CHAPTER

Rigid Body Dynamics

9

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9.1 Introduction

Just as Chapter 1 provides a foundation for the dof the equations of orbital mechanics, this chapter serves as a basis for developing the equations of satellite attitude dynamics. Chapter 1 deals with particles, whereas here we are concerned with rigid bodies. Those familiar with rigid body dynamics can move on to the next chapter, perhaps returning from time to time to review concepts.

The kinematics of rigid bodies is presented first. The subject depends on a theorem of the French mathematician Michel Chasles (1793–1880). Chasles' theorem states that the motion of a rigid body can be described by the displacement of any point of the body (the base point) plus a rotation about a unique axis through that point. The magnitude of the rotation does not depend on the base point. Thus, at any instant, a rigid body in a general state of motion has an angular velocity vector whose direction is that of the instantaneous axis of rotation. Describing the rotational component of the motion a rigid body in three dimensions requires taking advantage of the vector nature of angular velocity and knowing how to take the time derivative of moving vectors, which is explained in Chapter 1. Several examples illustrate how this is done.

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We then move on to study the interaction between the motion of a rigid body and the forces acting on it. Describing the translational component of the motion requires simply concentrating all of the mass at a point, the center of mass, and applying the methods of particle mechanics to determine its motion. Indeed, our study of the two-body problem up to this point has focused on the motion of their centers of mass without regard to the rotational aspect. Analyzing the rotational dynamics requires computing the body's angular momentum, and that in turn requires accounting for how the mass is distributed throughout the body. The mass distribution is described by the six components of the moment of inertia tensor.

Writing the equations of rotational motion relative to coordinate axes embedded in the rigid body and aligned with the principal axes of inertia yields the nonlinear Euler equations of motion, which are applied to a study of the dynamics of a spinning top (or one-axis gyro).

The expression for the kinetic energy of a rigid body is derived because it will be needed in the following chapter.

The chapter next describes the two sets of three angles commonly employed to specify the orientation of a body in three-dimensional space. One of these comprises the Euler angles, which are the same as the right ascension of the node (Ω), argument of periapsis (ω), and inclination (i) introduced in Chapter 4 to orient orbits in space. The other set comprises the yaw, pitch, and roll angles, which are suitable for describing the orientation of an airplane. Both the Euler angles and yaw, pitch, and roll angles will be employed in Chapter 10.

The chapter concludes with a brief discussion of quaternions and an example of how they are used to describe the evolution of the attitude of a rigid body.

9.2 Kinematics

Figure 9.1 shows a moving rigid body and its instantaneous axis of rotation, which defines the direction of the absolute angular velocity vector $\boldsymbol{\omega}$. The XYZ axes are a fixed, inertial frame of reference. The position vectors $\mathbf{R}_A$ and $\mathbf{R}_B$ of two points on the rigid body are measured in the inertial frame. The vector $\mathbf{R}_{B/A}$ drawn from point A to point B is the position vector of B relative to A. Since the body is rigid, $\mathbf{R}_{B/A}$ has a constant magnitude even though its direction is continuously changing. Clearly,

$$\mathbf{R}_B = \mathbf{R}_A + \mathbf{R}_{B/A}$$

Differentiating this equation through with respect to time, we get

$$\dot{\mathbf{R}}_B = \dot{\mathbf{R}}_A + \frac{\mathrm{d}\mathbf{R}_{B/A}}{\mathrm{d}t} \tag{9.1}$$

$\dot{\mathbf{R}}_A$ and $\dot{\mathbf{R}}_B$ are the absolute velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of points A and B. Because the magnitude of $\mathbf{R}_{B/A}$ does not change, its time derivative is given by Eqn (1.61), that is,

$$\frac{\mathrm{d}\mathbf{R}_{B/A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{R}_{B/A}$$

Thus, Eqn (9.1) becomes

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{R}_{B/A} \tag{9.2}$$

Taking the time derivative of Eqn (9.1) yields

$$\ddot{\mathbf{R}}_B = \ddot{\mathbf{R}}_A + \frac{\mathrm{d}^2\mathbf{R}_{B/A}}{\mathrm{d}t^2} \tag{9.3}$$

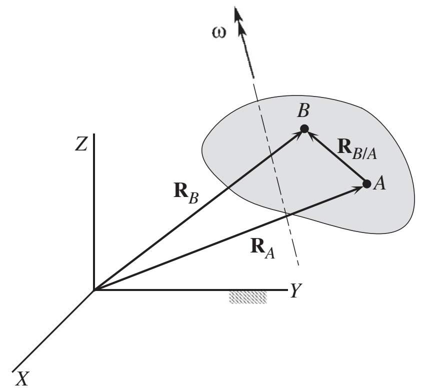


FIGURE 9.1

A rigid body and its instantaneous axis of rotation.

$\ddot{\mathbf{R}}_A$ and $\ddot{\mathbf{R}}_B$ are the absolute accelerations $\mathbf{a}_A$ and $\mathbf{a}_B$ of the two points of the rigid body, while from Eqn (1.62) we have

$$\frac{d^2\mathbf{R}_{B/A}}{dt^2} = \boldsymbol{\alpha} \times \mathbf{R}_{B/A} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{R}_{B/A}\right)$$

in which $\boldsymbol{\alpha}$ is the angular acceleration, $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$. Therefore, Eqn (9.3) can be written

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{R}_{B/A} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{R}_{B/A}\right) \tag{9.4}$$

Equations (9.2) and (9.4) are the relative velocity and acceleration formulas. Note that all quantities in these expressions are measured in the same inertial frame of reference.

When the rigid body under consideration is connected to and moving relative to another rigid body, computation of its inertial angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ must be done with care. The key is to remember that angular velocity is a vector. It may be found as the vector sum of a sequence of angular velocities, each measured relative to another, starting with one measured relative to an absolute frame, as illustrated in Figure 9.2. In that case, the absolute angular velocity $\boldsymbol{\omega}$ of body 4 is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_{2/1} + \boldsymbol{\omega}_{3/2} + \boldsymbol{\omega}_{4/3} \tag{9.5}$$

Each of these angular velocities is resolved into components along the axes of the moving frame of reference xyz as shown in Figure 9.2, so that the vector sum may be written as

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}} \tag{9.6}$$

The moving frame is chosen for convenience of the analysis, and its own inertial angular velocity is denoted as $\boldsymbol{\Omega}$, as discussed in Section 1.6. According to Eqn (1.65), the absolute angular acceleration $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$ is obtained from Eqn (9.6) by means of the following calculation:

$$\boldsymbol{\alpha} = \left.\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rel}} + \boldsymbol{\Omega} \times \boldsymbol{\omega} \tag{9.7}$$

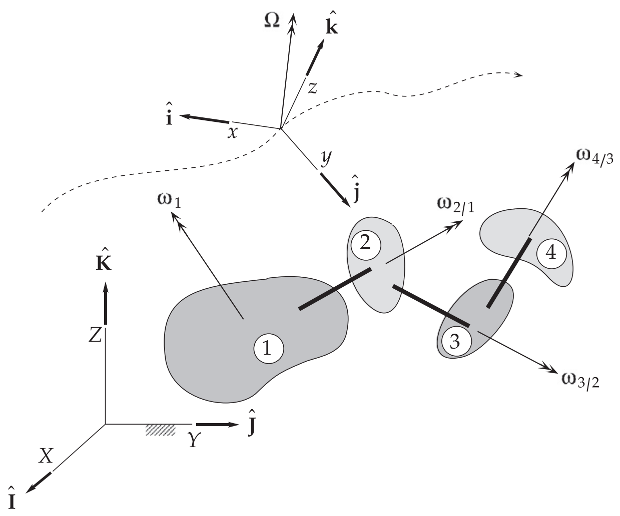


FIGURE 9.2

Angular velocity is the vector sum of the relative angular velocities starting with $\boldsymbol{\omega}_1$, measured relative to the inertial frame.

where

$$\left.\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rel}} = \dot{\omega}_x\hat{\mathbf{i}} + \dot{\omega}_y\hat{\mathbf{j}} + \dot{\omega}_z\hat{\mathbf{k}} \tag{9.8}$$

and $\dot{\omega}_x = d\omega_x/dt$, etc.

Being able to express the absolute angular velocity vector in an appropriately chosen moving reference frame, as in Eqn (9.6), is crucial to the analysis of rigid body motion. Once we have the components of $\boldsymbol{\omega}$, we simply differentiate them with respect to time to arrive at Eqn (9.8). Observe that the absolute angular acceleration $\boldsymbol{\alpha}$ and $d\boldsymbol{\omega}/dt)_{\text{rel}}$, the angular acceleration relative to the moving frame, are the same if and only if $\boldsymbol{\Omega} = \boldsymbol{\omega}$. That occurs if the moving reference is a body-fixed frame, that is, a set of *xyz* axes imbedded in the rigid body itself.

EXAMPLE 9.1

The airplane in Figure 9.3 flies at a constant speed v while simultaneously undergoing a constant yaw rate ω_{yaw} about a vertical axis and describing a circular loop in the vertical plane with a radius ρ. The constant propeller spin rate is ω_{spin} relative to the airframe. Find the velocity and acceleration of the tip P of the propeller relative to the hub H, when P is directly above H. The propeller radius is l.

Solution

The *xyz* axes are rigidly attached to the airplane. The *x*-axis is aligned with the propeller's spin axis. The *y* axis is vertical, and the *z* axis is in the spanwise direction, so that *xyz* forms a right-handed triad. Although the *xyz* frame is not inertial, we can imagine it to instantaneously coincide with an inertial frame.

The absolute angular velocity of the airplane has two components, the yaw rate and the counterclockwise pitch angular velocity v/ρ of its rotation in the circular loop,

$$\boldsymbol{\omega}_{\text{airplane}} = \omega_{\text{yaw}}\hat{\mathbf{j}} + \omega_{\text{pitch}}\hat{\mathbf{k}} = \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}$$

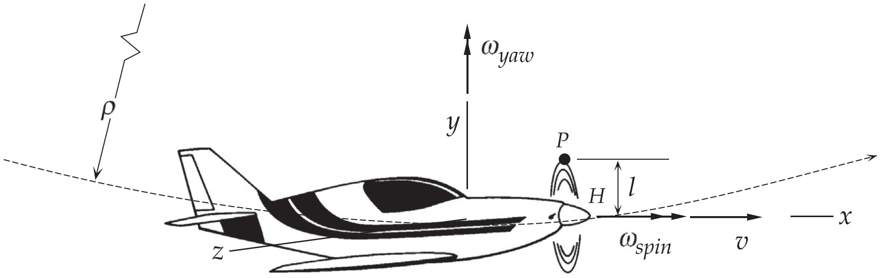


FIGURE 9.3

Airplane with an attached *xyz* body frame.

The angular velocity of the body-fixed moving frame is that of the airplane, $\boldsymbol{\Omega} = \boldsymbol{\omega}_{\text{airplane}}$, so that

$$\boldsymbol{\Omega} = \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}} \tag{a}$$

The absolute angular velocity of the propeller is that of the airplane plus the angular velocity of the propeller relative to the airplane

$$\boldsymbol{\omega}_{\text{prop}} = \boldsymbol{\omega}_{\text{airplane}} + \omega_{\text{spin}}\hat{\mathbf{i}} = \boldsymbol{\Omega} + \omega_{\text{spin}}\hat{\mathbf{i}}$$

which means

$$\boldsymbol{\omega}_{\text{prop}} = \omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}} \tag{b}$$

From Eqn (9.2), the velocity of point P on the propeller relative to H on the hub, $\mathbf{v}_{P/H}$, is given by

$$\mathbf{v}_{P/H} = \mathbf{v}_P - \mathbf{v}_H = \boldsymbol{\omega}_{\text{prop}} \times \mathbf{r}_{P/H}$$

where $\mathbf{r}_{P/H}$ is the position vector of P relative to H at this instant,

$$\mathbf{r}_{P/H} = l\hat{\mathbf{j}} \tag{c}$$

Thus, using Eqns (b) and (c),

$$\mathbf{v}_{P/H} = \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left(l\hat{\mathbf{j}}\right)$$

from which

$$\boxed{\mathbf{v}_{P/H} = -\frac{v}{\rho}l\hat{\mathbf{i}} + \omega_{\text{spin}}\, l\hat{\mathbf{k}}}$$

The absolute angular acceleration of the propeller is found by substituting Eqns (a) and (b) into Eqn (9.7),

$$\begin{aligned}\boldsymbol{\alpha}_{\text{prop}} &= \left.\frac{d\boldsymbol{\omega}_{\text{prop}}}{dt}\right)_{\text{rel}} + \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{prop}} \\ &= \left(\frac{d\omega_{\text{spin}}}{dt}\hat{\mathbf{i}} + \frac{d\omega_{\text{yaw}}}{dt}\hat{\mathbf{j}} + \frac{d(v/\rho)}{dt}\hat{\mathbf{k}}\right) + \left(\omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right)\end{aligned}$$

Since ω_{spin}, ω_{yaw}, v, and ρ are all constant, this reduces to

$$\boldsymbol{\alpha}_{\text{prop}} = \left(\omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right)$$

Carrying out the cross product yields

$$\boldsymbol{\alpha}_{\text{prop}} = \frac{v}{\rho}\omega_{\text{spin}}\hat{\mathbf{j}} - \omega_{\text{yaw}}\omega_{\text{spin}}\hat{\mathbf{k}} \tag{d}$$

From Eqn (9.4), the acceleration of P relative to H, $\mathbf{a}_{P/H}$, is given by

$$\mathbf{a}_{P/H} = \mathbf{a}_P - \mathbf{a}_H = \boldsymbol{\alpha}_{\text{prop}} \times \mathbf{r}_{P/H} + \boldsymbol{\omega}_{\text{prop}} \times \left(\boldsymbol{\omega}_{\text{prop}} \times \mathbf{r}_{P/H}\right)$$

Substituting Eqns (b), (c) and (d) into this expression yields

$$\mathbf{a}_{P/H} = \left(\frac{v}{\rho}\omega_{\text{spin}}\hat{\mathbf{j}} - \omega_{\text{yaw}}\omega_{\text{spin}}\hat{\mathbf{k}}\right) \times \left(l\hat{\mathbf{j}}\right) + \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left[\left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left(l\hat{\mathbf{j}}\right)\right]$$

From this we find that

$$\begin{aligned}\mathbf{a}_{P/H} &= \left(\omega_{\text{yaw}}\omega_{\text{spin}} l\hat{\mathbf{i}}\right) + \left(\omega_{\text{spin}}\hat{\mathbf{i}} + \omega_{\text{yaw}}\hat{\mathbf{j}} + \frac{v}{\rho}\hat{\mathbf{k}}\right) \times \left[-\frac{v}{\rho}l\hat{\mathbf{i}} + \omega_{\text{spin}} l\hat{\mathbf{k}}\right] \\ &= \left(\omega_{\text{yaw}}\omega_{\text{spin}} l\hat{\mathbf{i}}\right) + \left[\omega_{\text{yaw}}\omega_{\text{spin}} l\hat{\mathbf{i}} - \left(\frac{v^2}{\rho^2} + \omega_{\text{spin}}^2\right) l\hat{\mathbf{j}} + \omega_{\text{yaw}}\frac{v}{\rho}l\hat{\mathbf{k}}\right]\end{aligned}$$

so that finally,

$$\boxed{\mathbf{a}_{P/H} = 2\omega_{\text{yaw}}\omega_{\text{spin}} l\hat{\mathbf{i}} - \left(\frac{v^2}{\rho^2} + \omega_{\text{spin}}^2\right) l\hat{\mathbf{j}} + \omega_{\text{yaw}}\frac{v}{\rho}l\hat{\mathbf{k}}}$$

EXAMPLE 9.2

The satellite in Figure 9.4 is rotating about the z axis at a constant rate N. The xyz axes are attached to the spacecraft, and the z-axis has a fixed orientation in inertial space. The solar panels rotate at a constant rate $\dot{\theta}$ in the direction shown. Relative to point O, which lies at the center of the spacecraft and on the centerline of the panels, calculate for point A on the panel

(a) Its absolute velocity and
(b) Its absolute acceleration.

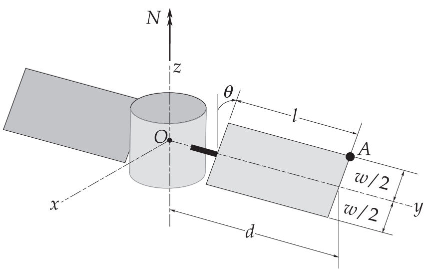


FIGURE 9.4

Rotating solar panel on a rotating satellite.

Solution

(a) Since the moving *xyz* frame is attached to the body of the spacecraft, its angular velocity is

$$\boldsymbol{\Omega} = N\hat{\mathbf{k}} \tag{a}$$

The absolute angular velocity of the panel is the absolute angular velocity of the spacecraft plus the angular velocity of the panel relative to the spacecraft,

$$\boldsymbol{\omega}_{\text{panel}} = -\dot{\theta}\hat{\mathbf{j}} + N\hat{\mathbf{k}} \tag{b}$$

The position vector of A relative to O is

$$\mathbf{r}_{A/O} = -\frac{w}{2}\sin\theta\hat{\mathbf{i}} + d\hat{\mathbf{j}} + \frac{w}{2}\cos\theta\hat{\mathbf{k}} \tag{c}$$

According to Eqn (9.2), the velocity of A relative to O is

$$\mathbf{v}_{A/O} = \mathbf{v}_A - \mathbf{v}_O = \boldsymbol{\omega}_{\text{panel}} \times \mathbf{r}_{A/O} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -\dot{\theta} & N \\ -\dfrac{w}{2}\sin\theta & d & \dfrac{w}{2}\cos\theta \end{vmatrix}$$

from which we get

$$\boxed{\mathbf{v}_{A/O} = -\left(\frac{w}{2}\dot{\theta}\cos\theta + Nd\right)\hat{\mathbf{i}} - \frac{w}{2}N\sin\theta\hat{\mathbf{j}} - \frac{w}{2}\dot{\theta}\sin\theta\hat{\mathbf{k}}}$$

(b) The absolute angular acceleration of the panel is found by substituting Eqns (a) and (b) into Eqn (9.7),

$$\begin{aligned}
\boldsymbol{\alpha}_{\text{panel}} &= \left.\frac{\mathrm{d}\boldsymbol{\omega}_{\text{panel}}}{\mathrm{d}t}\right)_{\text{rel}} + \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{panel}} \\
&= \left[\frac{\mathrm{d}(-\dot{\theta})}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}N}{\mathrm{d}t}\hat{\mathbf{k}}\right] + \left(N\hat{\mathbf{k}}\right) \times \left(-\dot{\theta}\hat{\mathbf{j}} + N\hat{\mathbf{k}}\right)
\end{aligned}$$

Since N and $\dot{\theta}$ are constants, this reduces to

$$\boldsymbol{\alpha}_{\text{panel}} = \dot{\theta}N\hat{\mathbf{i}} \tag{d}$$

To find the acceleration of A relative to O, we substitute Eqns (b)–(d) into Eqn (9.4),

$$\begin{aligned}
\mathbf{a}_{A/O} = \mathbf{a}_A - \mathbf{a}_O &= \boldsymbol{\alpha}_{\text{panel}} \times \mathbf{r}_{A/O} + \boldsymbol{\omega}_{\text{panel}} \times \left(\boldsymbol{\omega}_{\text{panel}} \times \mathbf{r}_{A/O}\right) \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dot{\theta}N & 0 & 0 \\ -\dfrac{w}{2}\sin\theta & d & \dfrac{w}{2}\cos\theta \end{vmatrix} + \left(-\dot{\theta}\hat{\mathbf{j}} + N\hat{\mathbf{k}}\right) \times \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -\dot{\theta} & N \\ -\dfrac{w}{2}\sin\theta & d & \dfrac{w}{2}\cos\theta \end{vmatrix} \\
&= \left(-\frac{w}{2}N\dot{\theta}\cos\theta\hat{\mathbf{j}} + N\dot{\theta}d\hat{\mathbf{k}}\right) + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -\dot{\theta} & N \\ -\dfrac{w}{2}\dot{\theta}\cos\theta - Nd & -N\dfrac{w}{2}\sin\theta & -\dfrac{w}{2}\dot{\theta}\sin\theta \end{vmatrix}
\end{aligned}$$

which leads to

$$\boxed{\mathbf{a}_{A/O} = \frac{w}{2}\left(N^2 + \dot{\theta}^2\right)\sin\theta\hat{\mathbf{i}} - N\left(Nd + w\dot{\theta}\cos\theta\right)\hat{\mathbf{j}} - \frac{w}{2}\dot{\theta}^2\cos\theta\hat{\mathbf{k}}}$$

EXAMPLE 9.3

The gyro rotor illustrated in Figure 9.5 has a constant spin rate ω_{spin} around axis *b*–*a* in the direction shown. The *XYZ* axes are fixed. The *xyz* axes are attached to the gimbal ring, whose angle θ with the vertical is increasing at a constant rate $\dot{\theta}$ in the direction shown. The assembly is forced to precess at a constant rate *N* around the vertical, For the rotor in the position shown, calculate

(a) The absolute angular velocity and
(b) The absolute angular acceleration.

Express the results in both the fixed *XYZ* frame and the moving *xyz* frame.

Solution

(a) We will need the instantaneous relationship between the unit vectors of the inertial *XYZ* axes and the comoving *xyz* frame, which on inspecting Figure 9.6 can be seen to be

$$\begin{aligned}\hat{\mathbf{I}} &= -\cos\theta\hat{\mathbf{j}} + \sin\theta\hat{\mathbf{k}}\\ \hat{\mathbf{J}} &= \hat{\mathbf{i}}\\ \hat{\mathbf{K}} &= \sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}\end{aligned} \tag{a}$$

so that the matrix of the transformation from *xyz* to *XYZ* is (Section 4.5)

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} 0 & -\cos\theta & \sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{b}$$

The absolute angular velocity of the gimbal ring is that of the base plus the angular velocity of the gimbal relative to the base given as

$$\boldsymbol{\omega}_{\text{gimbal}} = N\hat{\mathbf{K}} + \dot{\theta}\hat{\mathbf{i}} = N\left(\sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}\right) + \dot{\theta}\hat{\mathbf{i}} = \dot{\theta}\hat{\mathbf{i}} + N\sin\theta\hat{\mathbf{j}} + N\cos\theta\hat{\mathbf{k}} \tag{c}$$

where we made use of Eqn (a) above. Since the moving *xyz* frame is attached to the gimbal, $\boldsymbol{\Omega} = \boldsymbol{\omega}_{\text{gimbal}}$, so that

$$\boldsymbol{\Omega} = \dot{\theta}\hat{\mathbf{i}} + N\sin\theta\hat{\mathbf{j}} + N\cos\theta\hat{\mathbf{k}} \tag{d}$$

The absolute angular velocity of the rotor is its spin relative to the gimbal, plus the angular velocity of the gimbal,

$$\boldsymbol{\omega}_{\text{rotor}} = \boldsymbol{\omega}_{\text{gimbal}} + \omega_{\text{spin}}\hat{\mathbf{k}} \tag{e}$$

From Eqn (c), it follows that

$$\boxed{\boldsymbol{\omega}_{\text{rotor}} = \dot{\theta}\hat{\mathbf{i}} + N\sin\theta\hat{\mathbf{j}} + \left(N\cos\theta + \omega_{\text{spin}}\right)\hat{\mathbf{k}}} \tag{f}$$

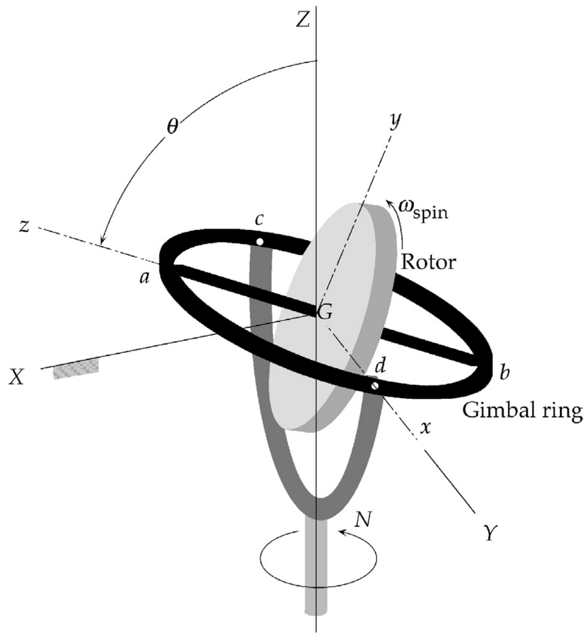


FIGURE 9.5

Rotating, precessing, nutating gyro.

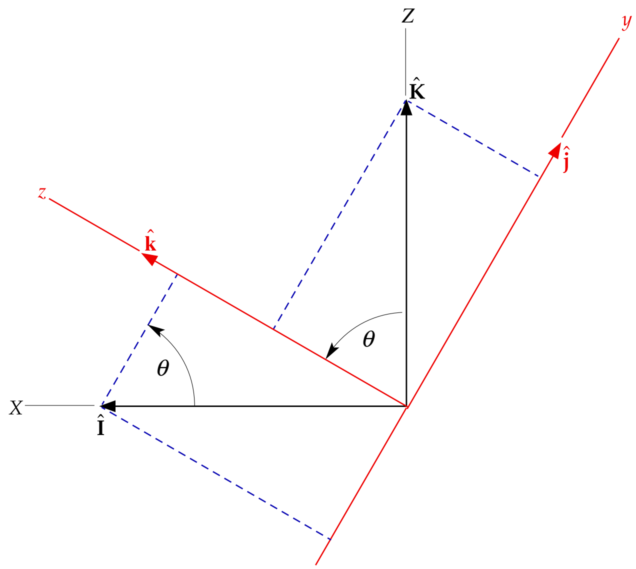


FIGURE 9.6

Orientation of the fixed *XZ* axes relative to the rotating *xz* axes.

Because $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ move with the gimbal, expression (f) is valid for any time, not just the instant shown in Figure 9.5. Alternatively, applying the vector transformation

$$\{\boldsymbol{\omega}_{\text{rotor}}\}_{XYZ} = [\mathbf{Q}]_{xX}\{\boldsymbol{\omega}_{\text{rotor}}\}_{xyz} \tag{g}$$

we obtain the angular velocity of the rotor in the inertial frame, but only at the instant shown in the figure, that is, when the x-axis aligns with the Y-axis.

$$\begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} = \begin{bmatrix} 0 & -\cos\theta & \sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ N\sin\theta \\ N\cos\theta + \omega_{\text{spin}} \end{Bmatrix} = \begin{Bmatrix} -N\sin\theta\cos\theta + N\sin\theta\cos\theta + \omega_{\text{spin}}\sin\theta \\ \dot{\theta} \\ N\sin^2\theta + N\cos^2\theta + \omega_{\text{spin}}\cos\theta \end{Bmatrix}$$

or

$$\boxed{\boldsymbol{\omega}_{\text{rotor}} = \omega_{\text{spin}}\sin\theta\hat{\mathbf{I}} + \dot{\theta}\hat{\mathbf{J}} + (N + \omega_{\text{spin}}\cos\theta)\hat{\mathbf{K}}} \tag{h}$$

(b) The angular acceleration of the rotor can be found by substituting Eqns (d) and (f) into Eqn (9.7), recalling that N, $\dot{\theta}$, and ω_{spin} are independent of time:

$$\begin{aligned}
\boldsymbol{\alpha}_{\text{rotor}} &= \left.\frac{d\boldsymbol{\omega}_{\text{rotor}}}{dt}\right)_{\text{rel}} + \boldsymbol{\Omega}\times\boldsymbol{\omega}_{\text{rotor}} \\
&= \left[\frac{d(\dot{\theta})}{dt}\hat{\mathbf{i}} + \frac{d(N\sin\theta)}{dt}\hat{\mathbf{j}} + \frac{d(N\cos\theta + \omega_{\text{spin}})}{dt}\hat{\mathbf{k}}\right] + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dot{\theta} & N\sin\theta & N\cos\theta \\ \dot{\theta} & N\sin\theta & N\cos\theta + \omega_{\text{spin}} \end{vmatrix} \\
&= \left(N\dot{\theta}\cos\theta\hat{\mathbf{j}} - N\dot{\theta}\sin\theta\hat{\mathbf{k}}\right) + \left[\hat{\mathbf{i}}(N\omega_{\text{spin}}\sin\theta) - \hat{\mathbf{j}}(\omega_{\text{spin}}\dot{\theta}) + \hat{\mathbf{k}}(0)\right]
\end{aligned}$$

Upon collecting terms, we get

$$\boxed{\boldsymbol{\alpha}_{\text{rotor}} = N\omega_{\text{spin}}\sin\theta\hat{\mathbf{i}} + \dot{\theta}(N\cos\theta - \omega_{\text{spin}})\hat{\mathbf{j}} - N\dot{\theta}\sin\theta\hat{\mathbf{k}}} \tag{i}$$

This expression, like Eqn (f), is valid at any time.

The components of $\boldsymbol{\alpha}_{\text{rotor}}$ along the XYZ axes are found in the same way as for $\boldsymbol{\omega}_{\text{rotor}}$,

$$\{\boldsymbol{\alpha}_{\text{rotor}}\}_{XYZ} = [\mathbf{Q}]_{xX}\{\boldsymbol{\alpha}_{\text{rotor}}\}_{xyz}$$

which means

$$\begin{aligned}
\begin{Bmatrix} \alpha_X \\ \alpha_Y \\ \alpha_Z \end{Bmatrix} &= \begin{bmatrix} 0 & -\cos\theta & \sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} N\omega_{\text{spin}}\sin\theta \\ \dot{\theta}(N\cos\theta - \omega_{\text{spin}}) \\ -N\dot{\theta}\sin\theta \end{Bmatrix} \\
&= \begin{Bmatrix} -N\dot{\theta}\cos^2\theta + \dot{\theta}\omega_{\text{spin}}\cos\theta - N\dot{\theta}\sin^2\theta \\ N\omega_{\text{spin}}\sin\theta \\ N\dot{\theta}\sin\theta\cos\theta - \dot{\theta}\omega_{\text{spin}}\sin\theta - N\dot{\theta}\sin\theta\cos\theta \end{Bmatrix}
\end{aligned}$$

or

$$\boxed{\boldsymbol{\alpha}_{\text{rotor}} = \dot{\theta}(\omega_{\text{spin}}\cos\theta - N)\hat{\mathbf{I}} + N\omega_{\text{spin}}\sin\theta\hat{\mathbf{J}} - \dot{\theta}\omega_{\text{spin}}\sin\theta\hat{\mathbf{K}}} \tag{j}$$

Note carefully that Eqn (j) is not simply the time derivative of Eqn (h). Equations (h) and (j) are valid only at the instant that the *xyz* and *XYZ* axes have the alignments shown in Figure 9.5.

9.3 Equations of translational motion

Figure 9.7 again shows an arbitrary, continuous, three-dimensional body of mass m. "Continuous" means that as we zoom in on a point it remains surrounded by a continuous distribution of matter having the infinitesimal mass dm in the limit. The point never ends up in a void. In particular, we ignore the actual atomic and molecular microstructure in favor of this continuum hypothesis, as it is called. Molecular microstructure does bear upon the overall dynamics of a finite body. We will use G to denote the center of mass. The position vectors of points relative to the origin of the inertial frame will be designated by capital letters. Thus, the position of the center of mass is $\mathbf{R}_G$ defined as

$$m\mathbf{R}_G = \int_m \mathbf{R}\mathrm{d}m \tag{9.9}$$

$\mathbf{R}$ is the position of a mass element dm within the continuum. Each element of mass is acted upon by a net external force $\mathrm{d}\mathbf{F}_{\text{net}}$ and a net internal force $\mathrm{d}\mathbf{f}_{\text{net}}$. The external force comes from direct contact with other objects and from action at a distance, such as gravitational attraction. The internal forces are those exerted from within the body by neighboring particles. These are the forces that hold the body together. For each mass element, Newton's second law, Eqn (1.47), is written as

$$\mathrm{d}\mathbf{F}_{\text{net}} + \mathrm{d}\mathbf{f}_{\text{net}} = \mathrm{d}m\ddot{\mathbf{R}} \tag{9.10}$$

Writing this equation for the infinite number of mass elements of which the body is composed, and then summing them all together leads, to the integral

$$\int \mathrm{d}\mathbf{F}_{\text{net}} + \int \mathrm{d}\mathbf{f}_{\text{net}} = \int_m \ddot{\mathbf{R}}\mathrm{d}m$$

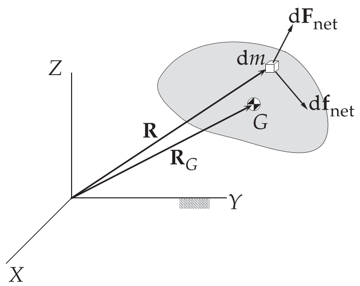


FIGURE 9.7

Forces on the mass element dm of a continuous medium.

Because the internal forces occur in action–reaction pairs, $\int \mathrm{d}\mathbf{f}_{\text{net}} = 0$. (External forces on the body are those without an internal reactant; the reactant lies outside the body and, hence, is outside our purview.) Thus,

$$\mathbf{F}_{\text{net}} = \int_m \ddot{\mathbf{R}} \mathrm{d}m \tag{9.11}$$

where $\mathbf{F}_{\text{net}}$ is the resultant external force on the body, $\mathbf{F}_{\text{net}} = \int \mathrm{d}\mathbf{F}_{\text{net}}$. From Eqn (9.9),

$$\int_m \ddot{\mathbf{R}} \mathrm{d}m = m\ddot{\mathbf{R}}_G$$

where $\ddot{\mathbf{R}}_G = \mathbf{a}_G$, the absolute acceleration of the center of mass. Therefore, Eqn (9.11) can be written as

$$\mathbf{F}_{\text{net}} = m\ddot{\mathbf{R}}_G \tag{9.12}$$

We are therefore reminded that the motion of the center of mass of a body is determined solely by the resultant of the external forces acting on it. So far, our study of orbiting bodies has focused exclusively on the motion of their centers of mass. In this chapter, we will turn our attention to rotational motion around the center of mass. To simplify things, we will ultimately assume that the body is not only continuous but that it is also rigid. This means all points of the body remain at a fixed distance from each other and there is no flexing, bending, or twisting deformation.

9.4 Equations of rotational motion

Our development of the rotational dynamics equations does not require at the outset that the body under consideration be rigid. It may be a solid, fluid, or gas.

Point P in Figure 9.8 is arbitrary; it need not be fixed in space nor attached to a point on the body. Then, the moment about P of the forces on mass element dm (cf. Figure 9.7) is

$$\mathrm{d}\mathbf{M}_P = \mathbf{r} \times \mathrm{d}\mathbf{F}_{\text{net}} + \mathbf{r} \times \mathrm{d}\mathbf{f}_{\text{net}}$$

where $\mathbf{r}$ is the position vector of the mass element dm relative to the point P. Writing the right-hand side as $\mathbf{r} \times (\mathrm{d}\mathbf{F}_{\text{net}} + \mathrm{d}\mathbf{f}_{\text{net}})$, substituting Eqn (9.10), and integrating over all the mass elements of the body yields

$$\mathbf{M}_{P_{\text{net}}} = \int_m \mathbf{r} \times \ddot{\mathbf{R}} \mathrm{d}m \tag{9.13}$$

where $\ddot{\mathbf{R}}$ is the absolute acceleration of dm relative to the inertial frame and

$$\mathbf{M}_{P_{\text{net}}} = \int \mathbf{r} \times \mathrm{d}\mathbf{F}_{\text{net}} + \int \mathbf{r} \times \mathrm{d}\mathbf{f}_{\text{net}}$$

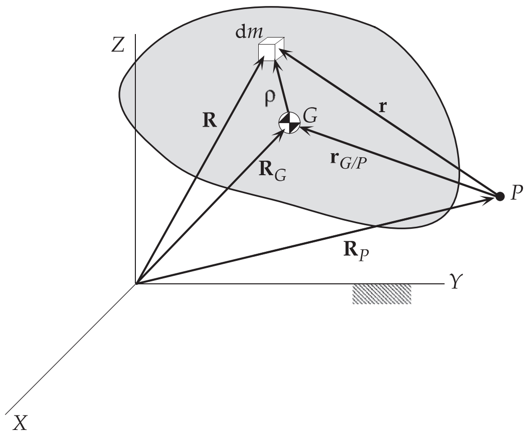


FIGURE 9.8

Position vectors of a mass element in a continuum from several key reference points.

But $\int \mathbf{r} \times \mathrm{d}\mathbf{f}_{\text{net}} = 0$ because the internal forces occur in action–reaction pairs. Thus,

$$\mathbf{M}_{P_{\text{net}}} = \int \mathbf{r} \times \mathrm{d}\mathbf{F}_{\text{net}}$$

which means the net moment includes only the moment of all of the external forces on the body.

From the product rule of calculus, we know that $\mathrm{d}(\mathbf{r} \times \dot{\mathbf{R}})/\mathrm{d}t = \mathbf{r} \times \ddot{\mathbf{R}} + \dot{\mathbf{r}} \times \dot{\mathbf{R}}$, so that the integrand in Eqn (9.13) may be written as

$$\mathbf{r} \times \ddot{\mathbf{R}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r} \times \dot{\mathbf{R}}\right) - \dot{\mathbf{r}} \times \dot{\mathbf{R}} \tag{9.14}$$

Furthermore, Figure 9.8 shows that $\mathbf{r} = \mathbf{R} - \mathbf{R}_P$, where $\mathbf{R}_P$ is the absolute position vector of P. It follows that

$$\dot{\mathbf{r}} \times \dot{\mathbf{R}} = \left(\dot{\mathbf{R}} - \dot{\mathbf{R}}_P\right) \times \dot{\mathbf{R}} = -\dot{\mathbf{R}}_P \times \dot{\mathbf{R}} \tag{9.15}$$

Substituting Eqn (9.15) into Eqn (9.14), then moving that result into Eqn (9.13), yields

$$\mathbf{M}_{P_{\text{net}}} = \frac{\mathrm{d}}{\mathrm{d}t}\int_m \mathbf{r} \times \dot{\mathbf{R}}\mathrm{d}m + \dot{\mathbf{R}}_P \times \int_m \dot{\mathbf{R}}\mathrm{d}m \tag{9.16}$$

Now, $\mathbf{r} \times \dot{\mathbf{R}}\mathrm{d}m$ is the moment of the absolute linear momentum of mass element dm about P. The moment of momentum, or angular momentum, of the entire body is the integral of this cross-product over all of its mass elements. That is, the absolute angular momentum of the body relative to point P is

$$\mathbf{H}_P = \int_m \mathbf{r} \times \dot{\mathbf{R}}\mathrm{d}m \tag{9.17}$$

Observing from Figure 9.8 that $\mathbf{r} = \mathbf{r}_{G/P} + \boldsymbol{\rho}$, we can write Eqn (9.17) as

$$\mathbf{H}_P = \int_m \left(\mathbf{r}_{G/P} + \boldsymbol{\rho}\right) \times \dot{\mathbf{R}}\mathrm{d}m = \mathbf{r}_{G/P} \times \int_m \dot{\mathbf{R}}\mathrm{d}m + \int_m \boldsymbol{\rho} \times \dot{\mathbf{R}}\mathrm{d}m \tag{9.18}$$

The last term is the absolute angular momentum relative to the center of mass G,

$$\mathbf{H}_G = \int \boldsymbol{\rho} \times \dot{\mathbf{R}}\mathrm{d}m \tag{9.19}$$

Furthermore, by the definition of center of mass, Eqn (9.9),

$$\int_m \dot{\mathbf{R}}\mathrm{d}m = m\dot{\mathbf{R}}_G \tag{9.20}$$

Equations (9.19) and (9.20) allow us to write Eqn (9.18) as

$$\mathbf{H}_P = \mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_G \tag{9.21}$$

This useful relationship shows how to obtain the absolute angular momentum about any point P once $\mathbf{H}_G$ is known.

For calculating the angular momentum about the center of mass, Eqn (9.19) can be cast in a much more useful form by making the substitution (cf. Figure 9.8) $\mathbf{R} = \mathbf{R}_G + \boldsymbol{\rho}$, so that

$$\mathbf{H}_G = \int_m \boldsymbol{\rho} \times \left(\dot{\mathbf{R}}_G + \dot{\boldsymbol{\rho}}\right)\mathrm{d}m = \int_m \boldsymbol{\rho} \times \dot{\mathbf{R}}_G\mathrm{d}m + \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\mathrm{d}m$$

In the two integrals on the right, the variable is $\boldsymbol{\rho}$. $\dot{\mathbf{R}}_G$ is fixed and can therefore be factored out of the first integral to obtain

$$\mathbf{H}_G = \left(\int_m \boldsymbol{\rho}\mathrm{d}m\right) \times \dot{\mathbf{R}}_G + \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\mathrm{d}m$$

By definition of the center of mass, $\int_m \boldsymbol{\rho}\mathrm{d}m = 0$ (the position vector of the center of mass relative to itself is zero), which means

$$\mathbf{H}_G = \int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\mathrm{d}m \tag{9.22}$$

Since $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ are the position and velocity relative to the center of mass G, $\int_m \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\mathrm{d}m$ is the total moment about the center of mass of the linear momentum relative to the center of mass, $\mathbf{H}_{G_{\mathrm{rel}}}$. In other words,

$$\mathbf{H}_G = \mathbf{H}_{G_{\mathrm{rel}}} \tag{9.23}$$

This is a rather surprising fact, hidden in Eqn (9.19), and is true in general for no other point of the body.

Another useful angular momentum formula, similar to Eqn (9.21), may be found by substituting $\mathbf{R} = \mathbf{R}_P + \mathbf{r}$ into Eqn (9.17),

$$\mathbf{H}_P = \int_m \mathbf{r} \times (\dot{\mathbf{R}}_P + \dot{\mathbf{r}})dm = \left(\int_m \mathbf{r}dm\right) \times \dot{\mathbf{R}}_P + \int_m \mathbf{r} \times \dot{\mathbf{r}}dm \tag{9.24}$$

The term on the far right is the net moment of relative linear momentum about P,

$$\mathbf{H}_{P_{\mathrm{rel}}} = \int_m \mathbf{r} \times \dot{\mathbf{r}}dm \tag{9.25}$$

Also, $\int_m \mathbf{r}dm = m\mathbf{r}_{G/P}$, where $\mathbf{r}_{G/P}$ is the position of the center of mass relative to P. Thus, Eqn (9.24) can be written as

$$\mathbf{H}_P = \mathbf{H}_{P_{\mathrm{rel}}} + \mathbf{r}_{G/P} \times m\mathbf{v}_P \tag{9.26}$$

Finally, substituting this into Eqn (9.21), solving for $\mathbf{H}_{P_{\mathrm{rel}}}$, and noting that $\mathbf{v}_G - \mathbf{v}_P = \mathbf{v}_{G/P}$ yields

$$\mathbf{H}_{P_{\mathrm{rel}}} = \mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_{G/P} \tag{9.27}$$

This expression is useful when the absolute velocity $\mathbf{v}_G$ of the center of mass, which is required in Eqn (9.21), is not available.

So far, we have written down some formulas for calculating the angular momentum about an arbitrary point in space and about the center of mass of the body itself. Let us now return to the problem of relating angular momentum to the applied torque. Substituting Eqns (9.17) and (9.20) into Eqn (9.16), we obtain

$$\mathbf{M}_{P_{\mathrm{net}}} = \dot{\mathbf{H}}_P + \dot{\mathbf{R}}_P \times m\dot{\mathbf{R}}_G$$

Thus, for an arbitrary point P,

$$\mathbf{M}_{P_{\mathrm{net}}} = \dot{\mathbf{H}}_P + \mathbf{v}_P \times m\mathbf{v}_G \tag{9.28}$$

where $\mathbf{v}_P$ and $\mathbf{v}_G$ are the absolute velocities of points P and G, respectively. This expression is applicable to two important special cases.

If the point P is at rest in inertial space ($\mathbf{v}_P = \mathbf{0}$), then Eqn (9.28) reduces to

$$\mathbf{M}_{P_{\mathrm{net}}} = \dot{\mathbf{H}}_P \tag{9.29}$$

This equation holds as well if $\mathbf{v}_P$ and $\mathbf{v}_G$ are parallel. If P is the point of contact of a wheel rolling while slipping in the plane. Note that the validity of Eqn (9.29) depends neither on the body's being rigid nor on its being in pure rotation about P. If point P is chosen to be the center of mass, then, since $\mathbf{v}_G \times \mathbf{v}_G = \mathbf{0}$, Eqn (9.28) becomes

$$\boxed{\mathbf{M}_{G_{\mathrm{net}}} = \dot{\mathbf{H}}_G} \tag{9.30}$$

This equation is valid for any state of motion.

If Eqn (9.30) is integrated over a time interval, then we obtain the angular impulse–momentum principle,

$$\int_{t_1}^{t_2} \mathbf{M}_{G_{\text{net}}} \mathrm{d}t = \mathbf{H}_{G_2} - \mathbf{H}_{G_1} \tag{9.31}$$

A similar expression follows from Eqn (9.29). $\int \mathbf{M}\mathrm{d}t$ is the angular impulse. If the net angular impulse is zero, then $\Delta\mathbf{H} = \mathbf{0}$, which is a statement of the conservation of angular momentum. Keep in mind that the angular impulse–momentum principle is not valid for just any reference point.

Additional versions of Eqns (9.29) and (9.30) can be obtained which may prove useful in special circumstances. For example, substituting the expression for $\mathbf{H}_P$ (Eqn (9.21)) into Eqn (9.28) yields

$$\begin{aligned}
\mathbf{M}_{P_{\text{net}}} &= \left[\dot{\mathbf{H}}_G + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}_{G/P} \times m\mathbf{v}_G\right)\right] + \mathbf{v}_P \times m\mathbf{v}_G \\
&= \dot{\mathbf{H}}_G + \frac{\mathrm{d}}{\mathrm{d}t}\left[(\mathbf{r}_G - \mathbf{r}_P) \times m\mathbf{v}_G\right] + \mathbf{v}_P \times m\mathbf{v}_G \\
&= \dot{\mathbf{H}}_G + (\mathbf{v}_G - \mathbf{v}_P) \times m\mathbf{v}_G + \mathbf{r}_{G/P} \times m\mathbf{a}_G + \mathbf{v}_P \times m\mathbf{v}_G
\end{aligned}$$

or, finally,

$$\mathbf{M}_{P_{\text{net}}} = \dot{\mathbf{H}}_G + \mathbf{r}_{G/P} \times m\mathbf{a}_G \tag{9.32}$$

This expression is useful when it is convenient to compute the net moment about a point other than the center of mass. Alternatively, by simply differentiating Eqn (9.27) we get

$$\dot{\mathbf{H}}_{P_{\text{rel}}} = \dot{\mathbf{H}}_G + \overbrace{\mathbf{v}_{G/P} \times m\mathbf{v}_{G/P}}^{=\mathbf{0}} + \mathbf{r}_{G/P} \times m\mathbf{a}_{G/P}$$

Solving for $\dot{\mathbf{H}}_G$, invoking Eqn (9.30), and using the fact that $\mathbf{a}_{P/G} = -\mathbf{a}_{G/P}$ leads to

$$\mathbf{M}_{G_{\text{net}}} = \dot{\mathbf{H}}_{P_{\text{rel}}} + \mathbf{r}_{G/P} \times m\mathbf{a}_{P/G} \tag{9.33}$$

Finally, if the body is rigid, the magnitude of the position vector $\boldsymbol{\rho}$ of any point relative to the center of mass does not change with time. Therefore, Eqn (1.52) requires that $\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \boldsymbol{\rho}$, leading us to conclude from Eqn (9.22) that

$$\mathbf{H}_G = \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})\mathrm{d}m \quad \text{(Rigid body)} \tag{9.34}$$

Again, the absolute angular momentum about the center of mass depends only on the absolute angular velocity and not on the absolute translational velocity of any point of the body.

No such simplification of Eqn (9.17) exists for an arbitrary reference point P. However, if the point P is fixed in inertial space and the rigid body is rotating about P, then the position vector $\mathbf{r}$ from P to any point of the body is constant. It follows from Eqn (1.52) that $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$. According to Figure 9.8,

$$\mathbf{R} = \mathbf{R}_P + \mathbf{r}$$

Differentiating with respect to time gives

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_P + \dot{\mathbf{r}} = \mathbf{0} + \boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}$$

Substituting this into Eqn (9.17) yields the formula for angular momentum in this special case as

$$\mathbf{H}_P = \int_m \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})dm \quad \text{(Rigid body rotating about fixed point } P) \tag{9.35}$$

Although Eqns (9.34) and (9.35) are mathematically identical, one must keep in mind the notation of Figure 9.8. Eqn (9.35) applies only if the rigid body is in pure rotation about a stationary point in inertial space, whereas Eqn (9.34) applies unconditionally to any situation.

9.5 Moments of inertia

To use Eqn (9.29) or Eqn (9.30) to solve problems, the vectors within them have to be resolved into components. To find the components of angular momentum, we must appeal to its definition. We will focus on the formula for angular momentum of a rigid body about its center of mass (Eqn (9.34)) because the expression for fixed-point rotation (Eqn (9.35)) is mathematically the same. The integrand of Eqn (9.34) can be rewritten using the bac–cab vector identity presented in Eqn (2.33),

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \boldsymbol{\omega}\rho^2 - \boldsymbol{\rho}(\boldsymbol{\omega}\cdot\boldsymbol{\rho}) \tag{9.36}$$

Let the origin of a comoving xyz coordinate system be attached to the center of mass G, as shown in Figure 9.9. The unit vectors of this frame are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. The vectors $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$ can be resolved into components in the xyz directions to get $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$. Substituting these vector expressions into the right side of Eqn (9.36) yields

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \left(\omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}\right)\left(x^2 + y^2 + z^2\right) - \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)\left(\omega_x x + \omega_y y + \omega_z z\right)$$

Expanding the right side and collecting the terms having the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in common, we get

$$\begin{aligned}\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = {} & \left[\left(y^2 + z^2\right)\omega_x - xy\omega_y - xz\omega_z\right]\hat{\mathbf{i}} + \left[-yx\omega_x + \left(x^2 + z^2\right)\omega_y - yz\omega_z\right]\hat{\mathbf{j}} \\ & + \left[-zx\omega_x - zy\omega_y + \left(x^2 + y^2\right)\omega_z\right]\hat{\mathbf{k}}\end{aligned} \tag{9.37}$$

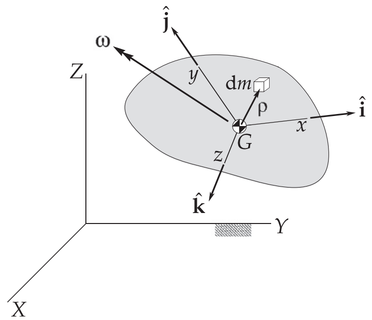


FIGURE 9.9

A comoving *xyz* frame used to compute the moments of inertia.

We put this result into the integrand of Eqn (9.34) to obtain

$$\mathbf{H}_G = H_x\hat{\mathbf{i}} + H_y\hat{\mathbf{j}} + H_z\hat{\mathbf{k}} \tag{9.38}$$

where

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \tag{9.39a}$$

or, in matrix notation,

$$\{\mathbf{H}\} = [\mathbf{I}]\{\boldsymbol{\omega}\} \tag{9.39b}$$

The nine components of the moment of inertia matrix $[\mathbf{I}]$ about the center of mass are

$$\begin{array}{lll} I_x = \int\left(y^2+z^2\right)\mathrm{d}m & I_{xy} = -\int xy\mathrm{d}m & I_{xz} = -\int xz\mathrm{d}m \\ I_{yx} = -\int yx\mathrm{d}m & I_y = \int\left(x^2+z^2\right)\mathrm{d}m & I_{yz} = -\int yz\mathrm{d}m \\ I_{zx} = -\int zx\mathrm{d}m & I_{zy} = -\int zy\mathrm{d}m & I_z = \int\left(x^2+y^2\right)\mathrm{d}m \end{array} \tag{9.40}$$

Since $I_{yx} = I_{xy}$, $I_{zx} = I_{xz}$, and $I_{zy} = I_{yz}$, it follows that $[\mathbf{I}]$ is a symmetric matrix $\left([\mathbf{I}]^T = [\mathbf{I}]\right)$. Therefore, $[\mathbf{I}]$ has six independent components instead of nine. Observe that, whereas the products of inertia I_{xy}, I_{xz}, and I_{yz} can be positive, negative, or zero, the moments of inertia I_x, I_y, and I_z are always positive (never zero or negative) for bodies of finite dimensions. For this reason, $[\mathbf{I}]$ is a symmetric positive-definite matrix. Keep in mind that Eqns (9.38) and (9.39) are valid as well for axes attached to a fixed point P about which the body is rotating.

The moments of inertia reflect how the mass of a rigid body is distributed. They manifest a body's rotational inertia, that is, its resistance to being set into rotary motion or stopped once rotation is underway. It is not an object's mass alone but how that mass is distributed that determines how the body will respond to applied torques.

If the xy plane is a plane of symmetry, then for any x and y within the body there are identical mass elements located at $+z$ and $-z$. This means the products of inertia with z in the integrand vanish. Similar statements are true if xz or yz are symmetry planes. In summary, we conclude

If the xy plane is a plane of symmetry of the body, then $I_{xz} = I_{yz} = 0$.
If the xz plane is a plane of symmetry of the body, then $I_{xy} = I_{yz} = 0$.
If the yz plane is a plane of symmetry of the body, then $I_{xy} = I_{xz} = 0$.

It follows that if the body has two planes of symmetry relative to the xyz frame of reference, then all three products of inertia vanish, and $[\mathbf{I}]$ becomes a diagonal matrix such that,

$$[\mathbf{I}] = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{9.41}$$

A, B, and C are the principal moments of inertia (all positive), and the xyz axes are the body's principal axes of inertia. In this case, relative to either the center of mass or a fixed point of rotation, we have

$$H_x = A\omega_x \quad H_y = B\omega_y \quad H_z = C\omega_z \tag{9.42}$$

In general, the angular velocity $\boldsymbol{\omega}$ and the angular momentum $\mathbf{H}$ are not parallel. However, if, for example, $\boldsymbol{\omega} = \omega\hat{\mathbf{i}}$, then according to Eqn (9.42), $\mathbf{H} = A\boldsymbol{\omega}$. In other words, if the angular velocity is aligned with a principal direction, so is the angular momentum. In that case, the two vectors $\boldsymbol{\omega}$ and $\mathbf{H}$ are indeed parallel.

Each of the three principal moments of inertia can be expressed as follows:

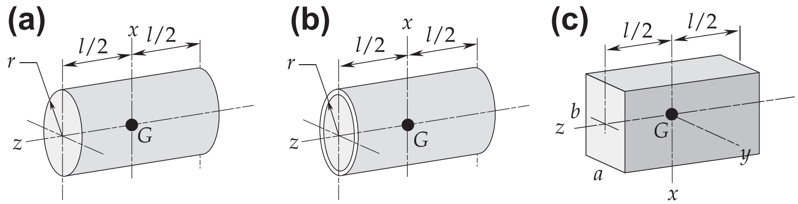
$$A = mk_x^2 \quad B = mk_y^2 \quad C = mk_z^2 \tag{9.43}$$

where m is the mass of the body and k_x, k_y, and k_z are the three radii of gyration. One may imagine the mass of a body to be concentrated around a principal axis at a distance equal to the radius of gyration.

The moments of inertia for several common shapes are listed in Figure 9.10. By symmetry, their products of inertia vanish for the coordinate axes used. Formulas for other solid geometries can be found in engineering handbooks and in dynamics textbooks.

For a mass concentrated at a point, the moments of inertia in Eqn (9.40) are just the mass times the integrand evaluated at the point. That is, the moment of inertia matrix $[\mathbf{I}^{(m)}]$ of a point mass m is given by

$$\left[\mathbf{I}^{(m)}\right] = \begin{bmatrix} m\left(y^2+z^2\right) & -mxy & -mxz \\ -mxy & m\left(x^2+z^2\right) & -myz \\ -mxz & -myz & m\left(x^2+y^2\right) \end{bmatrix} \tag{9.44}$$



(a)
$$I_x = \frac{1}{4}mr^2 + \frac{1}{12}ml^2$$
$$I_z = \frac{1}{2}mr^2$$

(b)
$$I_x = \frac{1}{2}mr^2 + \frac{1}{12}ml^2$$
$$I_z = mr^2$$

(c)
$$I_x = \frac{1}{12}m(a^2 + l^2)$$
$$I_y = \frac{1}{12}m(b^2 + l^2)$$
$$I_z = \frac{1}{12}m(a^2 + b^2)$$

FIGURE 9.10

Moments of inertia for three common homogeneous solids of mass m. (a) Solid circular cylinder. (b) Circular cylindrical shell. (c) Rectangular parallelepiped.

EXAMPLE 9.4

The following table lists out the mass and coordinates of seven point masses. Find the center of mass of the system and the moments of inertia about the origin.

Point, i	Mass m_i (kg)	x_i (m)	y_i (m)	z_i (m)
1	3	−0.5	0.2	0.3
2	7	0.2	0.75	−0.4
3	5	1	−0.8	0.9
4	6	1.2	−1.3	1.25
5	2	−1.3	1.4	−0.8
6	4	−0.3	1.35	0.75
7	1	1.5	−1.7	0.85

Solution

The total mass of this system is

$$m = \sum_{i=1}^{7} m_i = 28 \text{ kg}$$

For concentrated masses, the integral in Eqn (9.9) is replaced by the mass times its position vector. Therefore, in this case, the three components of the position vector of the center of mass are $x_G = (1/m)\sum_{i=1}^{7} m_i x_i$, $y_G = (1/m)\sum_{i=1}^{7} m_i y_i$, and $z_G = (1/m)\sum_{i=1}^{7} m_i z_i$, so that

$$\boxed{x_G = 0.35 \text{ m} \quad y_G = 0.01964 \text{ m} \quad z_G = 0.4411 \text{ m}}$$

The total moment of inertia is the sum over all the particles of Eqn (9.44) evaluated at each point. Thus,

$$[\mathbf{I}] = \overbrace{\begin{bmatrix} 0.39 & 0.3 & 0.45 \\ 0.3 & 1.02 & -0.18 \\ 0.45 & -0.18 & 0.87 \end{bmatrix}}^{(1)} + \overbrace{\begin{bmatrix} 5.0575 & -1.05 & 0.56 \\ -1.05 & 1.4 & 2.1 \\ 0.56 & 2.1 & 4.2175 \end{bmatrix}}^{(2)} + \overbrace{\begin{bmatrix} 7.25 & 4 & -4.5 \\ 4 & 9.05 & 3.6 \\ -4.5 & 3.6 & 8.2 \end{bmatrix}}^{(3)}$$
$$+ \overbrace{\begin{bmatrix} 19.515 & 9.36 & -9 \\ 9.36 & 18.015 & 9.75 \\ -9 & 9.75 & 18.78 \end{bmatrix}}^{(4)} + \overbrace{\begin{bmatrix} 5.2 & 3.64 & -2.08 \\ 3.64 & 4.66 & 2.24 \\ -2.08 & 2.24 & 7.3 \end{bmatrix}}^{(5)} + \overbrace{\begin{bmatrix} 9.54 & 1.62 & 0.9 \\ 1.62 & 2.61 & -4.05 \\ 0.9 & -4.05 & 7.65 \end{bmatrix}}^{(6)}$$
$$+ \overbrace{\begin{bmatrix} 3.6125 & 2.55 & -1.275 \\ 2.55 & 2.9725 & 1.445 \\ -1.275 & 1.445 & 5.14 \end{bmatrix}}^{(7)}$$

or

$$\boxed{[\mathbf{I}] = \begin{bmatrix} 50.56 & 20.42 & -14.94 \\ 20.42 & 39.73 & 14.90 \\ -14.94 & 14.90 & 52.16 \end{bmatrix} (\text{kg}\cdot\text{m}^2)}$$

EXAMPLE 9.5

Calculate the moments of inertia of a slender, homogeneous straight rod of length ℓ and mass m shown in Figure 9.11. One end of the rod is at the origin and the other has coordinates (a, b, c).

Solution

A slender rod is one whose cross-sectional dimensions are negligible compared with its length. The mass is concentrated along its centerline. Since the rod is homogeneous, the mass per unit length ρ is uniform and given by

$$\rho = \frac{m}{\ell} \tag{a}$$

The length of the rod is

$$\ell = \sqrt{a^2 + b^2 + c^2}$$

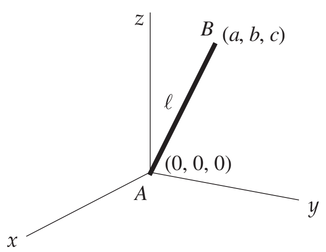


FIGURE 9.11

A uniform slender bar of mass m and length ℓ.

Starting with I_x, we have from Eqn (9.40),

$$I_x = \int_0^{\ell} \left(y^2 + z^2\right)\rho \mathrm{d}s$$

in which we replaced the element of mass dm by ρds, where ds is the element of length along the rod. The distance s is measured from end A of the rod, so that the x, y, and z coordinates of any point along it are found in terms of s by the following relations:

$$x = \frac{s}{\ell}a \quad y = \frac{s}{\ell}b \quad z = \frac{s}{\ell}c$$

Thus,

$$I_x = \int_0^{\ell} \left(\frac{s^2}{\ell^2}b^2 + \frac{s^2}{\ell^2}c^2\right)\rho \mathrm{d}s \;= \rho\frac{b^2 + c^2}{\ell^2}\int_0^{\ell} s^2 \mathrm{d}s \;= \frac{1}{3}\rho\left(b^2 + c^2\right)\ell$$

Substituting Eqn (a) yields

$$\boxed{I_x = \frac{1}{3}m\left(b^2 + c^2\right)}$$

In precisely the same way, we find

$$\boxed{I_y = \frac{1}{3}m\left(a^2 + c^2\right)} \quad \boxed{I_z = \frac{1}{3}m\left(a^2 + b^2\right)}$$

For I_{xy} we have

$$I_{xy} = -\int_0^{\ell} xy\rho \mathrm{d}s \;= -\int_0^{\ell} \frac{s}{\ell}a \cdot \frac{s}{\ell}b\rho \mathrm{d}s \;= -\rho\frac{ab}{\ell^2}\int_0^{\ell} s^2 \mathrm{d}s \;= -\frac{1}{3}\rho ab\ell$$

Once again using Eqn (a),

$$\boxed{I_{xy} = -\frac{1}{3}mab}$$

Likewise,

$$\boxed{I_{xz} = -\frac{1}{3}mac} \quad \boxed{I_{yz} = -\frac{1}{3}mbc}$$

EXAMPLE 9.6

The gyro rotor (Figure 9.12) in Example 9.3 has a mass m of 5 kg, radius r of 0.08 m, and thickness t of 0.025 m. If $N = 2.1$ rad/s, $\dot{\theta} = 4$ rad/s, $\omega = 10.5$ rad/s, and $\theta = 60°$, calculate

(a) The angular momentum of the rotor about its center of mass G in the body-fixed xyz frame.
(b) The angle between the rotor's angular velocity vector and its angular momentum vector.

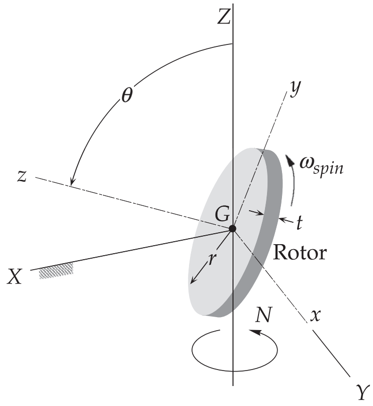


FIGURE 9.12

Rotor of the gyroscope in Figure 9.4.

Solution

Equation (f) from Example 9.3 gives the components of the absolute angular velocity of the rotor in the moving *xyz* frame.

$$\begin{aligned}\omega_x &= \dot{\theta} = 4 \ \text{rad/s}\\ \omega_y &= N\sin\theta = 2.1\cdot\sin 60^\circ = 1.819 \ \text{rad/s}\\ \omega_z &= \omega_{\text{spin}} + N\cos\theta = 10.5 + 2.1\cdot\cos 60^\circ = 11.55 \ \text{rad/s}\end{aligned} \tag{a}$$

Therefore,

$$\boldsymbol{\omega} = 4\hat{\mathbf{i}} + 1.819\hat{\mathbf{j}} + 11.55\hat{\mathbf{k}} \ \text{(rad/s)} \tag{b}$$

All three coordinate planes of the body-fixed *xyz* frame contain the center of mass G and all are planes of symmetry of the circular cylindrical rotor. Therefore, $\ell_{xy} = \ell_{zx} = \ell_{yz} = 0$.

From Figure 9.10(a), we see that the nonzero diagonal entries in the moment of inertia tensor are

$$\begin{aligned}A = B &= \frac{1}{12}mt^2 + \frac{1}{4}mr^2 = \frac{1}{12}\cdot 5\cdot 0.025^2 + \frac{1}{4}\cdot 5\cdot 0.08^2 = 0.008260 \ \text{kg}\cdot\text{m}^2\\ C &= \frac{1}{2}mr^2 = \frac{1}{2}\cdot 5\cdot 0.08^2 = 0.0160 \ \text{kg}\cdot\text{m}^2\end{aligned} \tag{c}$$

We can use Eqn (9.42) to calculate the angular momentum, because the origin of the *xyz* frame is the rotor's center of mass (which in this case also happens to be a fixed point of rotation, which is another reason why we can use Eqn (9.42)). Substituting Eqns (a) and (c) in Eqn (9.42) yields

$$\begin{aligned}H_x &= A\omega_x = 0.008260\cdot 4 &&= 0.03304 \ \text{kg}\cdot\text{m}^2/\text{s}\\ H_y &= B\omega_y = 0.008260\cdot 1.819 &&= 0.0150 \ \text{kg}\cdot\text{m}^2/\text{s}\\ H_z &= C\omega_z = 0.0160\cdot 11.55 &&= 0.1848 \ \text{kg}\cdot\text{m}^2/\text{s}\end{aligned} \tag{d}$$

so that

$$\boxed{\mathbf{H} = 0.03304\hat{\mathbf{i}} + 0.0150\hat{\mathbf{j}} + 0.1848\hat{\mathbf{k}} \ (\text{kg}\cdot\text{m}^2/\text{s})} \tag{e}$$

The angle ϕ between $\mathbf{H}$ and $\boldsymbol{\omega}$ is found by taking the dot product of the two vectors,

$$\phi = \cos^{-1}\left(\frac{\mathbf{H}\cdot\boldsymbol{\omega}}{H\omega}\right) = \cos^{-1}\left(\frac{2.294}{0.1883\cdot 12.36}\right) = \boxed{9.717^\circ} \tag{f}$$

As this problem illustrates, the angular momentum and the angular velocity are in general not collinear.

Consider a coordinate system $x'y'z'$ with the same origin as xyz but a different orientation. Let $[\mathbf{Q}]$ be the orthogonal matrix ($[\mathbf{Q}]^{-1} = [\mathbf{Q}]^T$) that transforms the components of a vector from the xyz system to the $x'y'z'$ frame. Recall from Section 4.5 that the rows of $[\mathbf{Q}]$ are the direction cosines of the $x'y'z'$ axes relative to xyz. If $\{\mathbf{H}'\}$ comprises the components of the angular momentum vector along the $x'y'z'$ axes, then $\{\mathbf{H}'\}$ is obtained from its components $\{\mathbf{H}\}$ in the xyz frame by the relation

$$\{\mathbf{H}'\} = [\mathbf{Q}]\{\mathbf{H}\}$$

From Eqn (9.39), we can write this as

$$\{\mathbf{H}'\} = [\mathbf{Q}][\mathbf{I}]\{\boldsymbol{\omega}\} \tag{9.45}$$

where $[\mathbf{I}]$ is the moment of inertia matrix (Eqn (9.39)) in the xyz coordinates. Like the angular momentum vector, the components $\{\boldsymbol{\omega}\}$ of the angular velocity vector in the xyz system are related to those in the primed system ($\{\boldsymbol{\omega}'\}$) by the expression

$$\{\boldsymbol{\omega}'\} = [\mathbf{Q}]\{\boldsymbol{\omega}\}$$

The inverse relation is simply

$$\{\boldsymbol{\omega}\} = [\mathbf{Q}]^{-1}\{\boldsymbol{\omega}'\} = [\mathbf{Q}]^T\{\boldsymbol{\omega}'\} \tag{9.46}$$

Substituting this into Eqn (9.45), we get

$$\{\mathbf{H}'\} = [\mathbf{Q}][\mathbf{I}][\mathbf{Q}]^T\{\boldsymbol{\omega}'\} \tag{9.47}$$

But the components of angular momentum and angular velocity in the $x'y'z'$ frame are related by an equation of the same form as Eqn (9.39), so that

$$\{\mathbf{H}'\} = [\mathbf{I}']\{\boldsymbol{\omega}'\} \tag{9.48}$$

where $[\mathbf{I}']$ comprises the components of the inertia matrix in the primed system. Comparing the right-hand sides of Eqns (9.47) and (9.48), we conclude that

$$[\mathbf{I}'] = [\mathbf{Q}][\mathbf{I}][\mathbf{Q}]^T \tag{9.49a}$$

that is,

$$\begin{bmatrix} I_{x'} & I_{x'y'} & I_{x'z'} \\ I_{y'x'} & I_{y'} & I_{y'z'} \\ I_{z'x'} & I_{z'y'} & I_{z'} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{32} & Q_{33} \end{bmatrix} \tag{9.49b}$$

This shows how to transform the components of the inertia matrix from the *xyz* coordinate system to any other orthogonal system with a common origin. Thus, for example,

$$I_{x'} = \overbrace{\lfloor Q_{11} \quad Q_{12} \quad Q_{13} \rfloor}^{\lfloor \text{Row } 1 \rfloor} \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \overbrace{\begin{Bmatrix} Q_{11} \\ Q_{12} \\ Q_{13} \end{Bmatrix}}^{\lfloor \text{Row } 1 \rfloor^T}$$

$$I_{y'z'} = \overbrace{\lfloor Q_{21} \quad Q_{22} \quad Q_{23} \rfloor}^{\lfloor \text{Row } 2 \rfloor} \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \overbrace{\begin{Bmatrix} Q_{31} \\ Q_{32} \\ Q_{33} \end{Bmatrix}}^{\lfloor \text{Row } 3 \rfloor^T} \tag{9.50}$$

etc.

Any object represented by a square matrix whose components transform according to Eqn (9.49) is called a second-order tensor. We may therefore refer to **[I]** as the inertia tensor.

EXAMPLE 9.7

Find the mass moment of inertia of the system of point masses in Example 9.4 about an axis from the origin through the point with coordinates (2 m, −3 m, 4 m).

Solution

From Example 9.4, the moment of inertia tensor for the system of point masses is

$$[\mathbf{I}] = \begin{bmatrix} 50.56 & 20.42 & -14.94 \\ 20.42 & 39.73 & 14.90 \\ -14.94 & 14.90 & 52.16 \end{bmatrix} \left(\text{kg}\cdot\text{m}^2\right)$$

The vector **V** connecting the origin with (2 m, −3 m, 4 m) is

$$\mathbf{V} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

The unit vector in the direction of **V** is

$$\hat{\mathbf{u}}_V = \frac{\mathbf{V}}{\|\mathbf{V}\|} = 0.3714\hat{\mathbf{i}} - 0.5571\hat{\mathbf{j}} + 0.7428\hat{\mathbf{k}}$$

We may consider $\hat{\mathbf{u}}_V$ as the unit vector along the x'-axis of a rotated Cartesian coordinate system. Then, from Eqn (9.50) we get

$$\ell_{V'} = \lfloor 0.3714 \quad -0.5571 \quad 0.7428 \rfloor \begin{bmatrix} 50.56 & 20.42 & -14.94 \\ 20.42 & 39.73 & 14.90 \\ -14.94 & 14.90 & 52.16 \end{bmatrix} \begin{Bmatrix} 0.3714 \\ -0.5571 \\ 0.7428 \end{Bmatrix}$$

$$= \lfloor 0.3714 \quad -0.5571 \quad 0.7428 \rfloor \begin{Bmatrix} -3.695 \\ -3.482 \\ 24.90 \end{Bmatrix} = \boxed{19.06 \ \text{kg}\cdot\text{m}^2}$$

EXAMPLE 9.8

For the satellite of Example 9.2, which is reproduced in Figure 9.13, the data are as follows: $N = 0.1$ rad/s and $\dot{\theta} = 0.01$ rad/s, in the directions shown: $\theta = 40°$ and $d_0 = 1.5$ m. The length, width, and thickness of the panel are $\ell = 6$ m, $w = 2$ m, and $t = 0.025$ m. The uniformly distributed mass of the panel is 50 kg. Find the angular momentum of the panel relative to the center of mass O of the satellite.

Solution

We can treat the panel as a thin parallelepiped. The panel's xyz axes have their origin at the center of mass G of the panel and are parallel to its three edge directions. According to Figure 9.10(c), the moments of inertia relative to the xyz coordinate system are

$$\begin{aligned}
I_{Gx} &= \frac{1}{12}m\left(\ell^2 + t^2\right) = \frac{1}{12}\cdot 50\cdot\left(6^2 + 0.025^2\right) = 150.0 \text{ kg}\cdot\text{m}^2 \\
I_{Gy} &= \frac{1}{12}m\left(w^2 + t^2\right) = \frac{1}{12}\cdot 50\cdot\left(2^2 + 0.025^2\right) = 16.67 \text{ kg}\cdot\text{m}^2 \\
I_{Gz} &= \frac{1}{12}m\left(w^2 + \ell^2\right) = \frac{1}{12}\cdot 50\cdot\left(2^2 + 6^2\right) = 166.7 \text{ kg}\cdot\text{m}^2 \\
I_{Gxy} &= I_{Gxz} = I_{Gyz} = 0
\end{aligned} \tag{a}$$

In matrix notation,

$$[\mathbf{I}_G] = \begin{bmatrix} 150.0 & 0 & 0 \\ 0 & 16.67 & 0 \\ 0 & 0 & 166.7 \end{bmatrix} \left(\text{kg}\cdot\text{m}^2\right) \tag{b}$$

The unit vectors of the satellite's $x'y'z'$ system are related to those of the panel's xyz frame. By inspection,

$$\begin{aligned}
\hat{\mathbf{i}}' &= -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{k}} = -0.6428\hat{\mathbf{i}} + 0.7660\hat{\mathbf{k}} \\
\hat{\mathbf{j}}' &= -\hat{\mathbf{j}} \\
\hat{\mathbf{k}}' &= \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{k}} = 0.7660\hat{\mathbf{i}} + 0.6428\hat{\mathbf{k}}
\end{aligned} \tag{c}$$

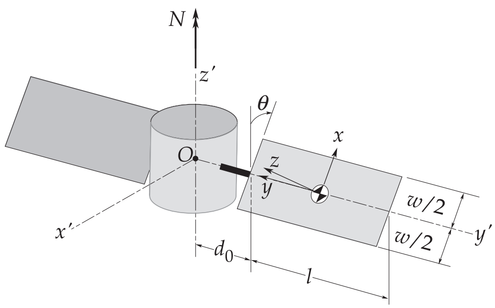


FIGURE 9.13

A satellite and solar panel.

The matrix $[\mathbf{Q}]$ of the transformation from xyz to $x'y'z'$ comprises the direction cosines of $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, and $\hat{\mathbf{k}}'$:

$$[\mathbf{Q}] = \begin{bmatrix} -0.6428 & 0 & 0.7660 \\ 0 & -1 & 0 \\ 0.7660 & 0 & 0.6428 \end{bmatrix} \tag{d}$$

In Example 9.2, we found that the absolute angular velocity of the panel, in the satellite's $x'y'z'$ frame of reference, is

$$\boldsymbol{\omega} = -\dot{\theta}\hat{\mathbf{j}}' + N\hat{\mathbf{k}}' = -0.01\hat{\mathbf{j}}' + 0.1\hat{\mathbf{k}}' \ \text{(rad/s)}$$

That is,

$$\{\boldsymbol{\omega}'\} = \begin{Bmatrix} 0 \\ -0.01 \\ 0.1 \end{Bmatrix} \text{(rad/s)} \tag{e}$$

To find the absolute angular momentum $\{\mathbf{H}'_G\}$ in the satellite system requires the use of Eqn (9.39),

$$\{\mathbf{H}'_G\} = [\mathbf{I}'_G]\{\boldsymbol{\omega}'\} \tag{f}$$

Before doing so, we must transform the components of the moments of the inertia tensor in Eqn (b) from the unprimed system to the primed system, by means of Eqn (9.49),

$$[\mathbf{I}'_G] = [\mathbf{Q}][\mathbf{I}_G][\mathbf{Q}]^T = \begin{bmatrix} -0.6428 & 0 & 0.7660 \\ 0 & -1 & 0 \\ 0.7660 & 0 & 0.6428 \end{bmatrix} \begin{bmatrix} 150.0 & 0 & 0 \\ 0 & 16.67 & 0 \\ 0 & 0 & 166.7 \end{bmatrix} \begin{bmatrix} -0.6428 & 0 & 0.7660 \\ 0 & -1 & 0 \\ 0.7660 & 0 & 0.6428 \end{bmatrix}$$

so that

$$[\mathbf{I}'_G] = \begin{bmatrix} 159.8 & 0 & 8.205 \\ 0 & 16.67 & 0 \\ 8.205 & 0 & 156.9 \end{bmatrix} \left(\text{kg}\cdot\text{m}^2\right) \tag{g}$$

Then Eqn (f) yields

$$\{\mathbf{H}'_G\} = \begin{bmatrix} 159.8 & 0 & 8.205 \\ 0 & 16.67 & 0 \\ 8.205 & 0 & 156.9 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.01 \\ 0.1 \end{Bmatrix} = \begin{Bmatrix} 0.8205 \\ -0.1667 \\ 15.69 \end{Bmatrix} \left(\text{kg}\cdot\text{m}^2/\text{s}\right)$$

or, in vector notation,

$$\mathbf{H}_G = 0.8205\hat{\mathbf{i}}' - 0.1667\hat{\mathbf{j}}' + 15.69\hat{\mathbf{k}}' \ \left(\text{kg}\cdot\text{m}^2/\text{s}\right) \tag{h}$$

This is the absolute angular momentum of the panel about its own center of mass G, and it is used in Eqn (9.27) to calculate the angular momentum $\mathbf{H}_{O_{\text{rel}}}$ relative to the satellite's center of mass O,

$$\mathbf{H}_{O_{\text{rel}}} = \mathbf{H}_G + \mathbf{r}_{G/O} \times m\mathbf{v}_{G/O} \tag{i}$$

$\mathbf{r}_{G/O}$ is the position vector from O to G,

$$\mathbf{r}_{G/O} = \left(d_O + \frac{\ell}{2}\right)\hat{\mathbf{j}}' = \left(1.5 + \frac{6}{2}\right)\hat{\mathbf{j}}' = 4.5\hat{\mathbf{j}}' \ \text{(m)} \tag{j}$$

The velocity of G relative to O, $\mathbf{v}_{G/O}$, is found from Eqn (9.2),

$$\mathbf{v}_{G/O} = \boldsymbol{\omega}_{\text{satellite}} \times \mathbf{r}_{G/O} = N\hat{\mathbf{k}}' \times \mathbf{r}_{G/O} = 0.1\hat{\mathbf{k}}' \times 4.5\hat{\mathbf{j}}' = -0.45\hat{\mathbf{i}}' \ \text{(m/s)} \tag{k}$$

Substituting Eqns (h), (j) and (k) into Eqn (i) finally yields

$$\begin{aligned}\mathbf{H}_{O_{\text{rel}}} &= \left(0.8205\hat{\mathbf{i}}' - 0.1667\hat{\mathbf{j}}' + 15.69\hat{\mathbf{k}}'\right) + 4.5\hat{\mathbf{j}}' \times \left[50\left(-0.45\hat{\mathbf{i}}'\right)\right] \\ &= \boxed{0.8205\hat{\mathbf{i}}' - 0.1667\hat{\mathbf{j}}' + 116.9\hat{\mathbf{k}}' \ (\text{kg}\cdot\text{m}^2/\text{s})}\end{aligned} \tag{l}$$

Note that we were unable to use Eqn (9.21) to find the absolute angular momentum $\mathbf{H}_O$ because that requires knowing the absolute velocity $\mathbf{v}_G$, which in turn depends on the absolute velocity of O, which was not provided.

How can we find the direction cosine matrix $[\mathbf{Q}]$ such that Eqn (9.49) will yield a moment of inertia matrix $[\mathbf{I}']$ that is diagonal, that is, of the form given by Eqn (9.41)? In other words, how do we find the principal directions ("eigenvectors") and the corresponding principal values ("eigenvalues") of the moment of inertia tensor?

Let the angular velocity vector $\boldsymbol{\omega}$ be parallel to the principal direction defined by the vector $\mathbf{e}$, so that $\boldsymbol{\omega} = \beta\mathbf{e}$, where β is a scalar. Since $\boldsymbol{\omega}$ points in the principal direction of the inertia tensor, so must $\mathbf{H}$, which means $\mathbf{H}$ is also parallel to $\mathbf{e}$. Therefore, $\mathbf{H} = \alpha\mathbf{e}$, where α is a scalar. From Eqn (9.39), it follows that

$$\alpha\{\mathbf{e}\} = [\mathbf{I}](\beta\{\mathbf{e}\})$$

or

$$[\mathbf{I}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$$

where $\lambda = \alpha/\beta$ (a scalar). That is,

$$\begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = \lambda \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix}$$

This can be written

$$\begin{bmatrix} I_x - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{9.51}$$

The trivial solution of Eqn (9.51) is $\mathbf{e} = \mathbf{0}$, which is of no interest. The only way that Eqn (9.51) will not yield the trivial solution is if the coefficient matrix on the left is singular. That will occur if its determinant vanishes, that is, if

$$\begin{vmatrix} I_x - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda \end{vmatrix} = 0 \tag{9.52}$$

Expanding the determinant, we find

$$\begin{vmatrix} I_x-\lambda & I_{xy} & I_{xz} \\ I_{xy} & I_y-\lambda & I_{yz} \\ I_{xz} & I_{yz} & I_z-\lambda \end{vmatrix} = -\lambda^3 + J_1\lambda^2 - J_2\lambda + J_3 \tag{9.53}$$

where

$$\begin{aligned} J_1 &= I_x + I_y + I_z \\ J_2 &= \begin{vmatrix} I_x & I_{xy} \\ I_{xy} & I_y \end{vmatrix} + \begin{vmatrix} I_x & I_{xz} \\ I_{xz} & I_z \end{vmatrix} + \begin{vmatrix} I_y & I_{yz} \\ I_{yz} & I_z \end{vmatrix} \\ J_3 &= \begin{vmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{vmatrix} \end{aligned} \tag{9.54}$$

J_1, J_2, and J_3 are invariants, that is, they have the same value in every Cartesian coordinate system.

Equations (9.52) and (9.53) yield the characteristic equation of the tensor [**I**],

$$\lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0 \tag{9.55}$$

The three roots λ_p ($p = 1, 2, 3$) of this cubic equation are real, since [**I**] is symmetric; furthermore, they are all positive, since [**I**] is a positive-definite matrix. We substitute each root, or eigenvalue, λ_p back into Eqn (9.51) to obtain

$$\begin{bmatrix} I_x-\lambda_p & I_{xy} & I_{xz} \\ I_{xy} & I_y-\lambda_p & I_{yz} \\ I_{xz} & I_{yz} & I_z-\lambda_p \end{bmatrix} \begin{Bmatrix} e_x^{(p)} \\ e_y^{(p)} \\ e_z^{(p)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad p = 1, 2, 3 \tag{9.56}$$

Solving this system yields the three eigenvectors $\mathbf{e}^{(p)}$ corresponding to each of the three eigenvalues λ_p. The three eigenvectors are orthogonal, also due to the symmetry of the matrix [**I**]. Each eigenvalue is a principal moment of inertia, and its corresponding eigenvector is a principal direction.

EXAMPLE 9.9

Find the principal moments of inertia and the principal axes of inertia of the inertia tensor.

$$[\mathbf{I}] = \begin{bmatrix} 100 & -20 & -100 \\ -20 & 300 & -50 \\ -100 & -50 & 500 \end{bmatrix} \text{ kg}\cdot\text{m}^2$$

Solution

We seek the nontrivial solutions of the system $[\mathbf{I}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$, that is,

$$\begin{bmatrix} 100-\lambda & -20 & -100 \\ -20 & 300-\lambda & -50 \\ -100 & -50 & 500-\lambda \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{a}$$

From Eqn (9.54),

$$\begin{aligned} J_1 &= 100 + 300 + 500 = 900 \\ J_2 &= \begin{vmatrix} 100 & -20 \\ -20 & 300 \end{vmatrix} + \begin{vmatrix} 100 & -100 \\ -100 & 500 \end{vmatrix} + \begin{vmatrix} 300 & -50 \\ -50 & 500 \end{vmatrix} = 217,100 \\ J_3 &= \begin{vmatrix} 100 & -20 & -100 \\ -20 & 300 & -50 \\ -100 & -50 & 500 \end{vmatrix} = 11,350,000 \end{aligned} \tag{b}$$

Thus, the characteristic equation is

$$\lambda^3 - 900\lambda^2 + 217{,}100\lambda - 11{,}350{,}000 = 0 \tag{c}$$

The three roots are the principal moments of inertia, which are found to be

$$\boxed{\lambda_1 = 532.052 \quad \lambda_2 = 295.840 \quad \lambda_3 = 72.1083} \tag{d}$$

Each of these is substituted, in turn, back into Eqn (a) to find its corresponding principal direction.

Substituting $\lambda_1 = 532.052\ \text{kg}\cdot\text{m}^2$ into Eqn (a) we obtain

$$\begin{bmatrix} -432.052 & -20.0000 & -100.0000 \\ -20.0000 & -232.052 & -50.0000 \\ -100.0000 & -50.0000 & -32.0519 \end{bmatrix} \begin{Bmatrix} e_x^{(1)} \\ e_y^{(1)} \\ e_z^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{e}$$

Since the determinant of the coefficient matrix is zero, at most two of the three equations in Eqn (e) are independent. Thus, at most, two of the three components of the vector $\mathbf{e}^{(1)}$ can be found in terms of the third. We can therefore arbitrarily set $e_x^{(1)} = 1$ and solve for $e_y^{(1)}$ and $e_z^{(1)}$ using any two of the independent equations in Eqn (e). With $e_x^{(1)} = 1$, the first two of Eqn (e) become

$$\begin{aligned} -20.0000e_y^{(1)} - 100.000e_z^{(1)} &= 432.052 \\ -232.052e_y^{(1)} - 50.000e_z^{(1)} &= 20.0000 \end{aligned} \tag{f}$$

Solving these two equations for $e_y^{(1)}$ and $e_z^{(1)}$ yields, together with the assumption that $e_x^{(1)} = 1$,

$$e_x^{(1)} = 1.00000 \quad e_y^{(1)} = 0.882793 \quad e_z^{(1)} = -4.49708 \tag{g}$$

The unit vector in the direction of $\mathbf{e}^{(1)}$ is

$$\hat{\mathbf{e}}_1 = \frac{\mathbf{e}^{(1)}}{\|\mathbf{e}^{(1)}\|} = \frac{1.00000\hat{\mathbf{i}} + 0.882793\hat{\mathbf{j}} - 4.49708\hat{\mathbf{k}}}{\sqrt{1.00000^2 + 0.882793^2 + (-4.49708)^2}}$$

or

$$\boxed{\hat{\mathbf{e}}_1 = 0.213186\hat{\mathbf{i}} + 0.188199\hat{\mathbf{j}} - 0.958714\hat{\mathbf{k}} \quad (\lambda_1 = 532.052 \ \text{kg}\cdot\text{m}^2)} \tag{h}$$

Substituting $\lambda_2 = 295.840\ \text{kg}\cdot\text{m}^2$ into Eqn (a) and proceeding as above we find that

$$\boxed{\hat{\mathbf{e}}_2 = 0.176732\hat{\mathbf{i}} - 0.972512\hat{\mathbf{j}} - 0.151609\hat{\mathbf{k}} \quad (\lambda_2 = 295.840 \ \text{kg}\cdot\text{m}^2)} \tag{i}$$

The two unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ define two of the three principal directions of the inertia tensor. Observe that $\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = 0$, as must be the case for symmetric matrices.

To obtain the third principal direction $\hat{\mathbf{e}}_3$, we can substitute $\lambda_3 = 72.1083\ \text{kg}\cdot\text{m}^2$ into Eqn (a) and proceed as above. However, since the inertia tensor is symmetric, we know that the three principal directions are mutually orthogonal, which means $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$. Substituting Eqns (h) and (i) into the crossproduct, we find that

$$\boxed{\hat{\mathbf{e}}_3 = -0.960894\hat{\mathbf{i}} - 0.137114\hat{\mathbf{j}} - 0.240587\hat{\mathbf{k}} \quad (\lambda_3 = 72.1083 \ \text{kg}\cdot\text{m}^2)} \tag{j}$$

We can check our work by substituting λ_3 and $\hat{\mathbf{e}}_3$ into Eqn (a) and verify that it is indeed satisfied:

$$\begin{bmatrix} 100 - 72.1083 & -20 & -100 \\ -20 & 300 - 72.1083 & -50 \\ -100 & -50 & 500 - 72.1083 \end{bmatrix} \begin{Bmatrix} -0.960894 \\ -0.137114 \\ -0.240587 \end{Bmatrix} \overset{\surd}{=} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{k}$$

The components of the vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ define the three rows of the orthogonal transformation $[\mathbf{Q}]$ from the xyz system into the $x'y'z'$ system aligned along the three principal directions:

$$[\mathbf{Q}] = \begin{bmatrix} 0.213186 & 0.188199 & -0.958714 \\ 0.176732 & -0.972512 & -0.151609 \\ -0.960894 & -0.137114 & -0.240587 \end{bmatrix} \tag{l}$$

Indeed, if we apply the transformation in Eqn (9.49), $[\mathbf{I}'] = [\mathbf{Q}][\mathbf{I}][\mathbf{Q}]^T$, we find

$$[\mathbf{I}'] = \begin{bmatrix} 0.213186 & 0.188199 & -0.958714 \\ 0.176732 & -0.972512 & -0.151609 \\ -0.960894 & -0.137114 & -0.240587 \end{bmatrix} \begin{bmatrix} 100 & -20 & -100 \\ -20 & 300 & -50 \\ -100 & -50 & 500 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.213186 & 0.176732 & -0.960894 \\ 0.188199 & -0.972512 & -0.137114 \\ -0.958714 & -0.151609 & -0.240587 \end{bmatrix}$$

$$= \begin{bmatrix} 532.052 & 0 & 0 \\ 0 & 295.840 & 0 \\ 0 & 0 & 72.1083 \end{bmatrix} \left(\text{kg}\cdot\text{m}^2\right)$$

An alternative to the above hand calculations in Example 9.9 is to type the following lines in the MATLAB Command Window:

```
I = [ 100  -20  -100
       -20  300   -50
     -100  -50   500];
[eigenVectors, eigenValues] = eig(I)
```

Hitting the Enter (or Return) key yields the following output to the Command Window:

```
eigenVectors =
    0.9609    0.1767   -0.2132
    0.1371   -0.9725   -0.1882
    0.2406   -0.1516    0.9587
eigenValues =
   72.1083         0         0
         0  295.8398         0
         0         0  532.0519
```

Two of the eigenvectors delivered by MATLAB are opposite in direction to those calculated in Example 9.9. This illustrates the fact that we can determine an eigenvector only to within an arbitrary scalar factor. That is, suppose $\mathbf{e}$ is an eigenvector of the tensor $[\mathbf{I}]$ so that $[\mathbf{I}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$. Multiplying this equation through by an arbitrary scalar a yields $[\mathbf{I}]\{\mathbf{e}\}a = \lambda\{\mathbf{e}\}a$, or $[\mathbf{I}]\{a\mathbf{e}\} = \lambda\{a\mathbf{e}\}$, which means that $\{a\mathbf{e}\}$ is an eigenvector corresponding to the same eigenvalue λ.

Parallel axis theorem

Suppose the rigid body in Figure 9.14 is in pure rotation about point P. Then, according to Eqn (9.39),

$$\{\mathbf{H}_{P_{\mathrm{rel}}}\} = [\mathbf{I}_P]\{\boldsymbol{\omega}\} \tag{9.57}$$

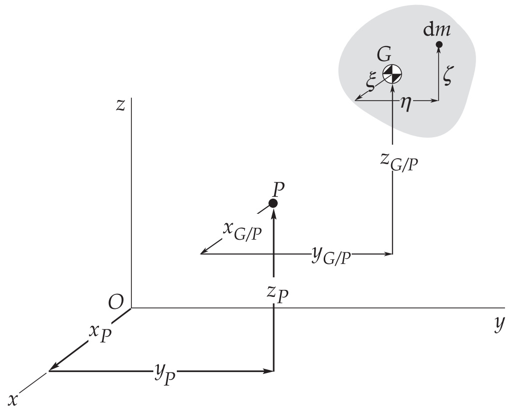


FIGURE 9.14

The moments of inertia are to be computed at P, given their values at G.

where $[\mathbf{I}_P]$ is the moment of inertia tensor about P, given by Eqn (9.40) with

$$x = x_{G/P} + \xi \quad y = y_{G/P} + \eta \quad z = z_{G/P} + \zeta$$

On the other hand, we have from Eqn (9.27) that

$$\mathbf{H}_{P_{\mathrm{rel}}} = \mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_{G/P} \tag{9.58}$$

The vector $\mathbf{r}_{G/P} \times m\mathbf{v}_{G/P}$ is the angular momentum about P of the concentrated mass m located at the center of mass G. Using matrix notation, it is computed as follows:

$$\left\{\mathbf{r}_{G/P} \times m\mathbf{v}_{G/P}\right\} \equiv \left\{\mathbf{H}_{P_{\mathrm{rel}}}^{(m)}\right\} = \left[\mathbf{I}_P^{(m)}\right]\{\boldsymbol{\omega}\} \tag{9.59}$$

where $\left[\mathbf{I}_P^{(m)}\right]$, the moment of inertia of the point mass m about P, is obtained from Eqn (9.44), with $x = x_{G/P}$, $y = y_{G/P}$, and $z = z_{G/P}$. That is,

$$[\mathbf{I}_P^{(m)}] = \begin{bmatrix} m\left(y_{G/P}^2 + z_{G/P}^2\right) & -mx_{G/P}\, y_{G/P} & -mx_{G/P}\, z_{G/P} \\ -mx_{G/P}\, y_{G/P} & m\left(x_{G/P}^2 + z_{G/P}^2\right) & -my_{G/P}\, z_{G/P} \\ -mx_{G/P}\, z_{G/P} & -my_{G/P}\, z_{G/P} & m\left(x_{G/P}^2 + y_{G/P}^2\right) \end{bmatrix} \tag{9.60}$$

Of course, Eqn (9.39) requires

$$\{\mathbf{H}_G\} = [\mathbf{I}_G]\{\boldsymbol{\omega}\}$$

Substituting this together with Eqns (9.57) and (9.59) into Eqn (9.58) yields

$$[\mathbf{I}_P]\{\boldsymbol{\omega}\} = [\mathbf{I}_G]\{\boldsymbol{\omega}\} + \left[\mathbf{I}_P^{(m)}\right]\{\boldsymbol{\omega}\} = \left([\mathbf{I}_G] + \left[\mathbf{I}_P^{(m)}\right]\right)\{\boldsymbol{\omega}\}$$

From this, we may infer the parallel axis theorem,

$$[\mathbf{I}_P] = [\mathbf{I}_G] + \left[\mathbf{I}_P^{(m)}\right] \tag{9.61}$$

The moment of inertia about P is the moment of inertia about the parallel axes through the center of mass plus the moment of inertia of the center of mass about P. That is,

$$\begin{aligned} &I_{P_x} = I_{G_x} + m\left(y_{G/P}^2 + z_{G/P}^2\right) \quad && I_{P_y} = I_{G_y} + m\left(y_{G/P}^2 + x_{G/P}^2\right) \quad && I_{P_z} = I_{G_z} + m\left(x_{G/P}^2 + y_{G/P}^2\right) \\ &I_{P_{xy}} = I_{G_{xy}} - mx_{G/P}\, y_{G/P} && I_{P_{xz}} = I_{G_{xz}} - mx_{G/P}\, z_{G/P} && I_{P_{yz}} = I_{G_{yz}} - my_{G/P}\, z_{G/P} \end{aligned} \tag{9.62}$$

EXAMPLE 9.10

Find the moments of inertia of the rod in Example 9.5 (Figure 9.15) about its center of mass G.

Solution

From Example 9.5,

$$[\mathbf{I}_A] = \begin{bmatrix} \frac{1}{3}m\left(b^2+c^2\right) & -\frac{1}{3}mab & -\frac{1}{3}mac \\ -\frac{1}{3}mab & \frac{1}{3}m\left(a^2+c^2\right) & -\frac{1}{3}mbc \\ -\frac{1}{3}mac & -\frac{1}{3}mbc & \frac{1}{3}m\left(a^2+b^2\right) \end{bmatrix}$$

Using Eqn (9.62)$_1$, and noting the coordinates of the center of mass in Figure 9.15,

$$I_{G_x} = I_{A_x} - m\left[(y_G-0)^2 + (z_G-0)^2\right] = \frac{1}{3}m\left(b^2+c^2\right) - m\left[\left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2\right] = \frac{1}{12}m\left(b^2+c^2\right)$$

Equation (9.62)$_4$ yields

$$I_{G_{xy}} = I_{A_{xy}} + m(x_G-0)(y_G-0) = -\frac{1}{3}mab + m\cdot\frac{a}{2}\cdot\frac{b}{2} = -\frac{1}{12}mab$$

The remaining four moments of inertia are found in a similar fashion, so that

$$\boxed{[\mathbf{I}_G] = \begin{bmatrix} \frac{1}{12}m\left(b^2+c^2\right) & -\frac{1}{12}mab & -\frac{1}{12}mac \\ -\frac{1}{12}mab & \frac{1}{12}m\left(a^2+c^2\right) & -\frac{1}{12}mbc \\ -\frac{1}{12}mac & -\frac{1}{12}mbc & \frac{1}{12}m\left(a^2+b^2\right) \end{bmatrix}} \tag{9.63}$$

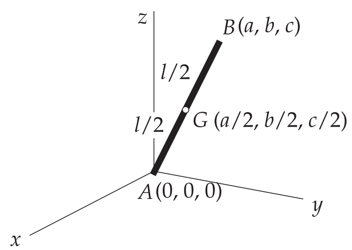


FIGURE 9.15

A uniform slender rod.

EXAMPLE 9.11

Calculate the principal moments of inertia about the center of mass and the corresponding principal directions for the bent rod in Figure 9.16. Its mass is uniformly distributed at 2 kg/m.

Solution

The mass of each of the rod segments is

$$m_1 = 2\cdot 0.4 = 0.8\ \text{kg} \quad m_2 = 2\cdot 0.5 = 1\ \text{kg} \quad m_3 = 2\cdot 0.3 = 0.6\ \text{kg} \quad m_4 = 2\cdot 0.2 = 0.4\ \text{kg} \tag{a}$$

The total mass of the system is

$$m = \sum_{i=1}^{4} m_i = 2.8\ \text{kg} \tag{b}$$

The coordinates of each segment's center of mass are

$$\begin{array}{lll} x_{G_1} = 0 & y_{G_1} = 0 & z_{G_1} = 0.2\ \text{m} \\ x_{G_2} = 0 & y_{G_2} = 0.25\ \text{m} & z_{G_2} = 0.2\ \text{m} \\ x_{G_3} = 0.15\ \text{m} & y_{G_3} = 0.5\ \text{m} & z_{G_3} = 0 \\ x_{G_4} = 0.3\ \text{m} & y_{G_4} = 0.4\ \text{m} & z_{G_4} = 0 \end{array} \tag{c}$$

If the slender rod in Figure 9.15 is aligned with, say, the x-axis, then $a = \ell$ and $b = c = 0$, so that according to Eqn (9.63),

$$[\mathbf{I}_G] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{1}{12}m\ell^2 & 0 \\ 0 & 0 & \dfrac{1}{12}m\ell^2 \end{bmatrix}$$

That is, the moment of inertia of a slender rod about the axes normal to the rod at its center of mass is $\frac{1}{12}m\ell^2$, where m and ℓ are the mass and length of the rod, respectively. Since the mass of a slender bar is assumed to be concentrated along the axis of the bar (its cross-sectional dimensions are infinitesimal), the moment of inertia

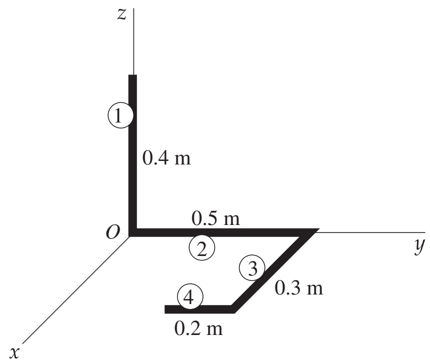


FIGURE 9.16

Bent rod for which the principal moments of inertia are to be determined.

about the centerline is zero. By symmetry, the products of inertia about the axes through the center of mass are all zero. Using this information and the parallel axis theorem, we find the moments and products of inertia of each rod segment about the origin O of the xyz system as follows:

Rod 1:

$$
\begin{aligned}
I_x^{(1)} &= \left(I_{G_1}^{(1)}\right)_x + m_1\left(y_{G_1}^2 + z_{G_1}^2\right) &&= \left(\frac{1}{12}\cdot 0.8\cdot 0.4^2\right) + \left[0.8\left(0+0.2^2\right)\right] = 0.04267\ \text{kg-m}^2\\
I_y^{(1)} &= \left(I_{G_1}^{(1)}\right)_y + m_1\left(x_{G_1}^2 + z_{G_1}^2\right) &&= \left(\frac{1}{12}\cdot 0.8\cdot 0.4^2\right) + \left[0.8\left(0+0.2^2\right)\right] = 0.04267\ \text{kg-m}^2\\
I_z^{(1)} &= \left(I_{G_1}^{(1)}\right)_z + m_1\left(x_{G_1}^2 + y_{G_1}^2\right) &&= 0 + [0.8(0+0)] = 0\\
I_{xy}^{(1)} &= \left(I_{G_1}^{(1)}\right)_{xy} - m_1 x_{G_1} y_{G_1} &&= 0 - [0.8(0)(0)] = 0\\
I_{xz}^{(1)} &= \left(I_{G_1}^{(1)}\right)_{xz} - m_1 x_{G_1} z_{G_1} &&= 0 - [0.8(0)(0.2)] = 0\\
I_{yz}^{(1)} &= \left(I_{G_1}^{(1)}\right)_{yz} - m_1 y_{G_1} z_{G_1} &&= 0 - [0.8(0)(0)] = 0
\end{aligned}
$$

Rod 2:

$$
\begin{aligned}
I_x^{(2)} &= \left(I_{G_2}^{(2)}\right)_x + m_2\left(y_{G_2}^2 + z_{G_2}^2\right) &&= \left(\frac{1}{12}\cdot 1.0\cdot 0.5^2\right) + \left[1.0\left(0+0.25^2\right)\right] = 0.08333\ \text{kg-m}^2\\
I_y^{(2)} &= \left(I_{G_2}^{(2)}\right)_y + m_2\left(x_{G_2}^2 + z_{G_2}^2\right) &&= 0 + [1.0(0+0)] = 0\\
I_z^{(2)} &= \left(I_{G_2}^{(2)}\right)_z + m_2\left(x_{G_2}^2 + y_{G_2}^2\right) &&= \left(\frac{1}{12}\cdot 1.0\cdot 0.5^2\right) + \left[1.0\left(0+0.5^2\right)\right] = 0.08333\ \text{kg-m}^2\\
I_{xy}^{(2)} &= \left(I_{G_2}^{(2)}\right)_{xy} - m_2 x_{G_2} y_{G_2} &&= 0 - [1.0(0)(0.5)] = 0\\
I_{xz}^{(2)} &= \left(I_{G_2}^{(2)}\right)_{xz} - m_2 x_{G_2} z_{G_2} &&= 0 - [1.0(0)(0)] = 0\\
I_{yz}^{(2)} &= \left(I_{G_2}^{(2)}\right)_{yz} - m_2 y_{G_2} z_{G_2} &&= 0 - [1.0(0.5)(0)] = 0
\end{aligned}
$$

Rod 3:

$$
\begin{aligned}
I_x^{(3)} &= \left(I_{G_3}^{(3)}\right)_x + m_3\left(y_{G_3}^2 + z_{G_3}^2\right) &&= 0 + \left[0.6\left(0.5^2+0\right)\right] = 0.15\ \text{kg-m}^2\\
I_y^{(3)} &= \left(I_{G_3}^{(3)}\right)_y + m_3\left(x_{G_3}^2 + z_{G_3}^2\right) &&= \left(\frac{1}{12}\cdot 0.6\cdot 0.3^2\right) + \left[0.6\left(0.15^2+0\right)\right] = 0.018\ \text{kg-m}^2\\
I_z^{(3)} &= \left(I_{G_3}^{(3)}\right)_z + m_3\left(x_{G_3}^2 + y_{G_3}^2\right) &&= \left(\frac{1}{12}\cdot 0.6\cdot 0.3^2\right) + \left[0.6\left(0.15^2+0.5^2\right)\right] = 0.1680\ \text{kg-m}^2\\
I_{xy}^{(3)} &= \left(I_{G_3}^{(3)}\right)_{xy} - m_3 x_{G_3} y_{G_3} &&= 0 - [0.6(0.15)(0.5)] = -0.045\ \text{kg-m}^2\\
I_{xz}^{(3)} &= \left(I_{G_3}^{(3)}\right)_{xz} - m_3 x_{G_3} z_{G_3} &&= 0 - [0.6(0.15)(0)] = 0\\
I_{yz}^{(3)} &= \left(I_{G_3}^{(3)}\right)_{yz} - m_3 y_{G_3} z_{G_3} &&= 0 - [0.6(0.5)(0)] = 0
\end{aligned}
$$

Rod 4:

$$I_x^{(4)} = \left.I_{G_4}^{(4)}\right)_x + m_4\left(y_{G_4}^2 + z_{G_4}^2\right) = \left(\frac{1}{12}\cdot 0.4\cdot 0.2^2\right) + \left[0.4\left(0.4^2 + 0\right)\right] = 0.06533\ \text{kg-m}^2$$

$$I_y^{(4)} = \left.I_{G_4}^{(4)}\right)_y + m_4\left(x_{G_4}^2 + z_{G_4}^2\right) = 0 + \left[0.4(0.3^2 + 0)\right] = 0.0360\ \text{kg-m}^2$$

$$I_z^{(4)} = \left.I_{G_4}^{(4)}\right)_z + m_4\left(x_{G_4}^2 + y_{G_4}^2\right) = \left(\frac{1}{12}\cdot 0.4\cdot 0.2^2\right) + \left[0.4\left(0.3^2 + 0.4^2\right)\right] = 0.1013\ \text{kg-m}^2$$

$$I_{xy}^{(4)} = \left.I_{G_4}^{(4)}\right)_{xy} - m_4 x_{G_4} y_{G_4} = 0 - [0.4(0.3)(0.4)] = -0.0480\ \text{kg-m}^2$$

$$I_{xz}^{(4)} = \left.I_{G_4}^{(4)}\right)_{xz} - m_4 x_{G_4} z_{G_4} = 0 - [0.4(0.3)(0)] = 0$$

$$I_{yz}^{(4)} = \left.I_{G_4}^{(4)}\right)_{yz} - m_4 y_{G_4} z_{G_4} = 0 - [0.4(0.4)(0)] = 0$$

The total moments of inertia for all the four rods about O are

$$\begin{array}{lll} I_x = \sum_{i=1}^{4} I_x^{(i)} = 0.3413\ \text{kg-m}^2 & I_y = \sum_{i=1}^{4} I_y^{(i)} = 0.09667\ \text{kg-m}^2 & I_z = \sum_{i=1}^{4} I_z^{(i)} = 0.3527\ \text{kg-m}^2 \\ I_{xy} = \sum_{i=1}^{4} I_{xy}^{(i)} = -0.0930\ \text{kg-m}^2 & I_{xz} = \sum_{i=1}^{4} I_{xz}^{(i)} = 0 & I_{yz} = \sum_{i=1}^{4} I_{yz}^{(i)} = 0 \end{array} \tag{d}$$

The coordinates of the center of mass of the system of four rods are, from Eqns (a)–(c),

$$\begin{aligned} x_G &= \frac{1}{m}\sum_{i=1}^{4} m_i x_{G_i} = \frac{1}{2.8}\cdot 0.21 = 0.075\ \text{m} \\ y_G &= \frac{1}{m}\sum_{i=1}^{4} m_i y_{G_i} = \frac{1}{2.8}\cdot 0.71 = 0.2536\ \text{m} \\ z_G &= \frac{1}{m}\sum_{i=1}^{4} m_i z_{G_i} = \frac{1}{2.8}\cdot 0.16 = 0.05714\ \text{m} \end{aligned} \tag{e}$$

We use the parallel axis theorems to shift the moments of inertia in Eqn (d) to the center of mass G of the system.

$$I_{G_x} = I_x - m(y_G^2 + z_G^2) = 0.3413 - 0.1892 = 0.1522\ \text{kg-m}^2$$

$$I_{G_y} = I_y - m(x_G^2 + z_G^2) = 0.09667 - 0.02489 = 0.07177\ \text{kg-m}^2$$

$$I_{G_z} = I_z - m(x_G^2 + y_G^2) = 0.3527 - 0.1958 = 0.1569\ \text{kg-m}^2$$

$$I_{G_{xy}} = I_{xy} + m x_G y_G = -0.093 + 0.05325 = -0.03975\ \text{kg-m}^2$$

$$I_{G_{xz}} = I_{xz} + m x_G z_G = 0 + 0.012 = 0.012\ \text{kg-m}^2$$

$$I_{G_{yz}} = I_{yz} + m y_G z_G = 0 + 0.04057 = 0.04057\ \text{kg-m}^2$$

Therefore, the inertia tensor, relative to the center of mass, is

$$[\mathbf{I}] = \begin{bmatrix} I_{G_x} & I_{G_{xy}} & I_{G_{xz}} \\ I_{G_{xy}} & I_{G_y} & I_{G_{yz}} \\ I_{G_{xz}} & I_{G_{yz}} & I_{G_z} \end{bmatrix} = \begin{bmatrix} 0.1522 & -0.03975 & 0.012 \\ -0.03975 & 0.07177 & 0.04057 \\ 0.012 & 0.04057 & 0.1569 \end{bmatrix} \left(\text{kg}\cdot\text{m}^2\right) \tag{f}$$

To find the three principal moments of inertia, we may proceed as in Example 9.9, or simply enter the following lines in the MATLAB Command Window:

```
I = [ 0.1522  -0.03975  0.012
     -0.03975  0.07177  0.04057
      0.012    0.04057  0.1569 ];

[eigenVectors, eigenValues] = eig(IG)
```

to obtain

```
eigenVectors =
    0.3469   -0.8482   -0.4003
    0.8742    0.1378    0.4656
   -0.3397   -0.5115    0.7893
eigenValues =
    0.0402         0         0
         0    0.1658         0
         0         0    0.1747
```

Hence, the three principal moments of inertia and their principal directions are

$\lambda_1 = 0.04023\ \text{kg-m}^2$	$\mathbf{e}^{(1)} = 0.3469\hat{\mathbf{i}} + 0.8742\hat{\mathbf{j}} - 0.3397\hat{\mathbf{k}}$
$\lambda_2 = 0.1658\ \text{kg-m}^2$	$\mathbf{e}^{(2)} = -0.8482\hat{\mathbf{i}} + 0.1378\hat{\mathbf{j}} - 0.5115\hat{\mathbf{k}}$
$\lambda_3 = 0.1747\ \text{kg-m}^2$	$\mathbf{e}^{(3)} = -0.4003\hat{\mathbf{i}} + 0.4656\hat{\mathbf{j}} + 0.7893\hat{\mathbf{k}}$

9.6 Euler's equations

For either the center of mass G or for a fixed point P about which the body is in pure rotation, we know from Eqns (9.29) and (9.30) that

$$\mathbf{M}_{\text{net}} = \dot{\mathbf{H}} \tag{9.64}$$

Using a comoving coordinate system, with angular velocity $\boldsymbol{\Omega}$ and its origin located at the point (G or P), the angular momentum has the analytical expression

$$\mathbf{H} = H_x\hat{\mathbf{i}} + H_y\hat{\mathbf{j}} + H_z\hat{\mathbf{k}} \tag{9.65}$$

We shall henceforth assume, for simplicity, that

(a) The moving xyz axes are the principal axes of inertia and (9.66a)

(b) The moments of inertia relative to xyz are constant in time. (9.66b)

Equations (9.42) and (9.66a) imply that

$$\mathbf{H} = A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}} \tag{9.67}$$

where A, B, and C are the principal moments of inertia.

According to Eqn (1.56), the time derivative of $\mathbf{H}$ is $\dot{\mathbf{H}} = \dot{\mathbf{H}})_{\text{rel}} + \boldsymbol{\Omega} \times \mathbf{H}$, so that Eqn (9.64) can be written as

$$\mathbf{M}_{\text{net}} = \dot{\mathbf{H}}\big)_{\text{rel}} + \boldsymbol{\Omega} \times \mathbf{H} \tag{9.68}$$

Keep in mind that, whereas $\boldsymbol{\Omega}$ (the angular velocity of the moving xyz coordinate system) and $\boldsymbol{\omega}$ (the angular velocity of the rigid body itself) are both absolute kinematic quantities, Eqn (9.68) contains their components as projected onto the axes of the noninertial xyz frame given by

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$$

$$\boldsymbol{\Omega} = \Omega_x\hat{\mathbf{i}} + \Omega_y\hat{\mathbf{j}} + \Omega_z\hat{\mathbf{k}}$$

The absolute angular acceleration $\boldsymbol{\alpha}$ is obtained using Eqn (1.56) as

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \overbrace{\frac{\mathrm{d}\omega_x}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}\omega_y}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}\omega_z}{\mathrm{d}t}\hat{\mathbf{k}}}^{\boldsymbol{\alpha}_{\text{rel}}} + \boldsymbol{\Omega} \times \boldsymbol{\omega}$$

that is,

$$\boldsymbol{\alpha} = \left(\dot{\omega}_x + \Omega_y\omega_z - \Omega_z\omega_y\right)\hat{\mathbf{i}} + \left(\dot{\omega}_y + \Omega_z\omega_x - \Omega_x\omega_z\right)\hat{\mathbf{j}} + \left(\dot{\omega}_z + \Omega_x\omega_y - \Omega_y\omega_x\right)\hat{\mathbf{k}} \tag{9.69}$$

Clearly, it is generally true that

$$\alpha_x \neq \dot{\omega}_x \quad \alpha_y \neq \dot{\omega}_y \quad \alpha_z \neq \dot{\omega}_z$$

From Eqn (1.57) and Eqn (9.67),

$$\dot{\mathbf{H}}\big)_{\text{rel}} = \frac{\mathrm{d}(A\omega_x)}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}\left(B\omega_y\right)}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}(C\omega_z)}{\mathrm{d}t}\hat{\mathbf{k}}$$

Since A, B, and C are constant, this becomes

$$\dot{\mathbf{H}}\big)_{\text{rel}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} \tag{9.70}$$

Substituting Eqns (9.67) and (9.70) into Eqn (9.68) yields

$$\mathbf{M}_{\text{net}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \Omega_x & \Omega_y & \Omega_z \\ A\omega_x & B\omega_y & C\omega_z \end{vmatrix}$$

Expanding the crossproduct and collecting the terms leads to

$$\begin{aligned} M_{x_{\text{net}}} &= A\dot{\omega}_x + C\Omega_y\omega_z - B\Omega_z\omega_y \\ M_{y_{\text{net}}} &= B\dot{\omega}_y + A\Omega_z\omega_x - C\Omega_x\omega_z \\ M_{z_{\text{net}}} &= C\dot{\omega}_z + B\Omega_x\omega_y - A\Omega_y\omega_x \end{aligned} \tag{9.71}$$

If the comoving frame is a rigidly attached body frame, then its angular velocity is the same as that of the body, that is, $\mathbf{\Omega} = \boldsymbol{\omega}$. In that case, Eqn (9.68) reduces to the classical Euler's equation of motion, namely,

$$\boxed{\mathbf{M}_{\text{net}} = \dot{\mathbf{H}}\big)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}} \tag{9.72a}$$

the three components of which are obtained from Eqn (9.71) as

$$\begin{aligned} M_{x_{\text{net}}} &= A\dot{\omega}_x + (C - B)\omega_y\omega_z \\ M_{y_{\text{net}}} &= B\dot{\omega}_y + (A - C)\omega_z\omega_x \\ M_{z_{\text{net}}} &= C\dot{\omega}_z + (B - A)\omega_x\omega_y \end{aligned} \tag{9.72b}$$

Equation (9.68) is sometimes referred to as the modified Euler equation.

When $\mathbf{\Omega} = \boldsymbol{\omega}$, it follows from Eqn (9.69) that

$$\dot{\omega}_x = \alpha_x \quad \dot{\omega}_y = \alpha_y \quad \dot{\omega}_z = \alpha_z \tag{9.73}$$

That is, the relative angular acceleration equals the absolute angular acceleration when $\mathbf{\Omega} = \boldsymbol{\omega}$. Rather than calculating the time derivatives $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$ for use in Eqn (9.72), we may in this case first compute $\boldsymbol{\alpha}$ in the absolute XYZ frame

$$\boldsymbol{\alpha} = \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \frac{\mathrm{d}\omega_X}{\mathrm{d}t}\hat{\mathbf{I}} + \frac{\mathrm{d}\omega_Y}{\mathrm{d}t}\hat{\mathbf{J}} + \frac{\mathrm{d}\omega_Z}{\mathrm{d}t}\hat{\mathbf{K}}$$

and then project these components onto the xyz body frame, so that

$$\begin{Bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{Bmatrix} = [\mathbf{Q}]_{Xx} \begin{Bmatrix} \mathrm{d}\omega_X/\mathrm{d}t \\ \mathrm{d}\omega_Y/\mathrm{d}t \\ \mathrm{d}\omega_Z/\mathrm{d}t \end{Bmatrix} \tag{9.74}$$

where $[\mathbf{Q}]_{Xx}$ is the time-dependent orthogonal transformation from the inertial XYZ frame to the noninertial xyz frame.

EXAMPLE 9.12

Calculate the net moment on the solar panel of Examples 9.2 and 9.8 (Figure 9.17).

Solution

Since the comoving frame is rigidly attached to the panel, Euler's equation (Eqn (9.72a)) applies to this problem, that is

$$\mathbf{M}_{G_{\text{net}}} = \dot{\mathbf{H}}_G\big)_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{H}_G \tag{a}$$

where

$$\mathbf{H}_G = A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}} \tag{b}$$

and

$$\dot{\mathbf{H}}_G\big)_{\text{rel}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} \tag{c}$$

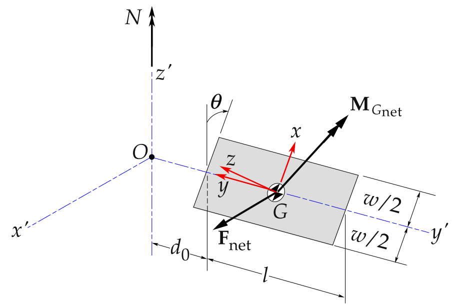


FIGURE 9.17

Free-body diagram of the solar panel in Examples 9.2 and 9.8.

In Example 9.2, the angular velocity of the panel in the satellite's $x'y'z'$ frame was found to be

$$\boldsymbol{\omega} = -\dot{\theta}\hat{\mathbf{j}}' + N\hat{\mathbf{k}}' \tag{d}$$

In Example 9.8, we showed that the transformation from the panel's xyz frame to that of the satellite is represented by the matrix

$$[\mathbf{Q}] = \begin{bmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & -1 & 0 \\ \cos\theta & 0 & \sin\theta \end{bmatrix} \tag{e}$$

We use the transpose of $[\mathbf{Q}]$ to transform the components of $\boldsymbol{\omega}$ into the panel frame of reference,

$$\{\boldsymbol{\omega}\}_{xyz} = [\mathbf{Q}]^T\{\boldsymbol{\omega}\}_{x'y'z'} = \begin{bmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & -1 & 0 \\ \cos\theta & 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} 0 \\ -\dot{\theta} \\ N \end{Bmatrix} = \begin{Bmatrix} N\cos\theta \\ \dot{\theta} \\ N\sin\theta \end{Bmatrix}$$

or

$$\omega_x = N\cos\theta \quad \omega_y = \dot{\theta} \quad \omega_z = N\sin\theta \tag{f}$$

In Example 9.2, N and $\dot{\theta}$ were said to be constant. Therefore, the time derivatives of Eqn (f) are

$$\begin{aligned} \dot{\omega}_x &= \frac{\mathrm{d}(N\cos\theta)}{\mathrm{d}t} = -N\dot{\theta}\sin\theta \\ \dot{\omega}_y &= \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = 0 \\ \dot{\omega}_z &= \frac{\mathrm{d}(N\sin\theta)}{\mathrm{d}t} = N\dot{\theta}\cos\theta \end{aligned} \tag{g}$$

In Example 9.8, the moments of inertia in the panel frame of reference were listed as

$$\begin{gathered} A = \frac{1}{12}m\left(\ell^2 + t^2\right) \quad B = \frac{1}{12}m\left(w^2 + t^2\right) \quad C = \frac{1}{12}m\left(w^2 + \ell^2\right) \\ \left(I_{Gxy} = I_{Gxz} = I_{Gyz} = 0\right) \end{gathered} \tag{h}$$

Substituting Eqns (b), (c), (f), (g) and (h) into Eqn (a) yields

$$\mathbf{M}_{G_{\text{net}}} = \frac{1}{12}m\left(\ell^2+t^2\right)(-N\dot{\theta}\sin\theta)\hat{\mathbf{i}} + \frac{1}{12}m\left(w^2+t^2\right)\cdot 0\cdot\hat{\mathbf{j}} + \frac{1}{12}m\left(w^2+\ell^2\right)(N\dot{\theta}\cos\theta)\hat{\mathbf{k}}$$
$$+\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ N\cos\theta & \dot{\theta} & N\sin\theta \\ \frac{1}{12}m\left(\ell^2+t^2\right)(N\cos\theta) & \frac{1}{12}m\left(w^2+t^2\right)\dot{\theta} & \frac{1}{12}m\left(w^2+\ell^2\right)(N\sin\theta) \end{vmatrix}$$

Upon expanding the cross product and collecting terms, this reduces to

$$\mathbf{M}_{G_{\text{net}}} = -\frac{1}{6}mt^2N\dot{\theta}\sin\theta\hat{\mathbf{i}} + \frac{1}{24}m\left(t^2-w^2\right)N^2\sin 2\theta\hat{\mathbf{j}} + \frac{1}{6}mw^2N\dot{\theta}\cos\theta\hat{\mathbf{k}}$$

Using the numerical data of Example 9.8 ($m = 50$ kg, $N = 0.1$ rad/s, $\theta = 40°$, $\dot{\theta} = 0.01$ rad/s, $\ell = 6$ m, $w = 2$ m, and $t = 0.025$ m), we find

$$\boxed{\mathbf{M}_{G_{\text{net}}} = -3.348\times10^{-6}\hat{\mathbf{i}} - 0.08205\hat{\mathbf{j}} + 0.02554\hat{\mathbf{k}}\ \ (\text{N}\cdot\text{m})}$$

EXAMPLE 9.13

Calculate the net moment on the gyro rotor of Examples 9.3 and 9.6.

Solution

Figure 9.18 is a free-body diagram of the rotor. Since in this case the comoving frame is not rigidly attached to the rotor, we must use Eqn (9.68) to find the net moment about G, that is

$$\mathbf{M}_{G_{\text{net}}} = \dot{\mathbf{H}}_G\Big)_{\text{rel}} + \mathbf{\Omega}\times\mathbf{H}_G \tag{a}$$

where

$$\mathbf{H}_G = A\omega_x\hat{\mathbf{i}} + B\omega_y\hat{\mathbf{j}} + C\omega_z\hat{\mathbf{k}} \tag{b}$$

and

$$\dot{\mathbf{H}}_G\Big)_{\text{rel}} = A\dot{\omega}_x\hat{\mathbf{i}} + B\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} \tag{c}$$

From Eqn (i) of Example 9.3, we know that the components of the angular velocity of the rotor in the moving reference frame are

$$\begin{aligned} \omega_x &= \dot{\theta} \\ \omega_y &= N\sin\theta \\ \omega_z &= \omega_{\text{spin}} + N\cos\theta \end{aligned} \tag{d}$$

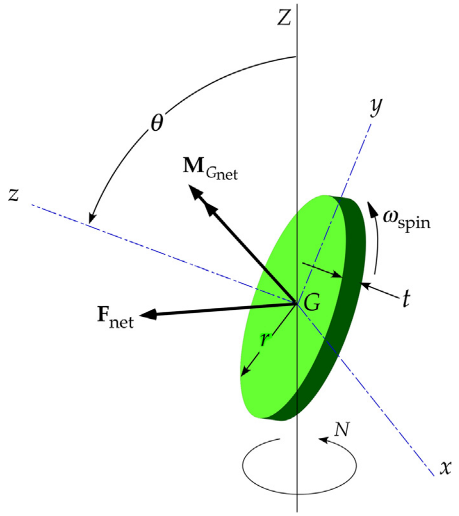


FIGURE 9.18

Free-body diagram of the gyro rotor of Examples 9.6 and 9.3.

Since, as specified in Example 9.3, $\dot{\theta}$, N, and ω_{spin} are all constant, it follows that

$$\begin{aligned}
\dot{\omega}_x &= \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = 0 \\
\dot{\omega}_y &= \frac{\mathrm{d}(N\sin\theta)}{\mathrm{d}t} = N\dot{\theta}\cos\theta \\
\dot{\omega}_z &= \frac{\mathrm{d}\left(\omega_{\text{spin}} + N\cos\theta\right)}{\mathrm{d}t} = -N\dot{\theta}\sin\theta
\end{aligned} \tag{e}$$

The angular velocity $\boldsymbol{\Omega}$ of the comoving xyz frame is that of the gimbal ring, which equals the angular velocity of the rotor minus its spin. Therefore,

$$\begin{aligned}
\Omega_x &= \dot{\theta} \\
\Omega_y &= N\sin\theta \\
\Omega_z &= N\cos\theta
\end{aligned} \tag{f}$$

In Example 9.6, we found that

$$\begin{aligned}
A = B &= \frac{1}{12}mt^2 + \frac{1}{4}mr^2 \\
C &= \frac{1}{2}mr^2
\end{aligned} \tag{g}$$

Substituting Eqns (b) through (g) into Eqn (a), we get

$$\mathbf{M}_{G_{\text{net}}} = \left(\frac{1}{12}mt^2 + \frac{1}{4}mr^2\right)\cdot 0\hat{\mathbf{i}} + \left(\frac{1}{12}mt^2 + \frac{1}{4}mr^2\right)(N\dot{\theta}\cos\theta)\hat{\mathbf{j}} + \frac{1}{2}mr^2(-N\dot{\theta}\sin\theta)\hat{\mathbf{k}}$$
$$+ \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dot{\theta} & N\sin\theta & N\cos\theta \\ \left(\frac{1}{12}mt^2 + \frac{1}{4}mr^2\right)\dot{\theta} & \left(\frac{1}{12}mt^2 + \frac{1}{4}mr^2\right)N\sin\theta & \frac{1}{2}mr^2(\omega_{\text{spin}} + N\cos\theta) \end{vmatrix}$$

Expanding the cross product, collecting terms, and simplifying leads to

$$\mathbf{M}_{G_{\text{net}}} = \left[\frac{1}{2}\omega_{\text{spin}} + \frac{1}{12}\left(3 - \frac{t^2}{r^2}\right)N\cos\theta\right]mr^2 N\sin\theta\hat{\mathbf{i}} + \left(\frac{1}{6}\frac{t^2}{r^2}N\cos\theta - \frac{1}{2}\omega_{\text{spin}}\right)mr^2\dot{\theta}\hat{\mathbf{j}} - \frac{1}{2}N\dot{\theta}\sin\theta mr^2\hat{\mathbf{k}} \quad \text{(h)}$$

In Example 9.3, the following numerical data were provided: $m = 5$ kg, $r = 0.08$ m, $t = 0.025$ m, $N = 2.1$ rad/s, $\theta = 60°$, $\dot{\theta} = 4$ rad/s, and $\omega_{\text{spin}} = 105$ rad/s. For this set of numbers, Eqn (h) becomes

$$\boxed{\mathbf{M}_{G_{\text{net}}} = 0.3203\hat{\mathbf{i}} - 0.6698\hat{\mathbf{j}} - 0.1164\hat{\mathbf{k}} \ \ (\text{N}\cdot\text{m})}$$

9.7 Kinetic energy

The kinetic energy T of a rigid body is the integral of the kinetic energy $\frac{1}{2}v^2\mathrm{d}m$ of its individual mass elements,

$$T = \int_m \frac{1}{2}v^2\mathrm{d}m = \int_m \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\mathrm{d}m \tag{9.75}$$

where $\mathbf{v}$ is the absolute velocity $\dot{\mathbf{R}}$ of the element of mass dm. From Figure 9.8, we infer that $\dot{\mathbf{R}} = \dot{\mathbf{R}}_G + \dot{\boldsymbol{\rho}}$. Furthermore, Eqn (1.52) requires that $\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \boldsymbol{\rho}$. Thus, $\mathbf{v} = \mathbf{v}_G + \boldsymbol{\omega} \times \boldsymbol{\rho}$, which means

$$\mathbf{v}\cdot\mathbf{v} = [\mathbf{v}_G + \boldsymbol{\omega} \times \boldsymbol{\rho}]\cdot[\mathbf{v}_G + \boldsymbol{\omega} \times \boldsymbol{\rho}] = {v_G}^2 + 2\mathbf{v}_G\cdot(\boldsymbol{\omega} \times \boldsymbol{\rho}) + (\boldsymbol{\omega} \times \boldsymbol{\rho})\cdot(\boldsymbol{\omega} \times \boldsymbol{\rho})$$

We can apply the vector identity introduced in Eqn (1.21), namely

$$\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = \mathbf{B}\cdot(\mathbf{C} \times \mathbf{A}) \tag{9.76}$$

to the last term to get

$$\mathbf{v}\cdot\mathbf{v} = {v_G}^2 + 2\mathbf{v}_G\cdot(\boldsymbol{\omega} \times \boldsymbol{\rho}) + \boldsymbol{\omega}\cdot[\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})]$$

Therefore, Eqn (9.75) becomes

$$T = \int_m \frac{1}{2}{v_G}^2\mathrm{d}m + \mathbf{v}_G\cdot\left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}\mathrm{d}m\right) + \frac{1}{2}\boldsymbol{\omega}\cdot\int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})\mathrm{d}m$$

Since $\boldsymbol{\rho}$ is measured from the center of mass, $\int\limits_m \boldsymbol{\rho}\mathrm{d}m = 0$. Recall that, according to Eqn (9.34),

$$\int\limits_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})\mathrm{d}m = \mathbf{H}_G$$

It follows that the kinetic energy may be written as

$$T = \frac{1}{2}m{v_G}^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_G \tag{9.77}$$

The second term is the rotational kinetic energy T_R,

$$T_R = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_G \tag{9.78}$$

If the body is rotating about a point P that is at rest in inertial space, we have from Eqn (9.2) and Figure 9.8 that

$$\mathbf{v}_G = \mathbf{v}_P + \boldsymbol{\omega} \times \mathbf{r}_{G/P} = \mathbf{0} + \boldsymbol{\omega} \times \mathbf{r}_{G/P} = \boldsymbol{\omega} \times \mathbf{r}_{G/P}$$

It follows that

$${v_G}^2 = \mathbf{v}_G\cdot\mathbf{v}_G = \left(\boldsymbol{\omega} \times \mathbf{r}_{G/P}\right)\cdot\left(\boldsymbol{\omega} \times \mathbf{r}_{G/P}\right)$$

Making use once again of the vector identity in Eqn (9.76), we find

$${v_G}^2 = \boldsymbol{\omega}\cdot\left[\mathbf{r}_{G/P} \times \left(\boldsymbol{\omega} \times \mathbf{r}_{G/P}\right)\right] = \boldsymbol{\omega}\cdot\left(\mathbf{r}_{G/P} \times \mathbf{v}_G\right)$$

Substituting this into Eqn (9.77) yields

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\left[\mathbf{H}_G + \mathbf{r}_{G/P} \times m\mathbf{v}_G\right]$$

Equation (9.21) shows that this can be written as

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_P \tag{9.79}$$

In this case, of course, all of the kinetic energy is rotational.

In terms of the components of $\boldsymbol{\omega}$ and $\mathbf{H}$, whether it is $\mathbf{H}_P$ or $\mathbf{H}_G$, the rotational kinetic energy expression becomes, with the aid of Eqn (9.39),

$$T_R = \frac{1}{2}\left(\omega_x H_x + \omega_y H_y + \omega_y H_z\right) = \frac{1}{2}\lfloor\, \omega_x \quad \omega_y \quad \omega_z \,\rfloor \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

Expanding, we obtain

$$T_R = \frac{1}{2}I_x{\omega_x}^2 + \frac{1}{2}I_y{\omega_y}^2 + \frac{1}{2}I_z{\omega_z}^2 + I_{xy}\omega_x\omega_y + I_{xz}\omega_x\omega_z + I_{yz}\omega_y\omega_z \tag{9.80}$$

Obviously, if the *xyz* axes are principal axes of inertia, then Eqn (9.80) simplifies considerably,

$$T_R = \frac{1}{2}A\omega_x{}^2 + \frac{1}{2}B\omega_y{}^2 + \frac{1}{2}C\omega_z{}^2 \tag{9.81}$$

EXAMPLE 9.14

A satellite in a circular geocentric orbit of 300-km altitude has a mass of 1500 kg, and the moments of inertia relative to a body frame with origin at the center of mass G are

$$[\mathbf{I}] = \begin{bmatrix} 2000 & -1000 & 2500 \\ -1500 & 3000 & -1500 \\ 2500 & -1500 & 4000 \end{bmatrix} \left(\mathrm{kg \cdot m^2}\right)$$

If at a given instant the components of angular velocity in this frame of reference are

$$\boldsymbol{\omega} = 1\hat{\mathbf{i}} - 0.9\hat{\mathbf{j}} + 1.5\hat{\mathbf{k}} \ \ (\mathrm{rad/s})$$

calculate the total kinetic energy of the satellite.

Solution

The speed of the satellite in its circular orbit is

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398,600}{6378 + 300}} = 7.7258 \ \ \mathrm{km/s}$$

The angular momentum of the satellite is

$$\{\mathbf{H}_G\} = [\mathbf{I}_G]\{\boldsymbol{\omega}\} = \begin{bmatrix} 2000 & -1000 & 2500 \\ -1500 & 3000 & -1500 \\ 2500 & -1500 & 4000 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.9 \\ 1.5 \end{Bmatrix} = \begin{Bmatrix} 6650 \\ -5950 \\ 9850 \end{Bmatrix} \left(\mathrm{kg \cdot m^2/s}\right)$$

Therefore, the total kinetic energy is

$$T = \frac{1}{2}mv_G{}^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_G = \frac{1}{2}\cdot 1500\cdot 7725.8^2 + \frac{1}{2}\lfloor 1 \quad -0.9 \quad 1.5 \rfloor \begin{Bmatrix} 6650 \\ -5950 \\ 9850 \end{Bmatrix} = 44.766 \times 10^6 + 13,390$$

$$\boxed{T = 44.766 \ \ \mathrm{MJ}}$$

Obviously, the kinetic energy is dominated by that due to the orbital motion.

9.8 The spinning top

Let us analyze the motion of the simple axisymmetric top in Figure 9.19. It is constrained to rotate about point O.

The moving coordinate system is chosen to have its origin at O. The z-axis is aligned with the spin axis of the top (the axis of rotational symmetry). The x-axis is the node line, which passes through O

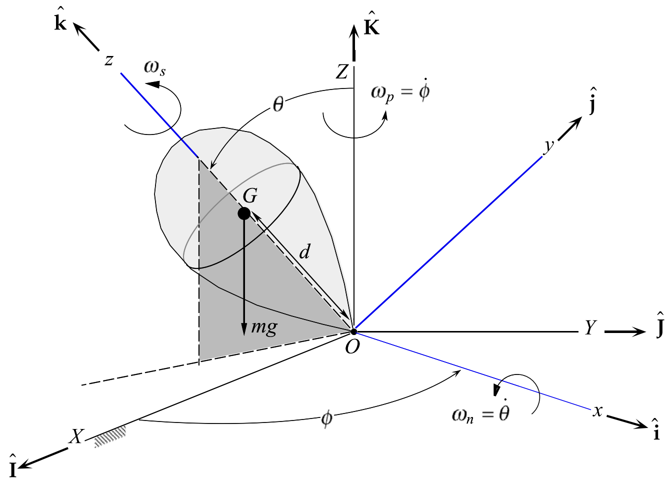


FIGURE 9.19

Simple top rotating about the fixed point O.

and is perpendicular to the plane defined by the inertial Z-axis and the spin axis of the top. The y-axis is then perpendicular to x and z, such that $\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}}$. By symmetry, the moment of inertia matrix of the top relative to the xyz frame is diagonal, with $I_x = I_y = A$ and $I_z = C$. From Eqns (9.68) and (9.70), we have

$$\mathbf{M}_{0_{\text{net}}} = A\dot{\omega}_x\hat{\mathbf{i}} + A\dot{\omega}_y\hat{\mathbf{j}} + C\dot{\omega}_z\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \Omega_x & \Omega_y & \Omega_z \\ A\omega_x & A\omega_y & C\omega_z \end{vmatrix} \tag{9.82}$$

The angular velocity $\boldsymbol{\omega}$ of the top is the vector sum of the spin rate ω_s and the rates of precession ω_p and nutation ω_n, where

$$\omega_p = \dot{\phi} \quad \omega_n = \dot{\theta} \tag{9.83}$$

Thus,

$$\boldsymbol{\omega} = \omega_n\hat{\mathbf{i}} + \omega_p\hat{\mathbf{K}} + \omega_s\hat{\mathbf{k}}$$

From the geometry, we see that

$$\hat{\mathbf{K}} = \sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \tag{9.84}$$

Therefore, relative to the comoving system,

$$\boldsymbol{\omega} = \omega_n\hat{\mathbf{i}} + \omega_p \sin\theta\hat{\mathbf{j}} + \left(\omega_s + \omega_p\cos\theta\right)\hat{\mathbf{k}} \tag{9.85}$$

From Eqn (9.85), we see that

$$\omega_x = \omega_n \quad \omega_y = \omega_p \sin\theta \quad \omega_z = \omega_s + \omega_p \cos\theta \tag{9.86}$$

Computing the time rates of these three expressions yields the components of angular acceleration relative to the xyz frame, given by

$$\dot{\omega}_x = \dot{\omega}_n \quad \dot{\omega}_y = \dot{\omega}_p \sin\theta + \omega_p\omega_n \cos\theta \quad \dot{\omega}_z = \dot{\omega}_s + \dot{\omega}_p \cos\theta - \omega_p\omega_n \sin\theta \tag{9.87}$$

The angular velocity $\mathbf{\Omega}$ of the xyz system is $\mathbf{\Omega} = \omega_p\hat{\mathbf{K}} + \omega_n\hat{\mathbf{i}}$, so that, using Eqn (9.84),

$$\mathbf{\Omega} = \omega_n\hat{\mathbf{i}} + \omega_p \sin\theta\hat{\mathbf{j}} + \omega_p \cos\theta\hat{\mathbf{k}} \tag{9.88}$$

From Eqn (9.88), we obtain

$$\Omega_x = \omega_n \quad \Omega_y = \omega_p \sin\theta \quad \Omega_z = \omega_p \cos\theta \tag{9.89}$$

In Figure 9.19, the moment about O is that of the weight vector acting through the center of mass G:

$$\mathbf{M}_{0_{\text{net}}} = \left(d\hat{\mathbf{k}}\right) \times \left(-mg\hat{\mathbf{K}}\right) = -mgd\hat{\mathbf{k}} \times \left(\sin\theta\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}\right)$$

or

$$\mathbf{M}_{0_{\text{net}}} = mgd \sin\theta\hat{\mathbf{i}} \tag{9.90}$$

Substituting Eqns (9.86), (9.87), (9.89), and (9.90) into Eqn (9.82), we get

$$\begin{aligned} mgd \sin\theta\hat{\mathbf{i}} &= A\dot{\omega}_n\hat{\mathbf{i}} + A\left(\dot{\omega}_p \sin\theta + \omega_p\omega_n \cos\theta\right)\hat{\mathbf{j}} + C\left(\dot{\omega}_s + \dot{\omega}_p \cos\theta - \omega_p\omega_n \sin\theta\right)\hat{\mathbf{k}} \\ &+ \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_n & \omega_p \sin\theta & \omega_p \cos\theta \\ A\omega_n & A\omega_p \sin\theta & C\left(\omega_s + \omega_p \cos\theta\right) \end{vmatrix} \end{aligned} \tag{9.91}$$

Let us consider the special case in which θ is constant, that is, there is no nutation, so that $\omega_n = \dot{\omega}_n = 0$. Then, Eqn (9.91) reduces to

$$mgd \sin\theta\hat{\mathbf{i}} = A\dot{\omega}_p \sin\theta\hat{\mathbf{j}} + C\left(\dot{\omega}_s + \dot{\omega}_p \cos\theta\right)\hat{\mathbf{k}} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & \omega_p \sin\theta & \omega_p \cos\theta \\ 0 & A\omega_p \sin\theta & C\left(\omega_s + \omega_p \cos\theta\right) \end{vmatrix} \tag{9.92}$$

Expanding the determinant yields

$$mgd \sin\theta\hat{\mathbf{i}} = A\dot{\omega}_p \sin\theta\hat{\mathbf{j}} + C\left(\dot{\omega}_s + \dot{\omega}_p \cos\theta\right)\hat{\mathbf{k}} + \left[C\omega_p\omega_s \sin\theta + (C-A){\omega_p}^2 \cos\theta \sin\theta\right]\hat{\mathbf{i}}$$

Equating the coefficients of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ on each side of the equation and assuming that $0 < \theta < 180°$ leads to

$$mgd = C\omega_p\omega_s + (C-A){\omega_p}^2 \cos\theta \tag{9.93a}$$

$$A\dot{\omega}_p = 0 \tag{9.93b}$$

$$C\left(\dot{\omega}_s + \dot{\omega}_p \cos\theta\right) = 0 \tag{9.93c}$$

Equation (9.93b) implies $\dot{\omega}_p = 0$, and from Eqn (9.93c), it follows that $\dot{\omega}_s = 0$. Therefore, the rates of spin and precession are both constant. From Eqn (9.93a), we find

$$(A - C)\cos\theta\omega_p{}^2 - C\omega_s\omega_p + mgd = 0 \tag{9.94}$$

If the spin rate is zero, Eqn (9.94) yields

$$\omega_p\big)_{\omega_s=0} = \pm\sqrt{\frac{mgd}{(C-A)\cos\theta}} \quad \text{if} \quad (C-A)\cos\theta > 0 \tag{9.95}$$

In this case, the top rotates about O at this rate, without spinning. If $A > C$ (prolate), its symmetry axis must make an angle between 90° and 180° to the vertical; otherwise ω_p is imaginary. On the other hand, if $A < C$ (oblate), the angle lies between 0° and 90°. Thus, in steady rotation without spin, the top's axis sweeps out a cone that lies either below the horizontal plane ($A > C$) or above the plane ($A < C$).

In the special case where $(A - C)\cos\theta = 0$, Eqn (9.94) yields a steady precession rate that is inversely proportional to the spin rate,

$$\omega_p = \frac{mgd}{C\omega_s} \quad \text{if} \quad (A-C)\cos\theta = 0 \tag{9.96}$$

If $A = C$, this precession apparently occurs irrespective of tilt angle θ. If $A \neq C$, this rate of precession occurs at $\theta = 90°$, that is, the spin axis is perpendicular to the precession axis.

In general, Eqn (9.94) is a quadratic equation in ω_p, so we can use the quadratic formula to find

$$\omega_p = \frac{C}{2(A-C)\cos\theta}\left(\omega_s \pm \sqrt{\omega_s{}^2 - \frac{4mgd(A-C)\cos\theta}{C^2}}\right) \tag{9.97}$$

Thus, for a given spin rate and tilt angle θ ($\theta \neq 90°$), there are two rates of precession $\dot{\phi}$.

Observe that if $(A - C)\cos\theta > 0$, then ω_p is imaginary when $\omega_s{}^2 < 4mgd(A - C)\cos\theta/C^2$. Therefore, the minimum spin rate required for steady precession at a constant inclination θ is

$$\omega_s\big)_{\min} = \frac{2}{C}\sqrt{mgd(A-C)\cos\theta} \quad \text{if} \quad (A-C)\cos\theta > 0 \tag{9.98}$$

If $(A - C)\cos\theta < 0$, the radical in Eqn (9.97) is real for all ω_s. In this case, as $\omega_s \to 0$, ω_p approaches the value given above in Eqn (9.95).

EXAMPLE 9.15

Calculate the precession rate ω_p for the top of Figure 9.19 if $m = 0.5$ kg, $A\ (= I_x = I_y) = 12 \times 10^{-4}$ kg m^2, $C\ (= I_z) = 4.5 \times 10^{-4}$ kg m^2, and $d = 0.05$ m.

Solution

For an inclination of, say, 60°, $(A - C)\cos\theta > 0$, so that Eqn (9.98) requires $\omega_s)_{\text{min}} = 407.01$ rpm. Let us choose the spin rate to be $\omega_s = 1000$ rpm $= 104.7$ rad/sec. Then, from Eqn (9.97), the precession rate as a function of the inclination θ is given by either one of the following formulas:

$$\boxed{\omega_p = 31.42\frac{1+\sqrt{1-0.3312\cos\theta}}{\cos\theta} \quad \text{and} \quad \omega_p = 31.42\frac{1-\sqrt{1-0.3312\cos\theta}}{\cos\theta}} \tag{a}$$

These are plotted in Figure 9.20.

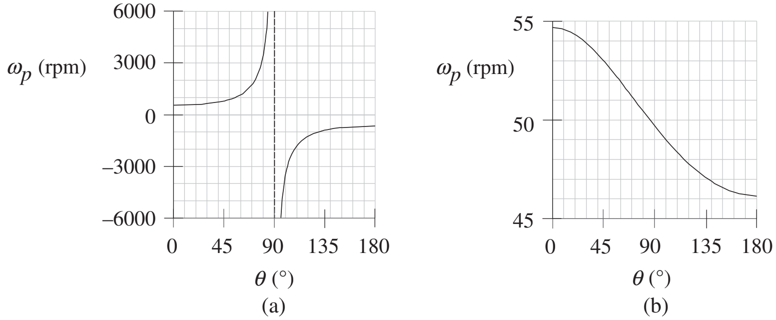


FIGURE 9.20

(a) High-energy precession rate (unlikely to be observed). (b) Low energy precession rate (the one most always seen).

Figure 9.21 shows an axisymmetric rotor mounted so that its spin axis (z) remains perpendicular to the precession axis (y). In that case, Eqn (9.85) with $\theta = 90°$ yields

$$\boldsymbol{\omega} = \omega_p\hat{\mathbf{j}} + \omega_s\hat{\mathbf{k}} \tag{9.99}$$

Likewise, from Eqn (9.88), the angular velocity of the comoving xyz system is $\boldsymbol{\Omega} = \omega_p\hat{\mathbf{j}}$. If we assume that the spin rate and precession rate are constant ($\mathrm{d}\omega_p/\mathrm{d}t = \mathrm{d}\omega_s/\mathrm{d}t = 0$), then Eqn (9.68), written for the center of mass G, becomes

$$\mathbf{M}_{G_{\text{net}}} = \boldsymbol{\Omega} \times \mathbf{H} = \left(\omega_p\hat{\mathbf{j}}\right) \times \left(A\omega_p\hat{\mathbf{j}} + C\omega_s\hat{\mathbf{k}}\right) \tag{9.100}$$

where A and C are the moments of inertia of the rotor about the x- and z-axes, respectively. Setting $C\omega_s\hat{\mathbf{k}} = \mathbf{H}_s$, the spin angular momentum, and $\omega_p\hat{\mathbf{j}} = \boldsymbol{\omega}_p$, we obtain

$$\mathbf{M}_{G_{\text{net}}} = \boldsymbol{\omega}_p \times \mathbf{H}_s \quad \left(\mathbf{H}_s = C\omega_s\hat{\mathbf{k}}\right) \tag{9.101}$$

Since the center of mass is the reference point, there is no restriction on the motion G for which Eqn (9.101) is valid. Observe that the net gyroscopic moment $\mathbf{M}_{G_{\text{net}}}$ exerted on the rotor by its supports is

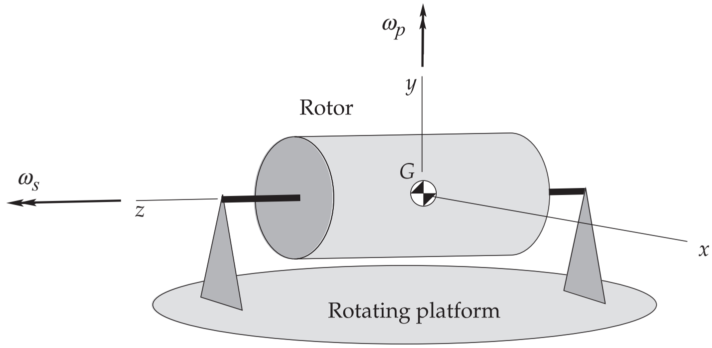


FIGURE 9.21

A spinning rotor on a rotating platform.

perpendicular to the plane of the spin and the precession vectors. If a spinning rotor is forced to precess, the gyroscopic moment $\mathbf{M}_{G_{\text{net}}}$ develops. Or, if a moment is applied normal to the spin axis of a rotor, it will precess so as to cause the spin axis to turn toward the moment axis.

EXAMPLE 9.16

A uniform cylinder of radius r, length L, and mass m spins at a constant angular velocity ω_s. It rests on simple supports (which cannot exert couples), mounted on a platform that rotates at an angular velocity of ω_p. Find the reactions at A and B. Neglect the weight (i.e., calculate the reactions due just to the gyroscopic effects).

Solution

The net vertical force on the cylinder is zero, so the reactions at each end must be equal and opposite in direction, as shown on the free-body diagram insert in Figure 9.22. Noting that the moment of inertia of a uniform cylinder about it axis of rotational symmetry is $\frac{1}{2}mr^2$, Eqn (9.101) yields

$$RL\hat{\mathbf{i}} = \left(\omega_p\hat{\mathbf{j}}\right) \times \left(\frac{1}{2}mr^2\omega_s\hat{\mathbf{k}}\right) = \frac{1}{2}mr^2\omega_p\omega_s\hat{\mathbf{i}}$$

so that

$$\boxed{R = \frac{mr^2\omega_p\omega_s}{2L}}$$

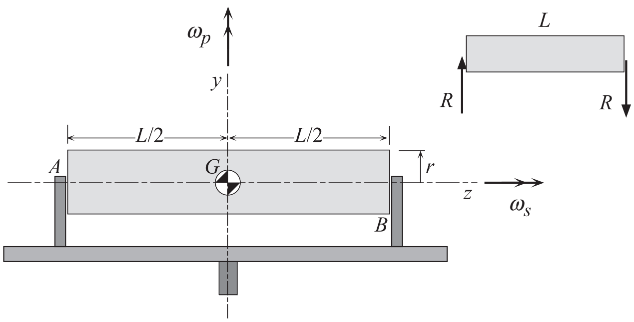


FIGURE 9.22

Illustration of the gyroscopic effect.

9.9 Euler angles

Three angles are required to specify the orientation of a rigid body relative to an inertial frame. The choice is not unique, but there are two sets in common use: the Euler angles and the yaw, pitch, and roll angles. We will discuss each of them in turn. The reader is urged to review Section 4.5 on orthogonal coordinate transformations and, in particular, the discussion of Euler angle sequences.

The three Euler angles ϕ, θ, and ψ shown in Figure 9.23 give the orientation of a body-fixed xyz frame of reference relative to the XYZ inertial frame of reference. The xyz frame is obtained from the XYZ frame by a sequence of rotations through each of the Euler angles in turn. The first rotation is around the Z $(=z_1)$ axis through the precession angle ϕ. This takes X into x_1 and Y into y_1. The second rotation is around the x_2 $(=x_1)$ axis through the nutation angle θ. This carries y_1 and z_1 into y_2 and z_2, respectively. The third and final rotation is around the z $(=z_2)$ axis through the spin angle ψ, which takes x_2 into x and y_2 into y.

The matrix $[\mathbf{Q}]_{Xx}$ of the transformation from the inertial frame to the body-fixed frame is given by the classical Euler angle sequence (Eqn (4.37)):

$$[\mathbf{Q}]_{Xx} = [\mathbf{R}_3(\psi)][\mathbf{R}_1(\theta)][\mathbf{R}_3(\phi)] \tag{9.102}$$

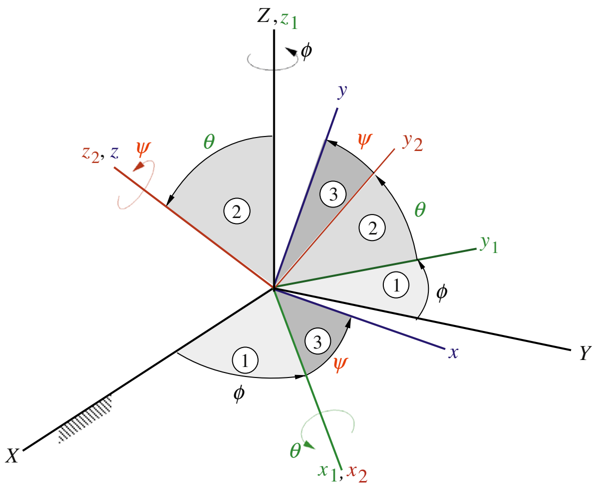


FIGURE 9.23

Classical Euler angle sequence. See also Figure 4.14.

From Eqns (4.32) and (4.34), we have

$$[\mathbf{R}_3(\psi)] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [\mathbf{R}_1(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$[\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.103}$$

According to Eqn (4.38), the direction cosine matrix is

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \sin\theta\sin\psi \\ -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{bmatrix} \tag{9.104}$$

Since this is an orthogonal matrix, the inverse transformation from xyz to XYZ is $[\mathbf{Q}]_{xX} = ([\mathbf{Q}]_{Xx})^T$,

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \sin\phi\sin\theta \\ \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix} \tag{9.105}$$

Algorithm 4.3 is used to find the three Euler angles θ, ϕ, and ψ from a given direction cosine matrix $[\mathbf{Q}]_{Xx}$.

EXAMPLE 9.17

The direction cosine matrix of an orthogonal transformation from XYZ to xyz is

$$[\mathbf{Q}] = \begin{bmatrix} -0.32175 & 0.89930 & -0.29620 \\ 0.57791 & -0.061275 & -0.81380 \\ -0.75000 & -0.43301 & -0.5000 \end{bmatrix}$$

Use Algorithm 4.3 to find the Euler angles ϕ, θ, and ψ for this transformation.

Solution

Step1 (precession angle):

$$\phi = \tan^{-1}\left(\frac{Q_{31}}{-Q_{32}}\right) = \tan^{-1}\left(\frac{-0.75000}{0.43301}\right) \quad (0 \le \phi < 360^\circ)$$

Since the numerator is negative and the denominator is positive, the angle ϕ lies in the fourth quadrant.

$$\boxed{\phi = \tan^{-1}(-1.7320) = 300°}$$

Step 2 (nutation angle):

$$\theta = \cos^{-1} Q_{33} = \cos^{-1}(-0.5000) \quad (0 \le \theta \le 180°)$$

$$\boxed{\theta = 120°}$$

Step 3 (spin angle):

$$\psi = \tan^{-1}\frac{Q_{13}}{Q_{23}} = \tan^{-1}\left(\frac{-0.29620}{-0.81380}\right) \quad (0 \le \psi < 360°)$$

Since both the numerator and denominator are negative, the angle ψ lies in the third quadrant.

$$\boxed{\psi = \tan^{-1}(0.36397) = 200°}$$

The time rates of change of the Euler angles ϕ, θ, and ψ are, respectively, the precession rate ω_p, the nutation rate ω_n, and the spin ω_s. That is,

$$\omega_p = \dot{\phi} \quad \omega_n = \dot{\theta} \quad \omega_s = \dot{\psi} \tag{9.106}$$

The absolute angular velocity $\boldsymbol{\omega}$ of a rigid body can be resolved into components ω_x, ω_y, and ω_z along the body-fixed xyz axes, so that

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}} \tag{9.107}$$

Figure 9.23 shows that precession is measured around the inertial Z-axis (unit vector $\hat{\mathbf{K}}$); nutation is measured around the intermediate x_1-axis (node line) with unit vector $\hat{\mathbf{i}}_1$; and spin is measured around the body-fixed z-axis (unit vector $\hat{\mathbf{k}}$). Therefore, the absolute angular velocity can alternatively be written in terms of the nonorthogonal Euler angle rates as

$$\boldsymbol{\omega} = \omega_p\hat{\mathbf{K}} + \omega_n\hat{\mathbf{i}}_1 + \omega_s\hat{\mathbf{k}} \tag{9.108}$$

In order to find the relationship between the body rates ω_x, ω_y, and ω_z and the Euler angle rates ω_p, ω_n, and ω_s, we must express $\hat{\mathbf{K}}$ and $\hat{\mathbf{i}}_1$ in terms of the unit vectors $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ of the body-fixed frame. To accomplish that, we proceed as follows.

The first rotation $[\mathbf{R}_3(\phi)]$ in Eqn (9.102) rotates the unit vectors $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ of the inertial frame into the unit vectors $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ of the intermediate $x_1y_1z_1$ axes in Figure 9.23. Hence $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ are rotated into $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ by the inverse transformation given by

$$\begin{Bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{Bmatrix} = [\mathbf{R}_3(\phi)]^T \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} \tag{9.109}$$

The second rotation $[\mathbf{R}_1(\theta)]$ rotates $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ into the unit vectors $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ of the second intermediate frame $x_2y_2z_2$ in Figure 9.23. The inverse transformation rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ back into $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$:

$$\begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = [\mathbf{R}_1(\theta)]^T \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} \tag{9.110}$$

Finally, the third rotation $[\mathbf{R}_3(\psi)]$ rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ into $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$, the target unit vectors of the body-fixed xyz frame. $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ are obtained from $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ by the reverse rotation,

$$\begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = [\mathbf{R}_3(\psi)]^T \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix} \tag{9.111}$$

From Eqns (9.109), (9.110) and (9.111), we observe that

$$\hat{\mathbf{K}} \overset{9.109}{\widehat{=}} \hat{\mathbf{k}}_1 \overset{9.110}{\widehat{=}} \sin\theta\hat{\mathbf{j}}_2 + \cos\theta\hat{\mathbf{k}}_2 \overset{9.111}{\widehat{=}} \sin\theta\left(\sin\psi\hat{\mathbf{i}} + \cos\psi\hat{\mathbf{j}}\right) + \cos\theta\hat{\mathbf{k}}$$

or

$$\hat{\mathbf{K}} = \sin\theta\sin\psi\hat{\mathbf{i}} + \sin\theta\cos\psi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \tag{9.112}$$

Similarly, Eqns (9.110) and (9.111) imply that

$$\hat{\mathbf{i}}_1 = \hat{\mathbf{i}}_2 = \cos\psi\hat{\mathbf{i}} - \sin\psi\hat{\mathbf{j}} \tag{9.113}$$

Substituting Eqns (9.112) and (9.113) into Eqn (9.108) yields

$$\boldsymbol{\omega} = \omega_p\left(\sin\theta\sin\psi\hat{\mathbf{i}} + \sin\theta\cos\psi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}\right) + \omega_n\left(\cos\psi\hat{\mathbf{i}} - \sin\psi\hat{\mathbf{j}}\right) + \omega_s\hat{\mathbf{k}}$$

or

$$\boldsymbol{\omega} = \left(\omega_p\sin\theta\sin\psi + \omega_n\cos\psi\right)\hat{\mathbf{i}} + \left(\omega_p\sin\theta\cos\psi - \omega_n\sin\psi\right)\hat{\mathbf{j}} + \left(\omega_s + \omega_p\cos\theta\right)\hat{\mathbf{k}} \tag{9.114}$$

Comparing Eqns (9.107) and (9.114), we see that

$$\begin{aligned} \omega_x &= \omega_p\sin\theta\sin\psi + \omega_n\cos\psi \\ \omega_y &= \omega_p\sin\theta\cos\psi - \omega_n\sin\psi \\ \omega_z &= \omega_s + \omega_p\cos\theta \end{aligned} \tag{9.115}$$

(Notice that the precession angle ϕ does not appear.) We can solve these three equations to obtain the Euler rates in terms of ω_x, ω_y, and ω_z:

$$\begin{aligned}
\omega_p &= \dot{\phi} = \frac{1}{\sin\theta}\left(\omega_x \sin\psi + \omega_y \cos\psi\right) \\
\omega_n &= \dot{\theta} = \omega_x \cos\psi - \omega_y \sin\psi \\
\omega_s &= \dot{\psi} = -\frac{1}{\tan\theta}\left(\omega_x \sin\psi + \omega_y \cos\psi\right) + \omega_z
\end{aligned} \tag{9.116}$$

Observe that if ω_x, ω_y, and ω_z are given functions of time, found by solving Euler's equations of motion (Eqn (9.72)), then Eqns (9.116) are three coupled differential equations that may be solved to obtain the three time-dependent Euler angles, namely

$$\phi = \phi(t) \quad \theta = \theta(t) \quad \psi = \psi(t)$$

With this solution, the orientation of the xyz frame, and hence the body to which it is attached, is known for any given time t. Note, however, that Eqns (9.116) "blow up" when $\theta = 0$, that is, when the xy plane is parallel to the XY plane.

EXAMPLE 9.18

At a given instant, the unit vectors of a body frame are

$$\begin{aligned}
\hat{\mathbf{i}} &= 0.40825\hat{\mathbf{I}} - 0.40825\hat{\mathbf{J}} + 0.81649\hat{\mathbf{K}} \\
\hat{\mathbf{j}} &= -0.10102\hat{\mathbf{I}} - 0.90914\hat{\mathbf{J}} - 0.40405\hat{\mathbf{K}} \\
\hat{\mathbf{k}} &= 0.90726\hat{\mathbf{I}} + 0.082479\hat{\mathbf{J}} - 0.41240\hat{\mathbf{K}}
\end{aligned} \tag{a}$$

and the angular velocity is

$$\boldsymbol{\omega} = -3.1\hat{\mathbf{I}} + 2.5\hat{\mathbf{J}} + 1.7\hat{\mathbf{K}} \ \text{(rad/s)} \tag{b}$$

Calculate ω_p, ω_n, and ω_s (the precession, nutation, and spin rates) at this instant.

Solution

We will ultimately use Eqn (9.116) to find ω_p, ω_n, and ω_s. To do so we must first obtain the Euler angles ϕ, θ, and ψ as well as the components of the angular velocity in the body frame.

The three rows of the direction cosine matrix $[\mathbf{Q}]_{Xx}$ comprise the components of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, respectively,

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0.40825 & -0.40825 & 0.81649 \\ -0.10102 & -0.90914 & -0.40405 \\ 0.90726 & 0.082479 & -0.41240 \end{bmatrix} \tag{c}$$

Therefore, the components of the angular velocity in the body frame are

$$\{\boldsymbol{\omega}\}_x = [\mathbf{Q}]_{Xx}\{\boldsymbol{\omega}\}_X = \begin{bmatrix} 0.40825 & -0.40825 & 0.81649 \\ -0.10102 & -0.90914 & -0.40405 \\ 0.90726 & 0.082479 & -0.41240 \end{bmatrix}\begin{Bmatrix} -3.1 \\ 2.5 \\ 1.7 \end{Bmatrix} = \begin{Bmatrix} -0.89817 \\ -2.6466 \\ -3.3074 \end{Bmatrix}$$

or

$$\omega_x = -0.89817 \text{ rad/s} \quad \omega_y = -2.6466 \text{ rad/s} \quad \omega_z = -3.3074 \text{ rad/s} \tag{d}$$

To obtain the Euler angles ϕ, θ, and ψ from the direction cosine matrix in Eqn (c), we use Algorithm 4.3, as illustrated in Example 9.17. That algorithm is implemented as the MATLAB function *dcm_to_Euler.m* in Appendix D.20. Typing the following lines in the MATLAB Command Window:

```
>>  Q = [ .40825 -.40825   .81649
        -.10102 -.90914  -.40405
         .90726  .082479 -.41240];
>> [phi theta psi] = dcm_to_euler(Q)
```

produces the following output:

```
phi =
   95.1945
theta =
  114.3557
psi =
  116.3291
```

Substituting $\theta = 114.36°$ and $\psi = 116.33°$ together with the angular velocities of Eqn (d) into Eqn (9.116) yields

$$\omega_p = \frac{1}{\sin 114.36°}[-0.89817\cdot\sin 116.33° + (-2.6466)\cdot\cos 116.33°] = \boxed{0.40492 \text{ rad/s}}$$

$$\omega_n = -0.89817\cdot\cos 116.33° - (-2.6466)\cdot\sin 116.33° = \boxed{2.7704 \text{ rad/s}}$$

$$\omega_s = -\frac{1}{\tan 114.36°}[-0.89817\cdot\sin 116.33° + (-2.6466)\cdot\cos 116.33°] + (-3.3074) = \boxed{-3.1404 \text{ rad/s}}$$

EXAMPLE 9.19

The mass moments of inertia of a body about the principal body frame axes with origin at the center of mass G are

$$A = 1000 \text{ kg}\cdot\text{m}^2 \quad B = 2000 \text{ kg}\cdot\text{m}^2 \quad C = 3000 \text{ kg}\cdot\text{m}^2 \tag{a}$$

The Euler angles in radians are given as functions of time in seconds as follows:

$$\begin{aligned} \phi &= 2te^{-0.05t} \\ \theta &= 0.02 + 0.3\sin 0.25t \\ \psi &= 0.6t \end{aligned} \tag{b}$$

At $t = 10$ s, find

(a) The net moment about G and
(b) The components α_X, α_Y, and α_Z of the absolute angular acceleration in the inertial frame.

Solution

(a) We must use Euler's equations (Eqns (9.72)) to calculate the net moment, which means we must first obtain ω_x, ω_y, ω_z, $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$. Since we are given the Euler angles as a function of time, we can compute their time derivatives and then use Eqn (9.115) to find the body frame angular velocity components and their derivatives. Starting with Eqn (b), we get

$$\omega_p = \frac{d\phi}{dt} = \frac{d}{dt}\left(2te^{-0.05t}\right) = 2e^{-0.05t} - 0.1te^{-0.05t}$$

$$\dot{\omega}_p = \frac{d\omega_p}{dt} = \frac{d}{dt}\left(2e^{-0.05t} - 0.1e^{-0.05t}\right) = -0.2e^{-0.05t} + 0.005te^{-0.05t}$$

Proceeding to the remaining two Euler angles leads to

$$\omega_n = \frac{d\theta}{dt} = \frac{d}{dt}(0.02 + 0.3\sin 0.25t) = 0.075\cos 0.25t$$

$$\dot{\omega}_n = \frac{d\omega_n}{dt} = \frac{d}{dt}(0.075\cos 0.25t) = -0.01875\sin 0.25t$$

$$\omega_s = \frac{d\psi}{dt} = \frac{d}{dt}(0.6t) = 0.6$$

$$\dot{\omega}_s = \frac{d\omega_s}{dt} = 0$$

Evaluating all these quantities, including those in Eqn (b), at $t = 10$ s yields

$$\begin{aligned}
&\phi = 335.03^\circ \quad && \omega_p = 0.60653 \text{ rad/s} \quad && \dot{\omega}_p = -0.09098 \text{ rad/s}^2 \\
&\theta = 11.433^\circ && \omega_n = -0.060086 \text{ rad/s} && \dot{\omega}_n = -0.011221 \text{ rad/s}^2 \\
&\psi = 343.77 && \omega_s = 0.6 \text{ rad/s} && \dot{\omega}_s = 0
\end{aligned} \tag{c}$$

Equation (9.115) relates the Euler angle rates to the angular velocity components,

$$\begin{aligned}
\omega_x &= \omega_p \sin\theta \sin\psi + \omega_n \cos\psi \\
\omega_y &= \omega_p \sin\theta \cos\psi - \omega_n \sin\psi \\
\omega_z &= \omega_s + \omega_p \cos\theta
\end{aligned} \tag{d}$$

Taking the time derivative of each of these equations in turn leads to the following three equations:

$$\begin{aligned}
\dot{\omega}_x &= \omega_p\omega_n \cos\theta \sin\psi + \omega_p\omega_s \sin\theta \cos\psi - \omega_n\omega_s \sin\psi + \dot{\omega}_p \sin\theta \sin\psi + \dot{\omega}_n \cos\psi \\
\dot{\omega}_y &= \omega_p\omega_n \cos\theta \cos\psi - \omega_p\omega_s \sin\theta \sin\psi - \omega_n\omega_s \cos\psi + \dot{\omega}_p \sin\theta \cos\psi - \dot{\omega}_n \sin\psi \\
\dot{\omega}_z &= -\omega_p\omega_n \sin\theta + \dot{\omega}_p \cos\theta + \dot{\omega}_s
\end{aligned} \tag{e}$$

Substituting the data in Eqn (c) into Eqns (d) and (e) yields

$$\begin{aligned}
&\omega_x = -0.091286 \text{ rad/s} \quad && \omega_y = 0.098649 \text{ rad/s} \quad && \omega_z = 1.1945 \text{ rad/s} \\
&\dot{\omega}_x = 0.063435 \text{ rad/s}^2 && \dot{\omega}_y = 2.2346 \times 10^{-5} \text{ rad/s}^2 && \dot{\omega}_z = -0.08195 \text{ rad/s}^2
\end{aligned} \tag{f}$$

With Eqns (a) and (f) we have everything we need for Euler's equations, namely,

$$\begin{aligned}
M_{x_{\text{net}}} &= A\dot{\omega}_x + (C - B)\omega_y\omega_z \\
M_{y_{\text{net}}} &= B\dot{\omega}_y + (A - C)\omega_z\omega_x \\
M_{z_{\text{net}}} &= C\dot{\omega}_z + (B - A)\omega_x\omega_y
\end{aligned}$$

from which we find

$$\boxed{\begin{aligned} M_{x_{\text{net}}} &= 181.27\ \text{N}\cdot\text{m} \\ M_{y_{\text{net}}} &= 218.12\ \text{N}\cdot\text{m} \\ M_{z_{\text{net}}} &= -254.86\ \text{N}\cdot\text{m} \end{aligned}}$$

(b) Since the comoving *xyz* frame is a body frame, rigidly attached to the solid, we know from Eqn (9.74) that

$$\begin{Bmatrix} \alpha_X \\ \alpha_Y \\ \alpha_Z \end{Bmatrix} = [\mathbf{Q}]_{xX} \begin{Bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{Bmatrix} \tag{g}$$

In other words, the absolute angular acceleration and the relative angular acceleration of the body are the same. All we have to do is project the components of relative acceleration in Eqn (f) onto the axes of the inertial frame. The required orthogonal transformation matrix is given in Eqn (9.105),

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \sin\phi\sin\theta \\ \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}$$

Upon substituting the numerical values of the Euler angles from Eqn (c), this becomes

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} 0.75484 & 0.65055 & -0.083668 \\ -0.65356 & 0.73523 & -0.17970 \\ -0.055386 & 0.19033 & 0.98016 \end{bmatrix}$$

Substituting this and the relative angular velocity rates from Eqn (f) into Eqn (g) yields

$$\begin{Bmatrix} \alpha_X \\ \alpha_Y \\ \alpha_Z \end{Bmatrix} = \begin{bmatrix} 0.75484 & 0.65055 & -0.083668 \\ -0.65356 & 0.73523 & -0.17970 \\ -0.055386 & 0.19033 & 0.98016 \end{bmatrix} \begin{Bmatrix} 0.063435 \\ 2.2345\times 10^{-5} \\ -0.08195 \end{Bmatrix} = \boxed{\begin{Bmatrix} 0.054755 \\ -0.026716 \\ -0.083833 \end{Bmatrix} (\text{rad/s}^2)}$$

EXAMPLE 9.20

Figure 9.24 shows a rotating platform on which is mounted a rectangular parallelepiped shaft (with dimensions b, h, and ℓ) spinning about the inclined axis *DE*. If the mass of the shaft is m, and the angular velocities ω_p and ω_s are constant, calculate the bearing forces at D and E as a function of ϕ and ψ. Neglect gravity, since we are interested only in the gyroscopic forces. (The small extensions shown at each end of the parallelepiped are just for clarity; the distance between the bearings at D and E is ℓ.)

Solution

The inertial *XYZ* frame is centered at O on the platform, and it is right handed ($\hat{\mathbf{I}} \times \hat{\mathbf{J}} = \hat{\mathbf{K}}$). The origin of the right-handed comoving body frame *xyz* is at the shaft's center of mass G, and it is aligned with the symmetry axes of the

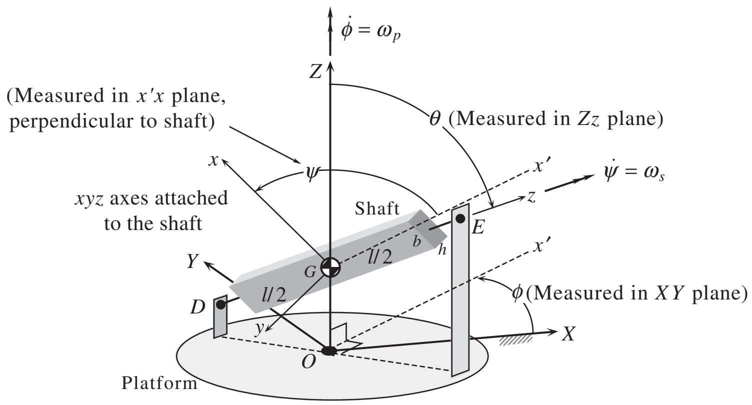


FIGURE 9.24

Spinning block mounted on a rotating platform.

parallelepiped. The three Euler angles ϕ, θ, and ψ are shown in Figure 9.24. Since θ is constant, the nutation rate is zero ($\omega_n = 0$). Thus, Eqn (9.115) reduces to

$$\omega_x = \omega_p \sin\theta \sin\psi \quad \omega_y = \omega_p \sin\theta \cos\psi \quad \omega_z = \omega_p \cos\theta + \omega_s \tag{a}$$

Since ω_p, ω_s, and θ are constant, it follows (recalling Eqn (9.106)) that

$$\dot{\omega}_x = \omega_p\omega_s \sin\theta \cos\psi \quad \dot{\omega}_y = -\omega_p\omega_s \sin\theta \sin\psi \quad \dot{\omega}_z = 0 \tag{b}$$

The principal moments of inertia of the parallelepiped are (Figure 9.10(c))

$$\begin{aligned} A = \ell_x &= \frac{1}{12}m\left(h^2 + \ell^2\right) \\ B = \ell_y &= \frac{1}{12}m\left(b^2 + \ell^2\right) \\ C = \ell_z &= \frac{1}{12}m\left(b^2 + h^2\right) \end{aligned} \tag{c}$$

Figure 9.25 is a free-body diagram of the shaft. Let us assume that the bearings at D and E are such as to exert just the six body frame components of force shown. Thus, D is a thrust bearing to which the axial torque T_D is applied from, say, a motor of some kind. At E, there is a simple journal bearing.

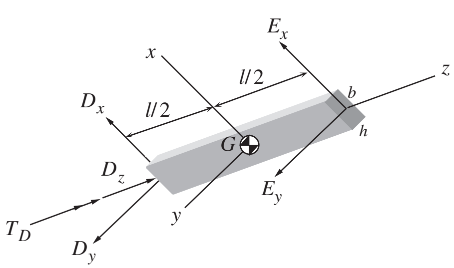


FIGURE 9.25

Free-body diagram of the block in Figure 9.24.

From Newton's laws of motion, we have $\mathbf{F}_{\text{net}} = m\mathbf{a}_G$. But G is fixed in inertial space, so $\mathbf{a}_G = \mathbf{0}$. Thus,

$$\left(D_x\hat{\mathbf{i}} + D_y\hat{\mathbf{j}} + D_z\hat{\mathbf{k}}\right) + \left(E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}}\right) = 0$$

It follows that

$$E_x = -D_x \quad E_y = -D_y \quad D_z = 0 \tag{d}$$

Summing moments about G we get

$$\begin{aligned}
\mathbf{M}_{G_{\text{net}}} &= \frac{\ell}{2}\hat{\mathbf{k}} \times \left(E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}}\right) + \left(-\frac{\ell}{2}\hat{\mathbf{k}}\right) \times \left(D_x\hat{\mathbf{i}} + D_y\hat{\mathbf{j}}\right) + T_D\hat{\mathbf{k}} \\
&= \left(D_y\frac{\ell}{2} - E_y\frac{\ell}{2}\right)\hat{\mathbf{i}} + \left(-D_x\frac{\ell}{2} + E_x\frac{\ell}{2}\right)\hat{\mathbf{j}} + T_D\hat{\mathbf{k}} \\
&= D_y\ell\hat{\mathbf{i}} - D_x\ell\hat{\mathbf{j}} + T_D\hat{\mathbf{k}}
\end{aligned}$$

where we made use of Eqn (d)$_2$. Thus,

$$M_{x_{\text{net}}} = D_y\ell \quad M_{y_{\text{net}}} = -D_x\ell \quad M_{z_{\text{net}}} = T_D \tag{e}$$

We substitute Eqns (a)–(c) and (e) into Euler's equations (Eqn (9.72)):

$$\begin{aligned}
M_{x_{\text{net}}} &= A\dot{\omega}_x + (C - B)\omega_y\omega_z \\
M_{y_{\text{net}}} &= B\dot{\omega}_y + (A - C)\omega_x\omega_z \\
M_{z_{\text{net}}} &= C\dot{\omega}_z + (B - A)\omega_x\omega_y
\end{aligned} \tag{f}$$

After making the substitutions and simplifying, the first Euler equation, Eqn (f)$_1$, becomes

$$D_x = \left\{\frac{1}{12}\frac{m}{\ell}\left[\left(\ell^2 - h^2\right)\omega_p\cos\theta - 2h^2\omega_s\right]\omega_p\sin\theta\right\}\cos\psi \tag{g}$$

Likewise, from Eqn (f)$_2$ we obtain

$$D_y = \left\{\frac{1}{12}\frac{m}{\ell}\left[\left(\ell^2 - b^2\right)\omega_p\cos\theta - 2b^2\omega_s\right]\omega_p\sin\theta\right\}\sin\psi \tag{h}$$

Finally, Eqn (f)$_3$ yields

$$T_D = \left[\frac{1}{24}m\left(b^2 - h^2\right)\omega_p^2\sin^2\theta\right]\sin 2\psi \tag{i}$$

This completes the solution, since $E_y = -D_y$ and $E_z = -D_z$. Note that the resultant transverse bearing load V at D (and E) is

$$V = \sqrt{D_x^2 + D_y^2} \tag{j}$$

As a numerical example, let

$$\ell = 1 \text{ m} \quad h = 0.1 \text{ m} \quad b = 0.025 \text{ m} \quad \theta = 30^\circ \quad m = 10 \text{ kg}$$

and

$$\omega_p = 100 \text{ rpm} = 10.47 \text{ rad/s} \quad \omega_s = 2000 \text{ rpm} = 209.4 \text{ rad/s}$$

For these numbers, the variation of V and T_D with ψ are as shown in Figure 9.26.

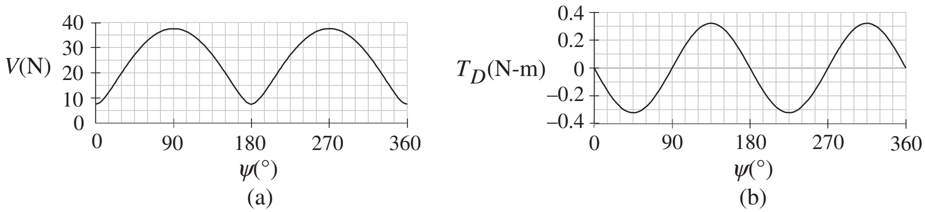


FIGURE 9.26

(a) Transverse bearing load. (b) Axial torque at *D*.

9.10 Yaw, pitch, and roll angles

The problem of the Euler angle relations (Eqn (9.116)) becoming singular when the nutation angle θ is zero can be alleviated by using the yaw, pitch, and roll angles discussed in Section 4.5. As in the classical Euler sequence, the yaw–pitch–roll sequence rotates the inertial XYZ axes into the body-fixed xyz axes triad by means of a series of three elementary rotations illustrated in Figure 9.27. Like the classical Euler sequence, the first rotation is around the Z $(=z_1)$ axis through the yaw angle ϕ. This takes X into x_1 and Y into y_1. The second rotation is around the y_2 $(=y_1)$ axis through the pitch angle θ. This carries x_1 and z_1 into x_2 and z_2, respectively. The third and final rotation is around the x $(=x_2)$ axis through the roll angle ψ, which takes y_2 into y and z_2 into z.

Equation (4.40) gives the matrix $[\mathbf{Q}]_{Xx}$ of the transformation from the inertial frame into the body-fixed frame,

$$[\mathbf{Q}]_{Xx} = [\mathbf{R}_1(\psi)][\mathbf{R}_2(\theta)][\mathbf{R}_3(\phi)] \tag{9.117}$$

From Eqns (4.32)–(4.34), the elementary rotation matrices are

$$[\mathbf{R}_1(\psi)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \quad [\mathbf{R}_2(\theta)] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.118}$$

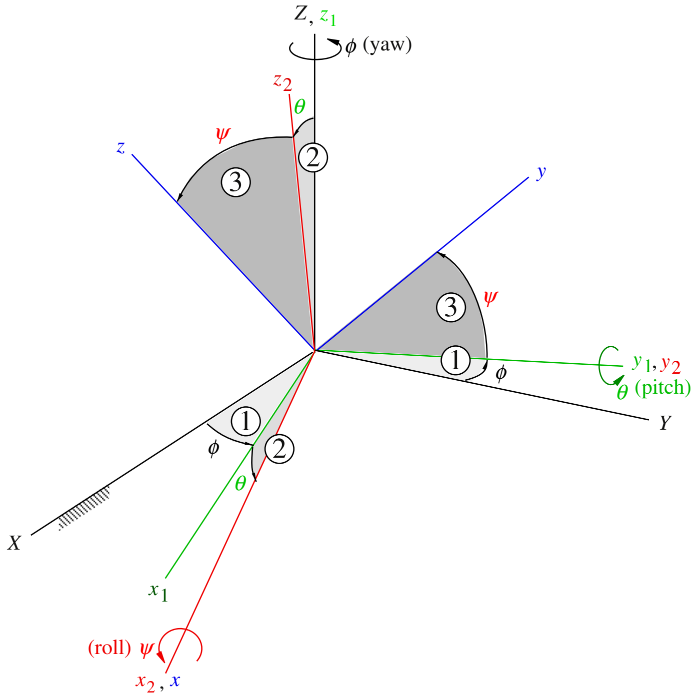


FIGURE 9.27

Yaw, pitch, and roll sequence. See also Figure 4.15.

According to Eqn (4.41), the multiplication on the right of Eqn (9.117) yields the following direction cosine matrix for the yaw, pitch, and roll sequence:

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta \\ \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\theta\sin\psi \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\theta\cos\psi \end{bmatrix} \tag{9.119}$$

The inverse matrix $[\mathbf{Q}]_{xX}$, which transforms xyz into XYZ, is just the transpose

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{bmatrix} \tag{9.120}$$

Algorithm 4.4 (*dcm_to_ypr.m* in Appendix D.21) is used to determine the yaw, pitch, and roll angles for a given direction cosine matrix. The following brief MATLAB session reveals that the yaw, pitch, and roll angles for the direction cosine matrix in Example 9.17 are $\phi = 109.69°$, $\theta = 17.230°$, and $\psi = 238.43°$.

```
>> Q = [-0.32175  0.89930  -0.29620
         0.57791 -0.061275 -0.81380
        -0.75000 -0.43301  -0.5000];
>> [yaw pitch roll] = dcm_to_ypr(Q)
yaw =
    109.6861
pitch =
     17.2295
roll =
    238.4334
```

Figure 9.27 shows that yaw ϕ is measured around the inertial Z-axis (unit vector $\hat{\mathbf{K}}$), pitch θ is measured around the intermediate y_1-axis (unit vector $\hat{\mathbf{j}}_1$), and roll ψ is measured around the body-fixed x-axis (unit vector $\hat{\mathbf{i}}$). The angular velocity $\boldsymbol{\omega}$, expressed in terms of the rates of yaw, pitch, and roll, is

$$\boldsymbol{\omega} = \omega_{\text{yaw}}\hat{\mathbf{K}} + \omega_{\text{pitch}}\hat{\mathbf{j}}_2 + \omega_{\text{roll}}\hat{\mathbf{i}} \tag{9.121}$$

in which

$$\omega_{\text{yaw}} = \dot{\phi} \quad \omega_{\text{pitch}} = \dot{\theta} \quad \omega_{\text{roll}} = \dot{\psi} \tag{9.122}$$

The first rotation $[\mathbf{R}_3(\phi)]$ in Eqn (9.117) rotates the unit vectors $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ of the inertial frame into the unit vectors $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ of the intermediate $x_1y_1z_1$ axes in Figure 9.27. Thus, $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ are rotated into $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ by the inverse transformation

$$\begin{Bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} \tag{9.123}$$

The second rotation $[\mathbf{R}_2(\theta)]$ rotates $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$ into the unit vectors $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ of the second intermediate frame $x_2y_2z_2$ in Figure 9.27. The inverse transformation rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ back into $\hat{\mathbf{i}}_1\hat{\mathbf{j}}_1\hat{\mathbf{k}}_1$:

$$\begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} \tag{9.124}$$

Lastly, the third rotation $[\mathbf{R}_1(\psi)]$ rotates $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ into $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$, the unit vectors of the body-fixed xyz frame. $\hat{\mathbf{i}}_2\hat{\mathbf{j}}_2\hat{\mathbf{k}}_2$ are obtained from $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ by the reverse transformation,

$$\begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix} \tag{9.125}$$

From Eqns (9.123)–(9.125), we see that

$$\hat{\mathbf{K}} \overbrace{=}^{9.123} \hat{\mathbf{k}}_1 \overbrace{=}^{9.124} -\sin\theta\hat{\mathbf{i}}_2 + \cos\theta\hat{\mathbf{k}}_2 \overbrace{=}^{9.125} -\sin\theta\hat{\mathbf{i}} + \cos\theta\left(\sin\psi\hat{\mathbf{j}} + \cos\psi\hat{\mathbf{k}}\right)$$

or

$$\hat{\mathbf{K}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\sin\psi\hat{\mathbf{j}} + \cos\theta\cos\psi\hat{\mathbf{k}} \tag{9.126}$$

From Eqn (9.125),

$$\hat{\mathbf{j}}_2 = \cos\psi\hat{\mathbf{j}} - \sin\psi\hat{\mathbf{k}} \tag{9.127}$$

Substituting Eqns (9.126) and (9.127) into Eqn (9.121) yields

$$\boldsymbol{\omega} = \omega_{\text{yaw}}\left(-\sin\theta\hat{\mathbf{i}} + \cos\theta\sin\psi\hat{\mathbf{j}} + \cos\theta\cos\psi\hat{\mathbf{k}}\right) + \omega_{\text{pitch}}\left(\cos\psi\hat{\mathbf{j}} - \sin\psi\hat{\mathbf{k}}\right) + \omega_{\text{roll}}\hat{\mathbf{i}}$$

or

$$\begin{aligned}\boldsymbol{\omega} = &\left(-\omega_{\text{yaw}}\sin\theta + \omega_{\text{roll}}\right)\hat{\mathbf{i}} + \left(\omega_{\text{yaw}}\cos\theta\sin\psi + \omega_{\text{pitch}}\cos\psi\right)\hat{\mathbf{j}} \\ &+ \left(\omega_{\text{yaw}}\cos\theta\cos\psi - \omega_{\text{pitch}}\sin\psi\right)\hat{\mathbf{k}}\end{aligned} \tag{9.128}$$

Comparing Eqns (9.107) and (9.128) we conclude that the body angular velocities are related to the yaw, pitch, and roll rates as follows:

$$\begin{aligned}\omega_x &= \omega_{\text{roll}} - \omega_{\text{yaw}}\sin\theta_{\text{pitch}} \\ \omega_y &= \omega_{\text{yaw}}\cos\theta_{\text{pitch}}\sin\psi_{\text{roll}} + \omega_{\text{pitch}}\cos\psi_{\text{roll}} \\ \omega_z &= \omega_{\text{yaw}}\cos\theta_{\text{pitch}}\cos\psi_{\text{roll}} - \omega_{\text{pitch}}\sin\psi_{\text{roll}}\end{aligned} \tag{9.129}$$

wherein the subscript on each symbol helps us remember the rotation it describes. The inverse of these equations is

$$\begin{aligned}\omega_{\text{yaw}} &= \frac{1}{\cos\theta_{\text{pitch}}}\left(\omega_y\sin\psi_{\text{roll}} + \omega_z\cos\psi_{\text{roll}}\right) \\ \omega_{\text{pitch}} &= \omega_y\cos\psi_{\text{roll}} - \omega_z\sin\psi_{\text{roll}} \\ \omega_{\text{roll}} &= \omega_x + \omega_y\tan\theta_{\text{pitch}}\sin\psi_{\text{roll}} + \omega_z\tan\theta_{\text{pitch}}\cos\psi_{\text{roll}}\end{aligned} \tag{9.130}$$

Notice that this system becomes singular ($\cos\theta_{\text{pitch}} = 0$) when the pitch angle is $\pm 90°$.

9.11 Quaternions

In Chapter 4, we showed that the transformation from any Cartesian coordinate frame to another having the same origin can be accomplished by three Euler angle sequences, each being an elementary rotation about one of the three coordinate axes. We have focused on the commonly

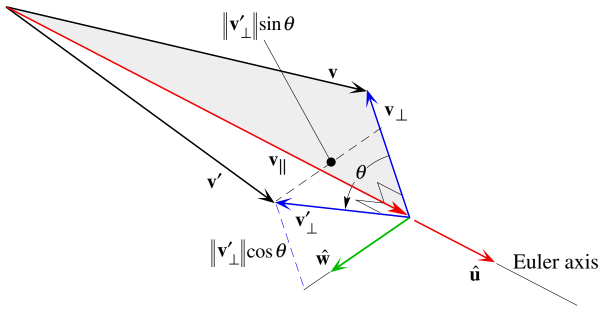


FIGURE 9.28

Rotation of a vector through an angle θ about an axis with unit vector û.

used classical Euler angle sequence ($[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]$) and the yaw–pitch–roll sequence ($[\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)]$).

Another of Euler's theorems, which we used in Section 1.6, states that any two Cartesian coordinate frames are related by a unique rotation about a single line through their common origin. This line is called the Euler axis and the angle is referred to as the principal angle.

Let $\hat{\mathbf{u}}$ be the unit vector along the Euler axis. A vector $\mathbf{v}$ can be resolved into orthogonal components $\mathbf{v}_\perp$ normal to $\hat{\mathbf{u}}$ and $\mathbf{v}_{||}$ parallel to $\hat{\mathbf{u}}$, so that we may write

$$\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_\perp \tag{9.131}$$

The component of $\mathbf{v}$ along $\hat{\mathbf{u}}$ is given by $\mathbf{v}\cdot\hat{\mathbf{u}}$. That is,

$$\mathbf{v}_{||} = (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} \tag{9.132}$$

From Eqns (9.131) and (9.132), we have

$$\mathbf{v}_\perp = \mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} \tag{9.133}$$

Let $\mathbf{v}'$ be the vector obtained by rotating $\mathbf{v}$ through an angle θ around $\hat{\mathbf{u}}$, as illustrated in Figure 9.28. This rotation leaves the magnitude of $\mathbf{v}_\perp$ and its component along $\hat{\mathbf{u}}$ is unchanged. That is

$$\|\mathbf{v}'_\perp\| = \|\mathbf{v}_\perp\| \tag{9.134}$$

$$\mathbf{v}'_{||} = (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} \tag{9.135}$$

$\mathbf{v}'_\perp$, having been rotated about $\hat{\mathbf{u}}$, has the component $\|\mathbf{v}'_\perp\|\cos\theta$ along the original vector $\mathbf{v}$ and the component $\|\mathbf{v}'_\perp\|\sin\theta$ along the vector normal to the plane of $\hat{\mathbf{u}}$ and $\mathbf{v}$. Let $\hat{\mathbf{w}}$ be the unit vector normal to that plane. Then,

$$\hat{\mathbf{w}} = \hat{\mathbf{u}} \times \frac{\mathbf{v}_\perp}{\|\mathbf{v}_\perp\|} \tag{9.136}$$

Thus,

$$\mathbf{v}'_{\perp} = \|\mathbf{v}'_{\perp}\|\cos\theta\frac{\mathbf{v}_{\perp}}{\|\mathbf{v}_{\perp}\|} + \|\mathbf{v}'_{\perp}\|\sin\theta\frac{\hat{\mathbf{u}}\times\mathbf{v}_{\perp}}{\|\mathbf{v}_{\perp}\|}$$

According to Eqn (9.134), this reduces to

$$\mathbf{v}'_{\perp} = \cos\theta\mathbf{v}_{\perp} + \sin\theta\hat{\mathbf{u}}\times\mathbf{v}_{\perp} \tag{9.137}$$

Observe that

$$\hat{\mathbf{u}}\times\mathbf{v}_{\perp} = \hat{\mathbf{u}}\times\left[\mathbf{v}-\mathbf{v}_{\|}\right] = \hat{\mathbf{u}}\times\mathbf{v}$$

since $\mathbf{v}_{\|}$ is parallel to $\hat{\mathbf{u}}$. This, together with Eqn (9.133), means we can write Eqn (9.137) as

$$\mathbf{v}'_{\perp} = \cos\theta[\mathbf{v}-(\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}] + \sin\theta(\hat{\mathbf{u}}\times\mathbf{v}) \tag{9.138}$$

Since $\mathbf{v}' = \mathbf{v}'_{\perp} + \mathbf{v}'_{\|}$, we find, upon substituting Eqns (9.137) and (9.138) and collecting terms, that

$$\mathbf{v}' = \cos\theta\mathbf{v} + (1-\cos\theta)(\hat{\mathbf{u}}\cdot\mathbf{v})\hat{\mathbf{u}} + \sin\theta(\hat{\mathbf{u}}\times\mathbf{v}) \tag{9.139}$$

This is a useful formula for determining the result of rotating a vector about a line.

We can obtain the body-fixed *xyz* Cartesian frame from the inertial *XYZ* frame by a single rotation through the principal angle θ about the Euler axis $\hat{\mathbf{u}}$. The unit vectors $\hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{K}}$ are thereby rotated into $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$. The two sets of unit vectors are related by Eqn (9.139). Thus,

$$\begin{aligned}
\hat{\mathbf{i}} &= \cos\theta\hat{\mathbf{I}} + (1-\cos\theta)\left(\hat{\mathbf{u}}\cdot\hat{\mathbf{I}}\right)\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}}\times\hat{\mathbf{I}}\\
\hat{\mathbf{j}} &= \cos\theta\hat{\mathbf{J}} + (1-\cos\theta)\left(\hat{\mathbf{u}}\cdot\hat{\mathbf{J}}\right)\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}}\times\hat{\mathbf{J}}\\
\hat{\mathbf{k}} &= \cos\theta\hat{\mathbf{K}} + (1-\cos\theta)\left(\hat{\mathbf{u}}\cdot\hat{\mathbf{K}}\right)\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}}\times\hat{\mathbf{K}}
\end{aligned} \tag{9.140}$$

Let us express the unit vector $\hat{\mathbf{u}}$ in terms of its direction cosines l, m, and n along the original *XYZ* axes, that is

$$\hat{\mathbf{u}} = l\hat{\mathbf{I}} + m\hat{\mathbf{J}} + n\hat{\mathbf{K}} \tag{9.141}$$

Substituting this into Eqn (9.140), carrying out the vector operations, and collecting the terms yields

$$\begin{aligned}
\hat{\mathbf{i}} &= \left[l^2(1-\cos\theta)+\cos\theta\right]\hat{\mathbf{I}} + [lm(1-\cos\theta)+n\sin\theta]\hat{\mathbf{J}} + [ln(1-\cos\theta)-m\sin\theta]\hat{\mathbf{K}}\\
\hat{\mathbf{j}} &= [lm(1-\cos\theta)-n\sin\theta]\hat{\mathbf{I}} + \left[m^2(1-\cos\theta)+\cos\theta\right]\hat{\mathbf{J}} + [mn(1-\cos\theta)+l\sin\theta]\hat{\mathbf{K}}\\
\hat{\mathbf{k}} &= [ln(1-\cos\theta)+m\sin\theta]\hat{\mathbf{I}} + [mn(1-\cos\theta)-l\sin\theta]\hat{\mathbf{J}} + \left[n^2(1-\cos\theta)+\cos\theta\right]\hat{\mathbf{K}}
\end{aligned} \tag{9.142}$$

Recall that the rows of the matrix $[\mathbf{Q}]_{Xx}$ of the transformation from XYZ to xyz comprise the direction cosines of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, respectively. That is,

$$[\mathbf{Q}] = \begin{bmatrix} l^2(1-\cos\theta)+\cos\theta & lm(1-\cos\theta)+n\sin\theta & ln(1-\cos\theta)-m\sin\theta \\ lm(1-\cos\theta)-n\sin\theta & m^2(1-\cos\theta)+\cos\theta & mn(1-\cos\theta)+l\sin\theta \\ ln(1-\cos\theta)+m\sin\theta & mn(1-\cos\theta)-l\sin\theta & n^2(1-\cos\theta)+\cos\theta \end{bmatrix} \tag{9.143}$$

The direction cosine matrix is thus expressed in terms of the Euler axis direction cosines and the principal angle.

Quaternions (also known as Euler symmetric parameters) were introduced in 1843 by the Irish mathematician Sir William R. Hamilton (1805–1865). They provide an alternative to the use of direction cosine matrices for describing the orientation of a body frame in three-dimensional space. Quaternions can be used to avoid encountering the singularities we observed for the classical Euler angle sequence when the nutation angle θ becomes zero (Eqn (9.116)) or for the yaw, pitch, and roll sequence when the pitch angle θ approaches 90° (Eqn (9.126)).

As the name implies, a quaternion $\{\widehat{\mathbf{q}}\}$ comprises four numbers:

$$\{\widehat{\mathbf{q}}\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \mathbf{q} \\ q_4 \end{Bmatrix} \tag{9.144}$$

$\mathbf{q}$ is called the vector part ($\mathbf{q} = q_1\hat{\mathbf{i}} + q_2\hat{\mathbf{j}} + q_3\hat{\mathbf{k}}$) and q_4 is the scalar part. (It is equally common to see the scalar part listed first, so that $\{\widehat{\mathbf{q}}\} = \lfloor q_4, \mathbf{q} \rfloor^T$ or $\{\widehat{\mathbf{q}}\} = \lfloor q_0, \mathbf{q} \rfloor^T$.)

The norm $\|\widehat{\mathbf{q}}\|$ of the quaternion $\{\widehat{\mathbf{q}}\}$ is defined as

$$\left\|\widehat{\mathbf{q}}\right\| = \sqrt{\mathbf{q}\cdot\mathbf{q} + q_4{}^2} = \sqrt{q_1{}^2 + q_2{}^2 + q_3{}^2 + q_4{}^2} \tag{9.145}$$

We will restrict our attention to unit quaternions, which are such that $\|\widehat{\mathbf{q}}\| = 1$. In that case

$$\mathbf{q} = \sin\frac{\theta}{2}\hat{\mathbf{u}} \quad q_4 = \cos\frac{\theta}{2} \tag{9.146}$$

$\hat{\mathbf{u}}$ is the unit vector along the Euler axis around which the inertial reference frame is rotated into the body-fixed frame. θ is the Euler principal rotation angle. Recalling Eqn (9.141), we observe that

$$q_1 = l\sin\frac{\theta}{2} \quad q_2 = m\sin\frac{\theta}{2} \quad q_3 = n\sin\frac{\theta}{2} \quad q_4 = \cos\frac{\theta}{2} \tag{9.147}$$

Employing these and the trigonometric identities

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

one can show that the direction cosine matrix $[\mathbf{Q}]_{Xx}$ of the body frame in Eqn (9.143) is obtained from the quaternion $\{\widehat{\mathbf{q}}\}$ by means of the following algorithm (Kuipers, 1999).

ALGORITHM 9.1

Obtain $[\mathbf{Q}]_{Xx}$ from the unit quaternion $\{\widehat{\mathbf{q}}\}$. This procedure is implemented in the MATLAB function *dcm_from_q.m* in Appendix D.37.

1. Write the quaternion as

$$\{\widehat{\mathbf{q}}\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

where $\lfloor q_1 \quad q_2 \quad q_3 \rfloor^T$ is the vector part, q_4 is the scalar part, and $\|\widehat{\mathbf{q}}\| = 1$.

2. Compute the direction cosine matrix of the transformation from XYZ to xyz as follows:

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} q_1{}^2 - q_2{}^2 - q_3{}^2 + q_4{}^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1{}^2 + q_2{}^2 - q_3{}^2 + q_4{}^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1{}^2 - q_2{}^2 + q_3{}^2 + q_4{}^2 \end{bmatrix} \tag{9.148}$$

One can verify by carrying out the matrix multiplication and using Eqn (9.145) that $[\mathbf{Q}]_{Xx}$ in Eqn (9.148) has the required orthogonality property,

$$[\mathbf{Q}]_{Xx}[\mathbf{Q}]_{Xx}{}^T = [\mathbf{Q}]_{Xx}{}^T[\mathbf{Q}]_{Xx} = [\mathbf{1}]$$

To find the unit quaternion $\left(q_1{}^2 + q_2{}^2 + q_3{}^2 + q_4{}^2 = 1\right)$ for a given direction cosine matrix, we observe from Eqn (9.148) that

$$\begin{aligned} q_4 &= \frac{1}{2}\sqrt{1 + Q_{11} + Q_{22} + Q_{33}} \\ q_1 &= \frac{Q_{23} - Q_{32}}{4q_4} \quad q_2 = \frac{Q_{31} - Q_{13}}{4q_4} \quad q_3 = \frac{Q_{12} - Q_{21}}{4q_4} \end{aligned} \tag{9.149}$$

This procedure obviously fails if $q_4 = 0$. The following algorithm (Bar-Itzhack, 2000) avoids having to deal with this situation.

ALGORITHM 9.2

Obtain the (unit) quaternion from the direction cosine matrix $[\mathbf{Q}]_{Xx}$. This procedure is implemented as the MATLAB function *q_from_dcm.m* in Appendix D.38.

1. Form the symmetric matrix

$$[\mathbf{K}] = \frac{1}{3}\begin{bmatrix} Q_{11}-Q_{22}-Q_{33} & Q_{21}+Q_{12} & Q_{31}+Q_{13} & Q_{23}-Q_{32} \\ Q_{21}+Q_{12} & -Q_{11}+Q_{22}-Q_{33} & Q_{32}+Q_{23} & Q_{31}-Q_{13} \\ Q_{31}+Q_{13} & Q_{32}+Q_{23} & -Q_{11}-Q_{22}+Q_{33} & Q_{12}-Q_{21} \\ Q_{23}-Q_{32} & Q_{31}-Q_{13} & Q_{12}-Q_{21} & Q_{11}+Q_{22}+Q_{33} \end{bmatrix} \tag{9.150}$$

2. Solve the eigenvalue problem $[\mathbf{K}]\{\mathbf{e}\} = \lambda\{\mathbf{e}\}$ for the largest eigenvalue λ_{max}. The corresponding eigenvector is the quaternion $\{\widehat{\mathbf{q}}\} = \{\mathbf{e}\}$.

The time derivative of a quaternion is given by (Sidi, 1997)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\widehat{\mathbf{q}}\right\} = \frac{1}{2}[\boldsymbol{\Omega}]\left\{\widehat{\mathbf{q}}\right\} \tag{9.151a}$$

where

$$[\boldsymbol{\Omega}] = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix} \tag{9.151b}$$

ω_x, ω_y, and ω_z are the components of the body frame's angular velocity.

EXAMPLE 9.21

(a) Write down the quaternion for a rotation about the x-axis through an angle θ. (b) Obtain the corresponding direction cosine matrix.

Solution

(a) According to Eqn (9.146),

$$\mathbf{q} = \sin(\theta/2)\hat{\mathbf{i}} \quad q_4 = \cos(\theta/2) \tag{a}$$

so that

$$\boxed{\left\{\widehat{\mathbf{q}}\right\} = \begin{Bmatrix} \sin(\theta/2) \\ 0 \\ 0 \\ \cos(\theta/2) \end{Bmatrix}} \tag{b}$$

(b) Substituting $q_1 = \sin(\theta/2)$, $q_2 = q_3 = 0$, and $q_4 = \cos(\theta/2)$ into Eqn (9.148) yields

$$[\mathbf{Q}] = \begin{bmatrix} \sin^2(\theta/2) + \cos^2(\theta/2) & 0 & 0 \\ 0 & -\sin^2(\theta/2) + \cos^2(\theta/2) & 2\sin(\theta/2)\cos(\theta/2) \\ 0 & -2\sin(\theta/2)\cos(\theta/2) & -\sin^2(\theta/2) + \cos^2(\theta/2) \end{bmatrix} \tag{c}$$

From trigonometry, we recall that

$$\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1 \quad 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta \quad \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$$

Therefore, Eqn (c) becomes

$$\boxed{[\mathbf{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}} \tag{d}$$

We recognize this as the direction cosine matrix $[\mathbf{R}_1(\theta)]$ for a rotation θ around the x-axis.

EXAMPLE 9.22

For the yaw–pitch–roll sequence $\phi_{\text{yaw}} = 50°$, $\theta_{\text{pitch}} = 90°$, and $\psi_{\text{roll}} = 120°$, calculate

(a) The quaternion and
(b) The rotation angle and the axis of rotation.

Solution

(a) Substituting the given angles into Eqn (9.119) yields the direction cosine matrix

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} 0 & 0 & -1 \\ 0.93969 & 0.34202 & 0 \\ 0.34202 & -0.93969 & 0 \end{bmatrix} \tag{a}$$

Substituting the components of $[\mathbf{Q}]_{Xx}$ into Eqn (9.145), we get

$$[\mathbf{K}] = \begin{bmatrix} -0.11401 & 0.31323 & -0.21933 & 0.31323 \\ 0.31323 & 0.11401 & -0.31323 & 0.44734 \\ -0.21933 & -0.31323 & 0.11401 & -0.31323 \\ 0.31323 & 0.44734 & -0.31323 & 0.11401 \end{bmatrix} \tag{b}$$

The eigenvalues of this matrix are $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = -1/3$. Hence, λ_1 is the largest eigenvalue. The quaternion is the vector $\{\mathbf{x}\}$ such that $[\mathbf{K}]\{\mathbf{x}\} = 1 \cdot \{\mathbf{x}\}$. Therefore,

$$\boxed{\left\{\widehat{\mathbf{q}}\right\} = \begin{Bmatrix} 0.40558 \\ 0.57923 \\ -0.40558 \\ 0.57923 \end{Bmatrix}}$$

(b) From Eqn (9.146), we find that the principal angle is

$$\theta = 2\cos^{-1}(q_4) = 2\cos^{-1}(0.57923) = \boxed{54.604°}$$

and the Euler axis is

$$\hat{\mathbf{u}} = \frac{0.40558\hat{\mathbf{I}} + 0.57923\hat{\mathbf{J}} - 0.40558\hat{\mathbf{K}}}{\sin(54.604°/2)} = \boxed{0.49754\hat{\mathbf{I}} + 0.71056\hat{\mathbf{J}} - 0.49754\hat{\mathbf{K}}}$$

EXAMPLE 9.23

Solve the spinning top problem of Example 9.15 numerically, using quaternions.

Solution

We will use Eqn (9.72b) (Euler's equations) to compute the angular velocity derivatives:

$$\begin{aligned}
\frac{d\omega_x}{dt} &= \frac{M_{x_{\text{net}}}}{A} - \frac{C-B}{A}\omega_y\omega_z \\
\frac{d\omega_y}{dt} &= \frac{M_{y_{\text{net}}}}{B} - \frac{A-C}{B}\omega_z\omega_x \\
\frac{d\omega_z}{dt} &= \frac{M_{z_{\text{net}}}}{C} - \frac{B-A}{C}\omega_x\omega_y
\end{aligned} \tag{a}$$

These require the moment to be expressed in components along the body-fixed axes. From Figure 9.19, the moment of the weight vector about O is

$$\mathbf{M}_{O_{\text{net}}} = d\hat{\mathbf{k}} \times \left(-mg\hat{\mathbf{K}}\right) = -mgd\left(\hat{\mathbf{k}} \times \hat{\mathbf{K}}\right) \tag{b}$$

where

$$\hat{\mathbf{k}} = Q_{31}\hat{\mathbf{I}} + Q_{32}\hat{\mathbf{J}} + Q_{33}\hat{\mathbf{K}} \tag{c}$$

The Qs are components of the direction cosine matrix $[\mathbf{Q}]_{Xx}$ in Eqn (9.148). Carrying out the cross product in Eqn (b) yields, in matrix form,

$$\{\mathbf{M}_O\}_X = \begin{Bmatrix} -mgdQ_{32} \\ mgdQ_{31} \\ 0 \end{Bmatrix} \tag{d}$$

To arrive at the components of $\{\mathbf{M}_O\}_x$ in the body frame, we perform the transformation

$$\{\mathbf{M}_O\}_x = [\mathbf{Q}]_{Xx}\{\mathbf{M}_O\}_X \tag{e}$$

which yields

$$M_{x_{\text{net}}} = mgdQ_{23} \quad M_{y_{\text{net}}} = -mgdQ_{13} \quad M_{z_{\text{net}}} = 0 \tag{f}$$

The MATLAB implementation of the following procedure is listed in Appendix D.39.

Step1:

Specify the initial orientation of the *xyz* axes of the body frame, thereby defining the initial value of the direction cosine matrix $[\mathbf{Q}]_{Xx}$.

Step 2:

Compute the initial quaternion $\{\widehat{\mathbf{q}}_0\}$ using Algorithm 9.2.

Step 3:

Specify the initial value of the angular velocity $\{\boldsymbol{\omega}_0\} = \lfloor \omega_{x0} \quad \omega_{y0} \quad \omega_{z0} \rfloor^T$ in body frame components.

Step 4:

Supply $\{\boldsymbol{\omega}_0\}$ and $\{\widehat{\mathbf{q}}_0\}$ as initial conditions to the Runge–Kutta–Fehlberg 4(5) numerical integration procedure (Algorithm 1.3). At each step of the numerical integration process:

(i) Use the current values of $\{\widehat{\mathbf{q}}\}$ to compute $[\mathbf{Q}]_{Xx}$ from Algorithm 9.1.

(ii) Use the current value of $[\mathbf{Q}]_{Xx}$ and $\{\boldsymbol{\omega}\}$ to compute $\mathrm{d}\{\boldsymbol{\omega}\}/\mathrm{d}t$ from (a) and (f) .

(iii) Use the current value of $\{\widehat{\mathbf{q}}\}$ and $\{\boldsymbol{\omega}\}$ to compute $\mathrm{d}\{\widehat{\mathbf{q}}\}/\mathrm{d}t$ from Eqns (9.151).

Step 5:

At each solution time:

(i) Use Algorithm 9.1 to compute the direction cosine matrix $[\mathbf{Q}]_{Xx}$.

(ii) Use Algorithm 4.3 to compute the Euler angles ϕ (presession), θ (nutation) and ψ (spin).

Step 6:

Plot the results.

Figure 9.29 shows the precession (ϕ), nutation (θ), and spin (ψ) angles as a function of time. We see that precession rate and spin rates are in agreement with Example 9.15. However, the nutation angle is

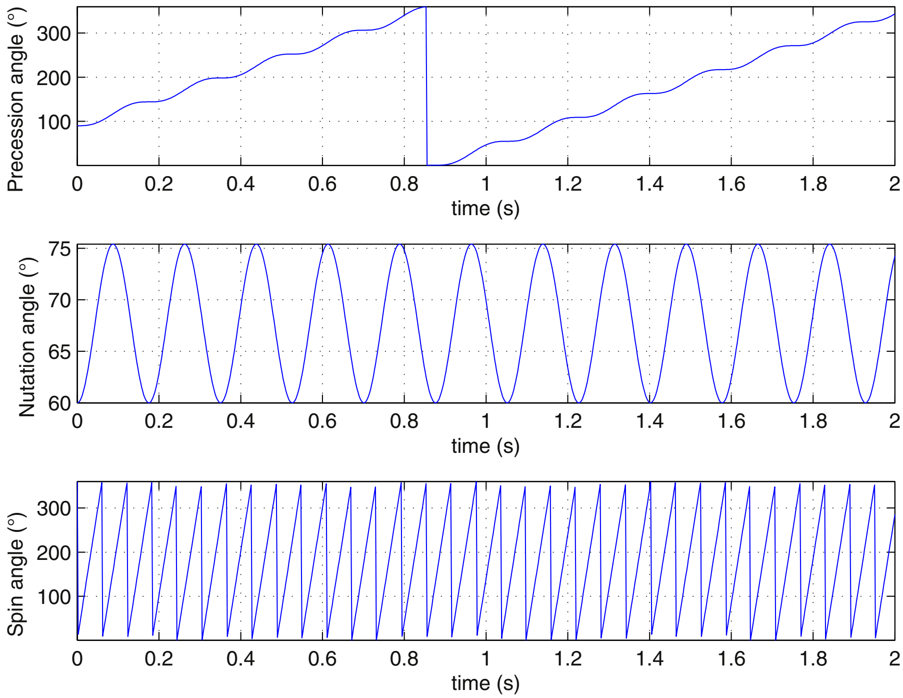


FIGURE 9.29

Precession, nutation, and spin angles of a prolate top ($A = B = 0.0012$ kg·m^2, $C = 0.00045$ kg·m^2).

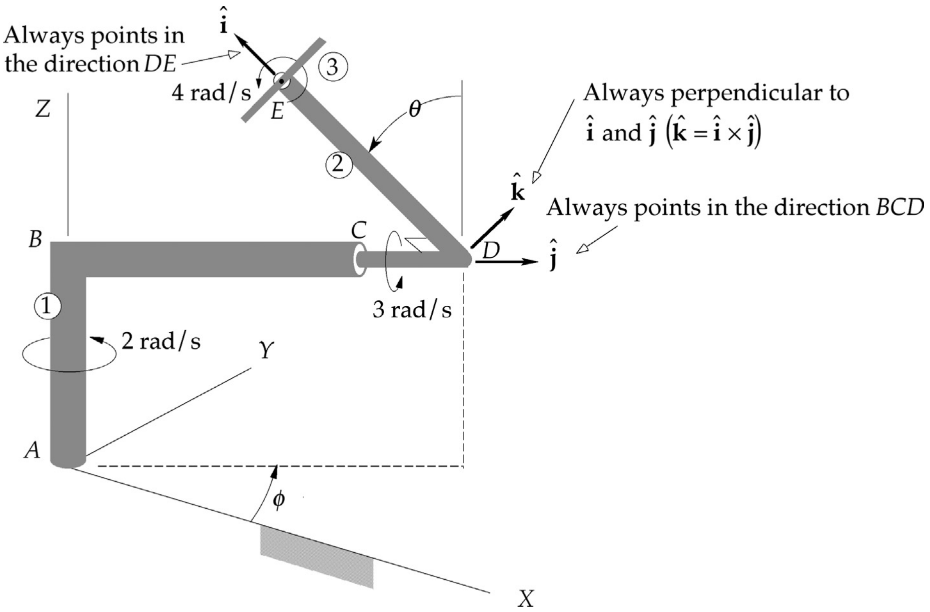
dramatically different. In Example 9.15, θ was assumed to be a constant 60°. Here we see that θ varies between 60° and 75° at a frequency of slightly less than 6 Hz.

PROBLEMS

Section 9.2

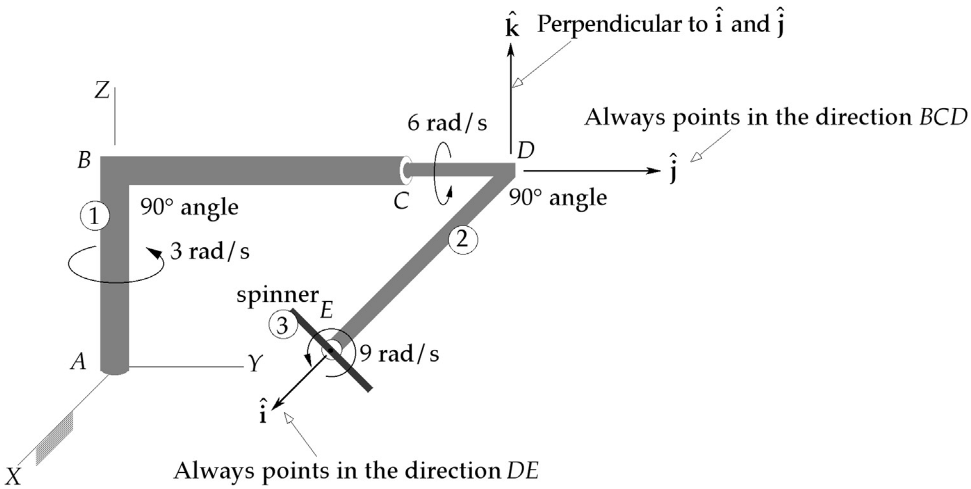
9.1 Rigid, bent shaft 1 (*ABC*) rotates at a constant angular velocity of $2\hat{\mathbf{K}}$ rad/s around the positive *Z*-axis of the inertial frame. Bent shaft 2 (*CDE*) rotates around *BC* with a constant angular velocity of $3\hat{\mathbf{j}}$ rad/s, relative to *BC*. Spinner 3 at *E* rotates around *DE* with a constant angular velocity of $4\hat{\mathbf{i}}$ rad/s relative to *DE*. Calculate the magnitude of the absolute angular acceleration $\boldsymbol{\alpha}_3$ of the spinner at the instant shown.

{Ans.: $\|\boldsymbol{\alpha}_3\| = \sqrt{180 + 64\sin^2\theta - 144\cos\theta}\,(\text{rad/s}^2)$}



9.2 All the spin rates shown are constant. Calculate the magnitude of the absolute angular acceleration $\boldsymbol{\alpha}_3$ of the spinner at the instant shown (i.e., at the instant when the unit vector $\hat{\mathbf{i}}$ is parallel to the *X*-axis and the unit vector $\hat{\mathbf{j}}$ is parallel to the *Y*-axis).

{Ans.: $\|\boldsymbol{\alpha}_3\| = 63 \ \text{rad/s}^2$}

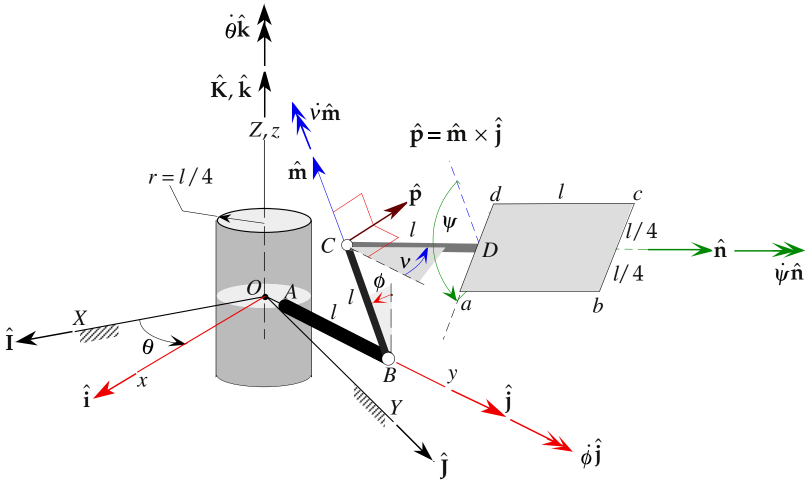


9.3 The body-fixed xyz frame is attached to the cylinder as shown. The cylinder rotates around the inertial Z-axis, which is collinear with the z-axis, with a constant absolute angular velocity $\dot{\theta}\hat{\mathbf{k}}$. Rod AB is attached to the cylinder and aligned with the y-axis. Rod BC is perpendicular to AB and rotates around AB with the constant angular velocity $\dot{\phi}\hat{\mathbf{j}}$ relative to the cylinder. Rod CD is perpendicular to BC and rotates around BC with the constant angular velocity $\dot{\nu}\hat{\mathbf{m}}$ relative to BC, where $\hat{\mathbf{m}}$ is the unit vector in the direction of BC. The plate $abcd$ rotates around CD with a constant angular velocity $\dot{\psi}\hat{\mathbf{n}}$ relative to CD, where the unit vector $\hat{\mathbf{n}}$ points in the direction of CD. Thus, the absolute angular velocity of the plate is $\boldsymbol{\omega}_{\text{plate}} = \dot{\theta}\hat{\mathbf{k}} + \dot{\phi}\hat{\mathbf{j}} + \dot{\nu}\hat{\mathbf{m}} + \dot{\psi}\hat{\mathbf{n}}$. Show that

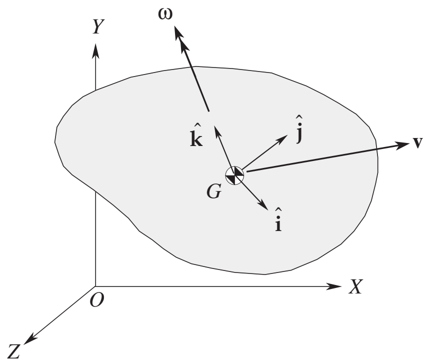
(a) $\boldsymbol{\omega}_{\text{plate}} = (\dot{\nu}\sin\phi - \dot{\psi}\cos\phi\sin\nu)\hat{\mathbf{i}} + (\dot{\phi} + \dot{\psi}\cos\nu)\hat{\mathbf{j}} + (\dot{\theta} + \dot{\nu}\cos\phi + \dot{\psi}\sin\phi\sin\nu)\hat{\mathbf{k}}$

(b)
$$\begin{aligned}\boldsymbol{\alpha}_{\text{plate}} = \frac{d\boldsymbol{\omega}_{\text{plate}}}{dt} &= [\dot{\nu}(\dot{\phi}\cos\phi - \dot{\psi}\cos\phi\cos\nu) + \dot{\psi}\dot{\phi}\sin\phi\sin\nu - \dot{\psi}\dot{\theta}\cos\nu - \dot{\phi}\dot{\theta}]\hat{\mathbf{i}} \\ &+ [\dot{\nu}(\dot{\theta}\sin\phi - \dot{\psi}\sin\nu) - \dot{\psi}\dot{\theta}\cos\phi\sin\nu]\hat{\mathbf{j}} \\ &+ [\dot{\psi}\dot{\nu}\cos\nu\sin\phi + \dot{\psi}\dot{\phi}\cos\phi\sin\nu - \dot{\phi}\dot{\nu}\sin\phi]\hat{\mathbf{k}}\end{aligned}$$

(c) $\mathbf{a}_C = -l(\dot{\phi}^2 + \dot{\theta}^2)\sin\phi\hat{\mathbf{i}} + \left(2l\dot{\phi}\dot{\theta}\cos\phi - \frac{5}{4}l\dot{\theta}^2\right)\hat{\mathbf{j}} - l\dot{\phi}^2\cos\phi\hat{\mathbf{k}}$



9.4 The mass center G of a rigid body has a velocity $\mathbf{v} = t^3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$ (m/s) and an angular velocity $\boldsymbol{\omega} = 2t^2\hat{\mathbf{k}}$ (rad/s), where t is time in seconds. The $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ unit vectors are attached to and rotate with the rigid body. Calculate the magnitude of the acceleration $\mathbf{a}_G$ of the center of mass at $t = 2$ s.
{Ans.: $\mathbf{a}_G = -20\hat{\mathbf{i}} + 64\hat{\mathbf{j}}$ (m/s^2)}



9.5 A rigid body is in pure rotation with angular velocity $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_y\hat{\mathbf{k}}$ about the origin of the inertial xyz frame. If point A with position vector $\mathbf{r}_A = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (m) has velocity $\mathbf{v}_A = 1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ (m/s), what is the magnitude of the velocity of the point B with position vector $\mathbf{r}_B = 1\hat{\mathbf{i}} + 1\hat{\mathbf{j}} - 1\hat{\mathbf{k}}$ (m)?
{Ans.: 1.871 m/s}

9.6 The inertial angular velocity of a rigid body is $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$, where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors of a comoving frame whose inertial angular velocity is $\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}}$. Calculate the components of angular acceleration of the rigid body in the moving frame, assuming that ω_x, ω_y, and ω_z are all constant.
{Ans.: $\boldsymbol{\alpha} = \omega_y\omega_z\hat{\mathbf{i}} - \omega_x\omega_z\hat{\mathbf{j}}$}

Section 9.5

9.7 Find the moments of inertia about the center of mass of the system of six point masses listed in the table.

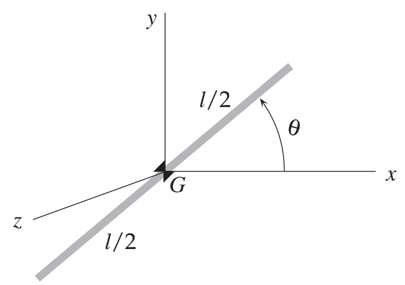
Point, *i*	**Mass m_i (kg)**	**x_i (m)**	**y_i (m)**	**z_i (m)**
1	10	1	1	1
2	10	−1	−1	−1
3	8	4	−4	4
4	8	−2	2	−2
5	12	3	−3	−3
6	12	−3	3	3

$$\{\text{Ans.: } [\mathbf{I}_G] = \begin{bmatrix} 783.5 & 351.7 & 40.27 \\ 351.7 & 783.5 & -80.27 \\ 40.27 & -80.27 & 783.5 \end{bmatrix} (\text{kg-m}^2)\}$$

9.8 Find the mass moment of inertia of the configuration of Problem 9.7 about an axis through the origin and the point with coordinates (1 m, 2 m, 2 m).
{Ans.: 898.7 kg-m^2}

9.9 A uniform slender rod of mass m and length l lies in the xy plane inclined to the x-axis by an angle θ. Use the results of Example 9.10 to find the mass moments of inertia about the xyz axes passing through the center of mass G.

$$\{\text{Ans.: } [\mathbf{I}_G] = \frac{1}{12}ml^2 \begin{bmatrix} \sin^2\theta & -\frac{1}{2}\sin 2\theta & 0 \\ -\frac{1}{2}\sin 2\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\}$$



9.10 The uniform rectangular box has a mass of 1000 kg. The dimensions of its edges are shown.

(a) Find the mass moments of inertia about the xyz axes.

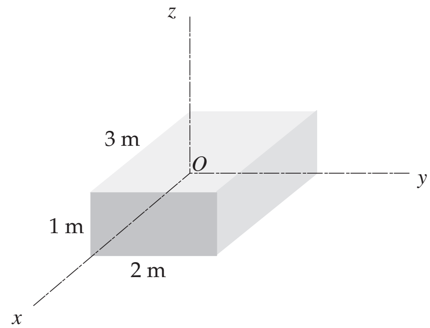
$$\{\text{Ans.: } [\mathbf{I}_O] = \begin{bmatrix} 1666.7 & -1500 & -750 \\ -1500 & 3333.3 & -500 \\ -750 & -500 & 4333.3 \end{bmatrix} (\text{kg-m}^2)\}$$

(b) Find the principal moments of inertia and the principal directions about the xyz axes through O.

{Partial Ans.: $I_1 = 568.9\ \text{kg-m}^2, \quad \hat{\mathbf{e}}_1 = 0.8366\hat{\mathbf{i}} + 0.4960\hat{\mathbf{j}} + 0.2326\hat{\mathbf{k}}$}

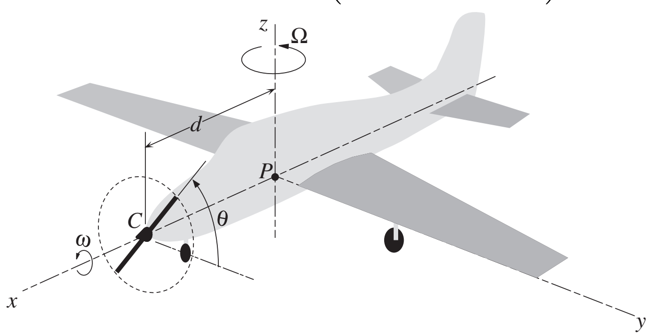
(c) Find the moment of inertia about the line through O and the point with coordinates (3 m, 2 m, 1 m).

{Ans.: $583.3\ \text{kg}\cdot\text{m}^2$}



9.11 A taxiing airplane turns about its vertical axis with an angular velocity Ω while its propeller spins at an angular velocity $\omega = \dot{\theta}$. Determine the components of the angular momentum of the propeller about the body-fixed xyz axes centered at P. Treat the propeller as a uniform slender rod of mass m and length l.

$$\{\text{Ans.: } \mathbf{H}_P = \frac{1}{12}m\omega l^2\hat{\mathbf{i}} - \frac{1}{24}m\Omega l^2 \sin 2\theta\hat{\mathbf{j}} + \left(\frac{1}{12}ml^2\cos^2\theta + md^2\right)\Omega\hat{\mathbf{k}}\}$$



9.12 Relative to an xyz frame of reference the components of angular momentum $\mathbf{H}$ are given by

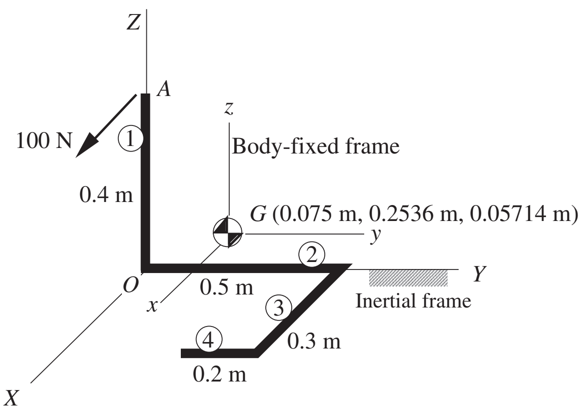
$$\{\mathbf{H}\} = \begin{bmatrix} 1000 & 0 & -300 \\ 0 & 1000 & 500 \\ -300 & 500 & 1000 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \left(\text{kg-m}^2/\text{s}\right)$$

where ω_x, ω_y, and ω_z are the components of the angular velocity $\boldsymbol{\omega}$. Find the components $\boldsymbol{\omega}$ such that $\{\mathbf{H}\} = 1000\{\boldsymbol{\omega}\}$, where the magnitude of $\boldsymbol{\omega}$ is 20 rad/s.
{Ans.: $\boldsymbol{\omega} = 17.15\hat{\mathbf{i}} + 10.29\hat{\mathbf{j}}$ (rad/s)}

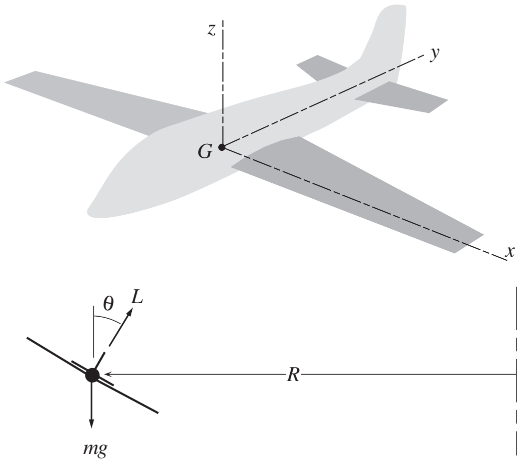
9.13 Relative to a body-fixed xyz frame $[\mathbf{I}_G] = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} (\text{kg}\cdot\text{m}^2)$ and

$\boldsymbol{\omega} = 2t^2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3t\hat{\mathbf{k}}$ (rad/s), where t is the time in seconds. Calculate the magnitude of the net moment about the center of mass G at $t = 3$ s.
{Ans.: 3374 N-m}

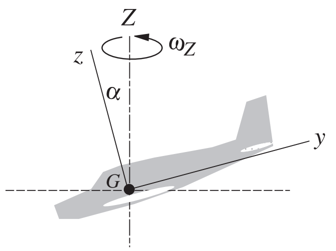
9.14 In Example 9.11, the system is at rest when a 100-N force is applied to point A as shown. Calculate the inertial components of angular acceleration at that instant.
{Ans.: $\alpha_X = 143.9\ \text{rad/s}^2,\ \ \alpha_Y = 553.1\ \text{rad/s}^2,\ \ \alpha_Z = 7.61\ \text{rad/s}^2$}



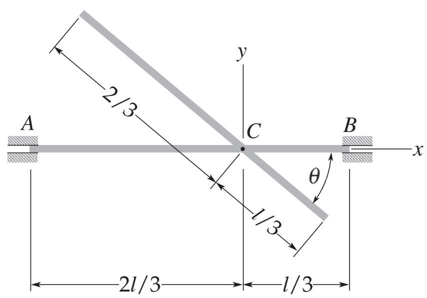
9.15 The body-fixed xyz axes pass through the center of mass G of the airplane and are the principal axes of inertia. The moments of inertia about these axes are A, B, and C, respectively. The airplane is in a level turn of radius R with a speed v.
(a) Calculate the bank angle θ.
(b) Use Euler's equations to calculate the rolling moment M_y that must be applied by the aerodynamic surfaces.
{Ans.: (a) $\theta = \tan^{-1}v^2/Rg$; (b) $M_y = v^2\sin 2\theta(C-A)/2R^2$}



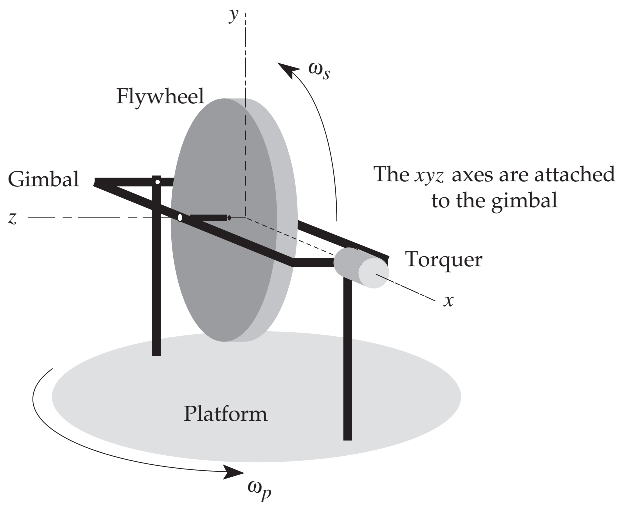
9.16 The airplane in Problem 9.15 is spinning with an angular velocity ω_Z about the vertical Z-axis. The nose is pitched down at the angle α. What external moments must accompany this maneuver?
{Ans.: $M_y = M_z = 0, \quad M_x = \omega_Z{}^2 \sin 2\alpha(C - B)/2$}



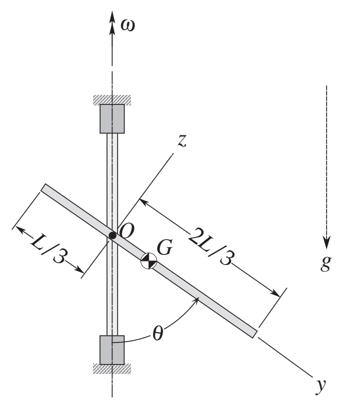
9.17 Two identical slender rods of mass m and length l are rigidly joined together at an angle θ at point C, their 2/3 point. Determine the bearing reactions at A and B if the shaft rotates at a constant angular velocity ω. Neglect gravity and assume that the only bearing forces are normal to rod AB.
{Ans.: $\|\mathbf{F}_A\| = m\omega^2 l \sin\theta(1 + 2\cos\theta)/18, \quad \|\mathbf{F}_B\| = m\omega^2 l \sin\theta(1 - \cos\theta)/9$}



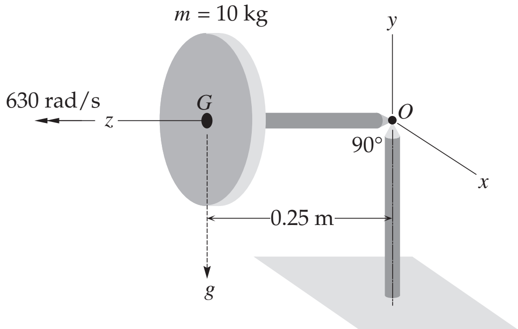
9.18 The flywheel ($A = B = 5$ kg-m^2, $C = 10$ kg-m^2) spins at a constant angular velocity of $\boldsymbol{\omega}_s = 100\hat{\mathbf{k}}$ (rad/s). It is supported by a massless gimbal that is mounted on the platform as shown. The gimbal is initially stationary relative to the platform, which rotates with a constant angular velocity of $\boldsymbol{\omega}_p = 0.5\hat{\mathbf{j}}$ (rad/s). What will be the gimbal's angular acceleration when the torquer applies a torque of $600\hat{\mathbf{i}}$ (N-m) to the flywheel?
{Ans.: $70\hat{\mathbf{i}}$ rad/s^2}



9.19 A uniform slender rod of length L and mass m is attached by a smooth pin at O to a vertical shaft that rotates at constant angular velocity ω. Use Euler's equations and the body frame shown to calculate ω at the instant shown.
{Ans.: $\omega = \sqrt{3g/(2L\cos\theta)}$}



9.20 A uniform, thin circular disk of mass 10 kg spins at a constant angular velocity of 630 rad/s about axis OG, which is normal to the disk, and pivots about the frictionless ball joint at O. Neglecting the mass of the shaft OG, determine the rate of precession if OG remains horizontal as shown. Gravity acts down, as shown. G is the center of mass, and the y-axis remains fixed in space. The moments of inertia about G are $I_{G_z} = 0.02812 \text{ kg} \cdot \text{m}^2$ and $I_{G_x} = I_{G_y} = 0.01406 \text{ kg} \cdot \text{m}^2$.
{Ans.: 1.38 rad/s}



Section 9.7

9.21 Consider a rigid body experiencing rotational motion associated with an angular velocity $\boldsymbol{\omega}$. The inertia tensor (relative to body-fixed axes though the center of mass G) is

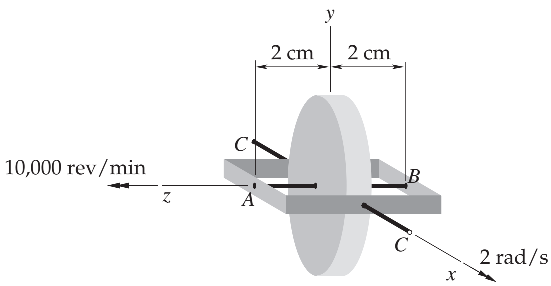
$$\begin{bmatrix} 20 & -10 & 0 \\ -10 & 30 & 0 \\ 0 & 0 & 40 \end{bmatrix} (\text{kg-m}^2)$$

and $\boldsymbol{\omega} = 10\hat{\mathbf{i}} + 20\hat{\mathbf{j}} + 30\hat{\mathbf{k}}$ (rad/s). Calculate
(a) the angular momentum $\mathbf{H}_G$ and
(b) the rotational kinetic energy (about G).
{Partial Ans.: (b) $T_R = 23{,}000$ J}

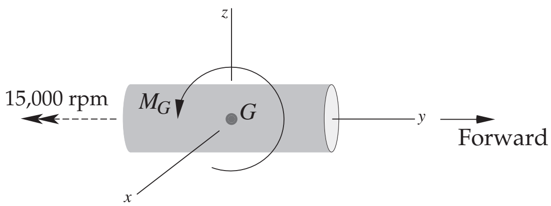
Section 9.8

9.22 At the end of its take-off run, an airplane with retractable landing gear leaves the runway with a speed of 130 km/h. The gear rotates into the wing with an angular velocity of 0.8 rad/s with the wheels still spinning. Calculate the gyroscopic bending moment in the wheel bearing B. The wheels have a diameter of 0.6 m, a mass of 25 kg, and a radius of gyration of 0.2 m.
{Ans.: 96.3 N m}

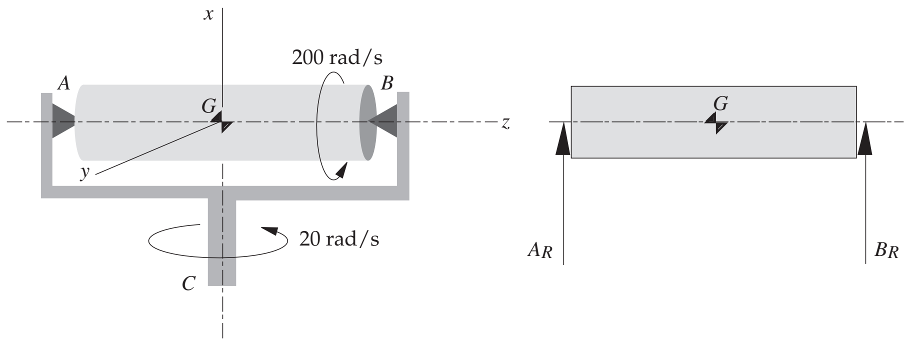
9.23 The gyro rotor, including shaft *AB*, has a mass of 4 kg and a radius of gyration 7 cm around *AB*. The rotor spins at 10,000 rpm while also being forced to rotate around the gimbal axis *CC* at 2 rad/s. What are the transverse forces exerted on the shaft at *A* and *B*? Neglect gravity.
{Ans.: 1.03 kN}



9.24 A jet aircraft is making a level, 2.5-km radius turn to the left at a speed of 650 km/h. The rotor of the turbojet engine has a mass of 200 kg, a radius of gyration of 0.25 m, and rotates at 15,000 rpm clockwise as viewed from the front of the airplane. Calculate the gyroscopic moment that the engine exerts on the airframe and specify whether it tends to pitch the nose up or down.
{Ans.: 1.418 kN-m; pitch down}



9.25 A cylindrical rotor of mass 10 kg, radius 0.05 m, and length 0.60 m is simply supported at each end in a cradle that rotates at a constant 20 rad/s counterclockwise as viewed from above. Relative to the cradle, the rotor spins at 200 rad/s counterclockwise as viewed from the right (from *B* toward *A*). Assuming that there is no gravity, calculate the bearing reactions R_A and R_B. Use the comoving *xyz* frame shown, which is attached to the cradle but not to the rotor.
{Ans.: $R_A = -R_B = 83.3$ N}



Section 9.9

9.26 The Euler angles of a rigid body are $\phi = 50°$, $\theta = 25°$, and $\psi = 70°$. Calculate the angle (a positive number) between the body-fixed x-axis and the inertial X-axis.
{Ans.: 115.6°}