CHAPTER

Preliminary Orbit Determination

5

CHAPTER OUTLINE

5.1 Introduction

In this chapter, we will consider some (by no means all) of the classical ways in which the orbit of a satellite can be determined from earth-bound observations. All the methods presented here are based on the two-body equations of motion. As such, they must be considered preliminary orbit determination techniques because the actual orbit is influenced over time by other phenomena (perturbations), such as the gravitational force of the moon and sun, atmospheric drag, solar wind, and the nonspherical shape and nonuniform mass distribution of the earth. We took a brief look at the dominant effects of the earth's oblateness in Section 4.7. To accurately propagate an orbit into the future from a set of initial observations requires taking the various perturbations, as well as instrumentation errors themselves, into account. More detailed considerations, including the means of updating the orbit based on additional observations, are beyond our scope. Introductory discussions may be found elsewhere. See Bate, Mueller, and White (1971), Boulet (1991), Prussing and Conway (1993), and Wiesel (1997), to name but a few.

We begin with the Gibbs method of predicting an orbit using three geocentric position vectors. This is followed by a presentation of Lambert's problem, in which an orbit is determined from two position

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vectors and the time between them. Both the Gibbs and Lambert procedures are based on the fact that two-body orbits lie in a plane. The Lambert problem is more complex and requires using the Lagrange f and g functions introduced in Chapter 2 as well as the universal variable formulation introduced in Chapter 3. The Lambert algorithm is employed in Chapter 8 to analyze interplanetary missions.

In preparation for explaining how satellites are tracked, the Julian day (JD) numbering scheme is introduced along with the notion of sidereal time. This is followed by a description of the topocentric coordinate systems and the relationships among topocentric right ascension/declension angles and azimuth/elevation angles. We then describe how orbits are determined from measuring the range and angular orientation of the line of sight together with their rates. The chapter concludes with a presentation of the Gauss method of angles-only orbit determination.

5.2 Gibbs method of orbit determination from three position vectors

Suppose that from the observations of a space object at the three successive times t_1, t_2, and t_3 ($t_1 < t_2 < t_3$) we have obtained the geocentric position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. The problem is to determine the velocities $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ at t_1, t_2, and t_3 assuming that the object is in a two-body orbit. The solution using purely vector analysis is due to J. W. Gibbs (1839–1903), an American scholar who is known primarily for his contributions to thermodynamics. Our explanation is based on that in Bate, Mueller, and White (1971).

We know that the conservation of angular momentum requires that the position vectors of an orbiting body must lie in the same plane. In other words, the unit vector normal to the plane of $\mathbf{r}_2$ and $\mathbf{r}_3$ must be perpendicular to the unit vector in the direction of $\mathbf{r}_1$. Thus, if $\hat{\mathbf{u}}_{r_1} = \mathbf{r}_1/r_1$ and $\hat{\mathbf{C}}_{23} = (\mathbf{r}_2 \times \mathbf{r}_3)/\|\mathbf{r}_2 \times \mathbf{r}_3\|$, then the dot product of these two unit vectors must vanish:

$$\hat{\mathbf{u}}_{r_1} \cdot \hat{\mathbf{C}}_{23} = 0$$

Furthermore, as illustrated in Figure 5.1, the fact that $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ lie in the same plane means we can apply scalar factors c_1 and c_3 to $\mathbf{r}_1$ and $\mathbf{r}_3$ so that $\mathbf{r}_2$ is the vector sum of $c_1\mathbf{r}_1$ and $c_3\mathbf{r}_3$:

$$\mathbf{r}_2 = c_1\mathbf{r}_1 + c_3\mathbf{r}_3 \tag{5.1}$$

The coefficients c_1 and c_3 are readily obtained from $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ as we shall see in Section 5.10 (Eqns (5.89) and (5.90)).

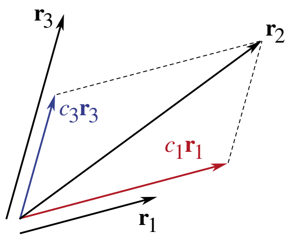

FIGURE 5.1

Any one of a set of three coplanar vectors ($\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$) can be expressed as the vector sum of the other two.

To find the velocity $\mathbf{v}$ corresponding to any of the three given position vectors $\mathbf{r}$, we start with Eqn (2.40), which may be written as

$$\mathbf{v} \times \mathbf{h} = \mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right)$$

where $\mathbf{h}$ is the angular momentum and $\mathbf{e}$ is the eccentricity vector. To isolate the velocity, take the crossproduct of this equation with the angular momentum,

$$\mathbf{h} \times (\mathbf{v} \times \mathbf{h}) = \mu\left(\frac{\mathbf{h} \times \mathbf{r}}{r} + \mathbf{h} \times \mathbf{e}\right) \tag{5.2}$$

By means of the bac–cab rule (Eqn (2.33)), the left side becomes

$$\mathbf{h} \times (\mathbf{v} \times \mathbf{h}) = \mathbf{v}(\mathbf{h}\cdot\mathbf{h}) - \mathbf{h}(\mathbf{h}\cdot\mathbf{v})$$

But $\mathbf{h}\cdot\mathbf{h} = h^2$, and $\mathbf{v}\cdot\mathbf{h} = 0$, since $\mathbf{v}$ is perpendicular to $\mathbf{h}$. Therefore,

$$\mathbf{h} \times (\mathbf{v} \times \mathbf{h}) = h^2\mathbf{v}$$

which means Eqn (5.2) may be written as

$$\mathbf{v} = \frac{\mu}{h^2}\left(\frac{\mathbf{h} \times \mathbf{r}}{r} + \mathbf{h} \times \mathbf{e}\right) \tag{5.3}$$

In Section 2.10, we introduced the perifocal coordinate system, in which the unit vector $\hat{\mathbf{p}}$ lies in the direction of the eccentricity vector $\mathbf{e}$ and $\hat{\mathbf{w}}$ is the unit vector normal to the orbital plane, in the direction of the angular momentum vector $\mathbf{h}$. Thus, we can write

$$\mathbf{e} = e\hat{\mathbf{p}} \tag{5.4a}$$

$$\mathbf{h} = h\hat{\mathbf{w}} \tag{5.4b}$$

so that Eqn (5.3) becomes

$$\mathbf{v} = \frac{\mu}{h^2}\left(\frac{h\hat{\mathbf{w}} \times \mathbf{r}}{r} + h\hat{\mathbf{w}} \times e\hat{\mathbf{p}}\right) = \frac{\mu}{h}\left[\frac{\hat{\mathbf{w}} \times \mathbf{r}}{r} + e(\hat{\mathbf{w}} \times \hat{\mathbf{p}})\right] \tag{5.5}$$

Since $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{w}}$ form a right-handed triad of unit vectors, it follows that $\hat{\mathbf{p}} \times \hat{\mathbf{q}} = \hat{\mathbf{w}}$, $\hat{\mathbf{q}} \times \hat{\mathbf{w}} = \hat{\mathbf{p}}$, and

$$\hat{\mathbf{w}} \times \hat{\mathbf{p}} = \hat{\mathbf{q}} \tag{5.6}$$

Therefore, Eqn (5.5) reduces to

$$\mathbf{v} = \frac{\mu}{h}\left(\frac{\hat{\mathbf{w}} \times \mathbf{r}}{r} + e\hat{\mathbf{q}}\right) \tag{5.7}$$

This is an important result, because if we can somehow use the position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ to calculate $\hat{\mathbf{q}}$, $\hat{\mathbf{w}}$, h, and e, then the velocities $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ will each be determined by this formula.

So far, the only condition we have imposed on the three position vectors is that they are coplanar (Eqn (5.1)). To bring in the fact that they describe an orbit, let us take the dot product of Eqn (5.1) with the eccentricity vector $\mathbf{e}$ to obtain the scalar equation

$$\mathbf{r}_2\cdot\mathbf{e} = c_1\mathbf{r}_1\cdot\mathbf{e} + c_3\mathbf{r}_3\cdot\mathbf{e} \tag{5.8}$$

According to Eqn (2.44)—the orbit equation—we have the following relations among h, e, and each of the position vectors:

$$\mathbf{r}_1 \cdot \mathbf{e} = \frac{h^2}{\mu} - r_1 \quad \mathbf{r}_2 \cdot \mathbf{e} = \frac{h^2}{\mu} - r_2 \quad \mathbf{r}_3 \cdot \mathbf{e} = \frac{h^2}{\mu} - r_3 \tag{5.9}$$

Substituting these equations into Eqn (5.8) yields

$$\frac{h^2}{\mu} - r_2 = c_1\left(\frac{h^2}{\mu} - r_1\right) + c_3\left(\frac{h^2}{\mu} - r_3\right) \tag{5.10}$$

To eliminate the unknown coefficients c_1 and c_2 from this expression, let us take the crossproduct of Eqn (5.1) first with $\mathbf{r}_1$ and then with $\mathbf{r}_3$. This results in two equations, both having $\mathbf{r}_3 \times \mathbf{r}_1$ on the right,

$$\mathbf{r}_2 \times \mathbf{r}_1 = c_3(\mathbf{r}_3 \times \mathbf{r}_1) \quad \mathbf{r}_2 \times \mathbf{r}_3 = -c_1(\mathbf{r}_3 \times \mathbf{r}_1) \tag{5.11}$$

Now multiply Eqn (5.10) through by the vector $\mathbf{r}_3 \times \mathbf{r}_1$ to obtain

$$\frac{h^2}{\mu}(\mathbf{r}_3 \times \mathbf{r}_1) - r_2(\mathbf{r}_3 \times \mathbf{r}_1) = c_1(\mathbf{r}_3 \times \mathbf{r}_1)\left(\frac{h^2}{\mu} - r_1\right) + c_3(\mathbf{r}_3 \times \mathbf{r}_1)\left(\frac{h^2}{\mu} - r_3\right)$$

Using Eqn (5.11), this becomes

$$\frac{h^2}{\mu}(\mathbf{r}_3 \times \mathbf{r}_1) - r_2(\mathbf{r}_3 \times \mathbf{r}_1) = -(\mathbf{r}_2 \times \mathbf{r}_3)\left(\frac{h^2}{\mu} - r_1\right) + (\mathbf{r}_2 \times \mathbf{r}_1)\left(\frac{h^2}{\mu} - r_3\right)$$

Observe that c_1 and c_2 have been eliminated. Rearranging the terms, we get

$$\frac{h^2}{\mu}(\mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1) = r_1(\mathbf{r}_2 \times \mathbf{r}_3) + r_2(\mathbf{r}_3 \times \mathbf{r}_1) + r_3(\mathbf{r}_1 \times \mathbf{r}_2) \tag{5.12}$$

This is an equation involving the given position vectors and the unknown angular momentum h. Let us introduce the following notation for the vectors on each side of Eqn (5.12),

$$\mathbf{N} = r_1(\mathbf{r}_2 \times \mathbf{r}_3) + r_2(\mathbf{r}_3 \times \mathbf{r}_1) + r_3(\mathbf{r}_1 \times \mathbf{r}_2) \tag{5.13}$$

and

$$\mathbf{D} = \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1 \tag{5.14}$$

Then, Eqn (5.12) may be written more simply as

$$\mathbf{N} = \frac{h^2}{\mu}\mathbf{D}$$

from which we obtain

$$N = \frac{h^2}{\mu}D \tag{5.15}$$

where $N = \|\mathbf{N}\|$ and $D = \|\mathbf{D}\|$. It follows from Eqn (5.15) that the angular momentum h is determined from $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ by the formula

$$h = \sqrt{\mu \frac{N}{D}} \tag{5.16}$$

Since $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are coplanar, all of the crossproducts $\mathbf{r}_1 \times \mathbf{r}_2$, $\mathbf{r}_2 \times \mathbf{r}_3$, and $\mathbf{r}_3 \times \mathbf{r}_1$ lie in the same direction, namely, normal to the orbital plane. Therefore, it is clear from Eqn (5.14) that $\mathbf{D}$ must be normal to the orbital plane. In the context of the perifocal frame, we use $\hat{\mathbf{w}}$ to denote the orbit unit normal. Therefore,

$$\hat{\mathbf{w}} = \frac{\mathbf{D}}{D} \tag{5.17}$$

So far, we have found h and $\hat{\mathbf{w}}$ in terms of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. We need likewise to find an expression for $\hat{\mathbf{q}}$ to use in Eqn (5.7). From Eqns (5.4a), (5.6), and (5.17), it follows that

$$\hat{\mathbf{q}} = \hat{\mathbf{w}} \times \hat{\mathbf{p}} = \frac{1}{De}(\mathbf{D} \times \mathbf{e}) \tag{5.18}$$

Substituting Eqn (5.14), we get

$$\hat{\mathbf{q}} = \frac{1}{De}\left[(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} + (\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{e} + (\mathbf{r}_3 \times \mathbf{r}_1) \times \mathbf{e}\right] \tag{5.19}$$

We can apply the bac–cab rule to the right side by noting

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$$

Using this vector identity, we obtain

$$\begin{aligned}
(\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{e} &= \mathbf{r}_3(\mathbf{r}_2\cdot\mathbf{e}) - \mathbf{r}_2(\mathbf{r}_3\cdot\mathbf{e}) \\
(\mathbf{r}_3 \times \mathbf{r}_1) \times \mathbf{e} &= \mathbf{r}_1(\mathbf{r}_3\cdot\mathbf{e}) - \mathbf{r}_3(\mathbf{r}_1\cdot\mathbf{e}) \\
(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} &= \mathbf{r}_2(\mathbf{r}_1\cdot\mathbf{e}) - \mathbf{r}_1(\mathbf{r}_2\cdot\mathbf{e})
\end{aligned}$$

Once again employing Eqn (5.9), these become

$$\begin{aligned}
(\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{e} &= \mathbf{r}_3\left(\frac{h^2}{\mu} - r_2\right) - \mathbf{r}_2\left(\frac{h^2}{\mu} - r_3\right) = \frac{h^2}{\mu}(\mathbf{r}_3 - \mathbf{r}_2) + r_3\mathbf{r}_2 - r_2\mathbf{r}_3 \\
(\mathbf{r}_3 \times \mathbf{r}_1) \times \mathbf{e} &= \mathbf{r}_1\left(\frac{h^2}{\mu} - r_3\right) - \mathbf{r}_3\left(\frac{h^2}{\mu} - r_1\right) = \frac{h^2}{\mu}(\mathbf{r}_1 - \mathbf{r}_3) + r_1\mathbf{r}_3 - r_3\mathbf{r}_1 \\
(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} &= \mathbf{r}_2\left(\frac{h^2}{\mu} - r_1\right) - \mathbf{r}_1\left(\frac{h^2}{\mu} - r_2\right) = \frac{h^2}{\mu}(\mathbf{r}_2 - \mathbf{r}_1) + r_2\mathbf{r}_1 - r_1\mathbf{r}_2
\end{aligned}$$

Summing up these three equations, collecting the terms, and substituting the result into Eqn (5.19) yields

$$\hat{\mathbf{q}} = \frac{1}{De}\mathbf{S} \tag{5.20}$$

where

$$\mathbf{S} = \mathbf{r}_1(r_2 - r_3) + \mathbf{r}_2(r_3 - r_1) + \mathbf{r}_3(r_1 - r_2) \tag{5.21}$$

Finally, we substitute Eqns (5.16), (5.17), and (5.20) into Eqn (5.7) to obtain

$$\mathbf{v} = \frac{\mu}{h}\left(\frac{\hat{\mathbf{w}} \times \mathbf{r}}{r} + e\hat{\mathbf{q}}\right) = \frac{\mu}{\sqrt{\mu \frac{N}{D}}}\left[\frac{\frac{\mathbf{D}}{D} \times \mathbf{r}}{r} + e\left(\frac{1}{De}\mathbf{S}\right)\right]$$

Simplifying this expression for the velocity yields

$$\mathbf{v} = \sqrt{\frac{\mu}{ND}}\left(\frac{\mathbf{D} \times \mathbf{r}}{r} + \mathbf{S}\right) \tag{5.22}$$

All the terms on the right depend only on the given position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$.

The Gibbs method may be summarized in the following algorithm:

ALGORITHM 5.1

Gibbs method of preliminary orbit determination. A MATLAB implementation of this procedure is found in Appendix D.24.

Given $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, the steps are as follows:

1. Calculate r_1, r_2, and r_3.
2. Calculate $\mathbf{C}_{12} = \mathbf{r}_1 \times \mathbf{r}_2$, $\mathbf{C}_{23} = \mathbf{r}_2 \times \mathbf{r}_3$, and $\mathbf{C}_{31} = \mathbf{r}_3 \times \mathbf{r}_1$.
3. Verify that $\hat{\mathbf{u}}_{r_1} \cdot \hat{\mathbf{C}}_{23} = 0$.
4. Calculate $\mathbf{N}$, $\mathbf{D}$, and $\mathbf{S}$ using Eqns (5.13), (5.14), and (5.21), respectively.
5. Calculate $\mathbf{v}_2$ using Eqn (5.22).
6. Use $\mathbf{r}_2$ and $\mathbf{v}_2$ to compute the orbital elements by means of Algorithm 4.2.

EXAMPLE 5.1

The geocentric position vectors of a space object at three successive times are

$$\mathbf{r}_1 = -294.32\hat{\mathbf{I}} + 4265.1\hat{\mathbf{J}} + 5986.7\hat{\mathbf{K}} \quad \text{(km)}$$
$$\mathbf{r}_2 = -1365.5\hat{\mathbf{I}} + 3637.6\hat{\mathbf{J}} + 6346.8\hat{\mathbf{K}} \quad \text{(km)}$$
$$\mathbf{r}_3 = -2940.3\hat{\mathbf{I}} + 2473.7\hat{\mathbf{J}} + 6555.8\hat{\mathbf{K}} \quad \text{(km)}$$

Determine the classical orbital elements using Gibbs method.

Solution

We employ Algorithm 5.1.

Step 1:

$$r_1 = \sqrt{(-294.32)^2 + 4265.1^2 + 5986.7^2} = 7356.5 \text{ km}$$
$$r_2 = \sqrt{(-1365.5)^2 + 3637.6^2 + 6346.8^2} = 7441.7 \text{ km}$$
$$r_3 = \sqrt{(-2940.3)^2 + 2473.7^2 + 6555.8^2} = 7598.9 \text{ km}$$

Step 2:

$$\mathbf{C}_{12}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ -294.32 & 4265.1 & 5986.7\\ -1365.5 & 3637.6 & 6346.8\end{vmatrix}=\left(5.2925\hat{\mathbf{I}}-6.3068\hat{\mathbf{J}}+4.7534\hat{\mathbf{K}}\right)\times10^{6}\quad(\mathrm{km}^2)$$

$$\mathbf{C}_{23}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ -1365.5 & 3637.6 & 6346.8\\ -2940.3 & 2473.7 & 6555.8\end{vmatrix}=\left(8.1473\hat{\mathbf{I}}-9.7096\hat{\mathbf{J}}+7.3178\hat{\mathbf{K}}\right)\times10^{6}\quad(\mathrm{km}^2)$$

$$\mathbf{C}_{31}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ -2940.3 & 2473.7 & 6555.8\\ -294.32 & 4265.1 & 5986.7\end{vmatrix}=\left(-1.3152\hat{\mathbf{I}}+1.5673\hat{\mathbf{J}}-1.1813\hat{\mathbf{K}}\right)\times10^{7}\quad(\mathrm{km}^2)$$

Step 3:

$$\hat{\mathbf{C}}_{23}=\frac{\mathbf{C}_{23}}{\|\mathbf{C}_{23}\|}=\frac{8.1473\hat{\mathbf{I}}-9.7096\hat{\mathbf{J}}+7.3178\hat{\mathbf{K}}}{\sqrt{8.1473^2+(-9.7096)^2+7.3178^2}}=0.55667\hat{\mathbf{I}}-0.66342\hat{\mathbf{J}}+0.5000\hat{\mathbf{K}}$$

Therefore,

$$\hat{\mathbf{u}}_{r_1}\cdot\hat{\mathbf{C}}_{23}=\frac{-294.32\hat{\mathbf{I}}+4265.1\hat{\mathbf{J}}+5986.7\hat{\mathbf{K}}}{7356.5}\cdot\left(0.55667\hat{\mathbf{I}}-0.66342\hat{\mathbf{J}}+0.5000\hat{\mathbf{K}}\right)=-6.1181\times10^{-6}$$

This is close enough to zero for our purposes. The three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are coplanar.

Step 4:

$$\begin{aligned}\mathbf{N}&=r_1\mathbf{C}_{23}+r_2\mathbf{C}_{31}+r_3\mathbf{C}_{12}\\&=7356.5\left[\left(8.1473\hat{\mathbf{I}}-9.7096\hat{\mathbf{J}}+7.3178\hat{\mathbf{K}}\right)\times10^{6}\right]+7441.7\left[\left(-1.3152\hat{\mathbf{I}}+1.5673\hat{\mathbf{J}}-1.1813\hat{\mathbf{K}}\right)\times10^{6}\right]\\&\quad+7598.9\left[\left(5.2925\hat{\mathbf{I}}-6.3068\hat{\mathbf{J}}+4.7534\hat{\mathbf{K}}\right)\times10^{6}\right]\end{aligned}$$

or

$$\mathbf{N}=\left(2.2811\hat{\mathbf{I}}-2.7186\hat{\mathbf{J}}+2.0481\hat{\mathbf{K}}\right)\times10^{9}\quad\left(\mathrm{km}^3\right)$$

so that

$$N=\sqrt{\left[2.2811^2+(-2.7186)^2+2.0481^2\right]\times10^{18}}=4.0975\times10^{9}\quad\left(\mathrm{km}^3\right)$$

$$\begin{aligned}\mathbf{D}&=\mathbf{C}_{12}+\mathbf{C}_{23}+\mathbf{C}_{31}\\&=\left[\left(5.295\hat{\mathbf{I}}-6.3068\hat{\mathbf{J}}+4.7534\hat{\mathbf{K}}\right)\times10^{6}\right]+\left[\left(8.1473\hat{\mathbf{I}}-9.7096\hat{\mathbf{J}}+7.3178\hat{\mathbf{K}}\right)\times10^{6}\right]\\&\quad+\left[\left(-1.3152\hat{\mathbf{I}}+1.5673\hat{\mathbf{J}}-1.1813\hat{\mathbf{K}}\right)\times10^{6}\right]\end{aligned}$$

or

$$\mathbf{D}=\left(2.8797\hat{\mathbf{I}}-3.4321\hat{\mathbf{J}}+2.5856\hat{\mathbf{K}}\right)\times10^{5}\quad\left(\mathrm{km}^2\right)$$

so that

$$D = \sqrt{\left[2.8797^2 + (-3.4321)^2 + 2.5856^2\right] \times 10^{10}} = 5.1728 \times 10^5 \quad \left(\text{km}^2\right)$$

Lastly,

$$\begin{aligned}
\mathbf{S} &= \mathbf{r}_1(r_2 - r_3) + \mathbf{r}_2(r_3 - r_1) + \mathbf{r}_3(r_1 - r_2) \\
&= \left(-294.32\hat{\mathbf{I}} + 4265.1\hat{\mathbf{J}} + 5986.7\hat{\mathbf{K}}\right)(7441.7 - 7598.9) \\
&\quad + \left(-1365.5\hat{\mathbf{I}} + 3637.6\hat{\mathbf{J}} + 6346.8\hat{\mathbf{K}}\right)(7598.9 - 7356.5) \\
&\quad + \left(-2940.3\hat{\mathbf{I}} + 2473.7\hat{\mathbf{J}} + 6555.8\hat{\mathbf{K}}\right)(7356.5 - 7441.7)
\end{aligned}$$

or

$$\mathbf{S} = -34{,}276\hat{\mathbf{I}} + 478.57\hat{\mathbf{J}} + 38{,}810\hat{\mathbf{K}} \quad \left(\text{km}^2\right)$$

Step 5:

$$\begin{aligned}
\mathbf{v}_2 &= \sqrt{\frac{\mu}{ND}}\left(\frac{\mathbf{D} \times \mathbf{r}_2}{r_2} + \mathbf{S}\right) \\
&= \sqrt{\frac{398{,}600}{(4.0971 \times 10^9)(5.1728 \times 10^3)}} \\
&\quad \times \left[\frac{\begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 2.8797 \times 10^6 & -3.4321 \times 10^6 & 2.5856 \times 10^6 \\ -1365.5 & 3637.6 & 6346.8 \end{vmatrix}}{7441.7} + \left(-34{,}276\hat{\mathbf{I}} + 478.57\hat{\mathbf{J}} + 38{,}810\hat{\mathbf{K}}\right) \right]
\end{aligned}$$

or

$$\mathbf{v}_2 = -6.2174\hat{\mathbf{I}} - 4.0122\hat{\mathbf{J}} + 1.5990\hat{\mathbf{K}} \quad (\text{km/s})$$

Step 6:

Using $\mathbf{r}_2$ and $\mathbf{v}_2$, Algorithm 4.2 yields the orbital elements:

$$\boxed{\begin{aligned} & a = 8000 \text{ km} \\ & e = 0.1 \\ & i = 60^\circ \\ & \Omega = 40^\circ \\ & \omega = 30^\circ \\ & \theta = 50^\circ (\text{for position vector } \mathbf{r}_2) \end{aligned}}$$

The orbit is sketched in Figure 5.2.

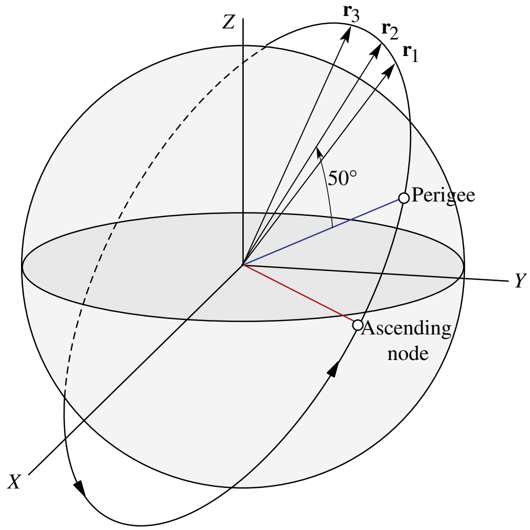

FIGURE 5.2

Sketch of the orbit of Example 5.1.

5.3 Lambert's problem

Suppose we know the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of two points P_1 and P_2 on the path of mass m around mass M, as illustrated in Figure 5.3. $\mathbf{r}_1$ and $\mathbf{r}_2$ determine the change in the true anomaly $\Delta\theta$, since

$$\cos\Delta\theta = \frac{\mathbf{r}_1\cdot\mathbf{r}_2}{r_1 r_2} \tag{5.23}$$

where

$$r_1 = \sqrt{\mathbf{r}_1\cdot\mathbf{r}_1} \quad r_2 = \sqrt{\mathbf{r}_2\cdot\mathbf{r}_2} \tag{5.24}$$

However, if $\cos\Delta\theta > 0$, then $\Delta\theta$ lies in either the first or fourth quadrant, whereas if $\cos\Delta\theta < 0$, then $\Delta\theta$ lies in the second or third quadrant (recall Figure 3.4). The first step in resolving this quadrant ambiguity is to calculate the Z component of $\mathbf{r}_1 \times \mathbf{r}_2$,

$$(\mathbf{r}_1\times\mathbf{r}_2)_Z = \hat{\mathbf{K}}\cdot(\mathbf{r}_1\times\mathbf{r}_2) = \hat{\mathbf{K}}\cdot\left(r_1 r_2 \sin\Delta\theta\hat{\mathbf{h}}\right) = r_1 r_2 \sin\Delta\theta\left(\hat{\mathbf{K}}\cdot\hat{\mathbf{w}}\right)$$

where $\hat{\mathbf{w}}$ is the unit normal to the orbital plane. Therefore, $\hat{\mathbf{K}}\cdot\hat{\mathbf{w}} = \cos i$, where i is the inclination of the orbit, so that

$$(\mathbf{r}_1\times\mathbf{r}_2)_Z = r_1 r_2 \sin\Delta\theta\cos i \tag{5.25}$$

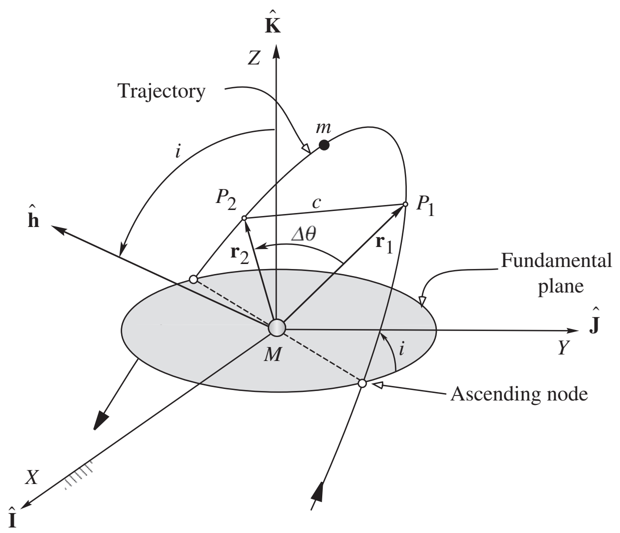

FIGURE 5.3

Lambert's problem.

We use the sign of the scalar $(\mathbf{r}_1 \times \mathbf{r}_2)_Z$ to determine the correct quadrant for $\Delta\theta$.

There are two cases to consider: prograde trajectories $(0 < i < 90°)$ and retrograde trajectories $(90° < i < 180°)$.

For prograde trajectories (like the one illustrated in Figure 5.3), $\cos\ i > 0$, so that if $(\mathbf{r}_1 \times \mathbf{r}_2)_Z > 0$, then Eqn (5.25) implies that $\sin\ \Delta\theta > 0$, which means $0° < \Delta\theta < 180°$. Since $\Delta\theta$ therefore lies in the first or second quadrant, it follows that $\Delta\theta$ is given by $\cos^{-1}(\mathbf{r}_1 \cdot \mathbf{r}_2 / r_1 r_2)$. On the other hand, if $(\mathbf{r}_1 \times \mathbf{r}_2)_Z < 0$, Eqn (5.25) implies that $\sin\ \Delta\theta < 0$, which means $180° < \Delta\theta < 360°$. In this case, $\Delta\theta$ lies in the third or fourth quadrant and is given by $360° - \cos^{-1}(\mathbf{r}_1 \cdot \mathbf{r}_2 / r_1 r_2)$. For retrograde trajectories, $\cos\ i < 0$. Thus, if $(\mathbf{r}_1 \times \mathbf{r}_2)_Z > 0$ then $\sin\ \Delta\theta < 0$, which places $\Delta\theta$ in the third or fourth quadrant. Similarly, if $(\mathbf{r}_1 \times \mathbf{r}_2)_Z > 0$, $\Delta\theta$ must lie in the first or second quadrant.

This logic can be expressed more concisely as follows:

$$\Delta\theta = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) & \text{if} \quad (\mathbf{r}_1 \times \mathbf{r}_2)_Z \geq 0 \\ & \qquad\qquad \text{prograde trajectory} \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) & \text{if} \quad (\mathbf{r}_1 \times \mathbf{r}_2)_Z < 0 \\ \hline \cos^{-1}\left(\dfrac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) & \text{if} \quad (\mathbf{r}_1 \times \mathbf{r}_2)_Z < 0 \\ & \qquad\qquad \text{retrograde trajectory} \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) & \text{if} \quad (\mathbf{r}_1 \times \mathbf{r}_2)_Z \geq 0 \end{cases} \tag{5.26}$$

J. H. Lambert (1728–1777) was a French-born German astronomer, physicist, and mathematician. Lambert proposed that the transfer time Δt from P_1 to P_2 in Figure 5.3 is independent of the orbit's eccentricity and depends only on the sum $r_1 + r_2$ of the magnitudes of the position vectors, the semimajor axis a and the length c of the chord joining P_1 and P_2. It is noteworthy that the period (of an ellipse) and the specific mechanical energy are also independent of the eccentricity (Eqns (2.83), (2.80), and (2.110)).

If we know the time of flight Δt from P_1 to P_2, then Lambert's problem is to find the trajectory joining P_1 and P_2. The trajectory is determined once we find $\mathbf{v}_1$, because, according to Eqns (2.135) and (2.136), the position and velocity of any point on the path are determined by $\mathbf{r}_1$ and $\mathbf{v}_1$. That is, in terms of the notation in Figure 5.3,

$$\mathbf{r}_2 = f\mathbf{r}_1 + g\mathbf{v}_1 \tag{5.27a}$$

$$\mathbf{v}_2 = \dot{f}\mathbf{r}_1 + \dot{g}\mathbf{v}_1 \tag{5.27b}$$

Solving the first of these for $\mathbf{v}_1$ yields

$$\mathbf{v}_1 = \frac{1}{g}(\mathbf{r}_2 - f\mathbf{r}_1) \tag{5.28}$$

Substitute this result into Eqn (5.27b) to get

$$\mathbf{v}_2 = \dot{f}\mathbf{r}_1 + \frac{\dot{g}}{g}(\mathbf{r}_2 - f\mathbf{r}_1) = \frac{\dot{g}}{g}\mathbf{r}_2 - \frac{f\dot{g} - \dot{f}g}{g}\mathbf{r}_1$$

However, according to Eqn (2.139), $f\dot{g} - \dot{f}g = 1$. Hence,

$$\mathbf{v}_2 = \frac{1}{g}(\dot{g}\mathbf{r}_2 - \mathbf{r}_1) \tag{5.29}$$

By means of Algorithm 4.2, we can find the orbital elements from either $\mathbf{r}_1$ and $\mathbf{v}_1$ or $\mathbf{r}_2$ and $\mathbf{v}_2$. Clearly, Lambert's problem is solved once we determine the Lagrange coefficients f, g, and $\dot{g}$.

The Lagrange f and g coefficients and their time derivatives are listed as functions of the change in true anomaly $\Delta\theta$ in Eqns (2.158),

$$f = 1 - \frac{\mu r_2}{h^2}(1 - \cos\Delta\theta) \qquad g = \frac{r_1 r_2}{h}\sin\Delta\theta \tag{5.30a}$$

$$\dot{f} = \frac{\mu}{h}\frac{1 - \cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1 - \cos\Delta\theta) - \frac{1}{r_0} - \frac{1}{r}\right] \qquad \dot{g} = 1 - \frac{\mu r_1}{h^2}(1 - \cos\Delta\theta) \tag{5.30b}$$

Equations (3.69) expresses these quantities in terms of the universal anomaly χ,

$$f = 1 - \frac{\chi^2}{r_1}C(z) \qquad g = \Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(z) \tag{5.31a}$$

$$\dot{f} = \frac{\sqrt{\mu}}{r_1 r_2}\chi[zS(z) - 1] \qquad \dot{g} = 1 - \frac{\chi^2}{r_2}C(z) \tag{5.31b}$$

where $z = \alpha\chi^2$. The f and g functions do not depend on the eccentricity, which would seem to make them an obvious choice for the solution of Lambert's problem.

The unknowns on the right of the above sets of equations are h, χ, and z, whereas $\Delta\theta$, Δt, r_1, and r_2 are given. Equating the four pairs of expressions for f, g, $\dot{f}$, and $\dot{g}$ in Eqns (5.30) and (5.31) yields four equations in the three unknowns h, χ, and z. However, because of the fact that $f\dot{g} - \dot{f}g = 1$, only three of these equations are independent. We must solve them for h, χ, and z in order to evaluate the Lagrange coefficients and thereby obtain the solution to Lambert's problem. We will follow the procedure presented by Bate, Mueller, and White (1971) and Bond and Allman (1996).

While $\Delta\theta$ appears throughout Eqn (5.30), the time interval Δt does not. However, Δt does appear in Eqn (5.31a). A relationship between $\Delta\theta$ and Δt can therefore be found by equating the two expressions for g,

$$\frac{r_1 r_2}{h}\sin\Delta\theta = \Delta t - \frac{1}{\sqrt{\mu}}\chi^3 S(z) \tag{5.32}$$

To eliminate the unknown angular momentum h, equate the expressions for f in Eqns (5.30a) and (5.31a),

$$1 - \frac{\mu r_2}{h^2}(1-\cos\Delta\theta) = 1 - \frac{\chi^2}{r_1}C(z)$$

Upon solving this for h, we obtain

$$h = \sqrt{\frac{\mu r_1 r_2(1-\cos\Delta\theta)}{\chi^2 C(z)}} \tag{5.33}$$

(Equating the two expressions for $\dot{g}$ leads to the same result.) Substituting Eqn (5.33) into Eqn (5.32), simplifying, and rearranging the terms yields

$$\sqrt{\mu}\Delta t = \chi^3 S(z) + \chi\sqrt{C(z)}\left(\sin\Delta\theta\sqrt{\frac{r_1 r_2}{1-\cos\Delta\theta}}\right) \tag{5.34}$$

The term in parentheses on the right is a constant that comprises solely the given data. Let us assign it the symbol A,

$$A = \sin\Delta\theta\sqrt{\frac{r_1 r_2}{1-\cos\Delta\theta}} \tag{5.35}$$

Then, Eqn (5.34) assumes the simpler form

$$\sqrt{\mu}\Delta t = \chi^3 S(z) + A\chi\sqrt{C(z)} \tag{5.36}$$

The right side of this equation contains both of the unknown variables χ and z. We cannot use the fact that $z = \alpha\chi^2$ to reduce the unknowns to one since α is the reciprocal of the semimajor axis of the unknown orbit.

In order to find a relationship between z and χ that does not involve orbital parameters, we equate the expressions for $\dot{f}$ (Eqns (5.30b) and (5.31b)) to obtain

$$\frac{\mu}{h}\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1-\cos\Delta\theta) - \frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{\sqrt{\mu}}{r_1 r_2}\chi[zS(z) - 1]$$

Multiplying through by $r_1 r_2$ and substituting for the angular momentum using Eqn (5.33) yields

$$\frac{\mu}{\sqrt{\dfrac{\mu r_1 r_2(1-\cos\Delta\theta)}{\chi^2 C(z)}}}\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{\dfrac{\mu r_1 r_2(1-\cos\Delta\theta)}{\chi^2 C(z)}}(1-\cos\Delta\theta)-r_1-r_2\right]=\sqrt{\mu}\chi[zS(z)-1]$$

Simplifying and dividing out the common factors leads to

$$\frac{\sqrt{1-\cos\Delta\theta}}{\sqrt{r_1 r_2}\sin\Delta\theta}\sqrt{C(z)}\left[\chi^2 C(z)-r_1-r_2\right]=zS(z)-1$$

We recognize the reciprocal of A on the left, so we can rearrange this expression to read as follows:

$$\chi^2 C(z)=r_1+r_2+A\frac{zS(z)-1}{\sqrt{C(z)}}$$

The right-hand side depends exclusively on z. Let us call that function $y(z)$, so that

$$\chi=\sqrt{\frac{y(z)}{C(z)}} \tag{5.37}$$

where

$$y(z)=r_1+r_2+A\frac{zS(z)-1}{\sqrt{C(z)}} \tag{5.38}$$

Equation (5.37) is the relation between χ and z that we were seeking. Substituting it back into Eqn (5.36) yields

$$\sqrt{\mu}\Delta t=\left[\frac{y(z)}{C(z)}\right]^{\frac{3}{2}}S(z)+A\sqrt{y(z)} \tag{5.39}$$

We can use this equation to solve for z, given the time interval Δt. It must be done iteratively.

Using Newton's method, we form the function

$$F(z)=\left[\frac{y(z)}{C(z)}\right]^{\frac{3}{2}}S(z)+A\sqrt{y(z)}-\sqrt{\mu}\Delta t \tag{5.40}$$

and its derivative

$$F'(z)=\frac{1}{2\sqrt{y(z)C^5(z)}}\left\{[2C(z)S'(z)-3C'(z)S(z)]y^2(z)+\left[AC^{\frac{5}{2}}(z)+3C(z)S(z)y(z)\right]y'(z)\right\} \tag{5.41}$$

in which $C'(z)$ and $S'(z)$ are the derivatives of the Stumpff functions, which are given by Eqn (3.63). $y'(z)$ is obtained by differentiating $y\,(z)$ in Eqn (5.38),

$$y'(z)=\frac{A}{2C(z)^{\frac{3}{2}}}\left\{[1-zS(z)]C'(z)+2[S(z)+zS'(z)]C(z)\right\}$$

If we substitute Eqn (3.63) into this expression, a much simpler form is obtained, namely

$$y'(z) = \frac{A}{4}\sqrt{C} \tag{5.42}$$

This result can be worked out by using Eqns (3.52) and (3.53) to express $C(z)$ and $S(z)$ in terms of the more familiar trig functions. Substituting Eqn (5.42) along with Eqn (3.63) into Eqn (5.41) yields

$$F'(z) = \begin{cases} \left[\dfrac{y(z)}{C(z)}\right]^{\frac{3}{2}}\left\{\dfrac{1}{2z}\left[C(z) - \dfrac{3}{2}\dfrac{S(z)}{C(z)}\right] + \dfrac{3}{4}\dfrac{S(z)^2}{C(z)}\right\} + \dfrac{A}{8}\left[3\dfrac{S(z)}{C(z)}\sqrt{y(z)} + A\sqrt{\dfrac{C(z)}{y(z)}}\right] & (z \neq 0) \\ \dfrac{\sqrt{2}}{40}y(0)^{\frac{3}{2}} + \dfrac{A}{8}\left[\sqrt{y(0)} + A\sqrt{\dfrac{1}{2y(0)}}\right] & (z = 0) \end{cases} \tag{5.43}$$

Evaluating $F'(z)$ at $z = 0$ must be done carefully (and is therefore shown as a special case) because of the z in the denominator within the curly brackets. To handle $z = 0$, we assume that z is very small (almost but not quite zero) so that we can retain just the first two terms in the series expansions of $C(z)$ and $S(z)$ (Eqn (3.51)),

$$C(z) = \frac{1}{2} - \frac{z}{24} + \cdots \quad S(z) = \frac{1}{6} - \frac{z}{120} + \cdots$$

Then, we evaluate the term within the curly brackets as follows:

$$\begin{aligned} \frac{1}{2z}\left[C(z) - \frac{3}{2}\frac{S(z)}{C(z)}\right] &\approx \frac{1}{2z}\left[\left(\frac{1}{2} - \frac{z}{24}\right) - \frac{3}{2}\frac{\left(\frac{1}{6} - \frac{z}{120}\right)}{\left(\frac{1}{2} - \frac{z}{24}\right)}\right] \\ &= \frac{1}{2z}\left[\left(\frac{1}{2} - \frac{z}{24}\right) - 3\left(\frac{1}{6} - \frac{z}{120}\right)\left(1 - \frac{z}{12}\right)^{-1}\right] \\ &\approx \frac{1}{2z}\left[\left(\frac{1}{2} - \frac{z}{24}\right) - 3\left(\frac{1}{6} - \frac{z}{120}\right)\left(1 + \frac{z}{12}\right)\right] \\ &= \frac{1}{2z}\left(-\frac{7z}{120} + \frac{z^2}{480}\right) \\ &= -\frac{7}{240} + \frac{z}{960} \end{aligned}$$

In the third step, we used the familiar binomial expansion theorem,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots \tag{5.44}$$

to set $(1-z/12)^{-1} \approx 1 + z/12$, which is true if z is close to zero. Thus, when z is actually zero,

$$\frac{1}{2z}\left[C(z) - \frac{3}{2}\frac{S(z)}{C(z)}\right] = -\frac{7}{240}$$

Evaluating the other terms in $F'(z)$ presents no difficulties.

$F(z)$ in Eqn (5.40) and $F'(z)$ in Eqn (5.43) are used in Newton's formula, Eqn (3.16), for the iterative procedure,

$$z_{i+1} = z_i - \frac{F(z_i)}{F'(z_i)} \tag{5.45}$$

For choice of a starting value for z, recall that $z = (1/a)\chi^2$. According to Eqn (3.57), $z = E^2$ for an ellipse and $z = -F^2$ for a hyperbola. Since we do not know what the orbit is, setting $z_0 = 0$ seems a reasonable, simple choice. Alternatively, one can plot or tabulate $F(z)$ and choose z_0 to be a point near where $F(z)$ changes sign.

Substituting Eqns (5.37) and (5.39) into Eqn (5.31) yields the Lagrange coefficients as functions of z alone.

$$f = 1 - \frac{\left[\sqrt{\dfrac{y(z)}{C(z)}}\right]^2}{r_1} C(z) = 1 - \frac{y(z)}{r_1} \tag{5.46a}$$

$$g = \frac{1}{\sqrt{\mu}}\left\{\left[\frac{y(z)}{C(z)}\right]^{\frac{3}{2}} S(z) + A\sqrt{y(z)}\right\} - \frac{1}{\sqrt{\mu}}\left[\frac{y(z)}{C(z)}\right]^{\frac{3}{2}} S(z) = A\sqrt{\frac{y(z)}{\mu}} \tag{5.46b}$$

$$\dot{f} = \frac{\sqrt{\mu}}{r_1 r_2}\sqrt{\frac{y(z)}{C(z)}}[zS(z) - 1] \tag{5.46c}$$

$$\dot{g} = 1 - \frac{\left[\sqrt{\dfrac{y(z)}{C(z)}}\right]^2}{r_2} C(z) = 1 - \frac{y(z)}{r_2} \tag{5.46d}$$

We are now in a position to present the solution of Lambert's problem in universal variables, following Bond and Allman (1996).

ALGORITHM 5.2

Solve Lambert's problem. A MATLAB implementation appears in Appendix D.25.

Given $\mathbf{r}_1$, $\mathbf{r}_2$, and Δt, the steps are as follows:

1. Calculate r_1 and r_2 using Eqn (5.24).
2. Choose either a prograde or a retrograde trajectory and calculate $\Delta\theta$ using Eqn (5.26).
3. Calculate A in Eqn (5.35).
4. By iteration, using Eqns (5.40), (5.43), and (5.45), solve Eqn (5.39) for z. The sign of z tells us whether the orbit is a hyperbola ($z < 0$), parabola ($z = 0$), or ellipse ($z > 0$).
5. Calculate y using Eqn (5.38).
6. Calculate the Lagrange f, g, and $\dot{g}$ functions using Eqn (5.46).
7. Calculate $\mathbf{v}_1$ and $\mathbf{v}_2$ from Eqns (5.28) and (5.29).
8. Use $\mathbf{r}_1$ and $\mathbf{v}_1$ (or $\mathbf{r}_2$ and $\mathbf{v}_2$) in Algorithm 4.2 to obtain the orbital elements.

EXAMPLE 5.2

The position of an earth satellite is first determined to be $\mathbf{r}_1 = 5000\hat{\mathbf{I}} + 10{,}000\hat{\mathbf{J}} + 2100\hat{\mathbf{K}}$ (km). After one hour the position vector is $\mathbf{r}_2 = -14{,}600\hat{\mathbf{I}} + 2500\hat{\mathbf{J}} + 7000\hat{\mathbf{K}}$ (km). Determine the orbital elements and find the perigee altitude and the time since perigee passage of the first sighting.

Solution

We first must execute the steps of Algorithm 5.2 in order to find $\mathbf{v}_1$ and $\mathbf{v}_2$.

Step 1:

$$r_1 = \sqrt{5000^2 + 10{,}000^2 + 2100^2} = 11{,}375 \text{ km}$$

$$r_2 = \sqrt{(-14{,}600)^2 + 2500^2 + 7000^2} = 16{,}383 \text{ km}$$

Step 2: Assume a prograde trajectory.

$$\mathbf{r}_1 \times \mathbf{r}_2 = \left(64.75\hat{\mathbf{I}} - 65.66\hat{\mathbf{J}} + 158.5\hat{\mathbf{K}}\right) \times 10^6$$

$$\cos^{-1}\frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2} = 100.29°$$

Since the trajectory is prograde and the z component of $\mathbf{r}_1 \times \mathbf{r}_2$ is positive, it follows from Eqn (5.26) that

$$\Delta\theta = 100.29°$$

Step 3:

$$A = \sin\Delta\theta\sqrt{\frac{r_1 r_2}{1-\cos\Delta\theta}} = \sin 100.29°\sqrt{\frac{11{,}375 \times 16{,}383}{1-\cos 100.29°}} = 12{,}372 \text{ km}$$

Step 4:

Using this value of A and $\Delta t = 3600$ s, we can evaluate the functions $F(z)$ and $F'(z)$ given by Eqns (5.40) and (5.43), respectively. Let us first plot $F(z)$ to estimate where it crosses the z-axis. As can be seen from Figure 5.4, $F(z) = 0$ near $z = 1.5$. With $z_0 = 1.5$ as our initial estimate, we execute Newton's procedure (Eqn (5.45)):

$$z_{i+1} = z_i - \frac{F(z_i)}{F'(z_i)}$$

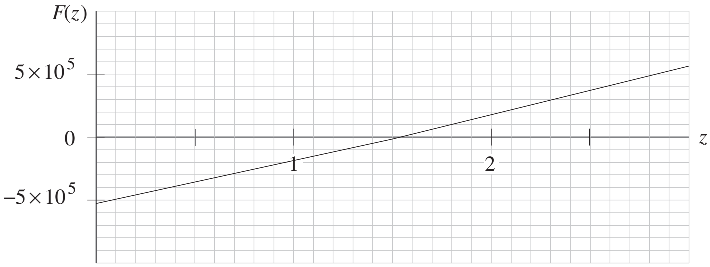

FIGURE 5.4

Graph of $F(z)$.

$$z_1 = 1.5 - \frac{-14{,}476.4}{362{,}642} = 1.53991$$

$$z_2 = 1.53991 - \frac{23.6274}{363{,}828} = 1.53985$$

$$z_3 = 1.53985 - \frac{6.29457 \times 10^{-5}}{363{,}826} = 1.53985$$

Thus, to five significant figures $z = 1.5398$. The fact that z is positive means the orbit is an ellipse.

Step 5:

$$y = r_1 + r_2 + A\frac{zS(z) - 1}{\sqrt{C(z)}} = 11{,}375 + 16{,}383 + 12{,}372\frac{1.5398S(1.5398)}{\sqrt{C(1.5398)}} = 13{,}523 \text{ km}$$

Step 6:

Equation (5.46) yields the Lagrange functions

$$f = 1 - \frac{y}{r_1} = 1 - \frac{13{,}523}{11{,}375} = -0.18877$$

$$g = A\sqrt{\frac{y}{\mu}} = 12{,}372\sqrt{\frac{13{,}523}{398{,}600}} = 2278.9 \text{ s}$$

$$\dot{g} = 1 - \frac{y}{r_2} = 1 - \frac{13{,}523}{16{,}383} = 0.17457$$

Step 7:

$$\mathbf{v}_1 = \frac{1}{g}(\mathbf{r}_2 - f\mathbf{r}_1) = \frac{1}{2278.9}\left[\left(-14{,}600\hat{\mathbf{I}} + 2500\hat{\mathbf{J}} + 7000\hat{\mathbf{K}}\right) - (-0.18877)\left(5000\hat{\mathbf{I}} + 10{,}000\hat{\mathbf{J}} + 2100\hat{\mathbf{K}}\right)\right]$$

$$\mathbf{v}_1 = -5.9925\hat{\mathbf{I}} + 1.9254\hat{\mathbf{J}} + 3.2456\hat{\mathbf{K}} \text{ (km)}$$

$$\mathbf{v}_2 = \frac{1}{g}(\dot{g}\mathbf{r}_2 - \mathbf{r}_1) = \frac{1}{2278.9}\left[(0.17457)\left(-14{,}600\hat{\mathbf{I}} + 2500\hat{\mathbf{J}} + 7000\hat{\mathbf{K}}\right) - \left(5000\hat{\mathbf{I}} + 10{,}000\hat{\mathbf{J}} + 2100\hat{\mathbf{K}}\right)\right]$$

$$\mathbf{v}_2 = -3.3125\hat{\mathbf{I}} - 4.1966\hat{\mathbf{J}} - 0.38529\hat{\mathbf{K}} \text{ (km)}$$

Step 8:

Using $\mathbf{r}_1$ and $\mathbf{v}_1$ Algorithm 4.2 yields the orbital elements:

$$\boxed{\begin{aligned} h &= 80{,}470 \text{ km}^2/\text{s} \\ a &= 20{,}000 \text{ km} \\ e &= 0.4335 \\ \Omega &= 44.60^\circ \\ i &= 30.19^\circ \\ \omega &= 30.71^\circ \\ \theta_1 &= 350.8^\circ \end{aligned}}$$

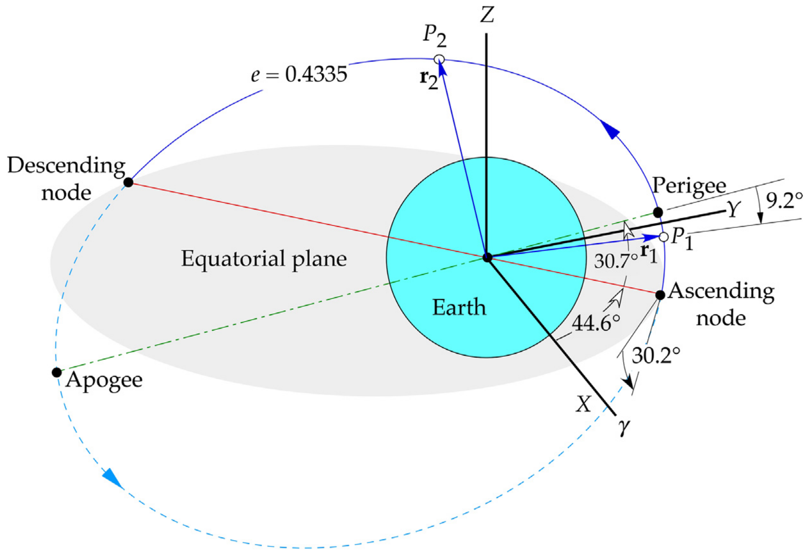

FIGURE 5.5

The solution of Example 5.2 (Lambert's problem).

This elliptical orbit is plotted in Figure 5.5. The perigee of the orbit is

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e\cos(0)} = \frac{80{,}470^2}{398{,}600}\frac{1}{1+0.4335} = 11{,}330 \text{ k}$$

Therefore, the perigee altitude is $11330 - 6378 = \boxed{4952 \text{ km}}$.

To find the time of the first sighting, we first calculate the eccentric anomaly by means of Eqn (3.13b),

$$E_1 = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) = 2\tan^{-1}\left(\sqrt{\frac{1-0.4335}{1+0.4335}}\tan\frac{350.8^\circ}{2}\right) = 2\tan^{-1}(-0.05041)$$
$$= -0.1007 \text{ rad}$$

Then using Kepler's equation for the ellipse (Eqn (3.14)), the mean anomaly is found to be

$$M_{e_1} = E_1 - e\sin E_1 = -0.1007 - 0.4335\sin(-0.1007) = -0.05715 \text{ rad}$$

so that from Eqn (3.7), the time since perigee passage is

$$t_1 = \frac{h^3}{\mu^2}\frac{1}{(1-e^2)^{\frac{3}{2}}}M_{e_1} = \frac{80{,}470^3}{398{,}600^2}\frac{1}{(1-.4335^2)^{\frac{3}{2}}}(-0.05715) = \boxed{-256.1 \text{ s}}$$

The minus sign means that there are 256.1 s until perigee encounter after the initial sighting.

EXAMPLE 5.3

A meteoroid is sighted at an altitude of 267,000 km. After 13.5 hours and a change in true anomaly of 5°, the altitude is observed to be 140,000 km. Calculate the perigee altitude and the time to perigee after the second sighting.

Solution

We have

$$P_1 : r_1 = 6378 + 267{,}000 = 273{,}378 \text{ km}$$
$$P_2 : r_2 = 6378 + 140{,}000 = 146{,}378 \text{ km}$$
$$\Delta t = 13.5 \times 3600 = 48{,}600 \text{ s}$$
$$\Delta\theta = 5^\circ$$

Since r_1, r_2, and $\Delta\theta$ are given, we can skip to Step 3 of Algorithm 5.2 and compute

$$A = 2.8263 \times 10^5 \text{ km}$$

Then, solving for z as in the previous example, we obtain

$$z = -0.17344$$

Since z is negative, the path of the meteoroid is a hyperbola.

With z available, we evaluate the Lagrange functions,

$$\begin{aligned} f &= 0.95846 \\ g &= 47{,}708 \text{ s} \\ \dot{g} &= 0.92241 \end{aligned} \tag{a}$$

Step 7 requires the initial and final position vectors. Therefore, for purposes of this problem, let us define a geocentric coordinate system with the x-axis aligned with $\mathbf{r}_1$ and the y-axis at 90° thereto in the direction of the motion (Figure 5.6). The z-axis is therefore normal to the plane of the orbit. Then,

$$\begin{aligned} \mathbf{r}_1 &= r_1\hat{\mathbf{i}} = 273{,}378\hat{\mathbf{i}} \quad \text{(km)} \\ \mathbf{r}_2 &= r_2 \cos\Delta\theta\hat{\mathbf{i}} + r_2 \sin\Delta\theta\hat{\mathbf{j}} = 145{,}820\hat{\mathbf{i}} + 12{,}758\hat{\mathbf{j}} \quad \text{(km)} \end{aligned} \tag{b}$$

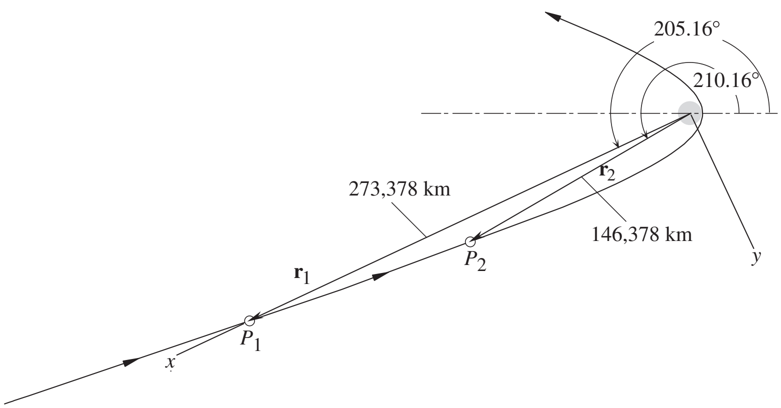

FIGURE 5.6

Solution of Example 5.3 (Lambert's problem).

With Eqns (a) and (b), we obtain the velocity at P_1,

$$\mathbf{v}_1 = \frac{1}{g}(\mathbf{r}_2 - f\mathbf{r}_1)$$

$$= \frac{1}{47{,}708}\left[\left(145{,}820\hat{\mathbf{i}} + 12{,}758\hat{\mathbf{j}}\right) - 0.95846\left(273{,}378\hat{\mathbf{i}}\right)\right]$$

$$= -2.4356\hat{\mathbf{i}} + 0.26741\hat{\mathbf{j}} \quad (\text{km/s})$$

Using $\mathbf{r}_1$ and $\mathbf{v}_1$, Algorithm 4.2 yields

$$h = 73{,}105 \text{ km}^2/\text{s}$$
$$e = 1.0506$$
$$\theta_1 = 205.16^\circ$$

The orbit is now determined except for its orientation in space, for which no information was provided. In the plane of the orbit, the trajectory is as shown in Figure 5.6.

The perigee radius is

$$r_\text{p} = \frac{h^2}{\mu}\frac{1}{1 + e\cos(0)} = 6538.2 \text{ km}$$

which means the perigee altitude is dangerously low for a large meteoroid,

$$z_\text{p} = 6538.2 - 6378 = \boxed{160.2 \text{ km}(100 \text{ miles})}$$

To find the time of flight from P_2 to perigee, we note that the true anomaly of P_2 is

$$\theta_2 = \theta_1 + 5^\circ = 210.16^\circ$$

The hyperbolic eccentric anomaly F_2 follows from Eqn (3.44a),

$$F_2 = 2\tan h^{-1}\left(\sqrt{\frac{e-1}{e+1}}\tan\frac{\theta_2}{2}\right) = -1.3347 \text{ rad}$$

Substituting this value into Kepler's equation (Eqn (3.40)) yields the mean anomaly M_h,

$$M_{h_2} = e\sin h(F_2) - F_2 = -0.52265 \text{ rad}$$

Finally, Eqn (3.34) yields the time

$$t_2 = \frac{M_{h_2} h^3}{\mu^2\left(e^2 - 1\right)^{\frac{3}{2}}} = -38{,}396 \text{ s}$$

The minus sign means that 38,396 s (a scant 10.6 h) remain until the meteoroid passes through perigee.

5.4 Sidereal time

To deduce the orbit of a satellite or celestial body from observations requires, among other things, recording the time of each observation. The time we use in everyday life, the time we set our clocks by, is the solar time. It is reckoned by the motion of the sun across the sky. A solar day is the time required for the sun to return to the same position overhead, that is, to lie on the same meridian. A solar day—from high noon to high noon—comprises 24 h. Universal time (UT) is determined by the sun's passage across the Greenwich meridian, which is zero degrees terrestrial longitude. See Figure 1.18. At noon UT, the sun lies on the Greenwich meridian. Local standard time, or civil time, is obtained from the UT by adding 1 h for each time zone between Greenwich and the site, measured westward.

Sidereal time is measured by the rotation of the earth relative to the fixed stars (i.e., the celestial sphere, Figure 4.3). The time it takes for a distant star to return to its same position overhead, that is, to lie on the same meridian, is one sidereal day (24 sidereal hours). As illustrated in Figure 4.20, the earth's orbit around the sun results in the sidereal day being slightly shorter than the solar day. One sidereal day is 23 hours and 56 minutes. To put it another way, the earth rotates 360° in one sidereal day, whereas it rotates 360.986° in a solar day.

Local sidereal time θ of a site is the time elapsed since the local meridian of the site passed through the vernal equinox. The number of degrees (measured eastward) between the vernal equinox and the local meridian is the sidereal time multiplied by 15. To know the location of a point on the earth at any given instant relative to the geocentric equatorial frame requires knowing its local sidereal time. The local sidereal time of a site is found by first determining the Greenwich sidereal time θ_G (the sidereal time of the Greenwich meridian) and then adding the east longitude (or subtracting the west longitude) of the site. Algorithms for determining sidereal time rely on the notion of Julian day.

The Julian day number is the number of days since noon UT on January 1, 4713 BC. The origin of this time scale is placed in antiquity so that, except for prehistoric events, we do not have to deal with positive and negative dates. The Julian day count is uniform and continuous and does not involve leap years or different numbers of days in different months. The number of days between two events is found by simply subtracting the Julian day of one from that of the other. The JD begins at noon rather than at midnight so that astronomers observing the heavens at night would not have to deal with a change of date during their watch.

The Julian day numbering system is not to be confused with the Julian calendar, which the Roman emperor Julius Caesar introduced in 46 BC. The Gregorian calendar, introduced in 1583, has largely supplanted the Julian calendar and is in common civil use today throughout much of the world.

J_0 is the symbol for the Julian day number at 0 h UT (which is halfway into the Julian day). At any other UT, the Julian day is given by

$$\mathrm{JD} = J_0 + \frac{\mathrm{UT}}{24} \tag{5.47}$$

Algorithms and tables for obtaining J_0 from the ordinary year (y), month (m), and day (d) exist in the literature and on the World Wide Web. One of the simplest formulas is found in Boulet (1991),

$$J_0 = 367y - \mathrm{INT}\left\{\frac{7\left[y + \mathrm{INT}\left(\frac{m+9}{12}\right)\right]}{4}\right\} + \mathrm{INT}\left(\frac{275m}{9}\right) + d + 1{,}721{,}013.5 \tag{5.48}$$

where y, m, and d are integers lying in the following ranges:

$$1901 \le y \le 2099$$

$$1 \le m \le 12$$

$$1 \le d \le 31$$

INT (x) means to retain only the integer portion of x, without rounding (or, in other words, round toward zero), that is, INT $(-3.9) = -3$ and INT $(3.9) = 3$. Appendix D.26 lists a MATLAB implementation of Eqn (5.48).

EXAMPLE 5.4

What is the Julian day number for May 12, 2004, at 14:45:30 UT?

Solution

In this case $y = 2004$, $m = 5$, and $d = 12$. Therefore, Eqn (5.48) yields the Julian day number at 0 h UT,

$$
\begin{aligned}
J_0 &= 367 \times 2004 - \mathrm{INT}\left\{\frac{7\left[2004 + \left(\dfrac{5+9}{12}\right)\right]}{4}\right\} + \mathrm{INT}\left(\frac{275 \times 5}{9}\right) + 12 + 1{,}721{,}013.5 \\
&= 735{,}468 - \mathrm{INT}\left\{\frac{7[2004+1]}{4}\right\} + 152 + 12 + 1{,}721{,}013.5 \\
&= 735{,}468 - 3508 + 152 + 12 + 1{,}721{,}013.5
\end{aligned}
$$

or

$$J_0 = 2{,}453{,}137.5 \text{ days}$$

The universal time, in hours, is

$$\mathrm{UT} = 14 + \frac{45}{60} + \frac{30}{3600} = 14.758 \text{ h}$$

Therefore, from Eqn (5.47), we obtain the Julian day number at the desired universal time,

$$\mathrm{JD} = 2{,}453{,}137.5 + \frac{14.758}{24} = \boxed{2{,}453{,}138.115 \text{ days}}$$

EXAMPLE 5.5

Find the elapsed time between October 4, 1957 UT 19:26:24, and the date of the previous example.

Solution

Proceeding as in Example 5.4 we find that the Julian day number of the given event (the launch of the first man-made satellite, Sputnik I) is

$$\mathrm{JD}_1 = 2{,}436{,}116.3100 \text{ days}$$

The JD of the previous example is

$$\mathrm{JD}_2 = 2{,}453{,}138.1149 \text{ days}$$

Hence, the elapsed time is

$$\Delta\mathrm{JD} = 2{,}453{,}138.1149 - 2{,}436{,}116.3100 = 17{,}021.805 \text{ days } (46 \text{ years, } 220 \text{ days})$$

The current Julian epoch is defined to have been noon on January 1, 2000. This epoch is denoted J2000 and has the exact Julian day number 2,451,545.0. Since there are 365.25 days in a Julian year, a Julian century has 36,525 days. It follows that the time T_0 in Julian centuries between the Julian day J_0 and J2000 is

$$T_0 = \frac{J_0 - 2{,}451{,}545}{36{,}525} \tag{5.49}$$

The Greenwich sidereal time θ_{G0} at 0 h UT may be found in terms of this dimensionless time (Seidelmann, 1992; Section 2.24). θ_{G0} is in degrees and is given by the series

$$\theta_{G0} = 100.4606184 + 36{,}000.77004T_0 + 0.000387933T_0^2 - 2.583\left(10^{-8}\right)T_0^3 \text{(degrees)} \tag{5.50}$$

This formula can yield a value outside the range $0 \le \theta_{G0} \le 360°$. If so, then the appropriate integer multiple of 360° must be added or subtracted to bring θ_{G0} into that range.

Once θ_{G0} has been determined, the Greenwich sidereal time θ_G at any other UT is found using the relation

$$\theta_G = \theta_{G0} + 360.98564724\frac{\text{UT}}{24} \tag{5.51}$$

where UT is in hours. The coefficient of the second term on the right is the number of degrees the earth rotates in 24 h (solar time).

Finally, the local sidereal time θ of a site is obtained by adding its east longitude Λ to the Greenwich sidereal time,

$$\theta = \theta_G + \Lambda \tag{5.52}$$

Here again, it is possible for the computed value of θ to exceed 360°. If so, it must be reduced to within that limit by subtracting the appropriate integer multiple of 360°. Figure 5.7 illustrates the relationship among θ_{G0}, θ_G, Λ, and θ.

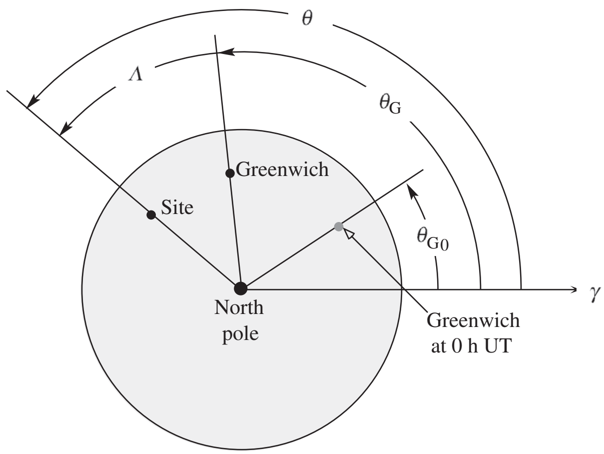

FIGURE 5.7

Schematic of the relationship among θ_{G_0}, θ_G, Λ, and θ.

ALGORITHM 5.3

Calculate the local sidereal time, given the date, the local time, and the east longitude of the site. This is implemented in MATLAB in Appendix D.27.

1. Using the year, month, and day, calculate J_0 using Eqn (5.48).
2. Calculate T_0 by means of Eqn (5.49).
3. Compute θ_{G0} from Eqn (5.50). If θ_{G0} lies outside the range $0° \leq \theta_{G0} \leq 360°$, then subtract the multiple of 360° required to place θ_{G0} in that range.
4. Calculate θ_G using Eqn (5.51).
5. Calculate the local sidereal time θ by means of Eqn (5.52), adjusting the final value so it lies between 0° and 360°.

EXAMPLE 5.6

Use Algorithm 5.3 to find the local sidereal time (in degrees) of Tokyo, Japan, on March 3, 2004, at 4:30:00 UT. The east longitude of Tokyo is 139.80°. (This places Tokyo nine time zones ahead of Greenwich, so the local time is 1:30 in the afternoon.)

Step 1:

$$J_0 = 367 \times 2004 - \mathrm{INT}\left\{\frac{7\left[2004 + \mathrm{INT}\left(\frac{3+9}{12}\right)\right]}{4}\right\} + \mathrm{INT}\left(\frac{275 \times 3}{9}\right) + 3 + 1{,}721{,}013.5$$

$$= 2{,}453{,}067.5 \text{ days}$$

Recall that the 0.5 means that we are halfway into the Julian day, which began at noon UT of the previous day.

Step 2:

$$T_0 = \frac{2{,}453{,}067.5 - 2{,}451{,}545}{36{,}525} = 0.041683778$$

Step 3:

$$\begin{aligned}\theta_{G0} &= 100.4606184 + 36{,}000.77004(0.041683778)\\ &\quad + 0.000387933(0.041683778)^2 - 2.583(10^{-8})(0.041683778)^3\\ &= 1601.1087°\end{aligned}$$

The right-hand side is too large. We must reduce θ_{G0} to an angle that does not exceed 360°. To that end, observe that

$$\mathrm{INT}(1601.1087/360) = 4$$

Hence,

$$\theta_{G0} = 1601.1087 - 4 \times 360 = 161.10873° \tag{a}$$

Step 4:

The UT of interest in this problem is

$$\mathrm{UT} = 4 + \frac{30}{60} + \frac{0}{3600} = 4.5 \ \mathrm{h}$$

Substitute this and Eqn (a) into Eqn (5.51) to get the Greenwich sidereal time.

$$\theta_G = 161.10873 + 360.98564724 \frac{4.5}{24} = 228.79354°$$

Step 5:

Add the east longitude of Tokyo to this value to obtain the local sidereal time,

$$\theta = 228.79354 + 139.80 = 368.59°$$

To reduce this result into the range $0 \le \theta \le 360°$, we must subtract 360° to get

$$\theta = 368.59 - 360 = \boxed{8.59° (0.573 \ \mathrm{h})}$$

Observe that the right ascension of a celestial body lying on Tokyo's meridian is 8.59°.

5.5 Topocentric coordinate system

A topocentric coordinate system is one that is centered at the observer's location on the surface of the earth. Consider an object B—a satellite or celestial body—and an observer O on the earth's surface, as illustrated in Figure 5.8. $\mathbf{r}$ is the position of the body B relative to the center of attraction C; $\mathbf{R}$ is the position vector of the observer relative to C; and ρ is the position of the body B relative to the observer. $\mathbf{r}$, $\mathbf{R}$, and ρ comprise the fundamental vector triangle. The relationship among these three vectors is

$$\mathbf{r} = \mathbf{R} + \rho \tag{5.53}$$

As we know, the earth is not a sphere but a slightly oblate spheroid. This ellipsoidal shape is exaggerated in Figure 5.8. The location of the observation site O is determined by specifying its east longitude Λ and latitude ϕ. East longitude Λ is measured positive eastward from the Greenwich meridian to the meridian through O. The angle between the vernal equinox direction (XZ plane) and the meridian of O is the local sidereal time θ. Likewise, θ_G is the Greenwich sidereal time. Once we know θ_G, then the local sidereal time is given by Eqn (5.52).

Latitude ϕ is the angle between the equator and the normal $\hat{\mathbf{n}}$ to the earth's surface at O. Since the earth is not a perfect sphere, the position vector $\mathbf{R}$, directed from the center C of the earth to O, does not point in the direction of the normal except at the equator and the poles.

The oblateness, or flattening f, was defined in Section 4.7,

$$f = \frac{R_e - R_p}{R_e}$$

where R_e is the equatorial radius and R_p is the polar radius. (Review from Table 4.3 that $f = 0.00335$ for the earth.) Figure 5.9 shows the ellipse of the meridian through O. Obviously,

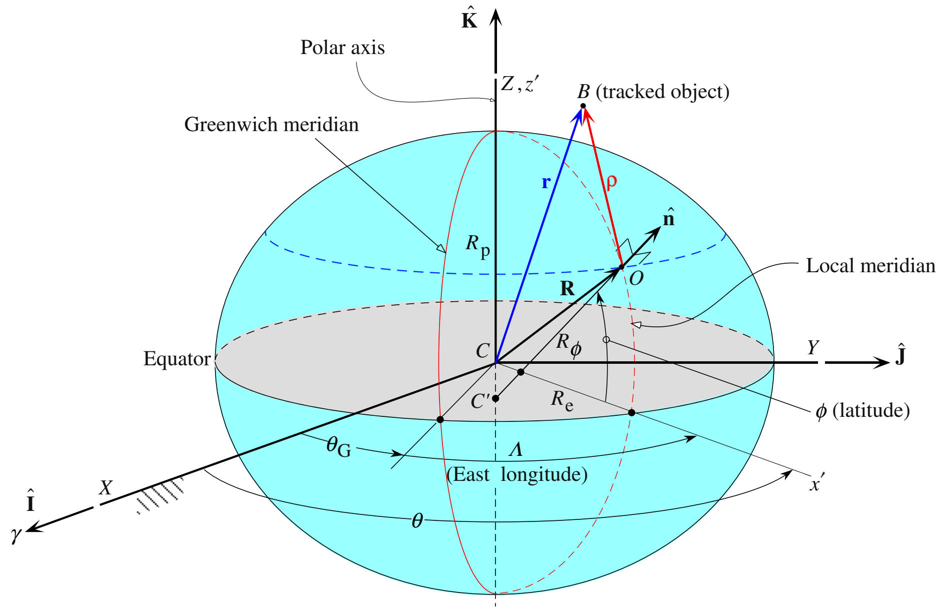

FIGURE 5.8

Oblate spheroidal earth (exaggerated).

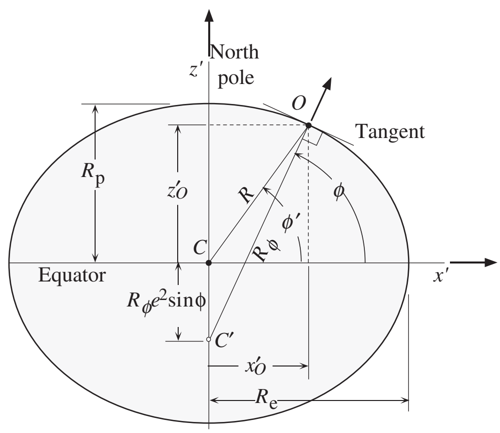

FIGURE 5.9

The relationship between geocentric latitude (ϕ') and geodetic latitude (ϕ).

R_e and R_p are, respectively, the semimajor and semiminor axes of the ellipse. According to Eqn (2.76),

$$R_p = R_e\sqrt{1-e^2}$$

It is easy to show from the above two relations that flattening and eccentricity are related as follows:

$$e = \sqrt{2f - f^2} \quad f = 1 - \sqrt{1-e^2}$$

As illustrated in Figure 5.8 and again in Figure 5.9, the normal to the earth's surface at O intersects the polar axis at a point C' that lies below the center C of the earth (if O is in the northern hemisphere). The angle ϕ between the normal and the equator is called the geodetic latitude, as opposed to geocentric latitude ϕ', which is the angle between the equatorial plane and the line joining O to the center of the earth. The distance from C to C' is $R_\phi e^2 \sin^2\phi$, where R_ϕ, the distance from C' to O, is a function of latitude (Seidelmann, 1992; Section 4.22)

$$R_\phi = \frac{R_e}{\sqrt{1-e^2\sin^2\phi}} = \frac{R_e}{\sqrt{1-(2f-f^2)\sin^2\phi}} \tag{5.54}$$

Thus, the meridional coordinates of O are

$$x'_O = R_\phi \cos\phi$$

$$z'_O = \left(1-e^2\right)R_\phi \sin\phi = (1-f)^2 R_\phi \sin\phi$$

If the observation point O is at an elevation H above the ellipsoidal surface, then we must add $H\cos\phi$ to x'_O and $H\sin\phi$ to z'_O to obtain

$$x'_O = R_c\cos\phi \quad z'_O = R_s\sin\phi \tag{5.55a}$$

where

$$R_c = R_\phi + H \quad R_s = (1-f)^2 R_\phi + H \tag{5.55b}$$

Observe that whereas R_c is the distance of O from point C' on the earth's axis, R_s is the distance from O to the intersection of the line OC' with the equatorial plane.

The geocentric equatorial coordinates of O are

$$X = x'_O\cos\theta \quad Y = x'_O\sin\theta \quad Z = z'_O$$

where θ is the local sidereal time given in Eqn (5.52). Hence, the position vector $\mathbf{R}$ shown in Figure 5.8 is

$$\mathbf{R} = R_c\cos\phi\cos\theta\hat{\mathbf{I}} + R_c\cos\phi\sin\theta\hat{\mathbf{J}} + R_s\sin\phi\hat{\mathbf{K}}$$

Substituting Eqns (5.54) and (5.55b) yields

$$\mathbf{R} = \left[\frac{R_e}{\sqrt{1-(2f-f^2)\sin^2\phi}} + H\right]\cos\phi\left(\cos\theta\hat{\mathbf{I}} + \sin\theta\hat{\mathbf{J}}\right) + \left[\frac{R_e(1-f)^2}{\sqrt{1-(2f-f^2)\sin^2\phi}} + H\right]\sin\phi\hat{\mathbf{K}} \tag{5.56}$$

In terms of the geocentric latitude ϕ',

$$\mathbf{R} = R_e \cos\phi' \cos\theta\hat{\mathbf{I}} + R_e \cos\phi' \sin\theta\hat{\mathbf{J}} + R_e \sin\phi'\hat{\mathbf{K}}$$

By equating these two expressions for $\mathbf{R}$ and setting $H = 0$, it is easy to show that at sea level the geodetic latitude is related to geocentric latitude ϕ' as follows:

$$\tan\phi' = (1 - f)^2 \tan\phi$$

5.6 Topocentric equatorial coordinate system

The topocentric equatorial coordinate system with the origin at point O on the surface of the earth uses a nonrotating set of xyz axes through O which coincide with the XYZ axes of the geocentric equatorial frame, as illustrated in Figure 5.10. As can be inferred from the figure, the relative position vector ρ in terms of the topocentric right ascension and declination is

$$\boldsymbol{\rho} = \rho\cos\delta\cos\alpha\hat{\mathbf{I}} + \rho\cos\delta\sin\alpha\hat{\mathbf{J}} + \rho\sin\delta\hat{\mathbf{K}}$$

since at all times, $\hat{\mathbf{i}} = \hat{\mathbf{I}}$, $\hat{\mathbf{j}} = \hat{\mathbf{J}}$, and $\hat{\mathbf{k}} = \hat{\mathbf{K}}$ for this frame of reference. We can write $\boldsymbol{\rho}$ as

$$\boldsymbol{\rho} = \rho\hat{\boldsymbol{\rho}}$$

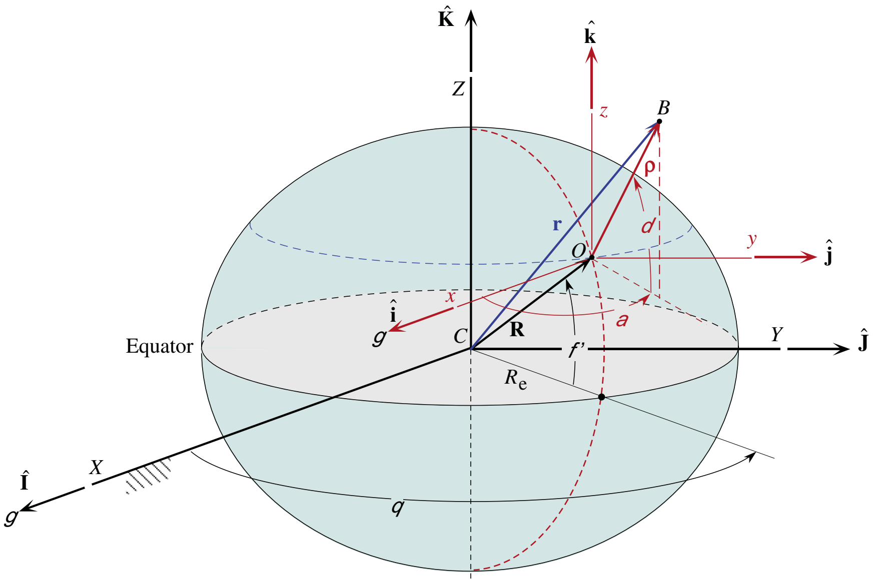

FIGURE 5.10

Topocentric equatorial coordinate system.

where ρ is the slant range and $\hat{\boldsymbol{\rho}}$ is the unit vector in the direction of the position vector $\boldsymbol{\rho}$,

$$\hat{\boldsymbol{\rho}} = \cos\delta\cos\alpha\hat{\mathbf{I}} + \cos\delta\sin\alpha\hat{\mathbf{J}} + \sin\delta\hat{\mathbf{K}} \tag{5.57}$$

Since the origins of the geocentric and topocentric systems do not coincide, the direction cosines of the position vectors $\mathbf{r}$ and $\boldsymbol{\rho}$ will in general differ. In particular the topocentric right ascension and declination of an earth-orbiting body B will not be the same as the geocentric right ascension and declination. This is an example of parallax. On the other hand, if $\|\mathbf{r}\| \gg \|\mathbf{R}\|$, then the difference between the geocentric and topocentric position vectors, and hence, the right ascension and declination, is negligible. This is true for the distant planets and stars.

EXAMPLE 5.7

At the instant when the Greenwich sidereal time is $\theta_G = 126.7°$, the geocentric equatorial position vector of the International Space Station is

$$\mathbf{r} = -5368\hat{\mathbf{I}} - 1784\hat{\mathbf{J}} + 3691\hat{\mathbf{K}} \quad \text{(km)}$$

Find its topocentric right ascension and declination at sea level ($H = 0$), latitude $\phi = 20°$, and east longitude $\Lambda = 60°$.

Solution

According to Eqn (5.52), the local sidereal time at the observation site is

$$\theta = \theta_G + \Lambda = 126.7 + 60 = 186.7°$$

Substituting $R_e = 6378$ km, $f = 0.003353$ (Table 4.3), $\theta = 189.7°$, and $\phi = 20°$ into Eqn (5.56) yields the geocentric position vector of the site,

$$\mathbf{R} = -5955\hat{\mathbf{I}} - 699.5\hat{\mathbf{J}} + 2168\hat{\mathbf{K}} \quad \text{(km)}$$

Having found $\mathbf{R}$, we obtain the position vector of the space station relative to the site from Eqn (5.53),

$$\begin{aligned}\boldsymbol{\rho} &= \mathbf{r} - \mathbf{R} \\ &= \left(-5368\hat{\mathbf{I}} - 1784\hat{\mathbf{J}} + 3691\hat{\mathbf{K}}\right) - \left(-5955\hat{\mathbf{I}} - 699.5\hat{\mathbf{J}} + 2168\hat{\mathbf{K}}\right) \\ &= 586.8\hat{\mathbf{I}} - 1084\hat{\mathbf{J}} + 1523\hat{\mathbf{K}} \quad \text{(km)}\end{aligned}$$

Applying Algorithm 4.1 to this vector yields

$$\boxed{\alpha = 298.4°} \quad \boxed{\delta = 51.01°}$$

Compare these with the geocentric right ascension α_0 and declination δ_0, which were computed in Example 4.1,

$$\alpha_0 = 198.4° \quad \delta_0 = 33.12°$$

5.7 Topocentric horizon coordinate system

The topocentric horizon coordinate system was introduced in Section 1.7 and is illustrated again in Figure 5.11. It is centered at the observation point O whose position vector is $\mathbf{R}$. The xy plane is the

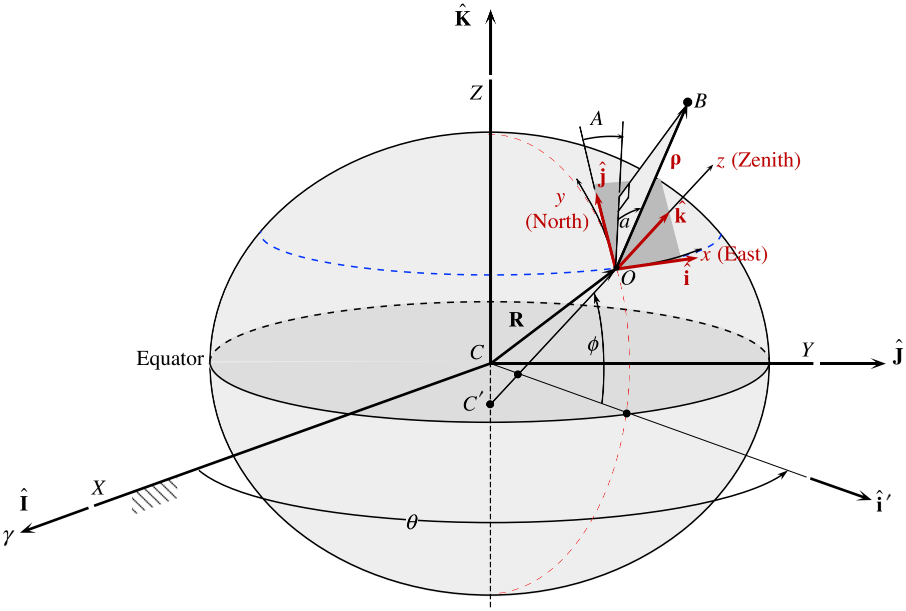

FIGURE 5.11

Topocentric horizon (*xyz*) coordinate system on the surface of the oblate earth.

local horizon, which is the plane tangent to the ellipsoid at point O. The z-axis is normal to this plane directed outward toward the zenith. The x-axis is directed eastward and the y-axis points north. Because the x-axis points east, this may be referred to as an *ENZ* (East–North–Zenith) frame. In the *SEZ* topocentric reference frame the x-axis points toward the south and the y-axis toward the east. The *SEZ* frame is obtained from *ENZ* by a 90° clockwise rotation around the zenith. Therefore, the matrix of the transformation from *NEZ* to *SEZ* is $[\mathbf{R}_3\ (-90°)]$, where $[\mathbf{R}_3\ (\phi)]$ is found in Eqn (4.34).

The position vector $\boldsymbol{\rho}$ of a body B relative to the topocentric horizon system in Figure 5.11 is

$$\boldsymbol{\rho} = \rho \cos a \sin A\hat{\mathbf{i}} + \rho \cos a \cos A\hat{\mathbf{j}} + \rho \sin a\hat{\mathbf{k}}$$

in which ρ is the range; A is the azimuth measured positive clockwise from due north $(0 \leq A \leq 360°)$; and a is the elevation angle or altitude measured from the horizontal to the line of sight of the body B $(-90° \leq a \leq 90)$. The unit vector $\hat{\rho}$ in the line of sight direction is

$$\hat{\boldsymbol{\rho}} = \cos a \sin A\hat{\mathbf{i}} + \cos a \cos A\hat{\mathbf{j}} + \sin a\hat{\mathbf{k}} \tag{5.58}$$

The transformation between geocentric equatorial and topocentric horizon systems is found by first determining the projections of the topocentric base vectors $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ onto those of the geocentric equatorial frame. From Figure 5.11, it is apparent that

$$\hat{\mathbf{k}} = \cos \phi\hat{\mathbf{i}}' + \sin \phi\hat{\mathbf{K}}$$

and

$$\hat{\mathbf{i}}' = \cos\theta\hat{\mathbf{I}} + \sin\theta\hat{\mathbf{J}}$$

where $\hat{\mathbf{i}}'$ lies in the local meridional plane and is normal to the Z-axis. Hence,

$$\hat{\mathbf{k}} = \cos\phi\cos\theta\hat{\mathbf{I}} + \cos\phi\sin\theta\hat{\mathbf{J}} + \sin\phi\hat{\mathbf{K}} \tag{5.59}$$

The eastward-directed unit vector $\hat{\mathbf{i}}$ may be found by taking the cross product of $\hat{\mathbf{K}}$ into the unit normal $\hat{\mathbf{k}}$,

$$\hat{\mathbf{i}} = \frac{\hat{\mathbf{K}}\times\hat{\mathbf{k}}}{\left\|\hat{\mathbf{K}}\times\hat{\mathbf{k}}\right\|} = \frac{-\cos\phi\sin\theta\hat{\mathbf{I}} + \cos\phi\cos\theta\hat{\mathbf{J}}}{\sqrt{\cos^2\phi(\sin^2\theta + \cos^2\theta)}} = -\sin\theta\hat{\mathbf{I}} + \cos\theta\hat{\mathbf{J}} \tag{5.60}$$

Finally, crossing $\hat{\mathbf{k}}$ into $\hat{\mathbf{i}}$ yields $\hat{\mathbf{j}}$,

$$\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = -\sin\phi\cos\theta\hat{\mathbf{I}} - \sin\phi\sin\theta\hat{\mathbf{J}} + \cos\phi\hat{\mathbf{K}} \tag{5.61}$$

Let us denote the matrix of the transformation from the geocentric equatorial to the topocentric horizon as $[\mathbf{Q}]_{Xx}$. Recall from Section 4.5 that the rows of this matrix comprise the direction cosines of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, respectively. It follows from Eqns (5.59)–(5.61) that

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \end{bmatrix} \tag{5.62a}$$

The reverse transformation, from the topocentric horizon to the geocentric equatorial, is represented by the transpose of this matrix,

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin\theta & -\sin\phi\cos\theta & \cos\phi\cos\theta \\ \cos\theta & -\sin\phi\sin\theta & \cos\phi\sin\theta \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \tag{5.62b}$$

Observe that these matrices also represent the transformation between the topocentric horizon and the topocentric equatorial frames, because the unit basis vectors of the latter coincide with those of the geocentric equatorial coordinate system.

EXAMPLE 5.8

The east longitude and latitude of an observer near San Francisco are $\Lambda = 238°$ and $\phi = 38°$, respectively. The local sidereal time is $\theta = 215.1°$ (14 h 20 min). At that time, the planet Jupiter is observed by means of a telescope to be located at azimuth $A = 214.3°$ and angular elevation $a = 43°$. What are Jupiter's right ascension and declination in the topocentric equatorial system?

Solution

The given information allows us to formulate the matrix of the transformation from the topocentric horizon to the topocentric equatorial using Eqn (5.62b),

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin 215.1° & -\sin 38°\cos 215.1° & \cos 38°\cos 215.1° \\ \cos 215.1° & -\sin 38°\sin 215.1° & \cos 38°\sin 215.1° \\ 0 & \cos 38° & \sin 38° \end{bmatrix} = \begin{bmatrix} 0.5750 & 0.5037 & -0.6447 \\ -0.8182 & 0.3540 & -0.4531 \\ 0 & 0.7880 & 0.6157 \end{bmatrix}$$

From Eqn (5.58), we have

$$\begin{aligned}\hat{\boldsymbol{\rho}} &= \cos a \sin A\hat{\mathbf{i}} + \cos a \cos A\hat{\mathbf{j}} + \sin a\hat{\mathbf{k}}\\ &= \cos 43^\circ \sin 214.3^\circ\hat{\mathbf{i}} + \cos 43^\circ \cos 214.3^\circ\hat{\mathbf{j}} + \sin 43^\circ\hat{\mathbf{k}}\\ &= -0.4121\hat{\mathbf{i}} - 0.6042\hat{\mathbf{j}} + 0.6820\hat{\mathbf{k}}\end{aligned}$$

Therefore, in matrix notation, the topocentric horizon components of ρ are

$$\{\hat{\rho}\}_x = \begin{Bmatrix} -0.4121 \\ -0.6042 \\ 0.6820 \end{Bmatrix}$$

We obtain the topocentric equatorial components $\{\hat{\rho}\}_X$ by the matrix operation

$$\{\hat{\rho}\}_X = [\mathbf{Q}]_{xX}\{\hat{\rho}\}_x = \begin{bmatrix} 0.5750 & 0.5037 & -0.6447 \\ -0.8182 & 0.3540 & -0.4531 \\ 0 & 0.7880 & 0.6157 \end{bmatrix}\begin{Bmatrix} -0.4121 \\ -0.6042 \\ 0.6820 \end{Bmatrix} = \begin{Bmatrix} -0.9810 \\ -0.1857 \\ -0.05621 \end{Bmatrix}$$

so that the topocentric equatorial line of the sight unit vector is

$$\hat{\boldsymbol{\rho}} = -0.9810\hat{\mathbf{I}} - 0.1857\hat{\mathbf{J}} - 0.05621\hat{\mathbf{K}}$$

Using this vector in Algorithm 4.1 yields the topocentric equatorial right ascension and declination,

$$\boxed{\alpha = 190.7^\circ \quad \delta = -3.222^\circ}$$

Jupiter is sufficiently far away that we can ignore the radius of the earth in Eqn (5.53). That is, to our level of precision, there is no distinction between the topocentric equatorial and geocentric equatorial systems:

$$\mathbf{r} \approx \boldsymbol{\rho}$$

Therefore, the topocentric right ascension and declination computed above are the same as the geocentric equatorial values.

EXAMPLE 5.9

At a given time, the geocentric equatorial position vector of the International Space Station is

$$\mathbf{r} = -2032.4\hat{\mathbf{I}} + 4591.2\hat{\mathbf{J}} - 4544.8\hat{\mathbf{K}} \quad \text{(km)}$$

Determine the azimuth and elevation angle relative to a sea-level ($H = 0$) observer whose latitude is $\phi = -40^\circ$ and local sidereal time is $\theta = 110^\circ$.

Solution

Using Eqn (5.56), we find the position vector of the observer to be

$$\mathbf{R} = -1673\hat{\mathbf{I}} + 4598\hat{\mathbf{J}} - 4078\hat{\mathbf{K}} \quad \text{(km)}$$

For the position vector of the space station relative to the observer, we have (Eqn (5.53))

$$\begin{aligned}\boldsymbol{\rho} &= \mathbf{r} - \mathbf{R}\\ &= \left(-2032\hat{\mathbf{I}} + 4591\hat{\mathbf{J}} - 4545\hat{\mathbf{K}}\right) - \left(-1673\hat{\mathbf{I}} + 4598\hat{\mathbf{J}} - 4078\hat{\mathbf{K}}\right)\\ &= -359.0\hat{\mathbf{I}} - 6.342\hat{\mathbf{J}} - 466.9\hat{\mathbf{K}} \quad \text{(km)}\end{aligned}$$

or, in matrix notation,

$$\{\boldsymbol{\rho}\}_X = \begin{Bmatrix} -359.0 \\ -6.342 \\ -466.9 \end{Bmatrix} \quad (\text{km})$$

To transform these geocentric equatorial components into the topocentric horizon system, we need the transformation matrix $[\mathbf{Q}]_{Xx}$, which is given by Eqn (5.62a),

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \end{bmatrix}$$

$$= \begin{bmatrix} -\sin 110^\circ & \cos 110^\circ & 0 \\ -\sin(-40^\circ)\cos 110^\circ & -\sin(-40^\circ)\sin 110^\circ & \cos(-40^\circ) \\ \cos(-40^\circ)\cos 110^\circ & \cos(-40^\circ)\sin 110^\circ & \sin(-40^\circ) \end{bmatrix}$$

Thus,

$$\{\boldsymbol{\rho}\}_x = [\mathbf{Q}]_{Xx}\{\boldsymbol{\rho}\}_X = \begin{bmatrix} -0.9397 & -0.3420 & 0 \\ -0.2198 & 0.6040 & 0.7660 \\ -0.2620 & 0.7198 & -0.6428 \end{bmatrix} \begin{Bmatrix} -359.0 \\ -6.342 \\ -466.9 \end{Bmatrix} = \begin{Bmatrix} 339.5 \\ -282.6 \\ 389.6 \end{Bmatrix} \quad (\text{km})$$

or, reverting to vector notation,

$$\boldsymbol{\rho} = 339.5\hat{\mathbf{i}} - 282.6\hat{\mathbf{j}} + 389.6\hat{\mathbf{k}} \quad (\text{km})$$

The magnitude of this vector is $\rho = 589.0$ km. Hence, the unit vector in the direction of ρ is

$$\hat{\boldsymbol{\rho}} = \frac{\boldsymbol{\rho}}{\rho} = 0.5765\hat{\mathbf{i}} - 0.4787\hat{\mathbf{j}} + 0.6615\hat{\mathbf{k}}$$

Comparing this with Eqn (5.58), we see that $\sin a = 0.6615$, so that the angular elevation is

$$a = \sin^{-1}0.6615 = \boxed{41.41^\circ}$$

Furthermore,

$$\sin A = \frac{0.5765}{\cos a} = 0.7687$$

$$\cos A = \frac{-0.4787}{\cos a} = -0.6397$$

It follows that

$$A = \cos^{-1}(-0.6397) = 129.8^\circ \ \ (\text{second quadrant}) \ \text{ or } \ 230.2^\circ \ \ (\text{third quadrant})$$

A must lie in the second quadrant because $\sin A > 0$. Thus, the azimuth is

$$\boxed{A = 129.8^\circ}$$

5.8 Orbit determination from angle and range measurements

We know that an orbit around the earth is determined once the state vectors $\mathbf{r}$ and $\mathbf{v}$ in the inertial geocentric equatorial frame are provided at a given instant of time (epoch). Satellites are of course observed from the earth's surface and not from its center. Let us briefly consider how the state vector is determined from the measurements by an earth-based tracking station.

The fundamental vector triangle formed by the topocentric position vector ρ of a satellite relative to a tracking station, the position vector $\mathbf{R}$ of the station relative to the center of attraction C, and the geocentric position vector $\mathbf{r}$ was illustrated in Figure 5.8 and is shown again schematically in Figure 5.12. The relationship among these three vectors is given by Eqn (5.53), which can be written as

$$\mathbf{r} = \mathbf{R} + \rho\hat{\boldsymbol{\rho}} \tag{5.63}$$

where the range ρ is the distance of the body B from the tracking site and $\hat{\rho}$ is the unit vector containing the directional information about B. By differentiating Eqn (5.63) with respect to time, we obtain the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{R}} + \rho\dot{\boldsymbol{\rho}} + \dot{\rho}\boldsymbol{\rho} \tag{5.64}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{\mathbf{R}} + \ddot{\rho}\hat{\boldsymbol{\rho}} + 2\dot{\rho}\dot{\hat{\boldsymbol{\rho}}} + \rho\ddot{\hat{\boldsymbol{\rho}}} \tag{5.65}$$

The vectors in these equations must all be expressed in the common basis $(\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}})$ of the inertial (nonrotating) geocentric equatorial frame.

Since $\mathbf{R}$ is a vector fixed in the earth, whose constant angular velocity is $\boldsymbol{\Omega} = \omega_E\hat{\mathbf{K}}$ (Eqn (2.67)), it follows from Eqns (1.61) and (1.62) that

$$\dot{\mathbf{R}} = \boldsymbol{\Omega} \times \mathbf{R} \tag{5.66}$$

$$\ddot{\mathbf{R}} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}) \tag{5.67}$$

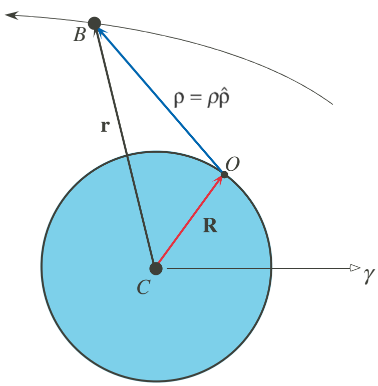

FIGURE 5.12

Earth-orbiting body B tracked by an observer O.

If L_x, L_Y, and L_Z are the topocentric equatorial direction cosines, then the direction cosine vector $\hat{\rho}$ is

$$\hat{\boldsymbol{\rho}} = L_X\hat{\mathbf{I}} + L_Y\hat{\mathbf{J}} + L_Z\hat{\mathbf{K}} \tag{5.68}$$

and its first and second derivatives are

$$\dot{\hat{\boldsymbol{\rho}}} = \dot{L}_X\hat{\mathbf{I}} + \dot{L}_Y\hat{\mathbf{J}} + \dot{L}_Z\hat{\mathbf{K}} \tag{5.69}$$

and

$$\ddot{\hat{\boldsymbol{\rho}}} = \ddot{L}_X\hat{\mathbf{I}} + \ddot{L}_Y\hat{\mathbf{J}} + \ddot{L}_Z\hat{\mathbf{K}} \tag{5.70}$$

Comparing Eqns (5.57) and (5.68) reveals that the topocentric equatorial direction cosines in terms of the topocentric right ascension α and declension δ are

$$\begin{Bmatrix} L_X \\ L_Y \\ L_Z \end{Bmatrix} = \begin{Bmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{Bmatrix} \tag{5.71}$$

Differentiating this equation twice yields

$$\begin{Bmatrix} \dot{L}_X \\ \dot{L}_Y \\ \dot{L}_Z \end{Bmatrix} = \begin{Bmatrix} -\dot{\alpha}\sin\alpha\cos\delta - \dot{\delta}\cos\alpha\sin\delta \\ \dot{\alpha}\cos\alpha\cos\delta - \dot{\delta}\sin\alpha\sin\delta \\ \dot{\delta}\cos\delta \end{Bmatrix} \tag{5.72}$$

and

$$\begin{Bmatrix} \ddot{L}_X \\ \ddot{L}_Y \\ \ddot{L}_Z \end{Bmatrix} = \begin{Bmatrix} -\ddot{\alpha}\sin\alpha\cos\delta - \ddot{\delta}\cos\alpha\sin\delta - \left(\dot{\alpha}^2 + \dot{\delta}^2\right)\cos\alpha\cos\delta + 2\dot{\alpha}\dot{\delta}\sin\alpha\sin\delta \\ \ddot{\alpha}\cos\alpha\cos\delta - \ddot{\delta}\sin\alpha\sin\delta - \left(\dot{\alpha}^2 + \dot{\delta}^2\right)\sin\alpha\cos\delta - 2\dot{\alpha}\dot{\delta}\cos\alpha\sin\delta \\ \ddot{\delta}\cos\delta - \dot{\delta}^2\sin\delta \end{Bmatrix} \tag{5.73}$$

Equations (5.71)–(5.73) show how the direction cosines and their rates are obtained from the right ascension and declination and their rates.

In the topocentric horizon system, the relative position vector is written as

$$\hat{\boldsymbol{\rho}} = l_x\hat{\mathbf{i}} + l_y\hat{\mathbf{j}} + l_z\hat{\mathbf{k}} \tag{5.74}$$

where, according to Eqn (5.58), the direction cosines l_x, l_y, and l_z are found in terms of the azimuth A and elevation a as

$$\begin{Bmatrix} l_x \\ l_y \\ l_z \end{Bmatrix} = \begin{Bmatrix} \sin A\cos a \\ \cos A\cos a \\ \sin a \end{Bmatrix} \tag{5.75}$$

L_x, L_Y, and L_Z are obtained from l_x, l_y, and l_z by the coordinate transformation

$$\begin{Bmatrix} L_X \\ L_Y \\ L_Z \end{Bmatrix} = [\mathbf{Q}]_{xX}\begin{Bmatrix} l_x \\ l_y \\ l_z \end{Bmatrix} \tag{5.76}$$

where $[\mathbf{Q}]_{XX}$ is given by Eqn (5.62b). Thus,

$$\begin{Bmatrix} L_X \\ L_Y \\ L_Z \end{Bmatrix} = \begin{bmatrix} -\sin\theta & -\cos\theta\sin\phi & \cos\theta\cos\phi \\ \cos\theta & -\sin\theta\sin\phi & \sin\theta\cos\phi \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{Bmatrix} \sin A\cos a \\ \cos A\cos a \\ \sin a \end{Bmatrix} \tag{5.77}$$

Substituting Eqn (5.71) we see that topocentric right ascension/declination and azimuth/elevation are related by

$$\begin{Bmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{Bmatrix} = \begin{bmatrix} -\sin\theta & -\cos\theta\sin\phi & \cos\theta\cos\phi \\ \cos\theta & -\sin\theta\sin\phi & \sin\theta\cos\phi \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{Bmatrix} \sin A\cos a \\ \cos A\cos a \\ \sin a \end{Bmatrix}$$

Expanding the right-hand side and solving for $\sin\delta$, $\sin\alpha$, and $\cos\alpha$, we get

$$\sin\delta = \cos\phi\cos A\cos a + \sin\phi\sin a \tag{5.78a}$$

$$\sin\alpha = \frac{(\cos\phi\sin a - \cos A\cos a\sin\phi)\sin\theta + \cos\theta\sin A\cos a}{\cos\delta} \tag{5.78b}$$

$$\cos\alpha = \frac{(\cos\phi\sin a - \cos A\cos a\sin\phi)\cos\theta - \sin\theta\sin A\cos a}{\cos\delta} \tag{5.78c}$$

We can simplify Eqns (5.78b) and (5.78c) by introducing the hour angle h,

$$h = \theta - \alpha \tag{5.79}$$

where h is the angular distance between the object and the local meridian. If h is positive, the object is west of the meridian; if h is negative, the object is east of the meridian.

Using well-known trig identities, we have

$$\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha \tag{5.80a}$$

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha \tag{5.80b}$$

Substituting Eqns (5.78b) and (5.78c) on the right of Eqn (5.80a) and simplifying yields

$$\sin(h) = -\frac{\sin A\cos a}{\cos\delta} \tag{5.81}$$

Likewise, Eqn (5.80b) leads to

$$\cos(h) = \frac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta} \tag{5.82}$$

We calculate h from this equation, resolving quadrant ambiguity by checking the sign of $\sin(h)$. That is,

$$h = \cos^{-1}\left(\frac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right)$$

if $\sin(h)$ is positive. Otherwise, we must subtract h from 360°. Since both the elevation angle a and the declension δ lie between $-90°$ and $+90°$, neither $\cos a$ nor $\cos\delta$ can be negative. It follows from Eqn (5.81) that the sign of $\sin(h)$ depends only on that of $\sin A$.

To summarize, given the topocentric azimuth A and altitude a of the target together with the sidereal time θ and latitude ϕ of the tracking station, we compute the topocentric declension δ and right ascension α as follows:

$$\delta = \sin^{-1}(\cos\phi\cos A\cos a + \sin\phi\sin a) \tag{5.83a}$$

$$h = \begin{cases} 360^\circ - \cos^{-1}\left(\dfrac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right) & 0^\circ < A < 180^\circ \\ \cos^{-1}\left(\dfrac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right) & 180^\circ \le A \le 360^\circ \end{cases} \tag{5.83b}$$

$$\alpha = \theta - h \tag{5.83c}$$

If A and a are provided as functions of time, then α and δ are found as functions of time by means of Eqn (5.83). The rates $\dot{\alpha}$, $\ddot{\alpha}$, $\dot{\delta}$, and $\ddot{\delta}$ are determined by differentiating $\alpha(t)$ and $\delta(t)$ and substituting the results into Eqns (5.68)–(5.73) to calculate the direction cosine vector $\hat{\boldsymbol{\rho}}$ and its rates $\dot{\hat{\boldsymbol{\rho}}}$ and $\ddot{\hat{\boldsymbol{\rho}}}$.

It is a relatively simple matter to find $\dot{\alpha}$ and $\dot{\delta}$ in terms of $\dot{A}$ and $\dot{a}$. Differentiating Eqn (5.78a) with respect to time yields

$$\dot{\delta} = \frac{1}{\cos\delta}\left[-\dot{A}\cos\phi\sin A\cos a + \dot{a}(\sin\phi\cos a - \cos\phi\cos A\sin a)\right] \tag{5.84}$$

Differentiating Eqn (5.81), we get

$$\dot{h}\cos(h) = -\frac{1}{\cos^2\delta}\left[\left(\dot{A}\cos A\cos a - \dot{a}\sin A\sin a\right)\cos\delta + \dot{\delta}\sin A\cos a\sin\delta\right]$$

Substituting Eqn (5.82) and simplifying leads to

$$\dot{h} = -\frac{\dot{A}\cos A\cos a - \dot{a}\sin A\sin a + \dot{\delta}\sin A\cos a\tan\delta}{\cos\phi\sin a - \sin\phi\cos A\cos a}$$

But $\dot{h} = \dot{\theta} - \dot{\alpha} = \omega_E - \dot{\alpha}$, so that, finally,

$$\dot{\alpha} = \omega_E + \frac{\dot{A}\cos A\cos a - \dot{a}\sin A\sin a + \dot{\delta}\sin A\cos a\tan\delta}{\cos\phi\sin a - \sin\phi\cos A\cos a} \tag{5.85}$$

ALGORITHM 5.4

Given the range ρ, azimuth A, angular elevation a together with the rates $\dot{\rho}$, $\dot{A}$, and $\dot{a}$ relative to an earth-based tracking station (for which the altitude H, latitude ϕ, and local sidereal time are known), calculate the state vectors $\mathbf{r}$ and $\mathbf{v}$ in the geocentric equatorial frame. A MATLAB script of this procedure appears in Appendix D.28.

1. Using the altitude H, latitude ϕ, and local sidereal time θ of the site, calculate its geocentric position vector $\mathbf{R}$ from Eqn (5.56):

$$\mathbf{R} = \left[\frac{R_e}{\sqrt{1-(2f-f^2)\sin^2\phi}} + H\right]\cos\phi\left(\cos\theta\hat{\mathbf{I}} + \sin\theta\hat{\mathbf{J}}\right) + \left[\frac{R_e(1-f)^2}{\sqrt{1-(2f-f^2)\sin^2\phi}} + H\right]\sin\phi\hat{\mathbf{K}}$$

   where f is the earth's flattening factor.

2. Calculate the topocentric declination δ using Eqn (5.83a).
3. Calculate the topocentric right ascension α from Eqns (5.83b) and (5.83c).
4. Calculate the direction cosine unit vector $\hat{\boldsymbol{\rho}}$ from Eqns (5.68) and (5.71),

$$\hat{\boldsymbol{\rho}} = \cos\delta\left(\cos\alpha\hat{\mathbf{I}} + \sin\alpha\hat{\mathbf{J}}\right) + \sin\delta\hat{\mathbf{K}}$$

5. Calculate the geocentric position vector $\mathbf{r}$ from Eqn (5.63),

$$\mathbf{r} = \mathbf{R} + \rho\hat{\boldsymbol{\rho}}$$

6. Calculate the inertial velocity $\dot{\mathbf{R}}$ of the site from Eqn (5.66).
7. Calculate the declination rate $\dot{\delta}$ using Eqn (5.84).
8. Calculate the right ascension rate $\dot{\alpha}$ by means of Eqn (5.85).
9. Calculate the direction cosine rate vector $\dot{\hat{\boldsymbol{\rho}}}$ from Eqns (5.69) and (5.72),

$$\dot{\hat{\boldsymbol{\rho}}} = \left(-\dot{\alpha}\sin\alpha\cos\delta - \dot{\delta}\cos\alpha\sin\delta\right)\hat{\mathbf{I}} + \left(\dot{\alpha}\cos\alpha\cos\delta - \dot{\delta}\sin\alpha\sin\delta\right)\hat{\mathbf{J}} + \dot{\delta}\cos\delta\hat{\mathbf{K}}$$

10. Calculate the geocentric velocity vector $\mathbf{v}$ from Eqn (5.64),

$$\mathbf{v} = \dot{\mathbf{R}} + \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\hat{\boldsymbol{\rho}}}$$

EXAMPLE 5.10

At $\theta = 300°$ local sidereal time a sea-level ($H = 0$) tracking station at a latitude of $\phi = 60°$ detects a space object and obtains the following data:

Slant range:	$\rho = 2551$ km
Azimuth:	$A = 90°$
Elevation:	$a = 30°$
Range rate:	$\dot{\rho} = 0$
Azimuth rate:	$\dot{A} = 1.973 \times 10^{-3}$ rad/s (0.1130°/s)
Elevation rate:	$\dot{a} = 9.864 \times 10^{-4}$ rad/s (0.05651°/s)

What are the orbital elements of the object?

Solution

We must first employ Algorithm 5.4 to obtain the state vectors **r** and **v** in order to compute the orbital elements by means of Algorithm 4.2.

Step 1:

The equatorial radius of the earth is $R_e = 6378$ km and the flattening factor is $f = 0.003353$. It follows from Eqn (5.56) that the position vector of the observer is

$$\mathbf{R} = 1598\hat{\mathbf{I}} - 2769\hat{\mathbf{J}} + 5500\hat{\mathbf{K}} \quad \text{(km)}$$

Step 2:

$$\begin{aligned}\delta &= \sin^{-1}(\cos\phi\cos A\cos a + \sin\phi\sin a)\\ &= \sin^{-1}(\cos 60°\cos 90°\cos 30° + \sin 60°\sin 30°)\\ &= 25.66\end{aligned}$$

Step 3:

Since the given azimuth lies between 0° and 180°, Eqn (5.83b) yields

$$\begin{aligned} h &= 360^\circ - \cos^{-1}\left(\frac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right) \\ &= 360^\circ - \cos^{-1}\left(\frac{\cos 60^\circ \sin 30^\circ - \sin 60^\circ \cos 90^\circ \cos 30^\circ}{\cos 25.66^\circ}\right) \\ &= 360^\circ - 73.90^\circ = 286.1^\circ \end{aligned}$$

Therefore, the right ascension is

$$\alpha = \theta - h = 300^\circ - 286.1^\circ = 13.90^\circ$$

Step 4:

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \cos\delta\left(\cos\alpha\hat{\mathbf{I}} + \sin\alpha\hat{\mathbf{J}}\right) + \sin\delta\hat{\mathbf{K}} \\ &= \cos 25.66^\circ\left(\cos 13.90^\circ\hat{\mathbf{I}} + \sin 13.90^\circ\hat{\mathbf{J}}\right) + \sin 25.66^\circ\hat{\mathbf{K}} \\ &= 0.8750\hat{\mathbf{I}} + 0.2165\hat{\mathbf{J}} + 0.4330 \end{aligned}$$

Step 5:

$$\begin{aligned} \mathbf{r} &= \mathbf{R} + \rho\hat{\boldsymbol{\rho}} \\ &= \left(1598\hat{\mathbf{I}} - 2769\hat{\mathbf{J}} + 5500\hat{\mathbf{K}}\right) + 2551\left(0.8750\hat{\mathbf{I}} + 0.2165\hat{\mathbf{J}} + 0.4330\hat{\mathbf{K}}\right) \\ &= 3831\hat{\mathbf{I}} - 2216\hat{\mathbf{J}} + 6605\hat{\mathbf{K}} \quad (\text{km}) \end{aligned}$$

Step 6:

Recalling from Eqn (2.67) that the angular velocity ω_E of the earth is 72.92×10^{-6} rad/s,

$$\begin{aligned} \dot{\mathbf{R}} &= \boldsymbol{\Omega} \times \mathbf{R} \\ &= \left(72.92 \times 10^{-6}\hat{\mathbf{K}}\right) \times \left(1598\hat{\mathbf{I}} - 2769\hat{\mathbf{J}} + 5500\hat{\mathbf{K}}\right) \\ &= 0.2019\hat{\mathbf{I}} + 0.1166\hat{\mathbf{J}} \quad (\text{km/s}) \end{aligned}$$

Step 7:

$$\begin{aligned} \dot{\delta} &= \frac{1}{\cos\delta}\left[-\dot{A}\cos\phi\sin A\cos a + \dot{a}(\sin\phi\cos a - \cos\phi\cos A\sin a)\right] \\ &= \frac{1}{\cos 25.66^\circ}\Big[-1.973 \times 10^{-3} \times \cos 60^\circ \sin 90^\circ \cos 30^\circ + 9.864 \times 10^{-4} \\ &\qquad (\sin 60^\circ \cos 30^\circ - \cos 60^\circ \cos 90^\circ \sin 30^\circ)\Big] \\ &= -1.2696 \times 10^{-4} \quad (\text{rad/s}) \end{aligned}$$

Step 8:

$$\dot{\alpha} - \omega_E = \frac{\dot{A}\cos A\cos a - \dot{a}\sin A\sin a + \dot{\delta}\sin A\cos a\tan\delta}{\cos\phi\sin a - \sin\phi\cos A\cos a}$$

$$= \frac{1.973 \times 10^{-3}\cos 90^\circ \cos 30^\circ - 9.864 \times 10^{-4}\sin 90^\circ \sin 30^\circ + \left(-1.2696 \times 10^{-4}\right)\sin 90^\circ \cos 30^\circ \tan 25.66^\circ}{\cos 60^\circ \sin 30^\circ - \sin 60^\circ \cos 90^\circ \cos 30^\circ}$$

$$= -0.002184$$

$$\dot{\alpha} = 72.92 \times 10^{-6} - 0.002184 = -0.002111 \ (\text{rad/s})$$

Step 9:

$$\dot{\hat{\boldsymbol{\rho}}} = \left(-\dot{\alpha}\sin\alpha\cos\delta - \dot{\delta}\cos\alpha\sin\delta\right)\hat{\mathbf{I}} + \left(\dot{\alpha}\cos\alpha\cos\delta - \dot{\delta}\sin\alpha\sin\delta\right)\hat{\mathbf{J}} + \dot{\delta}\cos\delta\hat{\mathbf{K}}$$

$$= [-(-0.002111)\sin 13.90°\cos 25.66° - (-0.1270)\cos 13.90°\sin 25.66°]\hat{\mathbf{I}}$$

$$+ [(-0.002111)\cos 13.90°\cos 25.66° - (-0.1270)\sin 13.90°\sin 25.66°]\hat{\mathbf{J}}$$

$$+ [-0.1270\cos 25.66°]\hat{\mathbf{K}}$$

$$\dot{\hat{\boldsymbol{\rho}}} = \left(0.5104\hat{\mathbf{I}} - 1.834\hat{\mathbf{J}} - 0.1144\hat{\mathbf{K}}\right)\left(10^{-3}\right)\ (\text{rad/s})$$

Step 10:

$$\mathbf{v} = \dot{\mathbf{R}} + \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\hat{\boldsymbol{\rho}}}$$

$$= \left(0.2019\hat{\mathbf{I}} + 0.1166\hat{\mathbf{J}}\right) + 0\cdot\left(0.8750\hat{\mathbf{I}} + 0.2165\hat{\mathbf{J}} + 0.4330\hat{\mathbf{K}}\right)$$

$$+ 2551\left(0.5104\times 10^{-3}\hat{\mathbf{I}} - 1.834\times 10^{-3}\hat{\mathbf{J}} - 0.1144\times 10^{-3}\hat{\mathbf{K}}\right)$$

$$\mathbf{v} = 1.504\hat{\mathbf{I}} - 4.562\hat{\mathbf{J}} - 0.2920\hat{\mathbf{K}}\ \ (\text{km/s})$$

Using the position and velocity vectors from Steps 5 and 10, the reader can verify that Algorithm 4.2 yields the following orbital elements of the tracked object:

$$\boxed{\begin{aligned} &a = 5170\ \text{km}\\ &i = 113.4°\\ &\Omega = 109.8°\\ &e = 0.6195\\ &\omega = 309.8°\\ &\theta = 165.3° \end{aligned}}$$

This is a highly elliptical orbit with a semimajor axis less than the earth's radius, so the object will impact the earth (at a true anomaly of 216°).

For objects orbiting the sun (planets, asteroids, comets, and man-made interplanetary probes), the fundamental vector triangle is as illustrated in Figure 5.13. The tracking station is on the earth but, of course, the sun rather than the earth is the center of attraction. The procedure for finding the heliocentric state vector **r** and **v** is similar to that outlined above. Because of the vast distances involved, the

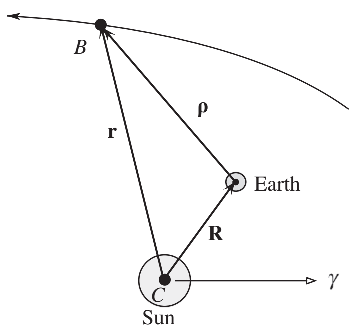

FIGURE 5.13

An object B orbiting the sun and tracked from earth.

observer can usually be imagined to reside at the center of the earth. Dealing with **R** is different in this case. The daily position of the sun relative to the earth ($-\mathbf{R}$ in Figure 5.13) may be found in ephemerides, such as *Astronomical Almanac* (US Naval Observatory, 2013). A discussion of interplanetary trajectories appears in Chapter 8 of this text.

5.9 Angles-only preliminary orbit determination

To determine an orbit requires specifying six independent quantities. These can be the six classical orbital elements or the total of six components of the state vector, **r** and **v**, at a given instant. To determine an orbit solely from observations therefore requires six independent measurements. In the previous section, we assumed the tracking station was able to measure simultaneously the six quantities range and range rate; azimuth and azimuth rate; plus elevation and elevation rate. These data lead directly to the state vector and, hence, to a complete determination of the orbit. In the absence of range and range rate measuring capability, as with a telescope, we must rely on measurements of just the two angles, azimuth, and elevation to determine the orbit. A minimum of three observations of azimuth and elevation is therefore required to accumulate the six quantities we need to predict the orbit. We shall henceforth assume that the angular measurements are converted to topocentric right ascension α and declination δ, as described in the previous section.

We shall consider the classical method of angles-only orbit determination due to Carl Friedrich Gauss (1777–1855), a German mathematician who many consider was one of the greatest mathematicians ever. This method requires gathering angular information over closely spaced intervals of time and yields a preliminary orbit determination based on those initial observations.

5.10 Gauss method of preliminary orbit determination

Suppose we have three observations of an orbiting body at times t_1, t_2, and t_3, as shown in Figure 5.14. At each time, the geocentric position vector **r** is related to the observer's position vector **R**, the slant range ρ, and the topocentric direction cosine vector $\hat{\rho}$ by Eqn (5.63),

$$\mathbf{r}_1 = \mathbf{R}_1 + \rho_1 \hat{\boldsymbol{\rho}}_1 \tag{5.86a}$$

$$\mathbf{r}_2 = \mathbf{R}_2 + \rho_2 \hat{\boldsymbol{\rho}}_2 \tag{5.86b}$$

$$\mathbf{r}_3 = \mathbf{R}_3 + \rho_3 \hat{\boldsymbol{\rho}}_3 \tag{5.86c}$$

The positions $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ of the observer O are known from the location of the tracking station and the time of the observations. $\hat{\boldsymbol{\rho}}_1$, $\hat{\boldsymbol{\rho}}_2$, and $\hat{\boldsymbol{\rho}}_3$ are obtained by measuring the right ascension α and declination δ of the body at each of the three times (recall Eqn (5.57)). Equations (5.86) are three vector equations, and therefore nine scalar equations, in 12 unknowns: the three components of each of the three vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, plus the three slant ranges ρ_1, ρ_2, and ρ_3.

An additional three equations are obtained by recalling from Chapter 2 that the conservation of angular momentum requires the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ to lie in the same plane. As in our discussion of the Gibbs method in Section 5.2, this means that $\mathbf{r}_2$ is a linear combination $\mathbf{r}_1$ and $\mathbf{r}_3$.

$$\mathbf{r}_2 = c_1 \mathbf{r}_1 + c_3 \mathbf{r}_3 \tag{5.87}$$

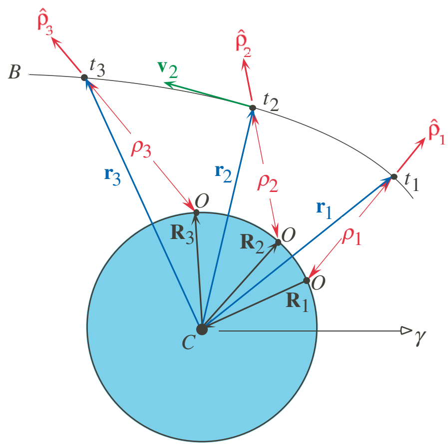

FIGURE 5.14

Center of attraction *C*, observer *O*, and tracked body *B*.

Adding this equation to those in Eqn (5.86) introduces two new unknowns, c_1 and c_3. At this point, we therefore have 12 scalar equations in 14 unknowns.

Another consequence of the two-body equation of motion (Eqn (2.22)) is that the state vectors $\mathbf{r}$ and $\mathbf{v}$ of the orbiting body can be expressed in terms of the state vectors at any given time by means of the Lagrange coefficients, Eqns (2.135) and (2.136). For the case at hand, this means we can express the position vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ in terms of the position $\mathbf{r}_2$ and velocity $\mathbf{v}_2$ at the intermediate time t_2 as follows:

$$\mathbf{r}_1 = f_1\mathbf{r}_2 + g_1\mathbf{v}_2 \tag{5.88a}$$

$$\mathbf{r}_3 = f_3\mathbf{r}_2 + g_3\mathbf{v}_2 \tag{5.88b}$$

where f_1 and g_1 are the Lagrange coefficients evaluated at t_1 while f_3 and g_3 are those same functions evaluated at time t_3. If the time intervals between the three observations are sufficiently small then Eqns (2.172) reveal that f and g depend approximately only on the distance from the center of attraction at the initial time. For the case at hand that means the coefficients in Eqn (5.88) depend only on r_2. Hence, Eqns (5.88) add six scalar equations to our previous list of 12 while adding to the list of 14 unknowns only four: the three components of $\mathbf{v}_2$ and the radius r_2. We have arrived at 18 equations in 18 unknowns, so the problem is well posed and we can proceed with the solution. The ultimate objective is to determine the state vector $\mathbf{r}_2$, $\mathbf{v}_2$ at the intermediate time t_2.

Let us start out by solving for c_1 and c_3 in Eqn (5.87). First, take the crossproduct of each term in that equation with $\mathbf{r}_3$,

$$\mathbf{r}_2 \times \mathbf{r}_3 = c_1(\mathbf{r}_1 \times \mathbf{r}_3) + c_3(\mathbf{r}_3 \times \mathbf{r}_3)$$

Since $\mathbf{r}_3 \times \mathbf{r}_3 = \mathbf{0}$, this reduces to

$$\mathbf{r}_2 \times \mathbf{r}_3 = c_1(\mathbf{r}_1 \times \mathbf{r}_3)$$

Taking the dot product of this result with $\mathbf{r}_1 \times \mathbf{r}_3$ and solving for c_1 yields

$$c_1 = \frac{(\mathbf{r}_2 \times \mathbf{r}_3)\cdot(\mathbf{r}_1 \times \mathbf{r}_3)}{\|\mathbf{r}_1 \times \mathbf{r}_3\|^2} \tag{5.89}$$

In a similar fashion, by forming the dot product of Eqn (5.87) with $\mathbf{r}_1$, we are led to

$$c_3 = \frac{(\mathbf{r}_2 \times \mathbf{r}_1)\cdot(\mathbf{r}_3 \times \mathbf{r}_1)}{\|\mathbf{r}_1 \times \mathbf{r}_3\|^2} \tag{5.90}$$

Let us next use Eqn (5.88) to eliminate $\mathbf{r}_1$ and $\mathbf{r}_3$ from the expressions for c_1 and c_3. First of all,

$$\mathbf{r}_1 \times \mathbf{r}_3 = (f_1\mathbf{r}_2 + g_1\mathbf{v}_2) \times (f_3\mathbf{r}_2 + g_3\mathbf{v}_2) = f_1g_3(\mathbf{r}_2 \times \mathbf{v}_2) + f_3g_1(\mathbf{v}_2 \times \mathbf{r}_2)$$

But $\mathbf{r}_2 \times \mathbf{v}_2 = \mathbf{h}$, where $\mathbf{h}$ is the constant angular momentum of the orbit (Eqn (2.28)). It follows that

$$\mathbf{r}_1 \times \mathbf{r}_3 = (f_1g_3 - f_3g_1)\mathbf{h} \tag{5.91}$$

and, of course,

$$\mathbf{r}_3 \times \mathbf{r}_1 = -(f_1g_3 - f_3g_1)\mathbf{h} \tag{5.92}$$

Therefore

$$\|\mathbf{r}_1 \times \mathbf{r}_3\|^2 = (f_1g_3 - f_3g_1)^2h^2 \tag{5.93}$$

Similarly

$$\mathbf{r}_2 \times \mathbf{r}_3 = \mathbf{r}_2 \times (f_3\mathbf{r}_2 + g_3\mathbf{v}_2) = g_3\mathbf{h} \tag{5.94}$$

and

$$\mathbf{r}_2 \times \mathbf{r}_1 = \mathbf{r}_2 \times (f_1\mathbf{r}_2 + g_1\mathbf{v}_2) = g_1\mathbf{h} \tag{5.95}$$

Substituting Eqns (5.91), (5.93), and (5.94) into Eqn (5.89) yields

$$c_1 = \frac{g_3\mathbf{h}\cdot(f_1g_3 - f_3g_1)\mathbf{h}}{(f_1g_3 - f_3g_1)^2h^2} = \frac{g_3(f_1g_3 - f_3g_1)h^2}{(f_1g_3 - f_3g_1)^2h^2}$$

or

$$c_1 = \frac{g_3}{f_1g_3 - f_3g_1} \tag{5.96}$$

Likewise, substituting Eqns (5.92), (5.93), and (5.95) into Eqn (5.90) leads to

$$c_3 = -\frac{g_1}{f_1g_3 - f_3g_1} \tag{5.97}$$

The coefficients in Eqn (5.87) are now expressed solely in terms of the Lagrange functions, and so far no approximations have been made. However, we will have to make some approximations in order to proceed.

We must approximate c_1 and c_2 under the assumption that the times between observations of the orbiting body are small. To that end, let us introduce the notation

$$\begin{aligned}\tau_1 &= t_1 - t_2\\ \tau_3 &= t_3 - t_2\end{aligned} \tag{5.98}$$

where τ_1 and τ_3 are the time intervals between the successive measurements of $\hat{\rho}_1$, $\hat{\rho}_2$, and $\hat{\rho}_3$. If the time intervals τ_1 and τ_3 are small enough, we can retain just the first two terms of the series expressions for the Lagrange coefficients f and g in Eqn (2.172), thereby obtaining the approximations

$$f_1 \approx 1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_1^2 \tag{5.99a}$$

$$f_3 \approx 1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_3^2 \tag{5.99b}$$

and

$$g_1 \approx \tau_1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_1^3 \tag{5.100a}$$

$$g_3 \approx \tau_3 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_3^3 \tag{5.100b}$$

We want to exclude all terms in f and g beyond the first two so that only the unknown r_2 appears in Eqns (5.99) and (5.100). One can see from Eqn (2.172) that the higher order terms include the unknown $\mathbf{v}_2$ as well.

Using Eqns (5.99) and (5.100), we can calculate the denominator in Eqns (5.96) and (5.97),

$$f_1 g_3 - f_3 g_1 = \left(1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_1^2\right)\left(\tau_3 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_3^3\right) - \left(1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_3^2\right)\left(\tau_1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_1^3\right)$$

Expanding the right side and collecting the terms yields

$$f_1 g_3 - f_3 g_1 = (\tau_3 - \tau_1) - \frac{1}{6}\frac{\mu}{r_2^3}(\tau_3 - \tau_1)^3 + \frac{1}{12}\frac{\mu^2}{r_2^6}\left(\tau_1^2\tau_3^3 - \tau_1^3\tau_3^2\right)$$

Retaining terms of at most the third order in the time intervals τ_1 and τ_3, and setting

$$\tau = \tau_3 - \tau_1 \tag{5.101}$$

reduces this expression to

$$f_1 g_3 - f_3 g_1 \approx \tau - \frac{1}{6}\frac{\mu}{r_2^3}\tau^3 \tag{5.102}$$

From Eqn (5.98) observe that τ is just the time interval between the first and last observations. Substituting Eqns (5.100b) and (5.102) into Eqn (5.96), we get

$$c_1 \approx \frac{\tau_3 - \dfrac{1}{6}\dfrac{\mu}{r_2^3}\tau_3^3}{\tau - \dfrac{1}{6}\dfrac{\mu}{r_2^3}\tau^3} = \frac{\tau_3}{\tau}\left(1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_3^2\right)\cdot\left(\frac{1}{1 - \dfrac{1}{6}\dfrac{\mu}{r_2^3}\tau^2}\right)^{-1} \tag{5.103}$$

We can use the binomial theorem to simplify (linearize) the last term on the right. Setting $a = 1$, $b = -\frac{1}{6}\frac{\mu}{r_2^3}\tau^2$, and $n = -1$ in Eqn (5.44), and neglecting terms of higher order than 2 in τ, yields

$$\left(1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau^2\right)^{-1} \approx 1 + \frac{1}{6}\frac{\mu}{r_2^3}\tau^2$$

Hence, Eqn (5.103) becomes

$$c_1 \approx \frac{\tau_3}{\tau}\left[1 + \frac{1}{6}\frac{\mu}{r_2^3}\left(\tau^2 - \tau_3^2\right)\right] \tag{5.104}$$

where only second-order terms in the time have been retained. In precisely the same way it can be shown that

$$c_3 \approx -\frac{\tau_1}{\tau}\left[1 + \frac{1}{6}\frac{\mu}{r_2^3}\left(\tau^2 - \tau_1^2\right)\right] \tag{5.105}$$

Finally, we have managed to obtain approximate formulas for the coefficients in Eqn (5.87) in terms of just the time intervals between observations and the as-yet unknown distance r_2 from the center of attraction at the central time t_2.

The next stage of the solution is to seek formulas for the slant ranges ρ_1, ρ_2, and ρ_3 in terms of c_1 and c_2. To that end, substitute Eqn (5.86) into Eqn (5.87) to get

$$\mathbf{R}_2 + \rho_2\hat{\boldsymbol{\rho}}_2 = c_1(\mathbf{R}_1 + \rho_1\hat{\boldsymbol{\rho}}_1) + c_3(\mathbf{R}_3 + \rho_3\hat{\boldsymbol{\rho}}_3)$$

which we rearrange into the form

$$c_1\rho_1\hat{\boldsymbol{\rho}}_1 - \rho_2\hat{\boldsymbol{\rho}}_2 + c_3\rho_3\hat{\boldsymbol{\rho}}_3 = -c_1\mathbf{R}_1 + \mathbf{R}_2 - c_3\mathbf{R}_3 \tag{5.106}$$

Let us isolate the slant ranges ρ_1, ρ_2, and ρ_3 in turn by taking the dot product of this equation with appropriate vectors. To isolate ρ_1 take the dot product of each term in this equation with $\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3$, which gives

$$\begin{aligned} &c_1\rho_1\hat{\boldsymbol{\rho}}_1\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) - \rho_2\hat{\boldsymbol{\rho}}_2\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) + c_3\rho_3\hat{\boldsymbol{\rho}}_3\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \\ &\quad = -c_1\mathbf{R}_1\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) + \mathbf{R}_2\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) - c_3\mathbf{R}_3\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \end{aligned}$$

Since $\hat{\boldsymbol{\rho}}_2\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) = \hat{\boldsymbol{\rho}}_3\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) = 0$, this reduces to

$$c_1\rho_1\hat{\boldsymbol{\rho}}_1\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) = (-c_1\mathbf{R}_1 + \mathbf{R}_2 - c_3\mathbf{R}_3)\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \tag{5.107}$$

Let D_0 represent the scalar triple product of $\hat{\boldsymbol{\rho}}_1$, $\hat{\boldsymbol{\rho}}_2$, and $\hat{\boldsymbol{\rho}}_3$,

$$D_0 = \hat{\boldsymbol{\rho}}_1\cdot(\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \tag{5.108}$$

We will assume that D_0 is not zero, which means that $\hat{\boldsymbol{\rho}}_1$, $\hat{\boldsymbol{\rho}}_2$, and $\hat{\boldsymbol{\rho}}_3$ do not lie in the same plane. Then, we can solve Eqn (5.107) for ρ_1 to get

$$\rho_1 = \frac{1}{D_0}\left(-D_{11} + \frac{1}{c_1}D_{21} - \frac{c_3}{c_1}D_{31}\right) \tag{5.109a}$$

where the D's stand for the scalar triple products

$$D_{11} = \mathbf{R}_1 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \quad D_{21} = \mathbf{R}_2 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \quad D_{31} = \mathbf{R}_3 \cdot (\hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3) \tag{5.109b}$$

In a similar fashion, by taking the dot product of Eqn (5.106) with $\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3$ and then $\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2$ we obtain ρ_2 and ρ_3,

$$\rho_2 = \frac{1}{D_0}\left(-c_1 D_{12} + D_{22} - c_3 D_{32}\right) \tag{5.110a}$$

where

$$D_{12} = \mathbf{R}_1 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3) \quad D_{22} = \mathbf{R}_2 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3) \quad D_{32} = \mathbf{R}_3 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3) \tag{5.110b}$$

and

$$\rho_3 = \frac{1}{D_0}\left(-\frac{c_1}{c_3} D_{13} + \frac{1}{c_3} D_{23} - D_{33}\right) \tag{5.111a}$$

where

$$D_{13} = \mathbf{R}_1 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2) \quad D_{23} = \mathbf{R}_2 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2) \quad D_{33} = \mathbf{R}_3 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2) \tag{5.111b}$$

To obtain these results, we used the fact that $\hat{\boldsymbol{\rho}}_2 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3) = -D_0$ and $\hat{\boldsymbol{\rho}}_3 \cdot (\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2) = D_0$ (Eqn (2.42)).

Substituting Eqns (5.104) and (5.105) into Eqn (5.110a) yields the slant range ρ_2,

$$\rho_2 = A + \frac{\mu B}{r_2^3} \tag{5.112a}$$

where

$$A = \frac{1}{D_0}\left(-D_{12}\frac{\tau_3}{\tau} + D_{22} + D_{32}\frac{\tau_1}{\tau}\right) \tag{5.112b}$$

$$B = \frac{1}{6D_0}\left[D_{12}\left(\tau_3^2 - \tau^2\right)\frac{\tau_3}{\tau} + D_{32}\left(\tau^2 - \tau_1^2\right)\frac{\tau_1}{\tau}\right] \tag{5.112c}$$

On the other hand, making the same substitutions into Eqns (5.109) and (5.111) leads to the following expressions for the slant ranges ρ_1 and ρ_3:

$$\rho_1 = \frac{1}{D_0}\left[\frac{6\left(D_{31}\dfrac{\tau_1}{\tau_3} + D_{21}\dfrac{\tau}{\tau_3}\right)r_2^3 + \mu D_{31}\left(\tau^2 - \tau_1^2\right)\dfrac{\tau_1}{\tau_3}}{6r_2^3 + \mu\left(\tau^2 - \tau_3^2\right)} - D_{11}\right] \tag{5.113}$$

$$\rho_3 = \frac{1}{D_0}\left[\frac{6\left(D_{13}\dfrac{\tau_3}{\tau_1} - D_{23}\dfrac{\tau}{\tau_1}\right)r_2^3 + \mu D_{13}\left(\tau^2 - \tau_3^2\right)\dfrac{\tau_3}{\tau_1}}{6r_2^3 + \mu\left(\tau^2 - \tau_1^2\right)} - D_{33}\right] \tag{5.114}$$

Equation (5.112a) is a relation between the slant range ρ_2 and the geocentric radius r_2. Another expression relating these two variables is obtained from Eqn (5.86b),

$$\mathbf{r}_2 \cdot \mathbf{r}_2 = (\mathbf{R}_2 + \rho_2 \hat{\boldsymbol{\rho}}_2) \cdot (\mathbf{R}_2 + \rho_2 \hat{\boldsymbol{\rho}}_2)$$

or

$$r_2^2 = \rho_2^2 + 2E\rho_2 + R_2^2 \tag{5.115a}$$

where

$$E = \mathbf{R}_2 \cdot \hat{\boldsymbol{\rho}}_2 \tag{5.115b}$$

Substituting Eqn (5.112a) into Eqn (5.115a) gives

$$r_2^2 = \left(A + \frac{\mu B}{r_2^3}\right)^2 + 2C\left(A + \frac{\mu B}{r_2^3}\right) + R_2^2$$

Expanding and rearranging terms leads to an eighth-order polynomial,

$$x^8 + ax^6 + bx^3 + c = 0 \tag{5.116}$$

where $x = r_2$ and the coefficients are

$$a = -\left(A^2 + 2AE + R_2^2\right) \quad b = -2\mu B(A + E) \quad c = -\mu^2 B^2 \tag{5.117}$$

We solve Eqn (5.116) for r_2 and substitute the result into Eqns (5.112)–(5.114) to obtain the slant ranges ρ_1, ρ_2, and ρ_3. Then Eqns (5.86) yield the position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. Recall that finding $\mathbf{r}_2$ was one of our objectives.

To attain the other objective, the velocity $\mathbf{v}_2$, we first solve Eqn (5.88a) for $\mathbf{r}_2$,

$$\mathbf{r}_2 = \frac{1}{f_1}\mathbf{r}_1 - \frac{g_1}{f_1}\mathbf{v}_2$$

Substitute this result into Eqn (5.88b) to get

$$\mathbf{r}_3 = \frac{f_3}{f_1}\mathbf{r}_1 + \left(\frac{f_1 g_3 - f_3 g_1}{f_1}\right)\mathbf{v}_2$$

Solving this for $\mathbf{v}_2$ yields

$$\mathbf{v}_2 = \frac{1}{f_1 g_3 - f_3 g_1}\left(-f_3 \mathbf{r}_1 + f_1 \mathbf{r}_3\right) \tag{5.118}$$

in which we employ the approximate Lagrange functions appearing in Eqns (5.99) and (5.100).

The approximate values we have found for $\mathbf{r}_2$ and $\mathbf{v}_2$ are used as the starting point for iteratively improving the accuracy of the computed $\mathbf{r}_2$ and $\mathbf{v}_2$ until convergence is achieved. The entire step-by-step procedure is summarized in Algorithms 5.5 and 5.6 presented below. See also Appendix D.29.

ALGORITHM 5.5

The Gauss method of preliminary orbit determination. Given the direction cosine vectors $\hat{\rho}_1$, $\hat{\rho}_2$, and $\hat{\rho}_3$ and the observer's position vectors $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ at the times t_1, t_2, and t_3, compute the orbital elements.

1. Calculate the time intervals τ_1, τ_3, and τ using Eqns (5.98) and (5.103).
2. Calculate the crossproducts $\mathbf{p}_1 = \hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3$, $\mathbf{p}_2 = \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3$, and $\mathbf{p}_3 = \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2$.
3. Calculate $D_0 = \hat{\boldsymbol{\rho}}_1 \cdot \mathbf{p}_1$ (Eqn (5.108)).
4. From Eqns (5.109b), (5.110b), and (5.111b) compute the six scalar quantities

$$\begin{array}{lll} D_{11} = \mathbf{R}_1 \cdot \mathbf{p}_1 & D_{12} = \mathbf{R}_1 \cdot \mathbf{p}_2 & D_{13} = \mathbf{R}_1 \cdot \mathbf{p}_3 \\ D_{21} = \mathbf{R}_2 \cdot \mathbf{p}_1 & D_{22} = \mathbf{R}_2 \cdot \mathbf{p}_2 & D_{23} = \mathbf{R}_2 \cdot \mathbf{p}_3 \\ D_{31} = \mathbf{R}_3 \cdot \mathbf{p}_1 & D_{32} = \mathbf{R}_3 \cdot \mathbf{p}_2 & D_{33} = \mathbf{R}_3 \cdot \mathbf{p}_3 \end{array}$$

5. Calculate A and B using Eqns (5.112b) and (5.112c).
6. Calculate E, using Eqn (5.115b), and $R_2^2 = \mathbf{R}_2 \cdot \mathbf{R}_2$.
7. Calculate a, b, and c from Eqn (5.117).
8. Find the roots of Eqn (5.116) and select the most reasonable one as r_2. Newton's method can be used, in which case Eqn (3.16) becomes

$$x_{i+1} = x_i - \frac{x_i^8 + ax_i^6 + bx_i^3 + c}{8x_i^7 + 6ax_i^5 + 3bx_i^2} \tag{5.119}$$

   One must first print or graph the function $F = x^8 + ax^6 + bx^3 + c$ for $x > 0$ and choose as an initial estimate a value of x near the point where F changes sign. If there is more than one physically reasonable root, then each one must be used and the resulting orbit checked against the knowledge that may already be available about the general nature of the orbit. Alternatively, the analysis can be repeated using additional sets of observations.
9. Calculate ρ_1, ρ_2, and ρ_3 using Eqns (5.113), (5.112a), and (5.114).
10. Use Eqn (5.86) to calculate $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$.
11. Calculate the Lagrange coefficients f_1, g_1, f_3, and g_3 from Eqns (5.99) and (5.100).
12. Calculate $\mathbf{v}_2$ using Eqn (5.118).
13. **(a)** Use $\mathbf{r}_2$ and $\mathbf{v}_2$ from Steps 10 and 12 to obtain the orbital elements from Algorithm 4.2.
    (b) Alternatively, proceed to Algorithm 5.6 to improve the preliminary estimate of the orbit.

ALGORITHM 5.6

Iterative improvement of the orbit determined by Algorithm 5.5.

Use the values of $\mathbf{r}_2$ and $\mathbf{v}_2$ obtained from Algorithm 5.5 to compute the "exact" values of the f and g functions from their universal formulation as follows:

1. Calculate the magnitude of $\mathbf{r}_2 (r_2 = \sqrt{\mathbf{r}_2 \cdot \mathbf{r}_2})$ and $\mathbf{v}_2 (v_2 = \sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2})$.
2. Calculate α, the reciprocal of the semimajor axis: $\alpha = \dfrac{2}{r_2} - \dfrac{v_2^2}{\mu}$.

3. Calculate the radial component of $\mathbf{v}_2$: $v_{r_2} = \mathbf{v}_2 \cdot \mathbf{r}_2 / r_2$.
4. Use Algorithm 3.3 to solve the universal Kepler's equation (Eqn (3.49)) for the universal variables χ_1 and χ_3 at times t_1 and t_3, respectively:

$$\sqrt{\mu}\tau_1 = \frac{r_2 v_{r2}}{\sqrt{\mu}} \chi_1^2 C\left(\alpha\chi_1^2\right) + (1 - \alpha r_2)\chi_1^3 S\left(\alpha\chi_1^2\right) + r_2\chi_1$$

$$\sqrt{\mu}\tau_3 = \frac{r_2 v_{r2}}{\sqrt{\mu}} \chi_3^2 C\left(\alpha\chi_3^2\right) + (1 - \alpha r_2)\chi_3^3 S\left(\alpha\chi_3^2\right) + r_2\chi_3$$

5. Use χ_1 and χ_3 to calculate f_1, g_1, f_3, and g_3 from Eqn (3.69):

$$f_1 = 1 - \frac{\chi_1^2}{r_2} C\left(\alpha\chi_1^2\right) \quad g_1 = \tau_1 - \frac{1}{\sqrt{\mu}} \chi_1^3 S\left(\alpha\chi_1^2\right)$$

$$f_3 = 1 - \frac{\chi_3^2}{r_2} C\left(\alpha\chi_3^2\right) \quad g_3 = \tau_3 - \frac{1}{\sqrt{\mu}} \chi_3^3 S\left(\alpha\chi_3^2\right)$$

6. Use these values of f_1, g_1, f_3, and g_3 to calculate c_1 and c_3 from Eqns (5.96) and (5.97).
7. Use c_1 and c_3 to calculate updated values of ρ_1, ρ_2, and ρ_3 from Eqns (5.109)–(5.111).
8. Calculate updated $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ from Eqn (5.86).
9. Calculate updated $\mathbf{v}_2$ using Eqn (5.118) and the f and g values computed in Step 5.
10. Go back to Step 1 and repeat until, to the desired degree of precision, there is no further change in ρ_1, ρ_2, and ρ_3.
11. Use $\mathbf{r}_2$ and $\mathbf{v}_2$ to compute the orbital elements by means of Algorithm 4.2.

EXAMPLE 5.11

A tracking station is located at $\phi = 40°$ north latitude at an altitude of $H = 1$ km. Three observations of an earth satellite yield the values for the topocentric right ascension and declination listed in the Table 5.1, which also shows the local sidereal time θ of the observation site.

Use the Gauss Algorithm 5.5 to estimate the state vector at the second observation time. Recall that $\mu = 398{,}600$ km^3/s^2.

Solution

Recalling that the equatorial radius of the earth is $R_e = 6378$ km and the flattening factor is $f = 0.003353$, we substitute $\phi = 40°$, $H = 1$ km, and the given values of θ into Eqn (5.56) to obtain the inertial position vector of the tracking station at each of the three observation times.

Table 5.1 Data for Example 5.11

Observation	Time (s)	Right Ascension, α (°)	Declension, δ (°)	Local Sidereal Time, θ (°)
1	0	43.537	−8.7833	44.506
2	118.10	54.420	−12.074	45.000
3	237.58	64.318	−15.105	45.499

$$\mathbf{R}_1 = 3489.8\hat{\mathbf{I}} + 3430.2\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}} \quad \text{(km)}$$

$$\mathbf{R}_2 = 3460.1\hat{\mathbf{I}} + 3460.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}} \quad \text{(km)}$$

$$\mathbf{R}_3 = 3429.9\hat{\mathbf{I}} + 3490.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}} \quad \text{(km)}$$

Using Eqn (5.57), we compute the direction cosine vectors at each of the three observation times from the right ascension and declination data

$$\begin{aligned}
\hat{\boldsymbol{\rho}}_1 &= \cos(-8.7833^\circ)\cos 43.537^\circ\hat{\mathbf{I}} + \cos(-8.7833^\circ)\sin 43.537^\circ\hat{\mathbf{J}} + \sin(-8.7833^\circ)\hat{\mathbf{K}} \\
&= 0.71643\hat{\mathbf{I}} + 0.68074\hat{\mathbf{J}} - 0.15270\hat{\mathbf{K}} \\
\hat{\boldsymbol{\rho}}_2 &= \cos(-12.074^\circ)\cos 54.420^\circ\hat{\mathbf{I}} + \cos(-12.074^\circ)\sin 54.420^\circ\hat{\mathbf{J}} + \sin(-12.074^\circ)\hat{\mathbf{K}} \\
&= 0.56897\hat{\mathbf{I}} + 0.79531\hat{\mathbf{J}} - 0.20917\hat{\mathbf{K}} \\
\hat{\boldsymbol{\rho}}_3 &= \cos(-15.105^\circ)\cos 64.318^\circ\hat{\mathbf{I}} + \cos(-15.105^\circ)\sin 64.318^\circ\hat{\mathbf{J}} + \sin(-15.105^\circ)\hat{\mathbf{K}} \\
&= 0.41841\hat{\mathbf{I}} + 0.87007\hat{\mathbf{J}} - 0.26059\hat{\mathbf{K}}
\end{aligned}$$

We can now proceed with Algorithm 5.5.

Step 1:

$$\begin{aligned}
\tau_1 &= 0 - 118.10 = -118.10 \text{ s} \\
\tau_3 &= 237.58 - 118.10 = 119.47 \text{ s} \\
\tau &= 119.47 - (-118.1) = 237.58 \text{ s}
\end{aligned}$$

Step 2:

$$\begin{aligned}
\mathbf{p}_1 &= \hat{\boldsymbol{\rho}}_2 \times \hat{\boldsymbol{\rho}}_3 = -0.025258\hat{\mathbf{I}} + 0.060753\hat{\mathbf{J}} + 0.16229\hat{\mathbf{K}} \\
\mathbf{p}_2 &= \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3 = -0.044538\hat{\mathbf{I}} + 0.12281\hat{\mathbf{J}} + 0.33853\hat{\mathbf{K}} \\
\mathbf{p}_3 &= \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2 = -0.020950\hat{\mathbf{I}} + 0.062977\hat{\mathbf{J}} + 0.18246\hat{\mathbf{K}}
\end{aligned}$$

Step 3:

$$D_0 = \hat{\boldsymbol{\rho}}_1 \cdot \mathbf{p}_1 = -0.0015198$$

Step 4:

$$\begin{aligned}
&D_{11} = \mathbf{R}_1 \cdot \mathbf{p}_1 = 782.15 \text{ km} \quad D_{12} = \mathbf{R}_1 \cdot \mathbf{p}_2 = 1646.5 \text{ km} \quad D_{13} = \mathbf{R}_1 \cdot \mathbf{p}_3 = 887.10 \text{ km} \\
&D_{21} = \mathbf{R}_2 \cdot \mathbf{p}_1 = 784.72 \text{ km} \quad D_{22} = \mathbf{R}_2 \cdot \mathbf{p}_2 = 1651.5 \text{ km} \quad D_{23} = \mathbf{R}_2 \cdot \mathbf{p}_3 = 889.60 \text{ km} \\
&D_{31} = \mathbf{R}_3 \cdot \mathbf{p}_1 = 787.31 \text{ km} \quad D_{32} = \mathbf{R}_3 \cdot \mathbf{p}_2 = 1656.6 \text{ km} \quad D_{33} = \mathbf{R}_3 \cdot \mathbf{p}_3 = 892.13 \text{ km}
\end{aligned}$$

Step 5:

$$A = \frac{1}{-0.0015198}\left[-1646.5\frac{119.47}{237.58} + 1651.5 + 1656.6\frac{(-118.10)}{237.58}\right] = -6.6858 \text{ km}$$

$$\begin{aligned}
B = \frac{1}{6(-0.0015198)}\Bigg\{&1646.5\left(119.47^2 - 237.58^2\right)\frac{119.47}{237.58} \\
&+1656.6\left[237.58^2 - (-118.10)^2\right]\frac{(-118.10)}{237.58}\Bigg\} = 7.6667 \times 10^9 \text{ km s}^2
\end{aligned}$$

Step 6:

$$E = \mathbf{R}_2 \cdot \hat{\boldsymbol{\rho}}_2 = 3867.5 \text{ km}$$
$$R_2^2 = \mathbf{R}_2 \cdot \mathbf{R}_2 = 4.058 \times 10^7 \text{ km}^2$$

Step 7:

$$a = -\left[(-6.6858)^2 + 2(-6.6858)(3875.8) + 4.058 \times 10^7\right] = -4.0528 \times 10^7 \text{ km}^2$$
$$b = -2(389{,}600)\left(7.6667 \times 10^9\right)(-6.6858 + 3875.8) = -2.3597 \times 10^{19} \text{ km}^5$$
$$c = -(398{,}600)^2\left(7.6667 \times 10^9\right)^2 = -9.3387 \times 10^{30} \text{ km}^8$$

Step 8:

$$F(x) = x^8 - 4.0528 \times 10^7 x^6 - 2.3597 \times 10^{19} x^3 - 9.3387 \times 10^{30} = 0$$

The graph of $F(x)$ in Figure 5.15 shows that it changes sign near $x = 9000$ km. Let us use that as the starting value in Newton's method for finding the roots of $F(x)$. For the case at hand, Eqn (5.119) is

$$x_{i+1} = x_i - \frac{x_i^8 - 4.0528 \times 10^7 x_i^6 - 2.3622 \times 10^{19} x_i^3 - 9.3186 \times 10^{30}}{8x_i^7 - 2.4317 \times 10^8 x_i^5 - 7.0866 \times 10^{19} x_i^2}$$

Stepping through Newton's iterative procedure yields

$$x_0 = 9000$$
$$x_1 = 9000 - (-276.93) = 9276.9$$
$$x_2 = 9276.9 - 34.526 = 9242.4$$
$$x_3 = 9242.4 - 0.63428 = 9241.8$$
$$x_4 = 9241.8 - 0.00021048 = 9241.8$$

Thus, after four steps we converge to

$$r_2 = 9241.8 \text{ km}$$

The other roots are either negative or complex and are therefore physically unacceptable.

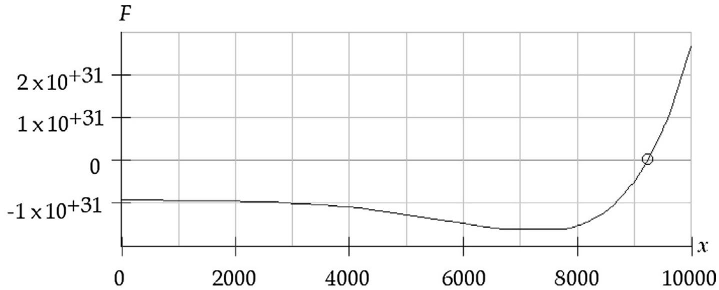

FIGURE 5.15

Graph of the polynomial $F(x)$ in Step 8.

Step 9:

$$\rho_1 = \frac{1}{-0.0015198} \times \left\{ \frac{6\left[787.31\dfrac{(-118.10)}{119.47} + 784.72\dfrac{237.58}{119.47}\right]9241.8^3 + 398{,}600\cdot 787.31\left[237.58^2 - (-118.10)^2\right]\dfrac{-118.10}{119.47}}{6\cdot 9241.8^3 + 398{,}600(237.58^2 - 119.47^2)} - 782.15 \right\}$$

$$= 3639.1 \text{ km}$$

$$\rho_2 = -6.6858 + \frac{398{,}600\cdot 7.6667\times 10^9}{9241.8^3} = 3864.8 \text{ km}$$

$$\rho_3 = \frac{1}{-0.0015198} \times \left[\frac{6\left(887.10\dfrac{119.47}{-118.10} - 889.60\dfrac{237.58}{-118.10}\right)9241.8^3 + 398{,}600\cdot 887.10\left(237.58^2 - 119.47^2\right)\dfrac{119.47}{-118.10}}{6\cdot 9241.8^3 + 398{,}600\left[237.58^2 - (-118.10)^2\right]} - 892.13 \right]$$

$$= 4172.8 \text{ km}$$

Step 10:

$$\mathbf{r}_1 = \left(3489.8\hat{\mathbf{I}} + 3430.2\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}\right) + 3639.1\left(0.71643\hat{\mathbf{I}} + 0.68074\hat{\mathbf{J}} - 0.15270\hat{\mathbf{K}}\right)$$
$$= 6096.9\hat{\mathbf{I}} + 5907.5\hat{\mathbf{J}} + 3522.9\hat{\mathbf{K}} \text{ (km)}$$
$$\mathbf{r}_2 = \left(3460.1\hat{\mathbf{I}} + 3460.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}\right) + 3864.8\left(0.56897\hat{\mathbf{I}} + 0.79531\hat{\mathbf{J}} - 0.20917\hat{\mathbf{K}}\right)$$
$$= 5659.1\hat{\mathbf{I}} + 6533.8\hat{\mathbf{J}} + 3270.1\hat{\mathbf{K}} \text{ (km)}$$
$$\mathbf{r}_3 = \left(3429.9\hat{\mathbf{I}} + 3490.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}\right) + 4172.8\left(0.41841\hat{\mathbf{I}} + 0.87007\hat{\mathbf{J}} - 0.26059\hat{\mathbf{K}}\right)$$
$$= 5175.8\hat{\mathbf{I}} + 7120.8\hat{\mathbf{J}} + 2991.1\hat{\mathbf{K}} \text{ (km)}$$

Step 11:

$$f_1 \approx 1 - \frac{1}{2}\frac{398{,}600}{9241.8^3}(-118.10)^2 = 0.99648$$
$$f_3 \approx 1 - \frac{1}{2}\frac{398{,}600}{9241.8^3}(119.47)^2 = 0.99640$$
$$g_1 \approx -118.10 - \frac{1}{6}\frac{398{,}600}{9241.8^3}(-118.10)^3 = -117.97$$
$$g_3 \approx 119.47 - \frac{1}{6}\frac{398{,}600}{9241.8^3}(119.47)^3 = 119.33$$

Step 12:

$$\mathbf{v}_2 = \frac{-0.99640\left(6096.9\hat{\mathbf{I}} + 5907.5\hat{\mathbf{J}} + 3522.9\hat{\mathbf{K}}\right) + 0.99648\left(5175.8\hat{\mathbf{I}} + 7120.8\hat{\mathbf{J}} + 2991.1\hat{\mathbf{K}}\right)}{0.99648\cdot 119.33 - 0.99640(-117.97)}$$
$$= -3.8800\hat{\mathbf{I}} + 5.1156\hat{\mathbf{J}} - 2.2397\hat{\mathbf{K}} \text{ (km/s)}$$

In summary, the state vector at time t_2 is, approximately,

$$\boxed{\begin{aligned}\mathbf{r}_2 &= 5659.1\hat{\mathbf{I}} + 6533.8\hat{\mathbf{J}} + 3270.1\hat{\mathbf{K}}\text{(km)}\\ \mathbf{v}_2 &= -3.8800\hat{\mathbf{I}} + 5.1156\hat{\mathbf{J}} - 2.2387\hat{\mathbf{K}}\text{(km/s)}\end{aligned}}$$

EXAMPLE 5.12

Starting with the state vector determined in Example 5.11, use Algorithm 5.6 to improve the vector to five significant figures.

Step 1:

$$r_2 = \|\mathbf{r}_2\| = \sqrt{5659.1^2 + 6533.8^2 + 3270.1^2} = 9241.8 \text{ km}$$

$$v_2 = \|\mathbf{v}_2\| = \sqrt{(-3.8800)^2 + 5.1156 + (-2.2397)^2} = 6.7999 \text{ km/s}$$

Step 2:

$$\alpha = \frac{2}{r_2} - \frac{v_2^2}{\mu} = \frac{2}{9241.8} - \frac{6.7999^2}{398{,}600} = 1.0154 \times 10^{-4} \text{ km}^{-1}$$

Step 3:

$$v_{r2} = \frac{\mathbf{v}_2 \cdot \mathbf{r}_2}{r_2} = \frac{(-3.8800)\cdot 5659.1 + 5.1156\cdot 6533.8 + (-2.2397)\cdot 3270.1}{9241.8} = 0.44829 \text{ km/s}$$

Step 4:

The universal Kepler's equation at times t_1 and t_3, respectively, becomes

$$\sqrt{398{,}600}\tau_1 = \frac{9241.8\cdot 0.44829}{\sqrt{398{,}600}}\chi_1^2 C\left(1.0040 \times 10^{-4}\chi_1^2\right) + \left(1 - 1.0040 \times 10^{-4}\cdot 9241.8\right)\chi_1^3 S\left(1.0040 \times 10^{-4}\chi_1^2\right) + 9241.8\chi_1$$

$$\sqrt{398{,}600}\tau_3 = \frac{9241.8\cdot 0.44829}{\sqrt{398{,}600}}\chi_3^2 C\left(1.0040 \times 10^{-4}\chi_3^2\right) + \left(1 - 1.0040 \times 10^{-4}\cdot 9241.8\right)\chi_3^3 S\left(1.0040 \times 10^{-4}\chi_3^2\right) + 9241.8\chi_3$$

or

$$631.35\tau_1 = 6.5622\chi_1^2 C(1.0040 \times 10^{-4}\chi_1^2) + 0.072085\chi_1^3 S(1.0040 \times 10^{-4}\chi_1^2) + 9241.8\chi_1$$

$$631.35\tau_3 = 6.5622\chi_3^2 C(1.0040 \times 10^{-4}\chi_3^2) + 0.072085\chi_1^3 S(1.0040 \times 10^{-4}\chi_3^2) + 9241.8\chi_3$$

Applying Algorithm 3.3 to each of these equations yields

$$\chi_1 = -8.0908\sqrt{\text{km}}$$
$$\chi_3 = 8.1375\sqrt{\text{km}}$$

Step 5:

$$f_1 = 1 - \frac{\chi_1^2}{r_2}C\left(\alpha\chi_1^2\right) = 1 - \frac{(-8.0908)^2}{9241.8}\cdot \overbrace{C\left(1.0040 \times 10^{-4}[-8.0908]^2\right)}^{0.49973} = 0.99646$$

$$g_1 = \tau_1 - \frac{1}{\sqrt{\mu}}\chi_1^3 S\left(\alpha\chi_1^2\right)$$

$$= -118.1 - \frac{1}{\sqrt{398{,}600}}(-8.0908)^3\cdot \overbrace{S\left(1.0040 \times 10^{-4}[-8.0908]^2\right)}^{0.16661} = -117.96 \text{ s}$$

and

$$f_3 = 1 - \frac{\chi_3^2}{r_2}C(\alpha\chi_3^2) = 1 - \frac{8.1375^2}{9241.8}\cdot \overbrace{C\left(1.0040\times 10^{-4}\cdot 8.1375^2\right)}^{0.49972} = 0.99642$$

$$g_3 = \tau_3 - \frac{1}{\sqrt{\mu}}\chi_3^3 S(\alpha\chi_3^2) = -118.1 - \frac{1}{\sqrt{398\ 600}}\, 8.1375^3\cdot \overbrace{S\left(1.0040\times 10^{-4}\cdot 8.1375^2\right)}^{0.16661} = 119.33$$

It turns out that the procedure converges more rapidly if the Lagrange coefficients are set equal to the average of those computed for the current step and those computed for the previous step. Thus, we set

$$f_1 = \frac{0.99648 + 0.99646}{2} = 0.99647$$

$$g_1 = \frac{-117.97 + (-117.96)}{2} = -117.96\ \text{s}$$

$$f_3 = \frac{0.99642 + 0.99641}{2} = 0.99641$$

$$g_3 = \frac{119.33 + 119.33}{2} = 119.34\ \text{s}$$

Step 6:

$$c_1 = \frac{119.33}{(0.99647)(119.33) - (0.99641)(-117.96\ \text{s})} = 0.50467$$

$$c_3 = -\frac{-117.96}{(0.99647)(119.33) - (0.99641)(-117.96)} = 0.49890$$

Step 7:

$$\rho_1 = \frac{1}{-0.001519\ 8}\left(-782.15 + \frac{1}{0.50467}784.72 - \frac{0.49890}{0.50467}787.31\right) = 3650.6\ \text{km}$$

$$\rho_2 = \frac{1}{-0.001519\ 8}(-0.50467\cdot 1646.5 + 1651.5 - 0.49890\cdot 1656.6) = 3877.2\ \text{km}$$

$$\rho_3 = \frac{1}{-0.001519\ 8}\left(-\frac{0.50467}{0.49890}887.10 + \frac{1}{0.49890}889.60 - 892.13\right) = 4186.2\ \text{km}$$

Step 8:

$$\begin{aligned}
\mathbf{r}_1 &= \left(3489.8\hat{\mathbf{I}} + 3430.2\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}\right) + 3650.6\left(0.71643\hat{\mathbf{I}} + 0.68074\hat{\mathbf{J}} - 0.15270\hat{\mathbf{K}}\right)\\
&= 6105.2\hat{\mathbf{I}} + 5915.3\hat{\mathbf{J}} + 3521.1\hat{\mathbf{K}}\quad (\text{km})\\
\mathbf{r}_2 &= \left(3460.1\hat{\mathbf{I}} + 3460.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}\right) + 3877.2\left(0.56897\hat{\mathbf{I}} + 0.79531\hat{\mathbf{J}} - 0.20917\hat{\mathbf{K}}\right)\\
&= 5666.6\hat{\mathbf{I}} + 6543.7\hat{\mathbf{J}} + 3267.5\hat{\mathbf{K}}\quad (\text{km})\\
\mathbf{r}_3 &= \left(3429.9\hat{\mathbf{I}} + 3490.1\hat{\mathbf{J}} + 4078.5\hat{\mathbf{K}}\right) + 4186.2\left(0.41841\hat{\mathbf{I}} + 0.87007\hat{\mathbf{J}} - 0.26059\hat{\mathbf{K}}\right)\\
&= 5181.4\hat{\mathbf{I}} + 7132.4\hat{\mathbf{J}} + 2987.6\hat{\mathbf{K}}\quad (\text{km})
\end{aligned}$$

Table 5.2 Key Results at Each Step of the Iterative Procedure

Step	χ_1	χ_3	f_1	g_1	f_3	g_3	ρ_1	ρ_2	ρ_3
1	−8.0908	8.1375	0.99647	−117.97	0.99641	119.33	3650.6	3877.2	4186.2
2	−8.0818	8.1282	0.99647	−117.96	0.996 42	119.33	3643.8	3869.9	4178.3
3	−8.0871	8.1337	0.99647	−117.96	0.996 42	119.33	3644.0	3870.1	4178.6
4	−8.0869	8.1336	0.99647	−117.96	0.996 42	119.33	3644.0	3870.1	4178.6

Step 9:

$$\mathbf{v}_2 = \frac{1}{0.99647\cdot 119.33 - 0.99641(-117.96)}$$

$$\times\left[-0.99641\left(6105.2\hat{\mathbf{I}} + 5915.3\hat{\mathbf{J}} + 3521.1\hat{\mathbf{K}}\right) + 0.99647\left(5181.4\hat{\mathbf{I}} + 7132.4\hat{\mathbf{J}} + 2987.6\hat{\mathbf{K}}\right)\right]$$

$$= -3.8856\hat{\mathbf{I}} + 5.1214\hat{\mathbf{J}} - 2.2434\hat{\mathbf{K}} \quad (\text{km/s})$$

This completes the first iteration.

The updated position $\mathbf{r}_2$ and velocity $\mathbf{v}_2$ are used to repeat the procedure beginning at Step 1. The results of the first and subsequent iterations are shown in Table 5.2. Convergence to five significant figures in the slant ranges ρ_1, ρ_2, and ρ_3 occurs in four steps, at which point the state vector is

$$\mathbf{r}_2 = 5662.1\hat{\mathbf{I}} + 6538.0\hat{\mathbf{J}} + 3269.0\hat{\mathbf{K}} \quad (\text{km})$$
$$\mathbf{v}_2 = -3.8856\hat{\mathbf{I}} + 5.1214\hat{\mathbf{J}} - 2.2433\hat{\mathbf{K}} \quad (\text{km/s})$$

Using $\mathbf{r}_2$ and $\mathbf{v}_2$ in Algorithm 4.2, we find that the orbital elements are

$$\begin{aligned} a &= 10{,}000 \text{ km} \quad \left(h = 62{,}818 \text{ km}^2/\text{s}\right) \\ e &= 0.1000 \\ i &= 30^\circ \\ \Omega &= 270^\circ \\ \omega &= 90^\circ \\ \theta &= 45.01^\circ \end{aligned}$$

PROBLEMS

Section 5.2

5.1 The geocentric equatorial position vectors of a satellite at three separate times are

$$\mathbf{r}_1 = 5887\hat{\mathbf{I}} - 3520\hat{\mathbf{J}} - 1204\hat{\mathbf{K}} \quad (\text{km})$$
$$\mathbf{r}_2 = 5572\hat{\mathbf{I}} - 3457\hat{\mathbf{J}} - 2376\hat{\mathbf{K}} \quad (\text{km})$$
$$\mathbf{r}_3 = 5088\hat{\mathbf{I}} - 3289\hat{\mathbf{J}} - 3480\hat{\mathbf{K}} \quad (\text{km})$$

Use Gibbs method to find $\mathbf{v}_2$.

{Partial Ans.: $v_2 = 7.59$ km/s}

5.2 Calculate the orbital elements and perigee altitude of the space object in the previous problem.

{Partial Ans.: $z_p = 567$ km}

Section 5.3

5.3 At a given instant, the altitude of an earth satellite is 400 km. Thirty minutes later, the altitude is 1000 km, and the true anomaly has increased by 120°. Find the perigee altitude.

{Ans.: 270.4 km}

5.4 At a given instant, the geocentric equatorial position vector of an earth satellite is

$$\mathbf{r}_1 = 3600\hat{\mathbf{I}} + 4600\hat{\mathbf{J}} + 3600\hat{\mathbf{K}} \quad \text{(km)}$$

Thirty minutes later, the position is

$$\mathbf{r}_2 = -5500\hat{\mathbf{I}} + 6240\hat{\mathbf{J}} - 5200\hat{\mathbf{K}} \quad \text{(km)}$$

Find the specific energy of the orbit.
{Ans.: -19.871 (km/s)^2}

5.5 Compute the perigee altitude and the inclination of the orbit in the previous problem.

{Ans.: 483.59 km, 44.17°}

5.6 At a given instant, the geocentric equatorial position vector of an earth satellite is

$$\mathbf{r}_1 = 5644\hat{\mathbf{I}} - 2830\hat{\mathbf{J}} - 4170\hat{\mathbf{K}} \quad \text{(km)}$$

Twenty minutes later, the position is

$$\mathbf{r}_2 = -2240\hat{\mathbf{I}} + 7320\hat{\mathbf{J}} - 4980\hat{\mathbf{K}} \quad \text{(km)}$$

Calculate $\mathbf{v}_1$ and $\mathbf{v}_2$.
{Partial Ans.: $v_1 = 10.84$ km/s, $v_2 = 9.970$ km/s}

5.7 Compute the orbital elements and perigee altitude for the previous problem.

{Partial Ans.: $z_p = 224$ km}

Section 5.4

5.8 Calculate the JD number for the following epochs:
(a) 5:30 UT on August 14, 1914.
(b) 14:00 UT on April 18, 1946.
(c) 0:00 UT on September 1, 2010.
(d) 12:00 UT on October 16, 2007.
(e) Noon today, your local time.

{Ans.: (a) 2,420,358.729, (b) 2,431,929.083, (c) 2,455,440.500, (d) 2,454,390.000}

5.9 Calculate the number of days from 12:00 UT on your date of birth to 12:00 UT on today's date.

5.10 Calculate the local sidereal time (in degrees) at

(a) Stockholm, Sweden (east longitude 18°03′) at 12:00 UT on January 1, 2008.
(b) Melbourne, Australia (east longitude 144°58′) at 10:00 UT on December 21, 2007.
(c) Los Angeles, California (west longitude 118°15′) at 20:00 UT on July 4, 2005.
(d) Rio de Janeiro, Brazil (west longitude 43°06′) at 3:00 UT on February 15, 2006.
(e) Vladivostok, Russia (east longitude 131°56′) at 8:00 UT on March 21, 2006.
(f) At noon today, your local time and place.

{Ans.: (a) 298.6°, (b) 24.6°, (c) 104.7°, (d) 146.9°, (e) 70.6°}

Section 5.8

5.11 Relative to a tracking station whose local sidereal time is 117° and latitude is +51°, the azimuth and elevation angle of a satellite are 27.5156 and 67.5556°, respectively. Calculate the topocentric right ascension and declination of the satellite.

{Ans.: $\alpha = 145.3°$, $\delta = 68.24°$}

5.12 A sea-level tracking station whose local sidereal time is 40° and latitude is 35° makes the following observations of a space object:

Azimuth: 36.0°.
Azimuth rate: 0.590°/s.
Elevation: 36.6°.
Elevation rate: −0.263°/s.
Range: 988 km.
Range rate: 4.86 km/s.
What is the state vector of the object?
{Partial Ans.: $r = 7003.3$ km, $v = 10.922$ km/s}

5.13 Calculate the orbital elements of the satellite in the previous problem.

{Partial Ans.: $e = 1.1$, $i = 40°$}

5.14 A tracking station at latitude −20° and elevation 500 m makes the following observations of a satellite at the given times.

Time (min)	Local Sidereal Time (°)	Azimuth (°)	Elevation Angle (°)	Range (km)
0	60.0	165.931	9.53549	1214.89
2	60.5014	145.967	45.7711	421.441
4	61.0027	2.40962	21.8825	732.079

Use the Gibbs method to calculate the state vector of the satellite at the central observation time.
{Partial Ans.: $r_2 = 6684$ km, $v_2 = 7.7239$ km/s}

5.15 Calculate the orbital elements of the satellite in the previous problem.

{Partial Ans.: $e = 0.001$, $i = 95°$}

Section 5.10

5.16 A sea-level tracking station at latitude +29° makes the following observations of a satellite at the given times.

Time (min)	Local Sidereal Time (°)	Topocentric Right Ascension (°)	Topocentric Declination (°)
0.0	0	0	51.5110
1.0	0.250684	65.9279	27.9911
2.0	0.501369	79.8500	14.6609

Use the Gauss method without iterative improvement to estimate the state vector of the satellite at the middle observation time.
{Partial Ans.: $r = 6700.9$ km, $v = 8.0757$ km/s}

5.17 Refine the estimate in the previous problem using iterative improvement.

{Partial Ans.: $r = 6701.5$ km, $v = 8.0881$ km/s}

5.18 Calculate the orbital elements from the state vector obtained in the previous problem.
{Partial Ans.: $e = 0.10$, $i = 30°$}

5.19 A sea-level tracking station at latitude +29° makes the following observations of a satellite at the given times.

Time (min)	Local Sidereal Time (°)	Topocentric Right Ascension (°)	Topocentric Declination (°)
0.0	90	15.0394	20.7487
1.0	90.2507	25.7539	30.1410
2.0	90.5014	48.6055	43.8910

Use the Gauss method without iterative improvement to estimate the state vector of the satellite.
{Partial Ans.: $r = 6999.1$ km, $v = 7.5541$ km/s}

5.20 Refine the estimate in the previous problem using iterative improvement.

{Partial Ans.: $r = 7000.0$ km, $v = 7.5638$ km/s}

5.21 Calculate the orbital elements from the state vector obtained in the previous problem.
{Partial Ans.: $e = 0.0048$, $i = 31°$}

5.22 The position vector $\mathbf{R}$ of a tracking station and the direction cosine vector $\hat{\boldsymbol{\rho}}$ of a satellite relative to the tracking station at three times are as follows:

$$t_1 = 0 \text{ min}$$
$$\mathbf{R}_1 = -1825.96\hat{\mathbf{I}} + 3583.66\hat{\mathbf{J}} + 4933.54\hat{\mathbf{K}} \quad \text{(km)}$$
$$\hat{\boldsymbol{\rho}}_1 = -0.301687\hat{\mathbf{I}} + 0.200673\hat{\mathbf{J}} + 0.932049\hat{\mathbf{K}}$$
$$t_2 = 1 \text{ min}$$
$$\mathbf{R}_2 = -1816.30\hat{\mathbf{I}} + 3575.63\hat{\mathbf{J}} + 4933.54\hat{\mathbf{K}} \quad \text{(km)}$$
$$\hat{\boldsymbol{\rho}}_2 = -0.793090\hat{\mathbf{I}} - 0.210324\hat{\mathbf{J}} + 0.571640\hat{\mathbf{K}}$$
$$t_3 = 2 \text{ min}$$
$$\mathbf{R}_3 = -1857.25\hat{\mathbf{I}} + 3567.54\hat{\mathbf{J}} + 4933.54\hat{\mathbf{K}} \quad \text{(km)}$$
$$\hat{\boldsymbol{\rho}}_3 = -0.873085\hat{\mathbf{I}} - 0.362969\hat{\mathbf{J}} + 0.325539\hat{\mathbf{K}}$$

Use the Gauss method without iterative improvement to estimate the state vector of the satellite at the central observation time.
{Partial Ans.: $r = 6742.3$ km, $v = 7.6799$ km/s}

5.23 Refine the estimate in the previous problem using iterative improvement.

{Partial Ans.: $r = 6743.0$ km, $v = 7.6922$ km/s}

5.24 Calculate the orbital elements from the state vector obtained in the previous problem.

{Partial Ans.: $e = 0.001$, $i = 52°$}

5.25 A tracking station at latitude 60°N and 500-m elevation obtains the following data:

Time (min)	Local Sidereal Time (°)	Topocentric Right Ascension (°)	Topocentric Declination (°)
0.0	150	157.783	24.2403
5.0	151.253	159.221	27.2993
10.0	152.507	160.526	29.8982

Use the Gauss method without iterative improvement to estimate the state vector of the satellite.
{Partial Ans.: $r = 25{,}132$ km, $v = 6.0588$ km/s}

5.26 Refine the estimate in the previous problem using iterative improvement.

{Partial Ans.: $r = 25{,}169$ km, $v = 6.0671$ km/s}

5.27 Calculate the orbital elements from the state vector obtained in the previous problem.

{Partial Ans.: $e = 1.09$, $i = 63°$}

5.28 The position vector $\mathbf{R}$ of a tracking station and the direction cosine vector $\hat{\boldsymbol{\rho}}$ of a satellite relative to the tracking station at three times are as follows:

$$\begin{aligned}
t_1 &= 0 \ \text{min} \\
\mathbf{R}_1 &= 5582.84\hat{\mathbf{I}} + 3073.90\hat{\mathbf{K}} \ \ (\text{km}) \\
\hat{\boldsymbol{\rho}}_1 &= 0.846\,428\hat{\mathbf{I}} + 0.532\,504\hat{\mathbf{K}} \\
t_2 &= 5 \ \text{min} \\
\mathbf{R}_2 &= 5581.50\hat{\mathbf{I}} + 122.122\hat{\mathbf{J}} + 3073.90\hat{\mathbf{K}} \quad (\text{km}) \\
\hat{\boldsymbol{\rho}}_2 &= 0.749290\hat{\mathbf{I}} + 0.463023\hat{\mathbf{J}} + 0.473470\hat{\mathbf{K}} \\
t_3 &= 10 \ \text{min} \\
\mathbf{R}_3 &= 5577.50\hat{\mathbf{I}} + 244.186\hat{\mathbf{J}} + 3073.90\hat{\mathbf{K}} \quad (\text{km}) \\
\hat{\boldsymbol{\rho}}_3 &= 0.529447\hat{\mathbf{I}} + 0.777163\hat{\mathbf{J}} + 0.340152\hat{\mathbf{K}}
\end{aligned}$$

Use the Gauss method without iterative improvement to estimate the state vector of the satellite. {Partial Ans.: $r = 9729.6$ km, $v = 6.0234$ km/s}

5.29 Refine the estimate in the previous problem using iterative improvement.

{Partial Ans.: $r = 9759.8$ km, $v = 6.0713$ km/s}

5.30 Calculate the orbital elements from the state vector obtained in the previous problem.

{Partial Ans.: $e = 0.1$, $i = 30°$}