Chapter 5 : **Mathematics**

Mathematical terms

Arithmetic Progression	Geometric Progression	Polynomial
Algebra	Logarithm/Power	BigInteger
Combinatorics	Fibonacci	Golden Ratio
Binet's formula	Zeckendorf's theorem	Catalan Numbers
Factorial	Derangement	Binomial Coefficients
Number Theory	Prime Number	Sieve of Eratosthenes
Modified Sieve	Miller-Rabin's	Euler Phi
Greatest Common Divisor	Lowest Common Multiple	Extended Euclid
Linear Diophantine Equation	Cycle-Finding	Probability Theory
Game Theory	Zero-Sum Game	Decision Tree
Perfect Play	Minimax	Nim Game

Table 5.1: List of *some* mathematical terms discussed in this chapter

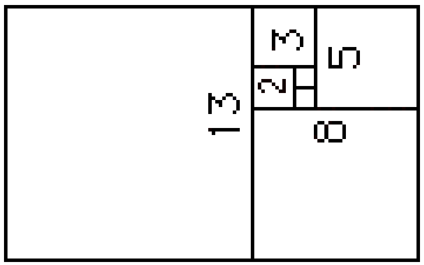
Ad Hoc Mathematics problems

- The simpler ones
- Brute force (mathematical simulation)
- Finding Pattern or Formula
- Grid
- Number Systems or Sequences

Combinatorics

- Fibonacci Numbers
- Binomial Coefficients
- Catalan Numbers

Fibonacci numbers



fib(0) = 0, *fib*(1) = 1,
and for n ≥ 2, *fib*(n) = *fib*(n-1) + *fib*(n-2)

- Thus the sequence begins as follows:
- 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144….

*F*ibo implementation

- DP technique : O(n)
- Compute the n-th *F*ibo using matrix power : O(log n)
- Compute the n-th *F*ibo using approximation technique : O(1)

Fibo with approximation technique

- Using Binet's formula

$$Fib(n) = \frac{\Phi^{n} - ((-\Phi)^{-n})}{\sqrt{5}}$$

$$Fib(n) = \frac{\Phi^{n} - ((-\phi)^{n})}{\sqrt{5}}$$

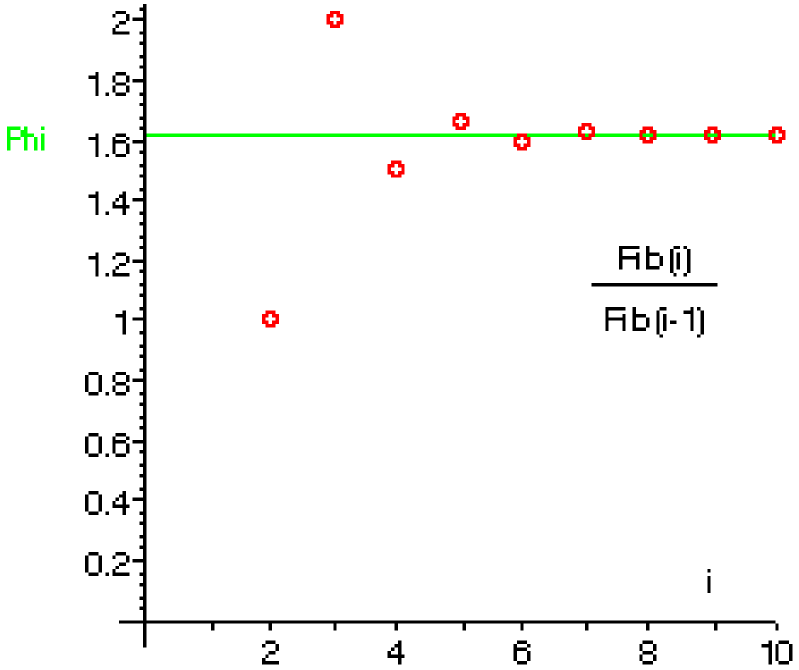
$$Fib(n) = \frac{\Phi^{n} - \frac{-1^{n}}{\Phi^{n}}}{\sqrt{5}}$$

Fibo with approximation technique

- Φ (Phi) : the limit of the ratio of a *fib*(i) and its predecessor, *fib*(i-1).
- Mathematically, Φ is equal to:

$$1+\sqrt{5}\Big/2$$

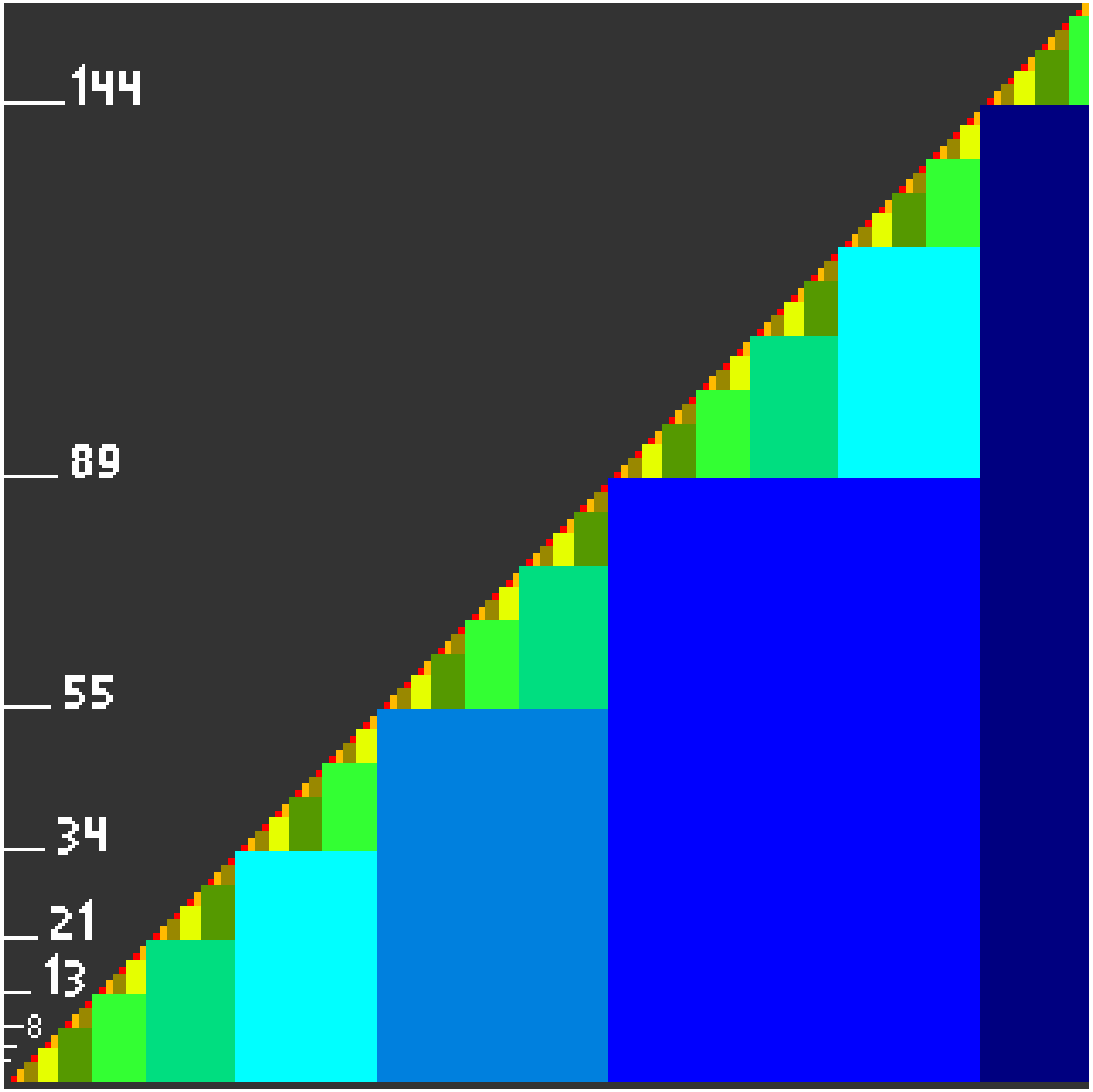
or approximately 1.618034.



Zeckendorf's theorem

The sum of
one or more
distinct fibo numbers
in such a way that
the sum
does not include
any 2 consecutive
Fibonacci numbers.

Fibo seq:
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144....



https://en.wikipedia.org/wiki/Zeckendorf%27s_theorem

Zeckendorf's theorem

- For example, the Zeckendorf of 100 is

  100 = 89 + 8 + 3.

- The other ways of representing 100 as the sum of Fibonacci numbers (NOT Zeckendorf) – for example

  100 = 89 + 8 + 2 + 1

  100 = 55 + 34 + 8 + 3

Pisano Period

The smallest r > 0 such that
$F_0 \equiv F_r \pmod{m}$ and $F_1 \equiv F_{r+1} \pmod{m}$,
where F_k is the k^{th} Fibonacci number,
is the Pisano period of m.
We denote this period with $\pi(m)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$F_n \pmod{2}$	**0**	**1**	**1**	0	1	1	0	1	1	0	1	1	0	1	1	0
$F_n \pmod{3}$	**0**	**1**	**1**	**2**	**0**	**2**	**2**	**1**	0	1	1	2	0	2	2	1
$F_n \pmod{4}$	**0**	**1**	**1**	**2**	**3**	**1**	0	1	1	2	3	1	0	1	1	2

$\pi(2) = 3$, $\pi(3)=8$, $\pi(4)=6$

Pisano Period

The last 1/ last 2/ last 3 / last 4 digits of Fibonacci numbers repeats with a period of

60 / 300 / 1500/ 15000

n	Fn	Fn mod 60	Fn mod 300	Fn mod 1500	Fn mod 15000
0	0	0	0	0	0
1	1	1	1	1	1
2	1	1	1	1	1
3	2	2	2	2	2
4	3	3	3	3	3
5	5	5	5	5	5
6	8	8	8	8	8
7	13	13	13	13	13
8	21	21	21	21	21
9	34	34	34	34	34
10	55	55	55	55	55
11	89	29	89	89	89
12	144	24	144	144	144
13	233	53	233	233	233
14	377	17	77	377	377
15	610	10	10	610	610
16	987	27	87	987	987
17	1597	37	97	97	1597
18	2584	4	184	1084	2584
19	4181	41	281	1181	4181
20	6765	45	165	765	6765
21	10946	26	146	446	10946
22	17711	11	11	1211	2711
23	28657	37	157	157	13657

5.4.2 Combination

The number of ***r***-*combinations of a set with* ***n*** *elements, where* ***n*** *is a nonnegative integer and* ***r*** *is an integer with* $0 \le r \le n$*, equals*

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

THE BINOMIAL THEOREM

- Let ***x*** *and* ***y*** *be variables,*
  and let ***n*** *be a nonnegative integer.*

  Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Pascal’s Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

Binomial coefficient

- Using top-down dynamic programming
- Using 2D memo table

$C(n,0) = C(n,n) = 1$ // base cases.
$C(n,k) = C(n-1,k-1) + C(n-1,k)$ // take or ignore an item, $n > k > 0$.

5.4.3 Catalan numbers

- the problem of completely equivalent
- involving recursively-defined objects
- Using bottom-up DP

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} = \prod_{k=2}^{n} \frac{n+k}{k} \qquad \text{for } n \geq 0.$$

Table 1: Balanced Parentheses

$n = 0$:	*	1 way
$n = 1$:	()	1 way
$n = 2$:	()(), (())	2 ways
$n = 3$:	()()(), ()(()), (())(), (()()), ((()))	5 ways
$n = 4$:	()()()(), ()()(()), ()(())(), ()(()()), ()((())), (())()(), (())(()), (()())(), ((()))(), (()()()), (()(())), ((())()), ((()())), (((())))	14 ways
$n = 5$:	()()()()(), ()()()(()), ()()(())(), ()()(()()), ()()((())), ()(())()(), ()(())(()), ()(()())(), ()((()))(), ()(()()()), ()(()(())), ()((())()), ()((()())), ()(((()))), (())()()(), (())()(()), (())(())(), (())(()()), (())((())), (()())()(), (()())(()), ((()))()(), ((()))(()), (()()())(), (()(()))(), ((())())(), ((()()))(), (((())))(), (()()()()), (()()(())), (()(())()), (()(()())), (()((()))), ((())()()), ((())(())), ((()())()), (((()))()), ((()()())), ((()(()))), (((())())), (((()()))), ((((()))))	42 ways

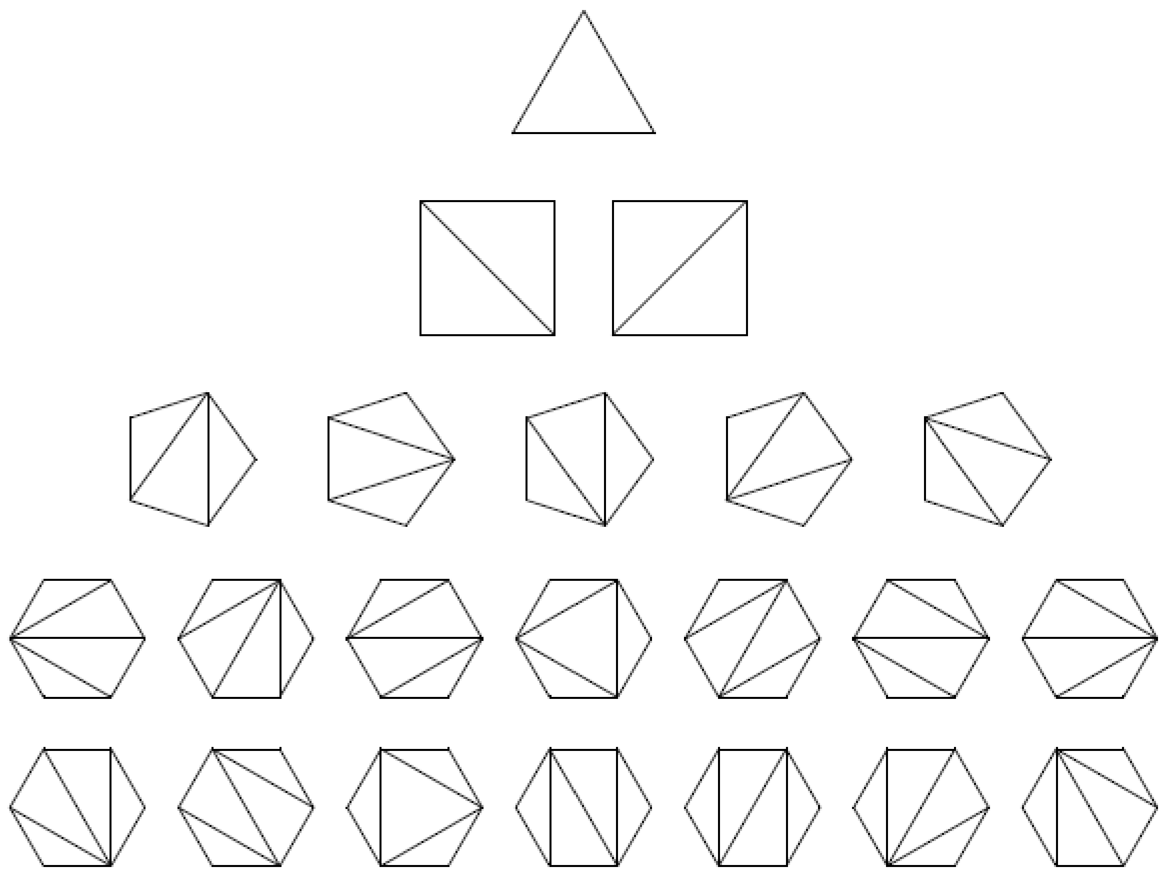
Mountain Ranges

How many “mountain ranges” can you form
with n upstrokes and n downstrokes
that all stay above the original line?

<table>
<tr><td>$n = 0$:</td><td><pre>*</pre></td><td>1 way</td></tr>
<tr><td>$n = 1$:</td><td><pre>/\</pre></td><td>1 way</td></tr>
<tr><td>$n = 2$:</td><td><pre>   /\
/\/\, / \</pre></td><td>2 ways</td></tr>
<tr><td>$n = 3$:</td><td><pre>   /\
           /\   /\   /\/\   / \
/\/\/\, /\/ \, / \/\, /   \, /   \</pre></td><td>5 ways</td></tr>
</table>

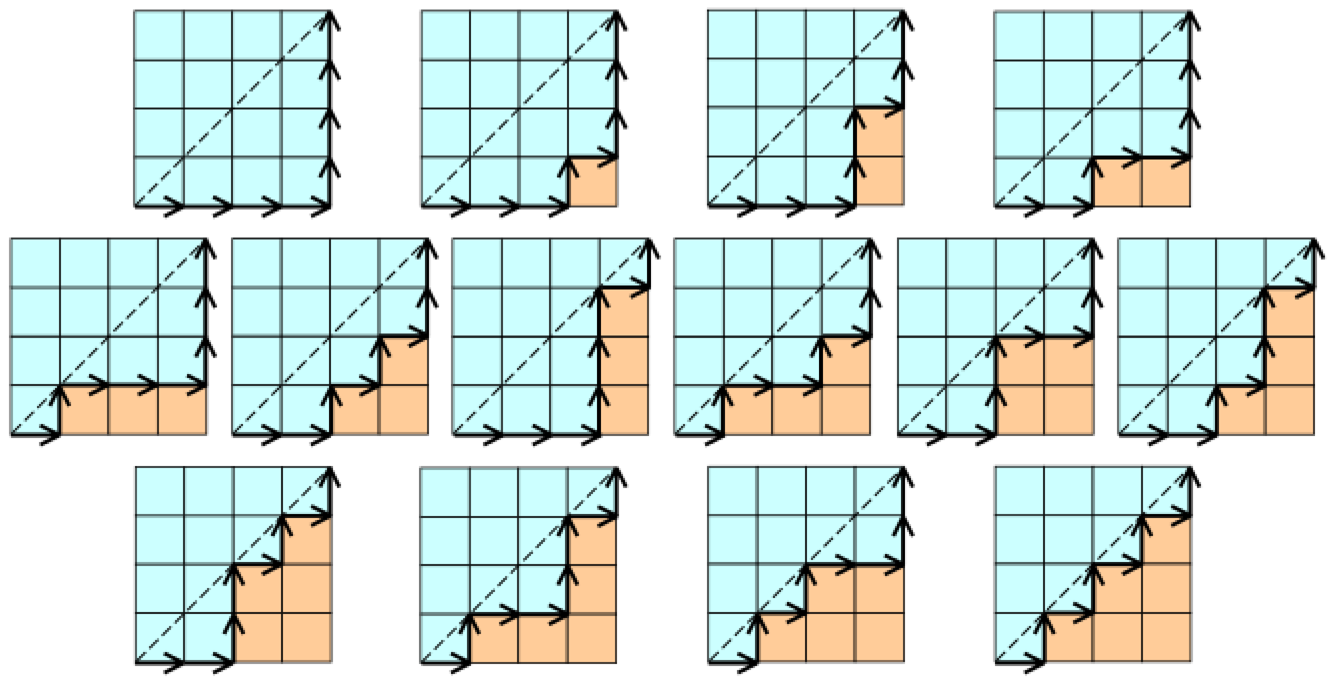
Polygon Triangulation

count the number of ways to triangulate
a regular polygon with n + 2 sides



Monotonic paths

counts the number of monotonic paths along the edges of an *n×n* grid, Which do not pass above the diagonal. A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards.



[0,0,0,0][0,0,0,1][0,0,0,2][0,0,1,1]
[0,1,1,1] [0,0,1,2] [0,0,0,3] [0,1,1,2][0,0,2,2][0,0,1,3]
[0,0,2,3][0,1,1,3] [0,1,2,2][0,1,2,3]

https://en.wikipedia.org/wiki/Catalan_number

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- https://sites.google.com/site/stevenhalim/home/material
- https://uva.onlinejudge.org/index.php?option=com_onlinejudge&Itemid=8&category=16