E_t, True Error = True value − Approximate value

Relative true error, $\epsilon_t = \dfrac{\text{True error}}{\text{True value}} = \dfrac{E_t}{\text{True Value}}$

E_a = approximate error = Present appxn. − Previous apprxn.

ϵ_a = relative approximate error = $\dfrac{\text{approximate error } (E_a)}{\text{Present approximation}}$

absolute relative approximate error = $|\epsilon_a|$

FDD $\quad f'(x) \approx \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

BDD $\quad f'(x) \approx \dfrac{f(x) - f(x-\Delta x)}{\Delta x}$

CDD $\quad f'(x) \approx \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$

$f''(x_i) \approx \dfrac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{(\Delta x)^2}$ [FDD] [not important]

$\lvert\epsilon_a\rvert$ %	m
≈ 5	1
≈ 0.5	2
≈ 0.05	3
≈ 0.005	4

Bisection Method

$x_L = \dots$

$x_u = \dots$

~~$if(f(x_L)$~~

if $f(x_L)\,f(x_u) > 0$: no root betwen x_L & x_u.
return

$$x_m = \frac{x_L + x_u}{2}$$

~~for i in range(∞):~~

while (!0):
$x^m_{old} = x_m$
if($f(x_L)\,f(x_m) < 0$):

$x_u = x_m$

$x_l = x_l$

elseif($f(x_L)\,f(x_m) > 0$):

$x_L = x_m$

else if($f(x_L)\,f(x_m) == 0$):

$x^m_{new} = \frac{x_l + x_u}{2}$ root found! break

$$|\epsilon_a| = \left|\frac{x^m_{new} - x^m_{old}}{x^m_{new}}\right| \times 100\,\%$$

$\boxed{\text{Newton-raphson}}$

$x_0 = \dots$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$|\epsilon_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100\%$$

if $|\epsilon_a| \leq \epsilon_s$, stop

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$|\epsilon_a|_1 = \left|\frac{x_1 - x_0}{x_1}\right| \times 100\%$$

$\boxed{\text{secant method}}$

$x_{-1} = \dots$ $\quad [x_0 > x_{-1}]$

$x_0 = \dots$

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

$$|\epsilon_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100\%$$

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

Advantage

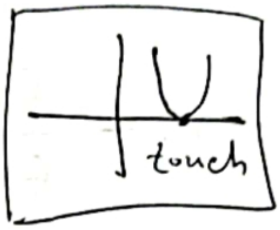
disadvantage

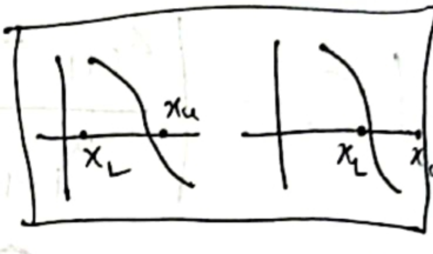
bisection

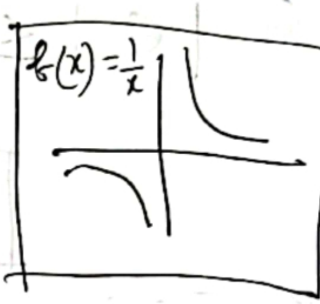
guarenteed to converge

bracket halbed at each iteration

slow convergence







Do I really look like a guy with a plan?

very poor choice of words

this city deserves a better class of criminals

It's not about the money

It's about sending a message

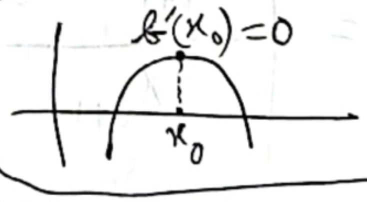
Newton-Raphson

Fast convergence

only one initial guess

divergence @ inflection point

division by zero



oscillation near local maxima / minima

root jumping

Advantage

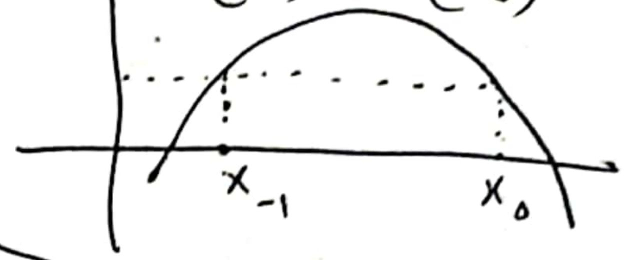
- fast convergence (if)
- two initial guesses don't have to bracket root

secant method

disadvantage

- division by zero

$f(x_{-1}) = f(x_0)$



- root jumping

$\frac{d}{dt} s = v$	$s = \int v\,dt$
$\frac{d}{dt} v = a$	
$v = \int a\,dt$	

Interpolation

- Direct method
- Newton's Divided difference method
- Lagrangean interpolation

আর কোনো কিছু (x, y) দেয়া থাকবে $x = \cdots$ এর জন্য interpolation করতে,

linear interpolation → first order polynomial → দুইটা point নিয়ে করতে হয়

quadratic interpolation → 2nd order polynomial → তিনটা point নিয়ে করতে হয়

cubic interpolation → 3rd order polynomial → চারটা point নিয়ে করতে হয়

Direct method n-th order polynomial নিয়ে করলে $n+1$টা point interpolant এ বসিয়ে system solve করে $a_0, a_1, \ldots, a_{n+1}$ বের করতে হয়,

order	interpolant
1st order	$v(t) = a_0 + a_1 t$
2nd order	$v(t) = a_0 + a_1 t + a_2 t^2$
3rd order	$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

Newton's Divided difference → linear interpolation

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$b_0 = f(x_0) = y_0$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

given points	
x_0	y_0
x_1	y_1

→ **quadratic interpolation**

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0) = y_0$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

given points	
x_0	y_0
x_1	y_1
x_2	y_2

$$\begin{array}{llllll}
 & & b_0 & b_1 & b_2 & b_3 \\
x_0 & & f(x_0) & & & \\
 & & & f[x_0, x_1] & f[x_0, x_1, x_2] & \\
x_1 & & f(x_1) & & & f[x_0, x_1, x_2, x_3] \\
 & & & f[x_1, x_2] & f[x_1, x_2, x_3] & \\
x_2 & & f(x_2) & & & \\
 & & & f[x_2, x_3] & & \\
x_3 & & f(x_3) & & &
\end{array}$$

Lagrange's Interpolation

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

linear interpolation $n=1$

$$v(t) = L_0(t) v(t_0) + L_1(t) v(t_1)$$

$$L_0(t) = \frac{t - t_1}{t_0 - t_1} \qquad L_1(t) = \frac{t - t_0}{t_1 - t_0}$$

quadratic interpolation $n=2$

$$v(t) = L_0(t) v(t) + L_1(t) v(t_1) + L_2(t) v(t_2)$$

$$L_0(t) = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right) \qquad L_1(t) = \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)$$

$$L_2(t) = \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)$$

2nd vs 1st $\longrightarrow$ $|\epsilon_a| = \left|\frac{2nd - 1st}{2nd}\right| \times 100\%$

division by zero

pitfalls

large round off errors

Naïve Gaussian Elimination

$A_{n\times n}$ কে U বানাতে forward elimination এ n-1 step লাগবে

$$\left.\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= C_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= C_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= C_3 \end{aligned}\right\} \text{System of linear equation}$$

$$\Rightarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad [\text{in } [A][X] = [C] \text{ form}]$$

$$\Rightarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A'_{22} & A'_{23} \\ 0 & A'_{32} & A'_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} C_1 \\ C'_2 \\ C'_3 \end{bmatrix} \quad \left[\begin{aligned} r'_2 &= r_2 - \frac{A_{21}}{A_{11}} \times r_1 \\ r'_3 &= r_3 - \frac{A_{31}}{A_{11}} \times r_1 \end{aligned} \right] \quad [\text{forward elimination step 1}]$$

$$\Rightarrow \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A'_{22} & A'_{23} \\ 0 & 0 & A''_{33} \end{bmatrix}}_{U \,\rightarrow\, \text{upper triangular matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} C_1 \\ C'_2 \\ C''_3 \end{bmatrix} \quad \left[r''_3 = r'_3 - \frac{A'_{32}}{A'_{22}} r'_2 \right] \quad [\text{forward elimination step 2}]$$

Back substitution: x_3 এর মান বের করে x_2, x_1 এর মান বের করা।

prevents division by zero

partial pivoting: forward elimination এর প্রতিটি step শুরুর আগে s তম step এ s তম column এর শেষ $n-(s-1)$ টা element এর পরমমান এর মধ্যে maximum টার row টা swap করে উপরে নিবো।

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}_{n\times n} \xrightarrow[\text{elimination}]{\text{naïve gussian}} U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} \quad \text{here, } U_{21}=0,\ U_{31}=0,\ U_{32}=0$$

$$\det(A) = |A| = \prod_{i=1}^{n} U_{ii}$$

in this case, $\det(A) = U_{11} \times U_{22} \times U_{33}$

$$\det(A) = \det(U)$$

LU decomposition

$[A][X]=[C]$ থেকে gaussian elimination করে $[U]$ বের করতে হবে,

lower triangular matrix

$$[U]=\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \qquad [L]=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

যেখানে $l_{21}=\dfrac{a_{21}}{a_{11}}$; $l_{31}=\dfrac{a_{31}}{a_{11}}$; $l_{32}=\dfrac{a_{32}}{a_{22}}$

How to solve SLE using LU decomposition

given, $[A][X]=[C]$

1. decompose, $[A]=[L][U]$
2. solve for $[Z]$ from $[L][Z]=[C]$
3. solve for $[X]$ from $[U][X]=[Z]$

$$B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Inverse of A

A^{-1}

B

How to find inverse of $A=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ using LU decomposition

1. decompose, $[A]=[L][U]$
2. find column-wise values of B

solve for $[Z]$ $[L]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow [U]\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ solve for $\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$

solve for $[Z]$ $[L]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow [U]\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix}=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ solve for $\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix}$

solve for $[Z]$ $[L]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow [U]\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix}=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ solve for $\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix}$

B

Gauss seidel method

কর্ণ বরাবর ভুক্তি গুলো অশূন্য হলে করা যাবে, না হলে row swap করে নিতে হবে,

step-1

given SLE,
$$ax_1 + bx_2 + cx_3 = d \quad \cdots\cdots (i)$$
$$ex_1 + fx_2 + gx_3 = h \quad \cdots\cdots (ii)$$
$$ix_1 + jx_2 + kx_3 = l \quad \cdots\cdots (iii)$$

(i) থেকে x_1 এর eq^n, $x_1 = \dfrac{d - bx_2 - cx_3}{a}$ —— ①

(ii) " x_2 " ", $x_2 = \dfrac{h - ex_1 - \cancel{k}x_3}{f}$ —— ②

(iii) " x_3 " ", $x_3 = \dfrac{l - ix_1 - jx_2}{k}$ —— ③

step-2 initial guess ধরে নিব / দেয়া থাকবে, যেমন $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

step-3 x_2, x_3 এর মান ① এ বসাবো,
x_1, x_3 এর latest মান ② তে বসাবো,
x_2, x_1 " " ③ এ বসাবো.

$|\epsilon_a| \le |\epsilon_s|$ হলে iteration থামাবো,

step-4 go to step 3

gauss seidel will converge only if $A = \begin{bmatrix} a & b & c \\ e & f & g \\ i & j & k \end{bmatrix}$ is diagonally dominant.

① $|a_{ij}| \ge \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|$ for all i

② $|a_{ij}| > \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|$ for at least one i

---

i-তম unknown এর absolute relative approximate error,

$$|\epsilon_a|_i = \left| \frac{x_i^{new} - x_i^{old}}{x_i^{new}} \right| \times 100$$

---

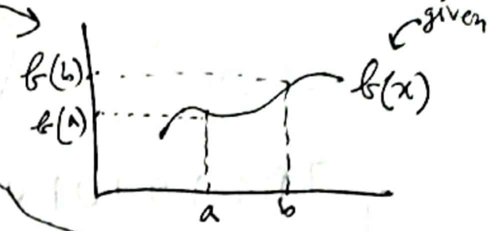
① $|a_{11}| \ge |a_{12}| + |a_{13}|$
& $|a_{22}| \ge |a_{21}| + |a_{23}|$
& $|a_{33}| \ge |a_{31}| + |a_{32}|$
} তিনটেই সত্য হতে হবে

② অন্তত একটি row তে (ধরি $i=2$)
$|a_{22}| > |a_{21}| + |a_{23}|$ হতে হবে,
① আর ② শর্ত মানলে A diagonally dominant.

$\boxed{\text{simpson's one third rule}}$

single segment

$$\longrightarrow x = \int_a^b f(x)\,dx$$



$$\approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Multiple segment / n-segment

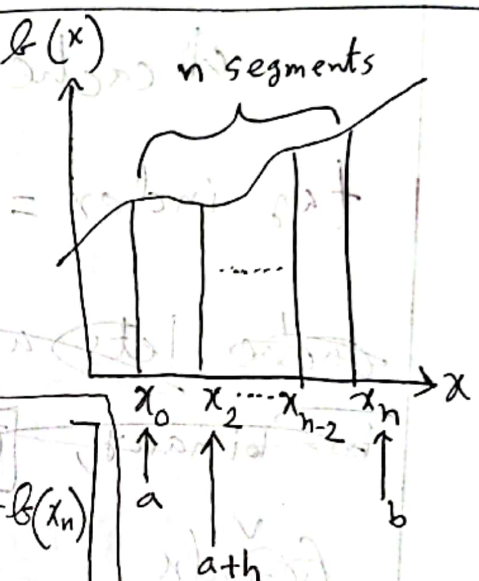
$$h = \frac{b-a}{n}$$

b ← upper limit

a ← lower limit

n ← number of segments

$x_0 = a \;;\; x_1 = a+h \;;\; x_2 = a+2h \ldots\ldots$



$$\int_a^b f(x)\,dx = \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1\\ i=\text{odd}}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ i=\text{even}}}^{n-2} f(x_i) + f(x_n)\right]$$

$\boxed{\text{Gauss quadrature rule}}$

one - point

$$\int_a^b f(x)\,dx \approx c_1 f(x_1) = (b-a)\, f\left(\frac{b+a}{2}\right)$$

two - point

$$\int_a^b f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$\to \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right)$$

n-point Gauss quadrature rule

প্রত্যেক n-point এর জন্য n টা c ভ্যালুস $c_1, c_2, \dots c_n$ এবং n টা x ভ্যালুস $x_1, x_2, \dots x_n$ এর মান দেওয়া থাকবে। given $\int_a^b f(x)\,dx$ কে $\int_{-1}^1 g(x)\,dx$ এ convert করে নিতে হবে এভাবে।—

$$\boxed{\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} \overbrace{f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)}^{g(x)} dt}$$

$$\approx \boxed{\frac{b-a}{2}c_1 g(x_1) + \frac{b-a}{2}c_2 g(x_2) + \dots + \frac{b-a}{2}c_n g(x_n)}$$

এখানে $c_1, c_2, \dots c_n$ আর $x_1, x_2 \dots x_n$ এর মান বসালেই ans. চলে আসবে।

Trapeziodal rule

single segment

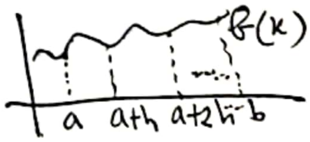
$$\int_a^b f(x)\,dx = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

multiple segment

$$\int_a^b f(x)\,dx = \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

$$\boxed{h = \frac{b-a}{n}}$$

number of segments



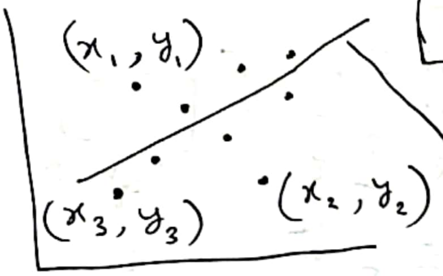
$\boxed{\text{Trapezoidal}}$

$$\int_a^b f(x)\,dx \approx (b-a)\left[\frac{f(a)+f(b)}{2}\right] \qquad h=\frac{b-a}{n}$$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(a)+2\sum_{i=1}^{n-1} f(a+ih)+f(b)\right]$$

$$\frac{b-a}{2n}\left[f(a)+2\sum_{i=1}^{n-1} f(a+ih)+f(b)\right] \qquad h=\frac{b-a}{n}$$

$\boxed{\text{linear regression}}$



x	y
x_1	y_1
x_2	y_2
...	...
x_n	y_n

n সংখ্যক point দেয়া

given data points,

$(x_1, y_1), (x_2, y_2), \ldots, (x_i, y_i), \ldots, (x_n, y_n)$

regression line, $\boxed{y = a_0 + a_1 x}$

$$a_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

$$\bar{x} = \frac{x_1 + x_2 + \ldots x_n}{n}$$

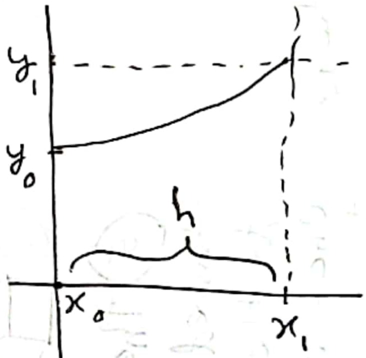
$$\bar{y} = \frac{y_1 + y_2 + \ldots y_n}{n}$$

$\boxed{\text{Euler's Method}}$

step size $= h \leftarrow$ given

given,

$$\underbrace{\frac{dy}{dx} = f(x, y)}_{\text{differential equation}}, \quad \underbrace{y(\overset{x_0}{0}) = y_0}_{\text{initial condition}}$$



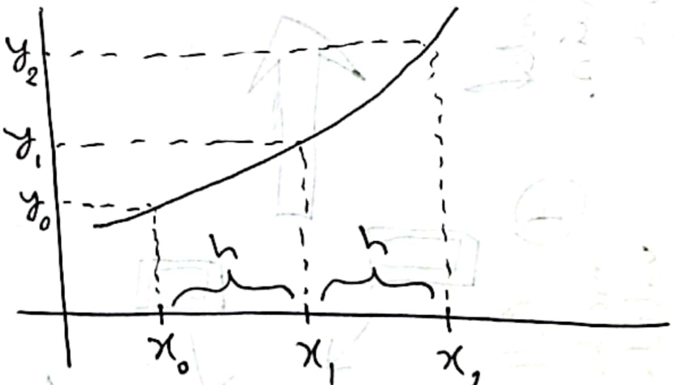
y_0 দেয়া থাকে, $\boxed{y_1}$ (at $x = x_1$) তারপর $\boxed{y_2}$ (at $x = x_2$) তারপর $\boxed{y_3}$ (at $x = x_3 = x_0 + 2h$) বের করতে হবে।

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$y_1 = y_0 + f(x_0, y_0)h$$

$$y_2 = y_1 + f(x_1, y_1)h$$

$$y_3 = y_2 + f(x_2, y_2)h$$



$\boxed{x_0 = x_0}$ $\boxed{x_1 = x_0 + h}$ $\boxed{x_2 = x_0 + 2h}$