma)^{-1}(s(g))). \quad (10)$$

- The *inner automorphism*

$$I_\sigma(g) = \sigma(t(g)) \cdot g \cdot (\sigma(s(g)))^{-1}. \quad (11)$$

The following properties can be proven straightforwardly:

$$\begin{aligned} L_{\sigma \star \tau} &= L_\sigma \circ L_\tau, \\ R_{\sigma \star \tau} &= R_\tau \circ R_\sigma, \\ I_{\sigma \star \tau} &= I_\sigma \circ I_\tau, \\ I_\sigma &= L_\sigma \circ R_{\sigma^{-1}} = R_{\sigma^{-1}} \circ L_\sigma, \end{aligned}$$

for any $\sigma, \tau \in \text{Bis}(\mathcal{G})$. Therefore, the sets of left-translations, right-translations, and inner automorphisms each form a group under composition; the maps $\sigma \mapsto L_\sigma$ and $\sigma \mapsto I_\sigma$ are group isomorphisms, while $\sigma \mapsto R_\sigma$ is an anti-isomorphism.

Definition 31. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and let $U \subseteq M$ be an open subset. A *local bisection* is a smooth map $\sigma: U \rightarrow \mathcal{G}$ such that $s \circ \sigma = \text{Id}_U$ and $t \circ \sigma|_U \rightarrow t \circ \sigma(U)$ is a diffeomorphism. The set of local bisections of \mathcal{G} on U is denoted by $\text{Bis}_U(\mathcal{G})$.

Each local bisection $\sigma \in \text{Bis}_U(\mathcal{G})$ defines:

- The *local left-translation* $L_\sigma: \mathcal{G}^U \rightarrow \mathcal{G}^V$, given by (9).
- The *local right-translation* $R_\sigma: \mathcal{G}_V \rightarrow \mathcal{G}_U$, given by (10).
- The *local inner automorphism* $I_\sigma: \mathcal{G}_U^U \rightarrow \mathcal{G}_V^V$, given by (11).

Proposition 32. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. For any $g \in \mathcal{G}$, there exists a sufficiently small open subset $U \subseteq M$ and a local bisection $\sigma \in \text{Bis}_U(\mathcal{G})$ such that $g = \sigma(U)$.

PROOF. Let C be a complementary vector subspace to both $\ker T_g s$ and $\ker T_g t$ in $T_g \mathcal{G}$, namely,

$$T_g \mathcal{G} = \ker T_g s \oplus C = \ker T_g t \oplus C.$$

Since the source map is a surjective submersion, it has local sections. On a sufficiently small neighbourhood $\tilde{U} \subseteq M$ of $s(g)$, we can choose (e.g. working in local coordinates) a smooth map $\sigma: \tilde{U} \rightarrow \mathcal{G}$ such that $s \circ \sigma = \text{Id}_{\tilde{U}}$ and

$$T_{s(g)} \sigma (T_{s(g)} M) = C.$$

Thus,

$$T_{s(g)}(t \circ \sigma): V \rightarrow T_{s(g)}(t \circ \sigma)(V)$$

is an isomorphism. Consequently, there exists a sufficiently small neighbourhood $U \subseteq \tilde{U}$ such that $t \circ \sigma|_U$ is a diffeomorphism onto its image. \square

Bibliography

- [1] K. C. H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*. in London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 2005. doi:10.1017/CBO9781107325883. (cited in p. 19)
- [2] R. L. Fernandes. Lecture notes from “Lie Groupoids and Lie Algebroids” 2021 course at U. Illinois. Available at <https://publish.illinois.edu/ruiloja/math-595-lie-groupoids-and-lie-algebroids-fall-2021/>. (cited in p. 5)
- [3] A. Weinstein. Groupoids: Unifying Internal and External Symmetry. A Tour through Some Examples. *Notices Amer. Math. Soc.* **43** (1996) 744–752. (cited in pp. 5 and 7)
- [4] F. M. Ciaglia, A. Ibort and G. Marmo. Schwinger’s picture of quantum mechanics I: Groupoids. *Int. J. Geom. Methods Mod. Phys.* **16** (2019) 1950119. doi:10.1142/S0219887819501196 (cited in p. 7)
- [5] A. Ibort, A. Mas and L. Schiavone. “Fields” in classical and quantum field theories. *Preprint ARXIV:* 2506.15327 [math-ph]. (cited in p. 7)
- [6] M. de León, M. Epstein and V. Jiménez. *Material Geometry: Groupoids in Continuum Mechanics*. (Singapore: World Scientific) 2021. doi:10.1142/12168 (cited in p. 7)
- [7] J. C. Marrero, D. M. de Diego and E. Martínez, Discrete Lagrangian and Hamiltonian mechanics on Lie groupoids. *Nonlinearity* **19** (2006) 1313. doi:10.1088/0951-7715/19/6/006 (cited in p. 7)
- [8] L. Colombo, D. M. de Diego. Second-order variational problems on Lie groupoids and optimal control applications. *Discrete Contin. Dyn. Syst.* **36** (2016) 6023–6064. doi:10.3934/dcds.2016064 (cited in p. 7)
- [9] R. Palais. A global formulation of the Lie theory of transformation groups. *Mem. Amer. Math. Soc.* **22** (1957). doi:10.1090/memo/0022 (cited in p. 8)
- [10] J.-P. Ortega and T. Ratiu. *Momentum Maps and Hamiltonian Reduction*. (Boston: Birkhäuser) 2004. doi:10.1007/978-1-4757-3811-7 (cited in p. 8)
- [11] A. Schmeding and C. Wockel, “The Lie group of bisections of a Lie groupoid,” *Annals of Global Analysis and Geometry*, vol. 48, no. 1, pp. 87–123, Jun. 2015. doi:10.1007/s10455-015-9459-z. (cited in pp. 19 and 20)
- [12] I. Moåcerdijk and J. Mrcun, *Introduction to Foliations and Lie Groupoids*. in Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2003. doi:10.1017/CBO9780511615450. (cited in p. 19)