LISTA DE EXERCÍCIOS – CÁLCULO DE GIBBS

Sandro Dias Pinto Vitenti

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1. Mostre que $\mathbb{R}^2$ e $\mathbb{R}^3$, são espaços vetoriais de dimensão dois e três respectivamente.
2. Mostre que linearidade a esquerda e simetria implicam linearidade a direita.
3. Escreva explicitamente os somatórios

$$\boldsymbol{v} \cdot \boldsymbol{u} = \left(\sum_{i=1}^{3} v^i \boldsymbol{e}_i\right) \cdot \left(\sum_{j=1}^{3} u^j \boldsymbol{e}_j\right) = \sum_{i,j=1}^{3} v^i u^j \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \sum_{i,j=1}^{3} v^i u^j \delta_{ij} = \sum_{i=1}^{3} v^i u^i \tag{1}$$

   e mostre que a presença do delta de Kronecker pode ser usado para eliminar um somatório.
4. Mostre que o produto externo de dois vetores $\boldsymbol{v}$ e $a\boldsymbol{v}$ é nulo, onde a é uma constante.
5. Calcule explicitamente a equação

$$\begin{aligned} \boldsymbol{v} \wedge \boldsymbol{u} &= \left(\sum_{i=1}^{3} v^i \boldsymbol{e}_i\right) \wedge \left(\sum_{j=1}^{3} u^j \boldsymbol{e}_j\right), \\ &= \left(v^1 u^2 - v^2 u^1\right) \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 - \left(v^1 u^3 - v^3 u^1\right) \boldsymbol{e}_3 \wedge \boldsymbol{e}_1 + \left(v^2 u^3 - v^3 u^2\right) \boldsymbol{e}_2 \wedge \boldsymbol{e}_3. \end{aligned} \tag{2}$$

6. Calcule explicitamente a equação

$$\boldsymbol{v} \times \boldsymbol{u} = \left(\sum_{i=1}^{3} v^i \boldsymbol{e}_i\right) \times \left(\sum_{j=1}^{3} u^j \boldsymbol{e}_j\right) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} v^i u^j \boldsymbol{e}_k. \tag{3}$$

7. Mostre que o produto externo de três vetores quaisquer $\boldsymbol{v} \wedge \boldsymbol{u} \wedge \boldsymbol{w}$ é proporcional a $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3$.

8. Mostre que

$$\boldsymbol{e}_i \cdot \mathrm{V}(\boldsymbol{e}_j \wedge \boldsymbol{e}_k) = \boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) = \Upsilon(\boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k) = \epsilon_{ijk} \tag{4}$$

e

$$(\boldsymbol{e}_i \times \boldsymbol{e}_j) \times \boldsymbol{e}_k = \mathrm{V}(\mathrm{V}(\boldsymbol{e}_i \wedge \boldsymbol{e}_j) \wedge \boldsymbol{e}_k) = (\boldsymbol{e}_k \cdot \boldsymbol{e}_i)\boldsymbol{e}_j - (\boldsymbol{e}_k \cdot \boldsymbol{e}_j)\boldsymbol{e}_i \tag{5}$$

são verdadeiras.

9. Partindo da Eq.(4) mostre que

$$\boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{w}) = \Upsilon(\boldsymbol{v} \wedge \boldsymbol{u} \wedge \boldsymbol{w}) \tag{6}$$

é verdadeira.

10. Usando a Eq. (4) mostre que

$$(\boldsymbol{v} \times \boldsymbol{u}) \times \boldsymbol{w} = \mathrm{V}(\mathrm{V}(\boldsymbol{v} \wedge \boldsymbol{u}) \wedge \boldsymbol{w}) = (\boldsymbol{w} \cdot \boldsymbol{v})\boldsymbol{u} - (\boldsymbol{w} \cdot \boldsymbol{u})\boldsymbol{v} \tag{7}$$

é válida.

11. Calcule explicitamente as séries e os limites:

(a)

$$\nabla_{\boldsymbol{v}} f(\boldsymbol{u}) = \lim_{\varepsilon \to 0} \frac{f(u^1 + \varepsilon v^1, u^2 + \varepsilon v^2, u^3 + \varepsilon v^3) - f(u^1, u^2, u^3)}{\varepsilon} = \sum_{i=1}^{3} v^i \frac{\partial f}{\partial u^i} \tag{8}$$

(b)

$$\begin{aligned} \mathrm{div}(\mathbf{F}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} \Big[& \mathbf{F}(x^1, x^2, x^3 + \varepsilon/2) \cdot \left(\varepsilon^2 \boldsymbol{e}_3\right) + \mathbf{F}(x^1, x^2, x^3 - \varepsilon/2) \cdot \left(-\varepsilon^2 \boldsymbol{e}_3\right) + \\ & \mathbf{F}(x^1, x^2 + \varepsilon/2, x^3) \cdot \left(\varepsilon^2 \boldsymbol{e}_2\right) + \mathbf{F}(x^1, x^2 - \varepsilon/2, x^3) \cdot \left(-\varepsilon^2 \boldsymbol{e}_2\right) + \\ & \mathbf{F}(x^1 + \varepsilon/2, x^2, x^3) \cdot \left(\varepsilon^2 \boldsymbol{e}_1\right) + \mathbf{F}(x^1 - \varepsilon/2, x^2, x^3) \cdot \left(-\varepsilon^2 \boldsymbol{e}_1\right) \Big] \\ = & \sum_{i=1}^{3} \frac{\partial \mathrm{F}^i}{\partial x^i}(\boldsymbol{x}) = \nabla \cdot \mathbf{F}(\boldsymbol{x}). \end{aligned} \tag{9}$$

(c)

$$
\begin{aligned}
rot(\mathbf{F}) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[&\mathbf{F}(x^1+\frac{\varepsilon}{2},x^2,x^3)\cdot(\varepsilon\boldsymbol{e}_2)+\mathbf{F}(x^1-\frac{\varepsilon}{2},x^2,x^3)\cdot(-q\varepsilon\boldsymbol{e}_2)\\
&+\mathbf{F}(x^1,x^2+\frac{\varepsilon}{2},x^3)\cdot(\varepsilon\boldsymbol{e}_1)+\mathbf{F}(x^1,x^2-\frac{\varepsilon}{2},x^3)\cdot(-q\varepsilon\boldsymbol{e}_1)\Big]\\
=&\left[\frac{\partial F^2}{\partial x^1}-\frac{\partial F^1}{\partial x^2}\right]\boldsymbol{e}_3-\left[\frac{\partial F^3}{\partial x^1}-\frac{\partial F^1}{\partial x^3}\right]\boldsymbol{e}_2+\left[\frac{\partial F^3}{\partial x^2}-\frac{\partial F^2}{\partial x^3}\right]\boldsymbol{e}_1
\end{aligned}
\tag{10}
$$

12. Calcular o divergente do gradiente de uma função de $f(\boldsymbol{x})\in C^\infty(\mathbb{R}^3)$.
13. Calcular o rotacional do gradiente de uma função de $f(\boldsymbol{x})\in C^\infty(\mathbb{R}^3)$.
14. Prove as equações

(a)

$$\nabla\left[f(\boldsymbol{x})g(\boldsymbol{x})\right]=g(\boldsymbol{x})\nabla f(\boldsymbol{x})+f(\boldsymbol{x})\nabla g(\boldsymbol{x}),\tag{11}$$

(b)

$$\nabla\cdot\left[f(\boldsymbol{x})\mathbf{F}(\boldsymbol{x})\right]=\nabla f(\boldsymbol{x})\cdot\mathbf{F}(\boldsymbol{x})+f(\boldsymbol{x})\nabla\cdot\mathbf{F}(\boldsymbol{x}),\tag{12}$$

(c)

$$\nabla\cdot\left[\mathbf{F}(\boldsymbol{x})\times\mathbf{G}(\boldsymbol{x})\right]=\mathbf{G}(\boldsymbol{x})\cdot\left[\nabla\times\mathbf{F}(\boldsymbol{x})\right]-\mathbf{F}(\boldsymbol{x})\cdot\left[\nabla\times\mathbf{G}(\boldsymbol{x})\right],\tag{13}$$

(d)

$$
\begin{aligned}
\nabla\times\left[\mathbf{F}(\boldsymbol{x})\times\mathbf{G}(\boldsymbol{x})\right]=&\left[\nabla\cdot\mathbf{G}(\boldsymbol{x})\right]\mathbf{F}(\boldsymbol{x})+\left[\mathbf{G}(\boldsymbol{x})\cdot\nabla\right]\mathbf{F}(\boldsymbol{x})\\
&-\left[\nabla\cdot\mathbf{F}(\boldsymbol{x})\right]\mathbf{G}(\boldsymbol{x})-\left[\mathbf{F}(\boldsymbol{x})\cdot\nabla\right]\mathbf{G}(\boldsymbol{x}).
\end{aligned}
\tag{14}
$$

onde $f(\boldsymbol{x}),g(\boldsymbol{x})\in C^\infty(\mathbb{R}^3)$ e $\mathbf{F}(\boldsymbol{x}),\mathbf{G}(\boldsymbol{x})\in\mathbb{V}$.

15. Calcular explicitamente a passagem da equação

$$\boldsymbol{x}_\lambda\wedge\boldsymbol{x}_\mu\wedge\boldsymbol{x}_\nu\,\nabla\cdot\mathbf{F}=\left[\frac{d\mathbf{F}(\boldsymbol{x})}{d\lambda}\right]\wedge\boldsymbol{x}_\mu\wedge\boldsymbol{x}_\nu-\left[\frac{d\mathbf{F}(\boldsymbol{x})}{d\mu}\right]\wedge\boldsymbol{x}_\lambda\wedge\boldsymbol{x}_\nu+\left[\frac{d\mathbf{F}(\boldsymbol{x})}{d\nu}\right]\wedge\boldsymbol{x}_\lambda\wedge\boldsymbol{x}_\mu.\tag{15}$$

para

$$\boldsymbol{x}_\lambda\wedge\boldsymbol{x}_\mu\wedge\boldsymbol{x}_\nu\,\nabla\cdot\mathbf{F}=\frac{d}{d\lambda}\left[\mathbf{F}(\boldsymbol{x})\wedge\boldsymbol{x}_\mu\wedge\boldsymbol{x}_\nu\right]-\frac{d}{d\mu}\left[\mathbf{F}(\boldsymbol{x})\wedge\boldsymbol{x}_\lambda\wedge\boldsymbol{x}_\nu\right]+\frac{d}{d\nu}\left[\mathbf{F}(\boldsymbol{x})\wedge\boldsymbol{x}_\lambda\wedge\boldsymbol{x}_\mu\right].\tag{16}$$