Relative Motion and Rendezvous

7

CHAPTER OUTLINE

7.1 Introduction

Up to now, we have mostly referenced the motion of orbiting objects to a nonrotating coordinate system fixed to the center of attraction (e.g., the center of the earth). This platform served as an inertial frame of reference, in which Newton's second law can be written as

$$\mathbf{F}_{net} = m\mathbf{a}_{absolute}$$

An exception to this rule was the discussion of the restricted three-body problem at the end of Chapter 2, in which we made use of the relative motion equations developed in Chapter 1. In a rendezvous maneuver, two orbiting vehicles observe one another from each of their own free-falling, rotating, clearly noninertial frames of reference. To base impulsive maneuvers on observations made from a moving platform requires transforming relative velocity and acceleration measurements into an inertial frame. Otherwise, the true thrusting forces cannot be sorted out from the fictitious "inertial forces" that appear in Newton's law when it is written incorrectly as

$$\mathbf{F}_{net} = m\mathbf{a}_{rel}$$

The purpose of this chapter is to use relative motion analysis to gain some familiarity with the problem of maneuvering one spacecraft relative to another, especially when they are in close proximity.

7.2 Relative motion in orbit

A rendezvous maneuver usually involves a target vehicle A, which is passive and nonmaneuvering, and a chase vehicle B, which is active and performs the maneuvers required to bring itself alongside the target. An obvious example was the Space Shuttle, the chaser, rendezvousing with the International Space Station, the target. The position vector of target A in the geocentric equatorial frame is $\mathbf{r}_A$. This radial is sometimes called the "r-bar". The moving frame of reference has its origin at the target, as illustrated in Figure 7.1. The x-axis is directed along the outward radial $\mathbf{r}_A$ to the target. Therefore, the unit vector $\hat{\mathbf{i}}$ along the moving x-axis is

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_A}{r_A} \tag{7.1}$$

The z-axis is normal to the orbital plane of the target spacecraft and therefore lies in the direction of A's angular momentum vector. It follows that the unit vector along the z-axis of the moving frame is given by

$$\hat{\mathbf{k}} = \frac{\mathbf{h}_A}{h_A} \tag{7.2}$$

The y-axis is perpendicular to both $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ and points in the direction of the target satellite's local horizon. Therefore, both the x- and y-axes lie in the target's orbital plane, with the y unit vector completing a right triad, that is,

$$\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}} \tag{7.3}$$

We may refer to the comoving xyz frame defined here as a local vertical/local horizontal (LVLH) frame.

The position, velocity, and acceleration of B relative to A, measured in the comoving frame, are given by

$$\mathbf{r}_{\text{rel}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{7.4a}$$

$$\mathbf{v}_{\text{rel}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \tag{7.4b}$$

$$\mathbf{a}_{\text{rel}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \tag{7.4c}$$
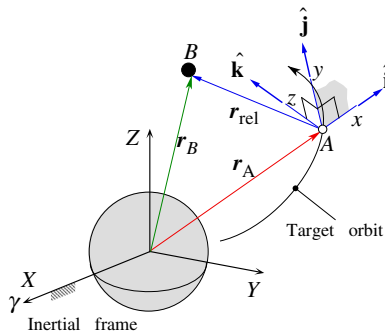

FIGURE 7.1

Comoving LVLH reference frame attached to A, from which the body B is observed.

The angular velocity Ω of the xyz axes attached to the target is just the angular velocity of the target's position vector. It is obtained with the aid of Eqns (2.31) and (2.46) from the fact that

$$\mathbf{h}_A = \mathbf{r}_A \times \mathbf{v}_A = (r_A v_{A_\perp})\hat{\mathbf{k}} = (r_A^2 \Omega)\hat{\mathbf{k}} = r_A^2 \Omega$$

from which we obtain

$$\Omega = \frac{\mathbf{h}_A}{r_A^2} = \frac{\mathbf{r}_A \times \mathbf{v}_A}{r_A^2} \tag{7.5}$$

To find the angular acceleration $\dot{\Omega}$ of the xyz frame, we take the time derivative of $\dot{\Omega}$ in Eqn (7.5) and use the fact that the angular momentum $\mathbf{h}_A$ of the passive target is constant,

$$\dot{\Omega} = \mathbf{h}_A \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r_A^2}\right) = -2\frac{\mathbf{h}_A}{r_A^3}\,\dot{r}_A$$

Recall from Eqn (2.35a) that $\dot{r}_A = \mathbf{v}_A \cdot \mathbf{r}_A / r_A$, so this may be written as

$$\dot{\Omega} = -2\frac{\mathbf{v}_A \cdot \mathbf{r}_A}{r_A^4}\mathbf{h}_A = -2\frac{\mathbf{v}_A \cdot \mathbf{r}_A}{r_A^2}\Omega \tag{7.6}$$

After first calculating

$$\mathbf{r}_{\mathrm{rel}} = \mathbf{r}_B - \mathbf{r}_A \tag{7.7}$$

we use Eqns (7.5) and (7.6) to determine the angular velocity and angular acceleration of the comoving frame, both of which are required in the relative velocity and acceleration formulas (Eqns (1.75) and (1.79)),

$$\mathbf{v}_{\mathrm{rel}} = \mathbf{v}_B - \mathbf{v}_A - \Omega \times \mathbf{r}_{\mathrm{rel}} \tag{7.8}$$

$$\mathbf{a}_{\mathrm{rel}} = \mathbf{a}_B - \mathbf{a}_A - \dot{\Omega} \times \mathbf{r}_{\mathrm{rel}} - \Omega \times (\Omega \times \mathbf{r}_{\mathrm{rel}}) - 2\Omega \times \mathbf{v}_{\mathrm{rel}} \tag{7.9}$$

The vectors in Eqn (7.7)–(7.9) are all referred to the inertial XYZ frame in Figure 7.1. In order to find their components in the accelerating xyz frame at any instant, we must first form the orthogonal transformation matrix $[\mathbf{Q}]_{Xx}$, as discussed in Section 4.5. The rows of this matrix comprise the direction cosines of each of the xyz axes with respect to the XYZ axes. That is, from Eqns (7.1), (7.2), and (7.3), we find

$$\hat{\mathbf{i}} = l_x\hat{\mathbf{I}} + m_x\hat{\mathbf{J}} + n_x\hat{\mathbf{K}}$$
$$\hat{\mathbf{j}} = l_y\hat{\mathbf{I}} + m_y\hat{\mathbf{J}} + n_y\hat{\mathbf{K}} \tag{7.10}$$
$$\hat{\mathbf{k}} = l_z\hat{\mathbf{I}} + m_z\hat{\mathbf{J}} + n_z\hat{\mathbf{K}}$$

where the ls, ms, and ns are the direction cosines. Then,

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix} \quad \begin{matrix} \leftarrow\text{components of } \hat{\mathbf{i}} \\ \leftarrow\text{components of } \hat{\mathbf{j}} \\ \leftarrow\text{components of } \hat{\mathbf{k}} \end{matrix} \tag{7.11}$$

The components of the relative position, velocity, and acceleration are computed as follows:

$$\{\mathbf{r}_{\text{rel}}\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{r}_{\text{rel}}\}_X \tag{7.12a}$$

$$\{\mathbf{v}_{\text{rel}}\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{v}_{\text{rel}}\}_X \tag{7.12b}$$

$$\{\mathbf{a}_{\text{rel}}\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{a}_{\text{rel}}\}_X \tag{7.12c}$$

in which

$$\{\mathbf{r}_{\text{rel}}\}_X = \begin{Bmatrix} X_B - X_A \\ Y_B - Y_A \\ Z_B - Z_A \end{Bmatrix} \tag{7.13a}$$

$$\{\mathbf{v}_{\text{rel}}\}_X = \begin{Bmatrix} \dot{X}_B - \dot{X}_A + \Omega_Z(Y_B - Y_A) - \Omega_Y(Z_B - Z_A) \\ \dot{Y}_B - \dot{Y}_A - \Omega_Z(X_B - X_A) + \Omega_X(Z_B - Z_A) \\ \dot{Z}_B - \dot{Z}_A + \Omega_Y(X_B - X_A) - \Omega_X(Y_B - Y_A) \end{Bmatrix} \tag{7.13b}$$

$$\{\mathbf{a}_{\text{rel}}\}_X = \begin{Bmatrix} \ddot{X}_B - \ddot{X}_A + 2\Omega_Z(\dot{Y}_B - \dot{Y}_A) - 2\Omega_Y(\dot{Z}_B - \dot{Z}_A)\cdots \\ -(\Omega_Y^2 + \Omega_Z^2)(X_B - X_A) + (\Omega_X\Omega_Y + a\Omega_Z)(Y_B - Y_A) + (\Omega_X\Omega_Z - a\Omega_Y)(Z_B - Z_A) \\ \ddot{Y}_B - \ddot{Y}_A - 2\Omega_Z(\dot{X}_B - \dot{X}_A) + 2\Omega_X(\dot{Z}_B - \dot{Z}_A)\cdots \\ +(\Omega_X\Omega_Y - a\Omega_Z)(X_B - X_A) - (\Omega_X^2 + \Omega_Z^2)(Y_B - Y_A) + (\Omega_Y\Omega_Z + a\Omega_X)(Z_B - Z_A) \\ \ddot{Z}_B - \ddot{Z}_A + 2\Omega_Y(\dot{X}_B - \dot{X}_A) - 2\Omega_X(\dot{Y}_B - \dot{Y}_A)\cdots \\ +(\Omega_X\Omega_Z + a\Omega_Y)(X_B - X_A) + (\Omega_Y\Omega_Z - a\Omega_X)(Y_B - Y_A) - (\Omega_X^2 - \Omega_Y^2)(Z_B - Z_A) \end{Bmatrix} \tag{7.13c}$$

The components of $\mathbf{\Omega}$ are obtained from Eqn (7.5), and $\dot{\mathbf{\Omega}} = a\mathbf{\Omega}$, where, according to Eqn (7.6), $a = -2\mathbf{v}_A \cdot \mathbf{r}_A/r_A^2$.

ALGORITHM 7.1

Given the state vectors $(\mathbf{r}_A, \mathbf{v}_A)$ of target spacecraft A and $(\mathbf{r}_B, \mathbf{v}_B)$ of chaser spacecraft B, find the position $\{\mathbf{r}_{\text{rel}}\}_x$, velocity $\{\mathbf{v}_{\text{rel}}\}_x$, and acceleration $\{\mathbf{a}_{\text{rel}}\}_x$ of B relative to A along the LVLH axes attached to A. See Appendix D.31 for an implementation of this procedure in MATLAB.

1. Calculate the angular momentum of A.

$$\mathbf{h}_A = \mathbf{r}_A \times \mathbf{v}_A$$

2. Calculate the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ of the comoving frame.

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_A}{r_A} \quad \hat{\mathbf{k}} = \frac{\mathbf{h}_A}{h_A} \quad \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}}$$

3. Calculate the orthogonal transformation matrix $[\mathbf{Q}]_{Xx}$ using Eqn (7.11).
4. Calculate $\mathbf{\Omega}$ and $\dot{\mathbf{\Omega}}$ from Eqns (7.5) and (7.6).
5. Calculate the absolute accelerations of A and B using Eqn (2.22).

$$\mathbf{a}_A = -\frac{\mu}{r_A^3}\mathbf{r}_A \quad \mathbf{a}_B = -\frac{\mu}{r_B^3}\mathbf{r}_B$$

6. Calculate $\mathbf{r}_{\text{rel}}$ using Eqn (7.6).
7. Calculate $\mathbf{v}_{\text{rel}}$ using Eqn (7.8).
8. Calculate $\mathbf{a}_{\text{rel}}$ using Eqn (7.9).
9. Calculate $\{\mathbf{r}_{\text{rel}}\}_x$, $\{\mathbf{v}_{\text{rel}}\}_x$, and $\{\mathbf{a}_{\text{rel}}\}_x$ using Eqn (7.13).

EXAMPLE 7.1

Spacecraft A is in an elliptical earth orbit having the following parameters:

$$h = 52{,}059 \text{ km}^2/\text{s} \quad e = 0.025724 \quad i = 60° \quad \Omega = 40° \quad \omega = 30° \quad \theta = 40° \tag{a}$$

Spacecraft B is likewise in an orbit with these parameters:

$$h = 52{,}362 \text{ km}^2/\text{s} \quad e = 0.0072696 \quad i = 50° \quad \Omega = 40° \quad \omega = 120° \quad \theta = 40° \tag{b}$$

Calculate the position $\mathbf{r}_{\text{rel}})_x$, velocity $\mathbf{v}_{\text{rel}})_x$, and acceleration $\mathbf{a}_{\text{rel}})_x$ of spacecraft B relative to spacecraft A, measured along the xyz axes of the comoving coordinate system of spacecraft A, as defined in Figure 7.1.
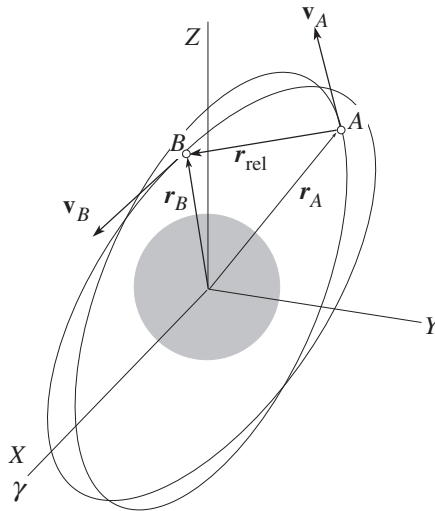

FIGURE 7.2

Spacecraft A and B in slightly different orbits.

Solution
From the orbital elements in Eqns (a) and (b), we can use Algorithm 4.5 to find the position and the velocity of both spacecraft relative to the geocentric equatorial reference frame. Omitting those familiar calculations here, the reader can verify that, for spacecraft A,

$$\mathbf{r}_A = -266.77\hat{\mathbf{I}} + 3865.8\hat{\mathbf{J}} + 5426.2\hat{\mathbf{K}} \ (\text{km}) \quad (r_A = 6667.8 \ \text{km}) \tag{c}$$

$$\mathbf{v}_A = -6.4836\hat{\mathbf{I}} - 3.6198\hat{\mathbf{J}} + 2.4156\hat{\mathbf{K}} \ (\text{km/s}) \tag{d}$$

and for spacecraft B,

$$\mathbf{r}_B = -5890.7\hat{\mathbf{I}} - 2979.8\hat{\mathbf{J}} + 1792.2\hat{\mathbf{K}} \ (\text{km}) \quad (r_B = 6840.4 \ \text{km}) \tag{e}$$

$$\mathbf{v}_B = 0.93583\hat{\mathbf{I}} - 5.2403\hat{\mathbf{J}} - 5.5009\hat{\mathbf{K}} \ \text{km/s} \tag{f}$$

Having found the state vectors, we can proceed with Algorithm 7.1.

Step 1:

$$\mathbf{h}_A = \mathbf{r}_A \times \mathbf{v}_A = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -266.77 & 3865.8 & 5426.2 \\ -6.4836 & -3.6198 & 2.4156 \end{vmatrix} = 28{,}980\hat{\mathbf{I}} - 34{,}537\hat{\mathbf{J}} + 26{,}029\hat{\mathbf{K}} \ (\text{km}^2/\text{s})$$

$$(h_A = 52{,}059 \ \text{km}^2/\text{s})$$

Step 2:

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_A}{r_A} = -0.040009\hat{\mathbf{I}} + 0.57977\hat{\mathbf{J}} + 0.81380\hat{\mathbf{K}}$$

$$\hat{\mathbf{k}} = \frac{\mathbf{h}_A}{h_A} = 0.55667\hat{\mathbf{I}} - 0.66341\hat{\mathbf{J}} + 0.5000\hat{\mathbf{K}}$$

$$\hat{\mathbf{j}} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.55667 & -0.66341 & 0.5000 \\ -0.040008 & 0.57977 & 0.81380 \end{vmatrix} = -0.82977\hat{\mathbf{I}} - 0.47302\hat{\mathbf{J}} + 0.29620\hat{\mathbf{K}}$$

Step 3:

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} -0.040009 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix}$$

Step 4:

$$\mathbf{\Omega} = \frac{\mathbf{h}_A}{r_A^2} = 0.00065183\hat{\mathbf{I}} - 0.00077682\hat{\mathbf{J}} + 0.00058547\hat{\mathbf{K}} \ (\text{rad/s})$$

$$\dot{\mathbf{\Omega}} = -2\frac{\mathbf{v}_A \cdot \mathbf{r}_A}{r_A^2}\mathbf{\Omega} = -2.47533(10^{-8})\hat{\mathbf{I}} + 2.9500(10^{-8})\hat{\mathbf{J}} - 2.2233(10^{-8})\hat{\mathbf{K}} \ (\text{rad/s}^2)$$

Step 5:

$$\mathbf{a}_A = -\mu \frac{\mathbf{r}_A}{r_A^3} = 0.00035870\hat{\mathbf{I}} - 0.0051980\hat{\mathbf{J}} - 0.0072962\hat{\mathbf{K}} \ (km/s^2)$$

$$\mathbf{a}_B = -\mu \frac{\mathbf{r}_B}{r_B^3} = 0.0073359\hat{\mathbf{I}} + 0.0037108\hat{\mathbf{J}} - 0.0022319\hat{\mathbf{K}} \ (km/s^2)$$

Step 6:

$$\mathbf{r}_{rel} = \mathbf{r}_B - \mathbf{r}_A = -5623.9\hat{\mathbf{I}} - 6845.5\hat{\mathbf{J}} - 3634.0\hat{\mathbf{K}} \ (km)$$

Step 7:

$$\mathbf{v}_{rel} = \mathbf{v}_B - \mathbf{v}_A - \mathbf{\Omega} \times \mathbf{r}_{rel}$$

$$= (0.93583\hat{\mathbf{I}} - 5.2403\hat{\mathbf{J}} - 5.5009\hat{\mathbf{K}}) - (-6.4836\hat{\mathbf{I}} - 3.6198\hat{\mathbf{J}} + 2.4156\hat{\mathbf{K}})$$

$$- \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.00065183 & -0.00077682 & 0.00058547 \\ -5623.9 & -6845.5 & -3634.0 \end{vmatrix}$$

$$\mathbf{v}_{rel} = 0.58855\hat{\mathbf{I}} - 0.69663\hat{\mathbf{J}} + 0.91436\hat{\mathbf{K}} \ (km/s)$$

Step 8:

$$\mathbf{a}_{rel} = \mathbf{a}_B - \mathbf{a}_A - \dot{\mathbf{\Omega}} \times \mathbf{r}_{rel} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{rel}) - 2\mathbf{\Omega} \times \mathbf{v}_{rel}$$

$$= (0.0073359\hat{\mathbf{I}} + 0.0037108\hat{\mathbf{J}} - 0.0022319\hat{\mathbf{K}}) - (0.00035870\hat{\mathbf{I}} - 0.0051980\hat{\mathbf{J}} - 0.0072962\hat{\mathbf{K}})$$

$$- \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -2.4753(10^{-8}) & 2.9500(10^{-8}) & -2.2233(10^{-8}) \\ -5623.9 & -6845.5 & -3634.0 \end{vmatrix}$$

$$-(0.00065183\hat{\mathbf{I}} - 0.00077682\hat{\mathbf{J}} + 0.00058547\hat{\mathbf{K}}) \times \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.00065183 & -0.00077682 & 0.00058547 \\ -5623.9 & -6845.5 & -3634.0 \end{vmatrix}$$

$$-2 \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.00065183 & -0.00077682 & 0.00058547 \\ 0.58855 & -0.69663 & 0.91436 \end{vmatrix}$$

$$\mathbf{a}_{rel} = 0.00044050\hat{\mathbf{I}} - 0.00037900\hat{\mathbf{J}} + 0.000018581\hat{\mathbf{K}} \ (km/s^2)$$

Step 9:

$$\mathbf{r}_{rel})_x = \begin{bmatrix} -0.040008 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix} \begin{Bmatrix} -5623.9 \\ -6845.5 \\ -3634.0 \end{Bmatrix} \Rightarrow \mathbf{r}_{rel})_x = \begin{Bmatrix} -6701.2 \\ 6828.3 \\ -406.26 \end{Bmatrix} \text{ (km)}$$

$$\mathbf{v}_{rel})_x = \begin{bmatrix} -0.040008 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix} \begin{Bmatrix} 0.58855 \\ -0.69663 \\ 0.91436 \end{Bmatrix} \Rightarrow \mathbf{v}_{rel})_x = \begin{Bmatrix} 0.31667 \\ 0.11199 \\ 1.2470 \end{Bmatrix} \text{ (km/s)}$$

$$\mathbf{a}_{rel})_x = \begin{bmatrix} -0.040008 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29620 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix} \begin{Bmatrix} 0.00044050 \\ -0.00037900 \\ 0.000018581 \end{Bmatrix} \Rightarrow \mathbf{a}_{rel})_x = \begin{Bmatrix} -0.00022222 \\ -0.00018074 \\ 0.00050593 \end{Bmatrix} \text{ (km/s}^2)$$

See Appendix D.31 for the MATLAB solution to this problem.

The motion of one spacecraft relative to another in orbit may be hard to visualize at first. Figure 7.3 is offered as an assist. Orbit 1 is circular, and orbit 2 is elliptical with an eccentricity of 0.125. Both orbits were chosen to have the same semimajor axis length, so they both have the same period. A comoving frame is shown attached to the observers A in circular orbit 1. At epoch I, the spacecraft B in elliptical orbit 2 is directly below the observers. In other words, A must draw an arrow in the negative local x direction to determine the position vector of B in the lower orbit. The figure shows eight different epochs (I, II, III, … ,VIII), equally spaced around the circular orbit, at which observers A construct the position vector pointing from them to B in the elliptical orbit. Of course, A's frame is rotating, because its x-axis must always be directed away from the earth. Observers A cannot sense this rotation and record the set of observations in their (to them) fixed xy coordinate system, as shown at the bottom of the figure. Coasting at a uniform speed along this circular orbit, observers A see the other vehicle orbiting them clockwise in a sort of bean-shaped path. The distance between the two spacecraft in this case never becomes so great that the earth intervenes.

If observers A declared theirs to be an inertial frame of reference, they would be faced with the task of explaining the physical origin of the force holding B in its bean-shaped orbit. Of course, there is no such force. The apparent path is due to the actual, combined motion of both spacecraft in their free fall around the earth. When B is below A (having a negative x coordinate),

As viewed in the inertial frame.

As viewed from the comoving frame in circular orbit **1**.
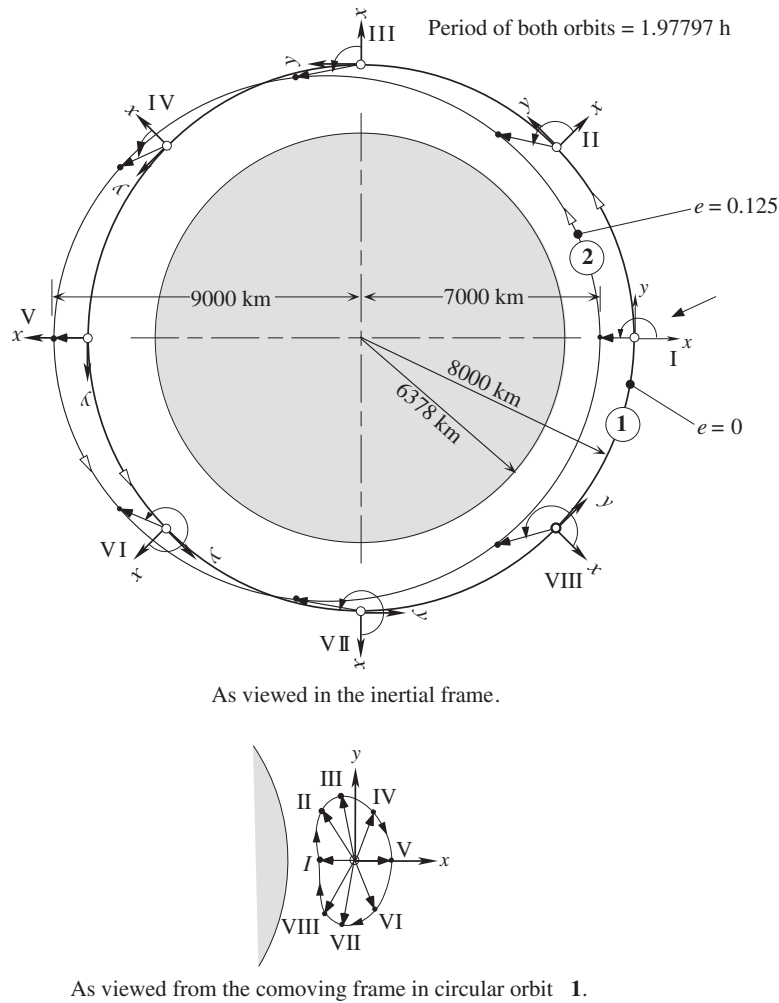
FIGURE 7.3

The spacecraft B in elliptical orbit 2 appears to orbit the observers A who are in circular orbit 1.

conservation of angular momentum demands that B move faster than A, thereby speeding up in A's positive y direction until the orbits cross ($x = 0$). When B's x coordinate become positive, that is, B is above A, the laws of momentum dictate that B slow down, which it does, progressing in A's negative y direction until the next crossing of the orbits. B then falls below and begins to pick up speed. The process repeats over and over again. From inertial space, the process is the motion of two satellites on intersecting orbits, appearing not at all like the orbiting motion seen by the moving observers A.
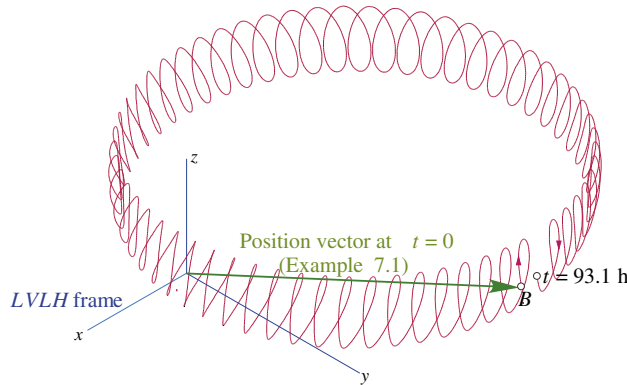
EXAMPLE 7.2

Plot the motion of spacecraft B relative to spacecraft A in Example 7.1.

Solution

In Example 7.1, we found $\mathbf{r}_{rel})_x$ at a single time. To plot the path of B relative to A we must find $\mathbf{r}_{rel})_x$ at a large number of times, so that when we "connect the dots" in three-dimensional space a smooth curve results. Let us outline an algorithm and implement it in MATLAB.

1. Given the orbital elements of spacecraft A and B, calculate their state vectors $(\mathbf{r}_{A_0}, \mathbf{v}_{A_0})$ and $(\mathbf{r}_{B_0}, \mathbf{v}_{B_0})$ at the initial time t_0 using Algorithm 4.5 (as we did in Example 7.1).
2. Calculate the period T_A of A's orbit from Eqn (2.82) (for the data of Example 7.1, $T_A = 5585$ s).
3. Let the final time t_f for the plot be $t_0 + mT_A$, where m is an arbitrary integer.
4. Let n be the number of points to be plotted, so that the time step is $\Delta t = (t_f - t_0)/n$.
5. At time $t \geq t_0$:
   a. Calculate the state vectors $(\mathbf{r}_A, \mathbf{v}_A)$ and $(\mathbf{r}_B, \mathbf{v}_B)$ using Algorithm 3.4.
   b. Calculate $\mathbf{r}_{rel})_x$ using Algorithm 7.1.
   c. Plot the point $(x_{rel}, y_{rel}, z_{rel})$.
6. Let $t \leftarrow t + \Delta t$ and repeat Step 5 until $t = t_f$.

This algorithm is implemented in the MATLAB script *Example_7_02.m* listed in Appendix D.32. The resulting plot of the relative motion for a time interval of 60 periods of spacecraft A is shown in Figure 7.4. The arrow drawn from A to B is the initial position vector $\mathbf{r}_{rel})_x$ found in Example 7.1. As can be seen, the trajectory of B is a looping, counterclockwise motion around a circular path about 14,000 km in diameter. The closest approach of B to A is 105.5 km at an elapsed time of 25.75 h.
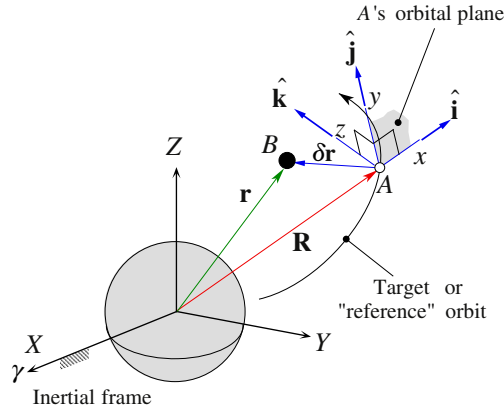

FIGURE 7.4

Trajectory of spacecraft B relative to spacecraft A for the data in Example 7.1. The total time is 60 periods of A's orbit (93.1 h).

7.3 Linearization of the equations of relative motion in orbit

Figure 7.5, like Figure 7.1 shows two spacecraft in earth orbit. Let the inertial position vector of the target vehicle A be denoted as $\mathbf{R}$ and that of the chase vehicle B be denoted as $\mathbf{r}$. The position vector of the chase vehicle relative to the target is $\delta\mathbf{r}$, so that

$$\mathbf{r} = \mathbf{R} + \delta\mathbf{r} \tag{7.14}$$

FIGURE 7.5

Position of chaser *B* relative to target *A*.

The symbol δ is used to represent the fact that the relative position vector has a magnitude that is very small compared to the magnitude of **R** (and **r**), that is,

$$\frac{\delta r}{R} << 1 \tag{7.15}$$

where $\delta r = \|\delta \mathbf{r}\|$ and $R = \|\mathbf{R}\|$. This is true if the two vehicles are in close proximity to each other, as is the case in a rendezvous maneuver or close formation flight. Our purpose in this section is to seek the equations of motion of the chase vehicle relative to the target when they are close together. Since the relative motion is seen from the target vehicle, its orbit is also called the reference orbit.

The equation of motion of the chase vehicle *B* relative to the inertial geocentric equatorial frame is

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r^3} \tag{7.16}$$

where $r = \|\mathbf{r}\|$. Substituting Eqn (7.14) into Eqn (7.16) and writing $\delta\ddot{\mathbf{r}} = (\mathrm{d}^2/\mathrm{d}t^2)\delta\mathbf{r}$ yields the equation of motion of the chaser relative to the target,

$$\delta\ddot{\mathbf{r}} = -\ddot{\mathbf{R}} - \mu \frac{\mathbf{R} + \delta\mathbf{r}}{r^3} \quad (\text{where } r = \|\mathbf{R} + \delta\mathbf{r}\|) \tag{7.17}$$

We will simplify this equation by making use of the fact that $\|\delta\mathbf{r}\|$ is very small, as expressed in Eqn (7.15).

First, note that

$$r^2 = \mathbf{r}\cdot\mathbf{r} = (\mathbf{R} + \delta\mathbf{r})\cdot(\mathbf{R} + \delta\mathbf{r}) = \mathbf{R}\cdot\mathbf{R} + 2\mathbf{R}\cdot\delta\mathbf{r} + \delta\mathbf{r}\cdot\delta\mathbf{r}$$

Since $\mathbf{R}\cdot\mathbf{R} = R^2$ and $\delta\mathbf{r}\cdot\delta\mathbf{r} = \delta r^2$, we can factor out R^2 on the right to obtain

$$r^2 = R^2\left[1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2} + \left(\frac{\delta r}{R}\right)^2\right]$$

By virtue of Eqn (7.15), we can neglect the last term in the brackets, so that

$$r^2 = R^2\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right) \tag{7.18}$$

In fact, we will neglect all powers of $\delta r/R$ greater than unity, wherever they appear. Since $r^{-3} = (r^2)^{-\frac{3}{2}}$, it follows from Eqn (7.18) that

$$r^{-3} = R^{-3}\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-\frac{3}{2}} \tag{7.19}$$

Using the binomial theorem (Eqn (5.44)) and neglecting terms of a higher order than 1 in $\delta r/R$, we obtain

$$\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-\frac{3}{2}} = 1 + \left(-\frac{3}{2}\right)\left(\frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)$$

Therefore, to our level of approximation, Eqn (7.19) becomes

$$r^{-3} = R^{-3}\left(1 - \frac{3}{R^2}\mathbf{R}\cdot\delta\mathbf{r}\right)$$

which can be written as

$$\frac{1}{r^3} = \frac{1}{R^3} - \frac{3}{R^5}\mathbf{R}\cdot\delta\mathbf{r} \tag{7.20}$$

Substituting Eqn (7.20) into Eqn (7.17) (the equation of motion), we get

$$\delta\ddot{\mathbf{r}} = -\ddot{\mathbf{R}} - \mu\left(\frac{1}{R^3} - \frac{3}{R^5}\mathbf{R}\cdot\delta\mathbf{r}\right)(\mathbf{R} + \delta\mathbf{r})$$

$$= -\ddot{\mathbf{R}} - \mu\left[\frac{\mathbf{R} + \delta\mathbf{r}}{R^3} - \frac{3}{R^5}(\mathbf{R}\cdot\delta\mathbf{r})(\mathbf{R} + \delta\mathbf{r})\right]$$

$$= -\ddot{\mathbf{R}} - \mu\left[\frac{\mathbf{R}}{R^3} + \frac{\delta\mathbf{r}}{R^3} - \frac{3}{R^5}(\mathbf{R}\cdot\delta\mathbf{r})\mathbf{R} + \overbrace{\text{terms of higher order than 1 in } \delta\mathbf{r}}^{\text{neglect}}\right]$$

That is, to our degree of approximation,

$$\delta\ddot{\mathbf{r}} = -\ddot{\mathbf{R}} - \mu\frac{\mathbf{R}}{R^3} - \frac{\mu}{R^3}\left[\delta\mathbf{r} - \frac{3}{R^2}(\mathbf{R}\cdot\delta r)\mathbf{R}\right] \tag{7.21}$$

But the equation of motion of the reference orbit is

$$\ddot{\mathbf{R}} = -\mu\frac{\mathbf{R}}{R^3} \tag{7.22}$$

Substituting this into Eqn (7.21) finally yields

$$\delta\ddot{\mathbf{r}} = -\frac{\mu}{R^3}\left[\delta\mathbf{r} - \frac{3}{R^2}(\mathbf{R}\cdot\delta r)\mathbf{R}\right] \tag{7.23}$$

This is the linearized version of Eqn (7.17), the equation that governs the motion of the chaser with respect to the target. The expression is linear because the unknown $\delta\mathbf{r}$ appears only in the numerator and only to the first power throughout. We achieved this by dropping a lot of terms that are insignificant when Eqn (7.15) is valid. Eqn (7.23) is nonlinear in $\mathbf{R}$, which is not an unknown because it is determined independently by solving Eqn (7.22).

In the comoving frame of Figure 7.4, the x-axis lies along the radial $\mathbf{R}$, so that

$$\mathbf{R} = R\hat{\mathbf{i}} \tag{7.24}$$

In terms of its components in the comoving frame, the relative position vector $\delta\mathbf{r}$ in Figure 7.4 is (cf. Eqn (7.4a))

$$\delta\mathbf{r} = \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}} \tag{7.25}$$

Substituting Eqns (7.24) and (7.25) into Eqn (7.23) yields

$$\delta\ddot{\mathbf{r}} = -\frac{\mu}{R^3}\left[\left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}\right) - \frac{3}{R^2}\left[\left(R\hat{\mathbf{i}}\right)\cdot\left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}\right)\right]\left(R\hat{\mathbf{i}}\right)\right]$$

After expanding the dot product on the right and collecting terms, we find that the linearized equation of relative motion takes a rather simple form when the components of $\mathbf{R}$ and $\delta\mathbf{r}$ are given in the comoving frame,

$$\delta\ddot{\mathbf{r}} = -\frac{\mu}{R^3}\left(-2\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}\right) \tag{7.26}$$

Recall that $\delta\ddot{\mathbf{r}}$ is the acceleration of chaser B relative to target A as measured in the inertial frame. That is,

$$\delta\ddot{\mathbf{r}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta\mathbf{r} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{r}_B - \mathbf{r}_A) = \ddot{\mathbf{r}}_B - \ddot{\mathbf{r}}_A = \mathbf{a}_B - \mathbf{a}_A$$

$\delta\ddot{\mathbf{r}}$ is not to be confused with $\delta\mathbf{a}_{\mathrm{rel}}$, the relative acceleration measured in the comoving frame. These two quantities are related by Eqn (7.9).

$$\delta\mathbf{a}_{\mathrm{rel}} = \delta\ddot{\mathbf{r}} - \dot{\boldsymbol{\Omega}} \times \delta\mathbf{r} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \delta\mathbf{r}) - 2\boldsymbol{\Omega} \times \delta\mathbf{v}_{\mathrm{rel}} \tag{7.27}$$

Since we arrived at an expression for $\delta\ddot{\mathbf{r}}$ in Eqn (7.26), let us proceed to evaluate each of the three terms on the right that involve Ω and $\dot{\Omega}$. First, recall that the angular momentum of A ($\mathbf{h} = \mathbf{R} \times \dot{\mathbf{R}}$) is normal to A's orbital plane, and so is the z-axis of the comoving frame. Therefore, $\mathbf{h} = h\hat{\mathbf{k}}$. It follows that Eqns (7.4) and (7.5) may be written as

$$\boldsymbol{\Omega} = \frac{h}{R^2}\hat{\mathbf{k}} \tag{7.28}$$

and

$$\dot{\boldsymbol{\Omega}} = -\frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\hat{\mathbf{k}} \tag{7.29}$$

where $\mathbf{V} = \dot{\mathbf{R}}$.

From Eqns (7.25), (7.28), and (7.29), we find

$$\dot{\boldsymbol{\Omega}} \times \delta\mathbf{r} = \left[-\frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\hat{\mathbf{k}} \right] \times \left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}} \right) = \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4} \left(\delta y\hat{\mathbf{i}} - \delta x\hat{\mathbf{j}} \right) \tag{7.30}$$

and

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \delta\mathbf{r}) = \frac{h}{R^2}\hat{\mathbf{k}} \times \left[\frac{h}{R^2}\hat{\mathbf{k}} \times \left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}} \right) \right] = -\frac{h^2}{R^4} \left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} \right) \tag{7.31}$$

According to Eqn (7.4b), $\delta\mathbf{v}_{\text{rel}} = \delta\dot{x}\hat{\mathbf{i}} + \delta\dot{y}\hat{\mathbf{j}} + \delta\dot{z}\hat{\mathbf{k}}$, where $\delta\dot{x} = (\mathrm{d}/\mathrm{d}t)\delta x$, etc. It follows that

$$2\boldsymbol{\Omega} \times \delta\mathbf{v}_{\text{rel}} = 2\frac{h}{R^2}\hat{\mathbf{k}} \times \left(\delta\dot{x}\hat{\mathbf{i}} + \delta\dot{y}\hat{\mathbf{j}} + \delta\dot{z}\hat{\mathbf{k}} \right) = 2\frac{h}{R^2} \left(\delta\dot{x}\hat{\mathbf{j}} - \delta\dot{y}\hat{\mathbf{k}} \right) \tag{7.32}$$

Substituting Eqn (7.26) along with Eqns (7.30)–(7.32) into Eqn (7.27) yields

$$\delta\mathbf{a}_{\text{rel}} = \overbrace{-\frac{\mu}{R^3} \left(-2\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}} \right)}^{\delta\ddot{\mathbf{r}}} - \overbrace{\frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4} \left(\delta y\hat{\mathbf{i}} - \delta x\hat{\mathbf{j}} \right)}^{\dot{\boldsymbol{\Omega}} \times \delta\mathbf{r}} - \overbrace{\left[-\frac{h^2}{R^4} \left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} \right) \right]}^{\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \delta\mathbf{r})}$$

$$\overbrace{-2\frac{h}{R^2} \left(\delta\dot{x}\hat{\mathbf{j}} - \delta\dot{y}\hat{\mathbf{i}} \right)}^{2\boldsymbol{\Omega} \times \delta\mathbf{v}_{\text{rel}}}$$

Referring to Eqn (7.4c), we set $\delta\mathbf{a}_{\text{rel}} = \delta\ddot{x}\hat{\mathbf{i}} + \delta\ddot{y}\hat{\mathbf{j}} + \delta\ddot{z}\hat{\mathbf{k}}$ [where $\delta\ddot{x} = (\mathrm{d}^2/\mathrm{d}t^2)\delta x$, etc.] and collect the terms on the right to obtain

$$
\begin{aligned}
\delta\ddot{x}\hat{\mathbf{i}} + \delta\ddot{y}\hat{\mathbf{j}} + \delta\ddot{z}\hat{\mathbf{k}} = & \left[\left(\frac{2\mu}{R^3} + \frac{h^2}{R^4} \right)\delta x - \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta y + 2\frac{h}{R^2}\delta\dot{y} \right]\hat{\mathbf{i}} \\
& + \left[\left(\frac{h^2}{R^4} - \frac{\mu}{R^3} \right)\delta y + \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta x - 2\frac{h}{R^2}\delta\dot{x} \right]\hat{\mathbf{j}} \\
& - \frac{\mu}{R^3}\delta z\hat{\mathbf{k}}
\end{aligned}
\tag{7.33}
$$

Finally, by equating the coefficients of the three unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, this vector equation yields the three scalar equations,

$$\delta\ddot{x} - \left(\frac{2\mu}{R^3} + \frac{h^2}{R^4} \right)\delta x + \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta y - 2\frac{h}{R^2}\delta\dot{y} = 0 \tag{7.34a}$$

$$\delta\ddot{y} + \left(\frac{\mu}{R^3} - \frac{h^2}{R^4} \right)\delta y - \frac{2(\mathbf{V}\cdot\mathbf{R})h}{R^4}\delta x + 2\frac{h}{R^2}\delta\dot{x} = 0 \tag{7.34b}$$

$$\delta\ddot{z} + \frac{\mu}{R^3}\delta z = 0 \tag{7.34c}$$

This set of linear second-order differential equations must be solved in order to obtain the relative position coordinates δx, δy, and δz as a function of time. Eqns (7.34a) and (7.34b) are coupled since δx and δy appear in each one of them. δz appears by itself in Eqn (7.34c) and nowhere else, which means

the relative motion in the z direction is independent of that in the other two directions. If the reference orbit is an ellipse, then $\mathbf{R}$ and $\mathbf{V}$ vary with time (although the angular momentum h is constant). In that case, the coefficients in Eqn (7.34) are time dependent, so there is no easy analytical solution. However, we can solve Eqn (7.34) numerically using the methods in Section 1.8.

   To that end, we recast Eqn (7.34) as a set of first-order differential equations in the standard form

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}) \tag{7.35}$$

where

$$
\mathbf{y} == \left\{ \begin{array}{c} \delta x \\ \delta y \\ \delta z \\ \delta \dot{x} \\ \delta \dot{y} \\ \delta \dot{z} \end{array} \right\} \quad
\dot{\mathbf{y}} = \left\{ \begin{array}{c} \delta \dot{x} \\ \delta \dot{y} \\ \delta \dot{z} \\ \delta \ddot{x} \\ \delta \ddot{y} \\ \delta \ddot{z} \end{array} \right\} \quad
\mathbf{f}(t, \mathbf{y}) = \left\{ \begin{array}{c} y_4 \\[2mm] y_5 \\[2mm] y_6 \\[2mm] \left(\dfrac{2\mu}{R^3} + \dfrac{h^2}{R^4} \right) y_1 - \dfrac{2(\mathbf{V} \cdot \mathbf{R})h}{R^4} y_2 + 2\dfrac{h}{R^2} y_5 \\[4mm] \left(\dfrac{h^2}{R^4} - \dfrac{\mu}{R^3} \right) y_2 + \dfrac{2(\mathbf{V} \cdot \mathbf{R})h}{R^4} y_1 - 2\dfrac{h_0}{R^2} y_4 \\[4mm] -\dfrac{\mu}{R^3} y_3 \end{array} \right\} \tag{7.36}
$$

These can be solved by Algorithm 1.1 (Runge–Kutta), Algorithm 1.2 (Heun), or Algorithm 1.3 (Runge–Kutta–Fehlberg). In any case, the state vector of the target orbit must be updated at each time step to provide the current values of $\mathbf{R}$ and $\mathbf{V}$. This is done with the aid of Algorithm 3.4. (Alternatively, Eqn (7.22), the equations of motion of the target, can be integrated along with Eqn (7.36) in order to provide $\mathbf{R}$ and $\mathbf{V}$ as a function of time.)

EXAMPLE 7.3

At time $t = 0$, the orbital parameters of target vehicle A in an equatorial earth orbit are

$$r_p = 6678 \text{ km} \quad e = 0.1 \quad i = \Omega = \omega = \theta = 0° \tag{a}$$

where r_p is the perigee radius. At that same instant, the state vector of the chaser vehicle B relative to A is

$$\delta \mathbf{r}_0 = -1\hat{\mathbf{i}} \text{ (km)} \quad \delta \mathbf{v}_{\text{rel}})_0 = 2n\hat{\mathbf{j}} \text{ (km/s)} \tag{b}$$

where n is the mean motion of A. Plot the path of B relative to A in the comoving frame for five periods of the reference orbit.

Solution

1. Use Algorithm 4.5 to obtain the initial state vector ($\mathbf{R}_0$, $\mathbf{V}_0$) of the target vehicle from the orbital parameters given in Eqn (a).
2. Starting with the initial conditions given in Eqn (b), use Algorithm 1.3 to integrate Eqn (7.36) over the specified time interval. Use Algorithm 3.4 to obtain the reference orbit state vector ($\mathbf{R}$, $\mathbf{V}$) at each time step in order to evaluate the coefficients in Eqn (7.36).
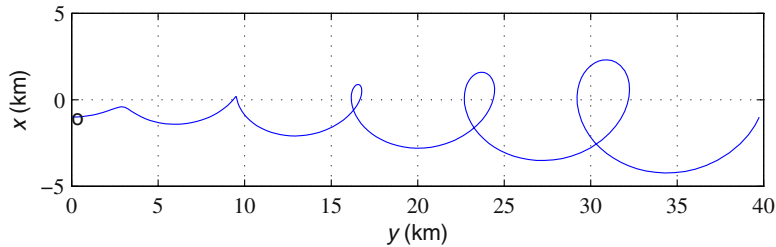

FIGURE 7.6

Trajectory of B relative to A in the comoving frame during five revolutions of the reference orbit. Eccentricity of the reference orbit $= 0.1$.
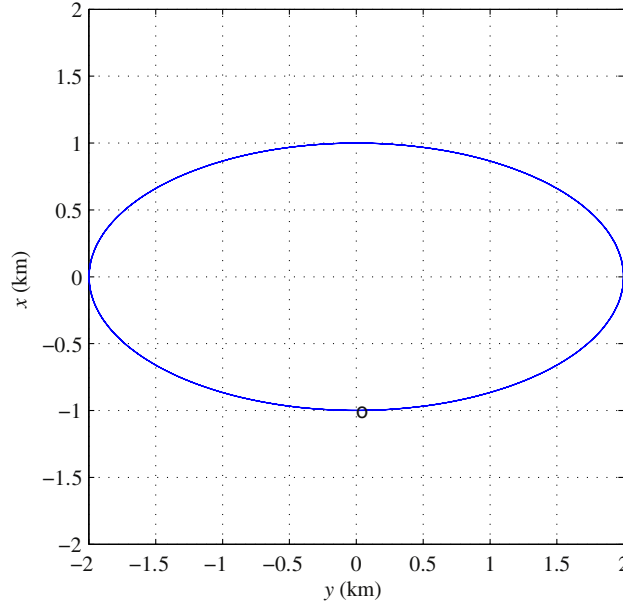

FIGURE 7.7

Trajectory of B relative to A in the comoving frame during five revolutions of the reference orbit. Eccentricity of reference orbit $= 0$.

3. Graph the trajectory $\delta y(t)$ versus $\delta x(t)$.

This procedure is implemented in the MATLAB function *Example_7_03.m* listed in Appendix D.33. The output of the program is shown in Figure 7.6. Observe that since $\delta z_0 = \delta \dot{z}_0 = 0$, no movement develops in the z direction. The motion of the chaser therefore lies in the plane of the target vehicle's orbit. Figure 7.6 shows that B rapidly moves away from A along the y direction and that the amplitude of its looping motion about the x-axis continuously increases. The accuracy of this solution degrades over time because eventually, the criterion in Eqn (7.15) is no longer satisfied.

It is interesting to note that if we change the eccentricity of A to zero, so that the reference orbit is a circle, then Figure 7.7 results. That is, for the same initial conditions, B orbits the target vehicle instead of drifting away from it.

7.4 Clohessy–Wiltshire equations

If the orbit of the target vehicle A in Figure 7.5 is a circle, then our LVLH frame is called a Clohessy–Wiltshire (CW) frame. In such a frame, Eqn (7.34) simplifies considerably. For a circular target orbit, $\mathbf{V} \cdot \mathbf{R} = 0$ and $h = \sqrt{\mu R}$. Substituting these into Eqn (7.34) yields

$$\delta \ddot{x} - 3\frac{\mu}{R^3}\delta x - 2\sqrt{\frac{\mu}{R^3}}\,\delta \dot{y} = 0$$

$$\delta \ddot{y} + 2\sqrt{\frac{\mu}{R^3}}\,\delta \dot{x} = 0 \tag{7.37}$$

$$\delta \ddot{z} + \frac{\mu}{R^3}\delta z = 0$$

It is furthermore true for circular orbits that the angular velocity (mean motion) is

$$n = \frac{V}{R} = \frac{\sqrt{\mu/R}}{R} = \sqrt{\frac{\mu}{R^3}}$$

Therefore, Eqn (7.37) may be written as

$$\delta \ddot{x} - 3n^2\delta x - 2n\delta \dot{y} = 0 \tag{7.38a}$$

$$\delta \ddot{y} + 2n\delta \dot{x} = 0 \tag{7.38b}$$

$$\delta \ddot{z} + n^2\delta z = 0 \tag{7.38c}$$

These are known as the Clohessy-Wiltshire (CW) equations. Unlike Eqn (7.34), where the target orbit is an ellipse, the coefficients in Eqn (7.38) are constant. Therefore, a straightforward analytical solution exists.

We start with the first two equations, which are coupled and define the motion of the chaser in the xy plane of the reference orbit. First, observe that Eqn (7.38b) can be written as $(\mathrm{d}/\mathrm{d}t)(\delta \dot{y} + 2n\delta x) = 0$, which means that $\delta \dot{y} + 2n\delta x = C_1$, where C_1 is a constant. Therefore,

$$\delta \dot{y} = C_1 - 2n\delta x \tag{7.39}$$

Substituting this expression into Eqn (7.38a) yields

$$\delta \ddot{x} + n^2\delta x = 2nC_1 \tag{7.40}$$

This familiar differential equation has the following solution, which can be easily verified by substitution:

$$\delta x = \frac{2}{n}C_1 + C_2 \sin nt + C_3 \cos nt \tag{7.41}$$

Differentiating this expression gives the x component of the relative velocity,

$$\delta \dot{x} = C_2 n \cos nt - C_3 n \sin nt \tag{7.42}$$

Substituting Eqn (7.41) into Eqn (7.39) yields the y component of the relative velocity.

$$\delta \dot{y} = -3C_1 - 2C_2 n \sin nt - 2C_3 n \cos nt \tag{7.43}$$

Integrating this equation with respect to time yields

$$\delta y = -3C_1 t + 2C_2 \cos nt - 2C_3 \sin nt + C_4 \tag{7.44}$$

The constants C_1 through C_4 are found by applying the initial conditions, namely,

$$\text{At } t = 0 \quad \delta x = \delta x_0 \quad \delta y = \delta y_0 \quad \delta \dot{x} = \delta \dot{x}_0 \quad \delta \dot{y} = \delta \dot{y}_0$$

Evaluating Eqns (7.41)–(7.44), respectively, at $t = 0$, we get

$$\frac{2}{n}C_1 + C_3 = \delta x_0$$

$$C_2 n = \delta \dot{x}_0$$

$$-3C_1 - 2C_3 n = \delta \dot{y}_0$$

$$2C_2 + C_4 = \delta y_0$$

Solving for C_1 through C_4 yields

$$C_1 = 2n\delta x_0 + \delta \dot{y}_0 \quad C_2 = \frac{1}{n}\delta \dot{x}_0 \quad C_3 = -3\delta x_0 - \frac{2}{n}\delta \dot{y}_0 \quad C_4 = -\frac{2}{n}\delta \dot{x}_0 + \delta y_0 \tag{7.45}$$

Finally, we turn our attention to Eqn (7.38c), which governs the relative motion normal to the plane of the circular reference orbit. Eqn (7.38c) has the same form as Eqn (7.40) with $C_1 = 0$. Therefore, its solution is

$$\delta z = C_5 \sin nt + C_6 \cos nt \tag{7.46}$$

It follows that the velocity normal to the reference orbit is

$$\delta \dot{z} = C_5 n \cos nt - C_6 n \sin nt \tag{7.47}$$

The initial conditions are $\delta z = \delta z_0$ and $\delta \dot{z} = \delta \dot{z}_0$ at $t = 0$, which means

$$C_5 = \frac{\delta \dot{z}_0}{n} \quad C_6 = \delta z_0 \tag{7.48}$$

Substituting Eqns (7.45) and (7.48) into Eqns (7.41), (7.44), and (7.46) yields the trajectory of the chaser in the CW frame,

$$\delta x = 4\delta x_0 + \frac{2}{n}\delta \dot{y}_0 + \frac{\delta \dot{x}_0}{n} \sin nt - \left(3\delta x_0 + \frac{2}{n}\delta \dot{y}_0 \right) \cos nt \tag{7.49a}$$

$$\delta y = \delta y_0 - \frac{2}{n}\delta\dot{x}_0 - 3(2n\delta x_0 + \delta\dot{y}_0)t + 2\left(3\delta x_0 + \frac{2}{n}\delta\dot{y}_0\right)\sin nt + \frac{2}{n}\delta\dot{x}_0 \cos nt \qquad (7.49b)$$

$$\delta z = \frac{1}{n}\delta\dot{z}_0 \sin nt + \delta z_0 \cos nt \qquad (7.49c)$$

Observe that all the three components of $\delta\mathbf{r}$ oscillate with a frequency equal to the frequency of revolution (mean motion n) of the CW frame. Only δy has a secular term, which grows linearly with time. Therefore, unless $2n\delta x_0 + \delta\dot{y}_0 = 0$, the chaser will drift away from the target and the distance δr will increase without bound. The accuracy of Eqn (7.49) will consequently degrade as the criterion (Eqn (7.15)) on which this solution is based eventually ceases to be valid. Figure 7.8 shows the motion of a particle relative to a CW frame of an orbital of radius 6678 km. The particle started at the origin with a velocity of 0.01 km/s in the negative y direction. This delta-v dropped the particle into a lower energy, a slightly elliptical orbit. The subsequent actual relative motion of the particle in the CW frame is graphed in Figure 7.8 as is the motion given by Eqn (7.49), the linearized CW solution. Clearly, the two solutions diverge markedly after one orbit of the reference frame, when the distance of the particle from the origin exceeds 150 km.

Now that we have finished solving the CW equations, let us simplify our notation a bit and denote the x, y, and z components of relative velocity in the moving frame as δu, δv, and δw, respectively. That is, let

$$\delta u = \delta\dot{x} \quad \delta v = \delta\dot{y} \quad \delta w = \delta\dot{z} \qquad (7.50a)$$

The initial conditions on the relative velocity components are then written as

$$\delta u_0 = \delta\dot{x}_0 \quad \delta v_0 = \delta\dot{y}_0 \quad \delta w_0 = \delta\dot{z}_0 \qquad (7.50b)$$

Using this notation in Eqn (7.49) and rearranging the terms, we get

$$\delta x = (4 - 3\cos nt)\delta x_0 + \frac{\sin nt}{n}\delta u_0 + \frac{2}{n}(1 - \cos nt)\delta v_0$$

$$\delta y = 6(\sin nt - nt)\delta x_0 + \delta y_0 + \frac{2}{n}(\cos nt - 1)\delta u_0 + \frac{1}{n}(4\sin nt - 3nt)\delta v_0 \qquad (7.51a)$$

$$\delta z = \cos nt\,\delta z_0 + \frac{1}{n}\,\sin nt\,\delta w_0$$
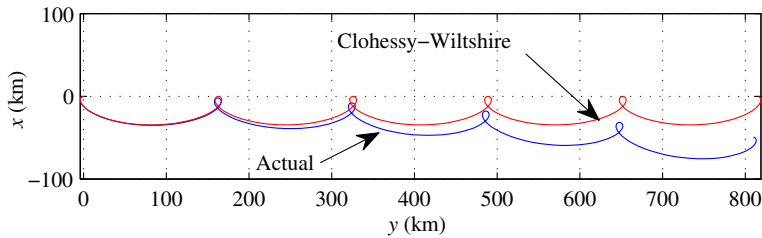

FIGURE 7.8

Relative motion of a particle and its CW approximation.

Differentiating each of these with respect to time and using Eqn (7.50a) yields

$$\delta u = 3n \sin nt \delta x_0 + \cos nt \delta u_0 + 2 \sin nt \delta v_0$$

$$\delta v = 6n(\cos nt - 1)\delta x_0 - 2 \sin nt \delta u_0 + (4 \cos nt - 3)\delta v_0 \qquad (7.51b)$$

$$\delta w = -n \sin nt \delta z_0 + \cos nt \delta w_0$$

Let us introduce matrix notation to define the relative position and velocity vectors

$$\{\delta \mathbf{r}(t)\} = \left\{ \begin{array}{c} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{array} \right\} \quad \{\delta \mathbf{v}(t)\} = \left\{ \begin{array}{c} \delta u(t) \\ \delta v(t) \\ \delta w(t) \end{array} \right\}$$

and their initial values (at $t = 0$)

$$\{\delta \mathbf{r}_0\} = \left\{ \begin{array}{c} \delta x_0 \\ \delta y_0 \\ \delta z_0 \end{array} \right\} \quad \{\delta \mathbf{v}_0\} = \left\{ \begin{array}{c} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{array} \right\}$$

Observe that we have dropped the subscript rel introduced in Eqn (7.17) because it is superfluous in the rendezvous analysis, where all kinematic quantities are relative to the CW frame. In matrix notation, Eqn (7.51) appears more compactly as

$$\{\delta \mathbf{r}(t)\} = [\Phi_{rr}(t)]\{\delta \mathbf{r}_0\} + [\Phi_{rv}(t)]\{\delta \mathbf{v}_0\} \qquad (7.52a)$$

$$\{\delta \mathbf{v}(t)\} = [\Phi_{vr}(t)]\{\delta \mathbf{r}_0\} + [\Phi_{vv}(t)]\{\delta \mathbf{v}_0\} \qquad (7.52b)$$

where, from Eqn (7.50), the CW matrices are

$$\left[\Phi_{rr}(t) \right] = \left[\begin{array}{cc|c} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ \hline 0 & 0 & \cos nt \end{array} \right] \qquad (7.53a)$$

$$\left[\Phi_{rv}(t) \right] = \left[\begin{array}{cc|c} \dfrac{1}{n}\sin nt & \dfrac{2}{n}(1 - \cos nt) & 0 \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{1}{n}(4\sin nt - 3nt) & 0 \\ \hline 0 & 0 & \dfrac{1}{n}\sin nt \end{array} \right] \qquad (7.53b)$$

$$\left[\Phi_{vr}(t)\right] = \left[\begin{array}{cc:c} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ \hdashline 0 & 0 & -n\sin nt \end{array}\right] \tag{7.53c}$$

$$\left[\Phi_{vv}(t)\right] = \left[\begin{array}{cc:c} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ \hdashline 0 & 0 & \cos nt \end{array}\right] \tag{7.53d}$$

The subscripts on Φ remind us which of the vectors $\delta\mathbf{r}$ and $\delta\mathbf{v}$ is related by that matrix to which of the initial conditions $\delta\mathbf{r}_0$ and $\delta\mathbf{v}_0$. For example, $[\Phi_{rv}]$ relates $\delta\mathbf{r}$ to $\delta\mathbf{v}_0$. The partition lines remind us that motion in the xy plane is independent of that in the z direction normal to the target's orbit. In problems where there is no motion in the z direction ($\delta z_0 = \delta w_0 = 0$), we need only use the upper 2 by 2 corners of the CW matrices. Finally, note also that

$$[\Phi_{vr}(t)] = \frac{d}{dt}[\Phi_{rr}(t)] \quad \text{and} \quad [\Phi_{vv}(t)] = \frac{d}{dt}[\Phi_{rv}(t)]$$

7.5 Two-impulse rendezvous maneuvers

Figure 7.9 illustrates the rendezvous problem. At time $t = 0^-$ (the instant preceding $t = 0$), the position $\delta\mathbf{r}_0$ and velocity $\delta\mathbf{v}_0^-$ of the chase vehicle B relative to target A are known. At $t = 0$, an impulsive maneuver instantaneously changes the relative velocity to $\delta\mathbf{v}_0^+$ at $t = 0^+$ (the instant after $t = 0$). The components of $\delta\mathbf{v}_0^+$ are shown in Figure 7.9. We must determine the values of δu_0^+, δv_0^+, and δw_0^+, at the beginning of the rendezvous trajectory, so that B will arrive at the target in a specified time t_f. The delta-v required to place B on the rendezvous trajectory is

$$\Delta\mathbf{v}_0 = \delta\mathbf{v}_0^+ - \delta\mathbf{v}_0^- = \left(\delta u_0^+ - \delta u_0^-\right)\hat{\mathbf{i}} + \left(\delta v_0^+ - \delta v_0^-\right)\hat{\mathbf{j}} + \left(\delta w_0^+ - \delta w_0^-\right)\hat{\mathbf{k}} \tag{7.54}$$

At time t_f, B arrives at A, at the origin of the CW frame, which means $\delta\mathbf{r}_f = \delta\mathbf{r}(t_f) = \mathbf{0}$. Evaluating Eqn (7.52a) at t_f, we find
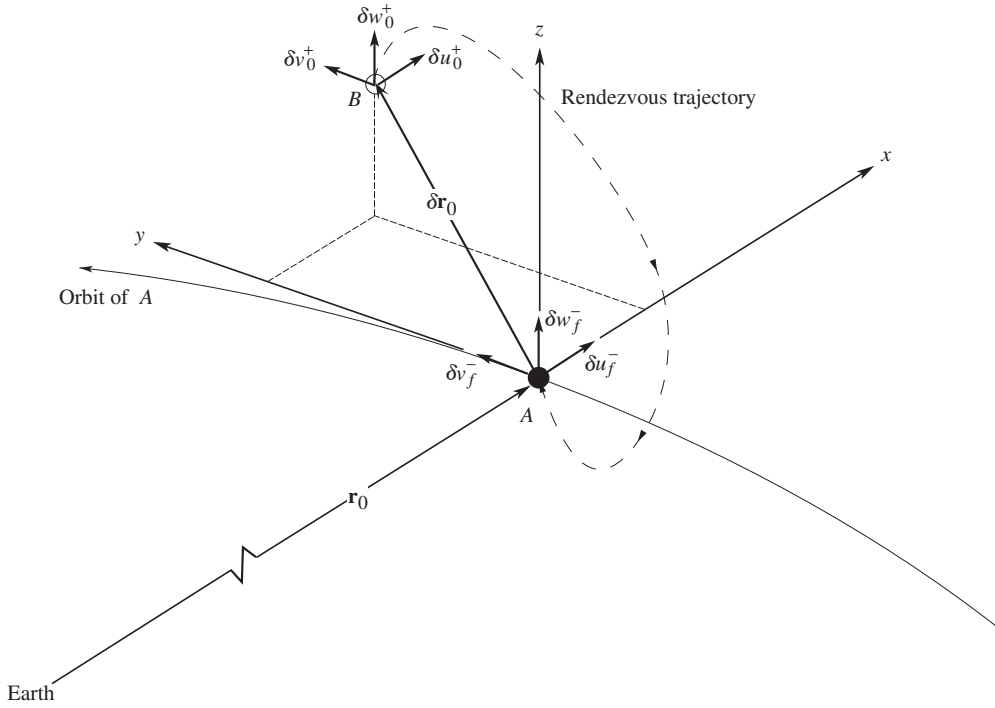
$$\{\mathbf{0}\} = \left[\Phi_{rr}(t_f)\right]\{\delta\mathbf{r}_0\} + \left[\Phi_{rv}(t_f)\right]\{\delta\mathbf{v}_0^+\} \tag{7.55}$$

Solving this for $\{\delta\mathbf{v}_0^+\}$ yields

$$\{\delta\mathbf{v}_0^+\} = -\left[\Phi_{rv}(t_f)\right]^{-1}\left[\Phi_{rr}(t_f)\right]\{\delta\mathbf{r}_0\} \quad \left(\delta\mathbf{v}_0^+ = \delta u_0^+\hat{\mathbf{i}} + \delta v_0^+\hat{\mathbf{j}} + \delta w_0^+\hat{\mathbf{k}}\right) \tag{7.56}$$

where $\left[\Phi_{rv}(t_f)\right]^{-1}$ is the matrix inverse of $\left[\Phi_{rv}(t_f)\right]$. Thus, we now have the velocity $\delta\mathbf{v}_0^+$ at the beginning of the rendezvous path. We substitute Eqn (7.56) into Eqn (7.52b) to obtain the velocity $\delta\mathbf{v}_f^-$ at $t = t_f^-$, when B arrives at target A:

$$\{\delta\mathbf{v}_f^-\} = \left[\Phi_{vr}(t_f)\right]\{\delta\mathbf{r}_0\} + \left[\Phi_{vv}(t_f)\right]\{\delta\mathbf{v}_0^+\}$$

$$= \left[\Phi_{vr}(t_f)\right]\{\delta\mathbf{r}_0\} + \left[\Phi_{vv}(t_f)\right]\left(-\left[\Phi_{rv}(t_f)\right]^{-1}\left[\Phi_{rr}(t_f)\right]\{\delta\mathbf{r}_0\}\right)$$

FIGURE 7.9

Rendezvous with a target *A* in the neighborhood of the chase vehicle *B*.

Collecting terms, we get

$$\{\delta\mathbf{v}_f^-\} = \left(\left[\Phi_{\mathrm{vr}}(t_f) \right] - \left[\Phi_{\mathrm{vv}}(t_f) \right] \left[\Phi_{\mathrm{rv}}(t_f) \right]^{-1} \left[\Phi_{\mathrm{rr}}(t_f) \right] \right) \{\delta\mathbf{r}_0\} \quad \left(\delta\mathbf{v}_f^- = \delta u_f^- \hat{\mathbf{i}} + \delta v_f^- \hat{\mathbf{j}} + \delta w_f^- \hat{\mathbf{k}} \right)$$

(7.57)

Obviously, an impulsive delta-v maneuver is required at $t = t_f$ to bring vehicle *B* to rest relative to *A* $(\delta\mathbf{v}_f^+ = \mathbf{0})$:

$$\Delta\mathbf{v}_f = \delta\mathbf{v}_f^+ - \delta\mathbf{v}_f^- = \mathbf{0} - \delta\mathbf{v}_f^- = -\delta\mathbf{v}_f^-$$

(7.58)

Note that in Eqns (7.54) and (7.58) we are using the difference between relative velocities to calculate delta-v, which is the difference in absolute velocities. To show that this is valid, use Eqn (1.75) to write

$$\mathbf{v}^- = \mathbf{v}_0^- + \mathbf{\Omega}^- \times \mathbf{r}_{\mathrm{rel}}^- + \mathbf{v}_{\mathrm{rel}}^-$$
$$\mathbf{v}^+ = \mathbf{v}_0^+ + \mathbf{\Omega}^+ \times \mathbf{r}_{\mathrm{rel}}^+ + \mathbf{v}_{\mathrm{rel}}^+$$

(7.59)

Since the target is passive, the impulsive maneuver has no effect on its state of motion, which means $\mathbf{v}_0^+ = \mathbf{v}_0^-$ and $\mathbf{\Omega}^+ = \mathbf{\Omega}^-$. Furthermore, by definition of an impulsive maneuver, there is no change in the position, that is, $\mathbf{r}_{\mathrm{rel}}^+ = \mathbf{r}_{\mathrm{rel}}^-$. It follows from Eqn (7.59) that

$$\mathbf{v}^+ - \mathbf{v}^- = \mathbf{v}_{\mathrm{rel}}^+ - \mathbf{v}_{\mathrm{rel}}^- \quad \text{or} \quad \Delta\mathbf{v} = \Delta\mathbf{v}_{\mathrm{rel}}$$

EXAMPLE 7.4

A space station and spacecraft are in orbits with the following parameters:

	Space Station	**Spacecraft**
Perigee × apogee (altitude)	300 km circular	320.06 km × 513.86 km
Period (computed using above data)	1.5086 h	1.5484 h
True anomaly, θ	60°	349.65°
Inclination, i	40°	40.130°
RA, Ω	20°	19.819°
Argument of perigee, ω	0° (arbitrary)	70.662°

Compute the total delta-v required for an 8-hour, two-impulse rendezvous trajectory.

Solution
We use the given data in Algorithm 4.5 to obtain the state vectors of the two spacecraft in the geocentric equatorial frame.

Space station:

$$\mathbf{r}_0 = 1622.39\hat{\mathbf{I}} + 5305.10\hat{\mathbf{J}} + 3717.44\hat{\mathbf{K}} \ \text{(km)}$$

$$\mathbf{v}_0 = -7.29936\hat{\mathbf{I}} + 0.492329\hat{\mathbf{J}} + 2.48304\hat{\mathbf{K}} \ \text{(km/s)}$$

Spacecraft:

$$\mathbf{r} = 1612.75\hat{\mathbf{I}} + 5310.19\hat{\mathbf{J}} + 3750.33\hat{\mathbf{K}} \ \text{(km)}$$

$$\mathbf{v} = -7.35170\hat{\mathbf{I}} + 0.463828\hat{\mathbf{J}} + 2.46906\hat{\mathbf{K}} \ \text{(km/s)}$$

The space station reference frame unit vectors (at this instant) are, by definition,

$$\hat{\mathbf{i}} = \frac{\mathbf{r}_o}{\|\mathbf{r}_o\|} = 0.242945\hat{\mathbf{I}} + 0.794415\hat{\mathbf{J}} + 0.556670\hat{\mathbf{K}}$$

$$\hat{\mathbf{j}} = \frac{\mathbf{v}_o}{\|\mathbf{v}_o\|} = -0.944799\hat{\mathbf{I}} + 0.063725\hat{\mathbf{J}} + 0.321394\hat{\mathbf{K}}$$

$$\hat{\mathbf{k}} = \hat{\mathbf{i}} \times \hat{\mathbf{j}} = 0.219846\hat{\mathbf{I}} - 0.604023\hat{\mathbf{J}} + 0.766044\hat{\mathbf{K}}$$

Therefore, the transformation matrix from the geocentric equatorial frame into the space station frame is (at this instant)

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix}$$

The position vector of the spacecraft relative to the space station (in the geocentric equatorial frame) is

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0 = -9.64015\hat{\mathbf{I}} + 5.08235\hat{\mathbf{J}} + 32.8822\hat{\mathbf{K}} \ \text{(km)}$$

The relative velocity is given by the formula (Eqn 1.75)

$$\delta\mathbf{v} = \mathbf{v} - \mathbf{v}_0 - \mathbf{\Omega}_{\text{space station}} \times \delta\mathbf{r}$$

where $\mathbf{\Omega}_{\text{space station}} = n\hat{\mathbf{k}}$ and n, the mean motion of the space station, is

$$n = \frac{v_0}{r_0} = \frac{7.72627}{6678} = 0.00115691 \ \text{rad/s} \tag{a}$$

Thus,

$$\delta\mathbf{v} = \overbrace{-7.35170\hat{\mathbf{I}} + 0.463828\hat{\mathbf{J}} + 2.46906\hat{\mathbf{K}}}^{\mathbf{v}} - \overbrace{\left(-7.29936\hat{\mathbf{I}} + 0.492329\hat{\mathbf{J}} + 2.48304\hat{\mathbf{K}}\right)}^{\mathbf{v}_0}$$

$$\overbrace{-(0.00115691)\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0.219846 & -0.604023 & 0.766044 \\ -9.64015 & 5.08235 & 32.8822 \end{vmatrix}}^{\mathbf{\Omega}_{\text{space station}} \times \delta\mathbf{r}}$$

so that

$$\delta\mathbf{v} = -0.024854\hat{\mathbf{I}} - 0.0115937\hat{\mathbf{J}} - 0.00853575\hat{\mathbf{K}} \ (\text{km/s})$$

In space station coordinates, the relative position vector $\delta\mathbf{r}_0$ at the beginning of the rendezvous maneuver is

$$\{\delta\mathbf{r}_0\} = [\mathbf{Q}]_{Xx}\{\delta\mathbf{r}\} = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix} \begin{Bmatrix} -9.64015 \\ 5.08235 \\ 32.8822 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix} \ (\text{km}) \quad (\text{b})$$

Likewise, the relative velocity $\delta\mathbf{v}_0^-$ just before launch into the rendezvous trajectory is

$$\{\delta\mathbf{v}_0^-\} = [\mathbf{Q}]_{Xx}\{\delta\mathbf{v}\} = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix} \begin{Bmatrix} -0.024854 \\ -0.0115937 \\ -0.00853575 \end{Bmatrix}$$

$$= \begin{Bmatrix} -0.02000 \\ 0.02000 \\ -0.005000 \end{Bmatrix} \ (\text{km/s})$$

The CW matrices, for $t = t_f = 8 \ \text{h} = 28{,}800 \ \text{s}$ and $n = 0.00115691 \ \text{rad/s}$ [from Eqn (a)], are

$$[\Phi_{rr}] = \begin{bmatrix} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 4.97849 & 0 & 0 \\ -194.242 & 1.000 & 0 \\ 0 & 0 & -0.326163 \end{bmatrix}$$

$$
[\Phi_{rv}] = \begin{bmatrix} \dfrac{1}{n}\sin nt & \dfrac{2}{n}(1-\cos nt) & 0 \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \dfrac{1}{n}\sin nt \end{bmatrix} = \begin{bmatrix} 817.102 & 2292.60 & 0 \\ -2292.60 & -83131.6 & 0 \\ 0 & 0 & 817.103 \end{bmatrix}
$$

$$
[\Phi_{vr}] = \begin{bmatrix} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n\sin nt \end{bmatrix} = \begin{bmatrix} 0.00328092 & 0 & 0 \\ -0.00920550 & 0 & 0 \\ 0 & 0 & -0.00109364 \end{bmatrix}
$$

$$
[\Phi_{vv}] = \begin{bmatrix} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} -0.326164 & 1.89063 & 0 \\ -1.89063 & -4.30466 & 0 \\ 0 & 0 & -0.326164 \end{bmatrix}
$$

From Eqns (7.56) and (b), we find $\delta\mathbf{v}_0^+$:

$$
\begin{Bmatrix} \delta u_o^+ \\ \delta v_o^+ \\ \delta w_o^+ \end{Bmatrix} = - \begin{bmatrix} 817.102 & 2292.60 & 0 \\ -2292.60 & -83131.6 & 0 \\ 0 & 0 & 817.103 \end{bmatrix}^{-1} \begin{bmatrix} 4.97849 & 0 & 0 \\ -194.242 & 1.000 & 0 \\ 0 & 0 & -0.326163 \end{bmatrix} \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix}
$$

$$
= - \begin{bmatrix} 817.102 & 2292.60 & 0 \\ -2292.60 & -83131.6 & 0 \\ 0 & 0 & 817.103 \end{bmatrix}^{-1} \begin{Bmatrix} 99.5698 \\ -3864.84 \\ -6.52386 \end{Bmatrix} = \begin{Bmatrix} 0.00930458 \\ -0.0467472 \\ 0.00798343 \end{Bmatrix} \; (\text{km/s})
$$

<div align="right">(c)</div>

From Eqn (7.52b), evaluated at $t = t_f$, we have

$$
\{\delta\mathbf{v}_f^-\} = [\Phi_{vr}(t_f)]\{\delta\mathbf{r}_0\} + [\Phi_{vv}(t_f)]\{\delta\mathbf{v}_0^+\}
$$

Substituting Eqns (b) and (c),

$$
\left\{ \begin{array}{c} \delta u_f^- \\ \delta v_f^- \\ \delta w_f^- \end{array} \right\} = \left[\begin{array}{ccc} 0.00328092 & 0 & 0 \\ -0.00920550 & 0 & 0 \\ 0 & 0 & -0.00109364 \end{array} \right] \left\{ \begin{array}{c} 20 \\ 20 \\ 20 \end{array} \right\}
$$

$$
+ \left[\begin{array}{ccc} -0.326164 & 1.89063 & 0 \\ -1.89063 & -4.30466 & 0 \\ 0 & 0 & -0.326164 \end{array} \right] \left\{ \begin{array}{c} 0.00930458 \\ -0.0467472 \\ 0.00798343 \end{array} \right\}
$$

$$
\left\{ \begin{array}{c} \delta u_f^- \\ \delta v_f^- \\ \delta w_f^- \end{array} \right\} = \left\{ \begin{array}{c} -0.0257978 \\ -0.000470870 \\ -0.0244767 \end{array} \right\} \text{ (km/s)} \qquad\qquad (d)
$$

The delta-v at the beginning of the rendezvous maneuver is found as

$$
\{\Delta \mathbf{v}_0\} = \{\delta \mathbf{v}_0^+\} - \{\delta \mathbf{v}_0^-\} = \left\{ \begin{array}{c} 0.00930458 \\ -0.0467472 \\ 0.00798343 \end{array} \right\} - \left\{ \begin{array}{c} -0.02 \\ 0.02 \\ -0.005 \end{array} \right\} = \left\{ \begin{array}{c} 0.0293046 \\ -0.0667472 \\ 0.0129834 \end{array} \right\} \text{ (km/s)}
$$

The delta-v at the conclusion of the maneuver is

$$
\{\Delta \mathbf{v}_f\} = \{\delta \mathbf{v}_f^+\} - \{\delta \mathbf{v}_f^-\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\} - \left\{ \begin{array}{c} -0.0257978 \\ -0.000470870 \\ -0.0244767 \end{array} \right\} = \left\{ \begin{array}{c} 0.0257978 \\ 0.000470870 \\ 0.0244767 \end{array} \right\} \text{ (km/s)}
$$

The total delta-v requirement is

$$
\Delta v_{\text{total}} = \|\Delta \mathbf{v}_0\| + \|\Delta \mathbf{v}_f\| = 0.0740440 + 0.0355649 = 0.109609 \text{ km/s} = \boxed{109.6 \text{ m/s}}
$$

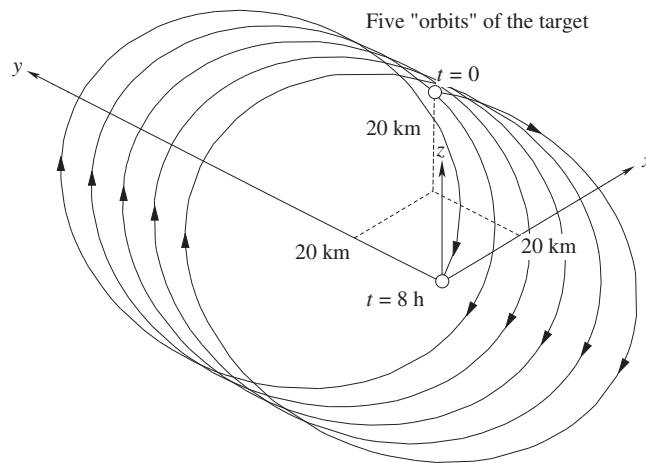From Eqn (7.42a), we have, for $0 < t < t_f$,

FIGURE 7.10

Rendezvous trajectory of the chase vehicle relative to the target.

$$\left\{\begin{array}{c} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{array}\right\} = \begin{bmatrix} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} \left\{\begin{array}{c} 20 \\ 20 \\ 20 \end{array}\right\}$$

$$+ \begin{bmatrix} \dfrac{1}{n}\sin nt & \dfrac{2}{n}(1 - \cos nt) & 0 \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \dfrac{1}{n}\sin nt \end{bmatrix} \left\{\begin{array}{c} 0.00930458 \\ -0.0467472 \\ 0.00798343 \end{array}\right\}$$

Substituting n from Eqn (a), we obtain the relative position vector as a function of time. It is plotted in Figure 7.10

EXAMPLE 7.5

A target and a chase vehicle are in the same 300-km circular earth orbit. The chaser is 2 km behind the target when the chaser initiates a two-impulse rendezvous maneuver so as to rendezvous with the target in 1.49 h. Find the total delta-v requirement.

Solution

For the circular reference orbit

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398,600}{6378 + 300}} = 7.7258 \text{ km/s} \tag{a}$$

so that the mean motion is

$$n = \frac{v}{r} = \frac{7.7258}{6678} = 0.0011569 \text{ rad/s} \tag{b}$$

For this mean motion and the rendezvous trajectory time $t = 1.49 \text{ h} = 5364 \text{ s}$, the CW matrices are

$$[\Phi_{rr}] = \begin{bmatrix} 1.0090 & 0 & 0 \\ -37.699 & 1 & 0 \\ \hline 0 & 0 & 0.99700 \end{bmatrix} \qquad [\Phi_{rv}] = \begin{bmatrix} -66.946 & 5.1928 & 0 \\ -5.1928 & -16360 & 0 \\ \hline 0 & 0 & -66.946 \end{bmatrix}$$

$$[\Phi_{vr}] = \begin{bmatrix} -2.6881 \times 10^{-4} & 0 & 0 \\ -2.0851 \times 10^{-5} & 0 & 0 \\ \hline 0 & 0 & 8.9603 \times 10^{-5} \end{bmatrix} \qquad [\Phi_{vv}] = \begin{bmatrix} 0.99700 & -0.15490 & 0 \\ 0.15490 & 0.98798 & 0 \\ \hline 0 & 0 & 0.99700 \end{bmatrix} \tag{c}$$

The initial and final positions of the chaser in the CW frame are

$$\{\delta r_0\} = \begin{Bmatrix} 0 \\ -2 \\ 0 \end{Bmatrix} \text{ (km)} \qquad \{\delta r_f\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ (km)} \tag{d}$$

Since $\delta z_0 = \delta w_0 = 0$, there is no motion in the z direction [$\delta z(t) = 0$], so we need employ only the upper left 2 by 2 corners of the CW matrices and treat this as a two-dimensional problem in the plane of the reference orbit. Thus, solving the first CW equation, $\{\delta r_f\} = [\Phi_{rr}]\{\delta r_0\} + [\Phi_{rv}]\{\delta v_0^+\}$, for $\{\delta v_0^+\}$, we get

$$\{\delta v_0^+\} = -[\Phi_{rv}]^{-1}[\Phi_{rr}]\{\delta r_0\} = -\begin{bmatrix} -0.014937 & -4.7412 \times 10^{-6} \\ 4.7412 \times 10^{-6} & -6.1124 \times 10^{-5} \end{bmatrix} \begin{bmatrix} 1.0090 & 0 \\ -37.699 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -2 \end{Bmatrix}$$

$$= \begin{Bmatrix} -9.4824 \times 10^{-6} \\ -1.2225 \times 10^{-4} \end{Bmatrix}$$

or

$$\delta v_0^+ = -9.4824 \times 10^{-6}\hat{i} - 1.2225 \times 10^{-4}\hat{j} \text{ (km/s)} \tag{e}$$

Therefore, the second CW equation, $\{\delta v_f^-\} = [\Phi_{vr}]\{\delta r_0\} + [\Phi_{vv}]\{\delta v_0^+\}$, yields

$$\{\delta v_f^-\} = \begin{bmatrix} -2.6881 \times 10^{-4} & 0 \\ -2.0851 \times 10^{-5} & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ -2 \end{Bmatrix} + \begin{bmatrix} 0.99700 & -0.15490 \\ 0.15490 & 0.98798 \end{bmatrix} \begin{Bmatrix} -9.4824 \times 10^{-6} \\ -1.2225 \times 10^{-4} \end{Bmatrix}$$

$$= \begin{Bmatrix} 9.4824 \times 10^{-6} \\ -1.2225 \times 10^{-4} \end{Bmatrix}$$
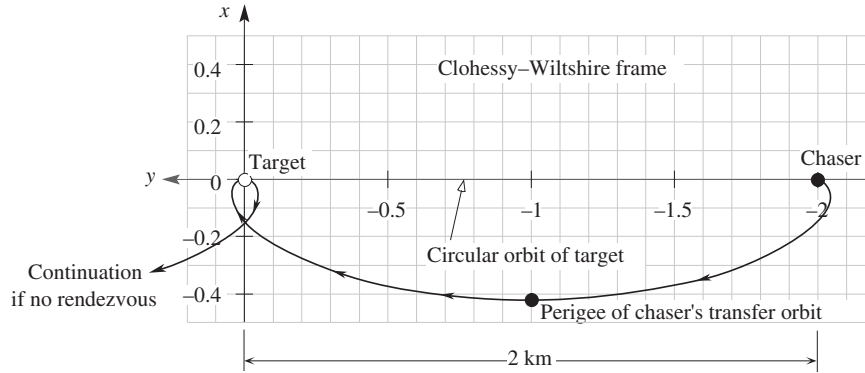
FIGURE 7.11

Motion of the chaser relative to the target.

or

$$\delta\mathbf{v}_f^- = 9.4824 \times 10^{-6}\hat{\mathbf{i}} - 1.2225 \times 10^{-4}\hat{\mathbf{j}} \quad (\text{km/s}) \tag{f}$$

Since the chaser is in the same circular orbit as the target, its relative velocity is initially zero, that is $\delta\mathbf{v}_0^- = \mathbf{0}$. (See also Eqn (7.68) at the end of the next section.) Thus,

$$\begin{aligned}\Delta\mathbf{v}_0 = \delta\mathbf{v}_0^+ - \delta\mathbf{v}_0^- &= (-9.4824 \times 10^{-6}\hat{\mathbf{i}} - 1.2225 \times 10^{-4}\hat{\mathbf{j}}) - \mathbf{0} \\ &= -9.4824 \times 10^{-6}\hat{\mathbf{i}} - 1.2225 \times 10^{-4}\hat{\mathbf{j}} \quad (\text{km/s})\end{aligned}$$

which implies

$$\|\Delta\mathbf{v}_0\| = 0.1226 \ \text{m/s} \tag{g}$$

At the end of the rendezvous maneuver, $\delta\mathbf{v}_f^+ = \mathbf{0}$, so that

$$\begin{aligned}\Delta\mathbf{v}_f = \delta\mathbf{v}_f^+ - \delta\mathbf{v}_f^- &= \mathbf{0} - (9.4824 \times 10^{-6}\hat{\mathbf{i}} - 1.2225 \times 10^{-4}\hat{\mathbf{j}}) \\ &= -9.4824 \times 10^{-6}\hat{\mathbf{i}} + 1.2225 \times 10^{-4}\hat{\mathbf{j}} \quad (\text{km/s})\end{aligned}$$

Therefore,

$$\|\Delta\mathbf{v}_f\| = 0.1226 \ \text{m/s} \tag{h}$$

The total delta-v required is

$$\Delta v_{\text{total}} = \|\Delta\mathbf{v}_0\| + \|\Delta\mathbf{v}_f\| = \boxed{0.2452 \ \text{m/s}} \tag{i}$$

The coplanar rendezvous trajectory relative to the CW frame is sketched in Figure 7.11.

7.6 Relative motion in close-proximity circular orbits

Figure 7.12 shows two spacecraft in coplanar circular orbits. Let us calculate the velocity $\delta\mathbf{v}$ of the chase vehicle B relative to target A when they are in close proximity. "Close proximity" means that

$$\frac{\delta r}{R} << 1$$

To solve this problem, we must use the relative velocity equation,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{\Omega} \times \delta\mathbf{r} + \delta\mathbf{v} \tag{7.60}$$

where $\mathbf{\Omega}$ is the angular velocity of the CW frame attached to A,

$$\mathbf{\Omega} = n\hat{\mathbf{k}}$$

n is the mean motion of the target vehicle,

$$n = \frac{v_A}{R} \tag{7.61}$$

where, by virtue of the circular orbit,

$$v_A = \sqrt{\frac{\mu}{R}} \tag{7.62}$$

Solving Eqn (7.60) for the relative velocity $\delta\mathbf{v}$ yields

$$\delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A - \left(n\hat{\mathbf{k}}\right) \times \delta\mathbf{r} \tag{7.63}$$
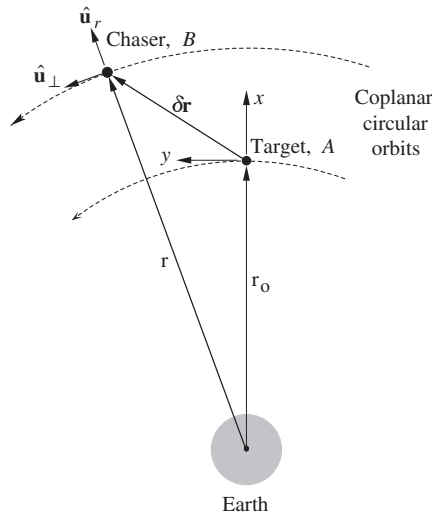
FIGURE 7.12

Two spacecraft in close proximity.

Since the chase orbit is circular, we have for the first term on the right-hand side of Eqn (7.63)

$$\mathbf{v}_B = \sqrt{\frac{\mu}{r}}\,\hat{\mathbf{u}}_\perp = \sqrt{\frac{\mu}{r}}\big(\hat{\mathbf{k}} \times \hat{\mathbf{u}}_r\big) = \sqrt{\mu}\,\hat{\mathbf{k}} \times \left(\frac{1}{\sqrt{r}}\frac{\mathbf{r}}{r}\right) \tag{7.64}$$

Since, as is apparent from Figure 7.9, $\mathbf{r} = \mathbf{R} + \delta\mathbf{r}$, we can write this expression for $\mathbf{v}_B$ as follows:

$$\mathbf{v}_B = \sqrt{\mu}\,\hat{\mathbf{k}} \times r^{-3/2}(\mathbf{R} + \delta\mathbf{r}) \tag{7.65}$$

Now

$$r^{-3/2} = \left(r^2\right)^{-3/4} = \left[\overbrace{R^2\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)}^{\text{See Eqn 7.18}}\right]^{-3/4} = R^{-3/2}\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-3/4} \tag{7.66}$$

Using the binomial theorem (Eqn (5.44)), and retaining terms at most linear in $\delta\mathbf{r}$, we get

$$\left(1 + \frac{2\mathbf{R}\cdot\delta\mathbf{r}}{R^2}\right)^{-3/4} = 1 - \frac{3}{2}\frac{\mathbf{R}\cdot\delta\mathbf{r}}{R^2}$$

Substituting this into Eqn (7.66) leads to

$$r^{-3/2} = R^{-3/2} - \frac{3}{2}\frac{\mathbf{R}\cdot\delta\mathbf{r}}{R^{7/2}}$$
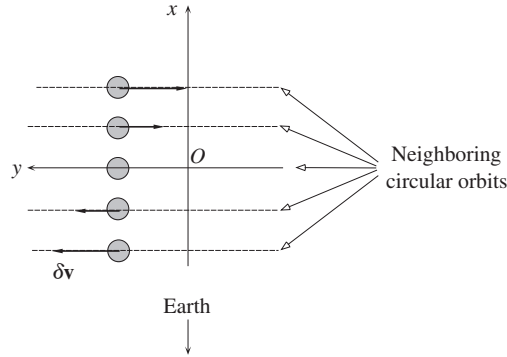
Upon substituting this result into Eqn (7.65), we get

$$\mathbf{v}_B = \sqrt{\mu}\,\hat{\mathbf{k}} \times (\mathbf{R} + \delta\mathbf{r})\left(R^{-3/2} - \frac{3}{2}\frac{\mathbf{R}\cdot\delta\mathbf{r}}{R^{7/2}}\right)$$

Retaining terms at most linear in $\delta\mathbf{r}$, we can write this as

$$\mathbf{v}_B = \hat{\mathbf{k}} \times \left\{\sqrt{\frac{\mu}{R}}\,\frac{\mathbf{R}}{R} + \frac{\sqrt{\mu/R}}{R}\delta\mathbf{r} - \frac{3}{2}\frac{\sqrt{\mu/R}}{R}\left[\left(\frac{\mathbf{R}}{R}\right)\cdot\delta\mathbf{r}\right]\frac{\mathbf{R}}{R}\right\}$$

Using Eqns (7.61) and (7.62), together with the facts that $\delta\mathbf{r} = \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}$ and $\mathbf{R}/R = \hat{\mathbf{i}}$, this reduces to

$$\mathbf{v}_B = \hat{\mathbf{k}} \times \left\{v_A\hat{\mathbf{i}} + \frac{v_A}{R}\left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}\right) - \frac{3}{2}\frac{v_A}{R}\left[\hat{\mathbf{i}}\cdot\left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}\right)\right]\hat{\mathbf{i}}\right\}$$

$$= v_A\hat{\mathbf{j}} + \left(-n\delta y\hat{\mathbf{i}} + n\delta x\hat{\mathbf{j}}\right) - \frac{3}{2}n\delta x\hat{\mathbf{j}}$$

FIGURE 7.13

Circular orbits, with relative velocity directions, in the vicinity of the CW frame.

so that

$$\mathbf{v}_B = -n\delta y\hat{\mathbf{i}} + \left(v_A - \frac{1}{2}n\delta x\right)\hat{\mathbf{j}} \tag{7.67}$$

This is the absolute velocity of the chaser resolved into components in the target's CW frame.

Substituting Eqn (7.67) into Eqn (7.63) and using the fact that $\mathbf{v}_A = v_A\hat{\mathbf{j}}$ yields

$$\delta\mathbf{v} = \left[-n\delta y\hat{\mathbf{i}} + \left(v_A - \frac{1}{2}n\delta x\right)\hat{\mathbf{j}}\right] - \left(v_A\hat{\mathbf{j}}\right) - \left(n\hat{\mathbf{k}}\right) \times \left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}}\right)$$

$$= -n\delta y\hat{\mathbf{i}} + v_A\hat{\mathbf{j}} - \frac{1}{2}n\delta x\hat{\mathbf{j}} - v_A\hat{\mathbf{j}} - n\delta x\hat{\mathbf{j}} + n\delta y\hat{\mathbf{i}}$$

so that

$$\delta\mathbf{v} = -\frac{3}{2}n\delta x\hat{\mathbf{j}} \tag{7.68}$$

This is the velocity of the chaser as measured in the moving reference frame of the neighboring target. Keep in mind that circular orbits were assumed at the outset.

In the CW frame, neighboring coplanar circular orbits appear to be straight lines parallel to the *y*-axis, which is the orbit of the origin. Figure 7.13 illustrates this point, showing also the linear velocity variation according to Eqn (7.68).

PROBLEMS

Section 7.2

7.1 Two manned spacecraft, *A* and *B* (see the figure), are in circular polar ($i = 90°$) orbits around the earth. *A*'s orbital altitude is 300 km; *B*'s is 250 km. At the instant shown (*A* over the equator,

B over the North Pole), calculate (a) the position, (b) velocity, and (c) the acceleration of *B* relative to *A*. *A*'s *y*-axis points always in the flight direction, and its *x*-axis is directed radially outward at all times.

{Ans.:   (a) $\mathbf{r}_{\text{rel}})_{xyz} = -6678\hat{\mathbf{i}} + 6628\hat{\mathbf{j}}$ km;   (b) $\mathbf{v}_{\text{rel}})_{xyz} = -0.08693\hat{\mathbf{i}}$ km/s;   (c) $\mathbf{a}_{\text{rel}})_{xyz} = -1.140 \times 10^{-6}\hat{\mathbf{j}}$ km/s^2}
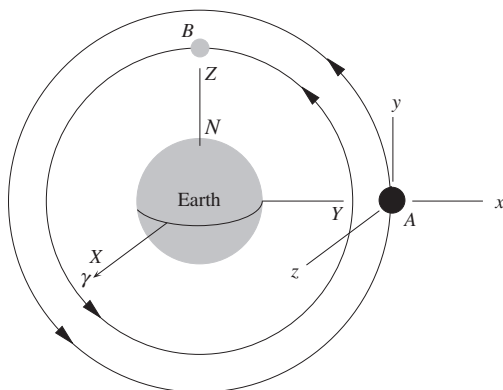
7.2 Spacecraft *A* and *B* are in coplanar, circular geocentric orbits. The orbital radii are shown in the figure. When *B* is directly below *A*, as shown, calculate *B*'s speed $v_{B_{\text{rel}}}$ relative to *A*.

{Ans. : $\mathbf{a}_{\text{rel}})_{xyz} = -0.268\hat{\mathbf{i}}$ $\left(\text{m/s}^2\right)$}
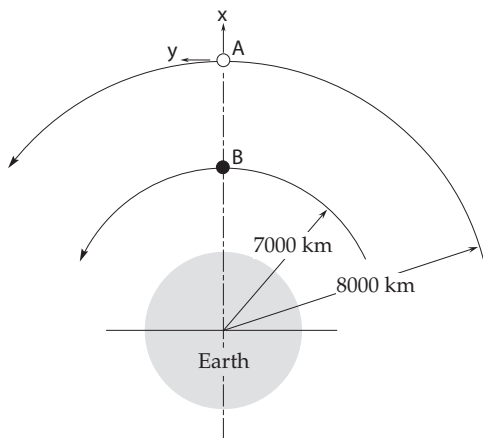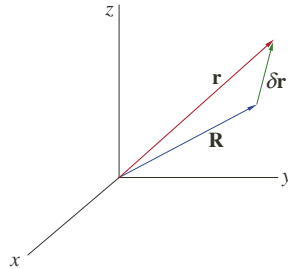

Section 7.3

7.3 Use the order of magnitude analysis in this chapter as a guide to answer the following questions:
   (a) If $\mathbf{r} = \mathbf{R} + \delta\mathbf{r}$, express $\sqrt{r}$ (where $r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$) to the first order in $\delta\mathbf{r}$ (i.e., to the first order in the components of $\delta\mathbf{r} = \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}$). In other words, find $O(\delta\mathbf{r})$, such that
   $\sqrt{r} = \sqrt{R} + O(\delta\mathbf{r})$, where $O(\delta\mathbf{r})$ is linear in $\delta\mathbf{r}$.

(b) For the special case $\mathbf{R} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ and $\delta\mathbf{r} = 0.01\mathbf{i} - 0.01\mathbf{j} + 0.03\mathbf{k}$, calculate $\sqrt{r} - \sqrt{R}$ and compare that result with $O(\delta\mathbf{r})$.

(c) Repeat Part (b) using $\delta\mathbf{r} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and compare the results.

{Ans.: (a) $O(\delta\mathbf{r}) = \mathbf{R} \cdot \delta\mathbf{r}/(2R^{\frac{3}{2}})$; (b) $O(\delta\mathbf{r})/(\sqrt{r} - \sqrt{R}) = 0.998$; (c) $O(\delta\mathbf{r})/ (\sqrt{r} - \sqrt{R}) = 0.903$}

7.4 Write the expression $r = \dfrac{a(1 - e^2)}{1 + e \cos \theta}$ as a linear function of e, valid for small values of e ($e \ll 1$).

Section 7.4

7.5 Given $\ddot{x} + 9x = 10$, with the initial conditions $x = 5$ and $\dot{x} = -3$ at $t = 0$, find x and $\dot{x}$ at $t = 1.2$.
{Ans.: $x(1.2) = -1.934$, $\dot{x}(1.2) = 7.853$}

**7.6** Given that

$$\ddot{x} + 10x + 2\dot{y} = 0$$

$$\ddot{y} + 3\dot{x} = 0$$

with initial conditions $x(0) = 1$, $y(0) = 2$, $\dot{x}(0) = -3$, and $\dot{y}(0) = 4$, find x and y at $t = 5$.
{Ans.: $x(5) = -6.460$, $y(5) = 97.31$}

7.7 A space station is in a 90-min period earth orbit. At $t = 0$, a satellite has the following position and velocity components relative to a CW frame attached to the space station: $\delta\mathbf{r} = \hat{\mathbf{i}}$ (km), $\delta\mathbf{v} = 10\hat{\mathbf{j}}$ (m/s). How far is the satellite from the space station 15 min later?
{Ans.: 11.2 km}

7.8 Spacecraft A and B are in the same circular earth orbit with a period of 2 h. B is 6 km ahead of A. At $t = 0$, B applies an in-track delta-v (retrofire) of 3 m/s. Using a CW frame attached to A, determine the distance between A and B at $t = 30$ min and the velocity of B relative to A.
{Ans.: $\delta r = 10.9$ km, $\delta v = 10.8$ m/s}

7.9 The CW coordinates and velocities of a spacecraft upon entering a rendezvous trajectory with the target vehicle are shown. The spacecraft orbits are coplanar. Calculate the distance d of the spacecraft from the target when $t = \pi/2n$, where n is the mean motion of the target's circular orbit.
{Ans.: $0.900\delta r$}
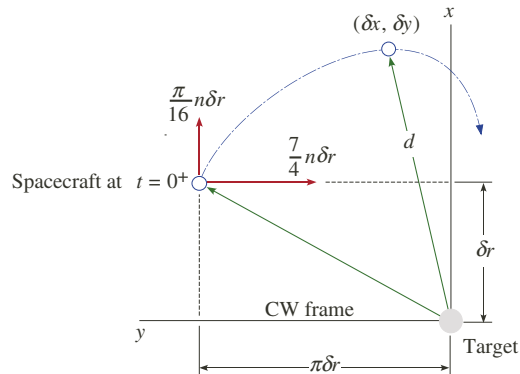
7.10 At time $t = 0$, a particle is at the origin of a CW frame with a relative velocity $\delta\mathbf{v}_0 = v\hat{\mathbf{j}}$. What will be the relative speed of the particle after one-half the orbital period of the CW frame? {Ans.: $7v$}

7.11 The chaser and the target are in close proximity, coplanar circular orbits. At $t = 0$, the position of the chaser relative to the target is $\delta\mathbf{r}_0 = r\hat{\mathbf{i}} + a\hat{\mathbf{j}}$, $\delta\mathbf{v}_0^+ = v_0\hat{\mathbf{j}}$ where a is given and r is unknown. The relative velocity at $t = 0^+$ is (v_0 is unknown), and the chaser ends up at $\delta\mathbf{r}_f = -a\,\hat{\mathbf{i}}$ when $t = \pi/n$, where n is the mean motion of the target. Use the CW equations to find the required value of the orbital spacing r. {Ans.: $1.424a$}
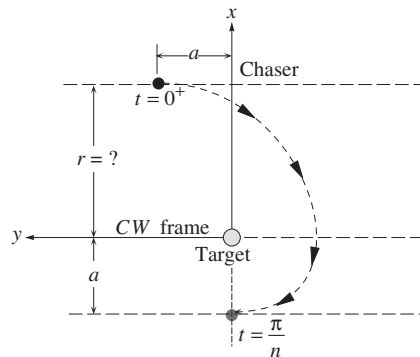
Section 7.5

7.12 A space station is in a circular earth orbit of radius 6600 km. An approaching spacecraft executes a delta-v burn when its position vector relative to the space station is $\delta\mathbf{r}_0 = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ (km). Just before the burn the relative velocity of the spacecraft was

$\delta \mathbf{v}_0^- = 5\hat{\mathbf{k}}$ (m/s). Calculate the total delta-v required for the spacecraft to rendezvous with the station in one-third period of the space station orbit.
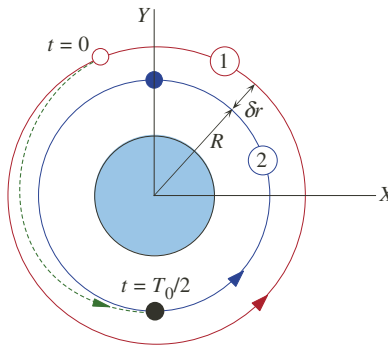{Ans.: 6.21 m/s}

7.13 A space station is in circular orbit 2 of radius R. A spacecraft is in coplanar circular orbit 1 of radius $R + \delta r$. At $t = 0$ the spacecraft executes an impulsive maneuver to rendezvous with the space station at time $t_f =$ one-half the period T_0 of the space station. If $\delta u_0^+ = 0$, find
(a) the initial position of the spacecraft relative to the space station and
(b) the relative velocity of the spacecraft when it arrives at the target. Sketch the rendezvous trajectory relative to the target.

$$\left\{ \text{Ans.: (a) } \delta \mathbf{r}_0 = \delta r \hat{\mathbf{i}} + \frac{3\pi \delta r}{4} \hat{\mathbf{j}}, \text{ (b) } \delta \mathbf{v}_f^- = \frac{\pi \delta r}{2 T_0} \hat{\mathbf{j}} \right\}$$

7.14 If $\delta u_0^+ = 0$, calculate the total delta-v required for rendezvous if $\delta \mathbf{r}_0 = \delta y_0 \hat{\mathbf{j}}$, $\delta \mathbf{v}_0^- = \mathbf{0}$, and $t_f =$ the period of the circular target orbit. Sketch the rendezvous trajectory relative to the target.
{Ans.: $\Delta v_{\text{tot}} = 2\delta y_o/(3T)$}

7.15 A GEO satellite strikes some orbiting debris and is found 2 h afterward to have drifted to the position $\delta \mathbf{r} = -10\hat{\mathbf{i}} + 10\hat{\mathbf{j}}$ (km) relative to its original location. At that time, the only slightly damaged satellite initiates a two-impulse maneuver to return to its original location in 6 h. Find the total delta-v for this maneuver.
{Ans.: 3.5 m/s}

7.16 A space station is in a 245-km circular earth orbit inclined at $30°$. The right ascension of its node line is $40°$. Meanwhile, a spacecraft has been launched into a 280-km by 250-km orbit inclined at $30.1°$, with a nodal right ascension of $40°$ and argument of perigee equal to $60°$. When the spacecraft's true anomaly is $40°$, the space station is $99°$ beyond its node line. At that instant, the spacecraft executes a delta-v burn to rendezvous with the space station in (precisely) t_f hours, where t_f is either selected by you or assigned by the instructor. Calculate the total delta-v required and sketch the projection of the rendezvous trajectory on the xy plane of the space station coordinates.

7.17 The target A is in a circular earth orbit with mean motion n. The chaser B is directly above A in a slightly larger circular orbit having the same plane as A. What relative initial velocity $\delta \mathbf{v}_0^+$ is required so that B arrives at target A at time $t_f =$ one-half the target's period?

$$\left\{ \text{Ans.: } \delta \mathbf{v}_0^+ = -0.589n\delta x_0\hat{\mathbf{i}} - 1.75n\delta x_0\hat{\mathbf{j}} \right\}$$
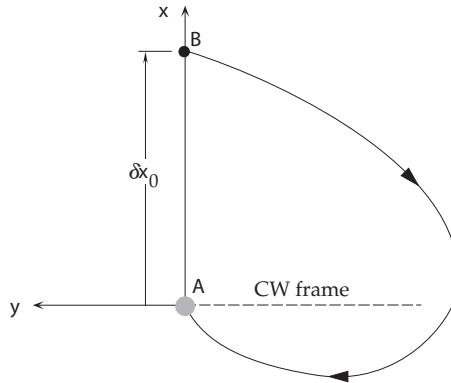
Section 7.6

7.18 The space station is in a circular earth orbit of radius 6600 km. A spacecraft is also in a circular orbit in the same plane as the space station. At the instant that the position of the spacecraft relative to the space station, in Clohessy-Wiltshire coordinates, is $\delta\mathbf{r} = 5\hat{\mathbf{i}}$ (km), what is the relative velocity $\delta\mathbf{v}$ of the spacecraft in meters/second?
{Ans.: 8.83 m/s}
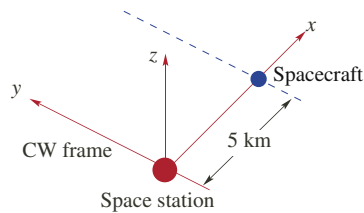
7.19 A spacecraft and the space station are in coplanar circular orbits. The space station has an orbital radius R and a mean motion n. The spacecraft's radius is $R - d$ ($d << R$). If a two-impulse rendezvous maneuver with $t_f = \pi/(4n)$ is initiated with zero relative velocity in the x direction ($\delta u_0^+ = 0$), calculate the total delta-v.
{Ans.: 4.406nd}
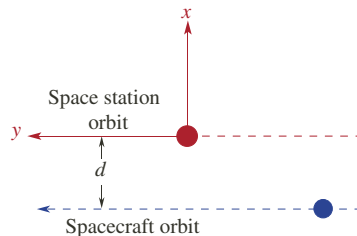
7.20 The chaser and the target are in close proximity, coplanar circular orbits. At $t = 0$, the position of the chaser relative to the target is $\delta\mathbf{r}_0 = a\hat{\mathbf{i}}$. Use the CW equations to find the total delta-v required for the chaser to end up in circular orbit 2 at $\delta\mathbf{r}_f = -a\hat{\mathbf{i}}$ when $t = \pi/n$, where n is the mean motion of the target.

{Ans.: na}