CHAPTER

The Two-Body Problem

2

CHAPTER OUTLINE

2.1 Introduction

This chapter presents the vector-based approach to the classical problem of determining the motion of two bodies due solely to their own mutual gravitational attraction. We show that the path of one of the masses relative to the other is a conic section (circle, ellipse, parabola, or hyperbola) whose shape is determined by the eccentricity. Several fundamental properties of the different types of orbits are developed with the aid of the laws of conservation of angular momentum and energy. These properties

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include the period of elliptical orbits, the escape velocity associated with parabolic paths, and the characteristic energy of hyperbolic trajectories. Following the presentation of the four types of orbits, the perifocal frame is introduced. This frame of reference is used to describe orbits in three dimensions, which is the subject of Chapter 4.

In this chapter, the perifocal frame provides the backdrop for developing the Lagrange f and g coefficients. By means of the Lagrange f and g coefficients, the position and velocity on a trajectory can be found in terms of the position and velocity at an initial time. These functions are needed in the orbit determination algorithms of Lambert and Gauss presented in Chapter 5.

The chapter concludes with a discussion of the restricted three-body problem in order to provide a basis for understanding the concepts of Lagrange points and the Jacobi constant. This material is optional.

In studying this chapter, it would be well from time to time to review the road map provided in Appendix B.

2.2 Equations of motion in an inertial frame

Figure 2.1 shows two-point masses acted upon only by the mutual force of gravity between them. The positions $\mathbf{R}_1$ and $\mathbf{R}_2$ of their centers of mass are shown relative to an inertial frame of reference XYZ. In terms of the coordinates of the two points

$$\begin{aligned}\mathbf{R}_1 &= X_1\hat{\mathbf{I}} + Y_1\hat{\mathbf{J}} + Z_1\hat{\mathbf{K}} \\ \mathbf{R}_2 &= X_2\hat{\mathbf{I}} + Y_2\hat{\mathbf{J}} + Z_2\hat{\mathbf{K}}\end{aligned} \tag{2.1}$$

The origin O of the inertial frame may move with a constant velocity (relative to the fixed stars), but the axes do not rotate. Each of the two bodies is acted upon by the gravitational attraction of the other. $\mathbf{F}_{12}$ is the force exerted on m_1 by m_2, and $\mathbf{F}_{21}$ is the force exerted on m_2 by m_1.

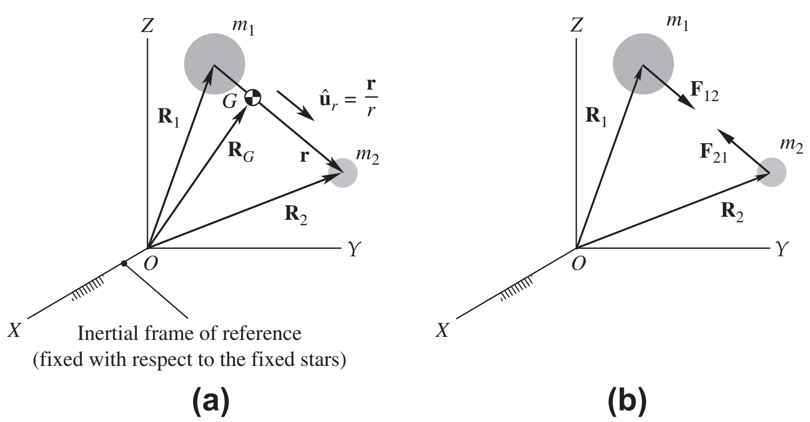


FIGURE 2.1

(a) Two masses located in an inertial frame. (b) Free-Body diagrams.

The position vector $\mathbf{R}_G$ of the center of mass G of the system in Figure 2.1(a) is defined by the formula

$$\mathbf{R}_G = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2} \tag{2.2}$$

Therefore, the absolute velocity and the absolute acceleration of G are

$$\mathbf{v}_G = \dot{\mathbf{R}}_G = \frac{m_1\dot{\mathbf{R}}_1 + m_2\dot{\mathbf{R}}_2}{m_1 + m_2} \tag{2.3}$$

$$\mathbf{a}_G = \ddot{\mathbf{R}}_G = \frac{m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2}{m_1 + m_2} \tag{2.4}$$

The adjective "absolute" means that the quantities are measured relative to an inertial frame of reference.

Let $\mathbf{r}$ be the position vector of m_2 relative to m_1. Then,

$$\mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 \tag{2.5}$$

Or, using Eqn (2.1),

$$\mathbf{r} = (X_2 - X_1)\hat{\mathbf{I}} + (Y_2 - Y_1)\hat{\mathbf{J}} + (Z_2 - Z_1)\hat{\mathbf{K}} \tag{2.6}$$

Furthermore, let $\hat{\mathbf{u}}_r$ be the unit vector pointing from m_1 toward m_2, so that

$$\hat{\mathbf{u}}_r = \frac{\mathbf{r}}{r} \tag{2.7}$$

where r is the magnitude of $\mathbf{r}$,

$$r = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \tag{2.8}$$

The body m_1 is acted upon only by the force of gravitational attraction toward m_2. The force of gravitational attraction, F_g, which acts along the line joining the centers of mass of m_1 and m_2, is given by Eqn (1.40). Therefore, the force exerted on m_1 by m_2 is

$$\mathbf{F}_{12} = \frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r \tag{2.9}$$

where $\hat{\mathbf{u}}_r$ accounts for the fact that the force vector $\mathbf{F}_{12}$ is directed from m_1 toward m_2. (Do not confuse the symbol G, used in this context to represent the universal gravitational constant, with its use elsewhere in the book to denote the center of mass.) By Newton's third law (the action–reaction principle), the force $\mathbf{F}_{21}$ exerted on m_2 by m_1 is $-\mathbf{F}_{12}$, so that

$$\mathbf{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r \tag{2.10}$$

Newton's second law of motion as applied to a body m_1 is $\mathbf{F}_{12} = m_1\ddot{\mathbf{R}}_1$, where $\ddot{\mathbf{R}}_1$ is the absolute acceleration of m_1. Combining this with Newton's law of gravitation (Eqn (2.9)) yields

$$m_1\ddot{\mathbf{R}}_1 = \frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r \tag{2.11}$$

Likewise, by substituting $\mathbf{F}_{21} = m_2\ddot{\mathbf{R}}_2$ into Eqn (2.10) we get

$$m_2\ddot{\mathbf{R}}_2 = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r \tag{2.12}$$

It is apparent upon forming the sum of Eqns (2.11) and (2.12) that $m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2 = 0$. According to Eqn (2.4), this means that the acceleration of the center of mass G of the system of two bodies m_1 and m_2 is zero. Therefore, as is true for any system that is free of external forces, G moves in a straight line through space with a constant velocity $\mathbf{v}_G$. Its position vector relative to XYZ is given by

$$\mathbf{R}_G = \mathbf{R}_{G_0} + \mathbf{v}_G t \tag{2.13}$$

where $\mathbf{R}_{G_0}$ is the position of G at time $t = 0$. The nonaccelerating center of mass of a two-body system may serve as the origin of an inertial frame.

EXAMPLE 2.1

Use the two-body equations of motion to show why orbiting astronauts experience weightlessness.

Solution

We sense weight by feeling the contact forces that develop wherever our body is supported. Consider an astronaut of mass m_A strapped into a spacecraft of mass m_S, in orbit about the earth. The distance between the center of the earth and the spacecraft is r, and the mass of the earth is M_E. Since the only external force is that of gravity, $\mathbf{F}_S)_g$, the equation of motion of the spacecraft is

$$\mathbf{F}_S)_g = m_S\mathbf{a}_S \tag{a}$$

where $\mathbf{a}_S$ is measured in an inertial frame. According to Eqn (2.6),

$$\mathbf{F}_S)_g = -\frac{GM_E m_S}{r^2}\hat{\mathbf{u}}_r \tag{b}$$

in which $\hat{\mathbf{u}}_r$ is the unit vector pointing outward from the earth toward the orbiting spacecraft. Thus, Eqns (a) and (b) imply that the absolute acceleration of the spacecraft is

$$\mathbf{a}_S = -\frac{GM_E}{r^2}\hat{\mathbf{u}}_r \tag{c}$$

The equation of motion of the astronaut is

$$\mathbf{F}_A)_g + C_A = m_A\mathbf{a}_A \tag{d}$$

In this expression $\mathbf{F}_A)_g$ is the force of gravity on (i.e., the weight of) the astronaut, $\mathbf{C}_A$ is the net contact force on the astronaut from restraints (e.g., seat, seat belt), and $\mathbf{a}_A$ is the astronaut's absolute acceleration. According to Eqn (2.6),

$$\mathbf{F}_A)_g = -\frac{GM_E m_A}{r^2}\hat{\mathbf{u}}_r \tag{e}$$

Since the astronaut is moving with the spacecraft, we have, noting Eqn (c),

$$\mathbf{a}_A = \mathbf{a}_S = -\frac{GM_E}{r^2}\hat{\mathbf{u}}_r \tag{f}$$

Substituting Eqns (e) and (f) into Eqn (d) yields

$$-\frac{GM_E m_A}{r^2}\hat{\mathbf{u}}_r + C_A = m_A\left(-\frac{GM_E}{r^2}\hat{\mathbf{u}}_r\right)$$

from which it is clear that

$$\boxed{\mathbf{C}_A = 0}$$

The net contact force on the astronaut is zero. With no reaction to the force of gravity exerted on the body, there is no sensation of weight.

The potential energy V of the gravitational force $\mathbf{F}$ between two point masses m_1 and m_2 separated by a distance r is given by

$$V = -\frac{Gm_1m_2}{r} \tag{2.14}$$

A conservative force like gravity can be obtained from its potential energy function V by means of the gradient operator,

$$\mathbf{F} = -\nabla V \tag{2.15}$$

where, in Cartesian coordinates,

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \tag{2.16}$$

For the two-body system in Figure 2.1 we have, by combining Eqns (2.8) and (2.14),

$$V = -\frac{Gm_1m_2}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}} \tag{2.17}$$

The attractive forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ in Eqn (2.6) are derived from Eqn (2.17) as follow:

$$\mathbf{F}_{12} = -\left(\frac{\partial V}{\partial X_2}\hat{\mathbf{i}} + \frac{\partial V}{\partial Y_2}\hat{\mathbf{j}} + \frac{\partial V}{\partial Z_2}\hat{\mathbf{k}}\right)$$

$$\mathbf{F}_{21} = -\left(\frac{\partial V}{\partial X_1}\hat{\mathbf{i}} + \frac{\partial V}{\partial Y_1}\hat{\mathbf{j}} + \frac{\partial V}{\partial Z_1}\hat{\mathbf{k}}\right)$$

In Appendix E, it is shown that the gravitational potential, V, and hence the gravitational force outside of a sphere with a spherically symmetric mass distribution M is the same as that of a point mass M located at the center of the sphere. Therefore, the two-body problem applies not only to point masses but also to spherical bodies (as long as, of course, they do not come into contact!).

Let us return to Eqn (2.7), the equations of motion of the two-body system relative to the XYZ inertial frame. We can divide m_1 out of Eqn (2.11) and m_2 out of Eqn (2.12) and then substitute Eqn (2.7) into both results to obtain

$$\ddot{\mathbf{R}}_1 = Gm_2\frac{\mathbf{r}}{r^3} \tag{2.18a}$$

$$\ddot{\mathbf{R}}_2 = -Gm_1\frac{\mathbf{r}}{r^3} \tag{2.18b}$$

These are the final forms of the equations of motion of the two bodies in inertial space. With the aid of Eqns (2.1), (2.6), and (2.8) we can express these equations in terms of the components of the position and acceleration vectors in the inertial XYZ frame:

$$\ddot{X}_1 = Gm_2\frac{X_2 - X_1}{r^3} \quad \ddot{Y}_1 = Gm_2\frac{Y_2 - Y_1}{r^3} \quad \ddot{Z}_1 = Gm_2\frac{Z_2 - Z_1}{r^3} \tag{2.19a}$$

$$\ddot{X}_2 = Gm_1\frac{X_1 - X_2}{r^3} \quad \ddot{Y}_2 = Gm_1\frac{Y_1 - Y_2}{r^3} \quad \ddot{Z}_2 = Gm_1\frac{Z_1 - Z_2}{r^3} \tag{2.19b}$$

where $r = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}$.

The position vector **R** and velocity vector **V** of a particle are referred to collectively as its state vector. The fundamental problem before us is to find the state vectors of both particles of the two-body system at a given time given the state vectors at an initial time. The numerical solution procedure is outlined in Algorithm 2.1.

ALGORITHM 2.1

Numerically compute the state vectors $\mathbf{R}_1,\mathbf{V}_1$ and $\mathbf{R}_2,\mathbf{V}_2$ of the two-body system as a function of time, given their initial values $\mathbf{R}_1^0, \mathbf{V}_1^0$ and $\mathbf{R}_2^0, \mathbf{V}_2^0$. This algorithm is implemented in MATLAB as the function *twobody3d.m*, which is listed in Appendix D.5.

1. Form the vector consisting of the components of the state vectors at time t_0,

$$\mathbf{y}_0 = \lfloor X_1^0 \quad Y_1^0 \quad Z_1^0 \quad X_2^0 \quad Y_2^0 \quad Z_2^0 \quad \dot{X}_1^0 \quad \dot{Y}_1^0 \quad \dot{Z}_1^0 \quad \dot{X}_2^0 \quad \dot{Y}_2^0 \quad \dot{Z}_2^0 \rfloor$$

2. Provide $\mathbf{y}_0$ and the final time t_f to Algorithm 1.1, 1.2, or 1.3, along with the vector that comprises the components are the state vector derivatives

$$\mathbf{f}(t, \mathbf{y}) = \lfloor \dot{X}_1 \quad \dot{Y}_1 \quad \dot{Z}_1 \quad \dot{X}_2 \quad \dot{Y}_2 \quad \dot{Z}_2 \quad \ddot{X}_1 \quad \ddot{Y}_1 \quad \ddot{Z}_1 \quad \ddot{X}_2 \quad \ddot{Y}_2 \quad \ddot{Z}_2 \rfloor$$

where the last six components, the accelerations, are given by Eqns (2.19).

3. The selected algorithm solves the system $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ for the system state vector

$$\mathbf{y} = \lfloor X_1 \quad Y_1 \quad Z_1 \quad X_2 \quad Y_2 \quad Z_2 \quad \dot{X}_1 \quad \dot{Y}_1 \quad \dot{Z}_1 \quad \dot{X}_2 \quad \dot{Y}_2 \quad \dot{Z}_2 \rfloor$$

at n discrete times t_n from t_0 through t_f.

4. The state vectors of m_1 and m_2 at the discrete times are

$$\begin{aligned} \mathbf{R}_1 &= X_1\hat{\mathbf{I}} + Y_1\hat{\mathbf{J}} + Z_1\hat{\mathbf{K}} \quad & \mathbf{V}_1 &= \dot{X}_1\hat{\mathbf{I}} + \dot{Y}_1\hat{\mathbf{J}} + \dot{Z}_1\hat{\mathbf{K}} \\ \mathbf{R}_2 &= X_2\hat{\mathbf{I}} + Y_2\hat{\mathbf{J}} + Z_2\hat{\mathbf{K}} \quad & \mathbf{V}_2 &= \dot{X}_2\hat{\mathbf{I}} + \dot{Y}_2\hat{\mathbf{J}} + \dot{Z}_2\hat{\mathbf{K}} \end{aligned}$$

EXAMPLE 2.2

A system consists of two massive bodies m_1 and m_2 each having a mass of 10^{26} kg. At time $t = 0$ the state vectors of the two particles in an inertial frame are

$$\begin{aligned} \mathbf{R}_1^{(0)} &= \mathbf{0} \quad & \mathbf{V}_1^{(0)} &= 10\hat{\mathbf{I}} + 20\hat{\mathbf{J}} + 30\hat{\mathbf{K}} \text{ (km/s)} \\ \mathbf{R}_2^{(0)} &= 3000\hat{\mathbf{I}} \text{ (km)} \quad & \mathbf{V}_2^{(0)} &= 40\hat{\mathbf{J}} \text{ (km/s)} \end{aligned}$$

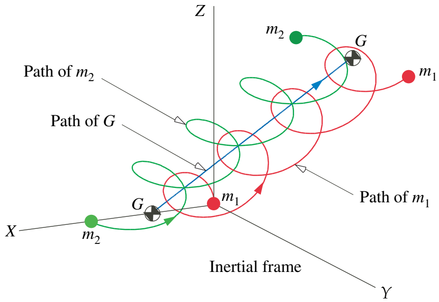


FIGURE 2.2

The motion of two identical bodies acted on only by their mutual gravitational attraction, as viewed from the inertial frame of reference.

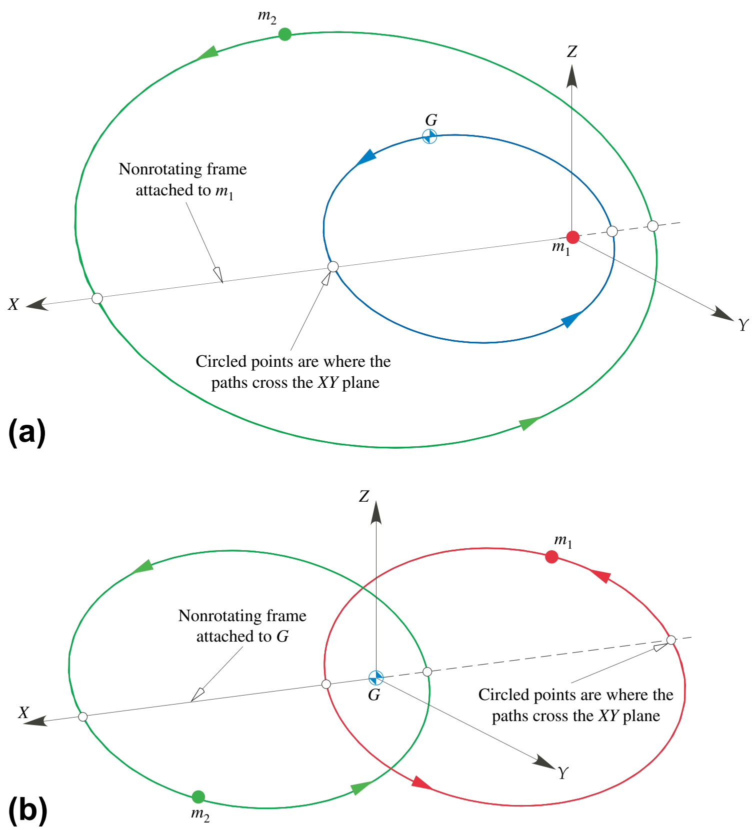


FIGURE 2.3

The Motion in Figure 2.2: (a) As viewed relative to m_1 (or m_2); (b) As viewed from the center of mass.

Use Algorithm 2.1 and the RKF4(5) method (Algorithm 1.3) to numerically determine the motion of the two masses due solely to their mutual gravitational attraction from $t=0$ to $t=480$ s.

(a) Plot the motion of m_1 and m_2 relative to the inertial frame.
(b) Plot the motion of m_2 and G relative to m_1.
(c) Plot the motion of m_1 and m_2 relative to the center of mass G of the system.

Solution

The MATLAB function *twobody3d.m* in Appendix D.5 contains within it the data for this problem. Embedded in the program are the subfunction *rates*, which compute the accelerations given by Eqns (2.19). *twobody3d.m* uses the solution vector from *rkf45.m* to plot Figures 2.2 and 2.3, which summarize the results requested in the problem statement.

In answer to part (a), Figure 2.2 shows the motion of the two-body system relative to the inertial frame. m_1 and m_2 are soon established in a periodic helical motion around the straight-line trajectory of the center of mass G through space. This pattern continues indefinitely.

Figure 2.3(a) relates to part (b) of the problem. The very same motion appears rather less complex when viewed from m_1. In fact we see that $\mathbf{R}_2(t)-\mathbf{R}_1(t)$, the trajectory of m_2 relative to m_1, appears to be an elliptical path. So does $\mathbf{R}_G(t)-\mathbf{R}_1(t)$, the path of the center of mass around m_1.

Finally, for part (c) of the problem, Figure 2.3(b) reveals that both m_1 and m_2 follow apparently elliptical paths around the center of mass.

One may wonder what the motion looks like if there are more than two bodies moving only under the influence of their mutual gravitational attraction. The n-body problem with $n>2$ has no closed form solution, which is complex and chaotic in nature. The three-body problem is briefly addressed in Appendix C, where the equations of motion of the system are presented. Appendix C lists the MATLAB program *threebody.m* that is used to solve the equations of motion for given initial conditions. Figure 2.4 shows the results for three particles of equal mass, equally spaced initially along the X-axis of an inertial frame. The central mass has an initial velocity in the XY plane, while the other two are at rest. As time progresses, we see no periodic behavior as was evident in the two-body motion in Figure 2.2. The chaos is more obvious if the motion is viewed from the center of mass of the three-body system, as shown in Figure 2.5. The computer simulation reveals that the masses all eventually collide.

2.3 Equations of relative motion

Let us differentiate Eqn (2.5) twice with respect to time in order to obtain the relative acceleration vector,

$$\ddot{\mathbf{r}}=\ddot{\mathbf{R}}_2-\ddot{\mathbf{R}}_1$$

Substituting Eqn (2.9) into the right side of this expression yields

$$\ddot{\mathbf{r}}=-\frac{G(m_1+m_2)}{r^2}\hat{\mathbf{u}}_r \tag{2.20}$$

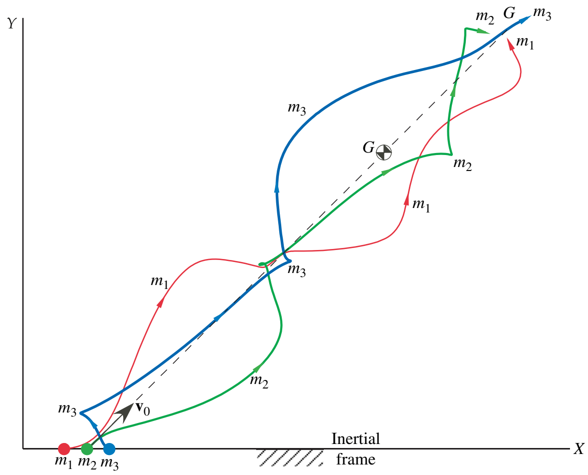


FIGURE 2.4

The motion of three identical masses as seen from the inertial frame in which m_1 and m_3 are initially at rest, while m_2 has an initial velocity $\mathbf{v}_0$ directed upward and to the right, as shown.

The gravitational parameter μ is defined as

$$\mu = G(m_1 + m_2) \tag{2.21}$$

The units of μ are cubic kilometers per square second. Using Eqn (2.21) together with Eqn (2.5), we can write Eqn (2.20) as

$$\boxed{\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}} \tag{2.22}$$

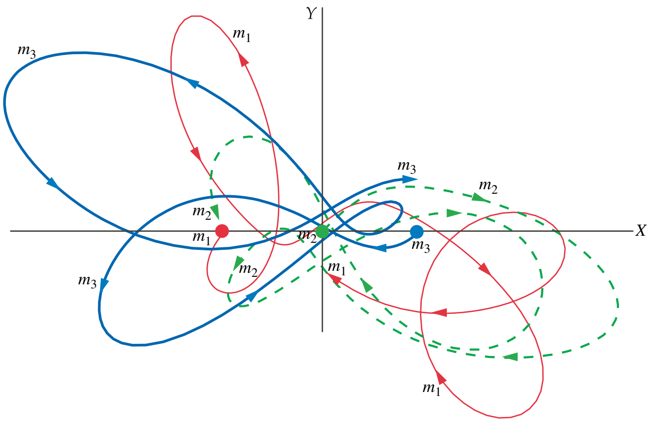


FIGURE 2.5

The same motion as in Figure 2.4, as viewed from the inertial frame attached to the center of mass G.

This nonlinear second-order differential equation governs the motion of m_2 relative to m_1. It has two vector constants of integration, each having three scalar components. Therefore, Eqn (2.22) has six constants of integration. Note that interchanging the roles of m_1 and m_2 amounts to simply multiplying Eqn (2.22) through by -1, which, of course, changes nothing. Thus, the motion of m_2 as seen from m_1 is precisely the same as the motion of m_1 as seen from m_2. The motion of the moon as observed from the earth appears the same as that of the earth as viewed from the moon.

The relative position vector $\mathbf{r}$ in Eqn (2.22) was originally defined in the inertial frame (Eqn (2.6)). It is convenient, however, to measure the components of $\mathbf{r}$ in a frame of reference attached to and moving with m_1. In a comoving reference frame, such as the xyz system illustrated in Figure 2.6, $\mathbf{r}$ has the expression

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

The relative velocity $\dot{\mathbf{r}}_{\text{rel}}$ and acceleration $\ddot{\mathbf{r}}_{\text{rel}}$ in the comoving frame are found by simply taking the derivatives of the coefficients of the unit vectors, which themselves are fixed in the moving xyz system. Thus,

$$\dot{\mathbf{r}}_{\text{rel}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \quad \ddot{\mathbf{r}}_{\text{rel}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}$$

From Eqn (1.77), we know that the relationship between absolute acceleration $\ddot{\mathbf{r}}$ and relative acceleration $\ddot{\mathbf{r}}_{\text{rel}}$ is

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{\text{rel}} + \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \ddot{\mathbf{r}}_{\text{rel}}$$

where $\boldsymbol{\Omega}$ and $\dot{\boldsymbol{\Omega}}$ are the absolute angular velocity and angular acceleration of the moving frame of reference. Thus, $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{\text{rel}}$ only if $\boldsymbol{\Omega} = \dot{\boldsymbol{\Omega}} = \mathbf{0}$. That is to say, the relative acceleration may be used on the left of Eqn (2.22) as long as the comoving frame in which it is measured is not rotating.

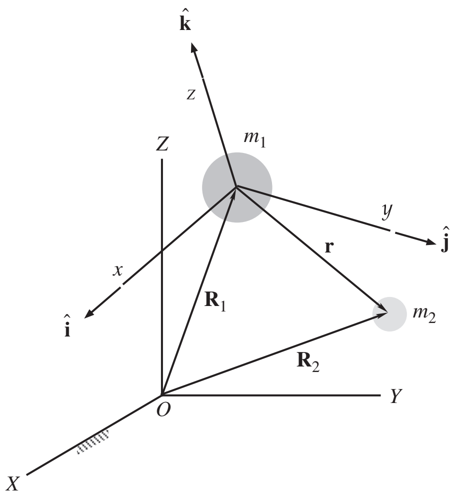


FIGURE 2.6

Moving reference frame xyz attached to the center of mass of m_1.

In the remainder of this chapter and those that follow, the analytical solution of the two-body equation of relative motion (Eqn (2.22)) will be presented and applied to a variety of practical problems in orbital mechanics. Pending an analytical solution, we can solve Eqn (2.22) numerically in a manner similar to Algorithm 2.1.

To begin, we imagine a nonrotating Cartesian coordinate system attached to m_1, as illustrated in Figure 2.6. Resolve $\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$ into components in this moving frame of reference to obtain the relative acceleration components

$$\ddot{x} = -\frac{\mu}{r^3}x \quad \ddot{y} = -\frac{\mu}{r^3}y \quad \ddot{z} = -\frac{\mu}{r^3}z \tag{2.23}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. The components of the state vector ($\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \quad \mathbf{v} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}$) are listed in the vector $\mathbf{y}$,

$$\mathbf{y} = \lfloor x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z} \rfloor$$

The time derivative of this vector comprises the state vector rates,

$$\dot{\mathbf{y}} = \lfloor \dot{x} \quad \dot{y} \quad \dot{z} \quad \ddot{x} \quad \ddot{y} \quad \ddot{z} \rfloor$$

where the last three components, the accelerations, are given by Eqn (2.23).

ALGORITHM 2.2

Numerically compute the state vector $\mathbf{r}$, $\mathbf{v}$ of m_1 relative to m_2 as a function of time, given the initial values $\mathbf{r}_0$, $\mathbf{v}_0$. This algorithm is implemented in MATLAB as the function *orbit.m*, which is listed in Appendix D.6.

1. Form the vector comprising the components of the state vector at time t_0,

$$\mathbf{y}_0 = \lfloor x_0 \quad y_0 \quad z_0 \quad \dot{x}_0 \quad \dot{y}_0 \quad \dot{z}_0 \rfloor$$

2. Provide the state vector derivatives

$$\mathbf{f}(t, \mathbf{y}) = \left\lfloor \dot{x} \quad \dot{y} \quad \dot{z} \quad -\frac{\mu}{r^3}x \quad -\frac{\mu}{r^3}y \quad -\frac{\mu}{r^3}z \right\rfloor$$

   together with $\mathbf{y}_0$ and the final time t_f to Algorithm 1.1, 1.2, or 1.3.
3. The selected algorithm solves the system $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ for the state vector

$$\mathbf{y} = \lfloor x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z} \rfloor$$

   at n discrete times t_n from t_0 through t_f.
4. The position and velocity at the discrete times are

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \quad \mathbf{v} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}$$

EXAMPLE 2.3

Relative to a nonrotating frame of reference with origin at the center of the earth, a 1000 kg satellite's initial position vector is $\mathbf{r} = 8000\hat{\mathbf{i}} + 6000\hat{\mathbf{k}}$ (km) and its initial velocity vector is $\mathbf{v} = 7\hat{\mathbf{j}}$ (km/s). Use Algorithm 2.2 and the RKF4(5) method to solve for the path of the spacecraft over the next 4 h. Determine its minimum and maximum distance from the earth's surface during that time.

Solution

The MATLAB function *orbit.m* in Appendix D.6 solves this problem. The initial value of the vector $\mathbf{y}$ is

$$\mathbf{y}_0 = \lfloor 8000\text{ km} \quad 0 \quad 6000\text{ km} \quad 0 \quad 5\text{ km/s} \quad 5\text{ km/s} \rfloor$$

The program provides these initial conditions to the function *rkf45* (Appendix D.4), which integrates the system $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$. *rkf45* uses the function *rates* embedded in *orbit.m* to calculate $\mathbf{f}(t,\mathbf{y})$ at each time step. The command window output of *orbit.m* in Appendix D.6 shows that

$$\boxed{\begin{array}{l}\text{The minimum altitude is 3622 km, and the speed at that point is 7 km/s.}\\ \text{The maximum altitude is 9560 km, and the speed at that point is 4.39 km/s.}\end{array}}$$

The minimum altitude in this case is at the starting point of the orbit. The maximum altitude occurs 2 h later on the opposite side of the earth.

orbit.m also uses some MATLAB plotting features to generate Figure 2.7. Observe that the orbit is inclined to the equatorial plane and has an apparently elliptical shape. The satellite moves eastwardly in the same direction as the earth's rotation.

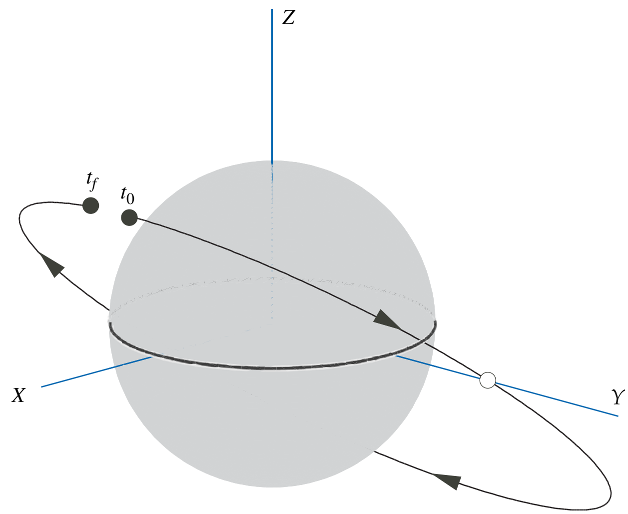


FIGURE 2.7

The computed earth orbit. the beginning of the path is marked by *o*, and *f* marks the end of the path 4 h later.

As pointed out earlier, since the center of mass G has zero acceleration, we can use it as the origin of an inertial reference frame. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of m_1 and m_2, respectively, relative to the center of mass G in Figure 2.1. The equation of motion of m_2 relative to the center of mass is

$$-G\frac{m_1 m_2}{r^2}\hat{\mathbf{u}}_r = m_2\ddot{\mathbf{r}}_2 \tag{2.24}$$

where, as before, r is the magnitude of $\mathbf{r}$, the position vector of m_2 relative to m_1. In terms of $\mathbf{r}_1$ and $\mathbf{r}_2$,

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \tag{2.25}$$

Since the position vector of the center of mass relative to itself is zero, it follows from Eqn (2.1) that

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \mathbf{0}$$

Therefore,

$$\mathbf{r}_1 = -\frac{m_2}{m_1}\mathbf{r}_2 \tag{2.26}$$

Substituting Eqn (2.26) into Eqn (2.25) yields

$$\mathbf{r} = \frac{m_1 + m_2}{m_1}\mathbf{r}_2$$

Substituting this back into Eqn (2.24) and using the fact that $\hat{\mathbf{u}}_r = \mathbf{r}_2/r_2$, we get

$$-G\frac{m_1^3 m_2}{(m_1+m_2)^2 r_2^3}\mathbf{r}_2 = m_2\ddot{\mathbf{r}}_2$$

Upon simplification, this becomes

$$-\left(\frac{m_1}{m_1+m_2}\right)^3\frac{\mu}{r_2^3}\mathbf{r}_2 = \ddot{\mathbf{r}}_2 \tag{2.27}$$

where μ is the gravitational parameter given by Eqn (2.21). If we let

$$\mu' = \left(\frac{m_1}{m_1+m_2}\right)^3\mu$$

then Eqn (2.27) reduces to

$$\ddot{\mathbf{r}}_2 = -\frac{\mu'}{r_2^3}\mathbf{r}_2$$

which is identical in form to Eqn (2.22).

In a similar fashion, the equation of motion of m_1 relative to the center of mass is found to be

$$\ddot{\mathbf{r}}_1 = -\frac{\mu''}{r_1^3}\mathbf{r}_1$$

in which

$$\mu'' = \left(\frac{m_2}{m_1+m_2}\right)^3\mu$$

Since the equations of motion of either particle relative to the center of mass have the same form as the equations of motion relative to either one of the bodies, m_1 or m_2, it follows that the relative motion as viewed from these different perspectives must be similar, as illustrated in Figure 2.3.

2.4 Angular momentum and the orbit formulas

The angular momentum of body m_2 relative to m_1 is the moment of m_2's relative linear momentum $m_2\dot{\mathbf{r}}$ (cf. Eqn (1.54)),

$$\mathbf{H}_{2/1} = \mathbf{r} \times m_2\dot{\mathbf{r}}$$

where $\dot{\mathbf{r}} = \mathbf{v}$ is the velocity of m_2 relative to m_1. Let us divide this equation through by m_2 and let $\mathbf{h} = \mathbf{H}_{2/1}/m_2$, so that

$$\boxed{\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}} \tag{2.28}$$

$\mathbf{h}$ is the relative angular momentum of m_2 per unit mass, that is, the specific relative angular momentum. The units of $\mathbf{h}$ are square kilometers per second.

Taking the time derivative of $\mathbf{h}$ yields

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}$$

But $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$. Furthermore, $\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$, according to Eqn (2.22), so that

$$\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \left(-\frac{\mu}{r^3}\mathbf{r}\right) = -\frac{\mu}{r^3}(\mathbf{r} \times \mathbf{r}) = 0$$

Therefore, angular momentum is conserved,

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = 0 \quad (\text{or } \mathbf{r} \times \dot{\mathbf{r}} = \text{constant}) \tag{2.29}$$

If the position vector $\mathbf{r}$ and the velocity vector $\dot{\mathbf{r}}$ are parallel, then it follows from Eqn (2.28) that the angular momentum is zero and, according to Eqn (2.29), it remains zero at all points of the trajectory. Zero angular momentum characterizes rectilinear trajectories whereon m_2 moves toward or away from m_1 in a straight line (see Example 1.20).

At any point of a curvilinear trajectory, the position vector $\mathbf{r}$ and the velocity vector $\mathbf{r}Y$ lie in the same plane, as illustrated in Figure 2.8. Their crossproduct $\mathbf{r} \times \mathbf{r}Y$ is perpendicular to that plane. Since $\mathbf{r} \times \mathbf{r}Y = \mathbf{h}$, the unit vector normal to the plane is

$$\hat{\mathbf{h}} = \frac{\mathbf{h}}{h} \tag{2.30}$$

By the conservation of angular momentum (Eqn (2.29)), this unit vector is constant. Thus, the path of m_2 around m_1 lies in a single plane.

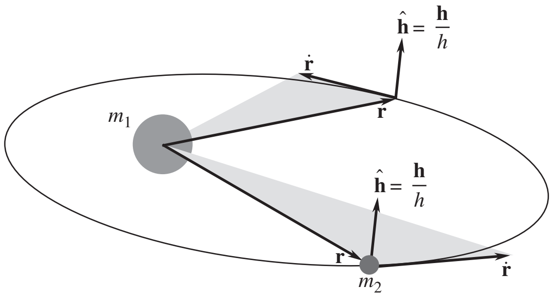


FIGURE 2.8

The path of m_2 around m_1 lies in a plane whose normal is defined by $\mathbf{h}$.

Since the orbit of m_2 around m_1 forms a plane, it is convenient to orient oneself above that plane and look down upon the path, as shown in Figure 2.9. Let us resolve the relative velocity vector $\mathbf{r}Y$ into components $\mathbf{v}_r = v_r\hat{\mathbf{u}}_r$ and $\mathbf{v}_\perp = v_\perp\hat{\mathbf{u}}_\perp$ along the outward radial from m_1 and perpendicular to it, respectively, where $\hat{\mathbf{u}}_r$ and $\hat{\mathbf{u}}_\perp$ are the radial and perpendicular (azimuthal) unit vectors. Then, we can write Eqn (2.28) as

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = r\hat{\mathbf{u}}_r \times (v_r\hat{\mathbf{u}}_r + v_\perp\hat{\mathbf{u}}_\perp) = rv_\perp\hat{\mathbf{h}}$$

That is,

$$\boxed{h = rv_\perp} \tag{2.31}$$

Clearly, the angular momentum depends only on the azimuthal component of the relative velocity.

During the differential time interval dt the position vector $\mathbf{r}$ sweeps out an area dA, as shown in Figure 2.10. From the figure it is clear that the triangular area dA is given by

$$\mathrm{d}A = \frac{1}{2} \times \text{base} \times \text{altitude} = \frac{1}{2} \times v\mathrm{d}t \times r\sin\phi = \frac{1}{2}r(v\sin\phi)\mathrm{d}t = \frac{1}{2}rv_\perp\mathrm{d}t$$

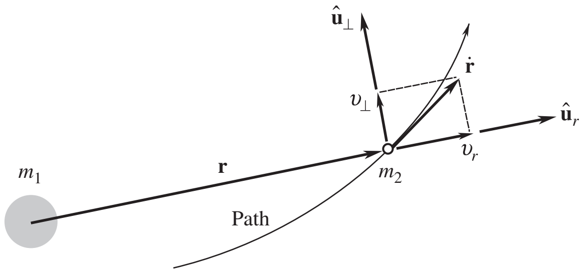


FIGURE 2.9

Components of the velocity of m_2, viewed above the plane of the orbit.

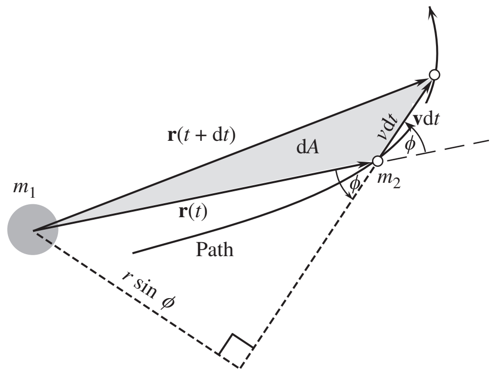


FIGURE 2.10

Differential area dA swept out by the relative position vector $\mathbf{r}$ during time interval dt.

Therefore, using Eqn (2.31) we have

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{h}{2} \tag{2.32}$$

dA/dt is called the areal velocity, and according to Eqn (2.32) it is constant. Named after the German astronomer Johannes Kepler (1571–1630), this result is known as Kepler's second law: equal areas are swept out in equal times.

Before proceeding with an effort to integrate Eqn (2.22), recall the bac–cab rule (Eqn (1.20)):

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{2.33}$$

Recall as well from Eqn (1.11) that

$$\mathbf{r}\cdot\mathbf{r} = r^2 \tag{2.34}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\cdot\mathbf{r}) = 2r\frac{\mathrm{d}r}{\mathrm{d}t}$$

But

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\cdot\mathbf{r}) = \mathbf{r}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\cdot\mathbf{r} = 2\mathbf{r}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

Thus, we obtain the important identity

$$\mathbf{r}\cdot\dot{\mathbf{r}} = r\dot{r} \tag{2.35a}$$

Since $\dot{\mathbf{r}} = \mathbf{v}$ and $r = \|\mathbf{r}\|$, this can be written alternatively as

$$\mathbf{r}\cdot\mathbf{v} = \|\mathbf{r}\|\frac{\mathrm{d}\|\mathbf{r}\|}{\mathrm{d}t} \tag{2.35b}$$

Now let us take the crossproduct of both sides of Eqn (2.22) $[\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}]$ with the specific angular momentum $\mathbf{h}$:

$$\ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} \tag{2.36}$$

Since $\dfrac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \ddot{\mathbf{r}} \times \mathbf{h} + \dot{\mathbf{r}} \times \dot{\mathbf{h}}$, the left-hand side can be written as

$$\ddot{\mathbf{r}} \times \mathbf{h} = \frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) - \dot{\mathbf{r}} \times \dot{\mathbf{h}}$$

But according to Eqn (2.29), the angular momentum is constant ($\dot{\mathbf{h}} = 0$), so this reduces to

$$\ddot{\mathbf{r}} \times \mathbf{h} = \frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) \tag{2.37}$$

The right-hand side of Eqn (2.36) can be transformed by the following sequence of substitutions:

$$\begin{aligned}
\frac{1}{r^3}\mathbf{r} \times \mathbf{h} &= \frac{1}{r^3}[\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})] \quad (\text{Eqn } (2.18)\ \ [\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}]) \\
&= \frac{1}{r^3}[\mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{r}}) - \dot{\mathbf{r}}(\mathbf{r}\cdot\mathbf{r})] \quad (\text{Eqn } (2.23)\ \ [\text{bac—cab rule}]) \\
&= \frac{1}{r^3}\left[\mathbf{r}(r\dot{r}) - \dot{\mathbf{r}}r^2\right] \quad (\text{Eqns } (2.24) \text{ and } (2.25)) \\
&= \frac{\mathbf{r}\dot{r} - \dot{\mathbf{r}}r}{r^2}
\end{aligned}$$

But

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{r\dot{\mathbf{r}} - \mathbf{r}\dot{r}}{r^2} = -\frac{\mathbf{r}\dot{r} - r\dot{\mathbf{r}}}{r^2}$$

Therefore,

$$\frac{1}{r^3}\mathbf{r} \times \mathbf{h} = -\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) \tag{2.38}$$

Substituting Eqns (2.37) and (2.38) into Eqn (2.36), we get

$$\frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \frac{d}{dt}\left(\mu\frac{\mathbf{r}}{r}\right)$$

or

$$\frac{d}{dt}\left(\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r}\right) = 0$$

That is,

$$\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r} = \mathbf{C} \tag{2.39}$$

where the vector $\mathbf{C}$, called the Laplace vector after the French mathematician Pierre-Simon Laplace (1749–1827), is a constant having the dimensions of μ. Equation (2.39) is the first integral of the equation of motion, $\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$. Taking the dot product of both sides of Eqn (2.39) with the vector $\mathbf{h}$ yields

$$(\dot{\mathbf{r}} \times \mathbf{h})\cdot\mathbf{h} - \mu\frac{\mathbf{r}\cdot\mathbf{h}}{r} = \mathbf{C}\cdot\mathbf{h}$$

Since $\dot{\mathbf{r}} \times \mathbf{h}$ is perpendicular to both $\dot{\mathbf{r}}$ and $\mathbf{h}$, it follows that $(\dot{\mathbf{r}} \times \mathbf{h})\cdot\mathbf{h} = 0$. Likewise, since $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ is perpendicular to both $\mathbf{r}$ and $\dot{\mathbf{r}}$, it is true that $\mathbf{r}\cdot\mathbf{h} = 0$. Therefore, we have $\mathbf{C}\cdot\mathbf{h} = 0$, that is, $\mathbf{C}$ is perpendicular to $\mathbf{h}$, which is normal to the orbital plane. That of course means that the Laplace vector must lie in the orbital plane.

Let us rearrange Eqn (2.39) and write it as

$$\frac{\mathbf{r}}{r} + \mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{\mu} \tag{2.40}$$

where $\mathbf{e} = \mathbf{C}/\mu$. The dimensionless vector $\mathbf{e}$ is called the eccentricity vector. The line defined by the vector $\mathbf{e}$ is commonly called the apse line. In order to obtain a scalar equation, let us take the dot product of both sides of Eqn (2.40) with $\mathbf{r}$:

$$\frac{\mathbf{r}\cdot\mathbf{r}}{r} + \mathbf{r}\cdot\mathbf{e} = \frac{\mathbf{r}\cdot(\dot{\mathbf{r}} \times \mathbf{h})}{\mu} \tag{2.41}$$

We can simplify the right-hand side by employing the vector identity presented in Eqn (1.21),

$$\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B})\cdot\mathbf{C} \tag{2.42}$$

from which we obtain

$$\mathbf{r}\cdot(\dot{\mathbf{r}} \times \mathbf{h}) = (\mathbf{r} \times \dot{\mathbf{r}})\cdot\mathbf{h} = \mathbf{h}\cdot\mathbf{h} = h^2 \tag{2.43}$$

Substituting this expression into the right-hand side of Eqn (2.41), and substituting $\mathbf{r}\cdot\mathbf{r} = r^2$ on the left yields

$$r + \mathbf{r}\cdot\mathbf{e} = \frac{h^2}{\mu} \tag{2.44}$$

Observe that by following the steps leading from Eqns (2.40) to (2.44) we have lost track of the variable time. This occurred at Eqn (2.43), because h is constant. Finally, from the definition of the dot product we have

$$\mathbf{r}\cdot\mathbf{e} = re\cos\theta$$

in which e is the eccentricity (the magnitude of the eccentricity vector $\mathbf{e}$) and θ is the true anomaly. θ is the angle between the fixed vector $\mathbf{e}$ and the variable position vector $\mathbf{r}$, as illustrated in Figure 2.11. (Other symbols used to represent true anomaly include the Greek letters ν and ϕ and the Latin letters f and v.) In terms of the eccentricity and the true anomaly, we may therefore write Eqn (2.44) as

$$r + re\cos\theta = \frac{h^2}{\mu}$$

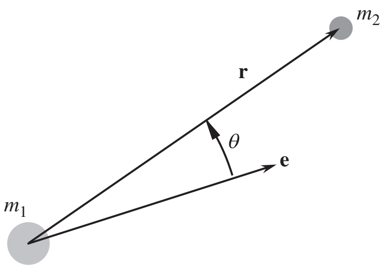


FIGURE 2.11

The true anomaly θ is the angle between the eccentricity vector **e** and the position vector **r**.

or

$$\boxed{r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}} \tag{2.45}$$

This is the orbit equation, and it defines the path of the body m_2 around m_1, relative to m_1. Remember that μ, h, and e are constants. Observe as well that there is no significance to negative values of eccentricity, that is, $e \geq 0$. Since the orbit equation describes conic sections, including ellipses, it is a mathematical statement of Kepler's first law, namely, that the planets follow elliptical paths around the sun. Two-body orbits are often referred to as Keplerian orbits.

In Section 2.3, it was pointed out that integration of the equation of relative motion (Eqn (2.22)) leads to six constants of integration. In this section, it would seem that we have arrived at those constants, namely, the three components of the angular momentum **h** and the three components of the eccentricity vector **e**. However, we showed that **h** is perpendicular to **e**. This places a condition, namely, $\mathbf{h}\cdot\mathbf{e} = 0$, on the components of **h** and **e**, so that we really have just five independent constants of integration. The sixth constant of the motion will arise when we work time back into the picture in the next chapter.

The angular velocity of the position vector **r** is $\dot{\theta}$, the rate of change of the true anomaly. The component of velocity normal to the position vector is found in terms of the angular velocity by the formula

$$v_\perp = r\dot{\theta} \tag{2.46}$$

Substituting this into Eqn (2.31) $(h = rv_\perp)$ yields the specific angular momentum in terms of the angular velocity,

$$h = r^2\dot{\theta} \tag{2.47}$$

It is convenient to have formulas for computing the radial and azimuthal components of velocity shown in Figure 2.12. From $h = rv_\perp$ we of course obtain

$$v_\perp = \frac{h}{r}$$

Substituting r from Eqn (2.45) readily yields

$$v_\perp = \frac{\mu}{h}(1 + e\cos\theta) \tag{2.48}$$

Since $v_r = \dot{r}$, we take the derivative of Eqn (2.45) to get

$$\dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left[\frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\right] = \frac{h^2}{\mu}\left[-\frac{e\left(-\dot{\theta}\sin\theta\right)}{(1+e\cos\theta)^2}\right] = \frac{h^2}{\mu}\frac{e\sin\theta}{(1+e\cos\theta)^2}\frac{h}{r^2}$$

where we made use of the fact that $\dot{\theta} = h/r^2$, from Eqn (2.47). Substituting Eqn (2.45) once again and simplifying finally yields

$$\boxed{v_r = \frac{\mu}{h}e\sin\theta} \tag{2.49}$$

We see from Eqn (2.45) that m_2 comes closest to m_1 (r is the smallest) when $\theta = 0$ (unless $e = 0$, in which case the distance between m_1 and m_2 is constant). The point of closest approach lies on the apse line and is called periapsis. The distance r_p to periapsis, as shown in Figure 2.12, is obtained by setting the true anomaly equal to zero,

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e} \tag{2.50}$$

From Eqn (2.49) it is clear that the radial component of velocity is zero at periapsis. For $0 < \theta < 180°$, v_r is positive, which means m_2 is moving *away* from periapsis. On the other hand, Eqn (2.49) shows that if $180° < \theta < 360°$, then v_r is negative, which means m_2 is moving *toward* periapsis.

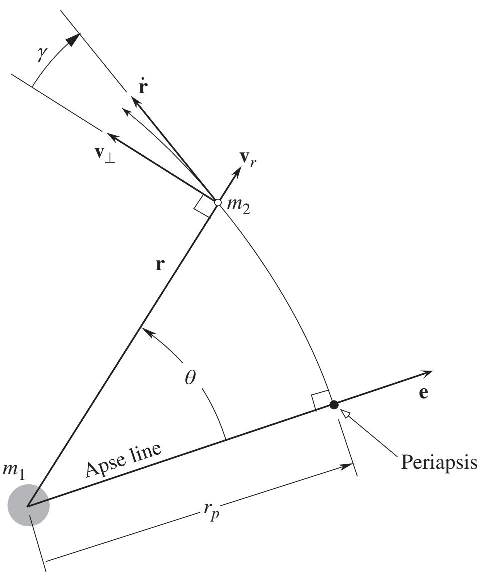


FIGURE 2.12

Position and velocity of m_2 in polar coordinates centered at m_1, with the eccentricity vector being the reference for true anomaly (polar angle) θ. γ is the flight path angle.

The flight path angle γ is illustrated in Figure 2.12. It is the angle that the velocity vector $\mathbf{v} = \dot{\mathbf{r}}$ makes with the normal to the position vector. The normal to the position vector points in the direction of $\mathbf{v}_\perp$, and it is called the local horizon. From Figure 2.12 it is clear that

$$\tan\gamma = \frac{v_r}{v_\perp} \tag{2.51}$$

Substituting Eqns (2.48) and (2.49) leads at once to the expression

$$\tan\gamma = \frac{e\sin\theta}{1 + e\cos\theta} \tag{2.52}$$

The flight path angle, like v_r, is positive (velocity vector directed above the local horizon) when the spacecraft is moving away from periapsis and is negative (velocity vector directed below the local horizon) when the spacecraft is moving toward periapsis.

Since $\cos(-\theta) = \cos\theta$, the trajectory described by the orbit equation is symmetric about the apse line, as illustrated in Figure 2.13, which also shows a chord, the straight line connecting any two points on the orbit. The latus rectum is the chord through the center of attraction perpendicular to the apse line. By symmetry, the center of attraction divides the latus rectum into two equal parts, each of length p, known historically as the semilatus rectum. In modern parlance, p is called the parameter of the orbit. From Eqn (2.45) it is apparent that

$$p = \frac{h^2}{\mu} \tag{2.53}$$

Since the curvilinear path of m_2 around m_1 lies in a plane, for the time being we will for simplicity continue to view the trajectory from above the plane. Unless there is a reason to do otherwise, we will assume that the eccentricity vector points to the right and that m_2 moves counterclockwise around m_1, which means that the true anomaly is measured positive counterclockwise, consistent with the usual polar coordinate sign convention.

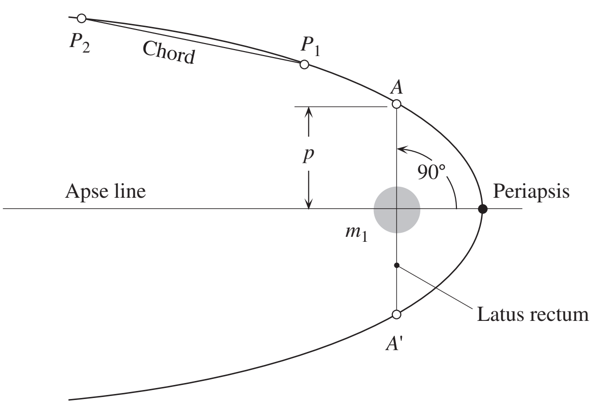


FIGURE 2.13

Illustration of latus rectum, semilatus rectum p, and the chord between any two points on an orbit.

2.5 The energy law

By taking the crossproduct of Eqn (2.22), $\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r}$ (Newton's second law of motion), with the relative angular momentum per unit mass $\mathbf{h}$, we were led to the vector Eqn (2.39), and from that we obtained the orbit formula, that is, Eqn (2.45). Now let us see what results from taking the *dot* product of Eqn (2.22) with the relative *linear* momentum per unit mass. The relative linear momentum per unit mass is just the relative velocity,

$$\frac{m_2\dot{\mathbf{r}}}{m_2} = \dot{\mathbf{r}}$$

Thus, carrying out the dot product in Eqn (2.22) yields

$$\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} = -\mu\frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r^3} \tag{2.54}$$

For the left-hand side we observe that

$$\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(v^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}\right) \tag{2.55}$$

For the right-hand side of Eqn (2.54) we have, recalling that $\mathbf{r}\cdot\mathbf{r} = r^2$ and that $\mathrm{d}(1/r)/\mathrm{d}t = (-1/r^2)(\mathrm{d}r/\mathrm{d}t)$,

$$\mu\frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r^3} = \mu\frac{r\dot{r}}{r^3} = \mu\frac{\dot{r}}{r^2} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{r}\right) \tag{2.56}$$

Substituting Eqns (2.55) and (2.56) into Eqn (2.54) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = 0$$

or

$$\frac{v^2}{2} - \frac{\mu}{r} = \varepsilon \quad \text{(constant)} \tag{2.57}$$

where ε is a constant. $v^2/2$ is the relative kinetic energy per unit mass. $(-\mu/r)$ is the potential energy per unit mass of the body m_2 in the gravitational field of m_1. The total mechanical energy per unit mass ε is the sum of the kinetic and potential energies per unit mass. Eqn (2.57) is a statement of the conservation of energy, namely, that the specific mechanical energy is the same at all points of the trajectory. Equation (2.57) is also known as the vis viva ("living force") equation. It is valid for any trajectory, including rectilinear ones.

For curvilinear trajectories, we can evaluate the constant ε at periapsis ($\theta = 0$),

$$\varepsilon = \varepsilon_p = \frac{v_p^2}{2} - \frac{\mu}{r_p} \tag{2.58}$$

where r_p and v_p are the position and speed at periapsis. Since $v_r = 0$ at periapsis, the only component of velocity is $v_\perp$, which means $v_p = v_\perp = h/r_p$. Thus,

$$\varepsilon = \frac{1}{2}\frac{h^2}{r_p^2} - \frac{\mu}{r_p} \tag{2.59}$$

Substituting the formula for periapse radius (Eqn (2.50)) into Eqn (2.59) yields an expression for the orbital specific energy in terms of the orbital constants h and e,

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}\left(1 - e^2\right) \tag{2.60}$$

Clearly, the orbital energy is not an independent orbital parameter.

Note that the energy $\mathcal{E}$ of a spacecraft of mass m is obtained from the specific energy ε by the formula

$$\mathcal{E} = m\varepsilon \tag{2.61}$$

2.6 Circular orbits ($e = 0$)

Setting $e = 0$ in the orbital equation $r = (h^2/\mu)/(1 + e\cos\theta)$ yields

$$r = \frac{h^2}{\mu} \tag{2.62}$$

That is, $r = \text{constant}$, which means the orbit of m_2 around m_1 is a circle. Since the radial velocity $\dot{r}$ is zero, it follows that $v = v_\perp$ so that the angular momentum formula $h = rv_\perp$ becomes simply $h = rv$ for a circular orbit. Substituting this expression for h into Eqn (2.62) and solving for v yields the velocity of a circular orbit,

$$v_{\text{circular}} = \sqrt{\frac{\mu}{r}} \tag{2.63}$$

The time T required for one orbit is known as the period. Because the speed is constant, the period of a circular orbit is easy to compute.

$$T = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi r}{\sqrt{\mu/r}}$$

so that

$$T_{\text{circular}} = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}} \tag{2.64}$$

The specific energy of a circular orbit is found by setting $e = 0$ in Eqn (2.60),

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}$$

Employing Eqn (2.62) yields

$$\varepsilon_{\text{circular}} = -\frac{\mu}{2r} \tag{2.65}$$

Obviously, the energy of a circular orbit is negative. As the radius goes up, the energy becomes less negative, that is, it increases. In other words, the larger the orbit is, the greater is its energy.

To launch a satellite from the surface of the earth into a circular orbit requires increasing its specific energy ε. This energy comes from the rocket motors of the launch vehicle. Since the energy of a satellite of mass m is $\mathcal{E} = m\varepsilon$, a propulsion system that can place a large mass in a low earth orbit (LEO) can place a smaller mass in a higher earth orbit.

The space shuttle orbiters are the largest man-made satellites so far placed in orbit with a single launch vehicle. For example, on NASA mission STS-82 in February 1997, the orbiter Discovery rendezvoused with the Hubble space telescope to repair and refurbish it. The altitude of the nearly circular orbit was 580 km (360 miles). Discovery's orbital mass early in the mission was 106,000 kg (117 t). That was only 6% of the total mass of the shuttle prior to launch (comprising the orbiter's dry mass, plus that of its payload and fuel, plus the two solid rocket boosters (SRBs), plus the external fuel tank filled with liquid hydrogen and oxygen). This mass of about two million kilograms (2200 t) was lifted off the launch pad by a total thrust in the vicinity of 35,000 kN (7.8 million pounds). Eighty-five percent of the thrust was furnished by the SRB's, which were depleted and jettisoned about 2 min into the flight. The remaining thrust came from the three liquid rockets (space shuttle main engines, or SSMEs) on the orbiter. These were fueled by the external tank, which was jettisoned just after the SSMEs were shut down at MECO (main engine cut off), about 8.5 min after liftoff.

Manned orbital spacecraft and a host of unmanned remote sensing, imaging and navigation satellites occupy nominally circular, low earth orbits (LEOs). A low earth orbit is one whose altitude lies between about 150 km (100 miles) and about 1000 km (600 miles). An LEO is well above the nominal outer limits of the drag-producing atmosphere (about 80 km or 50 miles), and well below the hazardous Van Allen radiation belts, the innermost of which begins at about 2400 km (1500 miles).

Nearly all our applications of the orbital equations will be for the analysis of man-made spacecraft, all of which have a mass that is insignificant compared to the sun and planets. For example, since the earth is nearly 20 orders of magnitude more massive than the largest conceivable artificial satellite, the center of mass of the two-body system lies at the center of the earth, and the constant μ in Eqn (2.21) becomes

$$\mu = G(m_{\text{earth}} + m_{\text{satellite}}) = Gm_{\text{earth}}$$

The value of the earth's gravitational parameter to be used throughout this book is found in Table A.2,

$$\mu_{\text{earth}} = 398{,}600\ \text{km}^3/\text{s}^2 \tag{2.66}$$

EXAMPLE 2.4

Plot the speed v and period T of a satellite in a circular LEO as a function of altitude z.

Solution

Eqns (2.63) and (2.64) give the speed and period, respectively, of the satellite:

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{R_E + z}} = \sqrt{\frac{398{,}600}{6378 + z}} \qquad T = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}} = \frac{2\pi}{\sqrt{398{,}600}}(6378 + z)^{\frac{3}{2}}$$

These relations are graphed in Figure 2.14.

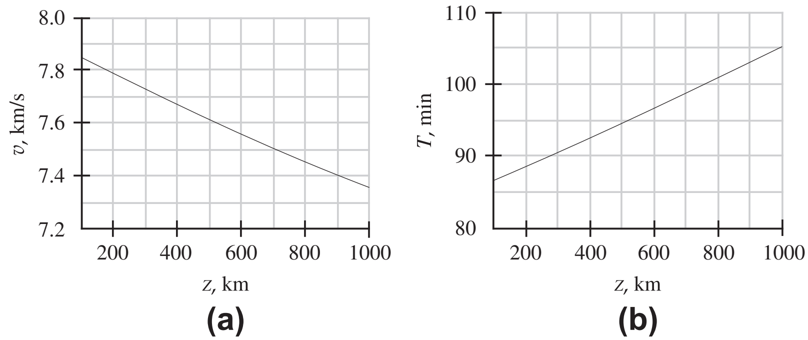


FIGURE 2.14

Circular orbital speed (a) and period (b) as a function of altitude.

If a satellite remains always above the same point on the earth's equator, then it is in a circular, geostationary equatorial orbit or GEO. For GEO, the radial from the center of the earth to the satellite must have the same angular velocity as the earth itself, namely, 2π radians per sidereal day. The sidereal day is the time it takes the earth to complete one rotation relative to inertial space (the fixed stars). The ordinary 24-h day, or synodic day, is the time it takes the sun to apparently rotate once around the earth, from high noon one day to high noon the next. The synodic and sidereal days would be identical if the earth stood still in space. However, while the earth makes one absolute rotation around its axis, it advances $2\pi/365.26$ rad along its solar orbit. Therefore, its inertial angular velocity ω_E is $[(2\pi + 2\pi/365.26)\ \text{rad}]/(24\ \text{h})$, that is,

$$\boxed{\omega_E = 72.9217 \times 10^{-6}\ \text{rad/s}} \tag{2.67}$$

Communication satellites and global weather satellites are placed in geostationary orbit because of the large portion of the earth's surface visible from that altitude and the fact that ground stations do not have to track the satellite, which appears motionless in the sky.

EXAMPLE 2.5

Calculate the altitude z_{GEO} and speed v_{GEO} of a geostationary earth satellite.

Solution

From Eqn (2.63), the speed of the satellite in its circular GEO of radius r_{GEO} is

$$v_{\text{GEO}} = \sqrt{\frac{\mu}{r_{\text{GEO}}}} \tag{a}$$

On the other hand, the speed v_{GEO} along its circular path is related to the absolute angular velocity ω_E of the earth by the kinematics formula

$$v_{\text{GEO}} = \omega_E r_{\text{GEO}}$$

Equating these two expressions and solving for r_{GEO} yields

$$r_{\text{GEO}} = \sqrt[3]{\frac{\mu}{\omega_E{}^2}}$$

Substituting Eqns (2.66) and (2.67), we get

$$r_{GEO} = \sqrt[3]{\frac{398{,}600}{(72.9217 \times 10^{-6})^2}} = 42{,}164 \text{ km} \tag{2.68}$$

Therefore, the distance of the satellite above the earth's surface is

$$z_{GEO} = r_{GEO} - R_E = 42{,}164 - 6378$$

$$\boxed{z_{GEO} = 35{,}786 \text{ km (22,241 miles)}}$$

Substituting Eqn (2.68) into Eqn (a) yields the speed,

$$v_{GEO} = \sqrt{\frac{398{,}600}{42{,}164}} = \boxed{3.075 \text{ km/s}} \tag{2.69}$$

EXAMPLE 2.6

Calculate the maximum latitude and the percentage of the earth's surface visible from GEO.

Solution

To find the maximum viewable latitude ϕ, use Figure 2.15, from which it is apparent that

$$\phi = \cos^{-1}\frac{R_E}{r} \tag{a}$$

where $R_E = 6378$ km and, according to Eqn (2.68), $r = 42{,}164$ km. Therefore,

$$\phi = \cos^{-1}\frac{6378}{42{,}164}$$

$$\boxed{\phi = 81.30°} \text{ Maximum visible north or south latitude} \tag{b}$$

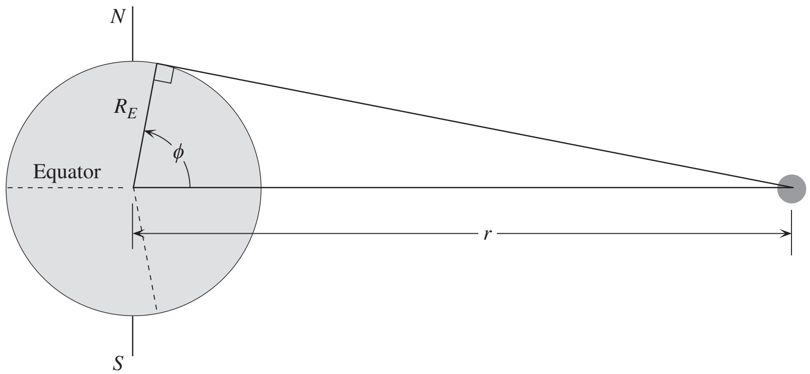


FIGURE 2.15

Satellite in GEO.

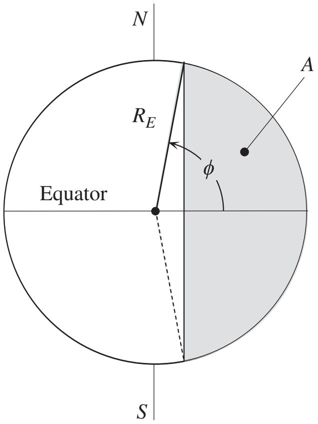


FIGURE 2.16

Surface area A visible from GEO.

The surface area A visible from the GEO is the shaded region illustrated in Figure 2.16. It can be shown that the area A is given by

$$A = 2\pi R_E^2 (1 - \cos\phi)$$

where $2\pi R_E^2$ is the area of the hemisphere. Therefore, the percentage of the hemisphere visible from the GEO is

$$\frac{A}{2\pi R_E^2} \times 100 = (1 - \cos 81.30°) \times 100 = 84.9\%$$

which of course means that $\boxed{42.4\%}$ of the total surface of the earth can be seen from the GEO.

Figure 2.17 is a photograph taken from geosynchronous equatorial orbit by one of the National Oceanic and Atmospheric Administration's Geostationary Operational Environmental Satellites (GOES).

2.7 Elliptical orbits $(0 < e < 1)$

If $0 < e < 1$, then the denominator of Eqn (2.45) varies with the true anomaly θ, but it remains positive, never becoming zero. Therefore, the relative position vector remains bounded, having its smallest magnitude at the periapsis r_p, given by Eqn (2.50). The maximum value of r is reached when the denominator of $r = (h^2/\mu)/(1 + e\cos\theta)$ obtains its minimum value, which occurs at $\theta = 180°$. That point is called the apoapsis, and its radial coordinate, denoted by r_a, is

$$r_a = \frac{h^2}{\mu}\frac{1}{1-e} \tag{2.70}$$

The curve defined by Eqn (2.45) in this case is an ellipse.

Let $2a$ be the distance measured along the apse line from periapsis P to apoapsis A, as illustrated in Figure 2.18. Then,

$$2a = r_p + r_a$$

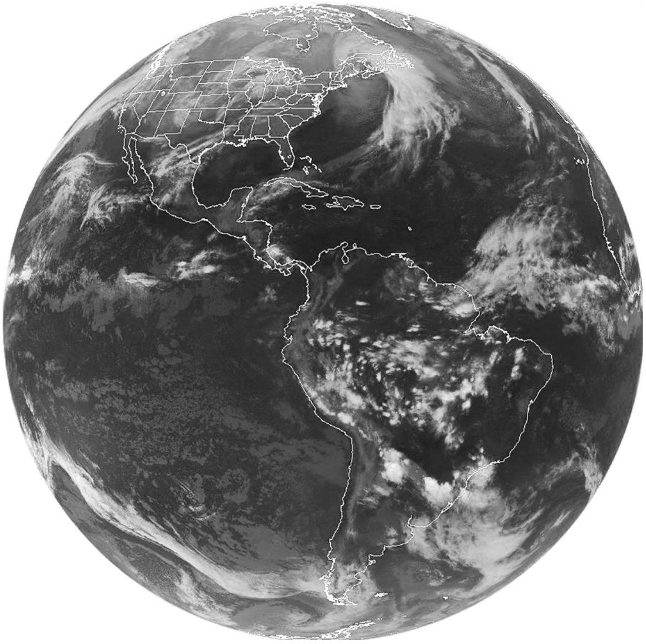


FIGURE 2.17

The view from GEO.

Substituting Eqns (2.50) and (2.70) into this expression we get

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} \tag{2.71}$$

where a is the semimajor axis of the ellipse. Solving Eqn (2.71) for h^2/μ and putting the result into Eqn (2.45) yields an alternative form of the orbit equation,

$$r = a\frac{1-e^2}{1+e\cos\theta} \tag{2.72}$$

In Figure 2.18, let F denote the location of the body m_1, which is the origin of the r,θ polar coordinate system. The center C of the ellipse is the point lying midway between the apoapsis and the periapsis. The distance CF from the center C to the focus F is

$$CF = a - FP = a - r_p$$

But from Eqn (2.72), evaluated at $\theta = 0$,

$$r_p = a(1-e) \tag{2.73}$$

Therefore, $CF = ae$, as indicated in Figure 2.18.

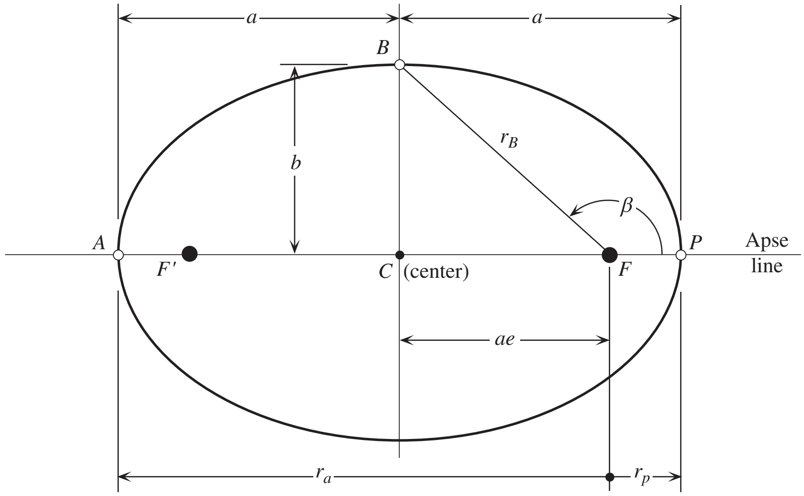


FIGURE 2.18

Elliptical orbit. m_1 is at the focus F. F' Is the unoccupied empty focus.

Let B be the point on the orbit that lies directly above C, on the perpendicular bisector of the major axis AP. The distance b from C to B is the semiminor axis. If the true anomaly of point B is β, then according to Eqn (2.72), the radial coordinate of B is

$$r_B = a\frac{1-e^2}{1+e\cos\beta} \tag{2.74}$$

The projection of r_B onto the apse line is ae, that is,

$$ae = r_B\cos(180-\beta) = -r_B\cos\beta = -\left(a\frac{1-e^2}{1+e\cos\beta}\right)\cos\beta$$

Solving this expression for e, we obtain

$$e = -\cos\beta \tag{2.75}$$

Substituting this result into Eqn (2.74) reveals the interesting fact that

$$r_B = a$$

According to the Pythagorean theorem,

$$b^2 = r_B^2 - (ae)^2 = a^2 - a^2e^2$$

which means that the semiminor axis is found in terms of the semimajor axis and the eccentricity of the ellipse as

$$\boxed{b = a\sqrt{1-e^2}} \tag{2.76}$$

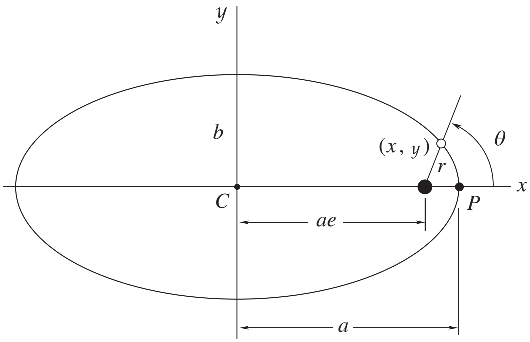


FIGURE 2.19

Cartesian coordinate description of the orbit.

Let an xy Cartesian coordinate system be centered at C, as shown in Figure 2.19. In terms of r and θ, we see from the figure that the x-coordinate of a point on the orbit is

$$x = ae + r\cos\theta = ae + \left(a\frac{1-e^2}{1+e\cos\theta}\right)\cos\theta = a\frac{e+\cos\theta}{1+e\cos\theta}$$

From this, we have

$$\frac{x}{a} = \frac{e+\cos\theta}{1+e\cos\theta} \tag{2.77}$$

For the y-coordinate, we make use of Eqn (2.76) to obtain

$$y = r\sin\theta = \left(a\frac{1-e^2}{1+e\cos\theta}\right)\sin\theta = b\frac{\sqrt{1-e^2}}{1+e\cos\theta}\sin\theta$$

Therefore,

$$\frac{y}{b} = \frac{\sqrt{1-e^2}}{1+e\cos\theta}\sin\theta \tag{2.78}$$

Using Eqns (2.77) and (2.78), we find that

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{1}{(1+e\cos\theta)^2}\left[(e+\cos\theta)^2 + \left(1-e^2\right)\sin^2\theta\right] \\
&= \frac{1}{(1+e\cos\theta)^2}\left[e^2 + 2e\cos\theta + \cos^2\theta + \sin^2\theta - e^2\sin^2\theta\right] \\
&= \frac{1}{(1+e\cos\theta)^2}\left[e^2 + 2e\cos\theta + 1 - e^2\sin^2\theta\right] \\
&= \frac{1}{(1+e\cos\theta)^2}\left[e^2\left(1-\sin^2\theta\right) + 2e\cos\theta + 1\right] \\
&= \frac{1}{(1+e\cos\theta)^2}\left[e^2\cos^2\theta + 2e\cos\theta + 1\right] \\
&= \frac{1}{(1+e\cos\theta)^2}(1+e\cos\theta)^2
\end{aligned}$$

That is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.79}$$

This is the familiar Cartesian coordinate formula for an ellipse centered at the origin, with x-intercepts at $\pm a$ and y-intercepts at $\pm b$. If $a = b$, Eqn (2.79) describes a circle, which is really an ellipse whose eccentricity is zero.

The specific energy of an elliptical orbit is negative, and it is found by substituting the angular momentum and eccentricity into Eqn (2.60),

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}\left(1 - e^2\right)$$

According to Eqn (2.71), $h^2 = \mu a(1 - e^2)$, so that

$$\boxed{\varepsilon = -\frac{\mu}{2a}} \tag{2.80}$$

This shows that the specific energy is independent of the eccentricity and depends only on the semimajor axis of the ellipse. For an elliptical orbit, the conservation of energy (Eqn (2.57)) may therefore be written

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \tag{2.81}$$

The area of an ellipse is found in terms of its semimajor and semiminor axes by the formula $A = \pi ab$ (which reduces to the formula for the area of a circle if $a = b$). To find the period T of the elliptical orbit, we employ Kepler's second law, $dA/dt = h/2$, to obtain

$$\Delta A = \frac{h}{2}\Delta t$$

For one complete revolution, $\Delta A = \pi ab$ and $\Delta t = T$. Thus, $\pi ab = (h/2)T$, or

$$T = \frac{2\pi ab}{h}$$

Substituting Eqns (2.71) and (2.76), we get

$$T = \frac{2\pi}{h}a^2\sqrt{1 - e^2} = \frac{2\pi}{h}\left(\frac{h^2}{\mu}\frac{1}{1 - e^2}\right)^2\sqrt{1 - e^2}$$

so that the formula for the period of an elliptical orbit, in terms of the orbital parameters h and e, becomes

$$T = \frac{2\pi}{\mu^2}\left(\frac{h}{\sqrt{1 - e^2}}\right)^3 \tag{2.82}$$

We can once again appeal to Eqn (2.71) to substitute $h = \sqrt{\mu a(1 - e^2)}$ into this equation, thereby obtaining an alternative expression for the period,

$$\boxed{T = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}}} \tag{2.83}$$

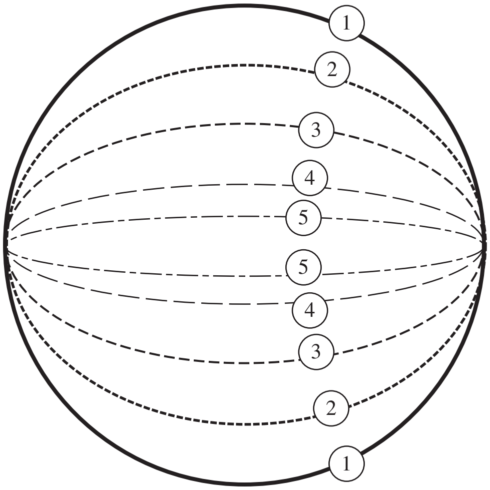


FIGURE 2.20

Since all five ellipses have the same major axis, their periods and energies are identical.

This expression, which is identical to that of a circular orbit of radius a (Eqn (2.64)), reveals that, like the energy, the period of an elliptical orbit is independent of the eccentricity (Figure 2.20). Equation (2.83) embodies Kepler's third law: the period of a planet is proportional to the three-half power of its semimajor axis.

Finally, observe that dividing Eqn (2.50) by Eqn (2.70) yields

$$\frac{r_p}{r_a} = \frac{1-e}{1+e}$$

Solving this for e results in a useful formula for calculating the eccentricity of an elliptical orbit, namely,

$$\boxed{e = \frac{r_a - r_p}{r_a + r_p}} \tag{2.84}$$

From Figure 2.18, it is apparent that $r_a - r_p = \overline{F'F}$ is the distance between the foci. As previously noted, $r_a + r_p = 2a$. Thus, Eqn (2.84) has the geometrical interpretation,

$$\text{eccentricity} = \frac{\text{distance between the foci}}{\text{length of the major axis}}$$

A rectilinear ellipse is characterized as having a zero angular momentum and an eccentricity of 1. That is, the distance between the foci equals the finite length of the major axis, along which the relative motion occurs. Since only the length of the semimajor axis determines the orbital specific energy, Eqn (2.80) applies to rectilinear ellipses as well.

What is the average distance of m_2 from m_1 in the course of one complete orbit? To answer this question, we divide the range of the true anomaly (2π) into n equal segments $\Delta\theta$, so that

$$n = \frac{2\pi}{\Delta\theta}$$

We then use $r = (h^2/\mu)/(1 + e\cos\theta)$ to evaluate $r(\theta)$ at the n equally spaced values of the true anomaly, starting at the periapsis:

$$\theta_1 = 0, \quad \theta_2 = \Delta\theta, \quad \theta_3 = 2\Delta\theta, \cdots, \quad \theta_n = (n-1)\Delta\theta$$

The average of this set of n values of r is given by

$$\bar{r}_\theta = \frac{1}{n}\sum_{i=1}^{n} r(\theta_i) = \frac{\Delta\theta}{2\pi}\sum_{i=1}^{n} r(\theta_i) = \frac{1}{2\pi}\sum_{i=1}^{n} r(\theta_i)\Delta\theta \tag{2.85}$$

Now let n become very large, so that $\Delta\theta$ becomes very small. In the limit as $n \to \infty$, Eqn (2.85) becomes

$$\bar{r}_\theta = \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\mathrm{d}\theta \tag{2.86}$$

Substituting Eqn (2.72) into the integrand yields

$$\bar{r}_\theta = \frac{1}{2\pi}a\left(1 - e^2\right)\int_0^{2\pi}\frac{\mathrm{d}\theta}{1 + e\cos\theta}$$

The integral in this expression can be found in integral tables (e.g., Beyer, 1991), from which we obtain

$$\bar{r}_\theta = \frac{1}{2\pi}a\left(1 - e^2\right)\left(\frac{2\pi}{\sqrt{1 - e^2}}\right) = a\sqrt{1 - e^2} \tag{2.87}$$

Comparing this result with Eqn (2.76), we see that the true anomaly-averaged orbital radius equals the length of the semiminor axis b of the ellipse. Thus, the semimajor axis, which is the average of the maximum and minimum distances from the focus, is not the mean distance. Since, from Eqn (2.72), $r_p = a(1 - e)$, and $r_a = a(1 + e)$, Eqn (2.87) also implies that

$$\bar{r}_\theta = \sqrt{r_p r_a} \tag{2.88}$$

The mean distance is the one-half power of the product of the maximum and minimum distances from the focus and not one-half of their sum.

EXAMPLE 2.7

An earth satellite is in an orbit with a perigee altitude $z_p = 400$ km and an apogee altitude $z_a = 4000$ km, as shown in Figure 2.21. Find each of the following quantities:

(a) Eccentricity, e;
(b) Angular momentum, h;
(c) Perigee velocity, v_p;
(d) Apogee velocity, v_a;
(e) Semimajor axis, a;

(f) Period of the orbit, T;
(g) True anomaly-averaged radius $\overline{r}_\theta$;
(h) True anomaly when $r = \overline{r}_\theta$;
(i) Satellite speed when $r = \overline{r}_\theta$;
(j) Flight path angle γ when $r = \overline{r}_\theta$;
(k) Maximum flight path angle γ_{max} and the true anomaly at which it occurs.

Recall from Eqn (2.66) that $\mu = 398{,}600\ \text{km}^3/\text{s}^2$ and also that R_E, the radius of the earth, is 6378 km.

Solution

The strategy is always to seek the primary orbital parameters (eccentricity e and angular momentum h) first. All the other orbital parameters are obtained from these two.

(a) A formula that involves the unknown eccentricity e as well as the given perigee and apogee data is Eqn (2.84). We must not forget to convert the given altitudes to radii:

$$r_p = R_E + z_p = 6378 + 400 = 6778\ \text{km}$$
$$r_a = R_E + z_a = 6378 + 4000 = 10{,}378\ \text{km}$$

Then

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{10{,}378 - 6778}{10{,}378 + 6778}$$
$$\boxed{e = 0.2098}$$

(b) Now that we have the eccentricity, we need an expression containing it and the unknown angular momentum h and any other given data. That would be Eqn (2.50), the orbit formula evaluated at perigee ($\theta = 0$),

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e}$$

We use this to compute the angular momentum

$$6778 = \frac{h^2}{398{,}600}\frac{1}{1+0.2098}$$
$$\boxed{h = 57{,}172\ \text{km}^2/\text{s}}$$

(c) The angular momentum h and the perigee radius r_p can be substituted into the angular momentum formula (Eqn (2.31)) to find the perigee velocity v_p,

$$(v_p = v_\perp)_{\text{perigee}} = \frac{h}{r_p} = \frac{57{,}172}{6778}$$
$$\boxed{v_p = 8.435\ \text{km/s}}$$

(d) Since h is a constant, the angular momentum formula can also be employed to obtain the apogee speed v_a,

$$v_a = \frac{h}{r_a} = \frac{57{,}172}{10{,}378}$$
$$\boxed{v_a = 5.509\ \text{km/s}}$$

(e) The semimajor axis is the average of the perigee and apogee radii (Figure 2.18),

$$a = \frac{r_p + r_a}{2} = \frac{6778 + 10{,}378}{2}$$
$$\boxed{a = 8578\ \text{km}}$$

(f) Since the semimajor axis a has been found, we can use Eqn (2.83) to calculate the period T of the orbit:

$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} = \frac{2\pi}{\sqrt{398{,}600}} 8578^{\frac{3}{2}} = 7907 \text{ s}$$
$$\boxed{T = 2.196 \text{ h}}$$

Alternatively, we could have used Eqn (2.82) for T, since both h and e were calculated above.

(g) Either Eqn (2.87) or Eqn (2.88) may be used at this point to find the true anomaly-averaged radius. Choosing the latter, we get

$$\bar{r}_\theta = \sqrt{r_p r_a} = \sqrt{6778 \cdot 10{,}378}$$
$$\boxed{\bar{r}_\theta = 8387 \text{ km}}$$

(h) To find the true anomaly when $r = \bar{r}_\theta$, we have only one choice, namely, the orbit formula (Eqn (2.45)):

$$\bar{r}_\theta = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta}$$

Substituting h and e, the primary orbital parameters found above, together with $\bar{r}_\theta$, we get

$$8387 = \frac{57{,}172^2}{398{,}600} \frac{1}{1 + 0.2098\cos\theta}$$

from which

$$\cos\theta = -0.1061$$

This means that the true anomaly-averaged radius occurs at $\boxed{\theta = 96.09°}$, where the satellite passes through $\bar{r}_\theta$ on its way *from* the perigee, and at $\boxed{\theta = 263.9°}$, where the satellite passes through $\bar{r}_\theta$ on its way *toward* the perigee.

(i) To find the speed of the satellite when $r = \bar{r}_\theta$, it is the simplest to use the energy equation for the ellipse (Eqn (2.81)),

$$\frac{v^2}{2} - \frac{\mu}{\bar{r}_\theta} = -\frac{\mu}{2a}$$
$$\frac{v^2}{2} - \frac{398{,}600}{8387} = -\frac{398{,}600}{2 \cdot 8578}$$
$$\boxed{v = 6.970 \text{ km/s}}$$

(j) Equation (2.52) gives the flight path angle in terms of the true anomaly of the average radius $\bar{r}_\theta$. Substituting the smaller of the two angles found in part (h) above yields

$$\tan\gamma = \frac{e\sin\theta}{1 + e\cos\theta} = \frac{0.2098\sin 96.09°}{1 + 0.2098\cos 96.09°} = 0.2134$$

This means that $\boxed{\gamma = 12.05°}$ when the satellite passes through $\bar{r}_\theta$ on its way *from* the perigee.

(k) To find where γ is a maximum, we must take the derivative of

$$\gamma = \tan^{-1}\frac{e\sin\theta}{1 + e\cos\theta} \tag{a}$$

with respect to θ and set the result equal to zero. Using the rules of calculus,

$$\frac{d\gamma}{d\theta} = \frac{1}{1 + \left(\dfrac{e\sin\theta}{1 + e\cos\theta}\right)^2} \frac{d}{d\theta}\left(\frac{e\sin\theta}{1 + e\cos\theta}\right) = \frac{e(e + \cos\theta)}{(1 + e\cos\theta)^2 + e^2\sin^2\theta}$$

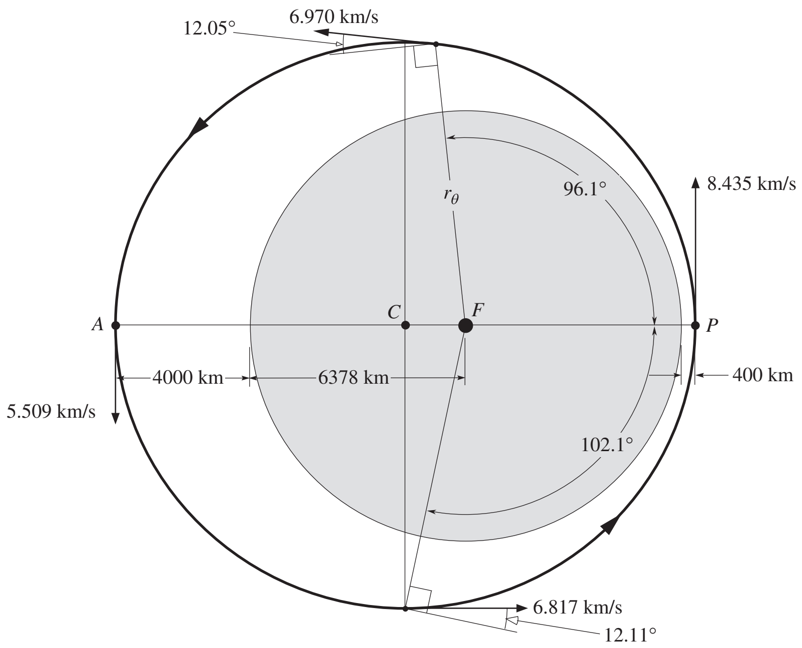


FIGURE 2.21

The orbit of Example 2.7.

For $e < 1$, the denominator is nonzero for all values of θ. Therefore, $d\gamma/d\theta = 0$ only if the numerator vanishes, that is, if $\cos\theta = -e$. Recall from Eqn (2.75) that this true anomaly locates the end point of the minor axis of the ellipse. The maximum positive flight path angle therefore occurs at the true anomaly,

$$\theta = \cos^{-1}(-0.2098)$$

$$\boxed{\theta = 102.1^\circ}$$

Substituting this into Eqn (a), we find the value of the flight path angle to be

$$\gamma_{max} = \tan^{-1}\frac{0.2098 \sin 102.1^\circ}{1 + 0.2098 \cos 102.1^\circ}$$

$$\boxed{\gamma_{max} = 12.11^\circ}$$

After attaining this greatest magnitude, the flight path angle starts to decrease steadily toward its value of zero at the apogee.

EXAMPLE 2.8

At two points on a geocentric orbit, the altitude and true anomaly are $z_1 = 1545$ km, $\theta_1 = 126^\circ$ and $z_2 = 852$ km, $\theta_2 = 58^\circ$, respectively. Find (a) the eccentricity, (b) the altitude of perigee, (c) the semimajor axis, and (d) the period.

Solution

The first objective is to find the primary orbital parameters e and h, since all other orbital data can be deduced from them.

(a) Before proceeding, we must remember to add the earth's radius to the given altitudes so that we are dealing with orbital radii. The radii of the two points are

$$r_1 = R_E + z_1 = 6378 + 1545 = 7923 \text{ km}$$
$$r_2 = R_E + z_2 = 6378 + 852 = 7230 \text{ km}$$

The only formula we have that relates the orbital position to the orbital parameters e and h is the orbit formula, Eqn (2.45). Writing that equation down for each of the two given points on the orbit yields two equations for e and h. For point 1, we obtain

$$r_1 = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_1}$$
$$7923 = \frac{h^2}{398{,}600}\frac{1}{1+e\cos 126°}$$
$$h^2 = 3.158\times 10^9 - 1.856\times 10^9 e \tag{a}$$

For point 2,

$$r_2 = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta_2}$$
$$7230 = \frac{h^2}{398{,}600}\frac{1}{1+e\cos 58°}$$
$$h^2 = 2.882\times 10^9 + 1.527\times 10^9 e \tag{b}$$

Equating Eqns (a) and (b), the two expressions for h^2, yields a single equation for the eccentricity e,

$$3.158\times 10^9 - 1.856\times 10^9 e = 2.882\times 10^9 + 1.527\times 10^9 e$$

or

$$3.384\times 10^9 e = 276.2\times 10^6$$

Therefore,

$$\boxed{e = 0.08164} \text{ (an ellipse)} \tag{c}$$

By substituting the eccentricity back into Eqn (a) [or Eqn (b)], we find the angular momentum,

$$h^2 = 3.158\times 10^9 - 1.856\times 10^9\cdot 0.08164 \Rightarrow h = 54{,}830 \text{ km}^2/\text{s} \tag{d}$$

(b) With the eccentricity and angular momentum available, we can use the orbit equation to obtain the perigee radius (Eqn (2.50)),

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{54{,}830^2}{398{,}600}\frac{1}{1+0.08164} = 6974 \text{ km} \tag{e}$$

From that we find the perigee altitude,

$$z_p = r_p - R_E = 6974 - 6378$$
$$\boxed{z_p = 595.5 \text{ km}}$$

(c) The semimajor axis is the average of the perigee and apogee radii. We just found the perigee radius above in Eqn (e). Thus, we need only to compute the apogee radius and that is accomplished by using Eqn (2.70), which is the orbit formula evaluated at the apogee.

$$r_a = \frac{h^2}{\mu}\frac{1}{1-e} = \frac{54{,}830^2}{398{,}600}\frac{1}{1-0.08164} = 8213\text{ km} \tag{f}$$

From Eqns (e) and (f) it follows that

$$a = \frac{r_p + r_a}{2} = \frac{8213+6974}{2}$$
$$\boxed{a = 7593\text{ km}}$$

(d) Since the semimajor axis has been determined, it is convenient to use Eqn (2.84) to find the period.

$$T = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}} = \frac{2\pi}{\sqrt{398{,}600}}7593^{\frac{3}{2}} = 6585\text{ s}$$
$$\boxed{T = 1.829\text{ h}}$$

2.8 Parabolic trajectories ($e = 1$)

If the eccentricity equals 1, then the orbit equation (Eqn (2.45)) becomes

$$r = \frac{h^2}{\mu}\frac{1}{1+\cos\theta} \tag{2.89}$$

As the true anomaly θ approaches 180°, the denominator approaches zero, so that r tends toward infinity. According to Eqn (2.60), the energy of a trajectory for which $e = 1$ is zero, so that for a parabolic trajectory the conservation of energy (Eqn (2.57)) is

$$\frac{v^2}{2} - \frac{\mu}{r} = 0$$

In other words, the speed anywhere on a parabolic path is

$$v = \sqrt{\frac{2\mu}{r}} \tag{2.90}$$

If the body m_2 is launched on a parabolic trajectory, it will coast to infinity, arriving there with zero velocity relative to m_1. It will not return. Parabolic paths are therefore called escape trajectories. At a given distance r from m_1, the escape velocity is given by Eqn (2.90),

$$v_{\text{esc}} = \sqrt{\frac{2\mu}{r}} \tag{2.91}$$

Let v_c be the speed of a satellite in a circular orbit of radius r. Then, from Eqns (2.63) and (2.91), we have

$$v_{\text{esc}} = \sqrt{2}v_c \tag{2.92}$$

That is, to escape from a circular orbit requires a velocity boost of 41.4%. However, remember our assumption is that m_1 and m_2 are the only objects in the universe. A spacecraft launched from the earth with a velocity v_{esc} (relative to the earth) will not coast to infinity (i.e., leave the solar system) because it will eventually succumb to the gravitational influence of the sun and, in fact, end up in the same orbit as the earth. This will be discussed in more detail in Chapter 8.

For the parabola, Eqn (2.52) for the flight path angle takes the form

$$\tan\gamma = \frac{\sin\theta}{1+\cos\theta}$$

Using the trigonometric identities

$$\sin\theta = 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$

$$\cos\theta = \cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2} = 2\cos^2\tfrac{\theta}{2} - 1$$

we can write

$$\tan\gamma = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

It follows that

$$\gamma = \frac{\theta}{2} \tag{2.93}$$

That is, on parabolic trajectories the flight path angle is always one-half of the true anomaly (Figure 2.22).

Equation (2.53) gives the parameter p of an orbit. Let us substitute that expression into Eqn (2.89) and then plot $r = p/(1 + \cos\theta)$ in a Cartesian coordinate system centered at the focus, as illustrated in Figure 2.23. From the figure, it is clear that

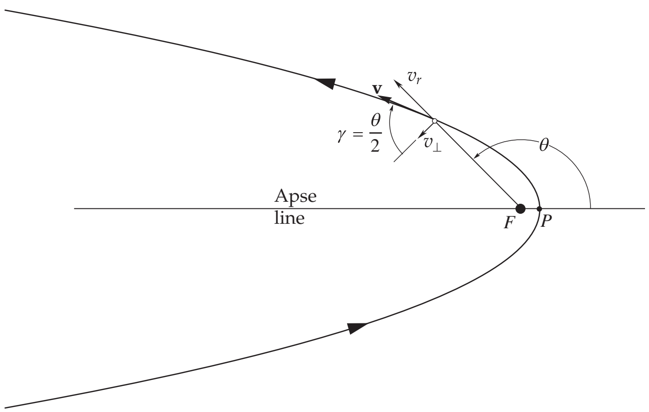


FIGURE 2.22

Parabolic trajectory around the focus F.

$$x = r\cos\theta = p\frac{\cos\theta}{1+\cos\theta} \tag{2.94a}$$

$$y = r\sin\theta = p\frac{\sin\theta}{1+\cos\theta} \tag{2.94b}$$

Therefore,

$$\frac{x}{p/2} + \left(\frac{y}{p}\right)^2 = 2\frac{\cos\theta}{1+\cos\theta} + \frac{\sin^2\theta}{(1+\cos\theta)^2}$$

Working to simplify the right-hand side, we get

$$\frac{x}{p/2} + \left(\frac{y}{p}\right)^2 = \frac{2\cos\theta(1+\cos\theta)+\sin^2\theta}{(1+\cos\theta)^2} = \frac{2\cos\theta + 2\cos^2\theta + \left(1-\cos^2\theta\right)}{(1+\cos\theta)^2}$$

$$= \frac{1+2\cos\theta+\cos^2\theta}{(1+\cos\theta)^2} = \frac{(1+\cos\theta)^2}{(1+\cos\theta)^2} = 1$$

It follows that

$$x = \frac{p}{2} - \frac{y^2}{2p} \tag{2.95}$$

This is the equation of a parabola in a Cartesian coordinate system whose origin serves as the focus.

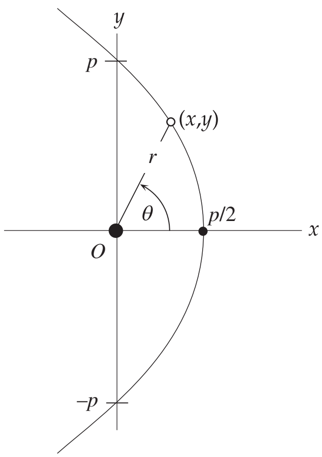


FIGURE 2.23

Parabola with focus at the origin of the cartesian coordinate system.

EXAMPLE 2.9

The perigee of a satellite in a parabolic geocentric trajectory of Figure 2.24 is 7000 km. Find the distance d between points P_1 and P_2 on the orbit, which are 8000 and 16,000 km, respectively, from the center of the earth.

Solution

This would be a simple trigonometry problem if we knew the angle $\Delta\theta$ between the radials to P_1 and P_2. We can find that angle by first determining the true anomalies of the two points. The true anomalies are obtained from the orbit formula, Eqn (2.89), once we have determined the angular momentum h.

We calculate the angular momentum of the satellite by evaluating the orbit equation at perigee,

$$r_p = \frac{h^2}{\mu}\frac{1}{1+\cos(0)} = \frac{h^2}{2\mu}$$

from which

$$h = \sqrt{2\mu r_p} = \sqrt{2\cdot 398{,}600\cdot 7000} = 74{,}700\ \text{km}^2/\text{s} \tag{a}$$

Substituting the radii and the true anomalies of points P_1 and P_2 into Eqn (2.89), we get

$$8000 = \frac{74{,}700^2}{398{,}600}\frac{1}{1+\cos\theta_1} \Rightarrow \cos\theta_1 = 0.75 \Rightarrow \theta_1 = 41.41^\circ$$

$$16{,}000 = \frac{74{,}700^2}{398{,}600}\frac{1}{1+\cos\theta_2} \Rightarrow \cos\theta_2 = -0.125 \Rightarrow \theta_2 = 97.18^\circ$$

The difference between these two angles is $\Delta\theta = 97.18^\circ - 41.41^\circ = 55.78^\circ$.

The length of the chord $\overline{P_1P_2}$ can now be found by using the law of cosines from trigonometry,

$$d^2 = 8000^2 + 16{,}000^2 - 2\cdot 8000\cdot 16{,}000\cos\Delta\theta$$

$$\boxed{d = 13{,}270\ \text{km}}$$

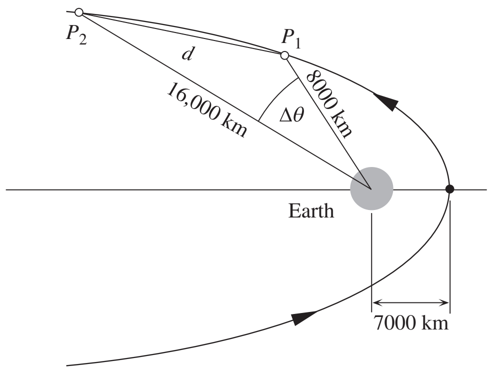


FIGURE 2.24

Parabolic geocentric trajectory.

2.9 Hyperbolic trajectories ($e > 1$)

If $e > 1$, the orbit formula,

$$r = \frac{h^2}{\mu} \frac{1}{1 + e \cos \theta} \tag{2.96}$$

describes the geometry of the hyperbola shown in Figure 2.25. The system consists of two symmetric curves. The orbiting body occupies one of them. The other one is its empty mathematical image. Clearly, the denominator of Eqn (2.96) goes to zero when $\cos \theta = -1/e$. We denote this value of true anomaly as

$$\theta_\infty = \cos^{-1}(-1/e) \tag{2.97}$$

since the radial distance approaches infinity as the true anomaly approaches θ_∞. θ_∞ is known as the true anomaly of the asymptote. Observe that θ_∞ lies between 90° and 180°. From the trig identity $\sin^2 \theta_\infty + \cos^2 \theta_\infty = 1$ it follows that

$$\sin \theta_\infty = \frac{\sqrt{e^2 - 1}}{e} \tag{2.98}$$

For $-\theta_\infty < \theta < \theta_\infty$, the physical trajectory is the occupied hyperbola *I* shown on the left in Figure 2.25. For $\theta_\infty < \theta < (360° - \theta_\infty)$, hyperbola *II*—the vacant orbit around the empty focus *F'*—is traced out. (The vacant orbit is physically impossible, because it would require a repulsive gravitational force.) Periapsis *P* lies on the apse line on the physical hyperbola *I*, whereas apoapsis

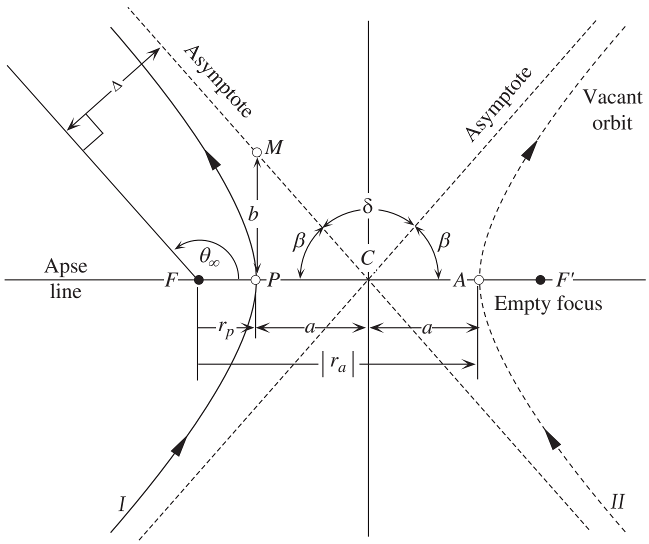


FIGURE 2.25

Hyperbolic trajectory.

A lies on the apse line on the vacant orbit. The point halfway between periapsis and apoapsis is the center C of the hyperbola. The asymptotes of the hyperbola are the straight lines toward which the curves tend as they approach infinity. The asymptotes intersect at C, making an acute angle β with the apse line, where $\beta = 180° - \theta_\infty$. Therefore, $\cos\beta = -\cos\theta_\infty$, which means

$$\beta = \cos^{-1}(1/e) \tag{2.99}$$

The angle δ between the asymptotes is called the turn angle. This is the angle through which the velocity vector of the orbiting body is rotated as it rounds the attracting body at F and heads back toward infinity. From the figure, we see that $\delta = 180° - 2\beta$, so that

$$\sin\frac{\delta}{2} = \sin\left(\frac{180° - 2\beta}{2}\right) = \sin(90° - \beta) = \cos\beta \overbrace{=}^{\text{Eqn (2.89)}} \frac{1}{e}$$

or

$$\delta = 2\sin^{-1}(1/e) \tag{2.100}$$

Equation (2.50) gives the distance r_p from the focus F to the periapsis,

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e} \tag{2.101}$$

Just as for an ellipse, the radial coordinate r_a of the apoapsis is found by setting $\theta = 180°$ in Eqn (2.45),

$$r_a = \frac{h^2}{\mu}\frac{1}{1-e} \tag{2.102}$$

Observe that r_a is negative, since $e > 1$ for the hyperbola. This means the apoapsis lies to the right of the focus F. From Figure 2.25 we see that the distance $2a$ from periapsis P to apoapsis A is

$$2a = |r_a| - r_p = -r_a - r_p$$

Substituting Eqns (2.101) and (2.102) yields

$$2a = -\frac{h^2}{\mu}\left(\frac{1}{1-e} + \frac{1}{1+e}\right)$$

From this it follows that a, the semimajor axis of the hyperbola, is given by an expression that is nearly identical to that for an ellipse (Eqn (2.72)),

$$a = \frac{h^2}{\mu}\frac{1}{e^2-1} \tag{2.103}$$

Therefore, Eqn (2.96) may be written for the hyperbola

$$r = a\frac{e^2-1}{1+e\cos\theta} \tag{2.104}$$

This formula is analogous to Eqn (2.72) for the elliptical orbit. Furthermore, from Eqn (2.104) it follows that

$$r_p = a(e-1) \tag{2.105a}$$

$$r_a = -a(e+1) \tag{2.105b}$$

The distance b from periapsis to an asymptote, measured perpendicular to the apse line, is the semiminor axis of the hyperbola. From Figure 2.25, we see that the length b of the semiminor axis $\overline{PM}$ is

$$b = a\tan\beta = a\frac{\sin\beta}{\cos\beta} = a\frac{\sin(180-\theta_\infty)}{\cos(180-\theta_\infty)} = a\frac{\sin\theta_\infty}{-\cos\theta_\infty} = a\frac{\frac{\sqrt{e^2-1}}{e}}{-\left(-\frac{1}{e}\right)}$$

so that for the hyperbola,

$$b = a\sqrt{e^2-1} \tag{2.106}$$

This relation is analogous to Eqn (2.76) for the semiminor axis of an ellipse.

The distance Δ between the asymptote and a parallel line through the focus is called the aiming radius, which is illustrated in Figure 2.25. From this figure we see that

$$\begin{aligned}
\Delta &= \left(r_p + a\right)\sin\beta \\
&= ae\sin\beta \quad (\text{Eqn (2.105a)}) \\
&= ae\frac{\sqrt{e^2-1}}{e} \quad (\text{Eqn (2.99)}) \\
&= ae\sin\theta_\infty \quad (\text{Eqn (2.98)}) \\
&= ae\sqrt{1-\cos^2\theta_\infty} (\text{trig identity}) \\
&= ae\sqrt{1-\frac{1}{e^2}} \quad (\text{Eqn (2.97)})
\end{aligned}$$

or

$$\Delta = a\sqrt{e^2-1} \tag{2.107}$$

Comparing this result with Eqn (2.106), it is clear that the aiming radius equals the length of the semiminor axis of the hyperbola.

As with the ellipse and the parabola, we can express the polar form of the equation of the hyperbola in a Cartesian coordinate system whose origin is in this case midway between the two foci, as illustrated in Figure 2.26. From the figure, it is apparent that

$$x = -a - r_p + r\cos\theta \tag{2.108a}$$

$$y = r\sin\theta \tag{2.108b}$$

Using Eqns (2.104) and (2.105a) in Eqn (2.108a), we obtain

$$x = -a - a(e-1) + a\frac{e^2-1}{1+e\cos\theta}\cos\theta = -a\frac{e+\cos\theta}{1+e\cos\theta}$$

Substituting Eqns (2.104) and (2.106) into Eqn (2.108b) yields

$$y = \frac{b}{\sqrt{e^2-1}}\,\frac{e^2-1}{1+e\cos\theta}\sin\theta = b\frac{\sqrt{e^2-1}\,\sin\theta}{1+e\cos\theta}$$

It follows that

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{e + \cos\theta}{1 + e\cos\theta}\right)^2 - \left(\frac{\sqrt{e^2 - 1}\sin\theta}{1 + e\cos\theta}\right)^2$$
$$= \frac{e^2 + 2e\cos\theta + \cos^2\theta - \left(e^2 - 1\right)\left(1 - \cos^2\theta\right)}{(1 + e\cos\theta)^2}$$
$$= \frac{1 + 2e\cos\theta + e^2\cos^2\theta}{(1 + e\cos\theta)^2} = \frac{(1 + e\cos\theta)^2}{(1 + e\cos\theta)^2}$$

That is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{2.109}$$

This is the familiar equation of a hyperbola that is symmetric about the x and y axes, with intercepts on the x-axis.

Equation (2.60) gives the specific energy of the hyperbolic trajectory. Substituting Eqn (2.103) into that expression yields

$$\varepsilon = \frac{\mu}{2a} \tag{2.110}$$

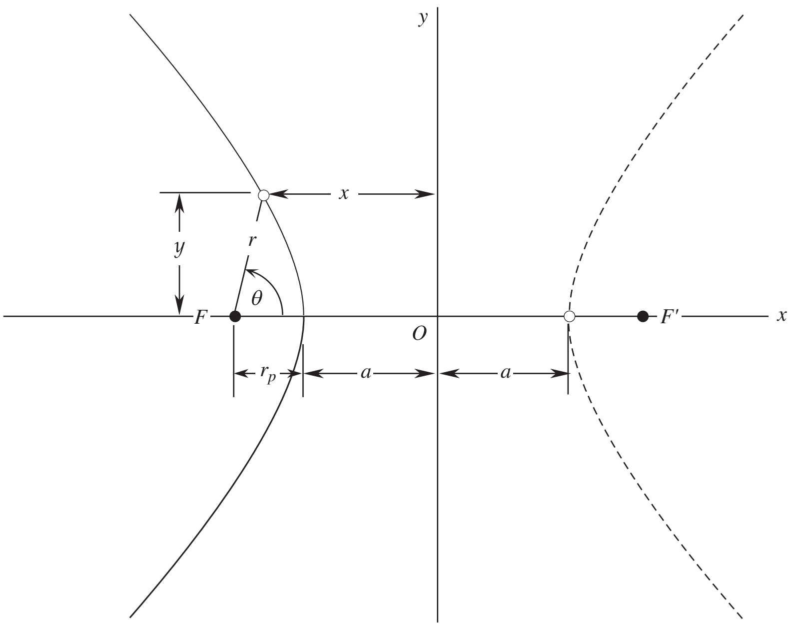


FIGURE 2.26

Plot of eqn (2.104) in a cartesian coordinate system with origin O midway between the two foci.

The specific energy of a hyperbolic orbit is clearly positive and independent of the eccentricity. The conservation of energy for a hyperbolic trajectory is

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a} \tag{2.111}$$

Let v_∞ denote the speed at which a body on a hyperbolic path arrives at infinity. According to Eqn (2.111)

$$v_\infty = \sqrt{\frac{\mu}{a}} \tag{2.112}$$

v_∞ is called the hyperbolic excess speed. In terms of v_∞ we may write Eqn (2.111) as

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{v_\infty^2}{2}$$

Substituting the expression for escape speed, $v_{\text{esc}} = \sqrt{2\mu/r}$ (Eqn (2.91)), we obtain for a hyperbolic trajectory

$$v^2 = v_{\text{esc}}^2 + v_\infty^2 \tag{2.113}$$

This equation clearly shows that the hyperbolic excess speed v_∞ represents the excess kinetic energy over that which is required to simply escape from the center of attraction. The square of v_∞ is denoted C_3, and is known as the characteristic energy,

$$C_3 = v_\infty^2 \tag{2.114}$$

C_3 is a measure of the energy required for an interplanetary mission, and C_3 is also a measure of the maximum energy a launch vehicle can impart to a spacecraft of a given mass. Obviously, to match a launch vehicle with a mission, $C_3)_{\text{launch vehicle}} > C_3)_{\text{mission}}$.

Note that the hyperbolic excess speed can also be obtained from Eqns (2.49) and (2.98),

$$v_\infty = \frac{\mu}{h} e \sin\theta_\infty = \frac{\mu}{h}\sqrt{e^2 - 1} \tag{2.115}$$

Finally, for purposes of comparison, Figure 2.27 shows a range of trajectories, from a circle through hyperbolas, all having a common focus and periapsis. The parabola is the demarcation between the closed, negative energy orbits (ellipses) and open, positive energy orbits (hyperbolas).

At this point, the reader may be understandably overwhelmed by the number of equations for Keplerian orbits (conic sections) that have been presented thus far in this chapter. As summarized in the Road Map in Appendix B, there is just a small set of equations from which all the others are derived.

Here is a "tool box" of the only equations necessary for solving two-dimensional curvilinear orbital problems that do not involve time, which is the subject of Chapter 3.

All orbits:

$$h = rv_\perp \qquad \text{Eqn (2.31)}$$

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta} \qquad \text{Eqn (2.45)}$$

$$v_r = \frac{\mu}{h} e \sin\theta \qquad \text{Eqn (2.49)}$$

$$\tan\gamma = \frac{v_r}{v_\perp} \qquad \text{Eqn (2.51)}$$

$$v = \sqrt{v_r^2 + v_\perp^2}$$

Ellipses ($0 \leq e < 1$):

$$a = \frac{r_p + r_a}{2} = \frac{h^2}{\mu}\frac{1}{1 - e^2} \quad \text{Eqn (2.71)}$$

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \quad \text{Eqn (2.81)}$$

$$T = \frac{2\pi}{\sqrt{\mu}}a^{3/2} \quad \text{Eqn (2.83)}$$

$$e = \frac{r_a - r_p}{r_a + r_p} \quad \text{Eqn (2.84)}$$

Parabolas ($e = 1$):

$$\frac{v^2}{2} - \frac{\mu}{r} = 0 \quad \text{Eqn (2.90)}$$

Hyperbolas ($e > 1$):

$$\theta_\infty = \cos^{-1}\left(-\frac{1}{e}\right) \quad \text{Eqn (2.97)}$$

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) \quad \text{Eqn (2.100)}$$

$$a = \frac{h^2}{\mu}\frac{1}{e^2 - 1} \quad \text{Eqn (2.103)}$$

$$\Delta = a\sqrt{e^2 - 1} \quad \text{Eqn (2.107)}$$

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a} \quad \text{Eqn (2.111)}$$

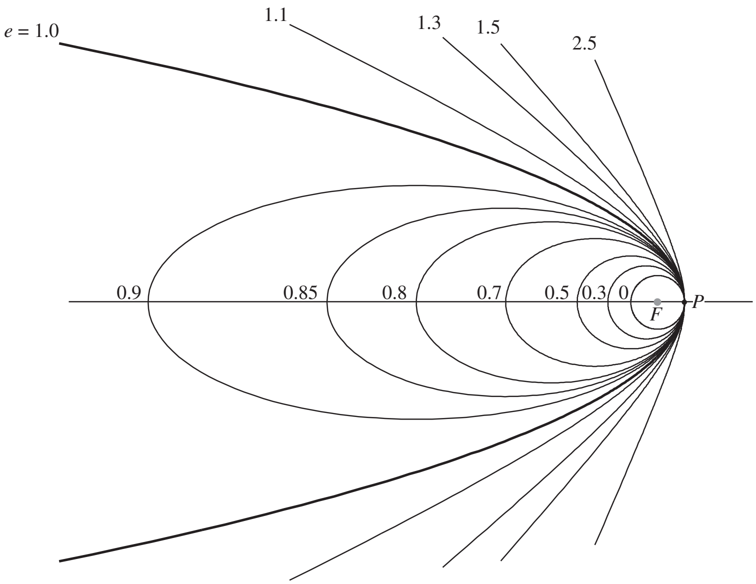


FIGURE 2.27

Orbits of various eccentricities, having a common focus *F* and periapsis *P*.

Note that we can rewrite Eqns (2.103) and (2.111) as follows (where a is positive),

$$-a = \frac{h^2}{\mu}\frac{1}{1-e^2} \qquad \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2(-a)}$$

That is, if we assume that the semimajor axis of a hyperbola has a negative value, then the semimajor axis formula and the vis viva equation become identical for ellipses and hyperbolas. There is no advantage at this point in requiring hyperbolas to have negative semimajor axes. However, doing so will be necessary for the universal variable formulation to be presented in the next chapter.

EXAMPLE 2.10

At a given point of a spacecraft's geocentric trajectory, the radius is 14,600 km, the speed is 8.6 km/s, and the flight path angle is 50°. Show that the path is a hyperbola and calculate the following:

(a) Angular momentum
(b) Eccentricity
(c) True anomaly
(d) Radius of the perigee
(e) Semimajor axis
(f) C_3
(g) Turn angle
(h) Aiming radius

This problem is illustrated in Figure 2.28.

Solution

Since both the radius and the speed are given, we can determine the type of trajectory by comparing the speed to the escape speed (of a parabolic trajectory) at the given radius:

$$v_{esc} = \sqrt{\frac{2\mu}{r}} = \sqrt{\frac{2\cdot 398{,}600}{14{,}600}} = 7.389 \text{ km/s}$$

The escape speed is less than the spacecraft's speed of 8.6 km/s, which means the path is a hyperbola.

(a) Before embarking on a quest for the required orbital data, remember that everything depends on the primary orbital parameters, angular momentum h and eccentricity e. These are among the list of five unknowns for this problem: h, e, θ, v_r, and $v_\perp$. From the "tool box" we have five equations involving these five quantities and the given data:

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} \tag{a}$$

$$v_r = \frac{\mu}{h}e\sin\theta \tag{b}$$

$$v_\perp = \frac{h}{r} \tag{c}$$

$$v = \sqrt{v_r^2 + v_\perp^2} \tag{d}$$

$$\tan\gamma = \frac{v_r}{v_\perp} \tag{e}$$

From Eqn (e)

$$v_r = v_\perp \tan 50° = 1.1918 v_\perp \tag{f}$$

Substituting this and the given speed into Eqn (d) yields

$$8.6^2 = (1.1918 v_\perp)^2 + v_\perp^2 \Rightarrow v_\perp = 5.528 \text{ km/s} \tag{g}$$

The angular momentum may now be found from Eqn (c),

$$h = 14{,}600 \cdot 5.528 = \boxed{80{,}708 \text{ km}^2/\text{s}}$$

(b) Substituting h into Eqn (f) we get the radial velocity component,

$$v_r = 1.1918 \cdot 5.528 = 6.588 \text{ km/s}$$

Substituting h and v_r into Eqn (b) yields an expression involving the eccentricity and the true anomaly,

$$6.588 = \frac{398{,}600}{80{,}708} e \sin\theta \Rightarrow e \sin\theta = 1.3339 \tag{h}$$

Similarly, substituting h and r into Eqn (a) we find

$$14{,}600 = \frac{80{,}708^2}{398{,}600} \frac{1}{1 + e\cos\theta} \Rightarrow e\cos\theta = 0.1193 \tag{i}$$

By squaring the expressions in Eqns (h) and (i) and then summing them, we obtain the eccentricity,

$$e^2 \overbrace{\left(\sin^2\theta + \cos^2\theta\right)}^{=1} = 1.7936$$

$$\boxed{e = 1.3393}$$

(c) To find the true anomaly, substitute the value of e into Eqn (i),

$$1.3393 \cos\theta = 0.1193 \Rightarrow \theta = 84.889^\circ \text{ or } \theta = 275.11^\circ$$

We choose the smaller of the angles because Eqns (h) and (i) imply that both $\sin\theta$ and $\cos\theta$ are positive, which means θ lies in the first quadrant ($\theta \le 90^\circ$). Alternatively, we may note that the given flight path angle (50°) is positive, which means the spacecraft is flying away from the perigee, so that the true anomaly must be $<180^\circ$. In any case, the true anomaly is given by $\boxed{\theta = 84.889^\circ}$.

(d) The radius of perigee can now be found from the orbit equation Eqn (a)

$$r_p = \frac{h^2}{\mu} \frac{1}{1 + e\cos(0)} = \frac{80{,}710^2}{398{,}600} \frac{1}{1 + 1.339} = \boxed{6986 \text{ km}}$$

(e) The semimajor axis of the hyperbola is found in Eqn (2.103),

$$a = \frac{h^2}{\mu} \frac{1}{e^2 - 1} = \frac{80{,}710^2}{398{,}600} \frac{1}{1.339^2 - 1} = \boxed{20{,}590 \text{ km}}$$

(f) The hyperbolic excess velocity is found using Eqn (2.113),

$$v_\infty^2 = v^2 - v_{esc}^2 = 8.6^2 - 7.389^2 = 19.36 \text{ km}^2/\text{s}^2$$

From Eqn (2.114) it follows that

$$\boxed{C_3 = 19.36 \text{ km}^2/\text{s}^2}$$

(g) The formula for turn angle is Eqn (2.100), from which

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = 2\sin^{-1}\left(\frac{1}{1.339}\right) = \boxed{96.60^\circ}$$

(h) According to Eqn (2.107), the aiming radius is

$$\Delta = a\sqrt{e^2-1} = 20{,}590\sqrt{1.339^2-1} = \boxed{18{,}340\ \text{km}}$$

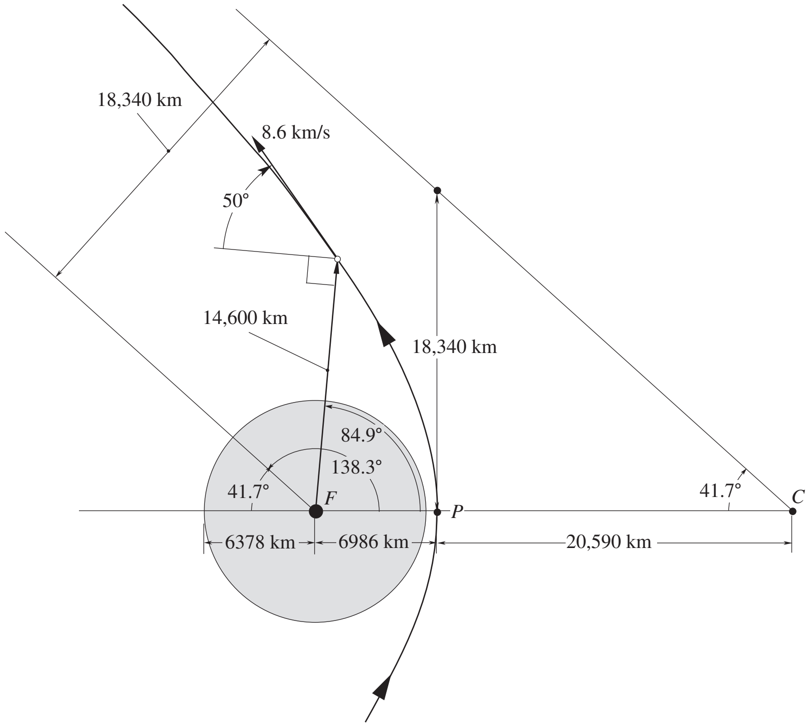


FIGURE 2.28

Solution of Example 2.10.

2.10 Perifocal frame

The perifocal frame is the "natural frame" for an orbit. It is a Cartesian coordinate system fixed in space and centered at the focus of the orbit. Its $\bar{x}\,\bar{y}$ plane is the plane of the orbit, and its $\bar{x}$-axis is directed from the focus through the periapsis, as illustrated in Figure 2.29. The unit vector along the $\bar{x}$-axis (the apse line) is denoted $\hat{\mathbf{p}}$. The $\bar{y}$-axis, with unit vector $\hat{\mathbf{q}}$, lies at 90° true anomaly to the $\bar{x}$-axis. The $\bar{z}$-axis is normal to the plane of the orbit in the direction of the angular momentum vector $\mathbf{h}$. The $\bar{z}$ unit vector is $\hat{\mathbf{w}}$,

$$\hat{\mathbf{w}} = \frac{\mathbf{h}}{h} \tag{2.116}$$

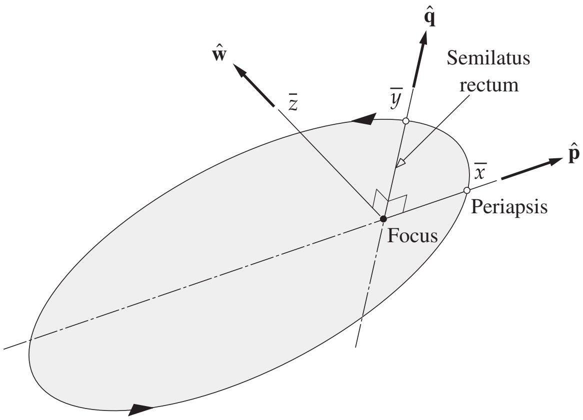


FIGURE 2.29

Perifocal frame $\hat{\mathbf{p}}\hat{\mathbf{q}}\hat{\mathbf{w}}$.

In the perifocal frame, the position vector $\mathbf{r}$ is written (Figure 2.30)

$$\mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} \tag{2.117}$$

where

$$\bar{x} = r\cos\theta \quad \bar{y} = r\sin\theta \tag{2.118}$$

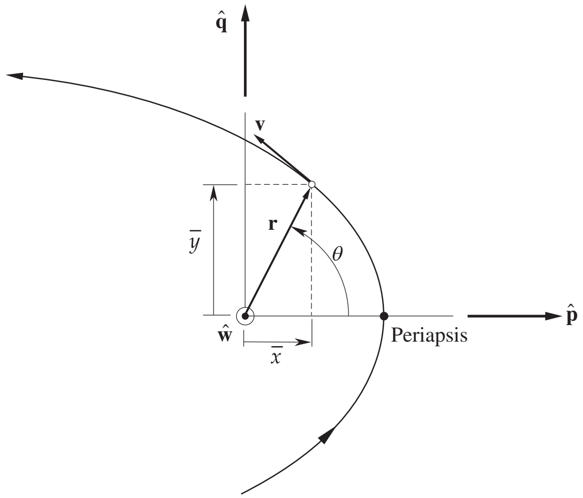


FIGURE 2.30

Position and velocity relative to the perifocal frame.

and r, the magnitude of $\mathbf{r}$, is given by the orbit equation, $r = (h^2/\mu)[1/(1 + e\cos\theta)]$. Thus, we may write Eqn (2.117) as

$$\mathbf{r} = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}(\cos\theta\hat{\mathbf{p}} + \sin\theta\hat{\mathbf{q}}) \tag{2.119}$$

The velocity is found by taking the time derivative of $\mathbf{r}$,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} \tag{2.120}$$

From Eqn (2.118) we obtain

$$\dot{\bar{x}} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \quad \dot{\bar{y}} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{2.121}$$

$\dot{r}$ is the radial component of velocity, v_r. Therefore, according to Eqn (2.49),

$$\dot{r} = \frac{\mu}{h}e\sin\theta \tag{2.122}$$

From Eqns (2.46) and (2.48), we have

$$r\dot{\theta} = v_\perp = \frac{\mu}{h}(1 + e\cos\theta) \tag{2.123}$$

Substituting Eqns (2.122) and (2.123) into Eqn (2.121) and simplifying the results yields

$$\dot{\bar{x}} = -\frac{\mu}{h}\sin\theta \quad \dot{\bar{y}} = \frac{\mu}{h}(e + \cos\theta) \tag{2.124}$$

Hence, Eqn (2.120) becomes

$$\mathbf{v} = \frac{\mu}{h}[-\sin\theta\hat{\mathbf{p}} + (e + \cos\theta)\hat{\mathbf{q}}] \tag{2.125}$$

Formulating the kinematics of orbital motion in the perifocal frame, as we have done here, is a prelude to the study of orbits in three dimensions (Chapter 4). We also need Eqns (2.117) and (2.120) in the next section.

EXAMPLE 2.11

An earth orbit has an eccentricity of 0.3, an angular momentum of 60,000 km^2/s, and a true anomaly of 120°. What are the position vector **r** and velocity vector **v** in the perifocal frame of reference?

Solution

From Eqn (2.119) we have

$$\mathbf{r} = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta}(\cos\theta\hat{\mathbf{p}} + \sin\theta\hat{\mathbf{q}}) = \frac{60{,}000^2}{398{,}600}\frac{1}{1 + 0.3\cos 120^\circ}(\cos 120^\circ\hat{\mathbf{p}} + \sin 120^\circ\hat{\mathbf{q}})$$

$$\boxed{\mathbf{r} = -5312.7\hat{\mathbf{p}} + 9201.9\hat{\mathbf{q}}\ (\mathrm{km})}$$

Substituting the given data into Eqn (2.125) yields

$$\mathbf{v} = \frac{\mu}{h}[-\sin\theta\hat{\mathbf{p}} + (e + \cos\theta)\hat{\mathbf{q}}] = \frac{398{,}600}{60{,}000}[-\sin 120^\circ\hat{\mathbf{p}} + (0.3 + \cos 120^\circ)\hat{\mathbf{q}}]$$

$$\boxed{\mathbf{v} = -5.7533\hat{\mathbf{p}} - 1.3287\hat{\mathbf{q}}\ (\mathrm{km/s})}$$

EXAMPLE 2.12

An earth satellite has the following position and velocity vectors at a given instant:

$$\mathbf{r} = 7000\hat{\mathbf{p}} + 9000\hat{\mathbf{q}}\ (\text{km})$$
$$\mathbf{v} = -5\hat{\mathbf{p}} + 7\hat{\mathbf{q}}\ (\text{km/s})$$

Calculate the specific angular momentum h, the true anomaly θ, and the eccentricity e.

Solution

This problem is obviously the reverse of the situation presented in the previous example. From Eqn (2.28) the angular momentum is

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{p}} & \hat{\mathbf{q}} & \hat{\mathbf{w}} \\ 7000 & 9000 & 0 \\ -5 & 7 & 0 \end{vmatrix} = 94{,}000\hat{\mathbf{w}}\left(\text{km}^2/\text{s}\right)$$

Hence, the magnitude of the angular momentum is

$$\boxed{h = 94{,}000\ \text{km}^2/\text{s}}$$

The true anomaly is measured from the positive $\bar{x}$-axis. By definition of the dot product, $\mathbf{r}\cdot\hat{\mathbf{p}} = r\cos\theta$. Thus,

$$\cos\theta = \frac{\mathbf{r}}{r}\cdot\hat{\mathbf{p}} = \frac{7000\hat{\mathbf{p}} + 9000\hat{\mathbf{q}}}{\sqrt{7000^2 + 9000^2}}\cdot\hat{\mathbf{p}} = \frac{7000}{11{,}402} = 0.61394$$

which means $\theta = 52.125°$ or $\theta = -52.125°$. Since the $\bar{y}$ component of $\mathbf{r}$ is positive, the true anomaly must lie between 0 and 180°. It follows that

$$\boxed{\theta = 52.125°}$$

Finally, the eccentricity may be found from the orbit formula, $r = (h^2/\mu)/(1 + e\cos\theta)$:

$$\sqrt{7000^2 + 9000^2} = \frac{94{,}000^2}{398{,}600}\,\frac{1}{1 + e\cos 52.125°}$$
$$\boxed{e = 1.538}$$

The trajectory is a hyperbola.

2.11 The Lagrange coefficients

In this section, we will establish what may seem intuitively obvious: if the position and velocity of an orbiting body are known at a given instant, then the position and velocity at any later time are found in terms of the initial values. Let us start with Eqns (2.117) and (2.120),

$$\mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} \tag{2.126}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} \tag{2.127}$$

Attach a subscript "zero" to quantities evaluated at time $t = t_0$. Then the expressions for $\mathbf{r}$ and $\mathbf{v}$ evaluated at $t = t_0$ are

$$\mathbf{r}_0 = \bar{x}_0\hat{\mathbf{p}} + \bar{y}_0\hat{\mathbf{q}} \tag{2.128}$$

$$\mathbf{v}_0 = \dot{\bar{x}}_0\hat{\mathbf{p}} + \dot{\bar{y}}_0\hat{\mathbf{q}} \tag{2.129}$$

The angular momentum $\mathbf{h}$ is constant, so let us calculate it using the initial conditions. Substituting Eqns (2.128) and (2.129) into Eqn (2.28) yields

$$\mathbf{h} = \mathbf{r}_0 \times \mathbf{v}_0 = \begin{vmatrix} \hat{\mathbf{p}} & \hat{\mathbf{q}} & \hat{\mathbf{w}} \\ \bar{x}_0 & \bar{y}_0 & 0 \\ \dot{\bar{x}}_0 & \dot{\bar{y}}_0 & 0 \end{vmatrix} = \hat{\mathbf{w}}\left(\bar{x}_0\dot{\bar{y}}_0 - \bar{y}_0\dot{\bar{x}}_0\right) \tag{2.130}$$

Recall that $\hat{\mathbf{w}}$ is the unit vector in the direction of $\mathbf{h}$ (Eqn (2.116)). Therefore, the coefficient of $\hat{\mathbf{w}}$ on the right side of Eqn (2.130) must be the magnitude of the angular momentum. That is,

$$h = \bar{x}_0\dot{\bar{y}}_0 - \bar{y}_0\dot{\bar{x}}_0 \tag{2.131}$$

Now let us solve the two vector Eqns (2.128) and (2.129) for the unit vectors $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ in terms of $\mathbf{r}_0$ and $\mathbf{v}_0$. From Eqn (2.128) we get

$$\hat{\mathbf{q}} = \frac{1}{\bar{y}_0}\mathbf{r}_0 - \frac{\bar{x}_0}{\bar{y}_0}\hat{\mathbf{p}} \tag{2.132}$$

Substituting this into Eqn (2.129), combining terms, and using Eqn (2.130) yields

$$\mathbf{v}_0 = \dot{\bar{x}}_0\hat{\mathbf{p}} + \dot{\bar{y}}_0\left(\frac{1}{\bar{y}_0}\mathbf{r}_0 - \frac{\bar{x}_0}{\bar{y}_0}\hat{\mathbf{p}}\right) = \frac{\bar{y}_0\dot{\bar{x}}_0 - \bar{x}_0\dot{\bar{y}}_0}{\bar{y}_0}\hat{\mathbf{p}} + \frac{\dot{\bar{y}}_0}{\bar{y}_0}\mathbf{r}_0 = -\frac{h}{\bar{y}_0}\hat{\mathbf{p}} + \frac{\dot{\bar{y}}_0}{\bar{y}_0}\mathbf{r}_0$$

Solve this for $\hat{\mathbf{p}}$ to obtain

$$\hat{\mathbf{p}} = \frac{\dot{\bar{y}}_0}{h}\mathbf{r}_0 - \frac{\bar{y}_0}{h}\mathbf{v}_0 \tag{2.133}$$

Putting this result back into Eqn (2.132) gives

$$\hat{\mathbf{q}} = \frac{1}{\bar{y}_0}\mathbf{r}_0 - \frac{\bar{x}_0}{\bar{y}_0}\left(\frac{\dot{\bar{y}}_0}{h}\mathbf{r}_0 - \frac{\bar{y}_0}{h}\mathbf{v}_0\right) = \frac{h - \bar{x}_0\dot{\bar{y}}_0}{\bar{y}_0}\mathbf{r}_0 + \frac{\bar{x}_0}{h}\mathbf{v}_0$$

Upon replacing h with the right-hand side of Eqn (2.131) we get

$$\hat{\mathbf{q}} = -\frac{\dot{\bar{x}}_0}{h}\mathbf{r}_0 + \frac{\bar{x}_0}{h}\mathbf{v}_0 \tag{2.134}$$

Equations (2.133) and (2.134) give $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ in terms of the initial position and velocity. Substituting those two expressions back into Eqns (2.126) and (2.127) yields, respectively

$$\mathbf{r} = \bar{x}\left(\frac{\dot{\bar{y}}_0}{h}\mathbf{r}_0 - \frac{\bar{y}_0}{h}\mathbf{v}_0\right) + \bar{y}\left(-\frac{\dot{\bar{x}}_0}{h}\mathbf{r}_0 + \frac{\bar{x}_0}{h}\mathbf{v}_0\right) = \frac{\bar{x}\,\dot{\bar{y}}_0 - \bar{y}\,\dot{\bar{x}}_0}{h}\mathbf{r}_0 + \frac{-\bar{x}\,\bar{y}_0 + \bar{y}\,\bar{x}_0}{h}\mathbf{v}_0$$

$$\mathbf{v} = \dot{\bar{x}}\left(\frac{\dot{\bar{y}}_0}{h}\mathbf{r}_0 - \frac{\bar{y}_0}{h}\mathbf{v}_0\right) + \dot{\bar{y}}\left(-\frac{\dot{\bar{x}}_0}{h}\mathbf{r}_0 + \frac{\bar{x}_0}{h}\mathbf{v}_0\right) = \frac{\dot{\bar{x}}\,\dot{\bar{y}}_0 - \dot{\bar{y}}\,\dot{\bar{x}}_0}{h}\mathbf{r}_0 + \frac{-\dot{\bar{x}}\,\bar{y}_0 + \dot{\bar{y}}\,\bar{x}_0}{h}\mathbf{v}_0$$

Therefore,

$$\mathbf{r} = f\mathbf{r}_0 + g\mathbf{v}_0 \tag{2.135}$$

$$\mathbf{v} = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0 \tag{2.136}$$

where f and g are given by

$$f = \frac{\bar{x}\,\dot{\bar{y}}_0 - \bar{y}\,\dot{\bar{x}}_0}{h} \tag{2.137a}$$

$$g = \frac{-\bar{x}\,\bar{y}_0 + \bar{y}\,\bar{x}_0}{h} \tag{2.137b}$$

together with their time derivatives

$$\dot{f} = \frac{\dot{\bar{x}}\,\dot{\bar{y}}_0 - \dot{\bar{y}}\,\dot{\bar{x}}_0}{h} \tag{2.138a}$$

$$\dot{g} = \frac{-\dot{\bar{x}}\,\bar{y}_0 + \dot{\bar{y}}\,\bar{x}_0}{h} \tag{2.138b}$$

The f and g functions are referred to as the Lagrange coefficients after Joseph-Louis Lagrange (1736–1813), a French mathematical physicist whose numerous contributions include calculations of planetary motion.

From Eqns (2.135) and (2.136) we see that the position and velocity vectors $\mathbf{r}$ and $\mathbf{v}$ are indeed linear combinations of the initial position and velocity vectors. The Lagrange coefficients and their time derivatives in these expressions are themselves functions of time and the initial conditions.

Before proceeding, let us show that the conservation of angular momentum $\mathbf{h}$ imposes a condition on f and g and their time derivatives $\dot{f}$ and $\dot{g}$. Calculate $\mathbf{h}$ using Eqns (2.135) and (2.136),

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = (f\mathbf{r}_0 + g\mathbf{v}_0) \times \left(\dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0\right)$$

Expanding the right-hand side yields

$$\mathbf{h} = \left(f\mathbf{r}_0 \times \dot{f}\mathbf{r}_0\right) + (f\mathbf{r}_0 \times \dot{g}\mathbf{v}_0) + \left(g\mathbf{v}_0 \times \dot{f}\mathbf{r}_0\right) + (g\mathbf{v}_0 \times \dot{g}\mathbf{v}_0)$$

Factoring out the scalars f, g, $\dot{f}$, and $\dot{g}$, we get

$$\mathbf{h} = f\dot{f}(\mathbf{r}_0 \times \mathbf{r}_0) + f\dot{g}(\mathbf{r}_0 \times \mathbf{v}_0) + \dot{f}g(\mathbf{v}_0 \times \mathbf{r}_0) + g\dot{g}(\mathbf{v}_0 \times \mathbf{v}_0)$$

But $\mathbf{r}_0 \times \mathbf{r}_0 = \mathbf{v}_0 \times \mathbf{v}_0 = \mathbf{0}$, so

$$\mathbf{h} = f\dot{g}(\mathbf{r}_0 \times \mathbf{v}_0) + \dot{f}g(\mathbf{v}_0 \times \mathbf{r}_0)$$

Since

$$\mathbf{v}_0 \times \mathbf{r}_0 = -(\mathbf{r}_0 \times \mathbf{v}_0)$$

this reduces to

$$\mathbf{h} = \left(f\dot{g} - \dot{f}g\right)(\mathbf{r}_0 \times \mathbf{v}_0)$$

or

$$\mathbf{h} = \left(f\dot{g} - \dot{f}g\right)\mathbf{h}_0$$

where $\mathbf{h}_0 = \mathbf{r}_0 \times \mathbf{v}_0$, which is the angular momentum at $t = t_0$. But the angular momentum is constant (recall Eqn (2.29)), which means $\mathbf{h} = \mathbf{h}_0$, so that

$$\mathbf{h} = \left(f\dot{g} - \dot{f}g\right)\mathbf{h}$$

Since $\mathbf{h}$ cannot be zero (unless the body is traveling in a straight line toward the center of attraction), it follows that

$$f\dot{g} - \dot{f}g = 1 \text{ (Conservation of angular momentum)} \tag{2.139}$$

Thus, if any three of the functions f, g, $\dot{f}$, and $\dot{g}$ are known, the fourth may be found from Eqn (2.139).

Let us use Eqns (2.137) and (2.138) to evaluate the Lagrange coefficients and their time derivative in terms of the true anomaly. First of all, note that evaluating Eqn (2.118) at time $t = t_0$ yields

$$\begin{aligned} \bar{x}_0 &= r_0 \cos\theta_0 \\ \bar{y}_0 &= r_0 \sin\theta_0 \end{aligned} \tag{2.140}$$

Likewise, from Eqn (2.124) we get

$$\begin{aligned} \dot{\bar{x}}_0 &= -\tfrac{\mu}{h}\sin\theta_0 \\ \dot{\bar{y}}_0 &= \tfrac{\mu}{h}(e + \cos\theta_0) \end{aligned} \tag{2.141}$$

To evaluate the function f, we substitute Eqns (2.118) and (2.141) into Eqn (2.137a),

$$\begin{aligned} f = \tfrac{\bar{x}\,\dot{\bar{y}}_0 - \bar{y}\,\dot{\bar{x}}_0}{h} &= \tfrac{1}{h}\left\{[r\cos\theta]\left[\tfrac{\mu}{h}(e + \cos\theta_0)\right] - [r\sin\theta]\left[-\tfrac{\mu}{h}\sin\theta_0\right]\right\} \\ &= \frac{\mu r}{h^2}[e\cos\theta + (\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0)] \end{aligned} \tag{2.142}$$

If we invoke the trig identity

$$\cos(\theta - \theta_0) = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0 \tag{2.143}$$

and let $\Delta\theta$ represent the difference between the current and initial true anomalies,

$$\Delta\theta = \theta - \theta_0 \tag{2.144}$$

then Eqn (2.142) reduces to

$$f = \frac{\mu r}{h^2}(e\cos\theta + \cos\Delta\theta) \tag{2.145}$$

Finally, from Eqn (2.45), we have

$$e\cos\theta = \frac{h^2}{\mu r} - 1 \tag{2.146}$$

Substituting this into Eqn (2.145) leads to

$$f = 1 - \frac{\mu r}{h^2}(1 - \cos\Delta\theta) \tag{2.147}$$

We obtain r from the orbit formula (Eqn (2.45)) in which the true anomaly θ appears, whereas the difference in the true anomalies occurs on the right-hand side of Eqn (2.147). However, we can express the orbit equation in terms of the difference in true anomalies as follows. From Eqn (2.144), we have $\theta = \theta_0 + \Delta\theta$, which means we can write the orbit equation as

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos(\theta_0 + \Delta\theta)} \tag{2.148}$$

By replacing θ_0 with $-\Delta\theta$ in Eqn (2.143), Eqn (2.148) becomes

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta_0 \cos\Delta\theta - e\sin\theta_0 \sin\Delta\theta} \tag{2.149}$$

To remove θ_0 from this expression, observe first of all that Eqn (2.146) implies that, at $t = t_0$,

$$e\cos\theta_0 = \frac{h^2}{\mu r_0} - 1 \tag{2.150}$$

Furthermore, from Eqn (2.49) for the radial velocity we obtain

$$e\sin\theta_0 = \frac{hv_{r0}}{\mu} \tag{2.151}$$

Substituting Eqns (2.150) and (2.151) into Eqn (2.149) yields

$$r = \frac{h^2}{\mu} \frac{1}{1 + \left(\frac{h^2}{\mu r_0} - 1\right)\cos\Delta\theta - \frac{hv_{r0}}{\mu}\sin\Delta\theta} \tag{2.152}$$

Using this form of the orbit equation, we can find r in terms of the initial conditions and the change in the true anomaly. Thus f in Eqn (2.147) depends only on $\Delta\theta$.

The Lagrange coefficient g is found by substituting Eqns (2.118) and (2.140) into Eqn (2.137b),

$$\begin{aligned} g &= \frac{-\bar{x}\,\bar{y}_0 + \bar{y}\,\bar{x}_0}{h} \\ &= \frac{1}{h}[(-r\cos\theta)(r_0\sin\theta_0) + (r\sin\theta)(r\cos\theta_0)] \\ &= \frac{rr_0}{h}(\sin\theta\cos\theta_0 - \cos\theta\sin\theta_0) \end{aligned} \tag{2.153}$$

Making use of the trig identity

$$\sin(\theta - \theta_0) = \sin\theta\cos\theta_0 - \cos\theta\sin\theta_0$$

together with Eqn (2.144), we find

$$g = \frac{rr_0}{h}\sin(\Delta\theta) \tag{2.154}$$

To obtain $\dot{g}$, substitute Eqns (2.124) and (2.140) into Eqn (2.138b),

$$\begin{aligned} \dot{g} = \frac{-\dot{\bar{x}}\,\bar{y}_0 + \dot{\bar{y}}\,\bar{x}_0}{h} &= \frac{1}{h}\left\{-\left[-\frac{\mu}{h}\sin\theta\right][r_0\sin\theta_0] + \left[\frac{\mu}{h}(e + \cos\theta)\right](r_0\cos\theta_0)\right\} \\ &= \frac{\mu r_0}{h^2}[e\cos\theta_0 + (\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0)] \end{aligned}$$

With the aid of Eqns (2.143) and (2.150), this reduces to

$$\dot{g} = 1 - \frac{\mu r_0}{h^2}(1 - \cos \Delta\theta) \tag{2.155}$$

$\dot{f}$ can be found using Eqn (2.139). Thus,

$$\dot{f} = \frac{1}{g}(f\dot{g} - 1) \tag{2.156}$$

Substituting Eqns (2.147), (2.153), and (2.155) results in

$$\begin{aligned}\dot{f} &= \frac{1}{\dfrac{rr_0}{h}\sin\Delta\theta}\left\{\left[1 - \frac{\mu r}{h^2}(1 - \cos\Delta\theta)\right]\left[1 - \frac{\mu r_0}{h^2}(1 - \cos\Delta\theta)\right] - 1\right\} \\ &= \frac{1}{\dfrac{rr_0}{h}\sin\Delta\theta}\frac{h^2\mu r r_0}{h^4}\left[(1 - \cos\Delta\theta)^2\frac{\mu}{h^2} - (1 - \cos\Delta\theta)\left(\frac{1}{r_0} + \frac{1}{r}\right)\right]\end{aligned}$$

or

$$\dot{f} = \frac{\mu}{h}\frac{1 - \cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1 - \cos\Delta\theta) - \frac{1}{r_0} - \frac{1}{r}\right] \tag{2.157}$$

To summarize, the Lagrange coefficients in terms of the change in true anomaly are:

$$f = 1 - \frac{\mu r}{h^2}(1 - \cos\Delta\theta) \tag{2.158a}$$

$$g = \frac{rr_0}{h}\sin\Delta\theta \tag{2.158b}$$

$$\dot{f} = \frac{\mu}{h}\frac{1 - \cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1 - \cos\Delta\theta) - \frac{1}{r_0} - \frac{1}{r}\right] \tag{2.158c}$$

$$\dot{g} = 1 - \frac{\mu r_0}{h^2}(1 - \cos\Delta\theta) \tag{2.158d}$$

where r is given by Eqn (2.152).

The implementation of these four functions in MATLAB is presented in Appendix D.7.

Observe that using the Lagrange coefficients to determine the position and velocity from the initial conditions does not require knowing the type of orbit we are dealing with (ellipse, parabola, and hyperbola), since the eccentricity does not appear in Eqns (2.152) and (2.158). However, the initial position and velocity give us that information. From $\mathbf{r}_0$ and $\mathbf{v}_0$ we obtain the angular momentum $h = \|\mathbf{r}_0 \times \mathbf{v}_0\|$. The initial radius r_0 is just the magnitude of the vector $\mathbf{r}_0$. The initial radial velocity v_{r0} is the projection of $\mathbf{v}_0$ onto the direction of $\mathbf{r}_0$,

$$v_{r0} = \mathbf{v}_0 \cdot \frac{\mathbf{r}_0}{r_0}$$

From Eqns (2.45) and (2.49) we have

$$r_0 = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta_0} \qquad v_{r0} = \frac{\mu}{h}e\sin\theta_0$$

These two equations can be solved for the eccentricity e and for the true anomaly of the initial point θ_0.

ALGORITHM 2.3

Given $\mathbf{r}_0$ and $\mathbf{v}_0$, find $\mathbf{r}$ and $\mathbf{v}$ after the true anomaly changes by $\Delta\theta$. See Appendix D.8 for an implementation of this procedure in MATLAB.

1. Compute the f and g functions and their derivatives by the following steps:
   (a) Calculate the magnitude of $\mathbf{r}_0$ and $\mathbf{v}_0$:

$$r_0 = \sqrt{\mathbf{r}_0 \cdot \mathbf{r}_0} \quad v_0 = \sqrt{\mathbf{v}_0 \cdot \mathbf{v}_0}$$

   (b) Calculate the radial component of $\mathbf{v}_0$ by projecting it onto the direction of $\mathbf{r}_0$:

$$v_{r0} = \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{r_0}$$

   (c) Calculate the magnitude of the constant angular momentum:

$$h = r_0 v_{\perp 0} = r_0\sqrt{v_0^2 - v_{r0}^2}$$

   (d) Substitute r_0, v_{r0}, h, and $\Delta\theta$ in Eqn (2.152) to calculate r.
   (e) Substitute r, r_0, h, and $\Delta\theta$ into Eqn (2.158) to find f, g, $\dot{f}$, and $\dot{g}$.
2. Use Eqns (2.135) and (2.136) to calculate $\mathbf{r}$ and $\mathbf{v}$.

EXAMPLE 2.13

An earth satellite moves in the xy plane of an inertial frame with the origin at the earth's center. Relative to that frame, the position and velocity of the satellite at time t_0 are

$$\begin{aligned} \mathbf{r}_0 &= 8182.4\hat{\mathbf{i}} - 6865.9\hat{\mathbf{j}}\ (\text{km}) \\ \mathbf{v}_0 &= 0.47572\hat{\mathbf{i}} + 8.8116\hat{\mathbf{j}}\ (\text{km/s}) \end{aligned} \tag{a}$$

Use Algorithm 2.3 to compute the position and velocity vectors after the satellite has traveled through a true anomaly of 120°.

Solution

Step 1:

(a) $r_0 = \sqrt{\mathbf{r}_0 \cdot \mathbf{r}_0} = 10{,}861\ \text{km} \quad v_0 = \sqrt{\mathbf{v}_0 \cdot \mathbf{v}_0} = 8.8244\ \text{km/s}$

(b) $$v_{r0} = \mathbf{v}_0 \cdot \frac{\mathbf{r}_0}{r_0} = \frac{\left(0.47572\hat{\mathbf{i}} + 8.8116\hat{\mathbf{j}}\right) \cdot \left(8182.4\hat{\mathbf{i}} - 6865.9\hat{\mathbf{j}}\right)}{10{,}681} = -5.2996\ \text{km/s}$$

(c) $$h = r_0\sqrt{v_0^2 - v_{r0}^2} = 10{,}861\sqrt{8.8244^2 - (-5.2996)^2} = 75{,}366\ \text{km}^2/\text{s}$$

(d)

$$r = \frac{h^2}{\mu}\,\frac{1}{1+\left(\frac{h^2}{\mu r_0}-1\right)\cos\Delta\theta - \frac{h v_{r0}}{\mu}\sin\Delta\theta}$$

$$= \frac{75{,}366^2}{398{,}600}\,\frac{1}{1+\left(\frac{75{,}366^2}{398{,}600\cdot 10{,}681}-1\right)\cos 120^\circ - \frac{75{,}366\cdot(-5.2996)}{398{,}600}\sin 120^\circ}$$

$$= 8378.8 \text{ km}$$

(e)

$$f = 1-\frac{\mu r}{h^2}(1-\cos\Delta\theta) = 1-\frac{398{,}600\cdot 8378.8}{75{,}366^2}(1-\cos 120^\circ) = 0.11802 \text{ (dimensionless)}$$

$$g = \frac{r r_0}{h}\sin(\Delta\theta) = \frac{8378.8\cdot 10{,}681}{75{,}366}\sin(120^\circ) = 1028.4 \text{ s}$$

$$\dot{f} = \frac{\mu}{h}\,\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1-\cos\Delta\theta)-\frac{1}{r_0}-\frac{1}{r}\right]$$

$$= \frac{398{,}600}{75{,}366}\,\frac{1-\cos 120^\circ}{\sin 120^\circ}\left[\frac{398{,}600}{75{,}366^2}(1-\cos 120^\circ)-\frac{1}{10{,}681}-\frac{1}{8378.9}\right]$$

$$= -9.8666\times 10^{-4} \text{ (dimensionless)}$$

$$\dot{g} = 1-\frac{\mu r_0}{h^2}(1-\cos\Delta\theta) = 1-\frac{398{,}600\cdot 10{,}681}{75{,}366^2}(1-\cos 120^\circ) = -0.12435 \text{ (dimensionless)}$$

Step 2:

$$\begin{aligned}
\mathbf{r} &= f\mathbf{r}_0 + g\mathbf{v}_0\\
&= 0.11802\left(8182.4\hat{\mathbf{i}} - 6865.9\hat{\mathbf{j}}\right) + 1028.4\left(0.47572\hat{\mathbf{i}} + 8.8116\hat{\mathbf{j}}\right)\\
&= \boxed{1454.9\hat{\mathbf{i}} + 8251.6\hat{\mathbf{j}} \text{ (km)}}\\
\mathbf{v} &= \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0\\
&= (-9.8666\times 10^{-4})\left(8182.4\hat{\mathbf{i}} - 6865.9\hat{\mathbf{j}}\right) + (-0.12435)\left(0.47572\hat{\mathbf{i}} + 8.8116\hat{\mathbf{j}}\right)\\
&= \boxed{-8.1323\hat{\mathbf{i}} + 5.6785\hat{\mathbf{j}} \text{ (km/s)}}
\end{aligned}$$

These results are shown in Figure 2.31.

EXAMPLE 2.14

Find the eccentricity of the orbit in Example 2.13 as well as the true anomaly at the initial time t_0 and, hence, the location of the perigee for this orbit.

Solution

In Example 2.13, we found

(a)
$$r_0 = 10{,}861 \text{ km}$$
$$v_{r0} = -5.2996 \text{ km/s}$$
$$h = 75{,}366 \text{ km}^2/\text{s}$$

Since v_{r0} is negative, we know that the spacecraft is approaching the perigee, which means that

(b) $180^\circ < \theta_0 < 360^\circ$

The orbit formula and the radial velocity formula (Eqns (2.45) and (2.49)) evaluated at t_0 are

$$r_0 = \frac{h^2}{\mu}\frac{1}{1 + e\cos\theta_0} \qquad v_{r0} = \frac{\mu}{h} e \sin\theta_0$$

Substituting the numerical values from Eqns (a) into these formulas yields

$$10{,}861 = \frac{75{,}366^3}{398{,}600}\frac{1}{1 + e\cos\theta_0} \qquad -5.2996 = \frac{398{,}600}{75{,}366} e\sin\theta_0$$

From these, we obtain two equations for the two unknowns e and θ_0:

(c) $e\cos\theta_0 = 0.3341 \quad e\sin\theta_0 = -1.002$

Squaring these two expressions and then summing them gives

$$e^2\left(\sin^2\theta_0 + \cos^2\theta_0\right) = 1.1157$$

Recalling the trig identity $\sin^2\theta_0 + \cos^2\theta_0 = 1$, we get

$$\boxed{e = 1.0563} \quad \text{(hyperbola)}$$

The eccentricity may be substituted back into either of the two expressions in Eqn (c) in order to find the true anomaly θ_0. Choosing Eqn $(c)_1$, we find

$$\cos\theta_0 = \frac{0.3341}{1.0563} = 0.3163$$

This means either $\theta_0 = 71.56^\circ$ (moving away from the perigee) or $\theta_0 = 288.44^\circ$ (moving toward the perigee). From Eqn (a) we know the motion is toward perigee, so that

$$\boxed{\theta_0 = 288.44^\circ}$$

Figure 2.31 shows the computed location of the perigee relative to the initial and final position vectors.

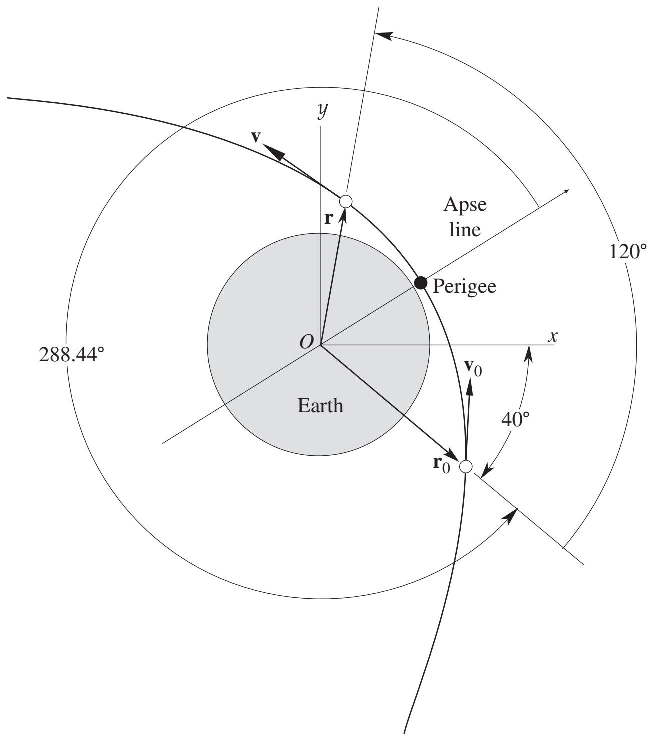


FIGURE 2.31

The initial and final position and velocity vectors and the perigee location for examples 2.13 and 2.14.

In order to use the Lagrange coefficients to find the position and velocity as a function of time instead of true anomaly, we need to come up with a relation between $\Delta\theta$ and time. We will deal with that complex problem in the next chapter. Meanwhile, for times t that are close to the initial time t_0, we can obtain polynomial expressions for f and g in which the variable $\Delta\theta$ is replaced by the time interval $\Delta t = t - t_0$.

To do so, we expand the position vector $\mathbf{r}(t)$, considered to be a function of time, in a Taylor series about $t = t_0$. As pointed out previously (Eqns (1.90) and (1.91)), the Taylor series is given by

$$\mathbf{r}(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{r}^{(n)}(t_0)(t - t_0)^n \tag{2.159}$$

where $\mathbf{r}^{(n)}(t_0)$ is the nth time derivative of $\mathbf{r}(t)$, evaluated at t_0,

$$\mathbf{r}^{(n)}(t_0) = \left(\frac{\mathrm{d}^n\mathbf{r}}{\mathrm{d}t^n}\right)_{t=t_0} \tag{2.160}$$

Let us truncate this infinite series at four terms. Then, to that degree of approximation,

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{t=t_0}\Delta t + \frac{1}{2}\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{t=t_0}\Delta t^2 + \frac{1}{6}\left(\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}\right)_{t=t_0}\Delta t^3 + \frac{1}{24}\left(\frac{\mathrm{d}^4\mathbf{r}}{\mathrm{d}t^4}\right)_{t=t_0}\Delta t^4 \tag{2.161}$$

where $\Delta t = t - t_0$. To evaluate the four derivatives, we note first that $(\mathrm{d}\mathbf{r}/\mathrm{d}t)_{t=t_0}$ is just the velocity $\mathbf{v}_0$ at $t = t_0$,

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{t=t_0} = \mathbf{v}_0 \tag{2.162}$$

$(\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2)_{t=t_0}$ is evaluated using Eqn (2.22),

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{2.163}$$

Thus,

$$\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{t=t_0} = -\frac{\mu}{r_0^3}\mathbf{r}_0 \tag{2.164}$$

$(\mathrm{d}^3\mathbf{r}/\mathrm{d}t^3)_{t=t_0}$ is evaluated by differentiating Eqn (2.163),

$$\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3} = -\mu\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r^3}\right) = -\mu\left(\frac{r^3\mathbf{v} - 3\mathbf{r}r^2\dot{r}}{r^6}\right) = -\mu\frac{\mathbf{v}}{r^3} + 3\mu\frac{\dot{r}\mathbf{r}}{r^4} \tag{2.165}$$

From Eqn (2.35a) we have

$$\dot{r} = \frac{\mathbf{r}\cdot\mathbf{v}}{r} \tag{2.166}$$

Hence, Eqn (2.165), evaluated at $t = t_0$, is

$$\left(\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}\right)_{t=t_0} = -\mu\frac{\mathbf{v}_0}{r_0^3} + 3\mu\frac{\mathbf{r}_0\cdot\mathbf{v}_0}{r_0^5}\mathbf{r}_0 \tag{2.167}$$

Finally, $(\mathrm{d}^4\mathbf{r}/\mathrm{d}t^4)_{t=t_0}$ is found by first differentiating Eqn (2.165),

$$\frac{\mathrm{d}^4\mathbf{r}}{\mathrm{d}t^4} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\mu\frac{\dot{\mathbf{r}}}{r^3} + 3\mu\frac{\dot{r}\mathbf{r}}{r^4}\right) = -\mu\left(\frac{r^3\ddot{\mathbf{r}} - 3r^2\dot{r}\dot{\mathbf{r}}}{r^6}\right) + 3\mu\left[\frac{r^4(\ddot{r}\mathbf{r} + \dot{r}\dot{\mathbf{r}}) - 4r^3\dot{r}^2\mathbf{r}}{r^8}\right] \tag{2.168}$$

$\ddot{r}$ is found in terms of $\mathbf{r}$ and $\mathbf{v}$ by differentiating Eqn (2.166) and making use of Eqn (2.163). This leads to the expression

$$\ddot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r}\right) = \frac{v^2}{r} - \frac{\mu}{r^2} - \frac{(\mathbf{r}\cdot\mathbf{v})^2}{r^3} \tag{2.169}$$

Substituting Eqns (2.163), (2.166), and (2.169) into Eqn (2.168), combining terms, and evaluating the result at $t = t_0$ yields

$$\left(\frac{\mathrm{d}^4\mathbf{r}}{\mathrm{d}t^4}\right)_{t=t_0} = \left[-2\frac{\mu^2}{r_0^6} + 3\mu\frac{v_0^2}{r_0^5} - 15\mu\frac{(\mathbf{r}_0\cdot\mathbf{v}_0)^2}{r_0^7}\right]\mathbf{r}_0 + 6\mu\frac{(\mathbf{r}_0\cdot\mathbf{v}_0)}{r_0^5}\mathbf{v}_0 \tag{2.170}$$

After substituting Eqns (2.162), (2.164), (2.167), and (2.170) into Eqn (2.161) and rearranging and collecting terms, we obtain

$$\begin{aligned}\mathbf{r}(t) = &\left\{1 - \frac{\mu}{2r_0^3}\Delta t^2 + \frac{\mu}{2}\frac{\mathbf{r}_0\cdot\mathbf{v}_0}{r_0^5}\Delta t^3 + \frac{\mu}{24}\left[-2\frac{\mu}{r_0^6} + 3\frac{v_0^2}{r_0^5} - 15\frac{(\mathbf{r}_0\cdot\mathbf{v}_0)^2}{r_0^7}\right]\Delta t^4\right\}\mathbf{r}_0 \\ &+ \left[\Delta t - \frac{1}{6}\frac{\mu}{r_0^3}\Delta t^3 + \frac{\mu}{4}\frac{(\mathbf{r}_0\cdot\mathbf{v}_0)}{r_0^5}\Delta t^4\right]\mathbf{v}_0\end{aligned} \tag{2.171}$$

Comparing this expression with Eqn (2.135), we see that, to the fourth order in Δt,

$$f = 1 - \frac{\mu}{2r_0^3}\Delta t^2 + \frac{\mu}{2}\frac{\mathbf{r}_0\cdot\mathbf{v}_0}{r_0^5}\Delta t^3 + \frac{\mu}{24}\left[-2\frac{\mu}{r_0^6} + 3\frac{v_0^2}{r_0^5} - 15\frac{(\mathbf{r}_0\cdot\mathbf{v}_0)^2}{r_0^7}\right]\Delta t^4$$
$$g = \Delta t - \frac{1}{6}\frac{\mu}{r_0^3}\Delta t^3 + \frac{\mu}{4}\frac{\mathbf{r}_0\cdot\mathbf{v}_0}{r_0^5}\Delta t^4 \tag{2.172}$$

For small values of elapsed time Δt these f and g series may be used to calculate the position of an orbiting body from the initial conditions.

EXAMPLE 2.15

The orbit of an earth satellite has an eccentricity $e = 0.2$ and a perigee radius of 7000 km. Starting at the perigee, plot the radial distance as a function of time using the f and g series and compare the curve with the exact solution.

Solution

Since the satellite starts at the perigee, $t_0 = 0$, and we have, using the perifocal frame,

$$\mathbf{r}_0 = 7000\hat{\mathbf{p}}\ (\text{km}) \tag{a}$$

The orbit equation evaluated at the perigee is Eqn (2.50), which in the present case becomes

$$7000 = \frac{h^2}{398{,}600}\frac{1}{1+0.2}$$

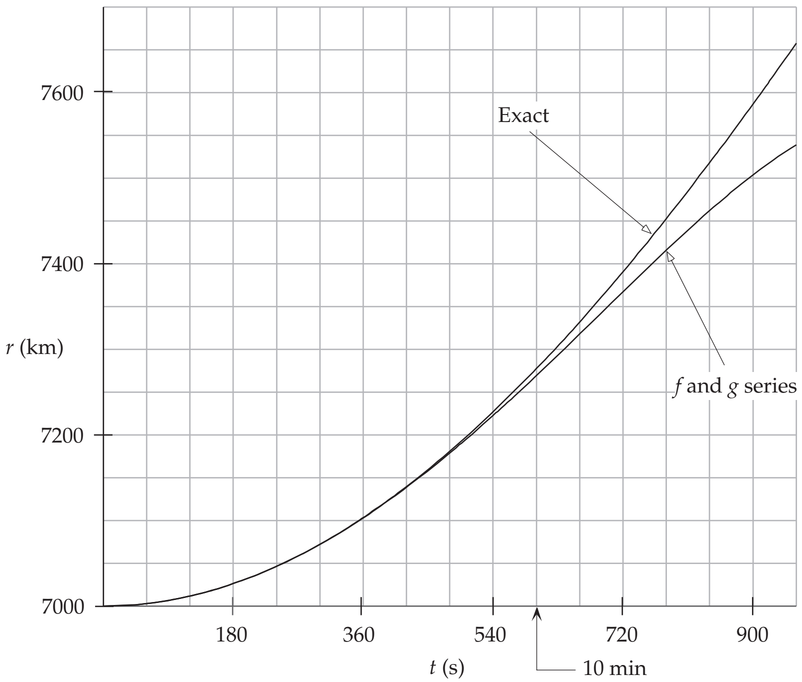


FIGURE 2.32

Exact and series solutions for the radial position of the satellite.

Solving for the angular momentum, we get $h = 57{,}864\ \text{km}^2/\text{s}$. Then, using the angular momentum formula, Eqn (2.31), we find that the speed at the perigee is $v_0 = 8.2663$ km/s, so that

$$\mathbf{v}_0 = 8.2663\hat{\mathbf{q}}\ (\text{km/s}) \tag{b}$$

Clearly, $\mathbf{r}_0 \cdot \mathbf{v}_0 = 0$. Hence, with $\mu = 398{,}600\ \text{km}^3/\text{s}^2$, the two Lagrange series in Eqn (2.172) become (setting $\Delta t = t$)

$$\begin{aligned} f &= 1 - 5.8105\left(10^{-7}\right)t^2 + 9.0032\left(10^{-14}\right)t^4 \\ g &= t - 1.9368\left(10^{-7}\right)t^3 \end{aligned}$$

where the units of t are seconds. Substituting f and g into Eqn (2.135) yields

$$\mathbf{r} = \left[1 - 5.8105\left(10^{-7}\right)t^2 + 9.0032\left(10^{-14}\right)t^4\right](7000\hat{\mathbf{p}}) + \left[t - 1.9368\left(10^{-7}\right)t^3\right](8.2663\hat{\mathbf{q}})$$

From this we obtain

$$\boxed{r = \|\mathbf{r}\| = \sqrt{49\left(10^6\right) + 11.389t^2 - 1.103\left(10^{-6}\right)t^4 - 2.5633\left(10^{-12}\right)t^6 + 3.9718\left(10^{-19}\right)t^8}} \tag{c}$$

For the exact solution of r versus time we must appeal to the methods presented in the next chapter. The exact solution and the series solution (Eqn (c)) are plotted in Figure 2.32. As can be seen, the series solution begins to seriously diverge from the exact solution after about 10 min.

If we include terms of fifth and higher orders in the f and g series (Eqn (2.172)), then the approximate solution in the above example will agree with the exact solution for a longer time interval than that indicated in Figure 2.32. However, there is a time interval beyond which the series solution will diverge from the exact one no matter how many terms we include. This time interval is called the radius of convergence. According to Bond and Allman (1996), for the elliptical orbit of Example 2.15, the radius of convergence is 1700 s (not quite half an hour), which is one-fifth of the period of that orbit. This further illustrates the fact that the series forms of the Lagrange coefficients are applicable only over small time intervals. For arbitrary time intervals, the closed form of these functions, presented in Chapter 3, must be employed.

2.12 Restricted three-body problem

Consider two bodies m_1 and m_2 moving under the action of just their mutual gravitation, and let their orbit around each other be a circle of radius r_{12}. Consider a noninertial, comoving frame of reference xyz whose origin lies at the center of mass G of the two-body system, with the x-axis directed toward m_2, as shown in Figure 2.33. The y-axis lies in the orbital plane, to which the z-axis is perpendicular. In this rotating frame of reference, m_1 and m_2 appear to be at rest, the force of gravity on each one seemingly balanced by the fictitious centripetal force required to hold it in its circular path around the system center of mass.

The constant, inertial angular velocity $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} = \Omega\hat{\mathbf{k}} \tag{2.173}$$

where

$$\Omega = \frac{2\pi}{T}$$

and T is the period of the orbit (Eqn (2.64)),

$$T = \frac{2\pi}{\sqrt{\mu}} r_{12}^{\frac{3}{2}}$$

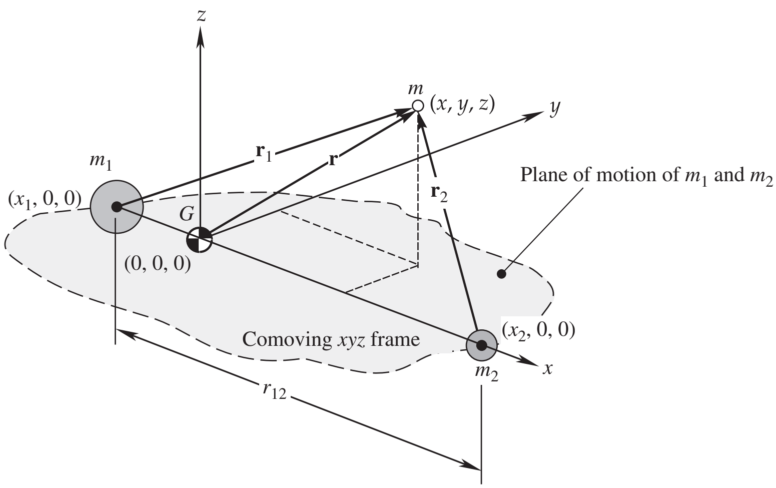


FIGURE 2.33

Primary bodies m_1 and m_2 in circular orbit around each other, plus a secondary mass m.

Thus,

$$\Omega = \sqrt{\frac{\mu}{r_{12}^3}} \tag{2.174}$$

Recall that if M is the total mass of the system,

$$M = m_1 + m_2 \tag{2.175}$$

then

$$\mu = GM \tag{2.176}$$

m_1 and m_2 lie in the orbital plane, so that their y and z coordinates are zero. To determine their locations on the x-axis, we use the definition of the center of mass (Eqn (2.1)) to write

$$m_1x_1 + m_2x_2 = 0$$

Since m_2 is at a distance of r_{12} from m_1 in the positive x-direction, it is also true that

$$x_2 = x_1 + r_{12}$$

From these two equations, we obtain

$$x_1 = -\pi_2 r_{12} \tag{2.177a}$$

$$x_2 = \pi_1 r_{12} \tag{2.177b}$$

where the dimensionless mass ratios π_1 and π_2 are given by

$$\begin{aligned} \pi_1 &= \frac{m_1}{m_1 + m_2} \\ \pi_2 &= \frac{m_2}{m_1 + m_2} \end{aligned} \tag{2.178}$$

Since m_1 and m_2 have the same period in their circular orbits around G, the larger mass (the one closest to G) has the greater orbital speed and hence the greatest centripetal force.

We now introduce a third body of mass m, which is vanishingly small compared to the primary masses m_1 and m_2—like the mass of a spacecraft compared to that of a planet or moon of the solar system. This is called the restricted three-body problem, because the mass m is assumed to be so small that it has no effect on the motion of the primary bodies. We are interested in the motion of m due to the gravitational fields of m_1 and m_2. Unlike the two-body problem, there is no general, closed form solution for this motion. However, we can set up the equations of motion and draw some general conclusions from them.

In the comoving coordinate system, the position vector of the secondary mass m relative to m_1 is given by

$$\mathbf{r}_1 = (x - x_1)\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = (x + \pi_2 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{2.179}$$

Relative to m_2 the position of m is

$$\mathbf{r}_2 = (x - \pi_1 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{2.180}$$

Finally, the position vector of the secondary body relative to the center of mass is

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{2.181}$$

The inertial velocity of m is found by taking the time derivative of Eqn (2.181). However, relative to inertial space, the xyz coordinate system is rotating with the angular velocity $\boldsymbol{\Omega}$, so that the time derivatives of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are not zero. To account for the rotating frame, we use Eqn (1.75) to obtain

$$\dot{\mathbf{r}} = \mathbf{v}_G + \boldsymbol{\Omega} \times \mathbf{r} + \mathbf{v}_{\text{rel}} \tag{2.182}$$

$\mathbf{v}_G$ is the inertial velocity of the center of mass (the origin of the xyz frame), and $\mathbf{v}_{\text{rel}}$ is the velocity of m as measured in the moving xyz frame, namely,

$$\mathbf{v}_{\text{rel}} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}} \tag{2.183}$$

The absolute acceleration of m is found using the "five-term" relative acceleration formula (Eqn (1.79))

$$\ddot{\mathbf{r}} = \mathbf{a}_G + \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}} \tag{2.184}$$

Recall from Section 2.2 that the velocity $\mathbf{v}_G$ of the center of mass is constant, so that $\mathbf{a}_G = 0$. Furthermore, $\dot{\boldsymbol{\Omega}} = 0$ since the angular velocity of the circular orbit is constant. Therefore, Eqn (2.184) reduces to

$$\ddot{\mathbf{r}} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}} \tag{2.185}$$

where

$$\mathbf{a}_{\text{rel}} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \tag{2.186}$$

Substituting Eqns (2.173), (2.181), (2.183), and 2.186 into Eqn (2.185) yields

$$\begin{aligned}\ddot{\mathbf{r}} &= (\Omega\mathbf{k}) \times \left[\left(\Omega\hat{\mathbf{k}}\right) \times \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)\right] + 2\left(\Omega\hat{\mathbf{k}}\right) \times \left(\dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}\right) + \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \\ &= -\Omega^2\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right) + 2\Omega\dot{x}\hat{\mathbf{j}} - 2\Omega\dot{y}\hat{\mathbf{i}} + \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}\end{aligned}$$

Collecting terms, we find

$$\ddot{\mathbf{r}} = \left(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x\right)\hat{\mathbf{i}} + \left(\ddot{y} + 2\Omega\dot{x} - \Omega^2 y\right)\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} \tag{2.187}$$

Now that we have an expression for the inertial acceleration in terms of quantities measured in the rotating frame, let us observe that Newton's second law for the secondary body is

$$m\ddot{\mathbf{r}} = \mathbf{F}_1 + \mathbf{F}_2 \tag{2.188}$$

$\mathbf{F}_1$ and $\mathbf{F}_2$ are the gravitational forces exerted on m by m_1 and m_2, respectively. Recalling Eqn (2.6), we have

$$\begin{aligned}
\mathbf{F}_1 &= -\frac{Gm_1 m}{r_1^2}\mathbf{u}_{r_1} = -\frac{\mu_1 m}{r_1^3}\mathbf{r}_1 \\
\mathbf{F}_2 &= -\frac{Gm_2 m}{r_2^2}\mathbf{u}_{r_2} = -\frac{\mu_2 m}{r_2^3}\mathbf{r}_2
\end{aligned} \tag{2.189}$$

where

$$\mu_1 = Gm_1 \quad \mu_2 = Gm_2 \tag{2.190}$$

Substituting Eqn (2.189) into Eqn (2.188) and canceling out m yields

$$\ddot{\mathbf{r}} = -\frac{\mu_1}{r_1^3}\mathbf{r}_1 - \frac{\mu_2}{r_2^3}\mathbf{r}_2 \tag{2.191}$$

Finally, we substitute Eqn (2.187) on the left and Eqns (2.179) and (2.180) on the right to obtain

$$\begin{aligned}
\left(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x\right)\hat{\mathbf{i}} + \left(\ddot{y} + 2\Omega\dot{x} - \Omega^2 y\right)\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}} = &-\frac{\mu_1}{r_1^3}\left[(x + \pi_2 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right] \\
&-\frac{\mu_2}{r_2^3}\left[(x - \pi_1 r_{12})\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right]
\end{aligned}$$

Equating the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ on each side of this equation yields the three scalar equations of motion for the restricted three-body problem:

$$\ddot{x} - 2\Omega\dot{y} - \Omega^2 x = -\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(x - \pi_1 r_{12}) \tag{2.192a}$$

$$\ddot{y} + 2\Omega\dot{x} - \Omega^2 y = -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y \tag{2.192b}$$

$$\ddot{z} = -\frac{\mu_1}{r_1^3}z - \frac{\mu_2}{r_2^3}z \tag{2.192c}$$

Lagrange points

Although Eqn (2.192) has no closed form analytical solution, we can use it to determine the location of the equilibrium points. These are the locations in space where the secondary mass m would have zero velocity and zero acceleration, that is, where m would appear permanently at rest relative to m_1 and m_2 (and therefore appear to an inertial observer to move in circular orbits around m_1 and m_2).

Once placed at an equilibrium point (also called libration point or Lagrange point), a body will presumably stay there. The equilibrium points are therefore defined by the conditions

$$\dot{x} = \dot{y} = \dot{z} = 0 \text{ and } \ddot{x} = \ddot{y} = \ddot{z} = 0$$

Substituting these conditions into Eqn (2.192) yields

$$-\Omega^2 x = -\frac{\mu_1}{r_1^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(x - \pi_1 r_{12}) \tag{2.193a}$$

$$-\Omega^2 y = -\frac{\mu_1}{r_1^3} y - \frac{\mu_2}{r_2^3} y \tag{2.193b}$$

$$0 = -\frac{\mu_1}{r_1^3} z - \frac{\mu_2}{r_2^3} z \tag{2.193c}$$

From Eqn (2.193c), we have

$$\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right) z = 0 \tag{2.194}$$

Since $\mu_1/r_1^3 > 0$ and $\mu_2/r_2^3 > 0$, it must therefore be true that $z = 0$. That is, the equilibrium points lie in the orbital plane.

From Eqn (2.178), it is clear that

$$\pi_1 = 1 - \pi_2 \tag{2.195}$$

Using this, along with Eqn (2.174), and assuming $y \neq 0$, we can write Eqns (2.193a) and (2.193b) as

$$\begin{aligned} &(1 - \pi_2)(x + \pi_2 r_{12})\frac{1}{r_1^3} + \pi_2(x + \pi_2 r_{12} - r_{12})\frac{1}{r_2^3} = \frac{x}{r_{12}^3} \\ &(1 - \pi_2)\frac{1}{r_1^3} + \pi_2 \frac{1}{r_2^3} = \frac{1}{r_{12}^3} \end{aligned} \tag{2.196}$$

where we made use of the fact that

$$\pi_1 = \mu_1/\mu \quad \pi_2 = \mu_2/\mu \tag{2.197}$$

Treating Eqn (2.196) as two linear equations in $1/r_1^3$ and $1/r_2^3$, we solve them simultaneously to find that

$$\frac{1}{r_1^3} = \frac{1}{r_2^3} = \frac{1}{r_{12}^3}$$

or

$$r_1 = r_2 = r_{12} \tag{2.198}$$

Using this result, together with $z = 0$ and Eqn (2.195), we obtain from Eqns (2.179) and (2.180), respectively,

$$r_{12}^2 = (x + \pi_2 r_{12})^2 + y^2 \tag{2.199}$$

$$r_{12}^2 = (x + \pi_2 r_{12} - r_{12})^2 + y^2 \tag{2.200}$$

Equating the right-hand sides of these two equations leads at once to the conclusion that

$$x = \frac{r_{12}}{2} - \pi_2 r_{12} \tag{2.201}$$

Substituting this result into Eqn (2.199) or Eqn (2.200) and solving for y yields

$$y = \pm\frac{\sqrt{3}}{2} r_{12}$$

We have thus found two of the equilibrium points, the Lagrange points L_4 and L_5. As Eqn (2.198) shows, these points are at the same distance r_{12} from the primary bodies m_1 and m_2 that the primary bodies are from each other, and in the comoving coordinate system, their coordinates are

$$L_4, L_5 : x = \frac{r_{12}}{2} - \pi_2 r_{12}, \quad y = \pm\frac{\sqrt{3}}{2} r_{12}, \quad z = 0 \tag{2.202}$$

Therefore, the two primary bodies and these two Lagrange points lie at the vertices of equilateral triangles, as illustrated in Figure 2.36.

The remaining equilibrium points are found by setting $y = 0$ as well as $z = 0$, which satisfy both Eqns (2.193b) and (2.193c). For these values, Eqns (2.179) and (2.180) become

$$\begin{aligned}\mathbf{r}_1 &= (x + \pi_2 r_{12})\hat{\mathbf{i}} \\ \mathbf{r}_2 &= (x - \pi_1 r_{12})\hat{\mathbf{i}} = (x + \pi_2 r_{12} - r_{12})\hat{\mathbf{i}}\end{aligned}$$

Therefore

$$\begin{aligned}r_1 &= |x + \pi_2 r_{12}| \\ r_2 &= |x + \pi_2 r_{12} - r_{12}|\end{aligned}$$

Substituting these together with Eqns (2.174), (2.195), and (2.197) into Eqn (2.193a) yields

$$\frac{1-\pi_2}{|x + \pi_2 r_{12}|^3}(x + \pi_2 r_{12}) + \frac{\pi_2}{|x + \pi_2 r_{12} - r_{12}|^3}(x + \pi_2 r_{12} - r_{12}) - \frac{1}{r_{12}^3}x = 0 \tag{2.203}$$

Further simplification is obtained by nondimensionalizing x,

$$\xi = \frac{x}{r_{12}}$$

In terms of ξ, Eqn (2.203) becomes $f(\pi_2, \xi) = 0$, where

$$f(\pi_2, \xi) = \frac{1-\pi_2}{|\xi + \pi_2|^3}(\xi + \pi_2) + \frac{\pi_2}{|\xi + \pi_2 - 1|^3}(\xi + \pi_2 - 1) - \xi \tag{2.204}$$

Figure 2.34 is a contour plot showing the locus of points (π_2, ξ)at which f is zero. For a given value of the mass ratio π_2 $(0 < \pi_2 < 1)$, the chart shows that there are three values of ξ, corresponding to each of the three collinear Lagrange points L_1, L_2, and L_3.

We cannot read these values precisely off the figure, but we can use them as starting points to solve for the roots of the function $f(\pi_2, \xi)$ in Eqn (2.204). The bisection method is a simple, though not very

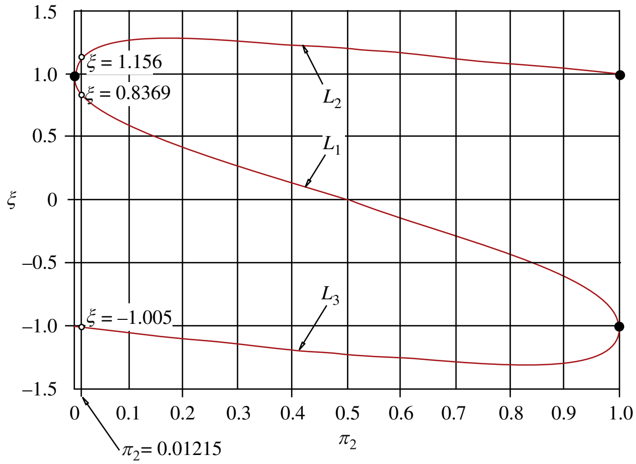


FIGURE 2.34

Contour plot of $f(\pi_2, \xi) = 0$ for the collinear equilibrium points of the restricted three-body problem. $\pi_2 = 0.01215$ for the earth–moon system

efficient, procedure that we can employ here as well as in other problems that require the root of a nonlinear function.

If r is a root of the function $f(x)$, then $f(r) = 0$. To find r by the bisection method, we first select two values of x that we know lie close to and on each side of the root. Label these values x_l and x_u, where $x_l < r$ and $x_u > r$. Since the function f changes sign at a root, it follows that $f(x_l)$ and $f(x_u)$ must be of opposite sign, which means $f(x_l) \cdot f(x_u) < 0$. For the sake of argument, suppose $f(x_l) < 0$ and $f(x_u) > 0$, as in Figure 2.35. Bisect the interval from x_l to x_u by computing $x_m = (x_l + x_u)/2$. If $f(x_m)$ is positive, then the root r lies between x_l and x_m, so (x_l, x_m) becomes our new search interval. If instead $f(x_m)$ is negative, then (x_m, x_u) becomes our search interval. In either case, we bisect the new search interval and repeat

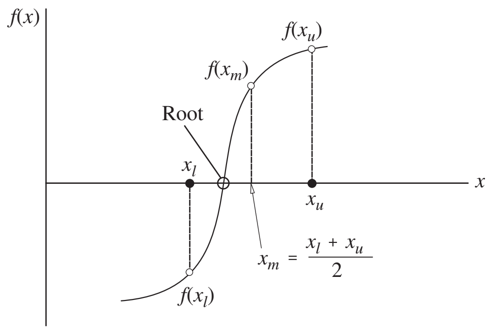


FIGURE 2.35

Determining a root by the bisection method.

the process over and over again, the search interval becoming smaller and smaller, until we eventually converge to r within a desired accuracy E. To achieve that accuracy from the starting values of x_l and x_u requires no more than n iterations, where n is the smallest integer such that (Hahn, 2002)

$$n > \frac{1}{\ln 2} \ln\left(\frac{|x_u - x_l|}{E}\right)$$

Let us summarize the procedure as follows:

ALGORITHM 2.4

Find a root r of the function $f(x)$ using the bisection method. See Appendix D.9 for a MATLAB implementation of this procedure in the script named *bisect*.

1. Select values x_l and x_u, which are known to be fairly close to r and such that $x_l < r$ and $x_u > r$.
2. Choose a tolerance E and determine the number of iterations n from the above formula.
3. Repeat the following steps n times:
   a. Compute $x_m = (x_l + x_u)/2$.
   b. If $f(x_l) \cdot f(x_u) > 0$ then $x_l \leftarrow x_m$; otherwise, $x_r \leftarrow x_m$.
   c. Return to a.
4. $r = x_m$.

EXAMPLE 2.16

Locate the five Lagrange points for the earth–moon system.

Solution

From Table A.1 we find

$$\begin{aligned} m_1 &= 5.974 \times 10^{24}\,\text{kg} \quad \text{(earth)} \\ m_2 &= 7.348 \times 10^{22}\,\text{kg} \quad \text{(moon)} \\ r_{12} &= 3.844 \times 10^{5}\,\text{km} \quad \text{(distance between the earth and moon)} \end{aligned} \tag{2.205}$$

We know that Lagrange points L_4 and L_5 lie on the moon's orbit around the earth. L_4 is 60° ahead of the moon and L_5 lies 60° behind the moon, as illustrated in Figure 2.36.

To find L_1, L_2, and L_3 requires finding the roots of Eqn (2.204) in which, for the case at hand, the mass ratio is

$$\pi_2 = \frac{m_2}{m_1 + m_2} = 0.01215$$

Using Algorithm 2.4, we proceed as follows.

Step 1:

For the above value of π_2, Figure 2.34 shows that L_3 lies near $\xi = -1$, whereas L_1 and L_2 lie on the low and high side, respectively, of $\xi = +1$. We cannot read these values precisely off the graph, but we can use them to select the starting values for the bisection method. For L_3, we choose $\xi_l = -1.1$ and $\xi_u = -0.9$.

Step 2:

Choose an error tolerance of $E = 10^{-6}$, which sets the number of iterations,

$$n > \frac{1}{\ln 2} \ln\left(\frac{|\xi_u - \xi_l|}{E}\right) = \frac{1}{\ln 2} \ln\left(\frac{|-0.9 - (-1.1)|}{10^{-6}}\right) = 17.61$$

That is, $n = 18$.

Table 2.1 Steps of the Bisection Method Leading to $\xi = 1.0050$ for L_3

n	ξ_l	ξ_u	ξ_m	Sign of $f(\pi_{12}, \xi_l) \cdot f(\pi_{12}, \xi_u)$
1	−1.1	−0.9	−1	<0
2	−1.1	−1	−1.05	>0
3	−1.05	−1	−1.025	>0
4	−1.025	−1	−1.0125	>0
5	−1.0125	−1	−1.00625	>0
6	−1.00625	−1	−1.003125	<0

(*continued on next page*)

Step 3:

This is summarized in the following table.

We conclude that, to five significant figures, $\xi_3 = -1.0050$.

The values of ξ for the Lagrange points L_1 and L_2 are found the same way using Algorithm 2.4, starting with the estimates obtained from Figure 2.34. Rather than repeating the lengthy hand computations, see instead Appendix D.9 for the MATLAB program *Example_2_16.m*, which carries out the calculations of all the three roots. It uses the program *bisect.m* to do the iterations, leading to $\xi_1 = 0.8369$ and $\xi_2 = 1.156$, as well as $\xi_3 = -1.005$ computed in Table 2.1.

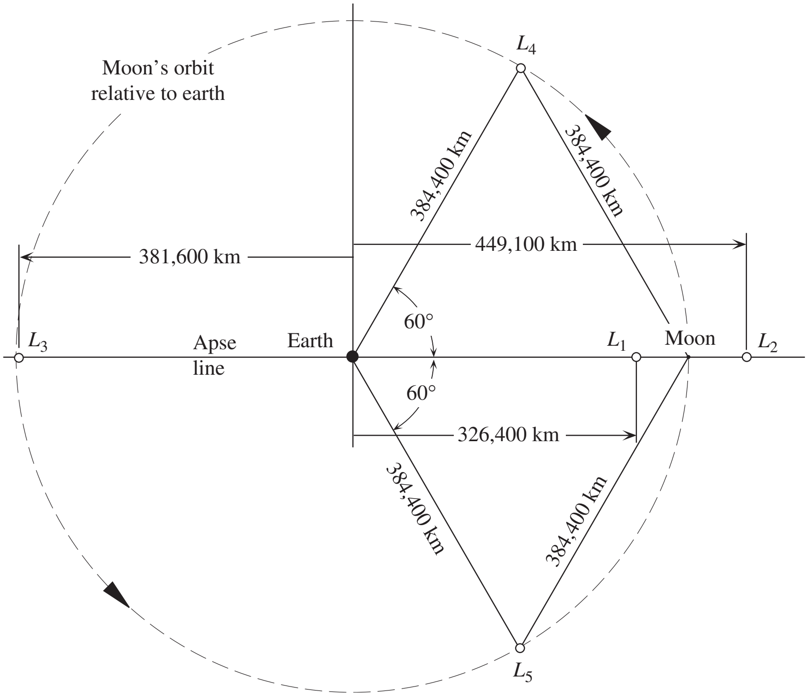


FIGURE 2.36

Location of the five lagrange points of the earth–moon system. these points orbit the earth with the same period as the moon.

Table 2.1 Steps of the Bisection Method Leading to $\xi = 1.0050$ for L_3 *(continued)*

n	ξ_l	ξ_u	ξ_m	Sign of $f(\pi_{12}, \xi_l)\cdot f(\pi_{12}, \xi_u)$
7	−1.00625	−1.003125	−1.0046875	<0
8	−1.00625	−1.0046875	−1.00546875	>0
9	−1.00546875	−1.0046875	−1.005078125	>0
10	−1.005078125	−1.0046875	−1.0049882812	<0
11	−1.005078125	−1.0049882812	−1.004980469	<0
12	−1.004980469	−1.0049882812	−1.005029297	>0
13	−1.005029297	−1.0049882812	−1.005004883	>0
14	−1.005004883	−1.0049882812	−1.004992676	<0
15	−1.005004883	−1.004992676	−1.004998779	>0
16	−1.004998779	−1.004992676	−1.004995728	>0
17	−1.004995728	−1.004992676	−1.004994202	<0
18	−1.004995728	−1.004994202	−1.004994965	>0

Multiplying each dimensionless root by r_{12} yields the x-coordinates of the collinear Lagrange points in kilometers.

$$\boxed{\begin{aligned} L_1 &: x = 0.8369r_{12} = 3.217\times10^5\text{km}\\ L_2 &: x = 1.156r_{12} = 4.444\times10^5\text{km}\\ L_3 &: x = -1.005r_{12} = -3.863\times10^5\text{km}\end{aligned}} \tag{2.206}$$

The locations of the five Lagrange points for the earth—moon system are shown in Figure 2.36. For convenience, all their positions are shown relative to the center of the earth, instead of the center of mass. As can be seen from Eqn (2.177a), the center of mass of the earth—moon system is only 4670 km from the center of the earth. That is, it lies within the earth at 73% of its radius. Since the Lagrange points are fixed relative to the earth and the moon, they follow circular orbits around the earth with the same period as the moon.

If an equilibrium point is stable, then a small mass occupying that point will tend to return to that point if nudged out of position. The perturbation results in a small oscillation (orbit) about the equilibrium point. Thus, objects can be placed in small orbits (called halo orbits) around stable equilibrium points without requiring much in the way of station keeping. On the other hand, if a body located at an unstable equilibrium point is only slightly perturbed, it will oscillate in a divergent fashion, drifting eventually completely away from that point. It turns out that the Lagrange points L_1, L_2, and L_3 on the apse line are unstable, whereas L_4 and L_5, which lie 60° ahead of m_1 and 60° behind m_1 in its orbit, are stable if the ratio m_2/m_1 exceeds 24.96. For the earth–moon system that ratio is 81.3. However, L_4 and L_5 are destabilized by the influence of the sun's gravity, so that in actuality station keeping would be required to maintain position in the neighborhood of those points of the earth–moon system.

Solar observation spacecraft have been placed in halo orbits around the L_1 point of the sun–earth system. L_1 lies about 1.5 million kilometers from the earth (1/100 the distance to the sun) and well outside the earth's magnetosphere. Three such missions were the International Sun–Earth Explorer 3 launched in August 1978; the Solar and Heliocentric Observatory launched in December 1995; and the Advanced Composition Explorer launched in August 1997.

In June 2001, the 830-kg Wilkinson Microwave Anisotropy Probe (WMAP) was launched aboard a Delta II rocket on a three-month journey to sun–earth Lagrange point L_2, which lies 1.5 million kilometers from the earth in the opposite direction from L_1. WMAP's several-year mission was to measure cosmic microwave background radiation. The 6200-kg James Webb Space Telescope is scheduled for a 2018 launch aboard an Arianne 5 to an orbit around L_2. This successor to the Hubble Space Telescope, which is in LEO, will use a 6.5-m mirror to gather data in the infrared spectrum over a period of 5–10 years.

Jacobi constant

Multiply Eqn (2.192a) by $\dot{x}$, Eqn (2.192b) by $\dot{y}$, and Eqn (2.192c) by $\dot{z}$ to obtain

$$\ddot{x}\dot{x} - 2\Omega\dot{x}\dot{y} - \Omega^2 x\dot{x} = -\tfrac{\mu_1}{r_1^3}(x\dot{x} + \pi_2 r_{12}\dot{x}) - \tfrac{\mu_2}{r_2^3}(x\dot{x} - \pi_1 r_{12}\dot{x})$$

$$\ddot{y}\dot{y} + 2\Omega\dot{x}\dot{y} - \Omega^2 y\dot{y} = -\tfrac{\mu_1}{r_1^3}y\dot{y} - \tfrac{\mu_2}{r_2^3}y\dot{y}$$

$$\ddot{z}\dot{z} = -\tfrac{\mu_1}{r_1^3}z\dot{z} - \tfrac{\mu_2}{r_2^3}z\dot{z}$$

Sum up the left and right sides of these equations to get

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} - \Omega^2(x\dot{x} + y\dot{y}) = -\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right)(x\dot{x} + y\dot{y} + z\dot{z}) + r_{12}\left(\frac{\pi_1\mu_2}{r_2^3} - \frac{\pi_2\mu_1}{r_1^3}\right)\dot{x}$$

or, by rearranging terms,

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} - \Omega^2(x\dot{x} + y\dot{y}) = -\frac{\mu_1}{r_1^3}(x\dot{x} + y\dot{y} + z\dot{z} + \pi_2 r_{12}\dot{x}) - \frac{\mu_2}{r_2^3}(x\dot{x} + y\dot{y} + z\dot{z} - \pi_1 r_{12}\dot{x}) \quad (2.207)$$

Note that

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}v^2 \quad (2.208)$$

where v is the speed of the secondary mass relative to the rotating frame. Similarly,

$$x\dot{x} + y\dot{y} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2 + y^2\right) \quad (2.209)$$

From Eqn (2.179) we obtain

$$r_1^2 = (x + \pi_2 r_{12})^2 + y^2 + z^2$$

Therefore

$$2r_1\frac{\mathrm{d}r_1}{\mathrm{d}t} = 2(x + \pi_2 r_{12})\dot{x} + 2y\dot{y} + 2z\dot{z}$$

or

$$\frac{\mathrm{d}r_1}{\mathrm{d}t} = \frac{1}{r_1}(\pi_2 r_{12}\dot{x} + x\dot{x} + y\dot{y} + z\dot{z})$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r_1} = -\frac{1}{r_1^2}\frac{\mathrm{d}r_1}{\mathrm{d}t} = -\frac{1}{r_1^3}(x\dot{x} + y\dot{y} + z\dot{z} + \pi_2 r_{12}\dot{x}) \quad (2.210)$$

In a similar fashion, starting with Eqn (2.180), we find

$$\frac{d}{dt}\frac{1}{r_2} = -\frac{1}{r_2^3}(x\dot{x} + y\dot{y} + z\dot{z} - \pi_1 r_{12}\dot{x}) \tag{2.211}$$

Substituting Eqns (2.208–2.211) into Eqn (2.207) yields

$$\frac{1}{2}\frac{dv^2}{dt} - \frac{1}{2}\Omega^2\frac{d}{dt}\left(x^2 + y^2\right) = \mu_1\frac{d}{dt}\frac{1}{r_1} + \mu_2\frac{d}{dt}\frac{1}{r_2}$$

Alternatively, upon rearranging terms

$$\frac{d}{dt}\left[\frac{1}{2}v^2 - \frac{1}{2}\Omega^2\left(x^2 + y^2\right) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}\right] = 0$$

which means the bracketed expression is a constant

$$\frac{1}{2}v^2 - \frac{1}{2}\Omega^2\left(x^2 + y^2\right) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} = C \tag{2.212}$$

$v^2/2$ is the kinetic energy per unit mass relative to the rotating frame. $-\mu_1/r_1$ and $-\mu_2/r_2$ are the gravitational potential energies of the two primary masses. $-\Omega^2(x^2 + y^2)/2$ may be interpreted as the potential energy of the centrifugal force per unit mass $\Omega^2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}})$ induced by the rotation of the reference frame. The constant C is known as the Jacobi constant, after the German mathematician Carl Jacobi (1804–1851), who discovered it in 1836. Jacobi's constant may be interpreted as the total energy of the secondary particle relative to the rotating frame. C is a constant of the motion of the secondary mass just like the energy and angular momentum are constants of the relative motion in the two-body problem.

Solving Eqn (2.212) for v^2 yields

$$v^2 = \Omega^2\left(x^2 + y^2\right) + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C \tag{2.213}$$

If we restrict the motion of the secondary mass to lie in the plane of motion of the primary masses, then

$$r_1 = \sqrt{(x + \pi_2 r_{12})^2 + y^2} \quad r_2 = \sqrt{(x - \pi_1 r_{12})^2 + y^2} \tag{2.214}$$

For a given value of the Jacobi constant, v^2 is a function only of position in the rotating frame. Since v^2 cannot be negative, it must be true that

$$\Omega^2\left(x^2 + y^2\right) + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C \geq 0 \tag{2.215}$$

Trajectories of the secondary body in regions where this inequality is violated are not allowed. The boundaries between forbidden and allowed regions of motion are found by setting $v^2 = 0$, that is

$$\Omega^2\left(x^2 + y^2\right) + \frac{2\mu_1}{r_1} + \frac{2\mu_2}{r_2} + 2C = 0 \tag{2.216}$$

For a given value of the Jacobi constant the curves of zero velocity are determined by this equation. These boundaries cannot be crossed by a secondary mass (spacecraft) moving within an allowed region.

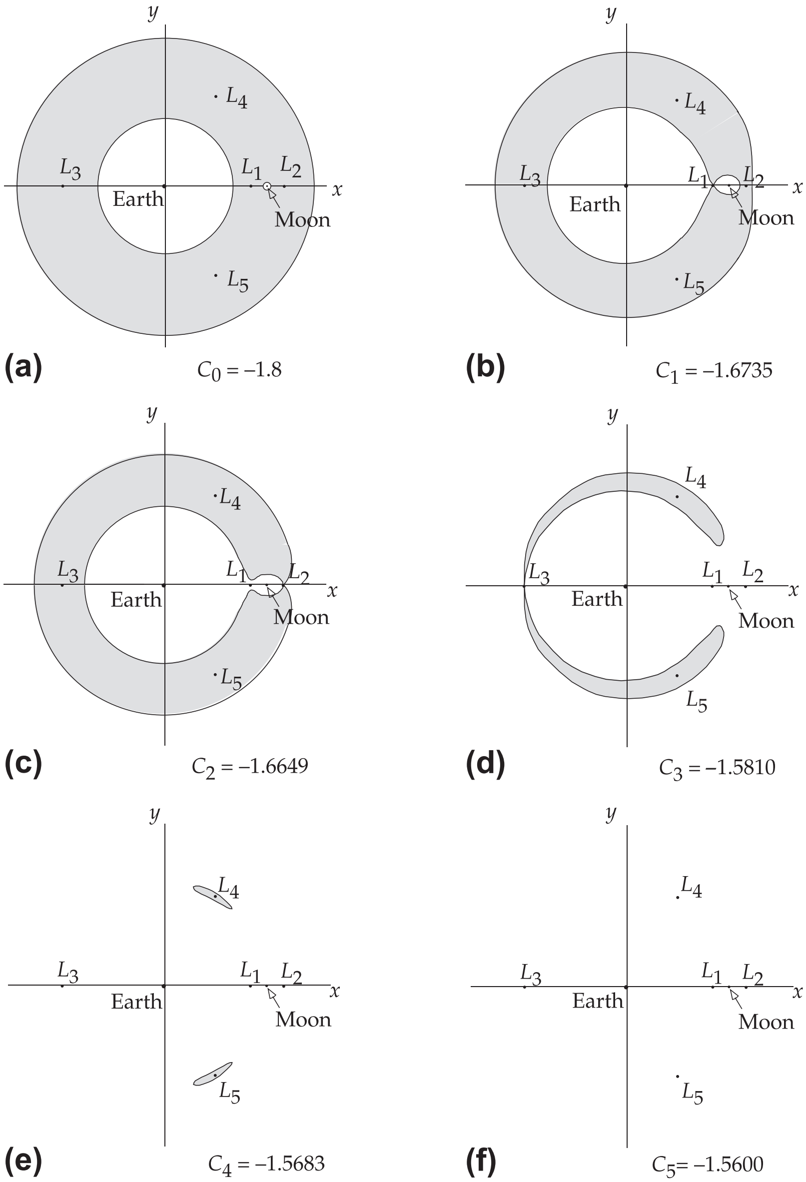


FIGURE 2.37

Forbidden regions (shaded) within the earth–moon system for increasing values of jacobi's constant (km^2/s^2).

Since the first three terms on the left of Eqn (2.216) are all positive, it follows that the zero velocity curves correspond to the negative values of the Jacobi constant. Large negative values of C mean that the secondary body is far from the system center of mass ($x^2 + y^2$ is large) or that the body is close to one of the primary bodies (r_1 is small or r_2 is small).

Let us consider again the earth–moon system. From Eqns (2.174–2.176), (2.190), and (2.205) we have

$$\Omega = \sqrt{\frac{G(m_1+m_2)}{r_{12}^3}} = \sqrt{\frac{6.67259\times10^{-20}\left(6.04748\times10^{24}\right)}{384,400^3}} = 2.66538\times10^{-6}\text{rad/s}$$
$$\mu_1 = Gm_1 = 6.67259\times10^{-20}\cdot5.9742\times10^{24} = 398,620\ \text{km}^3/\text{s}^2 \tag{2.217}$$
$$\mu_2 = Gm_2 = 6.67259\times10^{-20}\cdot7.348\times10^{22} = 4903.02\ \text{km}^3/\text{s}^2$$

Substituting these values into Eqn (2.216), we can plot the zero velocity curves for different values of Jacobi's constant. The curves bound regions in which the motion of a spacecraft is not allowed.

For $C = -1.8\ \text{km}^2/\text{s}^2$, the allowable regions are circles surrounding the earth and the moon, as shown in Figure 2.37(a). A spacecraft launched from the earth with this value of C cannot reach the moon, to say nothing of escaping the earth–moon system.

Substituting the coordinates of the Lagrange points L_1, L_2, and L_3 into Eqn (2.216), we obtain the successively larger values of the Jacobi constants C_1, C_2, and C_3, which are required to arrive at those points with zero velocity. These are shown along with the allowable regions in Figure 2.37. From part (c) of that figure we see that C_2 represents the minimum energy for a spacecraft to escape the earth–moon system via a narrow corridor around the moon. Increasing C widens that corridor, and at C_3 escape becomes possible in the opposite direction from the moon. The last vestiges of the forbidden regions surround L_4 and L_5. A further increase in Jacobi's constant makes the entire earth–moon system and beyond accessible to an earth-launched spacecraft.

For a given value of the Jacobi constant, the relative speed at any point within an allowable region can be found using Eqn (2.213).

EXAMPLE 2.17

The earth-orbiting spacecraft in Figure 2.38 has a relative burnout velocity v_{bo} at an altitude of $d = 200$ km on a radial for which $\varphi = -90°$. Find the value of v_{bo} for each of the scenarios depicted in Figure 2.37.

Solution

From Eqns (2.177) and (2.205), we have

$$\pi_1 = \frac{m_1}{m_1+m_2} = \frac{5.974\times10^{24}}{6.047\times10^{24}} = 0.9878 \quad \pi_2 = 1-\pi_1 = 0.1215$$
$$x_1 = -\pi_1 r_{12} = -0.9878\cdot384,400 = -4670.6\ \text{km}$$

Therefore, the coordinates of the burnout point are

$$x = -4670.6\ \text{km} \quad y = -6578\ \text{km}$$

Substituting these values along with the Jacobi constant into Eqns (2.213) and (2.214) yields the relative burnout speed v_{bo}. For the six Jacobi constants in Figure 2.38 we obtain

$$\boxed{\begin{array}{l} C_0 : v_{bo} = 10.84518\ \text{km/s} \\ C_1 : v_{bo} = 10.85683\ \text{km/s} \\ C_2 : v_{bo} = 10.85762\ \text{km/s} \\ C_3 : v_{bo} = 10.86535\ \text{km/s} \\ C_4 : v_{bo} = 10.86652\ \text{km/s} \\ C_5 : v_{bo} = 10.86728\ \text{km/s} \end{array}}$$

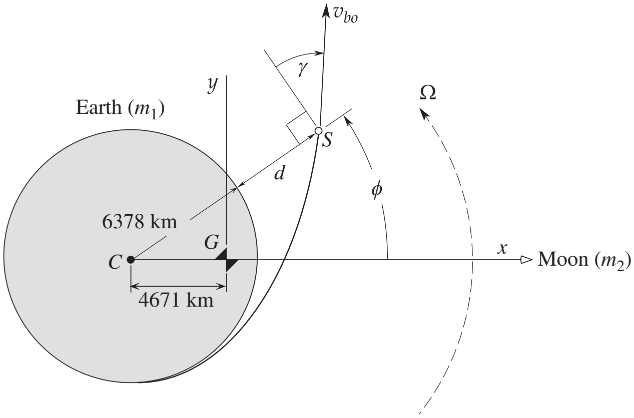


FIGURE 2.38

Spacecraft S burnout position and velocity relative to the rotating earth–moon frame.

These velocities are not substantially different from the escape velocity (Eqn (2.91)) at 200 km altitude,

$$v_{esc} = \sqrt{\frac{2\mu}{r}} = \sqrt{\frac{2 \cdot 398{,}600}{6578}} = 11.01 \text{ km/s}$$

Observe that a change in v_{bo} on the order of only $\leq$10 m/s can have a significant influence on the regions of the earth—moon space accessible to the spacecraft.

EXAMPLE 2.18

For the spacecraft in Figure 2.38 the initial conditions ($t=0$) are $d=200$ km, $\phi=-90°$, $\gamma=20°$, and $v_{bo}=10.9148$ km/s. Use Eqn (2.192), the restricted three-body equations of motion, to determine the trajectory and locate its position at $t=3.16689$ days.

Solution

Since z and $\dot{z}$ are initially zero, Eqn (2.192c) implies that z remains zero. The motion is therefore confined to the xy plane and is governed by Eqns (2.192a) and (2.192b). These have no analytical solution, so we must use a numerical approach.

In order to get Eqns (2.192a) and (2.192b) into the standard form for numerical solution (Section 1.8), we introduce the auxiliary variables

$$y_1 = x \quad y_2 = y \quad y_3 = \dot{x} \quad y_4 = \dot{y} \tag{a}$$

The time derivatives of these variables are

$$\begin{aligned}
\dot{y}_1 &= y_3 \\
\dot{y}_2 &= y_4 \\
\dot{y}_3 &= 2\Omega y_4 + \Omega^2 y_1 - \frac{\mu_1}{r_1^3}(y_1 + \pi_2 r_{12}) - \frac{\mu_2}{r_2^3}(y_1 - \pi_1 r_{12}) \quad \text{(Eqn (2.183a))} \\
\dot{y}_4 &= -2\Omega y_3 + \Omega^2 y_2 - \frac{\mu_1}{r_1^3} y_2 - \frac{\mu_2}{r_2^3} y_2 \quad \text{(Eqn (2.183b))}
\end{aligned} \tag{b}$$

where, from Eqns (2.179) and (2.180),

$$r_1 = \sqrt{(y_1 + \pi_2 r_{12})^2 + y_2^2} \quad r_2 = \sqrt{(y_1 - \pi_1 r_{12})^2 + y_2^2} \tag{c}$$

Equation (b) is of the form $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ given by Eqn (1.105).

To solve this system let us use the Runge–Kutta–Fehlberg 4(5) method and Algorithm 1.3, which is implemented in MATLAB as the program *rkf45.m* in Appendix D.4. The MATLAB function named *Example_2_18.m* in Appendix D.10 contains the data for this problem, the given initial conditions, and the time range. To perform the numerical integration, *Example_2_18.m* calls *rkf45.m*, which uses the subfunction *rates*, which is embedded within *Example_2_18.m*, to compute the derivatives in Eqn (b) above. Running *Example_2_18.m* yields the plot of the trajectory shown in Figure 2.39. After coasting 3.16689 days as specified in the problem statement,

$\boxed{\text{the spacecraft arrives at the far side of the moon on the earth–moon line at an altitude of 256 km.}}$

For comparison, the 1969 Apollo 11 translunar trajectory, which differed from this one in many details (including the use of midcourse corrections), required 3.04861 days to arrive at the lunar orbit insertion point.

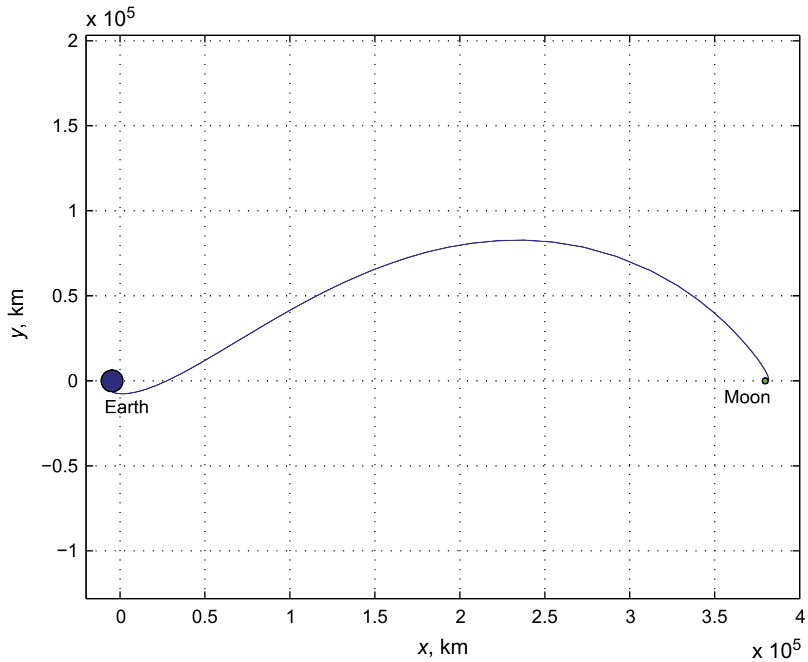


FIGURE 2.39

Translunar coast trajectory computed numerically from the restricted three-body differential equations using the *RKF4(5)* method.

PROBLEMS

For man-made earth satellites use $\mu = 398{,}600\ \text{km}^3/\text{s}^2$ and $R_E = 6378$ km (Tables A.1 and A.2).

Section 2.2

2.1 Two particles of identical mass m are acted on only by the gravitational force of one upon the other. If the distance d between the particles is constant, what is the angular velocity of the line joining them? Use Newton's second law with the center of mass of the system as the origin of the inertial frame.
{Ans.: $\omega = \sqrt{2Gm/d^3}$}

2.2 Three particles of identical mass m are acted on only by their mutual gravitational attraction. They are located at the vertices of an equilateral triangle with sides of length d. Consider the motion of any one of the particles about the system center of mass G and, using G as the origin of the inertial frame, employ Newton's second law to determine the angular velocity ω required for d to remain constant.
{Ans.: $\omega = \sqrt{3Gm/d^3}$}

Section 2.3

2.3 Consider the two-body problem illustrated in Figure 2.1. If a force $\mathbf{T}$ (such as rocket thrust) acts on m_2 in addition to the mutual force of gravitation $\mathbf{F}_{21}$, show that

(a) Equation (2.22) becomes

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \frac{\mathbf{T}}{m_2}$$

(b) If the thrust vector $\mathbf{T}$ has a magnitude T and is aligned with the velocity vector $\mathbf{v}$, then

$$\mathbf{T} = T\frac{\mathbf{v}}{v}$$

2.4 At a given instant t_0, a 1000-kg earth-orbiting satellite has the inertial position and velocity vectors $\mathbf{r}_0 = 3207\hat{\mathbf{i}} + 5459\hat{\mathbf{j}} + 2714\hat{\mathbf{k}}(\text{km})$ and $\mathbf{v}_0 = -6.532\hat{\mathbf{i}} + 0.7835\hat{\mathbf{j}} + 6.142\hat{\mathbf{k}}(\text{km/s})$. Solve Eqn (2.22) numerically to find the maximum altitude reached by the satellite and the time at which it occurs.
{Ans.: Using *ode45*, the maximum altitude = 9670 km at 1.66 h after t_0}

2.5 At a given instant, a 1000-kg earth-orbiting satellite has the inertial position and velocity vectors $\mathbf{r}_0 = 6600\hat{\mathbf{i}}(\text{km})$ and $\mathbf{v}_0 = 12\hat{\mathbf{j}}(\text{km/s})$. Solve Eqn (2.22) numerically to find the distance of the spacecraft from the center of the earth and its speed 24 h later.
{Ans.: Using *ode45*, distance = 456,500 km, speed = 5 km/s}

Section 2.4

2.6 If $\mathbf{r}$, in meters, is given by $\mathbf{r} = t\sin t\hat{\mathbf{I}} + t^2\cos t\hat{\mathbf{J}} + t^3\sin^2 t\hat{\mathbf{K}}$, where t is the time in seconds, calculate (a) $\dot{r}$ (where $r = \|\mathbf{r}\|$) and (b) $\|\dot{\mathbf{r}}\|$ at $t = 2$ s.

{Ans.: (a) $\dot{r} = 4.894$ m/s; (b) $\|\dot{\mathbf{r}}\| = 6.563$ m/s}

2.7 Starting with Eqn (2.35a), prove that $\dot{r} = \mathbf{v}\cdot\hat{\mathbf{u}}_r$ and interpret this result.

2.8 Show that $\hat{\mathbf{u}}_r\cdot d\hat{\mathbf{u}}_r/dt = 0$, where $\hat{\mathbf{u}}_r = \mathbf{r}/r$. Use only the fact that $\hat{\mathbf{u}}_r$ is a unit vector. Interpret this result.

2.9 Starting with Eqn (2.38), show that $\hat{\mathbf{u}}_r\cdot d\hat{\mathbf{u}}_r/dt = 0$.

2.10 Show that $v = \frac{\mu}{h}\sqrt{1 + 2e\cos\theta + e^2}$ for any orbit.

2.11 Relative to a nonrotating, earth-centered Cartesian coordinate system, the position and velocity vectors of a spacecraft are $\mathbf{r} = 7000\hat{\mathbf{i}} - 2000\hat{\mathbf{j}} - 4000\hat{\mathbf{k}}$(km) and $\mathbf{v} = 3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$(km/s). Calculate the orbit's (a) eccentricity vector and (b) the true anomaly.
{Ans.: (a) $\mathbf{e} = 0.2888\hat{\mathbf{i}} + 0.08523\hat{\mathbf{j}} - 0.3840\hat{\mathbf{k}}$; (b) $\theta = 33.32°$}

2.12 Show that the eccentricity is 1 for rectilinear orbits ($\mathbf{h} = \mathbf{0}$).

2.13 Relative to a nonrotating, earth-centered Cartesian coordinate system, the velocity of a spacecraft is $\mathbf{v} = -4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$(km/s) and the unit vector in the direction of the radius is $\hat{\mathbf{u}}_r = 0.26726\hat{\mathbf{i}} + 0.53452\hat{\mathbf{j}} + 0.80178\hat{\mathbf{k}}$. Calculate (a) the radial component of velocity v_r, (b) the azimuth component of velocity $v_\perp$, and (c) the flight path angle γ.
{Ans.: (a) -3.474 km/s; (b) 6.159 km/s; (c) $-29.43°$}

Section 2.5

2.14 If the specific energy ε of the two-body problem is negative, show that m_2 cannot move outside a sphere of radius $\mu/|\varepsilon|$ centered at m_1.

2.15 Relative to a nonrotating Cartesian coordinate frame with the origin at the center O of the earth, a spacecraft in a rectilinear trajectory has the velocity $\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$(km/s) when its distance from O is 10,000 km. Find the position vector $\mathbf{r}$ when the spacecraft comes to rest.
{Ans.: $\mathbf{r} = 5837.4\hat{\mathbf{i}} + 8756.1\hat{\mathbf{j}} + 11{,}675\hat{\mathbf{k}}$(km)}

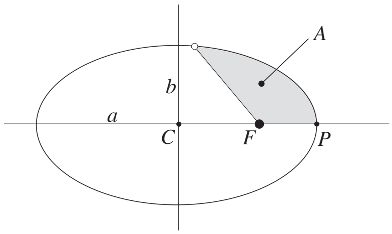
Section 2.6

2.16 The specific angular momentum of a satellite in circular earth orbit is 60,000 km^2/s. Calculate the period.
{Ans.: 2.372 h}

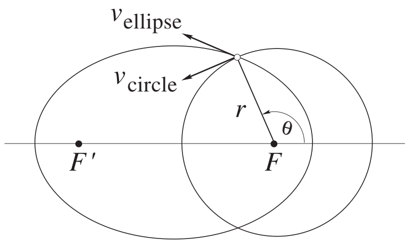
2.17 A spacecraft is in a circular orbit of Mars at an altitude of 200 km. Calculate its speed and its period.
{Ans.: 3.451 km/s; 1 h 49 min}

Section 2.7

2.18 Calculate the area A swept out during the time $t = T/4$ since periapsis, where T is the period of the elliptical orbit.
{Ans.: $0.7854ab$}



2.19 Determine the true anomaly θ of the point(s) on an elliptical orbit at which the speed equals the speed of a circular orbit with the same radius, that is, $v_{\text{ellipse}} = v_{\text{circle}}$.
{Ans.: $\theta = \cos^{-1}(-e)$, where e is the eccentricity of the ellipse}



2.20 Calculate the flight path angle at the locations found in Problem 2.19.
{Ans.: $\gamma = \tan^{-1}(e/\sqrt{1-e^2})$}

2.21 An unmanned satellite orbits the earth with a perigee radius of 10,000 km and an apogee radius of 100,000 km. Calculate (a) the eccentricity of the orbit; (b) the semimajor axis of the orbit (kilometers); (c) the period of the orbit (hours); (d) the specific energy of the orbit (kilometers squared per seconds squared); (e) the true anomaly at which the altitude is 10,000 km (degrees); (f) v_r and $v_\perp$ at the points found in part (e) (kilometers per second); and (g) the speed at perigee and apogee (kilometers per second).
{Partial Ans.: (c) 35.66 h; (e) 82.26°; (g) 8.513 km/s, 0.8513 km/s}

2.22 A spacecraft is in a 400-km by 600-km LEO. How long (in minutes) does it take to coast from the perigee to the apogee?
{Ans.: 48.34 min}

2.23 The altitude of a satellite in an elliptical orbit around the earth is 2000 km at apogee and 500 km at perigee. Determine (a) the eccentricity of the orbit; (b) the orbital speeds at perigee and apogee; and (c) the period of the orbit.
{Ans.: (a) 0.09832; (b) $v_P = 7.978$ km/s; $v_A = 6.550$ km/s; (c) $T = 110.5$ min}

2.24 A satellite is placed into an earth orbit at perigee at an altitude of 500 km with a speed of 10 km/s. Calculate the flight path angle γ and the altitude of the satellite at a true anomaly of 120°.
{Ans.: $\gamma = 44.60°$, $z = 12{,}247$ km}

2.25 A satellite is launched into the earth's orbit at an altitude of 1000 km with a speed of 10 km/s and a flight path angle of 15°. Calculate the true anomaly of the launch point and the period of the orbit.
{Ans.: $\theta = 32.48°$; $T = 30.45$ h}

2.26 A satellite has perigee and apogee altitudes of 500 and 21,000 km. Calculate the orbit period, eccentricity, and the maximum speed.
{Ans.: 6.20 h, 0.5984, 9.625 km/s}

2.27 A satellite is launched parallel to the earth's surface with a speed of 7.6 km/s at an altitude of 500 km. Calculate the period.
{Ans.: 1.61 h}

2.28 A satellite in orbit around the earth has a speed of 8 km/s at a given point of its orbit. If the period is 2 h, what is the altitude at that point?
{Ans.: 648 km}

2.29 A satellite in polar orbit around the earth comes within 200 km of the North Pole at its point of closest approach. If the satellite passes over the pole once every 100 min, calculate the eccentricity of its orbit.
{Ans.: 0.07828}

2.30 The following position data for an earth orbiter are given:
Altitude = 1000 km at a true anomaly of 40°.
Altitude = 2000 km at a true anomaly of 150°.
Calculate (a) The eccentricity, (b) the perigee altitude (kilometers), and (c) the semimajor axis (kilometers).
{Partial Ans.: (c) 7863 km}

2.31 An earth satellite has a speed of 7.5 km/s and a flight path angle of 10° when its radius is 8000 km. Calculate (a) the true anomaly (degrees) and (b) the eccentricity of the orbit.
{Ans.: (a) 63.82°; (b) 0.2151}

2.32 If, for an earth satellite, the specific angular momentum is 70,000 km^2/s and the specific energy is -10 km^2/s^2, calculate the apogee and perigee altitudes.
{Ans.: 25,889 and 1214.9 km}

2.33 A rocket launched from the surface of the earth has a speed of 7 km/s when the powered flight ends at an altitude of 1000 km. The flight path angle at this time is 10°. Determine the eccentricity and the period of the orbit.
{Ans.: 0.1963 and 92.0 min}

2.34 If the perigee velocity is c times the apogee velocity, calculate the eccentricity of the orbit in terms of c.
{Ans.: $e = (c-1)/(c+1)$}

Section 2.8

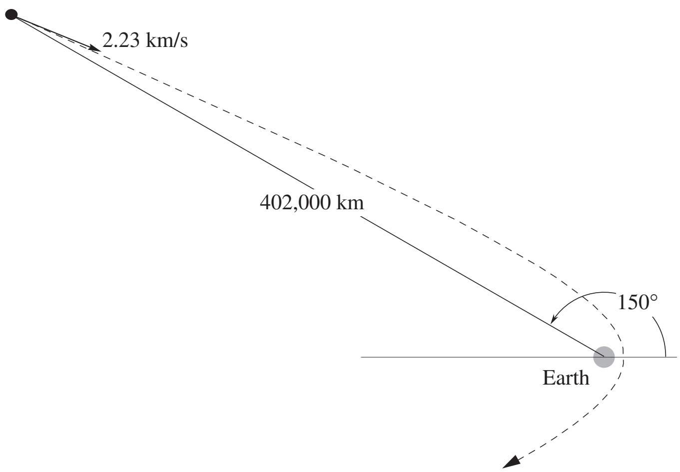
2.35 At what true anomaly does the speed on a parabolic trajectory equal α times the speed at the periapsis, where $\alpha \leq 1$?
{Ans.: $\cos^{-1}(2\alpha^2 - 1)$}

2.36 What velocity, relative to the earth, is required to escape the solar system on a parabolic path from earth's orbit?
{Ans.: 12.34 km/s}

Section 2.9

2.37 A hyperbolic earth departure trajectory has a perigee altitude of 250 km and a perigee speed of 11 km/s. (a) Calculate the hyperbolic excess speed (kilometers per second). (b) Find the radius (kilometers) when the true anomaly is 100°. (c) Find v_r and $v_\perp$ (kilometers per second) when the true anomaly is 100°.
{Partial Ans.: 16,179 km}

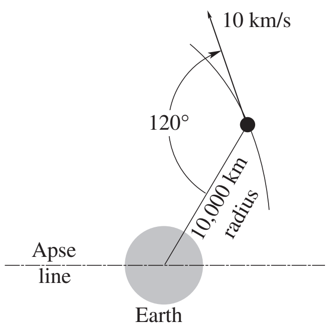
2.38 A meteoroid is first observed approaching the earth when it is 402,000 km from the center of the earth with a true anomaly of 150°. If the speed of the meteoroid at that time is 2.23 km/s, calculate (a) the eccentricity of the trajectory; (b) the altitude at closest approach; and (c) the speed at the closest approach.
{Ans.: (a) 1.086; (b) 5088 km; (c) 8.516 km/s}



2.39 If α is a number between 1 and $\sqrt{(1+e)/(1-e)}$, calculate the true anomaly at which the speed on a hyperbolic trajectory is α times the hyperbolic excess speed.
$\left\{\text{Ans.: } \cos^{-1}\left[\frac{(\alpha^2-1)(e^2-1)-2}{2e}\right]\right\}$

2.40 For a hyperbolic orbit, find the eccentricity in terms of the radius at periapsis r_p and the hyperbolic excess speed v_∞.
{Ans.: $e = 1 + r_p v_\infty^2/\mu$}

2.41 A space vehicle has a velocity of 10 km/s in the direction shown when it is 10,000 km from the center of the earth. Calculate its true anomaly.
{Ans.: 51°}



2.42 A spacecraft at a radius r has a speed v and a flight path angle γ. Find an expression for the eccentricity of its orbit.
{Ans.: $e = \sqrt{1 + \sigma(\sigma - 2)\cos^2\gamma}$ where $\sigma = rv^2/\mu$}

2.43 For an orbiting spacecraft, $r = r_1$ when $\theta = \theta_1$, and $r = r_2$ when $\theta = \theta_2$. What is the eccentricity?
{Ans.: $e = (r_1 - r_2)/(r_2 \cos\theta_2 - r_1 \cos\theta_1)$}

Section 2.11

2.44 At a given instant, a spacecraft has the position and velocity vectors $\mathbf{r}_0 = 7000\hat{\mathbf{i}}$(km) and $\mathbf{v}_0 = 7\hat{\mathbf{i}} + 7\hat{\mathbf{j}}$(km/s) relative to an earth-centered nonrotating frame. (a) What is the position vector after the true anomaly increases by 90°? (b) What is the true anomaly of the initial point?
{Ans.: (a) $\mathbf{r} = 43,180\hat{\mathbf{j}}$(km); (b) 99.21°}

2.45 Relative to an earth-centered, nonrotating frame the position and velocity vectors of a spacecraft are $\mathbf{r}_0 = 3450\hat{\mathbf{i}} - 1700\hat{\mathbf{j}} + 7750\hat{\mathbf{k}}$(km) and $\mathbf{v}_0 = 5.4\hat{\mathbf{i}} - 5.4\hat{\mathbf{j}} + 1.0\hat{\mathbf{k}}$(km/s), respectively. (a) Find the distance and speed of the spacecraft after the true anomaly changes by 82°. (b) Verify that the specific angular momentum h and total energy ε are conserved.
{Partial Ans.: (a) $r = 19{,}266$ km, $v = 2.925$ km/s}

2.46 Relative to an earth-centered, nonrotating frame the position and velocity vectors of a spacecraft are $\mathbf{r}_0 = 6320\hat{\mathbf{i}} + 7750\hat{\mathbf{k}}$(km) and $\mathbf{v}_0 = 11\hat{\mathbf{j}}$(km/s). (a) Find the position vector 10 min later. (b) Calculate the change in true anomaly over the 10-min time span.
{Ans.: (a) $\mathbf{r} = 5320\hat{\mathbf{i}} - 6194\hat{\mathbf{j}} + 3073\hat{\mathbf{k}}$(km); (b) 45°}

Section 2.12

2.47 For the sun–earth system, find the distance of the L_1, L_2, and L_3 Lagrange points from the center of mass of the system.
{Ans.: $x_1 = 151.101 \times 10^6$ km, $x_2 = 148.108 \times 10^6$ km, and $x_3 = -149.600 \times 10^6$ km (opposite side of the sun)}

2.48 Write a program like that for Example 2.18 to compute the trajectory of a spacecraft using the restricted three-body equations of motion. Use the program to design a trajectory from the earth to the earth–moon Lagrange point L4, starting at a 200-km altitude burnout point. The path should take the coasting spacecraft to within 500 km of L4 with a relative speed of not >1 km/s.